Viscosity, Reversibillity, Chaotic Hypothesis, Fluctuation Theorem and Lyapunov Pairing

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August 22, 2021

Abstract: *Incompressible fluid equations are studied with UV cut-off and in periodic boundary conditions. Properties of the resulting ODEs holding uniformly in the cut-off are considered and, in particular, are conjectured to be equivalent to properties of other time reversible equations. Reversible equations with the same regularization and describing equivalently the fluid, and the fluctuations of large classes of observables, are examined in the context of the "Chaotic Hypothesis", "Axiom C" and the "Fluctuation Theorem".*

I. ON THE EQUATIONS

The incompressible Euler equation, denoted **E**, in a periodic container $\mathcal{T}^d = [0, 2\pi]^d$, $d = 2, 3$, for a smooth velocity field $\mathbf{u}(x), x \in \mathcal{T}^d$ is:

$$\begin{aligned} \dot{\mathbf{u}}(x) =& -(\underset{\sim}{\mathbf{u}}(x) \cdot \underset{\sim}{\boldsymbol{\partial}}_x)\mathbf{u}(x) - \boldsymbol{\partial}_x P(x), \\ \boldsymbol{\partial}_x \cdot \mathbf{u}(x) =& 0, \quad \int_{\mathcal{T}^d} dx\, \mathbf{u}(x) = \mathbf{0} \end{aligned} \tag{1.1}$$

where $P = -\sum_{i,j=1}^d \Delta^{-1}(\partial_i u_j\, \partial_j u_i)$ is the 'pressure' and Δ=Laplace operator.

It is also useful to consider the **E** equations from the "Lagrangian viewpoint": a configuration of the fluid is described by assigning the dispacement $x = q_\xi$ of a fluid element, from the reference position $\xi \in \mathcal{T}^d$, and the velocity $\dot{q}_\xi$ of the same fluid element. So the state of the fluid is $(\mathbf{q}, \dot{\mathbf{q}})$ where $\mathbf{q}$ is a smooth map of $\mathcal{T}^d$ to itself and $\dot{\mathbf{q}}$ is a smooth vector field on $\mathcal{T}^d$ with 0 average. Denote $\mathcal{F}$ the space of the dynamical configurations $(\mathbf{q}, \dot{\mathbf{q}}) \in Dif(\mathcal{T}^d) \times Lin(\mathcal{T}^d) = \mathcal{F}$ where $Dif(\mathcal{T}^d)$ is the set of diffeomorphisms of $\mathcal{T}^d$ and $Lin(\mathcal{T}^d)$ the space of the vector fields on $\mathcal{T}^d$.

Actually we concentrate on the subspace of $(\mathbf{q}, \dot{\mathbf{q}}) \in (SDif(\mathcal{T}^d) \times SLin(\mathcal{T}^d)) \stackrel{def}{=} S\mathcal{F} \subset \mathcal{F}$ where the evolution of an *incompressible* fluid takes place: $SDif(\mathcal{T}^d)$ being the *volume preserving* diffeomorphisms and $SLin(\mathcal{T}^d)$ the 0-*divergence* vector fields.

A $(\mathbf{q}, \dot{\mathbf{q}}) = \{q_\xi, \dot{q}_\xi\}_{\xi \in \mathcal{T}^d} \in \mathcal{F}$ should be regarded as a set of Lagrangian coordinates labeled by $\xi \in \mathcal{T}^d$. And the equations Eq.(1.1) can be derived from a Hamiltonian in canonical coordinates $(\mathbf{q}, \mathbf{p}) \in \mathcal{F}$ which is *quadratic* in $\mathbf{p}$ and which generates motions in $\mathcal{F}$ evolving leaving $S\mathcal{F}$ invariant. Therefore the motion in $\mathcal{F}$ is a "geodesic motion" (*i.e.* a motion generated by a Hamiltonian quadratic in the momenta).

A key remark is that the motions that follow initial data in $S\mathcal{F}$ remain, *as long as the evolution is defined and smooth*,[1] in $S\mathcal{F}$, [3, 43], *i.e.* $S\mathcal{F}$ is an invariant surface in $\mathcal{F}$. And the equations of motion that H generates can be written (using incompressibility of $(\mathbf{q}, \mathbf{p}) \in S\mathcal{F}$) as:

$$\dot{p}_\xi = -\boldsymbol{\partial}_{q_\xi} Q(\mathbf{q}, \mathbf{p})_\xi, \quad \dot{q}_\xi = p_\xi \tag{1.2}$$

where Q_ξ is $\mathbf{q}$-dependent and quadratic in $\mathbf{p}$, see Appendix A.

Since $\dot{p}_\xi = \partial_t p_\xi + (\underset{\sim}{p}_\xi \cdot \underset{\sim}{\partial} p_\xi)$, setting $x = q_\xi, u(x) = p_\xi$ and $P(x) = \boldsymbol{\partial}_{q_\xi} Q(\mathbf{q}, \mathbf{p})_\xi$, the equations become:

$$\dot{q}_\xi = p_\xi, \quad \partial_t u(x) + ((\underset{\sim}{u} \cdot \underset{\sim}{\partial})u)(x) = -\partial P(x) \tag{1.3}$$

with $\boldsymbol{\partial} \cdot \mathbf{u} = 0$, and P as above. The *Lagrangian form* of Euler's equations, Eq.(1.2) or (1.3), will be called $\mathbf{E}^*$. See Appendix A.

The above "geodesic" formulation of $\mathbf{E}, \mathbf{E}^*$ will be used to exhibit symmetry properties of Euler's equation which may be relevant also for the IN (*irreversible Navier-Stokes*) equations:

$$\partial_t \mathbf{u}(x) + ((\underset{\sim}{\mathbf{u}} \cdot \underset{\sim}{\partial})\mathbf{u})(x) = \nu \Delta \mathbf{u}(x) - \partial P(x) + \mathbf{f}(x) \tag{1.4}$$

with the conditions $\boldsymbol{\partial} \cdot \mathbf{u} = 0$, $\int_{\mathcal{T}^d} \mathbf{u} = 0$.

II. ULTRAVIOLET REGULARIZATION

Here we study the regularized version, see below, of **E** or IN, Eq.(1.1),(1.4), obtained by requiring that the Fourier's transform $\mathbf{u_k}$ of $\mathbf{u}$ does not vanish only for modes $\mathbf{k}$ with components $\le N$.

We shall *focus on properties of the solutions which hold uniformly in the cut-off* N: the space of such $\mathbf{u}$'s with 0 divergence ($\boldsymbol{\partial} \cdot \mathbf{u} = 0$) and 0 average ($\int_{\mathcal{T}^d} \mathbf{u}(x) = 0$) will be denoted $\mathcal{C}_N$.

Therefore the equation in dimension $d = 2, 3$ is expressed in terms of complex scalars $u_{\beta,\mathbf{k}} = \overline{u}_{\beta,-\mathbf{k}}$, $\beta = 1, \ldots, d$, $\mathbf{k} \in Z^d$, $|\mathbf{k}_\beta| \le N$: thus the number of real coordinates is $\mathcal{N} = 4N(N+1)$ in 2D and $\mathcal{N} = 2(4N^3 + 6N^2 + 3N)$ in 3D and $\mathcal{N}$ will be the dimension of the phase space $\mathcal{C}_N$.

For instance in 3D choose, for each $\mathbf{k} \ne 0$, two unit vectors $\mathbf{e}_\beta(\mathbf{k}) = -\mathbf{e}_\beta(-\mathbf{k})$, $\beta = 1, 2$, mutually orthogonal and orthogonal to $\mathbf{k}$; data are combined to form a velocity field:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) =& \sum_{0<|\mathbf{k}|\le N} \mathbf{u_k} e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{k} = (k_\beta)_{\beta=1,2,3} \\ \mathbf{u_k} =& \sum_{\beta=1,2} i\, u_{\beta,\mathbf{k}}\, \mathbf{e}_\beta(\mathbf{k}), \qquad \mathbf{k} \cdot \mathbf{e}_\beta(\mathbf{k}) = 0 \end{aligned} \tag{2.1}$$

with $|\mathbf{k}| = \max_j |k_j|, u_{-\mathbf{k},j} = \overline{u}_{\mathbf{k},j}$.

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[1] This might be a very short time.

Define $D^{\beta_1,\beta_2,\beta}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} = -(\mathbf{e}_{\beta_1}(\mathbf{k}_1)\cdot\mathbf{k}_2)(\mathbf{e}_{\beta_2}(\mathbf{k}_2)\cdot\mathbf{e}_\beta(\mathbf{k}))$. Introduce also forcing $\mathbf{f}=\sum_{\mathbf{k},\beta} i f_{\mathbf{k},\beta}\mathbf{e}_\beta(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}$ and viscosity in the form $-\nu\mathbf{k}^2\mathbf{u}_\mathbf{k}$.

The IN equations Eq.(1.4) become, if $\mathbf{k}^2\stackrel{def}{=}k_1^2+k_2^2+k_3^2$ and the sum is restricted to $|\mathbf{k}|,|\mathbf{k}_1|,|\mathbf{k}_2|\le N$:

$$\dot u_{\beta,\mathbf{k}} = \sum_{\substack{\beta_1,\beta_2\\ \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2}} D^{\beta_1,\beta_2,\beta}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}}\, u_{\beta_1,\mathbf{k}_1}u_{\beta_2,\mathbf{k}_2} - \nu\mathbf{k}^2 u_{\beta,\mathbf{k}} + f_{\beta,\mathbf{k}} \quad (2.2)$$

which will define the *regularized IN equation*.

The 2D case is similar but simpler: no need for the labels β, and $\mathbf{e}(\mathbf{k})$ can be taken $\frac{\mathbf{k}^\perp}{||\mathbf{k}||}$.

The coefficients $D^{\beta_1,\beta_2,\beta_2}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ can be used to check that if $\nu=0,\mathbf{f}=\mathbf{0}$ then for all $\mathbf{u}\in\mathcal{C}_N$:[2]

$$\frac{d}{dt}\int_{\mathcal{T}^d}\mathbf{u}(\mathbf{x})^2 d\mathbf{x}=0,\ \frac{d}{dt}\int_{\mathcal{T}^d}\mathbf{u}(\mathbf{x})\cdot(\boldsymbol{\partial}\wedge\mathbf{u}(\mathbf{x}))d\mathbf{x}=0 \quad (2.3)$$

As is well known, the first of Eq.(2.3) leads to the *a priori*, N-independent, bounds for the solutions of the $\mathbf{E}$ and IN equations:

$$||\mathbf{u}^{X,N}(t)||_2^2 \le \max(E_0,(\frac{F_0}{\nu})^2),\ X=\mathbf{E},IN \qquad (2.4)$$

satisfied (for all X) by solutions $t\to\mathbf{u}^{X,N}(t)\stackrel{def}{=}S_t^{X,N}\mathbf{u}$, in terms of $E_0=||\mathbf{u}(0)||_2^2=\sum_{\beta,\mathbf{k}}|u_{\beta,\mathbf{k}}|^2$ and $F_0=||\mathbf{f}||_2$.

From now on the *cut-off N will be kept constant* and the solution of the equations will be denoted simply $S_t\mathbf{u}$ dropping the X,N as superscript of the solution map S_t. Only when not clear from the context a superscript $\mathbf{E}$ or IN or a label N will be added to clarify whether reference is made to the evolution, or to its properties, following $\mathbf{E}$ or IN equation with cut-off N.

By scaling, the equation can and will be written in a fully dimensionless form in which $||\mathbf{f}||_2=1$.

The Jacobian of the Euler flow $S_t^{\mathbf{E},\mathbf{N}}\mathbf{u}$ with UV cut-off N is more easily written, without using the Fourier's transform representation of $\mathbf{u}$, directly from Eq.(1.1) and, see Appendix B, is the *sum* of the following convolution operator on $(\varphi_j(x))_{j=1}^d=\boldsymbol{\varphi}\in L_2(\mathcal{T}^d)\times R^d$:

$$\frac{\partial\dot u_i(x)}{\partial u_j(y)} = -\mathcal{P}\delta(x-y)\partial_{y_j}u_i(y),\ i,j=1,..,d, \qquad (2.5)$$

plus an antisymmetric operator on the same space; here $\mathcal{P}$ is the orthogonal projection, in the $L_2(T^d)\times R^d$ metric, on the divergenceless fields $\boldsymbol{\varphi}$. [3]

The operator acts on the fields $\boldsymbol{\varphi}$ with 0 divergence (this is used in deriving Eq.(2.5) to discard contributions that vanish on the divergenceless fields $\boldsymbol{\varphi}$): and in the end its *symmetric* part is $\mathcal{P}$ times the multiplication operator, on 0-divergence fields $(\varphi_j(x))_{j=1}^d=\boldsymbol{\varphi}\in L_2(\mathcal{T}^d)\times R^d$, by:

$$W_{i,j}(x)=\frac12\Big(\partial_{x_j}u_i(x)+\partial_{x_i}u_j(x)\Big) \qquad (2.6)$$

i.e. $\mathcal{P}$ times the operator $(J\boldsymbol{\varphi})_i(x)=\sum_j W_{i,j}(x)\varphi_j(x)$.

Introducing also viscosity (and forcing, which however does not contribute) Eq.(2.6) immediately leads to express the symmetric part of the Jacobian of the regularized IN, irreversible Navier-Stokes, as $\mathcal{P}$ times $J^\nu_{i,j}=\nu\delta_{i,j}\Delta+W_{i,j}$.

Defining, for $d=2,3$, $w(x)^2=\frac{d-1}{4d}\sum_{i,j=1}^d W_{i,j}(x)^2$ the inequality $J^\nu\le\nu\Delta+w(x)$, derived in [28, 33] for the nonregularized IN equation, remains valid for the regularized one and leads to the bound, [28, 33]:

Theorem: the sum of the averages of the first p eigenvalues of the (Schrödinger operator) $\nu D+w(x)$ yields an upper bound to the sum of the first p Lyapunov exponents (of any invariant distribution on $\mathcal{F}$) of the flow S_t. [4]

III. REVERSIBLE EQUATIONS

The theory of nonequilibrium fluctuations has led to studying phenomena via equations considered equivalent (at least for some of the purposes of interest) to the "fundamental" ones.

Thus new non-Newtonian forces have been added to systems of particles claiming that the values of important quantities would have the same values as those implied by the fundamental equations, even in cases in which the modification was drastic: with the advantage, in several cases, of greatly facilitating simulations, [11, 25, 30].

At the same time the idea that modification of the equations would not affect, at least in some important cases, most of their predictions arose in other domains: it appeared for instance, in [39], to show that the Navier-Stokes (IN above) equation could be modified, into new reversible equations, still remaining consistent with selected predictions of the Obukov-Kolmogorov theory.

In [14, 15] an attempt was presented to link empirical equivalence observations to the well established theory of the equivalence of ensembles in Statistical Mechanics. [5]

[2] The symmetries of D arise from the identities $\int((\underset{\sim}{\mathbf{u}}\cdot\underset{\sim}{\boldsymbol{\partial}})\mathbf{u})\cdot\mathbf{u}=0$ and $\int(\mathbf{u}\cdot\boldsymbol{\partial}\underset{\sim}{\mathbf{u}})\cdot(\underset{\sim}{\boldsymbol{\partial}}\wedge\underset{\sim}{\mathbf{u}})=0$, by integration by parts.

[3] Other projections could be used: this is convenient to follow the analysis in [33].

[4] Lyapunov exponents depend on the invariant distribution used to select data: here they will be defined as the time averages of the eigenvalues of the symmetric part of the Jacobian of the evolution equation, [33]. The u-dependent non averaged eigenvalues will be called local Lyapunov exponents.

[5] A naive version would be to claim that modifications of equations describing given phenomena will not alter 'many other' properties of their solutions if the modifications have the effect that properties known to hold, by emprical or theoretical analysis, are *a priori* verified: of course the question is 'which are the other properties?' and 'are they interesting?'.

And a paradigmatic example was the NS incompressible fluid in the simple case of periodic boundary conditions and forcing acting at large scale (*i.e.* with a force with Fourier's coefficients non zero only for modes $|\mathbf{k}| < K_f$ for some K_f). In this case new equation proposed was:

$$\dot{\mathbf{u}} = -(\underset{\sim}{\mathbf{u}} \cdot \underset{\sim}{\boldsymbol{\partial}})\mathbf{u} + \alpha(\mathbf{u})\Delta\mathbf{u} + \mathbf{f} - \boldsymbol{\partial} P \tag{3.1}$$

with the multiplier $\alpha(\mathbf{u})$ so defined that a "global" quantity becomes a constant of motion: for instance the *energy* $\mathcal{E}(\mathbf{u}) = \sum_{\mathbf{k},i} \frac{1}{2}|u_{\mathbf{k},i}|^2$ or the *enstrophy* $\mathcal{D}(\mathbf{u}) = \sum_{\mathbf{k},i} \mathbf{k}^2|u_{\mathbf{k},i}|^2$.

In Statistical Mechanics global conserved quantities define the *ensembles*, which are collections of stationary probability distributions on phase space giving the statistical fluctuations of observables in the 'equilibrium states'.

The main property being that the "local" observables have in each state properties *independent* on the special global quantity that defines a given state, at least in some limiting situation (like in the "thermodynamic limit", in which the container volume $\to \infty$).

Distinction between local and global observables is essential: in particle systems global quantities can be the total energy (microcanonical ensemble) or the total kinetic energy (isokinetic ensemble) or the total potential energy or the average value of certain observables (like the kinetic energy, in the canonical ensemble).

Local observables, in such systems, are observables $O_{V_0}(\mathbf{q},\mathbf{p})$ whose value depends on the configuration of positions and velocities of particles located, at the time of observation, in a region V_0 of finite size compared to the total volume V of the system.[6]

And local observables, in most systems and in stationary states, evolve exhibiting statistical properties of the values of O_{V_0} which have a limit as $V \to \infty$, *for all* V_0, *i.e.* become independent of the "volume cut-off V".[7]

Local and global observables arise often also in connection with the theory of many systems whose evolution is controlled by differential equations.

In the next section the example of the fluid equations, always in presence of a UV cut-off N, in Eq.(1.4),(2.2)) will be analyzed choosing viscosity force as $\nu\Delta\mathbf{u}$ or $\alpha(\mathbf{u})\Delta\mathbf{u}$ as in Eq.(3.1) with:

$$\begin{aligned} &(1) \quad \alpha(\mathbf{u}) = \frac{\sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} \cdot \overline{\mathbf{u}}_{\mathbf{k}}}{\sum_{\mathbf{k}} \mathbf{k}^2 |\mathbf{u}_{\mathbf{k}}|^2} \\ &(2) \quad \alpha(\mathbf{u}) = \frac{\Lambda(\mathbf{u}) + \sum_{\mathbf{k}} \mathbf{k}^2 \mathbf{f}_{\mathbf{k}} \cdot \overline{\mathbf{u}}_{\mathbf{k}}}{\sum_{\mathbf{k}} \mathbf{k}^4 |\mathbf{u}_{\mathbf{k}}|^2} \end{aligned} \tag{3.2}$$

where, with D introduced in Eq.(2.2):

$$\Lambda(\mathbf{u}) = \sum_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0} D^{i,j,r}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \mathbf{k}_3^2 \, u_{\mathbf{k}_1,i} u_{\mathbf{k}_2,j} u_{\mathbf{k}_3,r} \tag{3.3}$$

With the choice (1) the equation Eq.(3.1) generates evolutions *conserving exactly* the energy $\mathcal{E}(\mathbf{u})$, considered in [40], while with the choice (2) evolution *conserves exactly* the enstrophy $\mathcal{D}(\mathbf{u})$, considered in [18, 19]. But remark that $\Lambda \equiv 0$ in 2D, implied by Eq.(2.3).

IV. ENSEMBLES

In the case of the fluid equations in Eq.(1.4) and Eq.(3.1) define:

Local observables: are functions of the velovity fields $\mathbf{u}$ *which depend on the Fourier's modes* $\mathbf{u}_{\mathbf{k}}$ *with* $|\mathbf{k}| < \overline{K}$ *with* $\overline{K} \ll N$, *i.e. of finite size compared to the maximum value* N *(UV cut-off) used to make the equations meaningful* [8]

It can be said that local observables refer to measurements that can be effected looking at large scale properties of the fluid.

While in Statistical Mechanics locality refers to events in regions in position space small with respect to the volume cut-off V, in fluid mechanics locality refers to events measurable in regions in Fourier's space small compared to the ultraviolet cut-off N. Hence locality has a physical meaning when the aim of the theory is to study properties of "large scale" observables (*i.e.* expressible in terms of Fourier's components $\mathbf{u}_{\mathbf{k}}$ of the velocity fields $\mathbf{u}$ with $|\mathbf{k}|^{-1}$ of the order of the linear size of the container).

Hereafter consider Eq.(3.1) and Eq.(1.4) with $\mathbf{f}$ fixed and with only few Fourier's components non zero, say $|\mathbf{k}| < K_f$ with K_f fixed, and $\|\mathbf{f}\|_2 = 1$: such $\mathbf{f}$ will be called a "large scale forcing".

Having named (2.2) "irreversible" IN, consistently the Eq.(3.2) will be named "reversible" RE in case (1), or "reversible" RN in case (2). Properties of RE,RN are:

(1) they generate reversible evolutions $\mathbf{u} \to S_t\mathbf{u}$: *i.e.* if $I\mathbf{u} = -\mathbf{u}$ is the "time reversal" then $IS_t = S_{-t}I$.
(2) RE evolutions conserve exactly energy $E = \mathcal{E}(\mathbf{u})$ and RN conserve exactly enstrophy $D = \mathcal{D}(\mathbf{u})$.

Stationary distributions are usually associated with chaotic evolutions: therefore the multipliers $\alpha(\mathbf{u}(t))$ in Eq.(3.1) should show, at large E or D, chaotic fluctuations and behave effectively as a constants: this leads to several "equivalence conjectures".

For clarity we reintroduce a label N as a reminder that all quantities considered so far were defined in presence of a UV cut-off N and to discuss variations of N.

Collect the invariant (*i.e.* stationary) distributions for IN,RE,RN and denote the collections $\mathcal{E}^{IN}_N, \mathcal{E}^{RE}_N, \mathcal{E}^{RN}_N$ respectively: we call each such collection an *ensemble*.

[6] Always to be thought as $\gg V_0$.

[7] *Necessary in almost all cases* because of lack of existence-uniqueness of solutions of the equations of motion in infinite volume, just as in the IN equations in 3D with infinite cut-off.

[8] The latter value N, "ultraviolet cut-off", is certainly necessary in 3D $\mathbf{E}$, [8], just to make sense of the equations, and "might" be necessary in 3D IN, [12].

The stationary distributions are parameterized by the viscosity ν in $\mathcal{E}^{IN}$ or by the energy E in the $\mathcal{E}^{RE}$ or by the enstrophy D in $\mathcal{E}^{IN}$.

Denoting as $\mu_N^{IN,\nu}, \mu_N^{RE,E}, \mu_N^{RN,D}$ the stationary distributions, respectively, in the ensembles $\mathcal{E}_N^{IN}\mathcal{E}_N^{RE}, \mathcal{E}_N^{RN}$, we shall try to establish a correspondence between the elements $\mu_N^{IN,\nu}$, $\mu_N^{RE,E}$, $\mu_N^{RN,D}$ so that corresponding distributions can be called "equivalent" in the sense discussed below.

To fix the ideas we focus first on the correspondence between the distributions in $\mathcal{E}_N^{IN}$ and $\mathcal{E}_N^{RN}$: the simplest situation arises when above equations, for each ν small or D large, admit a unique stable invariant distribution, *i.e.* a unique "natural stationary distribution" in the sense of [34, 35, 37],[9] a key concept whose relevance has been stressed since [32].

For an observable $O(\mathbf{u})$ define $\langle O\rangle_N^{IN,\nu} \stackrel{def}{=} \mu_N^{IN,\nu}(O)$, $\langle O\rangle_N^{RN,D} \stackrel{def}{=} \mu_N^{RN,D}(O)$ the respective time averages of $O(\mathbf{u}(t))$ observed under the (N-regularized) IN and RN evolutions.

Define also the *work per unit time* done by the forcing:

$$L(\mathbf{u}) = \int_{\mathcal{T}^d} \mathbf{f}(x)\cdot\mathbf{u}(x)\frac{dx}{(2\pi)^d} = \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}\cdot\overline{\mathbf{u}}_{\mathbf{k}}, \tag{4.1}$$

So the average work per unit time in the stationary states with parameters ν or D of the ensembles $\mathcal{E}_N^{IN}, \mathcal{E}_N^{RN}$ is, respectively, $\langle L\rangle_N^{IN,\nu} \equiv \mu_N^{IN,\nu}(L)$ or $\langle L\rangle_N^{RN,D} \equiv \mu_N^{RN,D}(L)$.

Given $\mu_N^{RN,D}, \mu_N^{IN,\nu}$: define $\mu_N^{IN,\nu}$ to be *correspondent* to $\mu_N^{RN,D}$, denote this by $\mu_N^{IN,\nu} \sim \mu_N^{RN,D}$, if the time average of the *enstrophy* is equal in the two distributions:

$$\langle \mathcal{D}\rangle_N^{IN,\nu} = D \tag{4.2}$$

The natural distributions, see footnote9, are associated with chaotic evolutions: therefore the multipliers $\alpha(\mathbf{u}(t))$ should show, at large D, chaotic fluctuations and behave effectively as constants equal to their average.

Hence the proposal, [14, 15]: for an observable $O(\mathbf{u})$ define $\langle O\rangle_\nu^N \stackrel{def}{=} \mu_N^{IN,\nu}(O)$, $\langle O\rangle_D^N \stackrel{def}{=} \mu^{RN,D}(O)$ the respective time averages of $O(\mathbf{u}(t))$ observed under the IN and RN evolutions; then:

Conjecture 1: *Under the equivalence condition Eq.(4.2), equal average enstrophy, if $O(\mathbf{u})$ is an observable, then:*

$$\lim_{\nu\to 0}\langle O\rangle_\nu^N = \lim_{\nu\to 0}\langle O\rangle_D^N \tag{4.3}$$

The collection of stationary distributions $\mu \in \mathcal{E}^{IN}$ can be assimilated to the distributions of Statistical Mechanics canonical ensemble and the distributions $\mu \in \mathcal{E}^{RN}$ can be assimilated to the distributions of the microcanonical ensemble. The regularization N plays the role of the volume and the friction ν that of temperature, the enstrophy that of energy.

So there is ´some´ similarity between the equilibrium states equivalence in Statistical Mechanics and the equivalence proposed by the conjecture 1 about averages observed following the two different evolutions IN and RN, under the condition of equal average enstrophy.

[9] *I.e.* almost all initial data selected via a probability with continuous density $\rho(\mathbf{u})d\mathbf{u}$ on the $\mathcal{N}$-dimensional phase space $\mathcal{C}_N$, as in Sec.II, assign the same statistics to the time-fluctuations of the observables.

V. ENSEMBLES IN FLUIDS AND STATISTICAL MECHANICS

However the need to consider the limit as $\nu \to 0$ in Eq.(4.2) limits strongly the analogy: the Statistical Mechanics theory of equivalence of the ensembles requires considering the thermodynamic limit $V \to \infty$ of the volume of the system container and *restricting* the observables O to be *local*.

In the conjecture in Sec.IV, instead, the observables are *unrestricted* and the role of the volume V is played by the cut-off N. Clearly for a full analogy equivalence should hold for ν *fixed* as $N \to \infty$, provided the observables are suitably restricted.[10]

To see what has to be understood to try to establish a closer connection between the theory of the ensembles in Statistical Mechanics and the proposed fluid equations equivalence the key remark is that the conjectured equivalence is based on the chaoticity of the evolution, which is ensured by the $\nu \to 0$ condition in Eq.(4.2).

So the same argument can simply be extended to many other equations in which the size of a parameter controls the increasingly "chaotic" motion of a system. Examples of this phenomenon have been explicitly considered adding new examples to a wide literature of *homogenization* phenomena: see [16, 17, 24, 26] for fluid equations or [5, 17, 22]. Thus the conjecture in Sec.IV although quite unsatisfactory, as pointed out, seems to hold in its generality, [22, 24].

Far more interesting would be to dispose of the condition $\nu \to 0$ and to realize a stronger analogy with Statistical Mechanics. The idea is that some, by far not all, equations describing macroscopic phenomena arise as scalingn limits of fundamental equations governing evolutions of systems of particles interacting via forces veryfying all principles and symmetries of Physics: staying within classical systems among these are Newton's laws, time reversal and parity and charge symmetry...

The evolutions, at so fundamental a level, are certainly chaotic and the ergodic hypothesis epitomizes this property: from them, via approximations and/or heuristic

[10] For instance in Statistical Mechanics microcanonical and canonical ensembles are equivalent unless, of course, one is interested in the fluctuations of the (global) observable 'total energy'.

arguments, arise simplified equations (*models*) that generate motions apt to describe many of the features found in the observations. One of the first examples is in the derivation of the (compressible) Navier-Stokes equations in [29].

A model can even fail to respect one or more of the fundamental laws or symmetries: like the time reversal symmetry breaking which accounts phenomenologically for dissipation. This has never been considered a violation of the basic principles: it has been always clear that it was simply due to the procedure followed in the derivations.

Then the idea arises that there could (should ?) exist models representing the same phenomena at the same level of accuracy and preserving some of the properties that other models do not respect, but which are properties on which there is a minor interest in the context on which one is working, [18, 19].

The case of the Navier-Stokes equation has been proposed as an example of the possibility of describing an incompressible fluid via a reversible equation, without the need (as in conjecture 1 above) of taking the limit $\nu\to 0$ but paying the price of restricting attention to a suitable (large) family of observables.

In the NS case the equations of motion are irreversible but they arise from a fundamental microscopic representation which is reversible and chaotic. If, as in most experimental studies, interest is on properties of "large scale" then it is natural to extend the conjecture 1 to the NS evolution restricting attention to the case in which the macroscopic forces act at large scale and whose results have to be observed also on large scale, [18, 19].

This can be formalized into the:

Conjecture 2: *Under the equal average enstrophy Eq.(4.2) and if O is a local observable, as defined in Sec.IV, then*

$$\lim_{N\to\infty}\langle O\rangle_\nu^N=\lim_{N\to\infty}\langle O\rangle_D^N \tag{5.1}$$

for all $\nu>0$.

The conjecture 2 therefore adds to conjecture 1 the restriction that the observables O *must be local* and replaces the equivalence condition $\nu\to0$ with the *condition* $N\to\infty$ (keeping equal average enstrophy).

The ensemble $\mathcal{E}_N^{IN,\nu}$ is analogous to the canonical ensemble with ν as temperature while $\mathcal{E}_N^{IN,D}$ is analogous to the microcanonical ensemble with the enstrophy D as the energy and $N\to\infty$ corresponds to $V\to\infty$, *i.e.* to the thermodynamic limit necessary for all local observables to show the same statistics.

The analogy with Statistical Mechanics is now 'essentially' complete (however see Sec.VI below) and provides an example of use of the 'thermodynamic limit' among the ideas emerging in nonequilibrium theory, [1, 36].

VI. EQUIVALENCE AND PHASE TRANSITIONS

Conjecture 2 of Sec.V leaves a gap in the strict analogy between Fluid Mechanics and Statistical Mechanics ensembles. Is there an analogue of the phase transitions?

So far we have considered the ensembles $\mathcal{E}^{IN,\nu},\mathcal{E}^{RN,D}$ assuming that for each pair of ν, D the equations IN and RN admit just one "natural" stationary distribution controlling the fluctuations of the (local) observables.

However it is possible that initial data chosen with a distribution density $\rho(\mathbf{u})>0$ generate a statistics which still depends on the initial $\mathbf{u}$ with positive probability: this case would be met if the evolution admitted several attracting sets in the phase space $\mathcal{C}_N$.

If so, label the "indecomposable" invariant distributions by $\mu_\theta\in\mathcal{E}_N^{IN,\nu}$, $\theta=1,2,\ldots,q_{\nu,N}$.[11] Likewise label the "indecomposable" invariant distributions by $\mu_\theta\in\mathcal{E}_N^{RN,D}$, $\theta=1,2,\ldots,p_{D,N}$. Each μ_θ will be called a "pure phase“.

For simplicity we assume that $q_\nu,p_D<\infty$ and say that at the values ν or D there are q_ν or p_D "pure phases".

Then, keeping in mind the theory of phase transitins in Statistical Mechanics, conjecture 2 should be modified as:

If under the equivalence condition between ν and D, Eq.(4.2), there are $q_{\nu,N}$ respectively $p_{D,N}$ pure phases, then $q_{\nu,N},p_{D,N}$ have the same limit $q\ge1$ as $N\to\infty$, and it is possible to establish a $1\longleftrightarrow1$ correspondence between the $\mu_j\in\mathcal{E}_N^{RN,\nu}$ and the $\mu_j\in\mathcal{E}_N^{RN,D}$ such that the distribution of the local observables become, in corresponding μ's and in the limit $N\to\infty$, the same.

If one thinks to the ferromagnetic Ising model in voume V at low temperature then there are two indecomposable pure phases in which the total magnetization or just its average is fixed to some $m=\pm m^*\neq0$, [23], whether the boundary conditions are periodic or free or whether the dynamics is of Glauber type or other. Make correspondent the phases with the same m then the local observables (*i.e.* the observables O which depend only on the spins located in a fixed region) have fluctuation with the same statistics in the thermodynamic limit, $V\to\infty$.

See comments following Eq.(9.1) for other analogies with phase transitions arising in RN and deveoped in [40].

[11] Indecomposable means that for each θ with probabiity 1 with respect to μ_θ initial data generate precisely μ_θ itself: synonimous of ergodic.

VII. CHAOTIC HYPOTHESIS AND REVERSIBILITY

In a general evolution equation $\dot{x} = g(x), x \in bR^n$ generating motions $t \to S_t x$ which lead to an attracting set[12] $\mathcal{A}$ on which they are chaotic (*i.e.* have positive Lyapunov exponents) the "chaotic hypothesis" is:

Chaotic hypothesis (CH): *The attracting sets can be considered smooth surfaces on which the motion is an Anosov flow, [13, 21].*[13]

The assumption implies the exsistence of a unique stationary probability distribution μ on $\mathcal{A}$ which is a natural distribution in the sense that it gives the statistical properties of the motions of almost all initial data chosen in the vicinity of $\mathcal{A}$ with a probability with density $\rho(x) > 0$.

This assumption is a generalization of the ergodic hypothesis in equilibrium problems, and examples which do not satisfy it are easy to find. Still it is an assumption that has been proposed to be applicable to most systems undergoing chaotic motions, in analogy with the ergodic hypothesis in equilibrium thermodynamics, [36].

The real problem is to show that it not only has the merit of providing a conceptual extension of ideas at the basis of equilibrium Statistical Mechanics to nonequilibrium and Fluid Mechanics but it has also predictive power on new observations.

The simplest applications of the CH deal with reversible evolutions; hence the equations RN or RE might offer insights.

Imagine to fix the UV cut-off N and that for some ν the evolution appears to generate trajectories of IN that visit densely the entire phase space. We expect that to be the case at small ν, at fixed N: and for $\nu = 0$ ergodicity is expected to hold. As ν increases the system develops an attracting set which, if the CH holds, should still be the full phase space (a consequence of the structural stability of Anosov systems[14]).

For such value of ν let D be the average enstrophy: we consider the RN evoution of initial data with enstrophy $\mathcal{D}(\mathbf{u}) = D$. The phase space "contracts" at a rate $\sigma(\mathbf{u})$, *i.e.* . if $u_{\beta,\mathbf{k}} = u_{r,\beta,\mathbf{k}} + i u_{i,\beta,\mathbf{k}}$, $\beta = 1, 2$, see (2.1), at a rate equal (by Liouville's theorem) to:

$$-\sigma(\mathbf{u}) = -\sum_{\mathbf{k},\beta}^{*} \Big(\frac{\partial \dot{u}_{r,\mathbf{k},\beta}}{\partial u_{r,\mathbf{k},\beta}} + \frac{\partial \dot{u}_{i,\mathbf{k},\beta}}{\partial u_{i,\mathbf{k},\beta}} \Big) \qquad (7.1)$$

where $\sum_{\mathbf{k}}^{*}$ denotes summation over the $\mathbf{k}$ so that only one $\mathbf{k}$ between $\pm\mathbf{k}$ contributes (the contribution is independent on which one is selected).

Let $F_4 = \sum_{\mathbf{k}}^{*} \mathbf{k}^4 \mathbf{f}_{\mathbf{k}} \overline{u}_{\mathbf{k}}$, $E_6 = \sum_{\mathbf{k}}^{*} \mathbf{k}^6 |\mathbf{u}_{\mathbf{k}}|^2$, $E_4 = \sum_{\mathbf{k}}^{*} \mathbf{k}^4 |\mathbf{u}_{\mathbf{k}}|^2$, $K_2 = \sum_{\mathbf{k}}^{*} \mathbf{k}^2$, then:

$$-\sigma(\mathbf{u}) = 2\Big(K_2 - \frac{E_6(\mathbf{u})}{E_4(\mathbf{u})}\Big)\,\alpha(\mathbf{u}) + \frac{F_4(\mathbf{u})}{E_4(\mathbf{u})} \qquad (7.2)$$

which has the same expression in dimension $2, 3$ (but the expression of α is of course different).

If CH holds the "Fluctuation theorem", FT, can be applied and the result is that it implies a simple prediction on the *non local* observable

$$p = \frac{1}{t} \int_0^t \frac{\sigma(\mathbf{u}(t'))}{\sigma_+} dt' \qquad (7.3)$$

where σ_+ is the average value of $\sigma(\mathbf{u}(t))$. The fluctuations of p in the stationary distribution $\mu_N^{RN,D}$ have the probability that $p \in [a, b]$ is $\exp\left(t \max_{p \in [a,b]} s(p) + o(t)\right)$ and the "large deviations rate" $s(p)$ has the symmetry property, [13, 20, 21]:

$$s(p) - s(-p) = p\, t\, \sigma_+ \qquad (7.4)$$

which follows combining CH and the time reversibility.

The observable $\sigma(\mathbf{u})$ *can be considered also as an observable for the* IN *evolution.* Although it is non local it has been tested in a few cases to see whether it nevertheless obeys the same fluctuation relation Eq.(7.4) in corresponding distributions, see [18] for a positive result. But ....

VIII. ATTRACTORS AND SMALL SCALES

However the assumption that at an enstrophy value D the stationary distribution $\mu_N^{RN,D}$ arises from an evolution which leads to an attracting set invariant under time reversal is too strong.

Certainly it does not cover the cases in which the UV cut-off N is large enough and the $\mathbf{u}_{\mathbf{k}}$ components are strongly damped for $|\mathbf{k}|$ large (as implied by the equivalence conjecture).

Hence if N is large the attracting set $\mathcal{A}$ will shrink and its time reversal image $I\mathcal{A}$ will become different from $\mathcal{A}$: a *spontaneous breaking of time reversal*.

The consequence is that the FT cannot be applied to the observable $\sigma(\mathbf{u})$, not even if the CH is assumed in the reversible RN equation.

Nevertheless FT could be applied, under the CH, to the motion on $\mathcal{A}$ **if** the time reversal I could be replaced by *another map* $\widetilde{I}$ which leaves $\mathcal{A}$ invariant and on $\mathcal{A}$ the $\widetilde{I} S_t = S_{-t} \widetilde{I}$ holds. Because by CH $\mathcal{A}$ is a surface on which the evolution is of Anosov type.

In this case the fluctuation relation will be applied no longer to $\sigma(\mathbf{u})$, but to the sum $\sigma_{\mathcal{A}}$ of the *local Lyapunov*

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[12] This is a closed set $\mathcal{A}$ such that all initial data x close enough to $\mathcal{A}$ are such that the distance $d(S_t x, \mathcal{A}) \xrightarrow[t\to\infty]{} 0$.

[13] Anosov evolutions are smooth flows on bounded smooth surfaces $\mathcal{A}$ such that at every point x the evolution is hyperbolic (*i.e.* in a system of coordinates following $S_t x$ as t varies the $S_t x$ is a hyperbolic fixed point); furthermore any open set $U \subset \mathcal{A}$ is such $S_t U$ covers any prefixed point $x \in \mathcal{A}$ for infinitely many $t > t_0$ and for all t_0 ("motion of most points covers densely $\mathcal{A}$", "recurrence"), [4, 27].

[14] Structural stability means here that small pertubations of Anosov systems are still Anosov systems.[2, 38, 41].

exponents [15] relative to the motion on $\mathcal{A}$: clearly the negative exponents pertaining to the attraction to $\mathcal{A}$ *should not* be counted.

Hence the question under which conditions a time reversal for the motions on $\mathcal{A}$ exists is preliminary to the second question of how to identify the Lyapunov exponents of the motions on $\mathcal{A}$.

Considering the RN equations with UV cut-off N and fixed enstrophy D. Suppose that for small N (*i.e.* at strong regularization) the motions invade densely the phase space $\mathcal{C}_N$: *i.e.* the attracting set $\mathcal{A}$ coincides with $\mathcal{C}_N$. Increasing N arrives a N_c beyond which the (average) viscosity affects the components $u_{\mathbf{k}}$ with large $\mathbf{k}$ so that $\mathcal{A}$ becomes smaller than $\mathcal{C}_N$.

So the evolution is reversible for all N, but for N large its restriction to the attracting set $\mathcal{A}$ is not.

In [6] the question of existence, as a "remnant" of the global symmetry I, of a time reversal $\widetilde{I}$ operating on $\mathcal{A}$ has been examined and a geometric property, named *Axiom C* property, leading to the existence of $\widetilde{I}$ was identified and shown to have a "structural stability" property (as demanded to properties of physical relevance).[16] The definition and main properties of Axiom C are described in Appendix D.

A scenario for the application to IN,RN (and more general) equations in which time reversal is a symmetry but $\mathcal{A}$ does not coincide with the full phase space can be the following.

Assume that Axiom C holds for RN, hence there is a map $\widetilde{I}:\mathcal{A}\to\mathcal{A}$ such that $\widetilde{I}S_t=S_{-t}\widetilde{I}$: to apply FT the problem still remains of identifying the phase space contraction $\sigma_{\mathcal{A}}$, *i.e.* the local Lyapunov exponents which contribute to the phase space contraction on the surface $\mathcal{A}$.

In studying the Lyapunov spectrum for IN, RN the following "pairing symmetry" has been tested and *approximately* verified in a **few** 2D simulations and for a few values of the ensembles parameters ν, D.

If the $\mathcal{N}$ *local* Lyapunov exponents are arranged indecreasing order and their time averages are $\lambda_0\ge\lambda_1\ge\ldots,\ge\lambda_{\mathcal{N}-1}$, then

$$(\lambda_k+\lambda_{\mathcal{N}-1-k})=n+O(k^{-1}),\ k=0,\ldots,\frac{\mathcal{N}}{2} \tag{8.1}$$

and the constant $n<0$ and the λ_k turned out to have, for each k, in IN and RN very different fluctuations but remarkably the *same average* in corresponding distributions $\mu_N^{IN,\nu}$ and $\mu_N^{RN,D}$: quite unexpected a result because the λ_k are not local observables. See figs.7,8 and, respectively, figs.5,6 in [18, 19]. Relations like Eq.(8.1) are called "pairing rules".

So among the $\mathcal{N}/2$ averages $\lambda_k, \lambda_{\mathcal{N}-1-k}$ there may be, depending on the values of ν, D, pairs in which both elements are <0: *i.e.* there may be $n^*\le\mathcal{N}/2$ pairs of opposite sign and $\mathcal{N}/2-n^*$ negative pairs.

A natural interpretation of the above pairing rule is that the pairs of exponents <0 represent the exponents controlling the approach to $\mathcal{A}$ while the other n^* pairs are associated with the chaotic motion on the attracting set; the phase space contraction on $\mathcal{A}$ would then be $\sigma_{\mathcal{A}}(\mathbf{u})=\sum_{k=0}^{n^*}(\lambda_k(\mathbf{u})+\lambda_{\mathcal{N}-1-k}(\mathbf{u}))$.

The interest of the above remarks is that **if** CH, axiom C and pairing are satisfied and if the $O(k^{-1})$ in Eq.(8.1) can be neglected the consequent relation:

$$\sigma_{\mathcal{A}}(\mathbf{u})=\frac{2n^*}{\mathcal{N}}\sigma(\mathbf{u}) \tag{8.2}$$

can be used to define the phase space contraction on $\mathcal{A}$. The advantage is that $\sigma_{\mathcal{A}}$ is measurable simply by measuring $\sigma(\mathbf{u})$ from the equations of motion using Eq.(8.2).

Then, applying FT, the relation Eq.(7.4) is simply changed into:

$$s(p)-s(-p)=p\,t\,\frac{2n^*}{\mathcal{N}}\,\sigma_+ \tag{8.3}$$

in the case of the RN evolution.

Furthermore *if* the equivalence conjecture can be *extended* to the non local observables $\sigma,\sigma_{\mathcal{A}}$ then the fluctuation relation gives a prediction on fluctuations of both IN and RN and, if $n^*<\mathcal{N}/2$, a test of the Axiom C.

The above scenario, proposed first in [7, p.445] and leading to the formulation of Axiom C, does not seem to have been tested, not even for simple test examples and it is certainly interesting if it can be confirmed in some instances: the only attempt to check Eq.(8.3) dealt, [18], with cases in which $n^*=\mathcal{N}/2$. Hence it does not deal with the most interesting part of the above scenario and in particular it does not test the Axiom C: however it did yield the result that the fluctuation relation holds in equivalent distributions, *i.e.* the observable $\sigma(\mathbf{u})$, Eq.(7.2), satisfies the same Eq.(7.4) even in the irreversible evolution IN.

There are cases in which the phase space contraction can be identified with entropy creation: this is important as the entropy production is accessible, in a laboratory experiment, to measurements of heat and work exchanges with the surroundings: however it is very difficult to perform complete analysis of such energy exchanges and among the many experimental works very few convincingly discuss the problem.

IX. OTHER ENSEMBLES

In statistical Mechanics there are several equivalent ensembles. The same should hold for the fluids considered

[15] The local exponents are defined as the eigenvalues of the symmetric part of the Jacobian of the motion on $\mathcal{A}$: their sum defines the contraction (or expansion) of the surface elements of $\mathcal{A}$.

[16] *i.e.* small pertubations of systems with the axiom C property still have the property, [6]. Persistence under perturbations is clearly essential in most Physics theories, [31].

above. For instance we could compare IN with the equation that will be called RE given by Eq.(3.1) with $a(\mathbf{u})$ given by the first of Eq.(3.2).

The RE is reversible and conserves the global quantity $\mathcal{E}(\mathbf{u})$, energy, instead of enstrophy. The ensemble is now the collection of the stationary states $\mu_N^{RE,E} \in \mathcal{E}_N^{RE}$.

The equivalence condition is equality of the average energy, hence Eq.(4.2) is modified as

$$\langle \mathcal{E} \rangle_N^{IN,\nu} = E \tag{9.1}$$

and the analysis of the previous sections can be repeated.

Care has to be exercized because the condition Eq.(4.2) is not the same as $E = \langle \mathcal{E} \rangle_N^{IN,\nu}$ (unlike the corresponding case of the RN equations).[17,18]

This implies that a first test of the conjecture in the case of RN is obtained by fixing ν and computing the average enstrophy D and checking that if ν, D correspond in the sense of Eq.(4.2):

$$\langle \alpha \rangle_N^{RN,D} D = \nu \langle \mathcal{D} \rangle_N^{IN,\nu}, \quad i.e. \; \langle \alpha \rangle^{RN,N} = \nu \tag{9.2}$$

where $\mathcal{D}(\mathbf{u})$ denotes, as above, the enstrophy. While for RE, if ν and E correspond in the sense of Eq.(9.1), the analogous test is to check

$$\frac{\langle \alpha \mathcal{D} \rangle^{RE,E}}{\langle \mathcal{D} \rangle^{RN,\nu}} = \nu \tag{9.3}$$

The above relations have been tested in several cases, with particular care and a few positive results for the equivalence between IN and RN in 2D; only in very few cases for the IN and RE equivalence.

Appendix A: Euler flow is geodesic

Here some details on the Hamiltonian representation Eq.(1.2) for the Euler flow are presented, listing again for the reader´s convenience, the conventions set in Sec.I.

It has to be kept in mind that in analytic mechanics the canonical coordinates for n-degrees of freedom systems are given as strings of $2n$ variables ($\{p_i, q_i\}_{i=1}^n$): particle i is located at position q_i and has momentum p_i.

In a Lagrangian description of a fluid, coordinates will be $(\mathbf{q}, \dot{\mathbf{q}}) = (\{q_\xi, \dot{q}_\xi\}_{\xi \in \mathcal{T}^d})$ with $(\mathbf{q}, \dot{\mathbf{q}})$ consisting in a diffeomorphism $\mathbf{q} : \xi \to q_\xi$ in the space $Dif(\mathcal{T}^d)$ of $\mathcal{C}^\infty$ diffeomorphisms of $\mathcal{T}^d$ and $\dot{\mathbf{q}} \in Lin(\mathcal{T}^d)$ where $Lin(\mathcal{T}^d)$ is the space of the $\mathcal{C}^\infty$ vector fields 'tangent' to $\mathbf{q}$: in a pair $(\mathbf{q}, \dot{\mathbf{q}})$ the vector $\dot{q}_\xi \in R^d$ is considered a vector applied at the point q_ξ.

Hence, given $\mathbf{q}, \dot{\mathbf{q}}$, the derivative $\partial_{q_{\xi,i}} \dot{q}_{\xi,i}$ is defined as well as the divergence $(\mathrm{div}\dot{\mathbf{q}})_\xi \stackrel{def}{=} \sum_{i=1}^d \partial_{q_{\xi,i}} \dot{q}_{\xi,i}$ of $\dot{q}_\xi$.

The space of the pairs $(\mathbf{q}, \dot{\mathbf{q}})$ will be called $\mathcal{F}$ and the points of $\mathcal{T}^d$ become labels of a fluid element located at the point q_ξ with velocity $\dot{q}_\xi$.

More formally $(\mathbf{q}, \dot{\mathbf{q}}) \in Dif(\mathcal{T}^d) \times Lin(\mathcal{T}^d) \stackrel{def}{=} \mathcal{F}$ where $Dif(\mathcal{T}^d)$ is the space of the $\mathcal{C}^\infty$ diffeomorphisms of $\mathcal{T}^d$ and $Lin(\mathcal{T}^d)$ the space of the $\mathcal{C}^\infty$ vector fields with 0 average: for each $(\mathbf{q}, \dot{\mathbf{q}})$ the vector $\dot{q}_\xi \in R^d$ is considered applied to the point q_ξ. and $(\mathrm{div}\dot{\mathbf{q}})_\xi = \sum_{i=1}^d \partial_{q_\xi} \dot{q}_\xi$.

Actually we concentrate on the subspace of $(\mathbf{q}, \dot{\mathbf{q}}) \in SDif(\mathcal{T}^d) \times (SLin(\mathcal{T}^d)) \stackrel{def}{=} S\mathcal{F} \subset \mathcal{F}$ where the evolution of an *incompressible* fluid takes place: $SDif(\mathcal{T}^d)$ being the *volume preserving* diffeomphisms and $SLin(\mathcal{T}^d)$ the 0-*divergence* vector fields, *i.e.* for each such pair $(\mathbf{q}, \dot{\mathbf{q}})$ it is $(\mathrm{div}\dot{\mathbf{q}})_\xi = 0$.

If the positions q_ξ are moved the variation of $\dot{q}_\xi$ is proportional to $\frac{\partial \dot{q}_\xi}{\partial q_\eta}$; the Lagrangian is:

$$\mathcal{L}(\dot{\mathbf{q}}, \mathbf{q}) = \int_{\mathcal{T}^d} \frac{1}{2} \Big(\dot{q}_\xi^2 - Q(\dot{\mathbf{q}}, \mathbf{q})_\xi \Big) d\xi \tag{A.1}$$

where Q is the quadratic form on $\mathcal{F}$:

$$-\frac{1}{4\pi} \int_{\mathcal{T}^d} d\gamma \sum_{r,s=1}^d \Big(\frac{1}{''|q_\xi - q_\gamma|''} \Big) \frac{\partial \dot{q}_{\gamma,r}}{\partial q_{\gamma,s}} \frac{\partial \dot{q}_{\gamma,s}}{\partial q_{\gamma,r}} \tag{A.2}$$

and $-(4\pi)^{-1}/''|x-y|''$ symbolizes the Green's function for the Laplacian on $\mathcal{T}^3$: [19] so that $\Delta_{q_\xi} Q_\xi = (\boldsymbol{\partial}\mathbf{p})_\xi (\boldsymbol{\partial}\mathbf{p})_\xi$. Hence $\mathcal{L}$ is a metric over $\mathcal{F}$ and the flow it generates runs over its geodesics.

Addition of Q corresponds to a force which has the property that it keeps data in $S\mathcal{F}$ inside $S\mathcal{F}$ as long as they evolve smoothly in time: this is checked in the following.

The Lagrangian $\mathcal{L}$ leads to the canonical variables via:

$$\begin{aligned} p_{\xi,i} \stackrel{def}{=} \dot{q}_{\xi,i} - \frac{1}{4\pi} \int_{\mathcal{T}^d} d\lambda \int_{\mathcal{T}^d} d\gamma \frac{1}{\{''|q_\lambda - q_\gamma|''\}} \\ \cdot\, \delta_{i,r} \frac{\partial \delta(q_\xi - q_\gamma)}{\partial q_{\gamma,s}} \frac{\partial \dot{q}_{\gamma,s}}{\partial q_{\gamma,r}} \stackrel{def}{=} \dot{q}_{\xi,i} + A(\mathbf{q}, \dot{\mathbf{q}})_{\xi,i} \end{aligned} \tag{A.3}$$

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[17] If the RN equation is multiplied side by side by $\overline{u}_{\beta,\mathbf{k}}$ and the result is summed ove $\mathbf{k}, \beta$, it immediately follows that both conjectures imply $D = \langle \mathcal{D} \rangle_N^{IN,\nu}$. And conversely the latter equality implies the equal dissipation property, Eq.(4.2), hence the condition for equivalence.

[18] The ensemble $\mathcal{E}_N^{RE}$, in which the global quantity conserved is the energy rather than the enstrophy, has been considered in detail in [40] where a different kind of very interesting phase transition phenomena occurring in the RE equations is studied. In the limiting case in which $\nu \to 0$ as well as in the analysis of the transition in [40] it is likely that the difference bewteen imposing the condition $E = \langle \mathcal{E} \rangle_N^{IN,\nu}$ instead of the equal average enstrophy, as in Eq.(9.1), is not appreciable.

[19] *i.e.* formally the summation over images $y + 2\pi\mathbf{n}$, $\mathbf{n} \in Z^d$; which makes sense if the kernel is applied to a smooth function with 0 average.

where in the last equality (defining A) $p_{\xi,i} = \dot q_{\xi,i}$ and $A = 0$ hold if $\mathbf{q} \in SDif(\mathcal{T}^d)$ and $\dot{\mathbf{q}} \in SLin(\mathcal{T}^d)$, *i.e.* if $\sum_{s=1}^d \frac{\partial \dot q_{\gamma,s}}{\partial q_{\gamma,s}} = 0$: so that the diffeomorphism $\mathbf{q}$ is incompressible, and also $\dot{\mathbf{q}}$ is divergenceless, $\mathrm{div}\dot{\mathbf{q}} = 0$, and the double integral vanishes because $\int d\lambda\{''|q_\lambda - q_\gamma|''\}^{-1}$ is a constant and integration by parts over γ becomes possible as $d\gamma = dq_\gamma$ if $\mathbf{q} \in SDif(\mathcal{T}^d)$.

Defining the linear operator $A^{-1}(\mathbf{q},\mathbf{p})$, obtained by inverting $\mathbf{p} = \dot{\mathbf{q}} + A(\mathbf{q},\dot{\mathbf{q}})$, the equation for $\dot{\mathbf{p}}$ is readily obtained, at least if $(\mathbf{p},\mathbf{q}) \in S\mathcal{F}$:

$$\dot q_\xi = p_\xi, \qquad \dot p_\xi = \partial_{q_\xi}\mathcal{L}(\dot{\mathbf{q}},\mathbf{q}) = \partial_{q_\xi}\frac{1}{2}Q(\dot{\mathbf{q}},\mathbf{q}) \tag{A.4}$$

and the Hamiltonian:

$$H(\mathbf{p},\mathbf{q}) \stackrel{def}{=} \int_{\mathcal{T}^d} (\mathbf{p}_\xi \cdot \dot{\mathbf{q}}_\xi - \mathcal{L}(\mathbf{p} - A^{-1}(\mathbf{q},\mathbf{p}),\mathbf{q})_\xi) d\xi \tag{A.5}$$

yields canonical equations, which *for data in $S\mathcal{F}$ are*:

$$\begin{aligned} &\dot q_\xi = p_\xi, \quad \dot p_\xi = -\partial_{q_\xi} P_\xi(\mathbf{q},\mathbf{p}) \\ &P_\xi \stackrel{def}{=} \int_{\mathcal{T}^d} d\gamma \sum_{r,s=1}^d \Big(\frac{-(4\pi)^{-1}}{''|q_\xi - q_\gamma|''}\Big) \frac{\partial p_{\gamma,r}}{\partial q_{\gamma,s}} \frac{\partial p_{\gamma,s}}{\partial q_{\gamma,r}} \end{aligned} \tag{A.6}$$

which hold only if $(\mathbf{p},\mathbf{q}) \in S\mathcal{F}$, while in $\mathcal{F}$ the equations would be more involved (but uninteresting for the present purposes) although still Hamiltonian.

The equations can be written *for data in $S\mathcal{F}$* setting $x = q_\xi, u(x) = p_\xi, P(x) = P_\xi$. Then $\frac{dp_{\xi,i}}{dt} = \partial_t p_{\xi,i} + \sum_j p_{\xi,j}\partial_{q_{\xi,j}} p_{\xi,i}$ and:

$$\begin{aligned} &\boldsymbol{\partial} \cdot u(x) = 0, \ P(x) = -\sum_{r,s=1}^d \Delta^{-1}(\partial_s u_r(x)\partial_r u_s(x)) \\ &\partial_t q_\xi = u(q_\xi), \quad \partial_t u(x) = -\underset{\sim}{u}(x)\cdot \underset{\sim}{\boldsymbol{\partial}} u(x) - \boldsymbol{\partial}P(x) \\ &\qquad\qquad + \mathbf{f}(x) + \nu\Delta u(x) \end{aligned} \tag{A.7}$$

where the terms in the third line are added in the case there is forcing and NS viscosity: the $\nu\Delta u$ will be replaced in the next Appendix C by Ekman's viscosity $-\nu u$ and the equation thus modified will be used to exhibit a special symmetry arising in this case.

Therefore the Eq.(A.7) coincide with the Navier Stokes equations and their solutions will remain in $S\mathcal{F}$ s long as they remain smooth: for data not in $S\mathcal{F}$ only solutions local in time can be envisaged and the equations would be more involved.[20]

[20]Representing a constrained motion as a special case of unconstrained motion subject suitable extra forces follows a familiar prototype. A point mass constrained on a circle of radius R, centered at the origin $O \in R^3$, can be seen as a point subject to a centripetal force evolving under the Lagrangian $\mathcal{L} = \frac{1}{2}\dot{\mathbf{q}}^2 - \frac{(|\mathbf{q}|-R)}{R}\dot{\mathbf{q}}^2$. This leads to $\mathbf{p} = \dot{\mathbf{q}}\theta$ with $\theta = (1 - 2\frac{|\mathbf{q}|-R}{R})$ and, for the Hamiltonian, $\mathcal{H} = \frac{1}{2}\frac{\mathbf{p}^2}{\theta}$, to the equations $\dot{\mathbf{q}} = \mathbf{p}\theta^{-1}$, $\dot{\mathbf{p}} = -\frac{\mathbf{p}^2}{\theta^2 R}\frac{\mathbf{q}}{|\mathbf{q}|}$. Thus it appears that the phase space R^6 is analogous to $\mathcal{F}$, the maps of the circle $\xi \to s$, mapping the arc ξ to the arc s, correspond with $SDif$ and the vectors tangent to the circle are analogous to $SLin$. The motion is in general a geodesic motion, as long as it is defined (*i.e.* as long as $|\mathbf{q}| \neq 0,\infty$), which for data initially on the circle and initial velocity tangent to it is a uniform rotation. On such motions the Hamiltonian value is $\frac{1}{2}\mathbf{p}^2$ as $\theta = 1$, alike the corresponding vanishing of Q on $S\mathcal{F}$ in Eq.(1.3),(A.5).

Appendix B: Euler´s equation Jacobian

The Jacobian is obtained by taking suitable functional derivatives of the transport term and the pressure term, before applying the projection operator $\mathcal{P}$ in Eq.(2.4). The contribution of the transport term is (before applying $\mathcal{P}$):

$$\frac{\partial \dot u_i(x)}{\partial u_j(y)} = -\delta(x-y)\partial_{x_j}u_i(x) - u_k(x)\delta_{i,j}\partial_{x_k}\delta(x-y) \tag{B.1}$$

where the second term is an *antisymmetric* operator in $L_2(\mathcal{T}^d)\times R^d$. The contribution from the pressure term is

$$\begin{aligned} \frac{\partial\, \partial_{x_i}P(x)}{\partial u_j(y)} =& -2\partial_{x_i}\int_{\mathcal{T}^d} dz\, \Delta^{-1}(x-z) \\ &\cdot \partial_{z_k}\delta(z-y)\partial_{z_j}u_k(z) \\ =& 2(\partial_{x_i}\partial_{y_k}\Delta^{-1}(x-y))\partial_{y_j}u_k(y) \end{aligned} \tag{B.2}$$

and in both Eq.(B.1),(B.2) summation over k is intended.

The latter operator does not contribute to the Jacobian because acting on a divergenceless field yields 0; therefore the *symmetric part* of the Jacobian is the multiplication operator, in $L_2(\mathcal{T}^d)\times R^d$, by:

$$W_{i,j}(x) = -\frac{1}{2}(\partial_{x_j}u_i(x) + \partial_{x_j}u_i(x)) \tag{B.3}$$

followed by the orthogonal projection $\mathcal{P}$ on the subspace of the divergenceless fields $SLin(\mathcal{T}^d) \subset L_2(\mathcal{T}^d)\times R^d$, Sec.I.

Appendix C: The pairing symmetry in $\mathbf{E}^*$

Symmetry on the Lyapunov spectrum for the fluid equations is not really surprising, at least not in the case of the equations obtained from Euler´s equations in Lagrangian form when forcing and viscosity of the form $-\nu\mathbf{u}$ (Ekman´s viscosity) are added. The ultimate reason is that the Euler flow is a geodesic flow as discussed in Sec. I and Appendix A. A heuristic analysis follows.

It is convenient to review the method discovered in [10]. Let $H(\mathbf{p},\mathbf{q}) \in R^{2n}$ be a n-degrees of freedom Hamiltonian and to its equations a friction $-\nu\mathbf{p}$ is introduced turning the equations of motion into $\dot{\mathbf{p}} = -\partial_{\mathbf{q}} H(\mathbf{p},\mathbf{q}) - \nu\mathbf{p}$, $\dot{\mathbf{q}} = \partial_{\mathbf{p}} H(\mathbf{p},\mathbf{q})$.

The Jacobian matrix J_ν (*i.e.* the linearization of the flow in phase space $(\mathbf{p},\mathbf{q}) \in R^{2n}$) is a symplectic matrix up to the contribution from the friction force: $J_\nu = J_0 +$ the contribution from the friction force.

The latter is a diagonal matrix with the first $\frac{n}{2}$ elements $= -\nu$ and the rest $= 0$: hence it is the *sum* of a diagonal matrix with the first half entries equal to $-\frac{\nu}{2}$ and the others equal to $+\frac{1}{2}\nu$ *plus* a second diagonal matrix with *all the* $2n$ *elements* $= -\frac{\nu}{2}$.

Hence the Jacobian matrix of the equations with friction is recognized to be the sum of the symplectic Jacobian matrix of the Hamiltonian $H(\mathbf{p},\mathbf{q}) + \frac{\nu}{2}\mathbf{p}\cdot\mathbf{q}$ and of the constant diagonal matrix $-\frac{\nu}{2}$: namely $J_\nu = J_0 - \frac{\nu}{2}$.

Therefore the eigenvalues will be equal to $-\frac{\nu}{2}$ plus those of a symplectic matrix: such eigenvalues arise in pairs of opposite sign. The same pairing will remain true for products of Jacobians $J_\nu(\mathbf{u}_i)$ and therefore the Lyapunov exponents will arise in pairs with sum exactly $= -\nu$.

In Sec.I the equations $\mathbf{E}^*$ in $\mathcal{F}$ defined by Eq.(1.3) are geodesic flows (Appendix A) which evolve in $\mathcal{F}$ *leaving invariant* the manifold $S\mathcal{F} \subset \mathcal{F}$. The Jacobian is therefore symplectic even if restricted to $S\mathcal{F}$.

Hence if viscosity and forcing are added, with force depending only on q_ξ and friction expressed as $-\nu p_\xi$ (Ekman´s viscosity), obtaining dissipative equations that can be called $\mathbf{IE}^*$ (*i.e.* Lagrangian Euler's with Ekman's viscosity), the pairing to $-\nu$ is valid by above argument on the geodesic flow.

The argument in [9] can also be applied to $\mathbf{RE}^*$, *i.e.* to the equation obtained by adding to $\mathbf{E}^*$ the forcing and a reversible viscosity $-\alpha(p)p_\xi$ with, if $x = q_\xi, u(x) = p_\xi$, and $\alpha(u)$ *given by item* (1) in Eq.(3.1).

The $\mathbf{RE}^*$ equation, in 2D and 3D, implies energy conservation: therefore there is a 0 Lyapunov exponent that can be added to the 0 exponent associated with the flow direction.

Discarding the latter 0 exponents the important new idea added in [9] to the above summarized work [10] will imply that the Lyapunov exponents of the Lagrangian motion of $\mathbf{RE}^*$ on $S\mathcal{F}$ are paired to the average of $\alpha(\mathbf{u})$ (if they exist, *i.e.* in 2D).[21]

The analysis does not apply to the $2\mathcal{N} - 2$ exponents of the truncated equations: because it is unclear whether the argument in [9] applies to them.

[21] In [9] the force is supposed conservative: in $\mathbf{RE}^*$ the force is solenoidal (divergenceless). Yet this does not affect the result because what really matters is that the Jacobian of the equation with 0 viscosity is a symplectic matrix; and this is implied by the Hamiltonian nature of the evolution. Note that in the above preliminary analysis of the method in [10] the pairing would also follow, and for the same reason, if a Lorentz or Coriolis type of force were added, like $\boldsymbol{\varphi}(\mathbf{q}) \wedge \dot{\mathbf{q}}$ with $\boldsymbol{\varphi}(\mathbf{q})$ an incompressible field, *i.e.* $\boldsymbol{\varphi}(\mathbf{q}) = \boldsymbol{\partial} \wedge \mathbf{A}(\mathbf{q})$.

So in 2D the Lyapunov exponents of the equations $\mathbf{IE}^*, \mathbf{RE}^*$ will verify a pairing symmetry by reflection at the level $-\nu$ or to the average of α respectively. But this neither applies to the Navier Stokes equations, because the viscosity in not a force proportional to $\mathbf{u}$, nor to $\mathbf{IE}^*$ in 3D because there is no global existence uniqueness result on the flow S_t. About the regularized $\mathbf{E}^*, \mathbf{IE}^*, \mathbf{RE}^*$ equations pairing might just be approximate.

See [15, Sec.6] for heuristic considerations about possible extensions of the above remarks to the NS equations considered as equations for the velocity field only. A particular intriguing question is: is there a relation between the Lyapunov spectrum of $\mathbf{E}^*$, Eq.(1.2), and the corresponding spectrum of the simple Euler equation $\mathbf{E}$, Eq.(1.1) ?

For instance is it possible that the $\mathbf{E}^*$ (Lagrangian Euler equations, Eq.(1.2)) has a Lyapunov spectrum which is just the one of $\mathbf{E}$ (the Euler equations $\mathbf{E}$, Eq.(1.1)) with each exponent counted twice, [19] ? If so the pairing in the simple Euler equations $\mathbf{E}$ with forcing and Ekman viscosity added would hold at the level $-\frac{1}{2}\nu$ and in the corresponding reversible equations at the level of the average of $-\frac{1}{2}\alpha(\mathbf{u})$.

Appendix D: The Axiom C.

To describe the main features of the Axiom C, [6], consider first the simpler case of a *reversible diffeomorphism* S, *i.e.* such that there is a diffeomorphism, I such that $IS = S^{-1}I, I^2 = 1$. Imagine that the attracting set $\mathcal{A}$ differs from its time reversal image $I\mathcal{A} = \mathcal{R}$ and that CH holds.

The tangent space at a generic point z is supposed to be smoothly decomposed as $T_u(z) \oplus T_s(z) \oplus T_m(z)$. If $z \in \mathcal{A}$ or $z \in \mathcal{R}$ then $T_s(z), T_u(z)$ coincide with the tangent, at z, to the stable manifold of S on $\mathcal{A}$ or $\mathcal{R}$ respectively; furthermore for each ball $U_\delta(x) \subset \mathcal{A}$, of radius δ, consider the manifolds $W_i(x) \cap U_\delta(x), i = u, s$, and assume that they can be continued into smooth manifolds W_+, W_- everywhere tangent $T_s \oplus T_m$ and $T_u \oplus T_m$ and which intersect $\mathcal{R}$ in a single point $\widetilde{x} = Px$ if δ is small enough: thus defining P as a map between $\mathcal{A}, \mathcal{R}$.

Finally, as the labels s,u suggest, the vectors in T_s, T_u uniformly contract exponentially as time tends to $+\infty$ or $-\infty$ respectively, while vectors in T_m contract exponentially as $t \to \pm\infty$ (*i.e.* in both directions, being ´squeezed´ on $\mathcal{A}$ and $\mathcal{R}$).

The case of a flow S_t can be described similarly by imagining that T_m contains also the neutral direction $\frac{d}{dt}S_t\mathbf{u}$ and contracts transversally to it. In this context Axiom C adds to the Axiom B, [42], the assumption of the existence of a repeller $\mathcal{R}$ intersected by the manifolds emerging from $\mathcal{A}$.

The latter property permits to establish the map P, thus allowing to define the composition $\widetilde{I} = PI$, acting as a time reversal on $\mathcal{A}$ and $\mathcal{R}$, because the invariance of the manifolds implies that on $\mathcal{A} \cup \mathcal{R}$ it is $PS_t = S_tP$:

so that $\widetilde{I}S_t = S_{-t}\widetilde{I}$ on $\mathcal{A}\cup\mathcal{R}$ (note that $\widetilde{I}$ is not defined outside $\mathcal{A}\cup\mathcal{R}$). See [6] for more details.

Acknowledgements: I am indebted to M. Cencini for pointing out an error on the first version of the paper.

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