

# FLUID FRICTION AND IRREVERSIBILITY

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ABSTRACT. *A review on the Navier-Stokes equation and the viscosity relation with irreversibility of fluid flows.*

## 1. NS-EQUATIONS

“... we know that in fluids the elasticity of form is evanescent, while that of volume considerable”, [?].

The Navier-Stokes (NS) equations, incompressible and in a periodic container  $\Omega = [-\pi, \pi]^d$ ,  $d = 2, 3$ , deal with a velocity field which can be expressed in terms of its Fourier's coefficients as:

$$(1.1) \quad \mathbf{u}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{1}{2}d}} \sum_{\substack{\mathbf{0} \neq \mathbf{k} \in \mathbb{Z}^d, \\ c=1, \dots, d-1}} u_{\mathbf{k}}^c \mathbf{e}_{\mathbf{k}}^c e^{-i\mathbf{k} \cdot \mathbf{x}}$$

where  $\mathbf{e}_{\mathbf{k}}^c$ ,  $c = 1, \dots, d-1$ , are  $d-1$  unit 'elicity' vectors, pairwise orthogonal and orthogonal to  $\mathbf{k}$ , with  $\mathbf{e}_{-\mathbf{k}}^c = -\mathbf{e}_{\mathbf{k}}^c$ , and  $u_{-\mathbf{k}}^c = \overline{u_{\mathbf{k}}^c}$  are the complex harmonics of  $\mathbf{u}$ , with  $|u_{\mathbf{0}}^c| \equiv 0$  (fixing the baricenter).

The NS equations are, formally,  $\partial_{\mathbf{x}} \cdot \mathbf{u}(\mathbf{x}) = 0$  (incompressibility) and:

$$(1.2) \quad \partial_t \mathbf{u}(\mathbf{x}) = -(\mathbf{u}(\mathbf{x}) \cdot \partial_{\mathbf{x}}) \mathbf{u}(\mathbf{x}) + \nu \Delta \mathbf{u}(\mathbf{x}) - \partial_{\mathbf{x}} p(\mathbf{x}) + \mathbf{f}(\mathbf{x})$$

with  $\nu =$  kinematic viscosity. <sup>1</sup> Or, in terms of the  $u_{\mathbf{k}}^c$ :

$$(1.3) \quad \dot{u}_{\mathbf{k}}^c = \sum_{\substack{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k} \\ a, b}} T_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}}^{a, b, c} u_{\mathbf{k}_1}^a u_{\mathbf{k}_2}^b - \nu \mathbf{k}^2 u_{\mathbf{k}}^c + f_{\mathbf{k}}^c$$

$$T_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}}^{a, b, c} = -(\mathbf{e}_{\mathbf{k}_1}^a \cdot \mathbf{k}_2) (\mathbf{e}_{\mathbf{k}_2}^b \cdot \mathbf{e}_{\mathbf{k}}^c), \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$$

with  $\mathbf{k}^2 \stackrel{def}{=} \sum_{i=1, \dots, d} k_i^2$  and the forcing  $\mathbf{f} \neq \mathbf{0}$  will be supposed fixed once and for all and to act only on 'large scale':  $f_{\mathbf{k}}^c = 0$  unless  $0 < |\mathbf{k}| = \max_{i=1, \dots, d} |k_i| \leq k_f < \infty$ .

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<sup>1</sup>Notation  $\underline{u} \cdot \underline{v}$ , means scalar product of vectors, as also  $\mathbf{u} \cdot \mathbf{v}$ .

Without restriction, suppose  $\|\mathbf{f}\|^2 = \sum_{\mathbf{k},c} |f_{\mathbf{k}}^c|^2 = 1$ : so the only free parameter in the equation is viscosity  $\nu$ , whose inverse will also be called Reynolds number  $R = \nu^{-1}$ . The equation is called “INS”, or irreversible NS, if  $\nu > 0$ . The equation with  $\nu = 0, f = 0$  is Euler’s equation.

Equations (??) will be called *regularized* if  $|\mathbf{k}_1|, |\mathbf{k}_2|, |\mathbf{k}|$  are restricted to be  $\leq N$  and  $u_{\mathbf{k}}^c = 0$  for  $|\mathbf{k}| > N$ ; they will be denoted  $INS^N$ .

These are ODE’s on  $\mathcal{R}^{D_N}$  with  $D_N = (d-1)((2N+1)^d - 1)$  degrees of freedom; in time  $t \in (-\infty, +\infty)$  evolve an initial velocity field  $u$  into  $u(t) = S_t^N u$ . The following is a heuristic analysis of the stationary properties of the solutions in the limit  $N \rightarrow \infty$ .

## 2. FLOW MODELS

The  $INS^N$  will be referred as a “flow model” and we shall study properties of “physical” stationary states  $\mu^N$  for  $INS^N$  <sup>2</sup> defined as:

**Definition 2.1.** A “physical state”  $\mu_\nu^N$  is an ergodic stationary distribution for  $INS^N$  generated by the evolution, at given viscosity  $\nu$ , of “typical” initial data  $u$ , i.e. data randomly selected via a probability with (any) smooth density on  $R^{D_N}$ , [?, ?, ?]. And, given  $\nu$ , it is assumed that such distributions are “finitely many”. More generally, given an evolution equation  $E^N$  on  $R^{D_N}$  admitting a constant of motion  $P(u)$ , a physical state at  $P = q$  is an ergodic stationary distribution  $\mu_q^N$  generated by the evolution of initial data on the surface (supposed smooth) defined by the value(s) of the constant(s) of motion  $P(u) = q$  and randomly selected via a probability with (any) smooth density on the surface (“typical” data).

Incompressibility can be considered in  $INS^N$  as a constraint imposing that the temperature  $\theta$  follows the equation of state  $\Theta$  of the considered fluid:  $\theta = \Theta(\rho, p)$ . The constraint  $\rho = const$  could be realized as holonomic ([?, ?, ?]) on a larger phase space; but viscosity  $\nu > 0$  is physically ascribed to stochastic equalization of microscopic average forces <sup>3</sup> and the  $INS^N$  is, even formally, far from Hamiltonian.

Therefore it seems natural to consider the possibility that viscosity could be associated with other forces (i.e. other ‘flow models’): the stationary fluctuations in the fluid chaotic evolutions should average with effects equivalent, in suitable limits (e.g.  $N \rightarrow \infty$ ) and for a large class of observables, to those of the constant friction model  $INS^N$ , including at least “local observables”

<sup>2</sup>A stationary state for  $INS^N$  is a probability distribution  $\mu(du)$ , on phase space  $R^{D_N}$ , invariant under the time evolution  $u \rightarrow u(t) = S_t^N u$  generated by  $INS^N$ , i.e.  $\mu(S_t^N A) \equiv \mu(A)$  for all  $t$  and open sets  $A$ .

<sup>3</sup>Maxwell: “. . . , however, this equalization of motion is not instantaneous, the pressures in all directions are perfectly equalized only in the case of a gas at rest, but when the gas is in a state of motion, the want of perfect equality in the pressures gives rise to the phenomena of viscosity or internal friction”, [?].

$O(u)$  depending only on components  $u_{\mathbf{k}}$  with  $|\mathbf{k}| \leq k_O$  for a  $N$ -independent scale  $k_O$  (often called “large scale” observables ).

In general, in analogy with equilibrium Statistical Mechanics (see appendix ?? below), several evolution models, *i.e.* different fluid evolution equations on  $R^{DN}$ , can be imagined. In spite of the differences, a correspondence between stationary probability distributions on phase space, called *states*, for different evolution equations *may* exist such that, in the limit  $N \rightarrow \infty$ , corresponding states attribute the same averages to each local observable, *i.e.* attribute the same statistics to local observables.

Let  $\mathcal{E}_{INS}^N$  be the collection of  $INS^N$  physical states  $\mu_\nu^{N,INS}(du)$  parameterized by  $\nu$ , and call it *viscosity ensemble* for an incompressible fluid. Set:

**Definition 2.2.** For an evolution equation on  $R^{DN}$  in which a few selected global (*i.e.* depending on all degrees of freedom) observables  $P(u)$  are constants of motion, call  $\mathcal{E}^N$  the collection of the physical states, ??,  $\mu_q^N \in \mathcal{E}^N$  labeled by the values  $q = P(u)$  of the selected observables.

The equation will be called a *flow model* if a correspondence, denoted symbolically  $\sim$ , established between pairs of states  $\mu_p^N \in \mathcal{E}^N$  and  $\mu_\nu^{N,INS} \in \mathcal{E}_{INS}^N$ , exists such that:

$$(2.1) \quad \mu_p^N \sim \mu_\nu^{N,INS} \quad \text{implies} \quad \mu_p^N(O) - \mu_\nu^{N,INS}(O) \xrightarrow{N \rightarrow \infty} 0.$$

for all local observables  $O$  ?? .

Questions: are there flow models other than  $INS^N$  itself? how to test whether an equation on  $R^{DN}$  is a flow model? can a flow model different from  $INS^N$  provide new information on turbulence or irreversibility? and on the “millennium problem”?

### 3. EQUIVALENCE CONJECTURES

Here three different ensembles will be considered, governed, respectively by different equations and conjectured to be “flow models” in the sense of Definition 2 ?? . They are listed (out of a far from exhaustive list) as follows.

- (1) Equation (or “flow model”) in which the *viscosity* is exactly constant is the regularized NS,  $INS^N$ , in Eq.(??) leading to the collection, called *viscosity ensemble*  $\mathcal{E}_{INS}^N$ , of the physical states  $\mu_\nu^N$  parameterized by the viscosity value  $\nu$ , [?].
- (2) Equation (or “flow model”) in which  $En = \sum_{|\mathbf{k}| \leq N,c} \mathbf{k}^2 |u_{\mathbf{k}}^c|^2$  is an exactly constant observable: it can be simply obtained by modifying  $\nu$  in Eq.(??) (with the  $\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2$  restricted by the cut off  $N$ ) into  $\alpha(u)$  as:

$$(3.1) \quad \alpha(u) = \frac{\sum_{\mathbf{k},c} \mathbf{k}^2 f_{\mathbf{k}}^c u_{\mathbf{k}}^c}{\sum_{\mathbf{k},c} (\mathbf{k}^2)^2 |u_{\mathbf{k}}^c|^2}, \quad \text{if } d = 2; \quad \text{or if } d = 3 :$$

$$\alpha(u) = \frac{\sum_{\substack{\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k} = 0 \\ a,b,c}} T_{\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}}^{a,b,c} \mathbf{k}^2 u_{\mathbf{k}_1}^a u_{\mathbf{k}_2}^b u_{-\mathbf{k}}^c + \sum_{\mathbf{k},c} \mathbf{k}^2 f_{\mathbf{k}}^c u_{-\mathbf{k}}^c}{\sum_{\mathbf{k},c} (\mathbf{k}^2)^2 |u_{\mathbf{k}}^c|^2},$$

where the sums run over  $|\mathbf{k}_1|, |\mathbf{k}_2|, |\mathbf{k}| \leq N$ ; if  $d = 2$  the (non local) observable  $\alpha$  is simpler because in it the term cubic in  $u$  is  $\equiv 0$ <sup>4</sup>.

The equations obtained by replacing  $\nu$  with  $\alpha$  as in Eq.(??), [?] will be called  $RNS^N$  and the collection of the physical distributions  $\mu_{En}^N$ , parameterized by the value  $En = \sum_{\mathbf{k},c} \mathbf{k}^2 |u_{\mathbf{k}}^c|^2$  of the enstrophy of the initial  $u$ , will be denoted  $\mathcal{E}_{RNS}^N$ , called *enstrophy ensemble*.

- (3) Equation (or “fluid model”) in which the *energy*,  $E = \sum_{\mathbf{k},c} |u_{\mathbf{k}}^c|^2$ , is an exactly constant global observable is simply obtained by modifying  $\nu$  in Eq.(??) with  $\alpha(u)$ , [?]:

$$(3.2) \quad \alpha(u) = \frac{\sum_{\mathbf{k} \leq N, c} f_{-\mathbf{k}}^c \cdot u_{\mathbf{k}}^c}{\sum_{\mathbf{k}, c} \mathbf{k}^2 |u_{\mathbf{k}}^c|^2}, \quad d = 2, 3$$

they will be called  $RNSE^N$ . The stationary distributions  $\tilde{\mu}_E^N$  of the new equation form a collection  $\tilde{\mathcal{E}}_{RNSE}^N$ , called *energy ensemble*, whose elements are parameterized by the value of the total energy  $E$  of the initial data.

In analogy with the theory of ensembles in Statistical Mechanics, see Appendix ??, each of the above three evolution equations are conjectured to provide alternative flow models describing equivalently (in the sense of Definition 2) a “viscous” incompressible fluid.

It should be stressed that the above conjectures put irreversibility in a different light: if  $S_t^N u$  is the evolution of  $u$  with any of the equations in the above items 2,3 and if  $Iu = -u$  is the “time-reversal”, it is:

$$(3.3) \quad S_t^N Iu = IS_{-t}^N u, \quad t \in (-\infty, +\infty)$$

Hence the conjectures imply that the same system can be equally considered to be controlled by a reversible or an irreversible dynamics<sup>5</sup> and might induce a complementary interpretation of viscosity.

#### 4. POSSIBLE TESTS

Tests of the conjecture of equivalence between  $INS^N$  and  $RNS^N$  have been attempted and published. Unfortunately still based on few simulations, mostly in 2 dimensions.

- (1) Fix  $\nu$  and choose randomly a typical, in the sense of ??, initial  $u$  and compute the  $INS^N$  average  $En(\nu, t, N)$  of the enstrophy over a long time interval  $[0, t]$ , so long that the  $(t = \infty)$ -average  $\overline{En}(\nu, N)$  can be considered reached. The observable  $O = \sum_{|\mathbf{k}| \leq k_f, c} \mathbf{k}^2 f_{\mathbf{k}}^c u_{\mathbf{k}}^c$  is a local observable (because by assumption the forcing  $f$  has only modes  $|\mathbf{k}| \leq k_f$  different from 0):

<sup>4</sup>As a consequence of the enstrophy conservation in Euler flows, implied by the identity  $\int [(u \cdot \partial)u] \cdot \Delta u \, dx \equiv 0$ , valid if  $d = 2$  and  $\partial \cdot u = 0$ .

<sup>5</sup>Perhaps not surprising, because fundamental microscopic equations are reversible.

therefore the average  $\mu_\nu^{N,INS}(O)$  and the average  $\mu_{\overline{En}(\nu,N)}^{N,RNS}(O)$  should agree in the limit  $N \rightarrow \infty$ .

The equations of motion imply that the average of  $O$  over  $t \in [0, \infty]$  is in the first case  $\nu \overline{En}(\nu, N)$  while in the second it is  $\mu_{\overline{En}(n,N)}^{N,RNS}(\alpha) \overline{En}(n, N)$ : *i.e.*

$$(4.1) \quad \lim_{N \rightarrow \infty} \mu_{\overline{En}(\nu,N)}^{N,RNS}(\alpha) = \nu$$

follows from the equivalence (in appendix ??, as an example of the test, in Fig.1 is presented the output that is obtained in a simple simulation, [?]).

(2) Fix a local observable, *e.g.*  $O(u) = |u_{\mathbf{k}_0}|^2$  selecting  $\mathbf{k}_0 \in Z^d$ , then

$$(4.2) \quad \lim_{N \rightarrow \infty} \mu_{\overline{En}(\nu,N)}^{N,RNS}(O) = \mu_\nu^{N,INS}(O)$$

is implied by the conjecture.

Easy tests of the type (1),(2) can be performed in several other cases in  $d = 2$  starting with a 1 mode forcing ( $f_{\mathbf{k}} = 0$  unless  $\mathbf{k} = \pm(2, -1)$ ) as in the example in appendix ?? Fig.1); for a few other cases see [?, ?, ?].

The case  $d = 3$  is studied, [?], in the ensemble  $\mathcal{E}_{RNS}^N$  in which the energy is kept constant with  $\alpha(u)$  defined via Eq.(??): this is much harder than the examples in appendix ?? because  $D_N$  goes up to  $\sim 1.2 \cdot 10^6$ . Compatibility with the conjecture is observed for  $\nu$  small (*i.e.* in presence of chaos).

Tests (1),(2) are also studied with  $d = 3$  in [?] with results, on models  $INS^N, RNS^N$  with 26-modes fixed forcing (random  $f_{\mathbf{k}} \neq 0$ , if  $\mathbf{k}^2 < 2$ ), and  $N = 21, 42, 85$  (*i.e.* up to  $\sim 1 \cdot 10^7$  modes) and  $\nu^{-1} = 2 \cdot 10^2, 10^3, 10^4, 10^5$ . The results in the tests are in good agreement with the conjectures if the average  $\bar{\alpha}$ , with  $\alpha(u)$  defined in Eq.(??), is compared with the corresponding  $\nu$  for several  $\nu, N$  (*i.e.* 1.%1000 accuracy for  $\nu < 1 \cdot 10^{-3}$  and for all  $N$  above), but are not easy to interpret for other observables (like the energy of the modes  $|\mathbf{k}| < k$  if  $k$  is below the Kolmogorov scale  $K_k$ <sup>6</sup> in cases in which  $\nu$  is so small that the cut-off  $N$  sets limits only to  $|\mathbf{k}|$ 's above  $K_k$  scale, as commented below.

- [i]: one possible interpretation is that the conjectures validity should be restricted to observables  $O(u)$  depending on  $u_{\mathbf{k}}^c$  with  $|\mathbf{k}| < const K_k$ . In other words the “local observable” definition, ??, has to be considerably restricted.

- [ii]: other interpretation: the average of  $\alpha$  in the  $RNS^N$  agrees with high accuracy with the  $\nu$  of the corresponding  $INS^N$  if  $10^2 < \nu^{-1} < 10^3$  and  $N$  is not too large, see the final table of data in [?]. The disagreement could be explained because  $N$  is not large enough or the integration step  $h$  of the simulations not small enough or the run time too short.

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<sup>6</sup>  $K_k = (\eta \nu^{-3})^{1/4}$ , with  $\eta = \nu \overline{En}$  being the energy dissipation rate, is the typical inverse length where inertial and *irreversible* dissipative effects balance.

Furthermore it is very strange that, in the  $RNS^N$  after an initial transient, in all cases  $\alpha(S_t^N u) \geq \varepsilon > 0$ , with  $\varepsilon > .8$ .

If really, after a transient, it is  $\alpha(S_t^{N,RMS} u) > \varepsilon > 0$  then an *autoregularization* phenomenon sets in and the evolving  $S_t^N u$  in  $RNS^N$  remains  $C^\infty$  with  $N$ -independent bounds, see the theorem in [?, appendix B] <sup>7</sup>

•[iii]: The  $\alpha(S_t^N u) > \varepsilon > 0$  in  $RNS^N$  in dimension 3, is observed after a transient time and for all later  $t$ , but can be expected to be very difficult to prove (if true) as long as the equivalence conjecture of  $INS$  and  $RNS$  holds, see Sec. ??.

## 5. CHAOTIC HYPOTHESIS

In view of the conjectures in Sec.?? the “chaotic hypothesis” establishes a link with the theory of dynamical systems which allows a better understanding of the conjectures implications:

**Chaotic hypothesis (CH):** *A hyperbolic evolution  $S_t$  on a smooth compact surface  $M$  (“phase space”), is called chaotic. If, in  $INS, RNS, RNSE$  systems,  $S_t$  is restricted to a transitive attracting set  $\mathcal{A}$ , <sup>8</sup> it can be regarded as an Anosov system, [?, ?], for the purpose of studying statistical properties of the stationary states. Furthermore the phase space is supposed to contain at most a finite number of attracting sets  $\mathcal{A}_i, i = 1, 2, \dots$  and all points in  $M$ , but a set of 0 volume, are attracted by  $\cup_i \mathcal{A}_i$ .*

It will supposed to hold for chaotic fluid evolution <sup>9</sup>: for each value of the parameter  $q$  defining chaotic stationary states  $\mu_q^N$ , in any of the ensembles  $\mathcal{E}_{NS}^N, \mathcal{E}_{RNS}^N, \mathcal{E}_{RNSE}^N$ ,  $N$  large enough, the number of attractors is finite and the dynamical systems  $(S_t^N, \mu_p^N)$  are “Anosov systems”, [?, ?].

If  $INS$  and  $RNS$  (or  $RNSE$ ) are equivalent (in the sense of Sec.??), then the same statistical properties of the local observables would be exhibited (in the  $N \rightarrow \infty$  limit) by an irreversible evolution as well as by a reversible evolution. Reversibility would, therefore, arise even in intrinsically irreversible evolutions like  $INS^N$ : the equivalence conjecture together with the chaotic hypothesis have interesting implications about the time reversal symmetry; for instance to “recover” reversibility it might be not necessary to appeal to

<sup>7</sup>Which is adapted from the  $INS$  result that if enstrophy has an upper bound then the solution is smooth, [?].

<sup>8</sup>Attracting set or surface: let  $\mathcal{A}$  be a smooth bounded surface in a phase space,  $M$  for a flow  $S_t$  or  $\Xi$  for a map  $S$ . Suppose existence of constants  $\varepsilon_0, \varepsilon, c > 0$  such that if  $x \in M$  and  $\delta(x, \mathcal{A}) \leq \varepsilon_0$  then  $\delta(S_t x, \mathcal{A}) \leq \varepsilon e^{-ct}$ ,  $t \geq 0$ , or if  $x \in \Xi$  then  $\delta(S^n x, \mathcal{A}) \leq \varepsilon e^{-cn}$ ,  $n \geq 0$ , furthermore  $\mathcal{A}$  contains a point with orbit dense in  $\mathcal{A}$ .  $\mathcal{A}$  will be called an *attracting surface* (or an *attracting set*) and the set  $U = \{x, \delta(x, \mathcal{A}) < \varepsilon_0\}$  will be an *attraction domain* for  $\mathcal{A}$ .

<sup>9</sup>Chaotic = hyperbolic = with positive as well as negative characteristic exponents.

microscopic reversibility of Newton's equations and to the continuum limit theory. Macroscopic evolution may be compatible with reversibility.

### 6. TIME REVERSAL

In the case of two corresponding stationary states  $\mu_{INS,\nu}^N$  and  $\mu_{RNS,\overline{En}}^N$  an immediate (too often not addressed) *warning* is necessary: even if the attractor is unique (as it is often the case) time reversal, violated in the  $INS^N$ -case, can remain a symmetry of the evolution on the attractor in  $RNS^N$  only if the attracting surface  $\mathcal{A}$  is the *entire phase space*, *i.e.* (dimension of  $\mathcal{A}$ )= $D_N$ . This can be expected if its dimension is  $\sim D_N$ , *i.e.* full: otherwise the set  $I\mathcal{A} \neq \mathcal{A}$  will be a repeller for  $S_t^N$ , and an attracting surface for  $S_{-t}^N$ : and time reversal symmetry  $I$  will be, manifestly, broken (on  $\mathcal{A}$ ).

A check of the validity of the chaotic hypothesis together with time-reversal can be based on a property of the phase space contraction  $\sigma(u) = \sum_{\mathbf{k},c} \frac{\partial \dot{u}_c}{\partial u_c} = \text{Tr} \frac{\partial \dot{u}}{\partial u}$  and of its time average  $\bar{\sigma} = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \sigma(S_\tau u) d\tau$ . A classic result on  $\sigma(u)$ , in a time reversal symmetric Anosov system, is that the variable:

$$(6.1) \quad p = \frac{1}{t} \int_0^t \frac{\sigma(S_\tau u)}{\bar{\sigma}} d\tau$$

has a probability distribution proportional to  $e^{-t\bar{\sigma}\zeta(p)+o(t)} dp$ , [?, ?, ?]. The chaotic hypothesis together with time-reversal symmetry imply:

**Fluctuation theorem (FT):** *Given a time reversible Anosov system with  $\bar{\sigma} > 0$ , there is  $\bar{p} > 1$  such that the large deviations function  $\zeta(p)$ , controlling the fluctuations of  $p$ , Eq.(??), verifies:*

$$(6.2) \quad \zeta(-p) = \zeta(p) - p, \quad \text{for all } p \in (-\bar{p}, \bar{p})$$

with  $\zeta(p)$  convex and smooth [?].

Therefore the CH ?? on the reversible dynamical systems  $RNS^N$  could be tested checking whether the variable  $\sigma^N(u)$  has a large deviation function with linear slope 1 (“no free parameters”), and simultaneously the test would also check whether the attracting set  $\mathcal{A}$  fills phase space and is time-reversal symmetric (because reversibility is an essential condition for validity of FT).

In the literature careful studies on the INS are available which analyze 2-dimensional models with truncations up to  $N = 98$  with  $\nu^{-1}$  up to 600 and fixed forcing (with 1 or 3 modes) and  $\sum_{\mathbf{k}} |f_{\mathbf{k}}| = 1$ , [?]. From the latter reference a picture emerges, if  $\nu$  is fixed large enough, in which observations performed at different  $N$ 's, as  $N$  increases, show attracting sets that may evolve from several fixed points, periodic or quasi periodic at small  $N$  but, as  $N$  grows, eventually reduce to an apparently unique chaotic attractor (provided  $\nu^{-1} \geq \nu_c^{-1} \sim 600$ ).

The equivalence conjectures suggest that, fixed  $\nu$  small enough, as  $N$  grows above a threshold  $N_c$  chaos arises and, at the onset of chaos the attractor (initially consisting of periodic or quasi periodic orbits) becomes chaotic and covers the full phase space, *i.e.* acquires full dimension  $D_N$ , and maintains it until, further increasing  $N$ , the dimension of the attractor (which if the CH?? is accepted is an integer) is expected to become  $< D_N$ , with the consequence that *time reversal symmetry is broken*.

The simulation in [?] shows that in a simple model with  $D_N = 48$ -modes, 'monochromatic' forcing on the  $\mathbf{k} = \pm(2, -1)$  and a large  $\nu^{-1} = 2048$ , the dimension of the attracting set  $\mathcal{A}$  (an integer by ??) is  $D_N$ , see next section. A working hypothesis for an interpretation is then: the fixed value of  $\nu$  is close to the onset of chaos for the chosen  $N$ , and  $\mathcal{A}$  is the full phase space (in  $RNS^N$  and, if the equivalence can be extended to the attractors, in corresponding states for  $INS^N$ ): time reversal symmetry holds in the RNS and the FT applies.[?]. Testing attracting sets dimensions and the FT is therefore interesting and discussed, more generally, in the following Sec.??.

## 7. ATTRACTING SURFACES DIMENSION

The dimension of the attracting set can be estimated via the average *local Lyapunov exponents*,<sup>10</sup>; in [?] it is seen to be  $D_N$  for  $RNS^N$  as well as for  $INS^N$ : easily inferred from the unexpected almost identity of the average local Lyapunov exponents, as reported in appendix?? Fig.2, for INS and RNS, and from the, also unexpected, "*pairing symmetry*" showing that the exponents can be arranged (approximately) in pairs  $\pm\lambda$  (hence implying dimension to be  $D_N$ , which is an integer by the CH??, and KY-dimension, [?], with integer part  $D_N$ ).

The simulation [?] also shows, surprisingly, that if the observable  $\sigma^N(u)$ , phase space contraction, defined for the  $RNS^N$  equation (hence depending on  $\alpha(u)$ <sup>11</sup>) is, instead, studied as an observable for the corresponding  $INS^N$  evolution, then it shows that the statistics of the new  $p$  has a large deviations function  $\tilde{\zeta}^N(p)$  verifying Eq.(??), *i.e.* a fluctuation relation like FT. Since the observable  $\sigma^N(u)$  is not local this is not implied by the equivalence conjecture and is non trivial.

**Remarks:** • The choice of the local exponents to test that the attracting set is the full phase space is natural as in the presence of pairing symmetry (as

<sup>10</sup> Local Lyapunov spectrum is defined by considering the Jacobian matrix of the evolution, formally the matrix  $J_{\mathbf{k},a;\mathbf{h},b} = \frac{\partial \dot{u}_{\mathbf{k}}^a}{\partial u_{\mathbf{h}}^b}$ , and then computing the eigenvalues of its symmetric part, in decreasing order, and averaging each one over the flow. A remarkable inequality links the average local exponents to the complete Lyapunov exponents, namely ordering both exponents in decreasing size  $\lambda_k$  and  $\lambda_k^{loc}$ , it is  $\sum_{k=1}^m \lambda_k \leq \sum_{k=1}^m \lambda_k^{loc}$ , [?, Eq.(1.7)],[?].

<sup>11</sup>For  $d = 2$  it is  $\sigma^N(u) = \alpha(u)(K2 - 2\frac{K6}{K4}) - \frac{FU}{K4}$ , with  $K2 = \sum_{\mathbf{k}} \mathbf{k}^2$ ,  $K6 = \sum_{\mathbf{k}} (\mathbf{k}^2)^3$ ,  $K4 = \sum_{\mathbf{k}} (\mathbf{k}^2)^2$ ,  $FU = \sum_{\mathbf{k}} (\mathbf{k}^2)^2 \bar{f}_{\mathbf{k}} u_{\mathbf{k}}$ .

in appendix??, Fig.2 and Fig.3, the attracting set cannot be time-reversal symmetric if the pairing level is not 0). For the same purpose (and conclusion) the real Lyapunov exponents can be used: however the computation time needed is much larger.

- From the graphs in appendix??, Fig.2 and Fig.3, it is easy to evaluate the Kaplan-Yorke dimension of the attracting set: which in Fig.2 is a fractal which is within the errors  $D_N$  (*i.e.* maximal) but it is  $< D_N$  in Fig.3.
- In [?] it is proposed that the attracting set dimension is maximal because of the *apparent* “pairing” between  $\lambda_k, \lambda_{n-1-k}$ : however this pairing is (approximately) realized only in a range of  $R = \nu^{-1}$  and  $N$ : if  $N$  is increased at fixed  $\nu$  the pairing line becomes  $< 0$  (from pairs  $\lambda, -\lambda$  to  $\lambda, -\lambda - \gamma$  with  $\gamma > 0$ ) still straight but with pairs reflection symmetric on a line at negative height  $-\frac{1}{2}\gamma$ , as in Fig.3 appendix??; but for  $N$  larger is expected to become *sensibly curved*.

The FT has been confirmed in the simple case of Fig.2 in appendix??, [?], but it could not yet be tested in the case of Fig.3 in appendix?? as it required more than the available computer power to achieve a reliable statistics. It is also puzzling to remark that the ideas that have permitted to prove the existence of a pairing symmetry for the characteristic exponents of several evolution equations, [?, ?], could not be used (so far) to obtain instances of pairing symmetry for fluid flows.

Performing the latter test in the case in Fig.3 will be interesting because time reversal is violated: although, by the chaotic hypothesis, it is supposed that the motion on the attracting set is an Anosov system, the FT holds for the phase space contraction rate *if measured on the attracting set: i.e.* discarding the contribution to the phase space contraction  $\sigma(u)$  from the directions contracting to the attracting set but external to it. Such rate is difficult to access because the attracting set is “strange” and so is the contraction of its surface elements.

However if the attracting set  $\mathcal{A}$  is not the full phase space (*i.e.* the KY dimension is  $< D_N - 1$ , as in Fig.3 in appendix??) there is still the possibility that FT can be applied, because it has been pointed out that time reversal  $I$  breakdown *could* arise together with the appearance of a new symmetry  $P$  such that  $PI = IP, S_t PI = PIS_{-t}$ : then  $PI$  maps  $\mathcal{A}$  into itself, hence acts as a time-reversal symmetry for the purpose of obtaining the FT. This has been formalized into an assumption called “Axiom C”, [?], (a sturcturally stable property, *i.e.* stable if the evolution is perturbed).

The FT for the PI symmetry, when the latter exists, would apply to the phase space contraction on  $\mathcal{A}$ : a property that can be tested because the Axiom C implies also that the phase space contraction on  $\mathcal{A}$  (in principle of difficult access, because “strange”) is proportional to the phase space contraction

on the full phase space via the factor  $\theta_{\mathcal{A}}$ =phase space dimension/dimension of  $\mathcal{A}$  and the FT should hold with slope replaced by  $\theta_{\mathcal{A}} > 1$ <sup>12</sup>.

It would be important to check validity of the FT in the case of appendix ?? Fig.3 and  $RNS^N$ : which shows that the attracting set has dimension  $< D_N$  and reversibility is a broken symmetry. So far I could not generate simulations long enough to obtain a statistically reliable error estimate on the  $\zeta^N(p)$ .

However there is a series of studies in which the  $INS^N$  is studied by restricting the  $\mathbf{k}$  values to a “log-lattice”, [?]: on such sparse lattices the equation can be studied under very large cut-off  $N$  and the check of the equivalence and of the FT can be studied in greater detail even in cases in which the attracting set dimension is smaller than that of phase space (and time-reversal breaks). The results are impressive and agree with the above relation between the slope  $\theta_{\mathcal{A}}$  and the KY-dimension of the attracting set in cases in which time-reversal is broken, [?, ?].

Similar studies could also be carried in the “shell models”: for such models, for which the analytical theory is well advanced [?] even in 3 dimensions, a systematic analysis of the conjectures as functions of  $N, \nu$  seems possible with modern computers. Likewise other models, like the Euler equation with Ekman viscosity, or even not dealing with fluids, like the Burgers’ equation or the Lorenz96 model, have been, or deserve to be, further investigated, [?, ?, ?, ?], in particular to see if Axiom C holds and plays any role in FT for systems which exhibit a time reversal broken symmetry.

## 8. GLOBAL NS EXISTENCE

In view of the analysis in [?] a natural question is: do the  $RNS_{E_n}^N$  evolutions remain smooth with derivatives uniformly bounded for all time?

As shown in the theorem in [?, appendix A]<sup>13</sup> the question could be (meaningfully) addressed if any physical initial  $u$  evolves under  $RNS^N$  so that after a transient time  $t_u$  the reversible viscosity  $\alpha(S_t^N u)$ , ?? is  $\alpha(S_t^N u) > \varepsilon$  for some  $\varepsilon > 0$  and  $\forall t \geq t_u$ .

If a  $N$ -independent constructive estimate of  $t_u$  for all physical  $u$  could be devised, it would be possible to have an entirely constructive algorithm for  $RNS^N$ , uniform in  $N$ , with obvious implications for the solutions of  $INS^N$ , and no need to consider the limit  $N \rightarrow \infty$  (uniform estimates being sufficient, like in Statistical Mechanics it is seldom necessary, or possible, to consider the limit of infinite volume).

By the conservation of difficulties principle, it is therefore not surprising that the existence of  $t_u$  is not proved yet, if at all true.

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<sup>12</sup>At least in the absence of correlation between the contraction to  $\mathcal{A}$  and the surface contraction on  $\mathcal{A}$ .

<sup>13</sup>A minor extension of the theorem that if enstrophy is uniformly bounded in  $t > 0$  then any initially smooth  $u$  remains uniformly smooth.

The question has been investigated numerically in 2 and 3 dimensions. Particularly in dimension 3 [?]: as mentioned above, in this case it seems that positivity of a lower bound  $\varepsilon > 0$  to  $\alpha(S_t^N u)$  might be related to the need that the conjecture in Sec.?? be amended by adding the condition that equivalence applies only to observables which depend on  $u_{\mathbf{k}}^c$  with  $\mathbf{k} < \text{const}K_k(\nu)$  where  $K_k(\nu)$  is Kolmogorov's scale?? associated with the dissipation. See two interpretations on the existence of the lower bound  $\varepsilon > 0$  on  $\alpha(u)$  in the remarks (i),(ii),(iii) in Sec.?. See also [?].

**Conclusions:** The equivalence conjectures in Sec.?? have been tested in  $d = 2$  in many cases: however a careful and more systematic study of the INS-RNS or INS-RNSE equivalence, at least for a few local observables, is needed in function of  $R, N$  and on the number of  $\mathbf{k}$ 's for which the driving force is  $f_{\mathbf{k}} \neq 0$ . The FT for RNS should be tested in cases in which the attracting set is not invariant under time reversal and the symmetry of the Lyapunov exponents, at least at the onset of chaos, should be understood. In  $d = 3$  it is essential to confirm that in RNS  $\alpha(S_t^{N,RNS} u) > 0$  is eventually true, a property left in an ambiguous state in [?].

## Appendices

### APPENDIX A. EQUIVALENCE IN STATISTICAL MECHANICS

- (1) for definitness consider the microcanonical and the isokinetic ensembles of  $N$  particles in a periodic container  $\Omega = [-L, L]^3$ , with  $N = \rho L^3$  for a fixed  $\rho$ . An initial datum  $X = (\mathbf{p}_i, \mathbf{q}_i)_{i=1}^N$  evolves into  $X(t)$  following

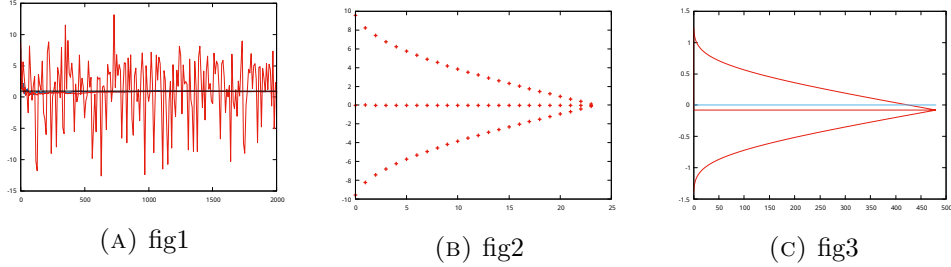
$$(A.1) \quad \dot{\mathbf{q}}_i = \mathbf{p}_i, \quad \dot{\mathbf{p}}_i = -\partial_{\mathbf{q}_i} V(\mathbf{q}) - \alpha \mathbf{p}_i$$

with  $V$  the total potential of the interparticle pair forces and  $-\alpha \mathbf{p}_i$  is a force chosen  $\alpha = 0$  in the microcanonical case and  $\alpha = -\sum_i (\mathbf{p}_i \cdot \partial_{\mathbf{q}_i} V(\mathbf{q})) / K$  with  $K = \frac{1}{2} \sum_i \mathbf{p}_i^2$  in the isokinetic case.

- (2) The time evolution, accepting the ergodic hypothesis, leads almost all physical data  $u$  with total energy  $E$  to a distribution  $\mu_E^L(d\mathbf{p}d\mathbf{q}) = Z^{-1} \delta(K + V - E) d\mathbf{p}d\mathbf{q}$ , while the isokinetic evolution leads (under an ergodic assumption and remarking that the isokinetic evolution rigorously conserves the total kinetic energy  $K$ ) to a stationary distribution, [?],  $\tilde{\mu}_K^L(\delta\mathbf{p}d\mathbf{q}) = \tilde{Z}^{-1} \delta(\sum_i \frac{1}{2} \mathbf{p}_i^2 - K) d\mathbf{p}d\mathbf{q}$  for almost all data with total kinetic energy  $K$ .
- (3) As  $E$  and  $K$  vary, the collections  $\mathcal{E}$  and  $\mathcal{K}$  of distributions  $\mu_E^L, \tilde{\mu}_K^L$  respectively parameterized by  $E, K$ , form the 'microcanonical' and 'isokinetic' ensembles in the volume  $\Omega = [-\frac{1}{2}L, \frac{1}{2}L]^3$ . Pairs  $\mu_E^L, \tilde{\mu}_K^L$  are said "corresponding" if  $\mu_E^L(\frac{1}{2} \sum_i \mathbf{p}_i^2) = K$ . The case of 'exceptional' pairs  $E, K$  which admit initial data  $X$  that evolve leading to different distributions are possible but will be discussed later.

- (4) It can be shown that corresponding  $\mu_E^L$  and  $\tilde{\mu}_K^L$  are equivalent as  $L \rightarrow \infty$  in the following sense: given any smooth observable  $O(\mathbf{p}, \mathbf{q})$  depending on  $(\mathbf{p}^B, \mathbf{q}^B)$  via coordinates of points located in  $B$ , where  $B$  is a ball centered at the origin, (“local observable”), then the averages  $\mu_E^L(O) - \tilde{\mu}_K^L(O) \xrightarrow{L \rightarrow \infty} 0$ .
- (5) Remark that there is no requirement of existence of the evolution of  $(\mathbf{p}, \mathbf{q})$  with infinitely many particles even if subject (as in the present case) to a constraint of bounded density. It has been a major success of Statistical Mechanics to achieve, in many physically important models, a rigorous proof that averages of local observables were related, in the infinite volume limit, to each other as required by Thermodynamic laws (when applicable). This sets aside the conceptual obstacle of having to deal with a theorem proving the existence of dynamics for systems with infinitely many particles. And even leads to situations in which, although no existence theorem of dynamics is available, physical consequences could simply be rigorously obtained via the, *far easier* to prove, existence of the limits in item (4). A few results studying simple cases of dynamics of infinitely many particles nevertheless exist<sup>14</sup>

#### APPENDIX B. SAMPLE SIMULATIONS



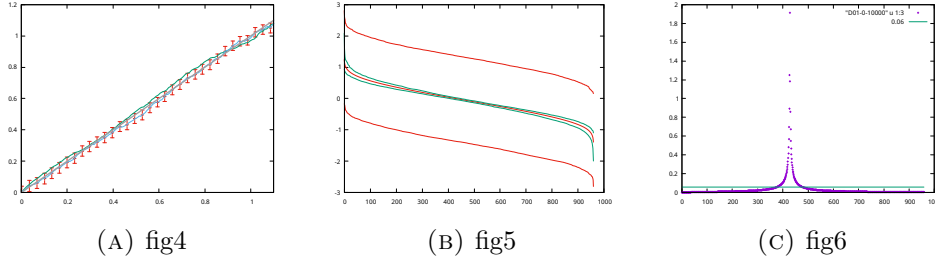
**Fig.1:** In evolution  $INS^N$  the observable  $\frac{\alpha(u(t))}{\nu} \stackrel{def}{=} \frac{2Re(f_{\mathbf{k}_0} u_{-\mathbf{k}_0}(t)) \mathbf{k}_0^2}{\nu \sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$  (red o.l.), strongly fluctuates as a function of time  $t$  (plotted at multiples of  $t$  every 10); superposed to its running average (also red o.l.); and also superposed to running average of  $\frac{\alpha}{\nu}$  in the alternative  $RNS^N$  evolution (blue o.l.); and finally the conjectured value 1 (black o.l.). Data:  $R = \nu^{-1} = 2048$ , 960 modes (i.e.  $N = 15$ ),  $\lambda_{max} = \max.$  local Lyapunov exp.  $\simeq 1.5$ , integration step  $h = 2^{-17}$ ,  $x$ -axis time unit  $4h$ , forcing  $f_{\mathbf{k}} = 0$  except for  $\mathbf{k} = \pm \mathbf{k}_0 = \pm(2, -1)$  where  $f_{\pm(2, -1)} = e^{\pm i\pi/3} / \sqrt{2}$ ; hence time  $t$  in abscissa corresponds to  $t * 2^{19}$  integration steps. The two running averages and the line 1 are not easy to distinguish on the graph scale, [?].

**Fig.2:** Local Lyapunov spectra for both  $NS_{irr}$  and  $NS_{rev}$  flows with  $d = 48$  modes ( $N = 7$ ),  $R = \nu^{-1} = 2048$ . Rapid computation with only 1000 samples taken every  $4/h$  time steps of time  $h = 2^{-13}$  and averaged: the upper and lower values give

<sup>14</sup>Like special 1-dimensional systems or systems of particles interacting via short range repulsive force even in 3-dimensions, [?, ?, ?, ?],[?],[?].

the  $d/2$  exponents  $\lambda_k, k = 0, \dots, d-1$  and respectively  $\lambda_{d-1-k}$ , while the middle values are  $\frac{1}{2}(\lambda_k + \lambda_{d-1-k})$ , not constant but close to  $\simeq -.01$ . So the KY dimension is  $\sim D_N = 48$  and the attracting set is taken to be the full phase space, [?]

**Fig.3:**The decrease of the attracting set dimension with  $N$ : for the same system with  $N$  increased to 960 the local Lyapunov spectrum (red o.l.) in a 960 modes in RNS flows at  $R = 2048$ . The  $n = 4N(N+1)$  exponents  $\lambda_0, \dots, \lambda_{n-1}$  are drawn reporting for each  $k = 0, \dots, k_{\frac{n}{2}-1}$  the values of  $\lambda_k, \lambda_{n-1-k}$  and the average  $\frac{1}{2}(\lambda_k + \lambda_{n-1-k})$  for each  $k = 0, \dots, \frac{n}{2}-1$  (“pairing curve”). The spectra are averaged over a time 800 units, sampled every 4 (quite short). The figure shows that the attracting set dimension is  $\sim 936$  sensibly below 960, implying breakdown of time reversal, [?].



**Fig.4:**Test the fluctuation relation in the flow  $NS_{irr}$  (red o.l.) and  $NS_{rev}$  (blue o.l.) flows with 48 modes,  $R = 2048$ . The  $\tau$  is chosen 8, the slope of the graph increases with  $\tau$  reaching 1 at  $\tau = 8$ . The graph is built with  $8 \cdot 10^4$  data, divided into  $2 \cdot 10^3$  bins, obtained sampling the flow every  $4/h$  time steps of size  $h = 2^{-13}$ . The keys AF0 and AF1 deal with  $NS_{irr}$  (red o.l.) and, respectively,  $NS_{rev}$  (blue o.l.); the error bars (red o.l.) deal with  $NS_{irr}$ ; the line  $f(x) = x$  is a visual aid, [?].

**Fig.5:**The upper (red o.l.) curve are the loci of the largest values per each  $k$  observed, in the time  $t \leq 10000$  considered in Fig.5, (960 modes,  $R = 2048$ ), of the RNS flow exponents; lower (red o.l.) curve are loci of smallest values per each  $k$  observed and the central (red o.l.) line is the actual Lyapunov spectrum for the reversible flow (in Fig.5 curve was drawn breaking it in two halves to exhibit pairing: and in the following fig.7 it is reproduced without breaking it). The two (green o.l.) central lines are the upper and lower values observed in the INS flow exponents: the drawing shows that the average of the reversible flow is between (actually covered by) upper and lower values of irreversible flow exponents (whose average values are not drawn but on drawing scale would coincide with the reversible flow exponents). Remark that while the RNS exponents fluctuate more than the INS ones their averages are superposed on the graph scale. [?].

**Fig.6:** The two spectra for RNS (red o.l.) and INS (green o.l.) whose averages coincide, at the scale of the graph, with the central (red o.l.) line in Fig.5 are here individually compared term by term, drawing for each  $k \in [0, 960)$  the difference  $\frac{|\lambda_k^{irr} - \lambda_k^{rev}|}{(|\lambda_k^{irr}| + |\lambda_k^{rev}|)/2}$ . The line marks 6%. The larger relative difference at the center of the spectrum mostly reflects that it is there that the exponents are close to zero so that the numerical errors are larger [?].

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