Equivalent Ensembles, Turbulence and Fluctuation Theorem

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Abstract: *Stationary states of Navier-Stokes fluids have been proposed to be described equivalently by several alternative equations, besides the NS equation itself. In particular equivalence between the NS evolution and a reversible. It is natural to test whether, assuming the Chaotic Hypothesis, the Fluctuation Theorem can be applied to the reversible flows. Here an example is provided which also leads to the possibility of testing the prediction of the fluctuation theorem even in systems evolving irreversibly.*

I. ENSEMBLES AND NONEQUILIBRIUM FLUIDS

Here we consider, reviewing [1, 2], the simple case of an incompressible fluid in a periodic container in dimension 2 (2D). Velocity $\mathbf{u}(\mathbf{x})$ can be expressed via a Fourier's series; if the container side is 2π then:

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{0}\neq\mathbf{k}=(k_1,k_2)\in Z^2} u_{\mathbf{k}} e(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \qquad e(\mathbf{k})\cdot\mathbf{k} = 0 \quad (1)$$

with $u_{\mathbf{k}} = \overline{u}_{-\mathbf{k}}$ scalars, $\mathbf{k}^\perp = (k_2, -k_1)$, $e(\mathbf{k}) = \frac{i\mathbf{k}^\perp}{|\mathbf{k}|}$

The NS equation for the components $u_{\mathbf{k}}$ is then:

$$\begin{aligned} \dot{u}_{\mathbf{k}} &= \mathcal{E}(u)_{\mathbf{k}} - \nu\mathbf{k}^2 u_{\mathbf{k}} + f_{\mathbf{k}}, \\ \mathcal{E}(u)_{\mathbf{k}} &= -\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \frac{(\mathbf{k}_2^2 - \mathbf{k}_1^2)\,(\mathbf{k}_1^\perp\cdot\mathbf{k}_2)}{2|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}|} u_{\mathbf{k}_1} u_{\mathbf{k}_2}, \end{aligned} \quad (2)$$

Forcing will be supposed to "act on large scale": $f_{\mathbf{k}} \equiv \mathbf{0}$ for $|\mathbf{k}| > K$ for some K. It is convenient to imagine that f is fixed *once and for all* and $\sum_{\mathbf{k}} |f_{\mathbf{k}}|^2 = 1$: below the case $f_{\mathbf{k}} = \overline{f}_{-\mathbf{k}} \neq 0$ only for $\mathbf{k} = \pm\mathbf{k}_0, \mathbf{k}_0 = (2, -1)$ and random phase will be considered.

Hence the *only dimensionless parameter* in the NS equation to which, for brevity, we refer as the "Reynolds number", is $R \equiv \nu^{-1}$.

The NS equations will be considered with ultraviolet regularization N, *i.e.* Eq.(2) in which all waves $\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2$ have components of modulus $\leq N$. Of course we are interested in properties which do not depend on N.

Notable cancellations are expressed by the identities:

$$\sum_{\mathbf{k}} \overline{u}_{\mathbf{k}} \mathcal{E}(u)_{\mathbf{k}} = 0, \quad \sum_{\mathbf{k}} \mathbf{k}^2 \overline{u}_{\mathbf{k}} \mathcal{E}(u)_{\mathbf{k}} = 0 \quad (3)$$

which, in the incompressible Euler flow with no stirring (*i.e.* $\nu = 0, f = 0$), imply conservation of energy E and enstrophy $\mathcal{D}$. [1] The identities Eq.(4) remain valid even in presence of the UV cut-off, *i.e.* if all components of $\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2$ in Eq.(2),(3) are restricted to be $\leq N$.

In Statistical Mechanics equilibrium states of a system can be *equivalently* described by a probability distribution in different ensembles (canonical, microcanonical and others). In the review [3] an analogous paradigm (evolved from the earlier work [2]) has been proposed to hold for stationary states in fluid mechanics (actually in more general stationary nonequilibria).

Returning to the NS equations, denote $t \to S_t^{irr,N}\mathbf{u} = \mathbf{u}(t)$ a solution to the NS equations with UV cut-off N. *Time reversal* $I\mathbf{u} = -\mathbf{u}$ is not a symmetry, *i.e.* $IS_t^{irr,N} \neq S_{-t}^{irr,N} I$, because of viscosity $\nu > 0$.

Consider also Eq.(2), with the same UV regularization:

$$\dot{u}_{\mathbf{k}} = \mathcal{E}(u)_{\mathbf{k}} - \alpha(u)\mathbf{k}^2 u_{\mathbf{k}} + f_{\mathbf{k}} \quad (4)$$

where the viscosity ν has been replaced by the multiplier $\alpha(u)$ designed so that the enstrophy $\mathcal{D}(u) = \sum_{\mathbf{k}} |\mathbf{k}|^2 |\mathbf{u}_{\mathbf{k}}|^2$ *is conserved*. In 2D the second Eq.(3) yields:

$$\alpha(\mathbf{u}) = \frac{\sum_{\mathbf{k}} \mathbf{k}^2 \overline{f}_{\mathbf{k}} \mathbf{u}_{\mathbf{k}}}{\sum_{\mathbf{k}} |\mathbf{k}|^4 |\mathbf{u}_{\mathbf{k}}|^2} \quad (5)$$

which also immediately implies that flows $t \to \mathbf{u}(t) = S_t^{rev,N}\mathbf{u}$ of Eq.(4) are *reversible i.e.* $IS_t^{rev,N} = S_{-t}^{rev,N} I$.

The well known difficulty with achieving control of the enstrophy is to be expected to correspond to an evolution of $\alpha(S_t^{rev,N}\mathbf{u})$ with extreme fluctuations at large Reynolds number $R = \frac{1}{\nu}$.[2] This in turn might produce a homogeneization phenomenon which could imply that α can be replaced, for practical purposes, by a constant: leading to statistical properties similar to those of the irreversible evolution $S_t^{irr,N}$, at least on *local observables*, *i.e.* observables $O(\mathbf{u})$ depending $\mathbf{u}$ via its Fourier components $u_{\mathbf{k}}$ with $|\mathbf{k}| < K \ll$ than the UV cut off N.

Possibility of equivalent descriptions of stationary states of turbulent fluids arose in the key work [5], where the NS equation has been shown to be describable, in simulations, by the stationary state of a different fluid equation obtained by imposing on the Euler equation the constraint that the energy content of "each shell" in $\mathbf{k}$-space is set to the value predicted by the 5/3-law.

In [5], at difference with Eq.(4), the constraint is imposed via as many multipliers as the number of inertial shells: yet it leads to reversible equations of motion. Stationary states of the new equations exhibited in [5], for several large scale observables, the same statistical properties obtained from the corresponding irreversible NS.

The following equivalent ensembles description was proposed, see [3] for a review, for the stationary states of the incompressible fluid.

[1] $E = \sum_{\mathbf{k}} |\mathbf{u}_k|^2, \mathcal{D} = \sum_{\mathbf{k}} |\mathbf{k}|^2 |\mathbf{u}_{\mathbf{k}}|^2$.
[2] In 2D (only) enstrophy can be controlled but it can grow up to $\nu^{-2} = R^2$, [4, Eq3.2.24].

Let $\mathcal{E}^{irr,N}$ be the family of stationary distributions that can be reached by evolving, via the usual NS Eq.(2), initial velocity fields $\mathbf{u}$ selected with probability 1 with respect to *(any)* distribution with density $\rho(\mathbf{u})d\mathbf{u}$ on the phase space M^N defined[3] by the Fourier's coefficients $\mathbf{u_k}$ of $\mathbf{u}$. The conceptual importance and the role of the selection criterion, adopted here, has been stressed and used by Ruelle, see the reviews [6, 7] and [3].

Existence of the stationary states will be, here, a consequence of a general assumption, *Chaotic Hypothesis*, on systems which are "chaotic" (*i.e.* have some positive Lyapunov exponents), supposed to hold throughout.[4]

At small viscosity $\nu = \frac{1}{R}$, *i.e.* large Reynolds number R, it is expected that there is a unique stationary state $\mu_R^{irr,N}(d\mathbf{u}) \in \mathcal{E}^{irr,N}$: discussion of the (well known) non uniqueness cases will also be considered later below.

Likewise let $\mathcal{E}^{rev,N}$ be the family of stationary distributions that can be built in the same way via Eq.(4): the distributions can be parameterized by the value of the enstrophy $\mathcal{D}$, which is a constant $D = \mathcal{D}(\mathbf{u})$, fixed by the initial datum enstrophy. And for large D it is expected that there will be a unique stationary state $\widetilde{\mu}_D^{rev,N}(d\mathbf{u}) \in \mathcal{E}$.

In [1] it is proposed, "Equivalence Hypothesis", that in a turbulent regime (*i.e.* at small ν or large D) the above $\mu_\nu^{irr,N}$ for the irreversible flow and $\widetilde{\mu}_D^{rev,N}$ for the reversible will be equivalent as $N \to \infty$ if

$$\mu_\nu^{irr,N}(\mathcal{D}) = D_N \tag{6}$$

i.e. if the enstrophy D_N in $\widetilde{\mu}_{D_N}^{rev,N}$ is the irreversible evolution average of the enstrophy $\mathcal{D}(S_t^{irr,N}\mathbf{u})$. Here D_N in the r.h.s will in general depend on N: remark, however, the *a priori* bound $D < R^2$ for all N (due to Eq.(3)).

The precise meaning is that, fixed ν, for any local observable $O(\mathbf{u})$ (*i.e.* of large scale, as defined in paragraph after Eq.(5)) it will be, under condition Eq.(6):

$$\lim_{N\to\infty} \mu_\nu^{irr,N}(O) = \lim_{N\to\infty} \widetilde{\mu}_{D_N}^{rev,N}(O) \tag{7}$$

This will be briefly denoted $\mu_\nu^{irr,N} \sim \widetilde{\mu}_D^{rev,N}$.

There is an analogy with the equivalence in Statistical Mechanics between the canonical and microcanonical ensembles: with ν playing the role of the inverse temperature β, the enstrophy that of the energy E and $N \to \infty$ that of the infinite volume ('thermofynamic') limit .

Remark that the work of the stirring force per unit time W is given by $W = \sum_{\mathbf{k}} f_{\mathbf{k}}\overline{u}_{\mathbf{k}}$ which, by the assumption that $f_{\mathbf{k}} = 0$ unless $|\mathbf{k}| < K$ for some fixed K, is a local observable. Hence from Eq.(7) applied to W it follows, see Eq.(10) below:

$$\mu_R^N(W) = \widetilde{\mu}_{D_N}^{rev,N}(W) \tag{8}$$

i.e. the stationary distributions, μ_R^N for the irreversible flow and $\widetilde{\mu}_{D_N}^N$ for the reversible one, measure the same average work per unit time of the stirring force.

The physically appealing relation Eq.(8) is a consequence of the equivalence hypothesis and leads to an important first test of it. Namely it implies that the multiplier α in Eq.(5) has an average $= \nu$:

$$\nu = \lim_{N\to\infty} \mu_{D_N}^{rev,N}(\alpha),\ \ i.e.\ \ \lim_{N\to\infty} R\mu_{D_N}^{rev,N}(\alpha) = 1 \tag{9}$$

which allows to interpret the equivalence hypothesis in terms of a "homogeneization property" and supports the above suggestion that equivalence relies on the chaotic evolution of the multiplier α.

Assuming the equivalence, the Eqs.(8),(9) can be easily checked: just multiply by $\overline{\mathbf{u}}_{\mathbf{k}}$ both sides of the equations Eq.(2) (irreversible NS equation) and Eq.(4) (reversible NS). Summing over $\mathbf{k}$ the result is:

$$\frac{d}{dt}\frac{1}{2}\sum_{\mathbf{k}} |\mathbf{u_k}|^2 = -\gamma\mathcal{D}(\mathbf{u}) + W(\mathbf{u}) \tag{10}$$

where γ is either ν, for $NS^{irr,N}$, or $\alpha(\mathbf{u})$, for $NS^{rev,N}$, and this is correct because the inertial terms cancel exactly as a consequence of the first identity in Eq.(3).

Averaging Eq.(10) over time yields:

$$\nu\mu_\nu^{irr,N}(\mathcal{D}) = \langle W\rangle, \quad \mu_{D_N}^{rev,N}(\alpha)D_N = \langle W\rangle \tag{11}$$

exactly, where $\langle W\rangle$ is the average of the work per unit time in the two cases.

The equivalence hypothesis (via Eq.(6)) yields that the $\langle W\rangle$ has the same limit as $N \to \infty$ and this leads to a first nontrivial test of equivalence: namely Eq.(4) follows from the equivalence condition Eq.(6).

This also shows that the equivalence hypothesis could be also formulated replacing Eq.(6) with $\nu_N = \mu_D^{rev,N}(\alpha)$ (this time ν will depend on N as D did in Eq.(6)) and, in this case, the relation $\lim_{N\to\infty} \mu_{\nu_N}^{irr,N}(\mathcal{D}) = D$ would be a nontrivial test.

The above analysis establishes a $1-1$ correspondence between the elements of the distributions in $\mathcal{E}^{irr}$ and $\mathcal{E}^{rev}$ of stationary states of the two equations in the region of parameters ν, D in which the equations admit a unique stationary distribution (with probability 1 with respect to the choice of initial data with a distribution with density on phase space, called SRB-distributions).

[3] If the UV cut-off is intended as contraining all components of $\mathbf{k} \neq \mathbf{0}$ to be $|k_i| \le N$ the real dimension of the space M^N is $4N(N+1)$ if $d = 2$, as for each $\mathbf{k} \neq \mathbf{0}$ there is one complex coordinate, and $\mathbf{u_k} = \overline{\mathbf{u}_{-\mathbf{k}}}$.

[4] The formmulation goes back to [8], for a review see [3, 9]: **Chaotic Hypothesis (CH):***Evolution of a chaotic system is attracted by a smooth surface in phase space and, on it, it is a smooth Anosov system.* It implies, [10], existence of a unique stationary state associated with each attractor: it is a "genericity" hypothesis and here it is supposed to hold for the evolutions considered. It is an interpretation of the (weaker) hypothesis that motion near the attractors is a Axiom A system, [6, 7].

However it is known that often, even with fixed and constant forcing, as it is the case here, the evolution may be attracted by different attracting sets, each with a probability > 0, particularly if ν is large (at fixed N and small Reynolds number), [11].

The hypothesis should then be extended. A natural extension is that the set of extremal (*i.e.* ergodic) stationary states with given ν or D are in $1-1$ correspondence and each pair $\mu^{irr}_{\nu,\eta}, \widetilde{\mu}^{rev}_{\eta,D}$, labeled by an extra index η, is reached as a limit as $N \to \infty$ of $\mu^{irr,N}_{\eta,\nu}, \widetilde{\mu}^{rev,N}_{\eta,D}$ (which might depend on the initial data or even on alternative ways of realizing the UV cut-off).

In other words the extra η is analogous to the role of the boundary conditions (which fix the pure phases) in the theory of the thermodynamic limit in presence of phase transitions, [12, 13]. However analysis of the problems related to cases in which there are several attractors will not be attacked here in any detail.

The argument about homogeneization based on the fluctuations of $\alpha(\mathbf{u})$ apparently fails when the stationary state is a periodic motion or just laminar. Nevertheless the microscopic model of the fluid remains always chaotic although at the same time, particularly if the macroscopic motion is laminar, the motion appears regular and develops on a very low dimensional attracting surface: hence I find it likely that equivalence could remain valid even in the laminar regimes.

Tests of equivalence can be found in several publications; to mention the most recent: [1, 3, 4, 9, 14, 15].

As an example one test of the key relation Eq.(9) is reported in Fig.1. Namely a check of Eq.(9): fixing the viscosity at $\nu = \frac{1}{R} = 1/2048$ and $N = 31$, *i.e.* 3968 modes:

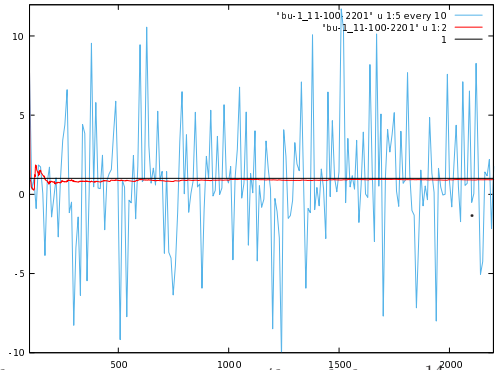

Fig.1: The axis is time in units $2/h$ with $h = 2^{-14}$ as integration time step, with Runge-Kutta-4 integrator. The (blue) fluctuating line yields the time evolution of the multiplier $R\alpha(t)$ in the reversible evolution (NS_{rev}); the (red) line yields at each time t the time average between the initial time up to t, which should be a line tending to 1. And the horizontal line, a visual aid, is the line at height 1. The total run is over $t \in [0, 2200]$; and the initial data are random while the forcing has only one complex mode with random phase, namely $\mathbf{k} = \pm(2, -1)$. Here $R = 2048, N = 31$, 3968 modes.

Fig.1 has been obtained via a semispectral code: in a non spectral method this should be comparable to a 63^2 discretization. Remarkably the same simulation, see [16, fig.2], can be done measuring the multiplier $\alpha(\mathbf{u})$ in the irreversible NS_{irr} evolution, regarding it as an observable defined by Eq.(4): this is not a local observable, still the result is very close to the one in Fig.1. In this case, although α, regarded as an observable for the irreversible NS_{irr} flows, is non local still, in corresponding distributions, its running average has the same average in NS_{rev} and NS_{irr}. This hints at the possible existence of families on non local observables which fall into the equivalence: a point on which we return below.

The same simulation for NS_{rev} can be performed at much larger friction, *e.g.* smaller $R \simeq 28$, and just 48 modes. This time the phenomenology is somewhat different and the variable α undergoes much smaller fluctuations becoming only rarely negative. Equivalence is however respected: increasing viscosity the multiplier α, while strongly fluctuating, will much less fluctuate relatively to its average. Eventually at very large viscosity the flow, in the stationary states, becomes laminar or periodic and fluctuations of α no longer extend to negative values.

Finally it has to be remarked that the 2D nature of the equations is not essential and all the general ideas carry unchanged to 3D: in particular the question of existence and uniqueness of the NS equation in 3D does not arise: the "only" difference is that attention should be really paid to the N dependence of D in Eq.(6). This depends on the absence of available *a priori* bounds on D (unlike the N independent bound $D < R^2$, valid in 2D). Studies of the 3D reversible NS and its relation with the 3D irreversible have been recently studied: see [17, 18] for NS and [19] for the shell model.

II. FLUCTUATION THEOREM

Consider now the main new question studied in this note. Assuming equivalence it is natural to ask whether the reversibility of the NS_{rev} evolution gives new insights in the corresponding NS_{irr} irreversible flows.

Consider the Fluctuation Theorem (FT): for reversible Anosov systems it deals with the phase space contraction (physically interpreted as "entropy production rate", [1]) whose fluctuations exhibit universal properties.

In the NS_{rev} evolution the non constant multiplier α leads to a phase space contraction (formally the divergence: $\sum_{\mathbf{k}} \frac{\partial \dot{u}_{\mathbf{k}}}{\partial u_{\mathbf{k}}}$) which, after a brief calculation, is:

$$\sigma(u) = \alpha(u)\Big(2K_2 - 2\frac{E_6(u)}{E_4(u)}\Big) + \frac{F(u)}{E_4(u)} \tag{1}$$

with α in Eq.(4) and $K_2, E_4(u), E_6(u), F(u)$ are:

$$\begin{aligned} 2K_2 &= \sum_{\mathbf{k}} \mathbf{k}^2, \;\; E_4(u) = \sum_{\mathbf{k}} (\mathbf{k}^2)^2 |u_{\mathbf{k}}|^2, \\ E_6(u) &= \sum_{\mathbf{k}} (\mathbf{k}^2)^3 |u_{\mathbf{k}}|^2, \;\; F(u) = \frac{\sum_{\mathbf{k}} (\mathbf{k}^2)^2 \overline{f}_{\mathbf{k}} u_{\mathbf{k}}}{E_4(u)} \end{aligned} \tag{2}$$

where the sums run over the $\mathbf{k}$ with $|k_i| \leq N$, $i = 1, 2$.

In *time reversible Anosov systems* the fluctuations of the divergence satisfy a general symmetry relation. Namely if S_t denotes the reversible or irreversible evolution and σ_+ the infinite time average of $\sigma(S_t u)$ and

$$p(u) = \frac{1}{\tau}\int_0^\tau \frac{\sigma(S_\theta u))}{\sigma_+} d\theta \tag{3}$$

then p has a probability distribution in the stationary state such that $p \in dp$ has density $P(p) = e^{s(p)\tau + O(1)}$, asymptotically as $\tau \to \infty$, with

$$s(-p) = s(p) - p\tau\sigma_+ \tag{4}$$

called the "Fluctuation Theorem" (FT), [8].

In applications it would be important to know that Eq.(4) holds: however in any experiment the relation cannot be considered satisfied because it is essentially impossible to check mathematically the CH and the reversibility. In the literature several attempts can be found studying empirically the relation Eq.(4) which, when it cannot be *a priori* proved, is called "Fluctuation Relation" (FR).

Before asking the key question "*is it meaningful to ask whether the FR holds in irreversible evolutions ?*" it is necessary studying, first:

1) the probability distribution P of p, defined by Eq.(3) both in the reversible and irreversible flows. Although this is not a local observable it might be among the non local observables, like the one illustrated in Fig.1, with equal or close corresponding distributions,[1, 3, 8].

2) the local Lyapunov spectrum: defined by considering the Jacobian matrix of the evolution, formally the matrix $J_{\mathbf{k},\mathbf{h}} = \frac{\partial \dot{u}_\mathbf{k}}{\partial u_\mathbf{h}}$, then computing its symmetric part eigenvalues, in decreasing order, and averaging each one over the flow. Whether the spectra of the reversible and irreversible evolutions are related is closely related to the key question: because in reversible Anosov systems the number of exponents ≥ 0 equals that of negative exponents. Hence their equality *indirectly tests CH.*

Preliminarily it should be asked whether the FR is even to be expected at least for the stationary flows obeying reversible NS_{rev}. The CH, which is assumed, will imply that the evolution is a Anosov flow on the attracting surface. However, to apply the theorem, it should also be time reversible: and if the attracting set is not the full phase space the FT cannot be applied, at least not without further work.[5]

Hence a simple check will be to count the numbers of positive and negative exponents: if the negative ones are more than the non negative the evolution *on the attracting manifold* cannot be reversible in spite of the time reversibility of NS_{rev} on the full phase space.

The test turns out to be possible, in a reasonable computer time, in the simple case of the NS equation with very few modes, 48 modes and $R = 2048$: the local Lyapunov spectrum can be computed, using the algorithm in [20, 21].[6]

The quick check in Fig.2 reports $\lambda_k, k = 0 \dots d/2 - 1$: the first half of the $d = 4N(N+1)$ exponents in decreasing order and the second half $\lambda_{d-1-k}, k = 0 \dots d/2 - 1$ as function of k (upper and lower curves), as well as $\frac{1}{2}(\lambda_k + \lambda_{d-1-k})$ (intermediate line). It yields other somewhat surprising results besides showing the equality of the numbers of positive and negative exponents which, as above, we take as evidence that the attracting set fills densely phase space so that the time reversal symmetry remains a symmetry of the attracting set. Figure draws *in the same panel*, spectra from both NS_{rev} and NS_{irr} flows under equivalence conditions, and also shows:

a) "coincidence" of the spectra of the NS_{rev} and NS_{irr} evolutions: quite surprising and justifying the following attempt to formulate and check the FR in the *irreversible* flows.

b) apparent "pairing": the exponents appear "paired", *i.e.* $\frac{1}{2}(\lambda_k + \lambda_{d-1-k})$ is k-independent. For further and more demanding results on pairing see Appendix below.

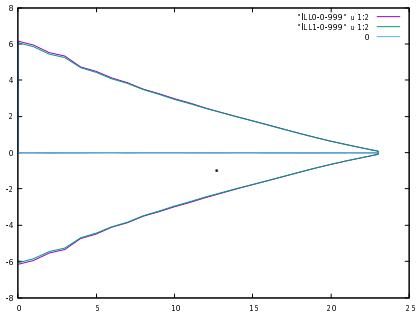

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Fig.2: Local Lyapunov spectra for the NS_{irr} and NS_{rev} flows with 48 modes, $R = 2048$. Rapid computation with only 1000 samples taken every $4/h$ time steps of time $h = 2^{-13}$ and averaged: this figure shows that the positive exponents are equally numerous as the negative ones and the features listed a),b) above.

Hence it stimulates testing the FT: and the result for the graph of $(s(p) - s(-p))/\sigma_+\tau$ in Eq.(4) is:

[5] If the attracting surface $\mathcal{A}$, see CH, is not the full phase space M^N then the time reversal image $I\mathcal{A}$ is likely to be disjoint from $\mathcal{A}$ and the motion restricted to $\mathcal{A}$ is not symmetric under the natural time reversal I.

[6] If $J(u)$ is the Jacobian matrix of the flow, formally $\partial \dot{u}_\kappa / \partial u_\mathbf{h}$, and $J^s(u)$ is its symmetric part, then the local Lyapunov exponents are defined as the eigenvalues of $J^s(u)$; they are here computed by iterating a large umber of times ($> h^{-1}$) the matrix $(1 + hJ^s(u))$ and applying the quoted method. The Lyapunov spectrum is related to the actual Lyapunov spectrum via interesting inequalities, [22, 23]: which can be used to test accuracy of simulations.

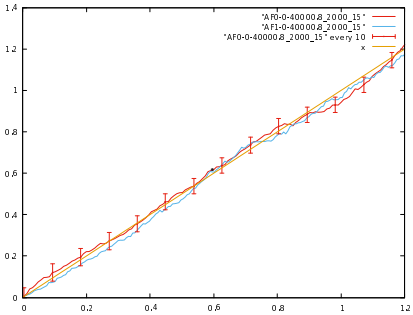


Fig.3: Test the fluctuation relation in the flow NS_{irr} (red) and NS_{rev} (blue) flows with 48 modes, $R = 2048$. The τ is chosen 8, the slope of the graph increases with τ reaching 1 at $\tau = 8$. The graph is built with $4 * 10^4$ data, divided into $2 \cdot 10^3$ bins, obtained sampling the flow every $4/h$ time steps of size $h = 2^{-13}$. The orange line is visual aid for $f(x) = x$.

The histogram of the PDF corresponding to Fig.3 is very close to a Gaussian centered at 1 and width yielding the slope of Fig.3:

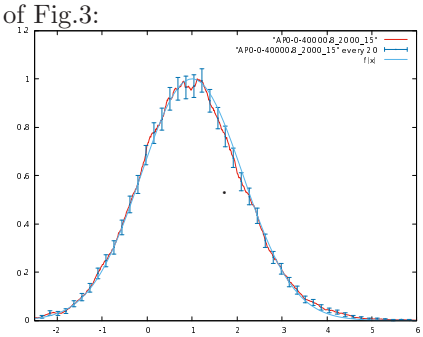


Fig.4: The histogram of the PDF *for the irreversible flow* of the variable p (red), with $\tau = 8$ generating the Fig.3 out of the $4 \cdot 10^4$ measurements of $\sigma(\mathbf{u})$ in the NS_{rev}, NS_{irr} equations. For each p the average of the number (and error bars) of points in $[p - \delta, p + \delta]$ is plotted (red) with $\delta = 15$ (corresponding to a small interval of p compared to the width $2\sqrt{\sigma+\tau}$) and the interpolating Gaussian (blue). The similar plot for the reversible flow (not drawn) would overlap with this within error bars.

Fig.3 shows that the proposed equivalence *extends* also to the phase space contraction ("entropy production rate", [1, 9]) as an observable defined for the reversible evolution but regarded as an observable for the irreversible NS: its interest is that observation of a *a priori* predicted fluctuation relation in a *irreversible* evolution has often been dismissed and not yet observed.[7]

III. PROBLEMS AT STRONG DISSIPATION

The results on the fluctuation relation (FR) are very special because the UV regularization is so small that the number of (local) Lyapunov exponents can be easily computed and checked to be the same for positive and nonpositive ones. This makes possible to suppose that CH holds and that the attracting surface is the entire phase space, so that time reversal is a symmetry for the evolution on the attractor: which implies that the FR follows from the FT and leads to the above test.

More interesting is the case of higher regularization: already at 224 models the number of negative exponents *exceeds* that of the positive ones. The first remark is that the (approximate) "pairing" between exponents already quite clear in Fig.2 remains a characteristic feature, and becomes more precise, as the cut-off N increases, see Fig.5 below.

Two objections can be raised before even beginning to attempt possible application of the FT to NS evolutions with strong dissipation and several momentum scales.

1) excess of negative Lyapunov exponents which indicates (if CH holds) that the flow evolves towards an attractor of dimension smaller than the full dimension of phase space: this breaks time reversal symmetry, see footnote[5],which ceases to be a symmetry of the evolution on the attractors (although it *remains* a global symmetry at least in the NS_{rev} flows).

2) if the attracting set dimension is lower than that phase space, the contraction to which the FT *might apply* is not the full divergence of the equations of motion: one should rather consider the contraction of the surface of the attracting set.

3) the fluctuation theorem does not apply to irreversible evolutions, like NS_{irr}, not even if CH holds.

The actual results on the determination of the local exponents spectrum in a 3968 truncation of the NS_{rev} and NS_{irr} equations at high Reynolds number $R = 2048$ are:

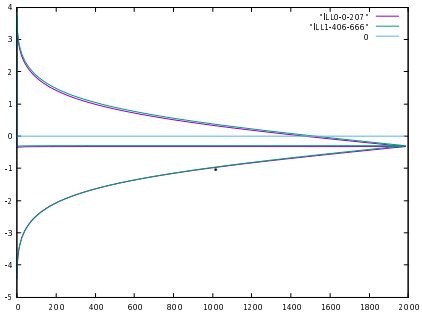
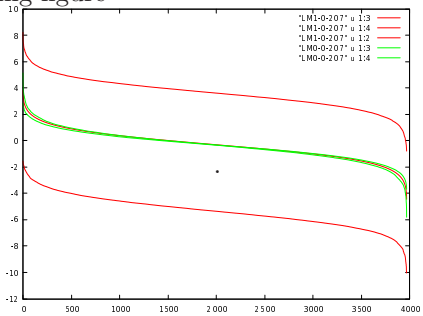


Fig.5: The local Lyapunov spectrum in a 3968 modes NS_{rev} and NS_{irr} flows at $R = 2048$. The $n = 4N(N+1)$ exponents $\lambda_0, \ldots \lambda_{n-1}$ are drawn reporting for each $k = 0, \ldots, k_{\frac{n}{2}-1}$ the vlues of $\lambda_k, \lambda_{n-1-k}$ and the average $\frac{1}{2}(\lambda_k + \lambda_{n-1-k})$ for each $k = 0, \ldots \frac{n}{2} - 1$. The spectra are averaged over a time 800 units sampled every 4: before reaching such times the running average values have become stable, although the individual exponents are still fluctuating. Also remarkable is the *apparent* "pairing" between $\lambda_k, \lambda_{n-1-k}$: however this pairing seems to be approximately realized only in a range of R and N: if R is lowered at fixed N the pairing line becomes sensibly curved (as we have checked) and the same should happen at fixed R and large N.

[7] In summary the prediction is based on CH, on the equality of numbers of negative and non negative exponents and on the extension of the equivalence hypothesis to the entropy production rate.

Fig.5 gives the spectra (in the same panel and *almost superposed* on the scale of the drawing) and shows their agreement in corresponding evolutions. The straight line at level 0 is a visual aid (it shows immediately that the sum of the exponents is < 0 and that time reversal I *is not a symmetry* on the attracting surface).

It is remarkable that the individual local exponents have fluctuations in the reversible flow much larger than those of the irreversible flow. This is clearly exhibited in the following figure



A

Fig.6: The upper red curve are the loci of the largest values observed, in the time $t \leq 200$ considered in Fig.5, (3968 modes, $R = 2048$), of the *reversible flow* exponents; the lower red curve are the loci of the smallest values observed and the central red line is the actual Lyapunov spectrum for the reversible flow (in the case of Fig.5 the curve was drawn differently, breaking it in two halves to exhibit the pairing). The two green lines are the upper and lower values observed in the *irreversible flow* exponents: the drawing shows that the average of the reversible flow is between the upper and lower values of the irreversible flow exponents (whose average values are not drawn but on the scale of the drawing would coincide).

Fig.5,6 exhibit a *large number* of observables which, although non local, have the "same average" values in corresponding stationary states: namely the 3868 local Lyapunov exponents in the 2D case of Fig.2 and the PDF's Fig.3 and Fig.4.

Returning to the FR and to the above objections the latter results on the Lyapunov spectrum suggest a new viewpoint.

In [1] is has been proposed that the first two objections do not apply to the cases considered here if the following interpretation of Fig.6 is accepted: the exponents which are part of the negative pairs have to be discarded being interpreted as the exponents controlling the uninsteresting attraction by the attracting surface. Hence one remains with an equal number of positive and negative exponents (*i.e.* only the pairs of opposite sign count to evaluate the phase space contraction on the attractor).

The lack of time reversal symmetry applies to the NS_{rev} whenever the attracting set is smaller than the full phase space (as in the case reported in Fig.5) and of course, always, to the NS_{irr}. A different time reversal symmetry mapping the attracting surface into itself, could be recovered if the assumption that the flow satisfies Axiom C is accepted, [9, 24].

This has not yet been tested: however the approximate (see caption to Fig.5) pairing would be very helpful because it establishes $\sim$-proportionality between the sum of the $2n^*$ exponents appearing in pairs of opposite sign and the sum of all $d = 4N(N+1) \stackrel{def}{=} 2n$ pairs: the latter is directly accessible from the total divergence and the sum of the opposite pairs is identified with the phase space contraction of the attracting set so that average of the latter will simply be

$$\sigma_{attractor,+} = \frac{num.\ of\ opposite\ sign\ pairs}{num.\ pairs}\sigma_+ \stackrel{def}{=} \frac{n^*}{n}\sigma_+ \tag{1}$$

The contraction on the surface of the attracting set at the configuration u is proposed to be identified with the sum $\sum_{k=0}^{n^*}(\lambda_k(u) + \lambda_{n-k-1}(u))$ of the local exponents.

The above comments on the problems 1,2,3 could then be tested, at the same time, by checking validity of the FR with slope $\frac{n^*}{n}\tau\sigma_+$ rather than $\tau\sigma_+$: this is a difficult (*i.e.* long computation time), not an impossible simulation task, but it has not been tested yet.

The properties in Fig.5,6 and a large scale representation of the apparent difference between corresponding reversible and irreversible cases is illustrated in the drawings in the appendix.

Appendix A: Extra plots

Complementary plots illustrate other aspects obtained in less ambitious, but more accurate, simulations

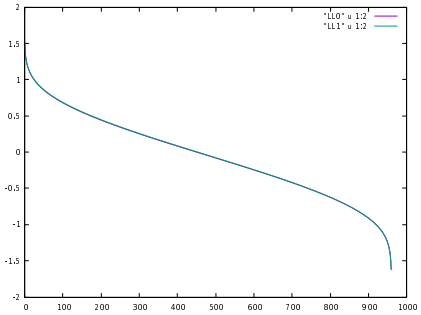


Fig.5': Local Lyapunov spectrum in a 960 modes NS_{rev} and NS_{irr} flows at $R = 2048$. The $n = 4N(N+1)$ exponents $\lambda_0, \ldots \lambda_{n-1}$ for the NS_{rev} and NS_{irr} fows are drawn and are apparently *superposed*. The spectra are averaged over a time $4 * 3600$ units of $4/h$ steps of size $h = 2^{-17}$, sampled every 4: before reaching such times the running average values have become stable, although the individual exponents are still fluctuating. See Fig.2' below.

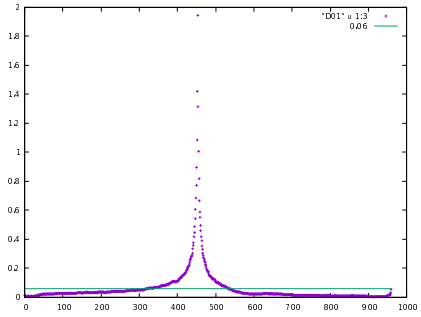


Fig.5": The two spectra in the previous figure are here individually compared

drawing for each $k \in [0, 960)$ the difference $\frac{|\lambda_k^{irr} - \lambda^{rev}|}{(|\lambda_k^{irr}| + |\lambda^{rev}|)/2}$. The line marks 6%. The larger relative difference at the center of the spectrum mostly reflects that it is there that the exponents are close to zero so that the numerical errors are larger.

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