

PRESERVATION OF CONDITIONALLY PERIODIC MOVEMENTS WITH SMALL CHANGE IN THE HAMILTON FUNCTION*

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ABSTRACT

This paper is a translation of Kolmogorov's original article announcing the theorem now known as the KAM theorem.

THEOREM AND DISCUSSION OF PROOF

Let us consider in the $2s$ -dimensional phase space of a dynamic system with s degrees of freedom the region G , represented as the product of an s -dimensional torus, T , and a region S , of a Euclidean s -dimensional space. We will designate the points of the torus, T , by the circular coordinates $q_1, \dots, q_s$ (replacing q_a with $q'_a = q_a + 2\pi$ does not change points q), and the coordinates of the points, p , of S we will designate as $p_1, \dots, p_s$. We will assume that in region G , in the coordinates $(q_1, \dots, q_s, p_1, \dots, p_s)$ the equations of motion have the canonical form

$$\frac{dq_a}{dt} = \frac{\partial}{\partial p_a} H(q, p), \quad \frac{dp_a}{dt} = - \frac{\partial}{\partial q_a} H(q, p). \quad (1)$$

The Hamilton function, H , is further assumed as dependent on the parameter θ and determined for all $(q, p) \in G$, $\theta \in (a, b)$, but not time-dependent. Moreover, further considerations require fairly significant restrictions on the smoothness of the function $H(q, p, \theta)$, stronger than infinite differentiability. For simplicity, in the following it is assumed that the function $H(q, p, \theta)$ is analytic over the set of variables (q, p, θ) .

Summation over the Greek indices is further assumed to be from 1 to s . The usual vector designations $(x, y) = \sum_a x_a y_a$, $|x| = + \sqrt{(x, x)}$ are used. A whole number vector indicates a vector for which all the components are whole numbers. The set of points (q, p) of G with

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$p = s$ are designated by T_0 . In theorem 1 it is assumed that S contains the point $p = 0$, i.e., $T_0 \subseteq S$.

Theorem 1. Let

$$H(q, p, 0) = m + \sum_a \lambda_a p_a + \frac{1}{2} \sum_{a, \beta} \varphi_{a\beta}(q) p_a p_\beta + o(|p|^3), \quad (2)$$

where m and λ_a are constants and for a certain choice of constants $c > 0$ and $n > 0$ for all whole-number vectors, n , the inequality

$$|(n, \lambda)| \geq \frac{c}{|n|^n} \quad (3)$$

is satisfied.

Let, moreover, the determinant composed of the average values

$$\varphi_{a\beta}(0) = \frac{1}{(2\pi)^S} \int_0^{2\pi} \dots \int_0^{2\pi} \varphi_{a\beta}(q) dq_1 \dots dq_S$$

of the functions

$$\varphi_{a\beta}(q) = \frac{\partial^2}{\partial p_a \partial p_\beta} H(q, 0, 0)$$

be different from zero:

$$|\varphi_{a\beta}(0)| \neq 0. \quad (4)$$

Then there exist analytical functions $F_a(Q, P, \theta)$ and $G_a(Q, P, \theta)$ which are determined for all sufficiently small θ and for all points (Q, P) of some neighborhood, V , of the set T_0 , which bring about a contact transformation

$$q_a = Q_a + \theta F_a(Q, P, \theta), \quad p_a = P_a + \theta G_a(Q, P, \theta)$$

of V into $V' \subseteq G$, which reduces H to the form

$$H = M(\theta) + \sum_a \lambda_a P_a + o(|P|^2) \quad (5)$$

($M(\theta)$ does not depend on Q and P).

It is easy to grasp the meaning of Theorem 1 for mechanics. It indicates that an s -parametric family of conditionally periodic motions

$$q_a = \lambda_a t + q_a^{(0)}, \quad p_a = 0,$$

which exists at $t = 0$ cannot, under conditions (3) and (4), disappear as a result of a small change in the Hamilton function H : there occurs only a displacement of the s -dimensional torus, T_0 , around which the trajectories of these motions run, into the torus $P = 0$, which remains filled by the trajectories of conditionally periodic motions with the same frequencies $\lambda_1, \dots, \lambda_s$.

The transformation

$$(Q, P) = K_0(q, p),$$

the existence of which is confirmed in Theorem 1, can be constructed in the form of the limit of the transformations

$$(Q^{(k)}, P^{(k)}) = K_0^{(k)}(q, p),$$

where the transformations

$$(Q^{(1)}, P^{(1)}) = L_0^{(1)}(q, p), \quad (Q^{(k+1)}, P^{(k+1)}) = L_0^{(k+1)}(Q^{(k)}, P^{(k)})$$

are found by the "generalized Newton method" (see Ref. 1). In this note we confine ourselves to the construction of the transformation: $K_0^{(1)} = L_0^{(1)}$, which itself permits grasping the role of conditions (3) and (4) of Theorem 1. Let us apply the transformation $L_0^{(1)}$ to the equations

$$\begin{aligned} \dot{Q}_\alpha^{(1)} &= q_\alpha + \epsilon Y_\alpha(q), \\ \dot{P}_\alpha &= P_\alpha^{(1)} - \epsilon \left\{ \sum_\beta P_\beta^{(1)} \frac{\partial Y_\beta}{\partial q_\alpha} + \xi_\alpha + \frac{\partial}{\partial q_\alpha} X(q) \right\} \end{aligned} \quad (6)$$

(it is easy to verify that this is a contact transformation) and seek the constants ξ_α and ϵ and the functions $X(q)$ and $Y_\beta(q)$, starting from the requirement that

$$\begin{aligned} H &= m + \sum_\alpha \lambda_\alpha P_\alpha + \frac{1}{2} \sum_{\alpha, \beta} a_{\alpha\beta}(q) P_\alpha P_\beta + \\ &+ \epsilon \left\{ \Lambda(q) + \sum_\alpha B_\alpha(q) P_\alpha \right\} + o(|P|^3 + o(|q|^2 + \epsilon^2)) \end{aligned} \quad (7)$$

take the form

$$H = \tilde{H} + \epsilon \zeta + \sum_\alpha \lambda_\alpha P_\alpha^{(1)} + o(|P^{(1)}|^2 + \epsilon^2). \quad (8)$$

Substituting (6) into (7), we get

$$m = m + \sum_{\alpha} \lambda_{\alpha} p_{\alpha}^{(1)} + \theta \left\{ A + \sum_{\alpha} \lambda_{\alpha} \left(\xi_{\alpha} + \frac{\partial X}{\partial q_{\alpha}} \right) \right\} + \\ + \theta \sum_{\alpha} p_{\alpha}^{(1)} \left\{ B_{\alpha} + \sum_{\beta} \varphi_{\alpha\beta}(q) \left(\xi_{\beta} + \frac{\partial X}{\partial q_{\beta}} \right) + \sum_{\beta} \lambda_{\beta} \frac{\partial Y_{\beta}}{\partial q_{\beta}} \right\} + O(|p^{(1)}|^2 + \theta^2).$$

Thus, our requirement (8) reduces to the equations

$$A + \sum_{\alpha} \lambda_{\alpha} \left(\xi_{\alpha} + \frac{\partial X}{\partial q_{\alpha}} \right) = \zeta \quad (9)$$

$$B_{\alpha} + \sum_{\beta} \varphi_{\alpha\beta} \left(\xi_{\beta} + \frac{\partial X}{\partial q_{\beta}} \right) + \sum_{\beta} \lambda_{\beta} \frac{\partial Y_{\beta}}{\partial q_{\beta}} = 0, \quad (10)$$

being fulfilled.

Let us introduce the functions

$$Z_{\alpha}(q) = \sum_{\beta} \varphi_{\alpha\beta}(q) \frac{\partial}{\partial q_{\beta}} X(q). \quad (11)$$

Expanding the functions $\varphi_{\alpha\beta}$, λ , B_{α} , X , Y_{α} , Z_{α} into a Fourier series of the type

$$X(q) = \sum x(n) e^{i(n, q)}$$

and assuming for definiteness that

$$x(0) = 0, \quad y(0) = 0, \quad (12)$$

we get for the remaining Fourier coefficients $x(n)$, $y_{\alpha}(n)$, and $z_{\alpha}(n)$ and constants ξ_0 and ζ of the equation which are relevant to the determination

$$a(0) + \sum_{\alpha} \lambda_{\alpha} \xi_{\alpha} = \zeta, \quad (13)$$

$$a(n) + (n, \lambda) x(n) = 0 \quad \text{for } n \neq 0, \quad (14)$$

$$b_{\alpha}(0) + \sum_{\beta} \varphi_{\alpha\beta}(0) \xi_{\beta} + z_{\alpha}(0) = 0, \quad (15)$$

$$b_{\alpha}(n) + \sum_{\beta} \varphi_{\alpha\beta}(n) \xi_{\beta} + z_{\alpha}(n) + (n, \lambda) y_{\alpha}(n) = 0 \quad \text{for } n \neq 0. \quad (16)$$

It is easy to see that the system (11) - (16) is unambiguously

solved under conditions (3) and (4). Condition (3) is important in the determination of $x(n)$ from (14), and in the determination of $y_\alpha(n)$ from (16). Condition (4) is important in the determination of x_p from (15). Since, as $|n|$ increases, the coefficients of the Fourier series of the analytical functions $\varphi_{\alpha\beta}$, A_n and B_n have an order of decrease not less than $O(|n|^{-s})$, $s > 1$, then from condition (3) there results not only the formal solvability of equations (13) - (16) but also the convergence of the Fourier series for the functions x , x_α , and y_α and the analyticity of these functions. The construction of further approximations is not associated with new difficulties. Only the use of condition (3) for proving the convergence of the recursions, $K_g^{(k)}$, to the analytical limit for the recursion K_g is somewhat more subtle.

The condition of the absence of "small denominators" (3) should be considered, "generally speaking," as fulfilled, since for any $n > s - 1$ for all points of an s -dimensional space $\lambda = (\lambda_1, \dots, \lambda_s)$ except the set of Lebesgue measure zero it is possible to find $c(\lambda)_n$ for which

$$(n, \lambda) > \frac{c(\lambda)}{|n|^s}$$

whatever the integers $n_1, n_2, \dots, n_s$ were². It is also natural to consider condition (4) as "generally speaking," fulfilled. Since

$$\varphi_{\alpha\beta}(0) = \frac{\partial}{\partial p_\alpha} \lambda_\beta(0),$$

where

$$\lambda_\beta(p) = \frac{1}{(2\pi)^s} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{dq_\beta}{ds} dq_1 \dots dq_s$$

is the frequency averaged over the coordinate q_β with fixed momenta $p_1, \dots, p_s$, condition (3) means that the Jacobian of the average frequencies over the momenta is different from zero.

Let us turn now to a consideration of the special case where $H(q, p, 0)$ depends only on p , i.e., $H(q, p, 0) = \pi(p)$. In this case, for $\theta = 0$ each torus, T_p , consists of the complete trajectories of the conditionally periodic movements with frequencies

$$\lambda_\alpha(p) = \frac{\partial W}{\partial p_\alpha}.$$

If the Jacobian

$$J = \left| \frac{\partial \Lambda}{\partial p} \right| = \left| \frac{\partial^2 W}{\partial p_a \partial p_b} \right| \quad (17)$$

is different from zero, then it is possible to apply Theorem 1 to almost all tori, T_p . There arises the natural hypothesis that at small θ the "displaced tori" obtained in accordance with Theorem 1 fill a larger part of region G. This is also confirmed by Theorem 2, pointed out later. In the formulation of this theorem we will consider the region S to be bounded and will introduce into the consideration the set, M_θ , of those points $(q^{(0)}, p^{(0)}) \in G$ for which the solution

$$q_a(t) = q_a(q^{(0)}, p^{(0)}, \theta), \quad p_a(t) = p_a(q^{(0)}, p^{(0)}, \theta)$$

of the system of equations (1) with initial conditions

$$q_a(0) = q_a^{(0)}, \quad p_a(0) = p_a^{(0)}$$

leads to trajectories not moving out of region G with change in t from $-\infty$ to $+\infty$, and conditionally periodic with periods $\lambda_a = \lambda_a(q^{(0)}, p^{(0)}, \theta)$, i.e., it has the form

$$q_a(t) = \varphi_a(e^{i\lambda_1 t}, \dots, e^{i\lambda_s t}), \quad p_a(t) = \psi_a(e^{i\lambda_1 t}, \dots, e^{i\lambda_s t}).$$

Theorem 2. If $H(q, p, \theta) = W(p)$ and determinant (17) is not equal to zero in region S, then for $\theta \rightarrow 0$ the Lebesgue degree of the set M_θ converges to the complete degree of region S.

Apparently, in the usual sense of the phrase, "general case" is when the set M_θ at all positive θ is everywhere dense. In such a case the complications arising in the theory of analytical dynamic systems are indicated more specifically in my note.³

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