Nonequilibrium and Fluctuation Relation

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Abstract: *A review on the fluctuation relation, fluctuation theorem and related topics.*

Contents

I. Introduction 1
II. Stationary Distributions (SRB) 3
III. Symmetries, (time reversal) 3
IV. Example: reversible dissipation 4
V. Hamiltonian dissipation? 5
VI. Fluctuation Relation (FR) 6
VII. Nonequilibrium ensembles. Ensembles equivalence 7
VIII. Strong Dissipation: attracting set size. Lyapunov pairs. 9
IX. Dissipation. Time Reversal & FR. 10
X. Fluctuation Dissipation Theorem 12
XI. Onsager's Reciprocity 13
XII. Fluctuation Patterns 14
XIII. Irreversibility time scale 15
XIV. Chaos. Structure of Anosov systems. Their digital codes. 16
XV. Volume as stochastic process. SRB as Ising spin chain equilibrium 18
XVI. Entropy ? Stationarity & Approach to it 20
XVII. Viscous Fluids and Reversibility 21
XVIII. Simulations on 2D-NS 23
XIX. Other relations. Comments. 25
A. Transient fluctuation theorem 25
B. The Jarzinsky relation 26
C. Ruelle-Lieb bounds 27
D. Wishes 27
A. About certain comments on CH 28
B. Local Fluctuations. An example. 28
C. Reversible heating 30
D. Arnold-Euler geodesics 31
References 32

I. INTRODUCTION

Many works investigate the Fluctuation Relation, (FR): but its interpretation is often very far from the original one proposed in [1] and in subsequent works, see for instance [2]. Here I present the original point of view referring also a few of its interpretations, consequences and related conjectures.

In this review only the foundations of the theory are discussed, as I cannot present the many (pertinent) developments that followed, starting with the early ones, [3–7].

The FR arises from a simple theorem on dynamical systems, the *Fluctuation Theorem*, (FT). The FT applies, under further suitable assumptions, to *Anosov systems*: which can be considered as playing a role analogous to that played for non chaotic systems by the harmonic oscillators, although sometimes they are considered an abstract mathematical notion.[1]

In natural observations initial data are generated by a well defined procedure, that is sometimes called a *protocol*, but are always affected by unavoidable errors, no matter how carefully one fixes the protocol.

Therefore initial data are generated with a probability distribution: defined by the protocol and *unknown*. Yet it is subject to the fundamental assumption that it is a probability distribution on the "phase space" M (a smooth Riemannian manifold here) which *admits a density* with respect to the volume of M. The probability of $x \in M$ being in a open set dx around x has the form $\rho(x)dx$ where ρ is some continuous function (or slightly more general). This is an assumption which should not be overlooked: it *cannot be proved* and is, therefore, a *law of nature*, with far reaching consequences.

The connection with Physics is established via the hypothesis (called *Chaotic Hypothesis*, (CH)) stating that "all" systems exhibiting chaotic motions can be treated for many purposes as *Anosov systems*. Informally, in such systems an observer co-moving, in phase space M, with a point x sees it as a "saddle point" (mathematically a "hyperbolic fixed point"), while it wanders invading an attracting surface $\mathcal{A}$ in M, The notion of Anosov maps and flows, [9, 10],[11, Ch.4], is briefly recalled in the footnote.[2]

[1] *e.g.* "Whether or not speculations concerning such hypothetical Anosov systems are an aid or a hindrance to understanding seems to be an aesthetic question", [8, p.221].

[2] If M is a smooth (*i.e.* ∞-differentiable) bounded manifold and S is an invertible smooth (*i.e.* ∞-differentiable together with the inverse S^{-1}) map on M, the system (M, S) is an Anosov map if
(a) at every point $x \in M$ there are two complementary tangent planes $T_s(x)$ and $T_u(x)$, transverse in x, which depend continuosly on x, are covariant in the sense that the Jacobian $\partial S(x)^{\pm 1}$ acts on the plane tangent to the attracting set so that $\partial S(x)^{\pm 1} T_\gamma(x) = T_\gamma(S(x)^{\pm 1}), \gamma = u, s$,
(b) furthermore there are $C > 0, \lambda < 1$ such that $|\partial S^n(x)v| < C\lambda^n|v|, n > 0$ if $v \in T_s(x)$ and $|\partial S^{-n}(x)v| < C\lambda^n|v|, n > 0$ if $v \in T_u(x)$,
(c) there is a point whose orbit is dense in M.
The definition of Anosov flow is similar: the covariant mutually transversal planes are now three: $T_u(x), T_s(x)$ on which expansion and contraction take place under action of the flow S_t as in the map case and a third 1-dimensional tangent plane parallel to $f(x)$, if $\dot{x} = f(x)$ is the differential equation defining the flow S_t; this is a tangent vector supposed not zero, $|f(x)| > 0$, and which, of course, neither expands nor contracts under the $\partial S_t(x)$.
The stable and unstable planes of Anosov systems (maps or

Suppose that evolution (∞-smooth, for simplicity) of a mechanical system is defined by a map S on a phase space M and is attracted by an ∞-smooth surface $\mathcal{A} \subset M$ on which S is an Anosov map $S\mathcal{A} \leftrightarrow \mathcal{A}$, as considered here in most cases (for simplicity). It transforms an initial datum x into the new datum Sx in a single time step; then the main property of the dynamical system (M, S) is that the evolution is chaotic and a phase space point, with exceptions forming a set of zero volume, moves accumulating at the attracting surface $\mathcal{A}$ densely.[3] Such S will be called a *map with an Anosov attractor*.[4]

The key property of maps with an Anosov attractor is that the fraction of time asymptotically spent in any open (or just measurable and with > 0 volume) phase space region dx, defines a stationary probability distribution $\mu(dx) = \mu(Sdx)$ which is independent of the initial x, again except for data x forming a set of 0 phase space volume, [12, 13]. The probability μ is concentrated on the attracting surface $\mathcal{A}$, *i.e.* $\mu(\mathcal{A}) = 1$; the data outside the attracting surface $\mathcal{A}$ evolve while exponentially attracted by $\mathcal{A}$, [14, 15].

Likewise if the evolution is instead described by a flow $x \to S_t x$, generated by a (∞-smooth) differential equation on M, $\dot{x} = f(x)$, which on $\mathcal{A} \subset M$ is an Anosov flow and $\mathcal{A}$ is an attractive smooth surface, then any initial point $x \in M$, with exceptions forming a set of zero volume, moves accumulating densely at $\mathcal{A}$ and spending a well defined fraction of time in every measurable region of M with positive volume: thus uniquely defining a stationary probability distribution $\mu(dx) = \mu(S_t dx)$, concentrated on $\mathcal{A}$, which is independent of the initial x, again except for data forming a set of 0 volume in phase space M, [12, 13, 16]: such flow S_t will be called a *flow with an Anosov attractor*.

Formally:

Theorem: Suppose that a smooth dynamical system is defined on a manifold M containing a globally attracting smooth surface $\mathcal{A}$ on which the motion is an Anosov map or flow. Then all initial data x, aside from a set of zero volume, evolve visiting open sets D with a time frequency $\mu(D)$: the probability distribution μ is independent on the protocol generating x. The probability μ is ergodic and mixing at exponential rate for all smooth observables.

Phase space M can be a smooth manifold or, more generally, an open set in R^n for some n, which is a domain of attraction of $\mathcal{A} \subset M$. For systems with a global Anosov attractor (flows or maps) the associated unique stationary probability distribution $\mu(dx)$ that defines the statistical properties of the evolution, (*i.e.* frequencies of visit to regions of M), is called the *SRB distribution*, [17–19].

The just mentioned theorems on maps or flows become relevant for systems evolving chaotically in the cases in which the following hypothesis holds *i.e.*, as its name suggests (as intended in [1, 20, 21], see also the warning in [1, endnote 18]), always when motions are *empirically chaotic*:

Chaotic hypothesis (CH): A chaotic evolution takes place on a phase space M being attracted by a bounded smooth attracting surface $\mathcal{A} \subset M$ and on $\mathcal{A}$ the map S (or the flow S_t) is an Anosov map (or flow).

The SRB distribution μ has support on the attracting set $\mathcal{A}$: so smoothness of $\mathcal{A}$ is a strong assumption.[5] Of course there is the possibility that $\mathcal{A}$ is a surface of dimension lower than that of M, as specified in the CH.[6]

The CH is a general and heuristic (more restrictive) interpretation of original ideas on turbulence phenomena, [22], and has been introduced in [1, 23] to interpret simulations on evolutions with attracting set coinciding with the full phase space, and extended to the more general case in which the attracting surface is lower dimensional, [20, 21]).

Hereafter the CH will be supposed to hold *for all dynamical systems considered*, unless stated otherwise.

The above formulation in which the Anosov system is realized on a attracting surface $\mathcal{A} \neq M$, rather than on the full phase space, already hinted in [1], has become relevant as soon as attempts were undertaken to apply CH to systems for which the strict inclusion $\mathcal{A} \subset M$ was manifest, [20, 21].

Besides smoothness of the attracting set the further strong assumption of CH is that the evolution on $\mathcal{A}$ is hyperbolic in the sense of Anosov. Both aspects of the CH can be simultaneously weakened by supposing that the motion has an attractor which satisfies the "Axiom

flows) can be *integrated* (*i.e.* there is a smooth surface everywhere tangent to them) to define stable and unstable manifolds $W_u(x), W_s(x)$ which are dense on M.

[3] This means that for all times t_0 the closure of the trajectory of $\{S^t x\}_{t>t_0}$ is $\mathcal{A}$.

[4] Evolutions defined by maps arise typically when studying evolutions in continuous time through observations triggered by "timing events", *i.e.* observations that are made every time a specified event takes place: for instance every time that the evolving trajectory crosses a given surface in phase space. These are referred as observations performed on a *Poincaré's section*.

[5] Which contrasts the picture of $\mathcal{A}$ as a fractal set. In systems of $\sim 10^{20}$ molecules with a fractal attractor of dimension $6 \cdot 10^{19} + 3.141$ this means that it 'behaves' as a smooth surface of dimension $6 \cdot 10^{19}$; or in a Navier-Stokes fluid (an ∞-dimensional system) at large Reynolds number R an attracting set of dimension $R^{\frac{9}{3}} + .33$ 'behaves' as a smooth surface of dimension integral part of $R^{\frac{9}{3}}$.

[6] Often, if the dynamical system depends on a parameter ε, the chaotic motion might occupy, asymptotically, an attracting set $\mathcal{A}_\varepsilon$ with a dimension dependent on ε and equal to that of the full phase space only for a small (if any) interval of variability of ε.

A", [16]; however such generality will not be envisaged here.

It will be convenient to distinguish between *attractor* and *attracting set* $\mathcal{A}$: the latter is an invariant set approached asymptotically by all points x in its *basin* of attraction (which is an open set around $\mathcal{A}$): $\lim_{t\to\infty} distance(S_t x, \mathcal{A}) = 0$; while an attractor $A \subset \mathcal{A}$ is an invariant subset dense on $\mathcal{A}$ which has full SRB measure (*i.e.* $\mu(A) = 1$) and minimal Hausdorff dimension, often smaller than the dimension of $\mathcal{A}$ which, in turn, could be *much smaller* than the dimension of M.

The SRB distributions have strong ergodic properties (see Sec.XIV for some details) and in particular the average value over time of a smooth observable $O(x)$ is reached exponentially fast: however, more generally, it is possible that in M there are several attracting sets $\mathcal{A}_i$, each with its own SRB distribution, just as in equilibrium statistical mechanics there are cases in which the Gibbs state is not unique and the extremal ones correspond to different phases. It will appear that the analogy is a deep one, see Sec.VII.

Anosov systems are chaotic systems whose properties can be studied in *great detail*: certainly they correspond to an idealization of chaos; but it should be kept in mind that Statistical Mechanics arose from the idealization, far more surprising, that microscopic motion could be regarded as periodic, [24–26], see also [27].

II. STATIONARY DISTRIBUTIONS (SRB)

Let time evolution on M be a map S_r, (or a flow $S_{r,t}$), that may depend on a parameter r (or on more but imagine, to simplify, that only r will be varied). Then as r changes the stationary SRB distribution $\mu_r(dx)$, for the system (M, S_r), changes and the *collection* of such distributions will be called $\mathcal{E}^{mc}$ and its elements will be thought of as *ensembles of stationary states*.

Volumes of regions in phase space M *change*. in general, when transformed by the discrete evolution map S or by the flow S_t generated by a differential equation $\dot{x} = f(x)$; and the rate of change per unit volume can be measured from the Jacobian matrix $J(x)_{ij} = \partial_i S(x)_j$ for maps (here $\partial_i \stackrel{def}{=} \partial_{x_i}$) or by the matrix $J_{ij} = \partial_i f_j(x)$ in the case of a flow. And the *phase space contraction rate* is defined by:

$$\begin{aligned} &\sigma(x) = -\log|\det J(x)|, && \text{discrete evolution} \\ &\sigma(x) = -\operatorname{div} f(x) \equiv -\mathrm{Tr} J(x), && \text{continuous evol.} \end{aligned} \tag{2.1}$$

Changing variables or the metric on phase space implies a change of $\sigma(x)$ into $\sigma(x) + u(S_r x) - u(x)$ for a suitable function $u(x)$:[7] which implies that the time average $\lim_{T\to\infty} \frac{1}{T}\sum_{j=0}^{T} \sigma(S_r^j x)$, if existing, does not depend on the metric used on M. Likewise and with the same implication on the averages, in the continuous evolution systems, $\sigma(x)$ changes into $\sigma(x) + \dot{u}(x)$ (where $\dot{u}(x) \stackrel{def}{=} \sum_j \partial_{x_j} U(x)\, f(x)_j$ for a suitable function $U(x)$).[8]

The time average σ_+ of σ *i.e.* of $n \to \sigma(S^n x)$ or $t \to \sigma(S_t x)$ coincides, except for a set of x's with 0 volume, with the average $\sigma_+ = \int \sigma(x)\mu(dx)$ with respect to the SRB distribution μ.

Remark that σ_+ is a quantity which has the dimension of an inverse time in the case of continuous systems while it is dimensionless for maps, (as time is an integer in the case of maps). It will play an important role in the following, particularly when $\sigma_+ \neq 0$, as it sets a time scale[9] that will be called the *dissipation time scale*.

III. SYMMETRIES, (TIME REVERSAL)

Trying to evince information from the statistical properties of the stationary distributions of a time evolution, discrete or continuous, it is important to take into account symmetries of the underlying equations of motion, [29].

There are not many such symmetries and a key role is played by the fundamental symmetries, like translation and rotation invariance or time reversal, often enjoyed by the molecular constituents of the systems of interest and perhaps reflected by their macroscopic properties.

Of particular interest will be systems in which the evolution is S, discrete in time, or is a continuous time evolution S_t, which satisfies a "*time reversal*" symmetry, *i.e.* such that there is an ∞-smooth map $x \to Ix$ on the phase space M, *independent or smoothly dependent* on any parameter that might affect the dynamics, with the property

$$\begin{aligned} &(a)\ IS^{-1} = SI, && I^2 = 1, && \text{discrete evol.} \\ &(a')\ IS_{-t} = S_t I, && I^2 = 1, && \text{cont. evol.} \\ &(b)\ I \text{ is isometric} \end{aligned} \tag{3.1}$$

For isolated particle systems I is just the reversal of all velocities and it is a basic law of nature in Newtonian physics.

A typical situation is described in the following section presenting a simple, but quite general, model of a nonequilibrium system.

The model will also illustrate the notion of *phase space contraction* and its relation with the thermodynamic notion of *entropy generation*. It will appear that although there is a relation between entropy creation rate and

[7] *e.g.* a change in variables $y = w(x)$ leads to $u(x) = -\log|\det \partial_{x_j} w_i|$.

[8] *e.g.* changing x into $y = w(x)$ leads to $U(x) = -\log|\partial_{x_j} w_j|$.

[9] Since the CH-evolutions that we consider proceed towards a bounded attracting set it is $\sigma_+ \geq 0$, [28].

phase space contraction, still the two notions are quite different. *Nevertheless* their difference can be expressed as a variation of a suitable phase space observable evaluated at successive map iterations or, in the cases of flows, as a *time derivative* of a suitable observable: therefore it has no influence, or a controlled one, on the *average phase space contraction*, [2, Ch2.5].

IV. EXAMPLE: REVERSIBLE DISSIPATION

The system consists in $N \equiv N_0$ particles in a container $\mathcal{C}_0$ and of N_a particles in n containers $\mathcal{C}_a$ which play the role of *thermostats*: their positions will be denoted $\mathbf{X}_a$, $a = 0, 1, \ldots, n$, and $\mathbf{X} \stackrel{def}{=} (\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_n)$. Interactions will be described by a potential energy

$$W(\mathbf{X}) = \sum_{a=0}^{n} U_a(\mathbf{X}_a) + \sum_{a=1}^{n} W_a(\mathbf{X}_0, \mathbf{X}_a) \tag{4.1}$$

i.e. particles in different thermostats only interact indirectly, via the system. All masses will be $m = 1$, for simplicity.

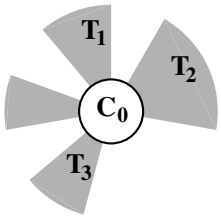

Fig.1: *The reservoirs occupy finite regions outside $\mathcal{C}_0$, e.g. sectors $\mathcal{C}_a \subset R^3$, $a = 1, 2 \ldots$. Their particles are constrained to have a total kinetic energy K_a constant, by suitable forces $\mathbf{F}_a$, so that the reservoirs "temperatures" T_a, are well defined, by $K_a = \sum_{j=1}^{N_a} \frac{1}{2} (\dot{\mathbf{X}}_{a,j})^2 \stackrel{def}{=} \frac{3}{2} N_a k_B T_a \stackrel{def}{=} \frac{3}{2} N_a \beta_a^{-1}$. The set-up, classical and quantum, is introduced in [30].*

Particles in $\mathcal{C}_0$ may also be subject to external, possibly non conservative, forces $\mathbf{F}(\mathbf{X}_0, \mathbf{E})$ depending on a few strength parameters $\mathbf{E} = (E_1, E_2, \ldots)$. It is convenient to imagine that the forces due to the confining potentials determining the geometrical shape of the region $\mathcal{C}_0$ are included in $\mathbf{F}$, so that one of the parameters is the volume $V = |\mathcal{C}_0|$. See Fig.1.

Following Sec.I the statistical properties of the stationary states of the system should be described, assuming the CH, by the SRB distributions $\mu_{\mathbf{E}}$ on phase space.

The equations of motion are:

$$\begin{aligned} \ddot{\mathbf{X}}_{0i} &= -\partial_i U_0(\mathbf{X}_0) - \sum_a \partial_i W_a(\mathbf{X}_0, \mathbf{X}_i) + \mathbf{F}_i \\ \ddot{\mathbf{X}}_{ai} &= -\partial_i U_a(\mathbf{X}_a) - \partial_i W_a(\mathbf{X}_0, \mathbf{X}_i) - \alpha_a \dot{\mathbf{X}}_a \end{aligned} \tag{4.2}$$

where the last term $-\alpha_a \dot{\mathbf{X}}_a$ is a *phenomenological* force that implies that thermostats particles keep constant total kinetic energies $K_a = \frac{3}{2} N_a k_B T_a$: α_a is therefore (as checked by direct computation of the time derivative of $K_a = \frac{1}{2} \dot{\mathbf{X}}_{a,j}^2$ defined by

$$\alpha_a \stackrel{def}{=} \frac{L_a - \dot{U}_a}{3 N_a k_B T_a} \tag{4.3}$$

where $L_a = -\partial_{\mathbf{X}_a} W_a(\mathbf{X}_0, \mathbf{X}_a) \cdot \dot{\mathbf{X}}_a$ is the work done per unit time by the forces that the particles in $\mathcal{C}_0$ exert on the particles in $\mathcal{C}_a, a > 0$; here k_B denotes Boltzmann's constant.

The exact form of the forces that have to be added in order to insure the kinetic energies constancy should not really matter, within wide limits. But this is a property that is not obvious and which is much debated.[10]

The work L_a in Eq.(4.3) will be interpreted as *heat* $\dot{Q}_a$ ceded, per unit time, by particles in $\mathcal{C}_0$ to the a-th thermostat. The *entropy production rate* due to heat exchanges between the system and the thermostats can, therefore, be naturally defined by

$$\sigma^0(\dot{\mathbf{X}}, \mathbf{X}) \stackrel{def}{=} \sum_{a=1}^{N_a} \frac{\dot{Q}_a}{k_B T_a} \tag{4.4}$$

because the "temperature" of $\mathcal{C}_a$ remains constant, and at stationarity the thermostats can be regarded in thermal equilibrium.

It should be stressed that here *no entropy notion* is introduced for the stationary state: only *variation* of the thermostats entropy is considered and it should not be regarded as a new quantity because, in the stationary states, the thermostats should be considered in equilibrium at a fixed temperature.

A question is whether there is any relation between σ^0 and the phase space contraction σ of Eq.(2.1) for the equations of motion in Eq.(4.2) (*i.e.* minus the divergence of the equations Eq.(4.2)).

The latter, in the recent literature, has been identified with the entropy production: and in the present case can be immediately computed by the appropriate differentiation of Eq.(4.2),(4.3) and is (neglecting $O(\min_{a>0} N_a^{-1})$)

$$\sigma(\dot{\mathbf{X}}, \mathbf{X}) = \sum_{a>0} \frac{3N_a - 1}{3N_a} \frac{\dot{Q}_a - \dot{U}_a}{k_B T_a} = \sum_{a>0} \frac{\dot{Q}_a}{k_B T_a} - \dot{U} \tag{4.5}$$

where $U = \sum_{a>0} \frac{3N_a - 1}{3N_a} \frac{U_a}{k_B T_a}$. Hence in this example, physically interesting, in which the thermostats are "external" to the system volume (unlike to what happens in several examples in which they act inside the volume of

[10] The above thermostatting forces choice can be seen to coincide with the ones obtained via Gauss' *least effort* principle for ideal anholonomic constraints applied to the constraints $K_a = const$, see [2, Ch.2]: this is a criterion that has been adopted in several simulations, [31, Sec.5.2,p103]. *Independently* of Gauss' principle it is immediate to check that if α_a is defined by Eq.(4.3) then the kinetic energies K_a are, strictly, constants of motion.

the system), the phase space contraction is not the entropy production rate, [2, Ch.2]. *However it differs from the entropy production rate by a total time derivative.*

Consequence: entropy creation rate σ^0 and phase space contraction σ differ, *but their time averages coincide.*

This is relevant because the definition Eq.((4.4)) *has meaning independently of the equations of motions* and can, therefore, be suitable for experimental tests, [2, Ch.5-2.Ch.3,Ch.4].

It should be stressed that the numbers N_a of particles in the reservoirs, $a > 0$, enters through $\frac{3N_a-1}{3N_a}$, hence is is essentially independent on the thermostat sizes (provided large).

Finally I mention that the identification, up to a total time derivative, of phase space contraction with entropy production rate can be shown, as discussed in Sec.VII , XV below, to cover the entropy production rate in systems whose evolution can be approximated by macroscopic continua equations, like fluids described by Navier-Stokes equations. In the sense that, again, phase space contraction of the interacting particles systems that underlay the macroscopic equations is related, in the stationary states, to the entropy production rate independently defined in classical nonequilibrium Thermodynamics, [29]: and, at most, it differs from it by a total time derivative, [2, Ch.4].

V. HAMILTONIAN DISSIPATION?

No entropy production is possible in a stationary state of a Hamiltonian system (*i.e.* an isolated system). At least not if it is finite. However things are different when the system is in contact with infinite systems.

As an example consider a Hamiltonian version of the model of Sec.IV, Fig.1 above. If the constraints on the kinetic energy in the containers $\mathcal{C}_a, a = 1, \dots, n$, are removed and the containers are extended to infinity, interesting stationary states can be obtained from initial configurations which, in each $\mathcal{C}_a$, are naturally chosen from the *canonical equilibrium ensemble* with density ρ_a and temperature T_a and from any distribution[11] for the particles in $\mathcal{C}_0$. The initial distribution will be called μ_0 and, although not invariant in time, it may evolve towards an invariant one, μ (as reasonable as this looks, however, a mathematical proof of this is *far from known*). The existence of μ will be assumed in this example.

Since the containers are infinite the stationary state that will be reached can be expected to keep an average kinetic energy per particle remaining $\frac{3}{2}k_BT_a$, identically equal to the initial value. For rather general models of microscopic interaction between particles, it can be shown, see for instance [32], that the time evolution of the particles in $\mathcal{C}_a$ that are far from the boundary of $\mathcal{C}_0$, are little affected by the interactions with the particles in $\mathcal{C}_0$ and the average kinetic energy per particle in each $\mathcal{C}_a$ will be an exact constant of motion, equal to the initial $\frac{3}{2}k_BT_a$, for all finite times (although it is still possible that in the limit of infinite time this might change).[12]

Consider as initial distribution $\mu_0(dx)$ which is a product of independent canonical distributions in each container $\mathcal{C}_a, a \geq 0$, with given densities and temperatures $\rho_a, T_a \equiv \frac{1}{k_B\beta_a}$.[13]

Although *now purely Hamiltonian* the system is infinite and the phase space volume *measured by the evolving distribution* $\mu_t(dx) = \mu_0(S_{-t}dx)$, changes per unit time by $\sigma(x)$ with:

$$\sigma(x) = \frac{d}{dt}\log\frac{\mu_0(S_{-t}dx)}{\mu_0(dx)} \equiv \sum_{a=1}^{n}\beta_a\dot{Q}_a + \beta_0\dot{Q}_0 \qquad (5.1)$$

where $\sigma(x)$ is computed from the equations of motion as:

$$\begin{aligned}\dot{Q}_a =& L_a = -\partial_{\mathbf{X}_a}W_a(\mathbf{X}_0, \mathbf{X}_a)\cdot\dot{\mathbf{X}}_a, \qquad a \geq 1\\ \dot{Q}_0 =& \dot{K}_0 + \dot{U}_0\end{aligned} \qquad (5.2)$$

and $\beta_0\dot{Q}_0 = \beta_0\left(\dot{K}_0 + \dot{U}_0\right) = \beta_0\,\mathbf{F}\cdot\dot{\mathbf{X}}_0 - \sum_{j>0}(\dot{U}_{0j} - \dot{Q}_j))$ is a time derivative so that it *does not contribute* to the time average $\sigma_+ = \lim_{T\to\infty}\frac{1}{T}\int\sigma(S_tx)dt$, [2, Ch.4].

Remark: If μ_0 is defined, as proposed above, as a product of independent canonical distributions it might be surprising that the system in $\mathcal{C}_0$ plays a special role: could one write the same formulae with $\mathcal{C}_1$ playing the role of $\mathcal{C}_0$ and find that $\dot{Q}_1$ is a total derivative of $U_1 + K_1$? However $U_1 + K_1$ is infinite unlike $U_0 + K_0$, so nothing can be concluded about its time derivative. If the regions $\mathcal{C}_j$ were finite then the system would evolve and all $\dot{U}_a + \dot{K}_a$ would uninterestingly average to 0: so $\mathcal{C}_0$ plays a special role and is the only container for which $U_0 + K_0$ and its time derivative are meaningful.

The expression Eq.(5.1), and the irrelevance of the contribution from $\dot{Q}_0$ to the average of σ (see Eq.(5.2)) is the key to the interpretation of fluctuations on σ in systems modeled by particles: even in cases (essentially in all experimental settings) in which the evolution is not describable in terms of equations of motion, in the sense that the equations of motion are not analytically known.

[11] With density on the phase space $\mathcal{C}_0 \times R^{3N_0}$.

[12] The physical picture is that the energy generated by work performed by the active forces on the particles in $\mathcal{C}_0$ and by the interactions between the particles in $\mathcal{C}_0$ and those in the thermostats is ceded to the thermostats creating in them heat currents J_a which decrease as the inverse of the square distance to $\mathcal{C}_0$: so the thermostats remain asymptotically, as the distance from $\mathcal{C}_0$ tends to ∞, in equilibrium.

[13] But the choice of T_0 has no particular physical meaning and the distribution in $\mathcal{C}_0$ could be replaced by "any" distribution, with some density on the $X_0, \dot{X}_0$ variables.

In such cases the average of $\sigma(x)$ (equal to that of $\sigma^0(x) = \sum_{a\geq 1} \beta_a \dot{Q}_a$) *is still accessible* via measurement of the heats exchanged and the temperatures of the reservoirs with which heat is exchanged. Of course it is a delicate and difficult task to measure *all* such quantities, *i.e.* the full entropy production.

It is remarkable, and it will be discussed in Sec.VI, that quite generally, under the Chaotic Hypothesis, that it becomes possible to obtain a general, universal, property of the (rare) fluctuations of the entropy production.

VI. FLUCTUATION RELATION (FR)

The *Fluctuation Relation* deals with the entropy production $\sigma^0(x) = \sum_a \frac{\dot{Q}_a}{k_B T_a}$, see (5.1),(5.2), or more generally, with the phase space volume contraction rate $\sigma(x) = -\sum_i \partial_{x_i} f_i(x)$ and it applies to *finite* systems whose evolution takes place on a phase space M, via an equation $\dot{x} = f(x)$, and is *time reversal symmetric* in the sense of Sec.III and hyperbolic on M (*i.e.* it is a Anosov system).[14] It equally deals with evolutions which are time reversible Anosov maps: attention will be mostly concentrated on the continuous time case, to avoid repetitions.

The $\sigma(x)$ is defined in terms of the metric on M and via appropriate covariant derivatives ∂_{x_i}: but it is simpler to imagine that M is a Euclidean space with coordinates measured in prefixed units, so that ∂_{x_i} are the usual partial derivatives.

In general $\sigma(x)$ *depends* on the metric and changing metric on M the expression of $\sigma(x)$ changes; however the variation can be, in general, expressed by the time derivative of a suitable observable so that the long time averages σ_+ of $\sigma(x)$ *do not depend on the metric*, like most physically interesting observables (*e.g.* the Lyapunov exponents). See comments following Eq.(2.1) and Eq.(5.2)

Given an attracting set $\mathcal{A}$ on which the evolution is an Anosov system, *i.e.* motion on $\mathcal{A}$ is a smooth continuous hyperbolic flow[2] $x \to S_t x$, or a smooth discrete hyperbolic map[2] $x \to Sx$; let $\sigma_{\mathcal{A}}(x)$ be the *surface contraction rate* on the surface $\mathcal{A}$ and consider the quantity

$$\begin{aligned} p =& \frac{1}{\tau}\int_0^\tau \frac{\sigma_{\mathcal{A}}(S_t)}{\sigma_+}dt \qquad \text{continuous time, } \tau > 0 \\ p =& \frac{1}{\tau}\sum_{k=0}^{\tau} \frac{\sigma_{\mathcal{A}}(S^k x)}{\sigma_+} \qquad \text{discrete time, } \tau \text{ integer} \end{aligned} \tag{6.1}$$

where σ_+ is the infinite time average of $\sigma_{\mathcal{A}}(x)$, which is x-independent, aside exceptional x's in a 0-volume set, and coincides with the average of σ with respect to the SRB distribution μ_{srb} on $\mathcal{A}$, *i.e.* the probability distribution generated by the motion of any point x chosen randomly with some density with respect to the volume, I.

Then consider the stationary μ_{srb}-probability of the above variable p, Eq.(6.1), and define its *large deviation rate* as a function $\zeta(p)$ such that:

$$P_{srb,\tau}(D) = e^{\tau \max_{p\in D}\zeta(p) + o(\tau)} \tag{6.2}$$

for all domains D (*i.e.* for all regions D which are closures of their interiors).

If the dynamical system is an Anosov system (or, more generally, if is satisfies the "axiom A") then

(a) the rate $\zeta(p)$ exists and is defined in the interior of an interval $[p', p'']$ containing $p = 1$,
(b) it is analytic in p if $p' < p''$ while it is $-\infty$ for $p \notin [p', p'']$
(c) $\sigma_+ = 0$ if and only if the SRB distribution μ admits a density over the attracting surface, [28].

A simple universal result for reversible Anosov systems, is the following *Fluctuation Theorem*, [1],[12, 33].

FT: *If the evolution is time reversal symmetric then the rate function*[15] *verifies the symmetry property:*

$$\zeta(-p) = \zeta(p) - p\sigma_+ \tag{6.3}$$

for $p \in [-p^*, p^*]$ *with* $p^* \geq 1$.

An immediate consequence is that if the Chaotic Hypothesis is considered valid, time reversibility holds and the attracting set can be supposed to coincide with the full phase space (so that $\mathcal{A} = M$ and $\sigma_{\mathcal{A}} \equiv \sigma$), then the entropy generation rate or more generally the phase space contraction rate are expected to satisfy the large deviation property which is, in this case, called *Fluctuation Relation*, FR, and it is *informally* written as

$$\log \frac{P_{srb}(p)}{P_{srb}(-p)} = \tau\, 1\, p\, \sigma_+ + o(\tau) \tag{6.4}$$

and more precisely formulated, as existence of a "large deviation" rate $\zeta(p)$ satisfying Eq.(6.3): where the 1 is inserted for later reference.

While the formal probability density for the events $\pm p$, *i.e.* $P_{srb}(\pm p)$, is a difficult quantity strongly dependent on the dynamical system, the interest of the FR is that Eq.(6.3),(6.4) are, under the above assumptions, an *exact symmetry* of $\zeta(p)$ and, at least in some cases, FR *deals with a quantity* ($\sigma_{\mathcal{A}}(x)$) *which has physical meaning* (entropy generation rate) and mathematical meaning (phase

[14] A positive $\sigma(x)$ means that the volume *contracts* near x: hence in stationary states the average σ_+ of σ must be ≥ 0, [28].

[15] The rate $\zeta(p)$ is defined so that the probability of finding $\frac{1}{\tau}\int_0^\tau \frac{\sigma(S_t x)}{\sigma_+}dtdt \in [p, p+\delta p]$ is $\exp \tau \max_{[p,p+\delta p]}\zeta(p)$ for $p \in (-p^*, p^*)$ where $p^* \geq 1$; it exists and is analytic if $\sigma(x)$ is the phase space contraction of an Anosov evolution.

space contraction rate): therefore a check of Eq.(6.4) can become a test of the chaotic hypothesis.

The fluctuations relation is, for time reversible evolutions, a symmetry of the SRB distributions. *However* it requires that:

(i) the motion on the attracting surface $\mathcal{A}$ has the Anosov property,
(ii) and *at the same time* it is reversible; hence, in the frequent cases in which $\mathcal{A}$ is not the full phase space but just a smooth surface in it, it should be $I\mathcal{A} = \mathcal{A}$, quite *unlikely* if I is the usual time reversal symmetry (*i.e.* velocities reversal),
(iii) furthermore, if (i) and (ii) hold, the FR concerns the fluctuations of the surface area of $\mathcal{A}$, and *not of the full volume*: which is very hard to access, as $\mathcal{A}$ itself.

and the three conditions strongly limit a literal applicability of FR and lead to the analysis of further properties of the considered evolutions, see Sec.IX , XVII.

Nevertheless it can be applied to systems that are only mildly out of equilibrium. If the system, remaining time reversal symmetric, is set out of equilibrium by the action of small forces and is in contact with thermostats with small differences of the respective temperatures, call ε a parameter measuring the size of the forces and of the temperature differences. Then, if for $\varepsilon = 0$ the system has the Anosov property on the full phase space, it will continue to have such property also for small $\varepsilon \neq 0$, because Anosov systems are structurally stable, [9]. Hence the attracting set $\mathcal{A}$ will remain identical to the full phase space and time reversal will be a symmetry of the motions on the attracting set and the FR assumptions will remain verified. This was the case of the systems to which the FR has been applied, [1], and tested, [21], to explain the fluctuations of the phase space contraction observed in the simulation in [34].

In attempting to test or use the FR in systems which are not very small it is not reasonable to hope that the smallness of the above ε does not depend on the system size, although important cases (lattices of coupled Anosov maps) are known in which ε can be taken independent of the size, [11, 35]: therefore it might be thought that FR becomes irrelevant in most interesting cases, [1]. Clearly more properties are needed to deal with the systems that are not small perturbations of Anosov systems. In Sec.IX the *applicability far beyond the latter cases* will be discussed.

Remark: Often the Eq.(4.5) raises the question "how can it be relevant" as the Boltzmann's constant in the denominator is likely to give a huge value to the inverse time scale $\langle\sigma\rangle$ which determines the time scale over which the FR yields predictions? For instance imagining to put $1\,cm^3$ of steel (with faces of $1\,cm^2$) in contact between two reservoirs at temperatures $T = 300^oK$ and $T + \delta T = 310^oK$ the average of the entropy production rate, $\dot{Q}\frac{\delta T}{k_B T^2}$, can be expressed via the steel thermal conductivity χ as $\chi(\frac{\delta T}{T})^2\frac{\Delta}{k_B}$: and the result is $\sim 10^{18}\,sec^{-1}$, see also [36, p.4]. If FR could be applied *literally* there would be no way to see a heat flow from cold to warm during $10^{-6}sec$ before "trying" to see it, say once every second, for at least $\sim 10^{18}/10^6$ times (*i.e.* $\sim 10^3$ billion days). See Sec.IX for a possible answer to the problem. [16]

The FR bears *formal* similarity with identities arising in the evolution of equilibrium states, or more generally with the evolution of initial distributions on phase space *which are symmetric under time reversal but not stationary*. The deep difference between the latter identities and the above FR is briefly commented in Sec.XIX) below.

Unfortunately the name "fluctuation relation" has been often used in all cases, causing great confusion to loom on the subject.

VII. NONEQUILIBRIUM ENSEMBLES. ENSEMBLES EQUIVALENCE

In general given an evolution equation on a phase space M depending on one or more parameters, denoted $\mathbf{E} = \{\nu, E, \ldots\}$, the SRB stationary states, *i.e.* the distributions that are generated by all points of M, excepting a subset of M with 0 volume, form a collection $\mathcal{E}^c$ of probability distributions $\mu_{\mathbf{E}}$ parameterized by the given parameters and each of which can be called an "ensemble".

Even in the cases, considered in this section, in which there is only one parameter ν and CH holds, there might be several distinct attracting surfaces and therefore more than a single SRB distribution μ_ν: if so, further parameters will have to be added to distinguish the various possibilities, *just* as done in equilibrium statistical mechanics in presence of phase transitions to distinguish the different pure phases, [37–39].

A key question is whether the same system can be described by *different* equations of motion. There are several instances in which this is possible: for instance a fluid motion can be equally well described by, say, a Navier-Stokes equation or by a (far more complex) collection of molecules, in contact with a thermostat and at given density, at least if attention is given to observations depending on large scale properties and performed over long time scales, [40].

Even for Navier-Stokes (NS) fluids there might be several different equations, simpler than the ultimate molecular models, that can describe the class of phenomena considered relevant in given physical situations.

For instance it has been convincingly argued that macroscopic transport coefficients can be obtained by replacing the equations of motion of molecules by simple(r)

[16] The problem is considered, by some colleagues, a "disaster" for FR, making it physically irrelevant.

models, suitable for simulations, obeying modified equations of motion which can even be non-Newtonian: in the context of molecular simulations this has been originated in the early '80s, [8, 31]. A first example of equations alternative to the NS equations to describe a developed turbulence flow is found in [41].

It is natural to consider, together with the collection of SRB distributions $\mu_\nu \in \mathcal{E}^c$ for a given equation depending on a parameter ν, the collection $\mu'_E \in \mathcal{E}'$ of SRB distributions corresponding to a different equation parameterized by a new parameter denoted E, which on physical or just heuristic grounds describes equivalently the same class of phenomena, *i.e.* predicts the same properties for large classes of observables.

Remarks: (i) The equivalence should mean that it is possible to establish a correspondence between the ensembles (*i.e.* the distributions) in $\mathcal{E}^c$ and the ones in $\mathcal{E}'$ so that for each ν there is a corresponding $E(\nu)$ and the average of "most" observables in the $\mu_\nu \in \mathcal{E}^c$ and $\mu'_{E(\nu)}$ in $\mathcal{E}'$ should coincide (or be close).
(ii) This is quite analogous to the description of equilibrium states in Statistical Mechanics (SM): the canonical distribution μ^V_β of $N = \rho V$ molecules of a gas in a container of volume V depends, at fixed density ρ, on a parameter (inverse temperature) $\beta = (k_B T)^{-1}$ and the microcanonical distribution $\mu^{'V}_E$ depends on a parameter (total energy) E. If E and β are so related that

$$\mu^{'V}_E(\sum_{i=1}^{N} \frac{1}{2m} p_i^2) = \frac{3}{2} N \beta^{-1} \tag{7.1}$$

then the average of "many" observables O, $\mu^{'V}_E(O)$, is equal or close to $\mu_\beta(O)$.
(iii) Actually, in SM, in the limit as $V \to \infty$, $\rho = N/V$ *fixed*, for any *local observable*. *i.e.* depending only on the configuration of the molecules located in a finite region, the canonical and microcanonical averages are not only close but *strictly equal*, at least in absence of long range forces or of phase transitions, [37, 39].
(iv) And, still considering (SM), in presence of phase transitions at β it will be necessary to label the distributions in $\mathcal{E}^c$ by further parameters a: in this case the distributions in the ensemble $\mathcal{E}'$ will also have to be distinguished by an equal number of parameters a' and a correspondence between a and a' can be established so that, under the condition Eq.(7.1), it is still $\mu^{'V}_{a',E}(O) = \mu^V_{a,\beta}(O)$, for local observables, in the limit $V \to \infty$.

A first example is obtained by considering a system described by equations on $x \in R^n$ which are obtained as follows

(a) let $\dot{x} = G(x)$ be a time reversible equation for the time reversal $Ix = -x$ (*i.e.* $G(x) = G(-x)$)
(b) add a reversible forcing $f(x)$, with $If = fI$ (*i.e.* $f(x) = f(-x)$)
(c) and a dissipative term, $-\nu Lx$, with L linear and positive ($Lx \cdot x > 0$, if $x \neq 0$) depending on a parameter ν, whose effect is to balance, in the average, the "energy" injected by the forcing

The complete equation has therefore the form

$$\dot{x} = G(x) + f(x) - \nu L x \tag{7.2}$$

At fixed f and for each "friction" $\nu > 0$ small enough, the evolution will be supposed to satisfy the CH and to lead to a unique stationary ("SRB") distribution μ_ν.

The collection of the SRB distributions μ_ν, as ν varies, will be denoted $\mathcal{E}^c$, and each of them defines a *nonequilibrium ensemble*.

Next consider a *different* equation in which the friction coefficient ν in Eq.(7.2) is replaced by a multiplier $\alpha(x)$ so defined that a selected observable Ω is an exact constant of motion. For instance the cases $\Omega(x) = x^2$ or $\Omega(x) = (x \cdot Lx)$ lead to new equations of motion $\dot{x} = G(x) + f(x) - \alpha(x) L x$ with, respectively:

$$\begin{aligned} \alpha(x) =& \frac{x \cdot G(x) + x \cdot f(x)}{x \cdot Lx} \qquad \text{or} \\ \alpha(x) =& \frac{Lx \cdot G(x) + Lx \cdot f(x)}{Lx \cdot Lx} \end{aligned} \tag{7.3}$$

Then the stationary states for the new equations form a collection of stationary states $\mathcal{E}'$ with elements parameterized by the value E of the constant of motion Ω introduced by the multiplier α.

Quite generally the motion generated by the new equations is eventually restricted to a bounded region, because of the action of the friction and of conservation laws possibly valid for the time reversible system in absence of forcing.

Therefore for ν small it can be expected that in the stationary states $\alpha(x)$ fluctuates leading to a *homogenization phenomenon*, *i.e.* to the property that in the stationary state for the new equation

$$\dot{x} = G(x) + f(x) - \alpha(x) L x \tag{7.4}$$

large classes of observables have the same averages in the distribution μ^c_ν and in the distribution μ'_E belonging to the new ensemble $\mathcal{E}'$, of stationary distributions for Eq.(7.4), *provided* ν and E are kept related by $\nu = \mu'_E(\alpha)$:

$$\lim_{\nu \to 0} \mu^c_\nu(O) = \lim_{\nu \to 0} \mu'_E(O), \tag{7.5}$$

or, equivalently, if $E = \mu_\nu(\Omega)$. More formally:

If motions following Eq.(7.2) eventually develop on a ball M generating a family $\mathcal{E}^c$ of stationary distributions parameterized by ν and if the motions following the Eq.(7.4) are also eventually confined in a ball M', generating a family $\mathcal{E}'$, then Eq.(7.5) holds for arbitrarily fixed observables O, provided the correspondence between μ^c_ν and $\mu'_{E(\nu)}$ is such that $\nu = \mu'_{E(\nu)}(\alpha)$ or, equivalently, $E(\nu) = \mu_\nu(\Omega)$.

As in the case of ensembles equivalence in equilibrium not all ensembles are equivalent, not even in the thermodynamic limit, therefore the observables Ω defining the ensemble have to be selected on a case by case basis.

The above statement has been tested in a few cases: involving strongly truncated NS equations, [42],[17] Lorenz96 equations, [44], shell model for turbulence, [45].

The conjecture will be analyzed in some detail, and considerably strengthened, in Sec.XVII for the stationary states of the incompressible NS equation with periodic boundary conditions. But it is convenient to discuss first in which sense the FR can be made relevant for systems irreversibly evolving in presence of strong friction, and to exhibit a few more applications of the FR to classical and new problems.

In particular a *key problem* is whether the FR can be of any utility if the attracting set $\mathcal{A}$ is a surface of dimension lower than that of M and, although the evolution equations remain reversible, it is $I\mathcal{A} \neq \mathcal{A}$, *i.e.* reversibility does not hold as a symmetry for motions on the attracting set (as, instead, required for the validity of the FR), see comments (i-iii) in Sec.VI .

VIII. STRONG DISSIPATION: ATTRACTING SET SIZE. LYAPUNOV PAIRS.

As forcing and dissipation increase the attracting set $\mathcal{A}$ may become a small subset of phase space: and, if the CH holds, it becomes a smooth surface of dimension *lower* than the full dimension of phase space.

In this case although the motion on $\mathcal{A}$ is a Anosov system it may appear, at first, that it does not even make sense to ask whether a FR holds because:

(1) it is not possible to think that it could express properties of the volume contraction on $\mathcal{A}$: the main difficulty is that time reversal symmetry, essential for the FR, is lost since, even if the equations of motion are time reversible, the action of the time reversal I will likely map the attracting set $\mathcal{A}$ into a "repelling" set $I\mathcal{A}$ disjoint from $\mathcal{A}$. The motion on $\mathcal{A}$ remains hyperbolic, as assumed by CH, but it no longer has the desired symmetry in time.
(2) Furthermore the FR deals with the volume contraction on the full phase space but the hyperbolic character (assumed by the CH) of the motion on the attracting set could establish, *if* for some reason a new time reversal $\widetilde{I}$ were spawned as a symmetry on $\mathcal{A}$, a property of the contraction of surface elements in $\mathcal{A}$; however their analysis would require a, highly unlikely, detailed understanding the geometry of the attracting surface $\mathcal{A}$.

[17] Doubts have been raised in [43]: which might be related to the use of a rather large value of ν in a strongly truncated NS equation in 2D: it is hoped that the latter results will be tested again at smaller ν (in spite of computational difficulties).

The latter two, seemingly insurmountable, difficulties are however intertwined and tend, in several cases, to "compensate". We begin with a simple case.

A remarkable property was discovered for a Hamiltonian evolution with n degrees of freedom for $x = (\mathbf{p}, \mathbf{q})$ with $H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{q})$ and *subject also* to a friction force $-\nu\mathbf{p}$. Namely, under very general conditions on the potential V (typically just boundedness of the surfaces $H = const$), the Lyapunov exponents $\lambda_0 \geq \lambda_1 \geq \lambda_{d/2-1} \geq \dots \lambda_{2n-1}$ of the motion are such that, [46],

$$\frac{1}{2}(\lambda_j + \lambda_{2n-1-j}) = -\frac{\nu}{2} \qquad j = 0, \dots, n-1 \qquad (8.1)$$

In other words the symplectic symmetry of the Hamiltonian systems (which in absence of friction implies $\lambda_j + \lambda_{2n-1-j} \equiv 0$) leaves, in presence of friction, Eq.(8.1) as a "remnant", at least *if the friction force has the simple form* $-\nu\mathbf{p}$. Furthermore Eq.(8.1) holds identically for the eigenvalues $\lambda_j(x)$ of the Jacobian matrix $J(x)$ of the flow at each point x.

Remarkably the relation Eq.(8.1) has been extended, [47], to the time reversible cases in which the friction $-\nu\mathbf{p}$ is replaced by a force $-\alpha(\mathbf{p}, \mathbf{q})\mathbf{p}$ with the multiplier α such that evolution conserves the total kinetic energy $\frac{1}{2}\mathbf{p}^2$ exactly, as in some of the simplest thermostat models, [48], *i.e.* $\alpha = -\frac{\mathbf{p}\cdot\partial_{\mathbf{q}}V(\mathbf{q})}{\mathbf{p}^2}$.

In the latter systems Eq.(8.1) not only holds with ν replaced by the time average of α but it follows from the stronger property that the evolution $t \to S_t x$ is such that, given $t_0 > 0$, the matrix $W = \partial_i (S_{t_0} x)_j$ has the property that the logarithms of the eigenvalues of $(W^T W)^{\frac{1}{2}}$ are t_0 times $\lambda_{t_0,j}(x) > 0$, $j = 0, \dots 2n$ (depending on t_0), which satisfy $\frac{1}{2}(\lambda_{t_0,j}(x) + \lambda_{t_0,2n-1-j}(x)) = -\frac{\nu}{2}$ or respectively $-\frac{1}{2}\langle\alpha\rangle$, where the average is intended over $S_t x$ for $t \in [0, t_0]$.[18]

[18] The proof of the pairing symmetry in the above mentioned cases is that the Jacobian matrix $\partial_i (S_t x)_j|_{t=0}$ is seen to be the sum of the Jacobian for the Hamiltonian flow of $H(\mathbf{p}, \mathbf{q}) - \frac{\nu}{2}(\mathbf{p}^2 + \mathbf{q}^2)$ plus the identity times $-\frac{\nu}{2}$, [46]. In the case of α a similar property holds replacing ν with α, as is seen via a calculation. If $J(t) = \partial_i (S_t x)_j$ then $J(t) P J(t)^T = P e^{-n\nu t}$ (or $P e^{-n\int_0^t \alpha(x(t))dt}$) where $P = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$ ($\mathbf{1}$ being the $n \times n$ identity) because $J(t) \stackrel{def}{=} J_0 e^{-\frac{1}{2}\nu t}$ with J_0 a symplectic matrix so that $J_0 P J_0^T = P$ (proposition 24, Sec.3.12 in [49]). Therefore let v be an eigenvector $J_0^T J_0 v = \lambda v$ then the following chain of identities, using $P^2 = -1$ shows that λ^{-1} is an eigenvalue, with eigenvector Pv:

$$J_0^T J_0 v = \lambda v \;\to\; P J_0^T J_0 v = \lambda P v \;\to\; -J_0^{-1} P J_0 P P v = \lambda P v$$
$$\to\; J_0^{-1} J_0^{-1T} P v = \lambda P v \;\to\; (J_0^T J)^{-1} P v = \lambda P v$$

impliyng pairing to $-\frac{\nu}{2}$ (respectively to $-t^{-1}\int_0^t \frac{1}{2}\alpha(x(t))dt$) for the matrix $(J(t)^T J(t))^{\frac{1}{2}}$. [46, 47].

Eq.(8.1), called *pairing symmetry*, is certainly very special, [8], but it suggests, [21], that the dimension of the attracting set $\mathcal{A}$ is equal to *twice the number of non negative Lyapunov* exponents: because it suggests that the pairs with two negative exponents simply correspond to the phase space compression in the directions that "stick out" of the surface $\mathcal{A}$.

If so the latter directions certainly do not contribute to the contraction of the surface of $\mathcal{A}$.[19]

An arbitrary number of negative exponents can be added to any spectrum by adding arbitrarily many dimensions whose coordinates contract to 0. It is only if there is a pairing symmetry that the negative pairs can be *conjectured* to be unambiguously identified: and it can be hoped that the same remains valid if the pairing is only approximate, which is a property that is *often encountered*, see Sec.XVIII for examples.

This idea has been discussed in the analysis of a simulation dedicated to tests of the CH and FR in a system with pairing symmetry, [21, Sec.6], and its relevance for strongly dissipative systems like the Navier-Stokes flows has been proposed in [51, Sec.5], see Sec.XVIII below.

Remarks: (1) Accepting the above proposal, the dimension of the attracting surface $\mathcal{A}$ is determined when the system has a (possibly approximate) pairing symmetry, and it is identified as twice the number of non negative Lyapunov exponents.
(2) It is worth stressing the general difference between the latter dimension, that will be called *fluctuation dimension* of $\mathcal{A}$ (or *fd-dimension*), and the Kaplan-Yorke dimension of $\mathcal{A}$: the Kaplan-Yorke dimension (or *ky-dimension*) is a measure of the fractal properties of the SRB attractor contained in the attracting set $\mathcal{A}$ and it is not larger than the fluctuation dimension; with which it coincides if the SRB distribution has a density on phase space. In general the ky-dimension is a fraction of the fd-dimension.
(3) The above discussion, *heuristically* proposes how to determine the dimension of the attracting surface under the CH when the pairing symmetry holds.

However it is unclear whether the pairing symmetry, exact or approximate, can be of any help to address the second of the above difficulties, *i.e.* the lower dimension of the attracting set and the accompanying breakdown of time reversal symmetry for the motions confined to $\mathcal{A}$, the only ones of statistical interest, and the consequent apparent irrelevance of the phase space contraction $\sigma(x)$, to which *also* contribute the contracting directions sticking out of the attracting surface.

The second of the two difficulties mentioned is addressed in the next section, on the basis of the proposal in [20, 51].

IX. DISSIPATION. TIME REVERSAL & FR.

Consider a time reversible evolution depending on a forcing parameter and, still assuming the CH, suppose the forcing, hence the dissipation, to grow so strong that the attracting set $\mathcal{A}$ becomes a surface of dimension smaller than that of phase space.

Then the time reversal symmetry I is *spontaneously broken* in the sense that it ceases to be a symmetry for the motions that develop on the attracting set. It remains a symmetry for the motions in phase space, but it has little relevance for the statistical properties (with respect to the SRB distribution) of the motions because, asymptotically, they are attracted to $\mathcal{A}$. The time reversal image $I\mathcal{A}$ of $\mathcal{A}$ is quite generally a *repeller* and no motion (except a set of data with 0 volume) evolves towards it.

Therefore a natural question is whether the continuing existence of the global time reversal symmetry I can be accompanied by a map $\widetilde{I}$ of $\mathcal{A}$ to itself which is still a smooth isometry, with $\widetilde{I}^2 \equiv 1$ and $S_t\widetilde{I} = \widetilde{I}S_{-t}$ in the flow case or $S\widetilde{I} = \widetilde{I}S^{-1}$ in the case of maps.

The question has been analyzed in [20] where a *geometric property* has been identified which, when holding, shows that a "local time reversal symmetry" $\widetilde{I}$, defined as a map of the attracting set $\mathcal{A}$ into itself, is *spawned out* of a global time reversal symmetry I, as a parameter varies and changes the dimension of $\mathcal{A}$ making it a surface of dimension smaller than the dimension of the phase space M.

The latter property will seem at first sight quite special. However it is enjoyed by a class of systems of interest in applications and at the same time is a *structurally stable property* (*i.e.* it remains valid under small perturbations of the dynamics). The property was named "Axiom C" because it is a modification of the "Axiom B" property introduced in [10].

To visualize the geometry of the Axiom C property consider the simpler case of a time reversible map S and imagine that the attracting surface $\mathcal{A}$ becomes disjoint from its time reversal image $I\mathcal{A}$, because a parameter controlling the evolution is raised above a critical value, see Fig.2.

Then the stable manifolds of the points in $\mathcal{A}$ are not entirely contained in $\mathcal{A}$ but extend out of $\mathcal{A}$ and intersect $I\mathcal{A}$ on manifolds which are *unstable manifolds* for the points of $I\mathcal{A}$. Likewise the evolution S^{-1} will have $I\mathcal{A}$ as

[19] Failure to realize the difference between Anosov motions on the attracting surface versus Anosov motions on the entire phase space in chaotic systems satisfying the CH together with time reversal symmetry is responsible for statements, [8, p.220], like: "*If there were such systems then it could be proved that they would generate relatively simple attractors, with equal numbers of positive and negative Lyapunov exponents. Because the simple geometric argument of Section 7.8 shows that nonequilibrium attractors are actually generated by any stable time reversible, steady dynamics, the applicability of the Anosov proofs is evidently rare to vanishing*", [1],[20],[50].

a attracting set out of which the S^{-1}-stable manifolds of the points of $I\mathcal{A}$ emerge and extend until they intersect the surface $\mathcal{A}$ on its unstable manifolds.

So out of each point x of $\mathcal{A}$ emerge two manifolds intersecting $\mathcal{A}$ respectively on the contracting and expanding manifolds at x restricted to $\mathcal{A}$ for S_t and *at the same time* the two manifolds intersect also $I\mathcal{A}$ and the intersections are the stable and unstable manifolds for S at some point $x' = Ix$ (linked by a 1-dimensional curve).

The correspondence $x' = Px$, thus established between $\mathcal{A}$ and $I\mathcal{A}$, *commutes* with the time evolution, because the manifolds whose intersection defines the correspondence $P : I\mathcal{A} \longleftrightarrow \mathcal{A}$ are covariant under the action of S: hence $PSx = SPx$ for all $x \in \mathcal{A}$ or $x \in I\mathcal{A}$.

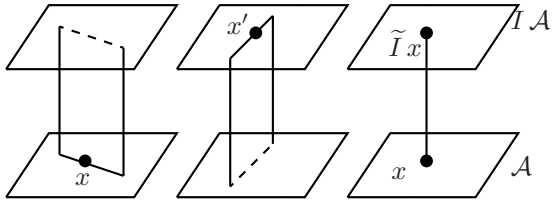

Fig.2: *Case of a map S. The first figure in Fig.2 illustrates a point $x \in \mathcal{A}$ and its attracting manifold, and a local part of its stable manifold that extends until $I\mathcal{A}$ intersecting it in the hatched line (stable manifold on $\mathcal{A}$ and unstable on $I\mathcal{A}$). Likewise the second figure describes a point x' on $I\mathcal{A}$ with a local part of its stable manifold for S^{-1} (extending to intersect $\mathcal{A}$ on a unstable manifold, hatched). The third figure shows the (1-dimensional) intersection between the stable manifold of a point $x \in \mathcal{A}$ and the unstable manifold of the point $x' \in I\mathcal{A}$: in the figure such intersection is a unidimensional curve that connects x with x' (uniquely determined by x) establishing the correspondence $P : \mathcal{A} \to I\mathcal{A}$ defining P, with $x' = Px$.*

The picture requires a few assumptions of technical nature to avoid occurrence of some more complex possibilities (for instance it is necessary to *exclude* that the contracting manifold emerging from $\mathcal{A}$ wraps around $I\mathcal{A}$ rather than meeting it transversally): the mathematical definition of the "axiom C" property can be found in [20] and is a modification of the notion of "axiom B", [10].

A consequence is that the map $\widetilde{I} = PI$ maps $\mathcal{A}$ into itself (as well as $I\mathcal{A}$ into itself) and is a time reversal symmetry for the restriction of S to $\mathcal{A}$ (and to $I\mathcal{A}$).

The above analysis exhibits a structurally stable mechanism, [20], which, if holding, implies that although time reversal is *lost as a symmetry* on an attracting set $\mathcal{A}$, it might be accompanied by a new map $\widetilde{I}$ on $\mathcal{A}$ which can be regarded as a *new* time reversal symmetry for motions evolving on $\mathcal{A}$. Therefore it is interesting to see whether a FR can also be established: heuristic ideas about such question, with attention to a few possible applications will now be presented in the rest of this section.

In presence of a pairing symmetry to the level $-\frac{1}{2}\nu$, Eq.(8.1), suppose that the pairs of negative exponents describe the approach to the attracting set and call n_+ the maximum number of *non negative* Lyapunov exponents. Then the local exponents with labels $j = 0, \dots, n_+ - 1$ and the corresponding negative ones contribute $\sigma_{\mathcal{A}}(x) = \sum_{j=0}^{n_+} \lambda_j(x) + \lambda_{2n-1-j}(x) = n_+\nu$ to the phase space contraction and $n - n_+$ pairs of negative exponents should be discarded in computing $\alpha_{\mathcal{A}}$; so that the total average phase space contraction on the attracting set $\sigma_{\mathcal{A},+}$ will be proportional to the total average phase space contraction (*i.e.* average σ_+ of the divergence of the equation of motion) $\sigma_{\mathcal{A},+} = n_+\nu = \frac{n_+}{n} n\nu = \frac{n_+}{n}\sigma_+$.

Remark that the number $n - n_+$ is defined in terms of the Lyapunov exponents: hence it does not depend on the point x. The conclusion is that in systems with *time reversal and pairing symmetry* satisfying the CH and axiom C, a fluctuation relation for the surface contraction of $\mathcal{A}$, $\sigma_{\mathcal{A}}(x) = \frac{n_+}{n}\sigma(x)$, holds.[20]

Set $\frac{n_+}{n} \equiv \frac{\mathcal{N}_{attr}}{\mathcal{N}}$ with $\mathcal{N}_{attr}$ = dimension of the attracting surface and $\mathcal{N}$ = dimension of the phase space. Then the CH combined with Axiom C and a parity property will give, for the probability of $\tau^{-1}\int_0^\tau \frac{\sigma(S_t x)}{\sigma_+} dt = \tau^{-1}\int_0^\tau \frac{\sigma_{\mathcal{A}}(S_t x)}{\sigma_{\mathcal{A},+}} dt$, the relation

$$\frac{P_{srb}(p)}{P_{srb}(-p)} = e^{\tau \frac{\mathcal{N}_{attr}}{\mathcal{N}}\sigma_+ p + o(\tau)} \tag{9.1}$$

in the notation of Eq.(6.4), [21], because $\sigma_{\mathcal{A},+} = \frac{\mathcal{N}_{attr}}{\mathcal{N}}\sigma_+$: *i.e.* the universal constant 1 in (6.4) is replaced by $\frac{\mathcal{N}_{attr}}{\mathcal{N}}$.

This covers the FR in systems verifying the pairing rule: *but, admittedly, such systems are not really common in the applications.*

A much larger class of systems can be imagined if the Lyapunov exponents, arranged as in Eq.(8.1), satisfy $\frac{1}{2}(\lambda_j + \lambda_{2n-1-j}) = c_j$ with c_j close to a constant (*i.e.* $c_j \sim C(\frac{j}{2n})$ with $C(\xi)$ a smooth function.

This property arises in a few important cases. For instance in simulations of reversible models for the 2D incompressible Navier-Stokes equation in periodic geometry, [42, 53].

Suppose that the local Lyapunov exponents are such that $\frac{1}{2}(\lambda_j(x) + \lambda_{2n-1-j}(x)) = c_j(x)$ and let $\overline{\lambda}_j, \overline{c}_j$ be the respective time averages. Then it can be still imagined that corresponding pairs $\overline{\lambda}_j, \overline{\lambda}_{2n-1-j}$ of exponents of *opposite sign* are exponents concerning the motion on the attracting set; particularly if the system is close to one for which the pairing rule holds and $\overline{c}_j$ is close to a constant.

By an argument similar to the one presented in the pairing symmetrical cases above, a large deviations relation might be obtained, for the total contraction $\sigma(x)$,

[20] It would seem that smoothness of $\widetilde{I}$ should also be required because the axiom C implies only Hölder continuity for $\widetilde{I}$, see [52]. However a careful examination of the FT proof shows that it is sufficient that the restrictions to $\mathcal{A}$ of the stable and unstable manifolds of the points $a \in \mathcal{A}$ are smooth manifolds and this is implied by the assumed smoothness of $\mathcal{A}$ itself (by CH) and by the smoothness of the global manifolds.

similar to Eq.(9.1) with $\frac{\mathcal{N}_{attr}}{\mathcal{N}}\sigma$ replaced by $\sigma_{\mathcal{A}}(x) = P\sigma(x)$ with $P \stackrel{def}{=} 1 - \frac{\sum_{j\in L_-}\overline{\lambda}_j}{\sum_{j\in L}\overline{\lambda}_j}$, where L is the set of Lyapunov exponents and L_- the subset formed by the pairs of negative exponents. Leading to a FR *with a controlled modification of the slope* in p, at least if the pairing functions $c_j(x)$ can be found, see Sec.XVIII for a non trivial example.

The above may apply to time reversible systems with Lyapunov spectrum obeying a pairing rule at least approximately; and could be extended, possibly, to irreversible ones if the latter fall under the equivalence properties mentioned in Sec.VII.

Remarks (i) In this respect it is worth coming back to the issue mentioned in the remark concluding Sec.VI. Which pointed out that when considering many particle systems, like for instance the steel cube brought up as an example, the phase space contraction might have an average too large, due to the size of the Boltzmann's constant. And consequently the FR would fail to be of any relevance for the fluctuations statistics.
(ii) However in such cases, as in any macroscopic system, the phase space contraction is very large: but the FR should be applied to the contraction of the surface of the attracting set. To do so the above axiom C and a pairing property may be of help: for instance in presence of exact pairing the FR holds with the σ_+ replaced by $\frac{\mathcal{N}_{attr}}{\mathcal{N}}\sigma_+$. The dimension $\mathcal{N}_{attr}$ in large systems will usually be $\ll \mathcal{N}$: and in order to apply the FR the average phase space contraction must be corrected by the factor $\frac{\mathcal{N}_{attr}}{\mathcal{N}}$.
(iii) Since $\mathcal{N}$ is, in the case (i), a number of the order of a multiple of Avogadro's number this is sufficient to turn k_BT_a into $R_0 = \mathcal{N}k_B$, with R_0 being of the order of the gas constant: and this converts Eq.(4.5) to $\mathcal{N}_{attr}\sum_a \frac{\dot{Q}_a}{R_0T_a}$ and R_0T_a is no longer very small.
(iv) Also $\mathcal{N}_{attr}$, in the systems like the ones in the example in Sec.IV, V, is typically *not of the order of Avogadro's number*: macroscopic systems often can be described by macroscopic equations and the number of positive exponents can be identified with the number of positive exponents of the "equivalent" macroscopic equation; the latter very often has a rather small number of positive exponents so $\mathcal{N}_{attr}$ can be small: in conclusion the FR could be applied to classes of system which admit microscopic and macroscopic equivalent representations.
(iv) A non trivial example is analyzed in Sec.XVII , XVIII.

X. FLUCTUATION DISSIPATION THEOREM

Suppose that a dynamical system equations

(a) depend on parameters $\mathbf{E} = (E_1, E_2, \ldots)$ and satisfy a $\mathbf{E}$-independent, smooth, time reversal symmetry I,[21]
(b) for $\mathbf{E} = \mathbf{0}$ the equations are supposed to satisfy the CH with attracting set coinciding with the full phase space (as in the cases in the footnote [21]).[22]

For $\mathbf{E} \neq \mathbf{0}$ the equations continue to be time reversal symmetric with the same symmetry map I.[23]

The dynamics is a Anosov system and it remains such at small $\mathbf{E}$: the attracting set coincides with the full phase space (by the structural stability of CH) and the FT holds for the SRB distributions.[24]

It is therefore interesting to find whether the average phase space contraction $\sigma_{\mathbf{E},+}$ is a function of $\mathbf{E}$ with interpretation that goes beyond its being a quantity associated with universal large fluctuations of the dissipation. In particular it is interesting to find an interpretation of the multiple derivatives $\boldsymbol{\partial}^{\mathbf{n}}_{\mathbf{E}}\sigma_{\mathbf{E},+}|_{\mathbf{E}=\mathbf{0}}$.

The phase space contraction $\sigma_{\mathbf{E}}(x)$, briefly $\sigma(x)$, will be supposed to have the Taylor expansion:

$$\sigma(x) = \sum_{i=1}^{s} E_i J_i^0(x) + O(\mathbf{E}^2) \tag{10.1}$$

and, having assumed CH, the large deviation rate $\zeta(p)$ exists (model dependent) and is analytic in p for p in the interval $(-p^*, p^*)$, $p^* \geq 1$, within which it can vary, [33, 55].

On general grounds the function $\zeta(p)$ is the Laplace transform of $\lambda(\beta) = \lim_{\tau\to\infty}\frac{1}{\tau}\log\int e^{\beta\tau(p-1)}P_\tau(dp)$ where $P_\tau(dp)$ is the PDF of the variable $p = \frac{1}{\tau\sigma_{\mathbf{E},+}}\int_0^t \sigma(S_t x)dt$ in the SRB distribution. Once $\lambda(\beta)$ is "known" then $\zeta(p)$ is recovered via a Legendre transform; $\zeta(p) = \max_\beta\left(\beta\langle\sigma\rangle_+(p-1) - \lambda(\beta)\right)$, [33, 56].

By using the cumulant expansion for $\lambda(\beta)$ we find that $\lambda(\beta) = \frac{1}{2!}\beta^2C_2 + \frac{1}{3!}\beta^3C_3 + \ldots$ where the coefficients C_j are $\int_{-\infty}^{\infty}\langle\sigma(S_{t_1}\cdot)\sigma(S_{t_2}\cdot)\ldots\sigma(S_{t_{j-1}}\cdot)\sigma(\cdot)\rangle_+^T\,dt_1\ldots$ if $\langle\ldots\rangle_+^T$ denote the cumulants of the variables $\sigma(x)$.

In our case the cumulants of order j have size $O(G^j)$ with $G \stackrel{def}{=} |\mathbf{E}|$, by Eq.(10.1), so that:

$$\zeta(p) = \frac{\langle\sigma\rangle_+^2}{2C_2}(p-1)^2 + O((p-1)^3G^3) \tag{10.2}$$

(remark that the first term in r.h.s. gives the central limit theorem). Eq.(10.2), together with the FR Eq.(6.3),

[21] To fix ideas think of a Hamiltonian system constrained to keep the total kinetic energy constant, for instance via a Gaussian constraint, as considered in many applications, [31]: in absence of external forcing, and assuming CH, the SRB distribution is quite generally explicitly known and equivalent to the canonical distribution, [31].
[22] In other words at $\mathbf{E} = \mathbf{0}$ the evolution is a Anosov system.
[23] The symmetry could also depend on $\mathbf{E}$, becoming $I_{\mathbf{E}}$, however further assumptions would be needed, like differentiability, [54].
[24] Remark that this is an important case whose occurrence has been considered "*rare to evanescent*" in [8, p.220].

yields at fixed p the key relations:

$$\langle\sigma\rangle_+ = \frac{1}{2}C_2 + O(G^3) \tag{10.3}$$

Define, [33, 56]: $J_i(x) = \partial_{E_i}\sigma(x)$ = *current*, $L_{ij} = \partial_{E_j}\langle J_i(x)\rangle_+|_{\mathbf{E}=\mathbf{0}}$ = *transport coefficients*; and study L_{ij}.

In the *r.h.s.*of the first of Eq.(10.3) discard $O(G^3)$: it becomes quadratic in $\mathbf{E}$ with coefficient $\frac{1}{2}C_2$, making use of the exponential decay of SRB-correlations in Anosov systems, given by:

$$\frac{1}{2}\int_{-\infty}^{\infty} dt\,\left(\langle J_i^0(S_t\cdot)J_j^0(\cdot)\rangle_+ - \langle J_i^0\rangle_+\langle J_j^0\rangle_+\right)\big|_{E=0} \tag{10.4}$$

where convergence is implied by the strong mixing properties of the SRB distribution due to the CH.

On the other hand the expansion of $\langle\sigma\rangle_+$ in the *l.h.s.*of Eq.(10.3) to second order in E gives:

$$\langle\sigma\rangle_+ = \frac{1}{2}\sum_{ij}\left(\partial_{E_i}\partial_{E_j}\langle\sigma\rangle_+\right)\big|_{E=0}E_iE_j \tag{10.5}$$

because the first order term vanishes, see Eq.(10.1).

If $\mu_+(dx)$ denotes the SRB distribution, the *r.h.s.*of Eq.(10.5) is the sum of $\frac{1}{2}E_iE_j$ times $\partial_{E_i}\partial_{E_j}\int\sigma(x)\mu_+(dx)$ which equals the sum of the following three terms:
(i) $\int\partial_{E_i}\partial_{E_j}\sigma(x)\mu_+(dx)$,
(ii) $\int\partial_{E_i}\sigma(x)\partial_{E_j}\mu_+(dx) + (i\leftrightarrow j)$
(iii) $\int\sigma(x)\partial_{E_i}\partial_{E_j}\mu_+(dx))$, all evaluated at $\mathbf{E}=\mathbf{0}$.

The first addend is 0 (by time reversal), the third addend is also 0 (as $\sigma = 0$ at $\mathbf{E}=\mathbf{0}$). Hence:

$$\partial_{E_i}\partial_{E_j}\langle\sigma\rangle_+|_{E=0} = \Big(\partial_{E_j}\langle J_i^0\rangle_+ + \partial_{E_i}\langle J_j^0\rangle_+\Big)|_{E=0} \tag{10.6}$$

and it is easy to check, again by using time reversal, that:

$$\partial_{E_j}\langle J_i^0\rangle_+|_{E=0} \;=\; \partial_{E_j}\langle J_i\rangle_+|_{E=0} = \; L_{ij} \tag{10.7}$$

Thus equating r.h.s and l.h.s. of Eq.(10.3), as expressed respectively by Eq.(10.4) and Eq.(10.6) the matrix $\frac{L_{ij}+L_{ji}}{2}$ is obtained, [57].

At least if $i=j$ this is a "Green-Kubo formula", a relation sometimes called "fluctuation dissipation theorem". It is however *very different* from "Onsager's reciprocity" which would be $L_{ij} = L_{ji}$. The latter will be discusses in the next section.

XI. ONSAGER'S RECIPROCITY

A far reaching extension is necessary to obtain $L_{ij} = L_{ji}$ which will lead to reciprocity, [55], and to further extensions, [58].

The main remark is that FT theorem can be *extended* to give properties of *joint* SRB distribution of $\sigma(x)$ and of the observable $q(x) = E_j\partial_{E_j}\sigma$. Defining *dimensionless* j-current $q = q_j(x)$ (at fixed j) as:

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} E_j\partial_{E_j}\sigma(S_tx)dt \stackrel{def}{=} q \tag{11.1}$$

where the factor E_j is there only to keep σ and $E_j\partial_{E_j}\sigma$ with the same dimensions, the really essential property of $q_j(x)$ is its odd *symmetry* under time reversal, as $\sigma(x)$.

Then if $P_\tau(dp,dq)$ is the joint PDF of p,q the *same* proof of the FT in [1, 12, 33] yields also the existence of a rate function $\zeta(p,q)$ for P_τ with the symmetry:

$$\zeta(p,q) = \zeta(-p,-q) \,+\, p\,\sigma_{\mathbf{E},+}, \qquad \text{for all } p,q \tag{11.2}$$

for the *joint large fluctuations* of the variables $\sigma_{\mathbf{E}}(x)$, $E_j\partial_{E_j}\sigma(x)$.

The $\zeta(p,q)$ can be computed, in the same way as $\zeta(p)$ in Sec.X, by considering first the transform $\lambda(\beta_1,\beta_2)$:

$$\lim_{\tau\to\infty}\tfrac{1}{\tau}\log\int e^{\tau(\beta_1\,(p-1)\langle\sigma\rangle_+ + \beta_2\,(q-1)\langle E_j\partial_{E_j}\sigma\rangle_+)}P_\tau(dp,dq) \tag{11.3}$$

and then the Legendre transform, abridging the SRB average $\langle\cdot\rangle_{\mathbf{E},+}$ with $\langle\cdot\rangle_+$,

$$\begin{aligned}\max_{\beta_1,\beta_2}\big(&\beta_1\,(p-1)\langle\sigma\rangle_+ + \beta_2\,(q-1)\langle E_j\partial_{E_j}\sigma\rangle_+ \\ &- \lambda(\beta_1,\beta_2)\big) \;=\; \zeta(p,q)\end{aligned} \tag{11.4}$$

The function $\lambda(\boldsymbol{\beta})$, $\boldsymbol{\beta} = (\beta_1,\beta_2)$, is evaluated by the cumulant expansion, as above, and one finds:

$$\lambda(\boldsymbol{\beta}) = \frac{1}{2}\left(\boldsymbol{\beta}, C\,\boldsymbol{\beta}\right) + O(E^3) \tag{11.5}$$

where C is the 2×2 matrix of the second order cumulants. The coefficient C_{11} is given by C_2 appearing in Eq.(10.3),(10.4); C_{22} is given by the same expression with σ replaced by $E_j\partial_{E_j}\sigma$ while C_{12} is the mixed cumulant:

$$\int_{-\infty}^{\infty}\left(\langle\sigma(S_t\cdot)\,E_j\partial_{E_j}\sigma(\cdot)\rangle_+ - \langle\sigma(S_t\cdot)\rangle_+\,\langle E_j\partial_{E_j}\sigma(\cdot)\rangle_+\right)dt \tag{11.6}$$

and convergence is again implied by the mixing properties of the SRB distributions due to the CH.

Hence if $\mathbf{w} = \begin{pmatrix}(p-1)\langle\sigma\rangle_+\\ (q-1)\langle E_j\partial_{E_j}\sigma\rangle_+\end{pmatrix}$ we get:

$$\zeta(p,q) = \frac{1}{2}\left(C^{-1}\mathbf{w},\mathbf{w}\right) + O(E^3) \tag{11.7}$$

completely analogous to Eq.(10.2). But the FT in Eq.(11.2), implies that $\zeta(p,q) - \zeta(-p,-q)$ is q *independent*: this immediately means:

$$-(C^{-1})_{22}\langle E_j\partial_{E_j}\sigma\rangle_+ - (C^{-1})_{21}\langle\sigma\rangle_+ = 0 + O(E^3) \quad (11.8)$$

which, from $(C^{-1})_{22} = C_{11}/\det C$, becomes the analogue of Eq.(10.3):

$$\langle E_j\partial_{E_j}\sigma\rangle_+ = \frac{1}{2}C_{12} + O(E^3) \quad (11.9)$$

Then, proceeding as in the derivation of Eq.(10.4) through (10.7) (*i.e.* expanding both sides of Eq.(11.9) *to first order* in the E_i's and using Eq.(11.8) we get that $\partial_{E_i}\langle\partial_{E_j}\sigma\rangle_+$ is given by the integral in Eq.(10.4). This means that $L_{ij} = L_{ji}$ and the general Green-Kubo formulae follow together with Onsager's reciprocity.

Thus GK, and OR, are in the cases considered here, a consequence of FT and of its extension, Eq.(11.2), in the limit $E \to 0$, when combined with the expansion Eq.(10.2) for entropy fluctuations. Those theorems and the fast decay of the $\sigma\sigma$ correlations, [12], are all natural consequences of (CH) for reversible systems (which are the starting point of our considerations). Reversibility is here assumed *both in equilibrium and in non equilibrium*: this is a feature of Gaussian thermostat models but by no means of all models; the **E**-independence of the reversibility map is also essential but in most reversible models it is just the velocity reversal map, which is independent of **E**.

Of course the OR and GK only hold around equilibrium, *i.e.* they are properties of **E**–derivatives evaluated at $\mathbf{E} = 0$; on the other hand the expansion for $\lambda(\beta)$ is a general consequence of the correlation decay and the FT also holds for *non equilibrium* stationary states, *i.e.* for $\mathbf{E} \neq \mathbf{0}$ small[25], and *can be considered a generalization of the OR and GK.*

Evidence for the relation between L_{ii}, Green-Kubo formulae, and FT was pointed out by P.Garrido in [21] in an effort to interpret results of various numerical experiments and an apparent incompatibility between the *a priori* known non Gaussian nature of the distribution $\pi_\tau(p)$ and the "Gaussian looking" empirical distributions; the extension to the reciprocity followed naturally (see also [55, 57]). In [21] the situation arising at really large fields, when the attractor is strictly smaller than the whole phase space, is also discussed (eventually leading to the analysis in Sec.IX above).

XII. FLUCTUATION PATTERNS

The derivation of Onsager's reciprocity for reversible Anosov systems with a time reversal map I smooth and parameters independent (usually just a "velocity reversal"), and therefore for systems verifying the Chaotic Hypothesis, suggests that the fluctuation relations might be extended to fluctuations of more general observables. At least for small perturbations of Anosov systems and for smooth Axiom C systems,[IX].

Consider first the fluctuations of the phase space contraction $\sigma(x)$ and those of a second observable $\varphi(x)$ with *definite parity* under time reversal: so $\sigma(Ix) = -\sigma(x)$ and $\varphi(x) = -\varphi(Ix)$ (or $\varphi(x) = \varphi(Ix)$).

Consider a SRB distribution μ_{srb} for the system: let $\langle\sigma\rangle_+ > 0, \langle\varphi\rangle_+$ be the SRB time averages of σ, φ. Call "fluctuation pattern" π a function on $[0,\tau]$: $t \to \pi(t) = (s(t), f(t))$.

The evolution of a point x in phase space such that

$$\begin{aligned} &|s(t) - \sigma(S_t x)| < \varepsilon, \\ &|f(t) - \varphi(S_t x))| < \eta \end{aligned} \qquad \text{for } t \in [0,\tau] \quad (12.1)$$

will be called a motion which *shadows the pattern* π *in the time interval* $[0,\tau]$ and it will be written $x \overset{\tau,\varepsilon,\eta}{\sim} \pi$.

The "time reversal" of the pattern π will be the pattern $I\pi = (-s(\tau - t), -f(\tau - t))$ (or if $\varphi(x)$ is even under time reversal $I\pi = (-s(\tau - t), f(\tau - t))$).

The SRB probability of a trajectory $x \to S_t x$ to follow a pattern π will be denoted $P_\tau(\{x \overset{\tau,\varepsilon,\eta}{\sim} \pi\})$; the argument at the basis of the Fluctuation Theorem can be applied to study the ratio:

$$\frac{1}{\tau}\log\frac{P_\tau(\{x \overset{\tau,\varepsilon,\eta}{\sim} \pi\})}{P_\tau(\{x \overset{\tau,\varepsilon}{\sim} I\pi\})} \quad (12.2)$$

and for reversible Anosov systems leads, at first surprisingly, immediately to the result:

$$\frac{P_\tau(\{x \overset{\tau,\varepsilon,\eta}{\sim} \pi\})}{P_\tau(\{x \overset{\tau,\varepsilon,\eta}{\sim} I\pi\})} = e^{\tau\,\langle\sigma\rangle_+\, p + o(\tau)} \quad (12.3)$$

asymptotically as $\tau \to \infty$ and to lowest order in the precision ε, η.

More generally several observables can be considered $\sigma, \varphi_1, \varphi_2, \ldots$ and the notion of pattern can be accordingly extended; with the same result that the ratio of the probability of a fluctuation pattern to that of the time reversed pattern is $e^{\tau\,\langle\sigma\rangle_+\, p + o(\tau)}$, to leading order as $\tau \to \infty$ and in the precision, *independent on the specification* of the fluctuations of $\varphi_1, \varphi_2, \ldots$.

Also the φ independence of Eq.(12.2) implies, given two fluctuation patterns π for the observables σ, φ and π' for the observables σ, ψ, to leading order in $\varepsilon, \delta, \tau^{-1}$:

$$\frac{P_\tau(\{x \overset{\tau,\varepsilon,\delta}{\sim} \pi\})}{P_\tau(\{x \overset{\tau,\varepsilon,\delta}{\sim} \pi'\})} = \frac{P_\tau(\{x \overset{\tau,\varepsilon,\delta}{\sim} I\pi\})}{P_\tau(\{x \overset{\tau,\varepsilon,\delta}{\sim} I\pi'\})} \quad (12.4)$$

The above relations show that *once the rare event of a sign change of the entropy production* is realized then

[25] *i.e.* as long as structural stability maintains the system an Anosov system.

the time reversed patterns have the *same relative probability* that they have when the entropy production has the opposite sign.

In other words to see that time reversed patterns occur it is "sufficient" to just *change the sign of entropy production* (!): "no further efforts" are needed.

In Sec.XI the Eq.(11.2) has been shown to be essentially equivalent to Onsager's reciprocity and it is a special case of the general Eq.(12.3): therefore the above Eq.(12.3) can be considered an extension of Onsager's reciprocity to stationary states of time reversible Anosov systems or more generally (if σ_+ is intended as the average area contraction of the attracting surface $\mathcal{A}$) to systems verifying the CH and the "axiom C", see Sec.IX and footnote[IX].

XIII. IRREVERSIBILITY TIME SCALE

The notion of "reversible transformation" between equilibrium states is defined (often) to be an *infinitely slow* transformation through a sequence of equilibrium states. The latter *oxymoron* is not really satisfactory:[26] it would be instead desirable to have a quantitative definition or, better, a way to associate with a transformation, defined by a given "protocol" leading from an equilibrium state to another, a time scale Θ whose size indicates how long it takes to realize that the process is irreversible.

Then a reversible transformation should be one with $\Theta = \infty$ (to be interpreted that irreversibility is *impossible* to detect). If $\Theta < \infty$ then it should be said that the evolution irreversible nature is revealed after time Θ which could be taken as the *irreversibility time scale*, [2, Ch.5-11].

So let μ_0 be the PDF of an equilibrium state and suppose that the protocol of action on the system is enforced by a change on the parameters on which the Hamiltonian depends: like the temperature of an external thermostat, or the volume available to the molecules, in the case of a gas enclosed in a container, or like the intensity of a volume force acting on an incompressible fluid.

The protocol has a duration τ and remains constant afterwards: during the time τ the system is no longer in equilibrium: the latter is reached after the time τ elapsed and the system remains isolated or in contact with thermostats at the same temperature reaching the new equilibrium on a characteristic time scale τ'.[27]

In the following the general system introduced in Sec.IV, see Fig.1, will be considered to fix ideas. The entropy production is given by Eq.(4.5). It is a quantity *with dimension of an inverse time*, coinciding with the phase space total contraction rate.

In nonequilibrium situations the thermostats temperatures can be time dependent and also the force f as well as the volume of the container $\mathcal{C}_0$ can be time dependent. The thermostats temperatures are fixed phenomenologically and the mechanism of variation of the stirring forces and of the geometric variation of the container shape or volume are more difficult to understand and to model physically.

For instance the variation of the force f can be imagined due to the varying speed of a paddle, rotating in the gas contained in $\mathcal{C}_0$, which in turn can be imagined to be controlled by a motor; but it is impossible to take into account, without a *Daemon* helping, how to keep control of the direction and intensity of the collisions on the paddle. Hence assuming that the paddle has constant speed, or that it follows a given protocol of variation, is a phenomenological assumption.

There are experimental setups in which a paddle, or varying forces, are present and act on the particles in the container $\mathcal{C}_0$. Or the external thermostats temperatures and the volume of $\mathcal{C}_0$ change following prescribed paths, *e.g.* in the case of volume variations due to a moving piston. Invariably the entropy production is measured via the amount of work that the motor and the forces perform maintaining (or trying to maintain) the external force constant, or constraining it to follow a prefixed protocol, and via the heat ceded to the thermostats.

Here few cases will be considered in which the protocol contemplates only variations of the external thermostats temperatures or of the volume of the containers.

Given the general interpretation of the entropy production rate in terms of phase space volume variation, the case of volume variation in the system of Fig.1 (Sec.IV) can be treated *phenomenologically* by simply adding to Eq.(4.5) the quantity $N\frac{\dot V}{V}$, which is the rate of variation of the phase space volume V^N allowed to the N particles in $\mathcal{C}_0$.

Consider the system in Fig.1, Sec.IV, and express the total phase space contraction per unit time, (4.5), as

$$\sigma(x) \equiv \sigma_{tot}(\dot{\mathbf{X}}, \mathbf{X}) = \sum_a \frac{\dot Q_a}{k_B T_a} - N\frac{\dot V_t}{V_t} - \dot U \qquad (13.1)$$

Let $[0,\tau]$ be the time during which a transformation protocol $\Gamma: t \to (T_a(t), V(t)) \equiv P(t)$ acts on an initial *equilibrium state* with SRB distribution $\mu_0(dx)$ (*e.g.* a canonical "Gibbs distribution"). Then it is possible to define

(1) $\mu_t(dx) = \mu_0(S_{-t}dx)$, *i.e.* the distribution into which μ_0 evolves in time t under the flow generated in phase space by the equations of motion (remark that S_t is not a group in t because the evolution is now time dependent).
(2) the *SRB* distribution $\mu_{srb,t}$ corresponding to the stationary distribution that corresponds to parameters

[26] "À la vérité, les choses ne peuvent pas se passer rigoureusement comme nons l'avons supposé ..., [59, p.13-14].

[27] Strictly speaking equilibrium will be reached after infinite time; however it can be considered reached for practical purposes after τ', which has the meaning of a time scale.

(T_a, V) fixed ("frozen") at their value at time t, $P(t) = (T_a(t), V(t))$.
(3) the "relative" phase space contraction

$$r(t) \stackrel{def}{=} (\overline{\sigma}_t - \overline{\sigma}_{srb,t}) \tag{13.2}$$

where $\overline{\sigma}_{srb,t}$ is the time average of the entropy production rate in the SRB distribution corresponding to the control parameters $P = (T_a, V)$ frozen at time t, while $\overline{\sigma}_t$ is the average phase space contraction in the non stationary distribution $\mu_t(dx)$ evolved from μ_0. [28]

Assuming the chaotic hypothesis the approach to the SRB states will be exponential: the state μ_t would evolve under the "frozen evolution" exponentially fast, on some time scale[29] κ^{-1} to $\mu_{srb,t}$. Therefore the integral (with a inverse time dimension):

$$\Theta(\Gamma)^{-1} = \int_0^\infty r(t)^2 dt \tag{13.3}$$

will converge, provided the final values of the control parameters are reached fast enough (*e.g.* in a finite time, as in actual protocols which last a prefixed finite time).

The *time scale of irreversibility* of the protocol could be defined by $\Theta(\Gamma)$: the larger Θ is, the closer to a quasi static one the transformation is, as suggested by the following remarks.

A physical definition of "quasi static" transformation protocol is a transformation that is "very slow" during its duration time τ. This can be translated mathematically into an evolution in which $P(t) \stackrel{def}{=} (T_a(t), V(t))$ evolves like, if not exactly, as

$$P(t) = P(0) + (1 - e^{-\varepsilon t})(P(\infty) - P(0)) \tag{13.4}$$

with $\varepsilon > 0$ small.

An evolution Γ "close to quasi static", but simpler for computing $\Theta(\Gamma)$, would proceed changing $P(0)$ into $P(\infty) = P(0) + \Delta$ by τ/δ steps of size δ, each of which has a time duration t_δ long enough so that, at the k-th step, the evolving system closely settles onto its stationary state $P(0) + k\delta$.

The t_δ can be defined[30] by $e^{-\kappa t_\delta} \ll \kappa\delta$ then by Eq.(13.3):

$$\Theta(\Gamma)^{-1} \simeq const\, \kappa^{-1} (\overline{\sigma}'\delta)^2 \log(\kappa\delta)^{-1} \tag{13.5}$$

where $\overline{\sigma}'$ is an estimate of $\partial_t \overline{\sigma}_{srb,t}$. Therefore the "slower" is the protocol Γ (*i.e.* the larger the time scale δ^{-1} is) the closer to ∞ is the irresversibility scale $\Theta(\Gamma)$.

[28] The two terms in Eq.(13.1) have in general different dependence on particles number N: as of $O(N)$ in the case of volume variations or $O(N^{\frac{2}{3}})$ if $P(t)$ only involves boundary temperature variations (hence the heat exchange is a boundary effect).
[29] Supposed to be the same for all t, for simplicity.
[30] Remark that the variation of $\overline{\sigma}_{(k+1)\delta,+} - \overline{\sigma}_{k\delta,+}$ is, in general, of order $\kappa\delta$ as a consequence of the differentiability, [60], of the SRB states with respect to the parameters.

Another way of reading the above: *the closer the actual entropy production $\overline{\sigma}_t$ is to the "ideal" $\overline{\sigma}_{srb,t}$ the longer is the irreversibility time scale, i.e. the time beyond which the process cannot be considered reversible.*

Remark: particularly interesting are adiabatic processes in which external forces vary remaining conservative:
(a) an example is an adiabatic expansion of a gas in a piston. The irreversibility time scale can be evaluated from the piston velocity, see [2, Ch.5].
(b) a second example is a rarefied gas, with mass m molecules in a *fixed* adiabatic container, subject to a force of potential mgz. At time 0 the gas is in equilibrium at temperature β^{-1} and the process G simply raises the acceleration g to $g' > g$ at time 0 and then decreases it back to g after a time $\tau > 0$ (or just stays g' forever).[31] Since $\sigma_t \equiv 0$ (by Liouville's theorem) and $\sigma_{srb,t} \equiv 0$ it is $\Theta = \infty$: *i.e.* the transformation is *reversible* according to the above proposal of reversibility time scale, as also suggested by Gibbs' entropy constancy in Hamiltonian evolutions (even when the Hamiltonian is time dependent). Nevertheless the cycle leads to an intermediate temperature *variation* δT (with $\frac{\delta T}{T} \simeq \frac{\delta g}{g}$, up to finite volume corrections): an apparent disagreement with the independence on the rapidity of the process, see Appendix C for details.

XIV. CHAOS. STRUCTURE OF ANOSOV SYSTEMS. THEIR DIGITAL CODES.

Since the early works on Statistical Mechanics the concept of *coarse graining* played a major role in relating macroscopic and microscopic descriptions of mechanical systems, [29].

Anosov systems, through the chaotic hypothesis, offer *new* perspectives. For simplicity here will be considered the case of a discrete time evolution via a map S on a phase space M, which could be a Poincaré's section of either a macroscopic model of evolution (possibly infinite dimensional, like Navier-Stokes equation) or a microscopic one (like Newton's equations for 10^{19} molecules) or a phenomenological model (like Lorenz96 or Lorenz63 models or GOY shell model).

What follows can be extended to the case of Anosov flows, [16, 56], essentially by reduction of the problem to the Anosov maps case by replacing the flow with a Poincaré's map between timed events (*i.e.* by fixing a surface in phase space and studying the return map to it). Extension to axiom A maps or flows is also possible, [16, 61, 62].

The discussion below is necessarily somewhat technical as it tries to convey the reason why Anosov maps lead to

[31] This means that the initial potential energy is $mgNh_0$, where h_0 is the height of the center of mass, and varies at time 0 to $mg'Nh_0$.

stationary states which can be identified with equilibria of *one dimensional spin chains with short range interactions*: the extreme simplicity of Anosov maps will be manifest after understanding the formalism. It will reward the necessary time, thus providing strong support to the statement (see Sec.I) that Anosov maps play, for chaotic systems, a role parallel to that of the harmonic oscillators for ordered dynamics.

A main feature of Anosov maps is that the stable and unstable manifolds of each point x are smooth manifolds which depend "almost" differentiably on x (they are Hölder continuous and the Hölder exponent can be taken as close to 1 as wished, paying the price of a larger Hölder constant). The manifolds can be used to build "cells" in M enclosed within boundaries which are unions of subsets of stable or unstable manifolds.

The key remark, [14, 15], is that the phase space M can be paved with cells, $\mathcal{P} = (P_1, \dots, P_N)$, which are connected sets, closures of their interiors, and are either pairwise disjoint or have only common boundary points; furthermore are "*covariant*" if transformed by the map S in the following sense:

(1) the boundary ∂P_j of P_j has the form $\partial_u P_j \cup \partial_s P_j$ with $\partial_u P_j$ consisting of surface elements which are unions of portions of unstable manifolds and $\partial_s P_j$ consisting of surface elements unions of portions stable manifolds: call $\partial_u \mathcal{P} = \cup_j \partial_u P_j$ and $\partial_s \mathcal{P} = \cup_j \partial_s P_j$.
(2) the images $S^{\pm 1} P_j$ of the cell P_j will have boundary still consisting of stable or unstable surface elements (because images of stable or unstable manifolds are still stable or unstable manifolds) and, furthermore, will have the *covariance property*, see Fig.3:

$$S\partial_s P_j \subset \partial_s \mathcal{P}, \quad S^{-1}\partial_u P_j \subset \partial_u \mathcal{P} \tag{14.1}$$

This means that the P_j are so deformed by S (resp. S^{-1}) that no new stable (resp. unstable) boundaries are created. Furthermore the points x in their evolution will never end up on any of the cells boundaries with the exception of a set of zero volume (*i.e.* the set $\cup_{i=-\infty}^{\infty} S^i \partial \mathcal{P}$).
(3) the $\partial_u \mathcal{P}, \partial_s \mathcal{P}$ have 0 volume

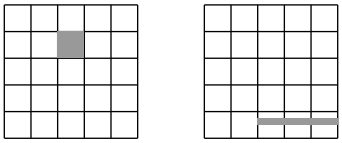

Fig.3: *Very symbolically, as 2-dimensional squares, a few elements of $\mathcal{P}$ are shown as an array of squares. An element P_i (shaded, left) of $\mathcal{P}$ is transformed by S into SP_i (shaded, right) in such a way that the part of the boundary that contracts ends up exactly on a boundary of some element among $P_1, P_2, \dots, P_n$. A similar figure with horizontal and vertical lines exchanged would illustate the action of S^{-1}.*

Obviously if $\mathcal{P}$ is a pavement of M with the above properties (1),(2) then also the pavement whose elements are $P_{i,j} = SP_i \cap P_j$ has the same properties: hence the hyperbolicity of the map yields that there exist pavements with elements with diameter smaller than a prefixed $\varepsilon > 0$.

Such pavements $\mathcal{P} = (P_0, \dots, P_n)$ of M are called *Markovian partitions* if the maximum diameter of the P_i's is so small that the intersection between $S^{\pm} P_i$ and P_j is a connected set: as inthe figure.[32]

The hyperbolicity of S implies existence of Markovian partitions and they can be constructed iteratively, [11, 15, 61, 63].

The elements of $\mathcal{P}$ are called "rectangles" as they have boundaries formed by portions of stable and unstable manifolds which in the case of the simplest Anosov maps, *i.e.* algebraic hyperbolic maps of the 2-dimensional torus, are really quadrilaterals with opposite sides parallel and equal: algrebraic means that the maps are defined by a constant matrix with integer entries, no eigenvalue with modulus 1 and determinant ± 1 (like $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$).

In dimension 2, in general, they have the aspect of deformed rectangles (as the manifolds constituting their boundaries are neither parallel nor flat) with smooth boundaries. If the map S is an algbraic map of the 2-torus (*i.e.* S is a 2×2 matrix with integer entries) they are rectangles, in the literal sense.

In ≥ 3 dimensions the intersections between the stable manifolds and the unstable manifolds meeting at the edges of the rectangle are not smooth: in general a portion of unstable manifold of dimension $u > 1$, contained in ∂P_i, may have a boundary[33] which does not contain a smooth surface of dimension $u - 1$ (*e.g.* in dimension 3 and if the unstable manifold had dimension 2 it does not contain a differentiable arc, as one might naively imagine, [64]: *i.e.* the edge is not a smooth line).

Likewise a portion of stable manifold, of dimension $s > 1$, contained in ∂P_i, may have a boundary which does not contain a smooth surface of dimension $s - 1$. So the rectangles edges may be *quite rugged*. Nevertheless the boundaries of the sets P_i can be shown to have *zero volume*.

Given a point $x \in M$ its *history* $\boldsymbol{\sigma} = (\sigma_k)_{k=-\infty}^{\infty}$, $\sigma_k = 1, 2, \dots, n$, on a Markovian partition $\mathcal{P}$ is defined by

$$S^k x \in P_{\sigma_k}, \quad \forall k \in (-\infty, \infty) \tag{14.2}$$

uniquely with the exception of the set, with zero volume in M, of the points $x \in \cup_{-\infty}^{\infty} S^k \partial \mathcal{P}$, *i.e.* except the set of points which in their evolution fall on the boundary of some of the $P_i \in \mathcal{P}$. The history is a *digital code* for the points of M and the labels k can, naturally, be called *"times"*.

The history is very convenient as it transforms the evolution $x \to Sx$ into the simple "translation": if x has history $\boldsymbol{\sigma} = \{\sigma_i\}_{i=-\infty}^{\infty}$ and x evolves into Sx then its history $\boldsymbol{\sigma}$ evolves into $\tau\boldsymbol{\sigma} = \{\sigma_{i+1}\}_{i=-\infty}^{\infty}$.

[32] Disconnected intersections may happen if the maximum diameter of the P_i can be dilated by the action of S or S^{-1} to become larger than the diameter of M.
[33] *i.e.* the intersection with the stable manifolds in ∂P_i.

Define the $n \times n$ "transitivity matrix" $T_{\sigma,\sigma'} = 1$ if there is an interior point $x \in P_\sigma$ whose image Sx is an interior point of $P_{\sigma'}$ and $T_{\sigma,\sigma'} = 0$ otherwise. Then only sequences $\boldsymbol{\sigma}$ with $T_{\sigma_k,\sigma_{k+1}} \equiv 1$, that will be called "$\mathcal{P}$-compatible", can arise as histories of points.

Viceversa given any $\mathcal{P}$-compatible history $\boldsymbol{\sigma}$ there is at least one $x \in M$ whose history is $\boldsymbol{\sigma}$ and the correspondence $x \longleftrightarrow \boldsymbol{\sigma}$ is one-to-one with the exception of points x in the above mentioned zero volume set $\cup_i S^i \partial\mathcal{P}$. This geometric property follows from hyperbolicity and the covariance of the boundaries of $\mathcal{P}$, [2]. The history $\boldsymbol{\sigma}$ determines the corresponding point x "exponentially fast", meaning that there is a constant $\kappa > 0$ such that the $\{\sigma_i\}_{i=-n}^{n}$ determines x within $const\, e^{-\kappa n}$ (and κ can be taken any smaller than the minimum of the expansion rates for S and S^{-1}).

In Anosov maps there are points with a dense trajectory (see [2] in Sec.I): hence the compatibility matrix T is "transitive", *i.e.* there is $K > 0$ such that $T^K_{\sigma,\sigma'} \geq 1$ for all pairs σ, σ': this means that among compatible histories it is possible that any symbol σ is followed by any other symbol σ' after at most K steps.

The symbolic history can, therefore be used to code the distributions $\mu_0(dx) = \rho(x)dx$, with density $\rho(x)$ with respect to the volume element dx in M, into *stochastic processes*, *i.e.* into probability distributions on the space of the compatible histories.[34]

XV. VOLUME AS STOCHASTIC PROCESS. SRB AS ISING SPIN CHAIN EQUILIBRIUM

The key to the theory of Anosov maps is the *representation of the volume measure* as a probability distribution on the set of compatible sequences, *i.e.* as a stochastic process, which in the above case has been proved to be a "Gibbs process" with a short range potential, which *however*, in general, *is not translation invariant*, [14, 15, 65], see below. A connection with the Gibbs processes emerges naturally also when attempting to interpret results of simulations, [21, Sec.3],[6, 7].

[34] The code that associates with $x \in M$ the history $\boldsymbol{\sigma}$ of x is closely related to the "structural stability" of Anosov maps. Structural stability of Anosov maps means that if an Anosov map S is smoothly perturbed to become a map S_ε, as a function of a parameter ε, with $S_0 = S$ then, if ε is small enough, also S_ε is an Anosov map and S_ε is conjugated to S in the sense that there is a Hölder continuous homeomorphism Θ of M such that $S_\varepsilon = \Theta\, S_0\, \Theta^{-1}$.
An essential step to prove this property is to show that a Markovian partition $\mathcal{P}$ for S can be deformed "by continuity" into a Markovian partition $\mathcal{P}_\varepsilon$ for S_ε *with the same transition matrix*: so that the conjugation is the map $x \longleftrightarrow x'$ associating pairs with the same histories: under S_0 on $\mathcal{P}$ and under S_ε on $\mathcal{P}_\varepsilon$.

Given a Anosov map S, its phase space M can be thought as the space of states of a spin system on a 1-dimensional lattice: evolution of $x \in M$ being just the shift of the history $\boldsymbol{\sigma}$ on a Markovian partition $\mathcal{P} = (P_1, P_2, \ldots, P_m)$, see Sec.XIV, into which x is coded. Therefore points of M are still digitally represented although *the usual digital sequences* for their Cartesian coordinates) are abandoned.

Remark: Representation via histories on Markovian partitions in not universal, like the the one via the digits of the cartesian coordinates, but is specifically adapted to the particular dynamical system (M, S).

The normalized volume is then coded into a probability distribution $\mu_{vol}(d\boldsymbol{\sigma})$ on the space $C(\mathbf{Z})$ of compatible strings. In the language of Statistical Mechanics, it would be an "Ising model", in which the $\boldsymbol{\sigma}$'s can be regarded as sequences of *spins*,[35] so that the time label i of $\sigma_i \in \boldsymbol{\sigma}$ becomes the location of the spin on a (*one dimensional*) lattice.

The μ_{vol} can be contructed via a function $\boldsymbol{\Phi}$, called "*potential*", defined for all integers $a \leq b$ on the finite strings $\boldsymbol{\sigma} = \{\sigma_a, \ldots, \sigma_b\} \in C([a, b])$ that are compatible (*i.e.* that are restrictions to $[a, b]$ of a string in $C(\mathbf{Z})$. The $\boldsymbol{\Phi}$ has the "short range" property, *i.e.* $\boldsymbol{\Phi}_{[a,b]}(\boldsymbol{\sigma})$ tends to 0 *exponentially* if $b - a \to \infty$ and *uniformly* in a as

$$||\boldsymbol{\Phi}|| = \sup_a \sum_{b \geq a} \sum_{\boldsymbol{\sigma} \in C([a,b])} |\boldsymbol{\Phi}_{[a,b]}(\boldsymbol{\sigma})| e^{\kappa|b-a|} < \infty \quad (15.1)$$

for some $\kappa > 0$: at fixed time a the potential $\boldsymbol{\Phi}$ is *exponentially localized* at time a.

The potential $\boldsymbol{\Phi}$ will attribute to spin configurations $\boldsymbol{\sigma} \in C([-\tau, \tau])$, an "energy":

$$U(\boldsymbol{\sigma}, \tau) = \sum_{B \subset [-\tau,\tau]} \boldsymbol{\Phi}_B(\boldsymbol{\sigma}_B) \quad (15.2)$$

where $\boldsymbol{\sigma}_B$ is the part of $\boldsymbol{\sigma}$ with time labels in the interval B and the summation is over the intervals B in $[-\tau, \tau]$.

The basic property concerns the set of x's whose history $\boldsymbol{\sigma}$ restricted to $i \in \Lambda = [-\ell, \ell]$ coincides with a given $\boldsymbol{\sigma}_\Lambda \in C(\Lambda)$: by the definition of history of x this set is simply $P_{\boldsymbol{\sigma}_\Lambda} = \cap_{k \in \Lambda} P_{\sigma_k}$. Fixed $\boldsymbol{\sigma}_\Lambda \in C(\Lambda)$ the normalized volume $\mu_{vol}(P_{\boldsymbol{\sigma}_\Lambda})$ is expressed in terms of the potential $\boldsymbol{\Phi}$ as:

$$\mu_{vol}(P_{\boldsymbol{\sigma}_\Lambda}) = \lim_{\tau \to \infty} \frac{\sum^{\Lambda}_{\boldsymbol{\sigma} \in C([-\tau,\tau])} e^{U(\boldsymbol{\sigma},\tau)}}{\sum_{\boldsymbol{\sigma} \in C([-\tau,\tau])} e^{U(\boldsymbol{\sigma},\tau)}} \quad (15.3)$$

where the superscript Λ restricts the sum in the numerator to the configurations $\boldsymbol{\sigma}$ coinciding with $\boldsymbol{\sigma}_\Lambda$ in the sites of Λ.

[35] Here a *spin* is a variable that can assume a finite number of values, *e.g.* $\sigma = \pm 1$ or $\sigma = 1, 2, .., m$.

The $\mathbf{\Phi}$ is a suitable potential, expressible in terms of the representation of the expansion and contraction rates at the point x coded into $\boldsymbol{\sigma}$, [2, 14, 15].

Eq.(15.3) can be fairly easily checked in systems of dimension 2 (particularly if S is an algebraic map of the torus, see also [63]) because the description, see Sec.XIV, of the Markovian partition can be well visualized via geometric drawings, see Fig.3 Sec.XIV, but requires some effort in higher dimension, [61].

Furthermore the potential $\mathbf{\Phi}$ tends asymptotically to become $\mathbf{\Phi}^+$ to the right of the origin but it becomes asymptotically $\mathbf{\Phi}^-$ to the left and $\mathbf{\Phi}^\pm$ are translationally invariant.[36] This means:

$$\sum_{B}^{\theta}\sum_{\boldsymbol{\sigma}_B}|\mathbf{\Phi}_B(\boldsymbol{\sigma}_B)-\mathbf{\Phi}_B^\pm(\boldsymbol{\sigma}_B)|e^{\kappa(|B|+d(0,B))},\ \theta=\pm \quad (15.4)$$

where B are intervals $[a,b]$ to the right of the origin if $\theta=+$ or to the left if $\theta=-$ (respectively) and $d(0,B)$ is the distance of B to the origin.

Hence, if $\boldsymbol{\sigma}=(\sigma_{-\tau},\dots,\sigma_\tau)=(\boldsymbol{\sigma}_-,\boldsymbol{\sigma}_+)$, with $\boldsymbol{\sigma}_-=\{\sigma_k\}_{k=-\tau}^0$ and $\boldsymbol{\sigma}_+=\{\sigma_k\}_{k=1}^\tau$, $U(\boldsymbol{\sigma})$ can be split as

$$U(\boldsymbol{\sigma},\tau)=U_-(\boldsymbol{\sigma}_-,\tau)+U_+(\boldsymbol{\sigma}_+,\tau)+\mathbf{\Psi}(\boldsymbol{\sigma},\tau) \quad (15.5)$$

where $U_\pm(\boldsymbol{\sigma}^\pm,\tau)=\sum_{B\subset[0,\pm\tau]}\mathbf{\Phi}_B^\pm(\boldsymbol{\sigma}_B)$ and $\mathbf{\Psi}(\boldsymbol{\sigma},\tau)$ can be expressed in terms of a potential $\mathbf{\Psi}$ which satisfies a bound like Eq.(15.1) with $\mathbf{\Psi}_B=0$ unless B contains at least one of the three sites $\pm\tau,0$: in other words $\mathbf{\Psi}$ is a *suitable interpolation* between $\mathbf{\Phi}^\pm$ and $\mathbf{\Phi}$.

The limit Eq.(15.3) exists as a consequence of the 1-dimensionality of the $\boldsymbol{\sigma}$'s, of the short range of $\mathbf{\Phi}^\pm,\mathbf{\Phi}$ and of the absence of phase transitions in stochastic processes with such potentials: the usual SM analysis is presented only for the case of translation invariant potentials, [39, Sec.5.8], but it works, essentially word-by-word, also for non translation invariant potentials like the above $\mathbf{\Phi}$'s.

The proof of the Eq.(15.3) is technical, [15],[14]: in *heuristic* form can be found in Ch.3 of [2] where it is discussed together with several important corollaries which are summarized in the following remarks, see also Ch.6 in [11].

Remarks (1) As a byproduct of the proof of Eq.(15.3) an interesting expression for the phase space contraction emerges. Let x be selected in $P_{\boldsymbol{\sigma}}$ with $\boldsymbol{\sigma}=(\boldsymbol{\sigma}^-,\boldsymbol{\sigma}^+)$ (as above), then the logarithm of the total phase space contraction in the interval $[-\tau,0]$ at $y=S^{-\tau}x$ can be expressed by

$$\begin{aligned}\tau\sigma_{t[-\tau,0]}(y)&\stackrel{def}{=}-\log|\det\partial_i S_j^\tau(y)|\\&=-U_-(\boldsymbol{\sigma}^-,\tau)+U_+(\boldsymbol{\sigma}^-,\tau)\end{aligned} \quad (15.6)$$

up to a correction $\mathbf{\Psi}'(\boldsymbol{\sigma},\tau)$ with $\mathbf{\Psi}'$ a potential with the same properties as $\mathbf{\Psi}$ in Eq.(15.5), hence up to a τ-independent constant.
(2) Eq.(15.6) is a function which has average towards the future equal to $\tau\sigma_+$ with σ_+ being the SRB average of the single step phase space contraction $-\log|\det\partial_i S_j x|$. Eq.(15.6) says that the r.h.s. $-U_-(\boldsymbol{\sigma}^-,\tau)+U_+(\boldsymbol{\sigma}^-,\tau)$ and be used to replace $\sum_{i=-\tau}^\tau\sigma(S^ix)$ up to a correction bounded by a τ independent constant.
(3) A second byproduct, see Eq.3.8.5, Eq.3.11.2 in [2],is

$$\frac{\mu_{srb}(P_{\boldsymbol{\sigma}})}{\mu_{vol}(P_{\boldsymbol{\sigma}})}=e^{-\tau\sigma_{[-\tau,0]}(S^{-\tau}x)+\dots} \quad (15.7)$$

where the dots indicate a correction which is bounded by a τ-independent constant: this gives details about the singularity of the SRB distribution with respect to the volume.

Existence of $\mathbf{\Phi}^+,\mathbf{\Phi}^-$ is behind the theorem on Anosov maps, [14, 15, 65], stating that the SRB distribution μ_{srb} can be naturally represented as a PDF on the set of compatible sequences associated with a Markovian pavement $\mathcal{P}$ (any one, as there are infinitely many of them to choose). From the general theory of the one-dimensional Gibbs states, and from Eq.(15.3), it can be read:

(a) the SRB is given by Eq.(15.3) with $\mathbf{\Phi}=\mathbf{\Phi}^+$,
(b) the volume distribution has the form in Eq.(15.3) with $\mathbf{\Phi}$ in general $\neq\mathbf{\Phi}^\pm$, and
(c) the SRB distribution for the *backward evolution*, S^{-1}, is given by Eq.(15.3) with $\mathbf{\Phi}=\mathbf{\Phi}^-$: (a),(b),(c) together imply the theorem of Sec.I.
(d) the phase space contraction $\sigma(x)$ is expressed in terms of the symbolic history $\boldsymbol{\sigma}$ of x and of the potentials $\mathbf{\Phi}^\pm$ via Eq.(15.6). This is the key to derive the FT.

With the above "Ising model interpretation" of the phase space volume, the short range nature of the potentials $\mathbf{\Phi},\mathbf{\Phi}^-,\mathbf{\Phi}^+$ and the 1-dimensionality of the time (*i.e.* of the labels of the strings $\boldsymbol{\sigma}$) imply, from a SM viewpoint and as a *theorem*, that the volume distribution is a simple stochastic process with very strong ergodicity properties.

Therefore a randomly chosen point x (except for a set of x in a set with zero volume) will have a well defined statistics, the SRB statistics, such that S^tx is coded into a string $\boldsymbol{\sigma}_t=\{\sigma_{i+t}\}$ which, for $t>0$ and large, is a *typical string* for the process with the "future potential" $\mathbf{\Phi}^+$, while for $t<0$ and large is a typical string for the process with the "past potential" $\mathbf{\Phi}^-$.

With probability 1, with respect to the volume measure, or to any one which has a density with respect to the volume, a point x will generate a well defined SRB statistics, *in general different* for the evolution S towards the future or for the evolution S^{-1} towards the past. This explains why in general the SRB distribution for S and that for S^{-1} are singular with respect to each other and to the volume.

The result can be suitably adapted to Anosov flows

[36] If $B=[a,b]$ and $B+t=[a+t,b+t]$ then $\mathbf{\Phi}_{B+t}^\pm(\boldsymbol{\sigma}_B)\equiv\mathbf{\Phi}_B^\pm(\boldsymbol{\sigma}_B)$ for all t.

and also extended to more general maps or flows, called Axiom A maps or flows, [16, 56].

The structure of Anosov systems as a stochastic process with potential $\mathbf{\Phi}$ is basic in the derivation of the fluctuation relation in Sec.VI ; it also indicates a urgent problem: namely that what said so far might be simply insufficient to define a local phase entropy production and to formulate a *local fluctuation relation* dealing with some of the fluctuations taking place in a small region.

The problem is interesting as the fluctuations of the phase space contraction, just because of its physical meaning, will be often macroscopic quantities which, therefore, will be difficult to observe in measurements.[37]

Nevertheless there is some relation that can be established between the latter problem and the structure of the just described global SRB distributions, and it indicates that a fluctuation relation valid for locally observed fluctuations (*i.e.* observed in small regions compared to the system size) might be possible:[38]: more details are deferred to Appendix B.

XVI. ENTROPY ? STATIONARITY & APPROACH TO IT

Boltzmann's H-theorem for *rarefied gases* led to the general definition of equilibrium entropy as $S = k_B \log W$, as written by Planck, where W is the volume of phase space where the equilibrium distribution is concentrated. In the H-theorem S is the limit value of the more general H-function, defined even for a nonequilibrium distribution of a rarefied gas, which reaches its maximum S on the equilibrium state.

Therefore H can be regarded as an extension to nonequilibrium evolutions (of rarefied gases *not in equilibrium*, but isolated and evolving towards equilibrium) with the main feature that it is a "Lyapunov function" varying with time and approaching (monotonically) a maximum value, namely the equilibrium entropy.

Recently the Boltzmann's formula $S = k_B \log W$ has been extended to general evolutions towards equilibrium, [67], defining appropriately the volume W as the volume in phase space of the macrostate associated with the initial microscopic state, determined by a local a coarse grained empirical density and by the total energy (initial data consisting of single (typical) phase space points and for a dense gas), and showing that the new quantity appears to increase monotonically in time (towards an equilibrium state).

This is different from a natural question arising here: namely whether an entropy function can be associated with a nonequilibrium stationary state, and if it even admits an extension to the evolution towards stationary states which plays the role of a Lyapunov function.

Going back to the origin of the ergodic hypothesis imagine the phase space compatible with the constraints *as a discrete set of points* located in the usual continuum phase space.

This is tempting as it would bring back the idea that a phase space point wanders visiting successively all other points: it would explain the existence of a unique stationary distribution, to be *therefore*, identified with the SRB distribution, which would be simply the distribution giving *equal weight to all points*, whether in isolated systems (*i.e.* Hamiltonian evolutions) or in systems out of equilibrium (*i.e.* under the action of non conservative forces and thermostats): thus a unification of the equilibrium and nonequilibrium phenomena would be achieved.

To discuss the latter question consider a chaotic system *defined by a map* on a manifold M (and satisfying the CH).

Form a Markovian partition $\mathcal{P}$ of the continuum phase space of a system into finitely many "cells" P_i and call $\mu_{srb}(P_i)$ the SRB probability of each set, *i.e.* the frequency of visit to P_i from a randomly chosen initial data. $\mu_{srb}(P_i)$ is well defined although singular, *i.e. not expressible* in general via an integral over P_i of a density function: hence it is different from the volume $\mu_{vol}(P_i)$ of P_i. Then replace the continuum phase space by a *finite number* of points $\mathcal{N}_0$, with $\mathcal{N}_0\mu_{srb}(P_i)$ of them in each $P_i \in \mathcal{P}$.

The evolution should be a *one cycle permutation* of the phase space points: in this way each cell P_i is visited, in a very long time, with a frequency which is, therefore, uniquely determined and is a representation of the SRB distribution, at least for the computation of the averages of observables whose variations in the cells P_i can be considered negligible. The time necessary might even be not be too long if the cells P_i are not too small and contain a order 1 fraction of the total number of (discretized) points.

But in the case of nonequilibrium the equations of motion are no longer Hamiltonian, and are dissipative. This means that, in general, the divergence $-\sigma$ of the equations of motion is not 0 (as it is for the isolated evolutions, *i.e.* in the Hamiltonian cases) and must have a non negative average $\langle\sigma\rangle \geq 0$.[39] If $\langle\sigma\rangle > 0$ this means that motion evolves towards an attracting set which has zero volume: it can be imagined (by CH) dense on a smooth surface $\mathcal{A}$ of dimension lower than that of phase space and initial data x starting out of it evolve in time with their distance to $\mathcal{A}$ tending to 0 exponentially fast.

A discretization of phase space should therefore be a discrete representation of the attracting set $\mathcal{A}$. Under the

[37] Very large fluctuations can hint at "violations" of the second principle, [34], hence cannot be observed in large systems.

[38] The importance of the problem is made obvious by a few recent experimental works, *e.g.* [36, 66]

[39] The average $\langle\sigma\rangle$ cannot be < 0, *i.e.* phase space cannot keep expanding forever if a stationary state can be reached, [28].

chaotic hypothesis heuristic arguments can be developed to *estimate the number* $\mathcal{N}$ *of discrete points* necessary to give an accurate description of the motions of data on the attracting set $\mathcal{A}$, [2, Ch.3.11].

The points on which the dynamics develops can be obtained by covering phase space with a *uniform lattice* with meshes $\delta p, \delta q$ in momentum and position and representing the dynamics as a map on the discrete set of $\mathcal{N}_0$ points so obtained: then *select* $\mathcal{N} \leq \mathcal{N}_0$ *of them which are recurrent*; such points exist, having supposed that the motion can be represented on a regular discrete lattice and the SRB is ergodic. Here one should have in mind the *numerical simulations* of chaotic dynamical systems: there the evolution is *literally* simulated as a map ("code") of a discrete set of points, digitally represented and regularly spaced.

Then it is natural to try to define entropy of the SRB state the quantity $S = k_B \log \mathcal{N}$. A *heuristic estimate* of $\mathcal{N}$, under the CH, has been proposed in [2, Ch3.11] as sketched below.

First refine the Markovian partition $\mathcal{P} = \{P_\sigma\}_{\sigma=1}^n$ into $\widehat{\mathcal{P}} = \vee_{-\tau}^{\tau} S^k \mathcal{P}$, *i.e.* define the partition whose elements have the form $\cap_{i=-\tau}^{\tau} P_{\sigma_i} = P_{\boldsymbol{\sigma}}$, choosing τ so large that the size of each element is so small that the *few observables* of interest have a *constant value* in each $\widehat{P} = P_{\boldsymbol{\sigma}}$.

Therefore choose τ so that $e^{-\lambda\tau}\delta = \delta'$ where δ is the maximal linear dimension of the $P \in \mathcal{P}$, δ' is the maximal linear dimension of $\widehat{P} \in \widehat{\mathcal{P}}$ and λ is the minimum Lyapunov exponent of S: thus τ depends on the precision $\tau = \lambda^{-1} \log \frac{\delta}{\delta'}$.[40]

Let $\mathcal{N}$ be the number of the points on the attractor and $\mathcal{N}_0$ be the number of points in the regular lattice over which the dynamics is discretized: then in a cell $P_{\boldsymbol{\sigma}}$ the numbers of points will be respectively $\mathcal{N}\mu_{srb}(P_{\boldsymbol{\sigma}})$ and $\mathcal{N}_0\mu_{vol}((P_{\boldsymbol{\sigma}}))$.

Therefore a simple estimate, [2], of the number points of the uniform lattice that must be recurrent to guarantee a "faithful" discrete representation of the dynamics over a time τ is

$$\mathcal{N} \leq \mathcal{N}_0 \min_{\boldsymbol{\sigma}} \frac{\mu_{srb}(P_{\boldsymbol{\sigma}})}{\mu_{vol}(P_{\boldsymbol{\sigma}})} \tag{16.1}$$

where the minimum is over the histories $\boldsymbol{\sigma} \in C([-\tau, \tau])$. The Eq.(15.7) leads to: $\mathcal{N} \leq \mathcal{N}_0 e^{-\langle\sigma\rangle\tau}$ with $\langle\sigma\rangle$ equal to the SRB average of the phase space contraction $\sigma(x)$. Hence:

$$S = k_B \log \mathcal{N} \leq k_B(\log \mathcal{N}_0 - \frac{\langle\sigma\rangle}{\lambda} \log \frac{\delta}{\delta'}) \tag{16.2}$$

and changing the precision of the observations, *i.e.* changing the observables determining δ', S changes by a quantity which depends on the SRB distribution, if $\langle\sigma\rangle > 0$, via $\langle\sigma\rangle$ and the smallest non zero Lyapunov exponent, except when $\langle\sigma\rangle = 0$, [28]). This provides some evidence that S is not defined just up to an additive constant.

This is in sharp contrast with the equilibrium result (in which $\langle\sigma\rangle = 0$) where changing the precision changes $\log \mathcal{N}$ by a constant *independent of the particular equilibrium state* studied. And, although the derivation of the estimate is heuristic (and is an inequality), it seems to indicate that entropy, as a *function of state*, might not be definable for stationary states out of equilibrium, [2, Sec.3.10,3.11].

Nevertheless one of the main features of the *extension* of entropy, as $S = k_B \log W$, to rarefied gases not in equilibrium but isolated and evolving towards equilibrium, is that it is a "Lyapunov function" varying with time and approaching (monotonically) a maximum value as a limit value, namely the equilibrium entropy, [67].

It is conceivable that *also* in the evolution to a stationary state it could be possible to define a Lyapunov function with the same property of evolving (possibly not monotonically) to a maximum which is reached at stationarity, [2, Ch3.11], as briefly discussed below.

Consider as an initial non stationary distribution *a delta function on a single point in phase space*, for simplicity. Then the fraction $P(\xi, t)$ of times that the point $\xi \in \mathcal{A}$ is visited tends to $\frac{1}{\mathcal{N}}$, as prescribed by the SRB distribution in the above discrete representation, where $\mathcal{N}$ is the number of points in $\mathcal{A}$. Therefore:

$$\begin{aligned} S(t) &= k_B \sum_\xi -\overline{P}(\xi, t) \log \overline{P}(\xi, t) \\ &\xrightarrow[t\to+\infty]{} S_\infty = k_B \sum_\xi -\frac{1}{\mathcal{N}} \log \frac{1}{\mathcal{N}} = k_B \log \mathcal{N} \end{aligned} \tag{16.3}$$

Hence S_∞ is the maximum value that $S(t)$ can reach: so that $S(t)$ can play the role of a Lyapunov function.

Although S_∞ depends *non trivially* on the precision of the discretisation used still, for all choices of the discretisation, the $S(t)$ will have the property of evolving to reach (*however not necessarily monotonically*) the maximum value on the SRB distribution, *i.e.* on the natural stationary state. Entropy might be not defined in general stationary states, as a function of state, although in the approach to stationarity it could be a Lyapunov function (not unique) extending the equilibrium entropy function, [2, Sec.3.12].

XVII. VISCOUS FLUIDS AND REVERSIBILITY

The analysis of the previous sections deal essentially with systems of particles and leaves out the important class of stationary distributions that arise in systems normally described via PDE's, but often can be also described by properties of assemblies of microscopic particles, via suitable scaling limits, [68].

[40] The CH implies that there is no vanishing exponent for the map.

This suggests that it should be possible to apply the same ideas to macroscopic systems, like fluids. Of course the theory of chaos was developed precisely for such systems, [17–19, 69]: however, if systems like fluids are considered, the reversibility is usually lost in the macroscopic descriptions.

Yet friction, responsible for the loss of reversibility, is a phenomenological notion and it can be thought that the same systems could admit equivalent descriptions via other equations, possibly even reversible.

A key might be the theory of "ensembles" for stationary non equilibrium states, following the proposals considered in Sec.VII - IX. An attempt in this direction is presented now focusing attention on the incompressible Navier-Stokes fluids. A first step is to propose, via the example of the NS equations, that the stationary states of macroscopic systems that are scaling limits derived from microscopic molecular evolutions can be described, in suitable circumstances, by reversible equations, and equally well.

In the case of the NS equations the proposal goes back, in a related context, to [41] and, in the form proposed below, to works summarized in [2]: it appeared already in [1, 21, 51, 70–74].

The classical NS equation in dimension $d = 2, 3$, for a velocity $\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}\neq\mathbf{0}} e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{u}_\mathbf{k}$, with periodic boundary conditions in $[0, 2\pi]^d$, is

$$\dot{\underset{\sim}{\mathbf{u}}} + (\mathbf{u}\cdot\boldsymbol{\partial})\underset{\sim}{\mathbf{u}} = \nu\Delta\underset{\sim}{\mathbf{u}} + \mathbf{f} - \boldsymbol{\partial} p\ , \quad \boldsymbol{\partial}\cdot\mathbf{u} = 0 \tag{17.1}$$

where the external forcing $\mathbf{f}$ is supposed to be *concentrated on the large scale* Fourier components, actually it will be supposed to have only one Fourier's component $f_{\pm\mathbf{k}_0}$ with $|f_{\pm\mathbf{k}_0}| = \frac{1}{\sqrt{2}}$ and $\mathbf{k}_0 = (2, -1)$, to fix ideas.[41] The equation is not reversible for the time reversal map $I\mathbf{u} = -\mathbf{u}$ and will be called INS.[42]

In the above dimensionless form the viscosity is written $\nu = \frac{1}{R}$, where R is usually called "Grashof's number". The viscosity is a phenomenological notion derived from reversible microscopic equations of motion, [40], and it is possible to think that the coefficient $\frac{1}{R}$ could be replaced by a Lagrange multiplier designed to hold constant a property characteristic of the flow.

The dissipation per unit time is $\nu\mathcal{D}(\mathbf{u}) \equiv \frac{1}{R}\mathcal{D}(\mathbf{u})$ with:

$$\mathcal{D}(\mathbf{u}) = \frac{1}{(2\pi)^2}\int(\boldsymbol{\partial}\underset{\sim}{\mathbf{u}})^2 d\mathbf{x} = \sum_{\mathbf{k}\neq\mathbf{0}}\mathbf{k}^2|u_\mathbf{k}|^2 \tag{17.2}$$

which is called *enstrophy*, controls statistical properties of the flow through its average. Therefore a first proposal is to replace the viscosity $\frac{1}{R}$ with a multiplier such that $\partial_t\mathcal{D}(\mathbf{u}) \equiv 0$. This leads immediately to

$$\begin{aligned}&\dot{\underset{\sim}{\mathbf{u}}} + (\mathbf{u}\cdot\boldsymbol{\partial})\underset{\sim}{\mathbf{u}} = \alpha(\underset{\sim}{\mathbf{u}})\Delta\underset{\sim}{\mathbf{u}} + \mathbf{f} - \underset{\sim}{\boldsymbol{\partial}} p\ , \quad \underset{\sim}{\boldsymbol{\partial}}\cdot\underset{\sim}{\mathbf{u}} = 0\\ &\alpha(\mathbf{u}) = \frac{\sum_\mathbf{k}\mathbf{k}^2\overline{\mathbf{f}}_\mathbf{k}\cdot\mathbf{u}_\mathbf{k}}{\sum_{\mathbf{k}\neq\mathbf{0}}\mathbf{k}^4|\mathbf{u}_\mathbf{k}|^2}\end{aligned} \tag{17.3}$$

in space dimension $d = 2$. The equation is reversible for the time reversal map $I\mathbf{u} = -\mathbf{u}$ and will be called RNS.[43]

Let N be a cut-off and consider the evolutions for INS and RNS above in $d = 2$ for simplicity. Then the evolution equation for $\mathbf{u}_\mathbf{k} = i\frac{\mathbf{k}^\perp}{|\mathbf{k}|}u_\mathbf{k}$ are

$$\begin{aligned}\dot{u}_\mathbf{k} = &-\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}\frac{(\mathbf{k}_1^\perp\cdot\mathbf{k}_2)(\mathbf{k}_2^2-\mathbf{k}_1^2)}{2\,|\mathbf{k}_1|\,|\mathbf{k}_2|\,|\mathbf{k}|}u_{\mathbf{k}_1}u_{\mathbf{k}_2}\\ &-\beta\,\mathbf{k}^2u_\mathbf{k} + f_\mathbf{k}, \quad |\mathbf{k}_2|, |\mathbf{k}_2|, |\mathbf{k}| \le N\end{aligned} \tag{17.4}$$

where $\beta = \frac{1}{R}$ in the case of INS and $\beta = \alpha(u)$ in the case of RNS and in both cases $|\mathbf{u}_\mathbf{k}| = 0$ for $|\mathbf{k}| > N$ or $|\mathbf{k}| = 0$.

The size of the parameter R controls the stability of the evolution: *for simplicity* it will be supposed that if R is large enough then all initial data, with the exception of a set of zero volume, evolve towards a unique attracting set and define a unique stationary distribution at least if N is not too small.

The stationary distributions of the two equations will be parameterized by R for INS and by D, the constant value of the enstrophy. And the question will be whether there is a correspondence $R\longleftrightarrow D$ which associates distributions which are "equivalent", *i.e.* assign equal averages to suitable classes of observables.

Remarks: (1) It is well known that at fixed N it is not true that there is a unique stationary distribution at given R or D: at small R (*e.g.* $R < 60$) by direct calculations by and accurate simulations this is shown. In [76–81] the phenomenon of "hysteresis", *i.e.* coexistence of several attracting sets, is discussed in detail. // (2) One of the reasons behind the phenomenon (but by no means the only one) is the "gauge" symmetry of the NS equations: if there is $\mathbf{a}$ such that $\mathbf{k}\cdot\mathbf{a} = 0$ for all $\mathbf{k}$ for which $f_\mathbf{k} \neq 0$, then if $u_\mathbf{k}$ is a solution also $e^{i\mathbf{k}\cdot\mathbf{a}}u_\mathbf{k}$ is a solution. Hence if $f_\mathbf{k} = c\,\delta_{\mathbf{k},\pm\mathbf{k}_0}e^{\pm i\gamma}$, $c \in R$ also $u_\mathbf{k}e^{i\theta_0\mathbf{k}\cdot\mathbf{k}_0^\perp}$ is a solution: for each θ_0 an invariant set of data is then defined.
(3) In the several cases some (or all) of the invariant sets

[41] More generally the forcing can be supposed to have $f_\mathbf{k} \neq 0$ only for $|\mathbf{k}| < F$, with F being a fixed cut-off. The cases $\mathbf{k} = (0, \pm1)$ and $\mathbf{k} = (\pm1, 0)$ are somewhat trivial, see [75].

[42] Viscosity plays the role of a model of thermostat: the fluid keeps a constant temperature in spite of the viscosity: therefore viscosity is a model for the undisclosed mechanism keeping the temperature constant.

[43] In dimension $d = 3$: $\alpha = \alpha(\mathbf{u})$ has to be modified by adding to the numerator of Eq.(17.3) be quantity $\sum_{\mathbf{k}_1,\mathbf{k}_2}(\mathbf{k}_1+\mathbf{k}_2)^2$ $(\underset{\sim}{\mathbf{u}}_{\mathbf{k}_1}\cdot i\,\underset{\sim}{\mathbf{k}}_2)(\overline{\mathbf{u}}_{\mathbf{k}_1+\mathbf{k}_2}\cdot\mathbf{u}_{\mathbf{k}_2})$.

may be stable and several stationary states will coexist.[44] (4) Symmetry, or more generally existence of more than one attracting surface, divides the stationary distributions into equivalence classes: and the particular stationary distribution that is reached starting from a given initial data u may depend on u, see remark (iv) in Sec.VI. In these instances equivalence may be difficult to check. Unless for all data u, aside a set of zero volume, the stationary distribution is unique and we are interested only in generic behavior in the space of velocity fields u *with complex components.*
(5) The simplification of uniqueness of the attracting set (on which the CH holds) that will be used below means that all invariant sets, except one, become unstable at large R.
(6) In the following we shall proceed under the above uniqueness assumption. However this is not essential: if there are several possible attracting sets then they will have to be distinguished by labels γ and the stationary distributions will be parameterized by the labels: equivalence becomes in this case the existence for both equations of an equal number of attracting sets and all parameters determining them can be put in correspondence so that the corresponding stationary distributions assign the same averages to the local observables.

Call $S_t^{irr,N}, S_t^{rev,N}$ the evolutions generated on the phase space (of dimension $4N(N+1)$ if $d=2$) by the two equations. The SRB distributions will be parameterized respectively by R, N or by D, N where $D = \mathcal{D}(u) \stackrel{def}{=} \sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$ is the (constant) enstrophy and constitute elements of the "ensembles" $\mathcal{E}^{irr,N}$ and $\mathcal{E}^{rev,N}$ respectively, whose elements will be denoted $\mu_\nu^{irr,N}$ and $\mu_D^{rev,N}$, respectively.

The discussion in Sec.VII-IX suggests considering the two collections of SRB distributions and estabilsh a correspondence $\sim$ between $\mu_R^{irr,N} \in \mathcal{E}^{irr,N}$ and $\mu_D^{rev,N} \in \mathcal{E}^{rev,N}$ by, see Eq.(17.2),

$$\mu_R^{irr,N} \sim \mu_D^{rev,N} \quad \text{if} \quad \mu_R^{irr,N}(\mathcal{D}) = D, \tag{17.5}$$

Conjecture: If $O(u)$ is a "local" observable, in the sense that O depends only on the components $u_{\mathbf{k}}$ with $|\mathbf{k}| < K$:

$$\lim_{N\to\infty} \mu_R^{irr,N}(O) = \lim_{N\to\infty} \mu_D^{rev,N}(O), \ \forall K \text{ prefixed} \tag{17.6}$$

provided the equality in Eq.(17.5) holds as a relation between $R \equiv \frac{1}{\nu}$ and D.

Multiplying both sides of Eq.(17.4) by $u_{-\mathbf{k}}$ yields that the time derivative $\dot{E}$ of the energy $E = \frac{1}{2}\sum_{\mathbf{k}} |u_{\mathbf{k}}|^2$ is given by $-\frac{1}{R}\mathcal{D}(u) + W(u)$ or $-\alpha(u)D + W(u)$ with $W(u) = \sum_{\mathbf{k}} f_{\mathbf{k}} u_{-\mathbf{k}}$: where W is the work done per unit time by the external force $\mathbf{f}$.

Hence since W is a local observable, as $f_{\mathbf{k}}$ has been supposed such, the average of W, which will be called W^a for $a = (rev, N), (irr, N)$ respectively, has to be the same in equivalent stationary states if $N \to \infty$, *i.e.* :

$$W^{irr,N} \equiv \nu \mu_R^{irr,N}(D), \quad W^{rev,N} \equiv \mu_D^{rev,N}(\alpha) En \tag{17.7}$$

because the average of $\dot{E}$ has to vanish. Hence the equivalence condition $\mu_R^{irr,N}(\mathcal{D}) = D$ immediately implies:

$$R\mu_D^{rev,N}(\alpha) \xrightarrow[N\to\infty]{} 1 \tag{17.8}$$

which becomes a key preliminary test of the conjecture when initial data are randomly chosen and the evolution has a unique stationary state.

And the equivalence condition, if the conjecture holds, receives the interpretation that the average work done by the forcing and dissipated per unit time *is the same in the two evolutions.*

In the cases in which there are several attracting sets, hence several SRB distributions, the conjecture has to be modified (see remarks (iv) in Sec.VI and (6) above) simply by saying that if γ, γ' are labels distinguishing the extremal distributions in $\mathcal{E}^{irr,N}, \mathcal{E}^{rev,N}$, with a given R and the corresponding D, then a correspondence between γ and γ' is eventually, for N large enough, possible so that Eq.(17.8) holds.

If holding, the conjecture would establish a strong analogy between, on one hand, the theory of the thermodynamic limit of the canonical and microcanonical equilibrium ensembles and, on the other hand, the above proposed equivalence of ensembles of SRB distributions for the INS and RNS equations. The $\mathcal{E}^{irr,N}$ is analogous to the canonical ensemble with $\nu = R^{-1}$ corresponding to $k_B T$ and $\mathcal{E}^{rev,N}$ is analogous to the microcanonical ensemble with D, the enstrophy, corresponding to the energy. The observables O play the role of the local observables and their localization in momentum corresponds to the localization in space in the thermodynamical equilibrium ensembles.

The above conjecture can be tested and some tests are being made in simulations. It is also emerging that the conjecture could be strengthened to cover also the Lyapunov spectra of equivalent elements of the two nonequilibrium ensembles.

XVIII. SIMULATIONS ON 2D-NS

Consider the two equations Eq.(17.4) and fix $R = 2048$: the conjecture stated in the previous section can be tested in simulations. The cut-off will be set, in the tests that follow, at 960 Fourier's modes *i.e.* $|k_i| \le 15$. The first test is to check the Eq.(17.8): in all cases below the evolution is empirically chaotic.

The figure, as well as the subsequent ones, is obtained after running the *irreversible evolution* at $R = 2048$, with

[44] For instance if $f_{\mathbf{k}}$ is real and $\mathbf{k}\cdot\mathbf{a} = 0$ if $f_{\mathbf{k}} \neq 0$, there is a solution with $u_{\mathbf{k}}$ real: and we have infinitely many invariant sets in which $u_{\mathbf{k}}$ has the form $v_{\mathbf{k}} e^{\theta_0 \mathbf{k}\cdot\mathbf{a}}$, parameterized by θ_0.

960 modes for a long time to obtain the average value D for the enstrophy: this realizes the equivalence condition Eq.(17.5). Then the conjecture would predict that in *reversible evolution*, run from an initial data with enstrophy D, the average of $\alpha(\mathbf{u})$ should be $\frac{1}{R}$. The first simulations yields Fig.4.

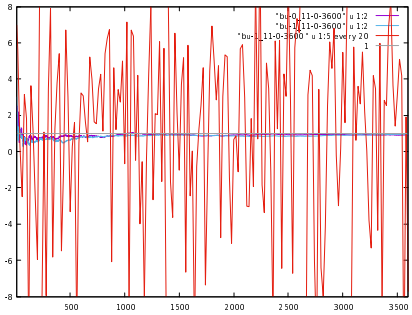

Fig.4: *Reversible evolution NS_{rev}: running average of the "reversible friction" $R\alpha(u) \equiv R\frac{2Re(f_{-\mathbf{k}_0} u_{\mathbf{k}_0})\mathbf{k}_0^2}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$, superposed to the conjectured value* 1 *and to the fluctuating values $R\alpha(u)$: R=2048, 960 modes, λ_{max}= max. Lyapunov exp.* $\simeq 1.5$*, integration step $h = 2^{-17}$, x-axis time unit $4h$, forcing $f_{\mathbf{k}} = 0$ except $f_{\pm(2,-1)} = e^{\pm i\pi/3}/\sqrt{2}$; hence time unit in abscissa corresponds to 2^{19} integration steps: data are plotted by lines at such time intervals. Superposed also to the running average of $R\alpha(u)$ in the equivalent irreversible NS eq. The two running averages and the line 1 are not easy to distinguish on the scale of the drawing.*

A daring test, which goes beyond the conjecture, deals with the equivalence of the exponents of the Jacobian matrix evaluated in the two equations under the equivalence conditions: the result for the same truncation of the equations (960 modes) are drawn, on the same frame which reports the exponents for the Jacobian matrix over a time of the order of λ_{max}^{-1} where λ_{max} is (an approximation of) the largest Lyapunov exponent. Such exponents will be called "local Lyapunov exponents",[45] and are different from the Lyapunov exponents whose evaluation would require substantially larger computation time), [82, 83].

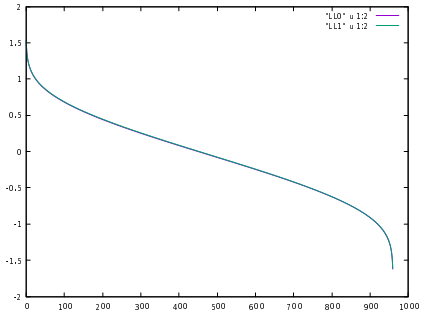

Fig.5: *The local, over a time step $4h^{-1}$, Lyapunov spectra for* 960 *modes truncation: reversible and irreversible superposed. The sum of the (local) exponents in Fig.4 is* < 0.

The two spectra look quite identical: and the relative difference of corresponding exponents $(|\lambda_i^{rev} - \lambda_i^{irr}|/\max(|\lambda_i^{rev}|, |\lambda_i^{irr}|)$ is perhaps more informative:

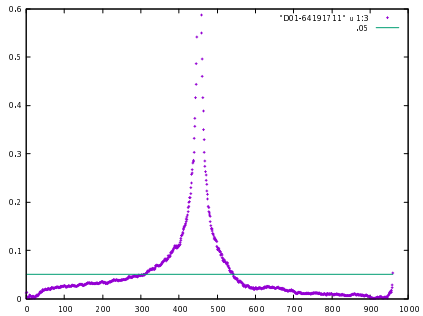

Fig.6: *Relative difference betweeen (local) Lyapunov exponents in the previous Fig.5; R=2048,* 960 *modes. The bar marks the* 5% *discrepancy, and the lines are visual aids.*

It is remarkable that the (local) Lyapunov exponents may provide an example of a pairing rule, see Sec.VIII:

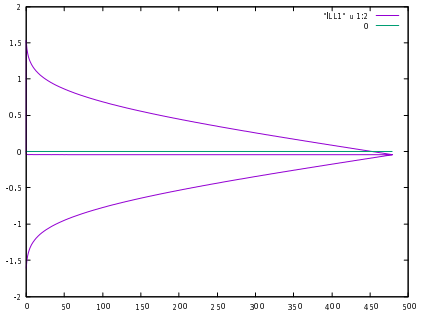

Fig.7: *The approximate pairing rule: graph of $\frac{1}{2}(\lambda_k + \lambda_{n-1-k})$, $n = 960$, with the λ_k the local exponents in the previous Fig.5; R=2048,* 960 *modes.*

A pairing rule emerges from Fig.7. This remarkable fact possibly *suggests* that the pairs consisting of two negative exponents are associated with the attraction by the attracting set and the dimension of the latter is therefore twice the number of exponents > 0, while the fractal dimension of the attractor is the KY dimension computed using only the pairs of exponents of opposite sign. In the case of the previous picture the following Fig.8 provides

[45] They are defined in terms of the diagonal elements of the QR decomposition of the matrix $\partial_u(S_\tau u)$, linearizing the flow, with τ fixed: in the picture $\tau = h2^{10}$ was chosen, which is a small fraction of the time unit fixed by the integration step (which is in the pictures $h = 2^{-17}$). Approximating the matrix $\partial_u(S_\tau u)$ as $V = (1 + hJ(u))^{\tau h^{-1}}$, where $J(u) = \partial_u \dot{u}$ is the Jacobian matrix of the flow, the QR decomposition of V gives a triangular matrix $R > 0$ and the logarithms of its eigenvalues divided by τ, denoted $\Lambda_k(u)$, $k = 1, 2, \ldots$, are sampled as $u(t)$ evolves in time every $4h^{-1}$ steps and their averages are the definition adopted here of the local Lyapunov exponents λ_k. A much more accurate definition would be replacing V with the time ordered product $\prod_{k=0}^{\tau h^{-1}}(1 + hJ(S_k u))$: but this greatly increases the computation time (by a factor 2^7 here). Alternatively one could consider the eigenvalues of the symmetric part of $J(u)$, [82]: but this also requires a large computation time.

a detail with a clearer pairing illustration:

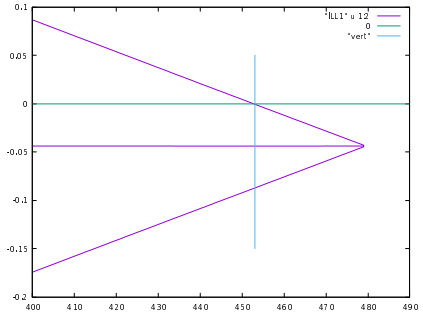

Fig.8: *Detail of Fig.7 showing the pairs of opposite sign and the ones of equal (negative) sign. The vertical line marks the* $k \simeq 452$ *where the negative pairs begin to appear: hence suggest a dimension of the attracting set* 904 *out of* 960. *Graph of* $\frac{1}{2}(\lambda_k + \lambda_{n-1-k})$, $n = 960$, *with the* λ_k *the local exponents in the previous Fig.5.*

The pairing appears exact, but is is not: as it could be seen by drawing the pairing line on a larger scale. Still even on the scale of Fig.5 it is not possible to distinguish the pairing line from an exactly horizontal line.

A pairing property, quite manifest in Fig.5,6, was proposed in [51] as possible in NS fluids. It could be an approximate pairing reflecting an exact one which should hold for the spectrum of the fluid equations with "Ekmann friction" (*i.e.* with viscosity force $-\nu\mathbf{u}$ instead of $\nu\Delta\mathbf{u}$).

For $\nu = 0$ the equation can be considered a Hamiltonian equation with conjugate variables $(\boldsymbol{\delta}, \mathbf{u})$, called Arnold-Euler equation, where $\boldsymbol{\delta}(x)$ is the displacement (with respect to an initial configuration of fluid particles) of the "fluid particle" that reaches the point x at the instant in which the fluid velocity at x is $\mathbf{u}(x)$. So $\mathbf{u}$ is a momentum variable while $\boldsymbol{\delta}$ is a position variable and $\frac{d}{dt}\boldsymbol{\delta}(x) = \mathbf{u}(x)$ (*e.g.* see [51]).

Formally the Ekmann equation, aside the infinite dimensionality, is covered by the pairing theorem [46]: hence its Lyapunov spectrum should have the exponents paired to $-\nu$, see Appendix D. In [51] the idea has been proposed that the Lyapunov spectrum of the $(\boldsymbol{\delta}, \mathbf{u})$-system and that of the $\mathbf{u}$ are related by a symmetry which would imply pairing also for the latter spectrum.

The NS equation viscosity is not proportional to $\mathbf{u}$ and the argument in [46] does not apply, not even formally. Still in [51] the possible pairing in the NS spectrum is discussed (called "barometric formula") and for large cut-off is proposed to pair $\lambda_k, \lambda_{n-1-k}$ to a suitable curve c_k (which would be close to a constant in large intervals of k).

A few more simulations have been performed to test the conjecture, all in 2D, because the 3D case is too demanding. For a few further tests in systems with $48, 224, 960, 3968$ modes (*i.e.* increasing the cut-off N) and for R up to 8192, see [84] where particular attention is dedicated to the approximate pairing, see Sec.VIII, of the Lyapunov exponents. Very few tests have been done for R small: but the conjecture should hold even in the laminar regimes; *i.e.* when at given forcing the attractors can be coexisting stable periodic motions.

Furthermore changing the forcing to allow a $\mathbf{f}$ with more than a single mode, but still keeping it acting only on the large scale $\mathbf{k}$'s and of size $||\mathbf{f}||_2 = 1$, the average enstrophy can change substantially but the results on the equivalence remain encouraging. Also the precision, *i.e.* the integration time step h, can have strong influence: even hysteresis may appear if h is not small enough even though it disappears for smaller h.

The results are still preliminary and hopefully will be continued not only to check those so far obtained but also to study further tests and refinements.

Remarks: (1) Since a real force $f_{\mathbf{k}}$ transforms real data $u_{\mathbf{k}}$ into real ones, there will be an invariant distribution concentrated on real velocity fields $u_{\mathbf{k}}$: it may, as N (or R or both) grows, become unstable to perturbations of u which break the symmetry (*i.e.* reality of $u_{\mathbf{k}}$). Nevertheless such distribution may be unique among those which are generated by a real initial $u_{\mathbf{k}}$: hence, as mentioned in Sec.XVII, to check equivalence it becomes necessary, in general, to identify other invariant conditions on u on which to base the selections of pairs of equivalent distributions besides the corresponding R and E; for instance compare only distributions concentrated on real $u_{\mathbf{k}}$'s.
(2) The simplest checks of the equivalence concern "gauge invariant" observables: at least possible different stationary states related by the symmetry (*i.e.* that can be transformed into each other by application of the symmetry) will attribute the same averages to such observable. The $R\langle\alpha\rangle_{rev} = 1$ is an example and the average work $\langle W\rangle_{rev} = \langle W\rangle_{irr}$ or also $\langle|u_{\mathbf{k}}|^2\rangle_{rev} = \langle|u_{\mathbf{k}}|^2\rangle_{irr}$ are examples.

XIX. OTHER RELATIONS. COMMENTS.

Several *universal* relations have been proposed in the recent literature. I select below two among them.

A. Transient fluctuation theorem

Deals, [85], with reversible evolutions starting from random initial data *chosen from an equilibrium distribution* of particles (hence, in the nontrivial cases, not stationary), of *Boltzmann-Gibbs kind*, or more generally from a distribution *symmetric under time reversal* (which, in most cases, is velocity reversal) and with density with respect to the volume.

In this case the statement is that the probability density that a phase space volume contracts by a factor e^A compared to the probability that it contracts by e^{-A} in a time interval τ is such that:

$$\frac{P(A)}{P(-A)} = e^A, \quad \forall\tau < \infty \tag{19.1}$$

which is an immediate consequence of the definition (*i.e.* of the above few lines preceding it): *no further assumption is necessary.*

Since Eq.(19.1) is sometimes compared to Eq.(6.3) (or Eq.(6.4)) then, for the purpose of comparison, the σ_+ should be defined as the average as $\tau \to \infty$ of $\frac{A}{\tau}$ and p should be set $p \stackrel{def}{=} \frac{A}{\tau\sigma_+}$. In terms of p, σ_+, τ Eq.(19.1) becomes formally identical to Eq.(6.4).

It is claimed that Eq.(19.1), being valid for all τ, will imply the fluctuation relation for the stationary state reached by the evolution at infinite time and for the variable $\frac{1}{\tau\sigma_+}A$. However the stationary state in nonequilibrium cases is singular with respect to the initial state and typical fluctuations observed in time τ have the form $p\sigma_+\tau$ so, if $\sigma_+ > 0$ as it is in nonequilibrium cases, the quantity A in Eq.(19.1) has an unclear meaning when the system has reached a stationary state and a time $\tau = +\infty$ has already elapsed.[46]

A proof of any relation between Eq.(19.1) and the FR discussed in the present review, in any event, has never been published, in spite of several announcements.

B. The Jarzinsky relation

The Eq.(19.1) is an identity but nevertheless it can be useful, as shown by its applications in various domains and this might be the explanation of the lack of interest on the FR and the Chaotic hypothesis.

In this respect there are other relations which are exact and useful identities with several interdisciplinary applications in nonequilibrium phenomena.

An example is provided by an implementation of the simplest "Monte Carlo method": here the general purpose of the Monte Carlo methods is intended as the use of a controlled random number generator to produce random events with a prescribed distribution.

For instance suppose that it is necessary to produce spin configurations on a N-points lattice $\mathcal{L}$ with a distribution proportional to $e^{-\beta U(\boldsymbol{\sigma})}$, $\beta > 0$, with $U(\boldsymbol{\sigma}) = \sum_R J_R\sigma_R$, where $J_R \geq 0$ are are given "couplings" for the spins $\sigma_r = \pm 1$ with r in a subset $R = (r_1, \dots, r_N) \subset \mathcal{L}$. Suppose available a random number generator G_0 able to generate a known distribution of $\boldsymbol{\sigma}$, for instance a Bernoulli shift $(\frac{1}{2}, \frac{1}{2})$ distribution; then follow the algorithm, also called a "protocol":

(1) generate a spin configuration $\boldsymbol{\sigma}$ for the Bernoulli distribution using a deterministic random number generator G_0 (initialized beforehand once and for all with a fixed number). This plays the role of selection of initial data from a known initial state (here a sample Bernoulli path). And compute the weight of the $\boldsymbol{\sigma}$ in the Bernoulli shift (which in the case under consideration would be $e^{-\beta U_0(\boldsymbol{\sigma})} \equiv 1$, *i.e.* probability $Z_0^{-1} = 2^{-N}$).
(2) compute $U(\boldsymbol{\sigma})$ and the ratio $e^{-\beta(U(\boldsymbol{\sigma})-U_0(\boldsymbol{\sigma}))}$
(3) attribute to $\boldsymbol{\sigma}$ the weight $e^{W(\boldsymbol{\sigma})} \equiv e^{-\beta(U(\boldsymbol{\sigma})-U_0(\boldsymbol{\sigma}))}$.

Repeat the protocol many times: the statistic determined on the outputs $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \dots$ by assigning them the weights $e^{-\beta(U(\boldsymbol{\sigma}_i)-U_0(\boldsymbol{\sigma}_i))}$ acquires eventually the PDF $P(\boldsymbol{\sigma}) = Z^{-1}\exp -\beta U(\boldsymbol{\sigma})$ and the relation

$$\langle e^W \rangle = \frac{Z}{Z_0} \tag{19.2}$$

holds with the average being taken with respect to the initial distribution (*i.e.* the Bernoulli shift in the present case).

The procedure can be used to generate the Gibbs distribution at temperature β and Hamiltonian $H_1(x)$ from a Gibbs distribution with Hamiltonian $H_0(x)$, [86]. Imagine to have at hand a system in equilibrium with Hamiltonian $H_0(x)$ and a way ("protocol") to force the evolution of a configuration via equations of motion following a time dependent Hamiltonian $H_t(x)$ which evolves from H_0 at time 0 to H_1 at time 1:

(1) generate an initial state x by picking a sample out of the initial distribution, and
(2) act, by changing the parameters of the Hamiltonian, so that x evolves with the time dependent Hamiltonian H_t as $x \to S_{0,t}x$ and keep track of the energy $W_t(x) = H(S_{0,t}x) - H_0(x)$
(3) weigh the output at time $t = 1$ with $e^{-\beta W_1(x)}$: eventually the statistics of the weighted outputs will be the distribution $Z^{-1}e^{-\beta H_1(x)}$, as it is immediately checked using the Liouville theorem $S_{0,t}dx = dx$ (where $dx = dpdq$ in canonical coordinates).

[46] Simple examples of the meaning of Eq.(19.1) compared to FR can be constructed: which exhibit systems, *as chaotic as wished*, evolving towards a stationary state with average phase space contraction $\sigma_+ > 0$ and which for every finite time satisfy Eq.(19.1) but at infinite time do not satisfy the FR. An example of such a map follows: let S_0 be a map on the unit circle $\mathcal{T}$ defined by the evolution at time $t = 1$ (say) of $\dot{\varphi} = -\sin\varphi$: it has $\varphi = \pi$ as an unstable fixed point and $\varphi = 0$ as a stable fixed point (with Lyapunov exponents $\lambda_0 = \pm 1$, respectively). Let I be the reflection of the point φ at the circle center. Then the evolution is I-reversible and the distribution $\mu_0(d\varphi) = \frac{d\varphi}{2\pi}$ is I-symmetric. Hence Eq.(19.1) holds for all finite τ: at $\tau = \infty$ the distribution of $p = \frac{1}{\tau}\sum_{k=0}^{\tau}\cos(S_0^k\varphi) = \frac{1}{\tau}A$ evolves to $\delta(p)$ which does not satisfy the FR for any $p > 0$ although $\sigma_+ = 1$. The example can be easily adapted to deal with a chaotic evolution: it is enough to consider the dynamical system acting on pairs (φ, x) which evolve in $(S_0\varphi, S_\Xi x)$, where Ξ is any Anosov map reversible under a map J. This is reversible under the time reversal $(\varphi, x) \to (I\varphi, Jx)$. Then Eq.(19.1) holds but leads to a relation with slope $\sigma_+ = \sigma(\Xi) + 1$, where $\sigma(\Xi)$ is the phase space contraction of the map Ξ), while FR predicts the correct slope $\sigma(\Xi)$, because the example is a simple example of a system with a smooth hyperbolic attracting set (*i.e.* the pairs $(0, x)$), hence it satisfies the Chaotic Hypothesis: a case in which the FR is a theorem. Likewise a flow example can be easily constructed. The example is due to F.Bonetto.

The above two protocols are realizations of a (naive) "Monte Carlo" method: the second can be particularly useful, aside numerical simulations, even in applications to bio-systems where it has been possible to find a way to measure W_1 at each run of the protocol.

The quantity W_1 has been identified, in several cases, with the work performed on the system during one iteration of the protocol: it has then been used particularly to measure the free energy variation between two different equilibria at the same temperature: $\beta\Delta F = -\log\langle e^{-\beta W_1}\rangle$ (with the average being over the statistics of the initial data), [86].

Notice that access to W_1 is the only requirement necessary: the random generator being the *initial equilibrium state* and the evolution H_t only needs to be always the same, each time the protocol is run. Of course it is necessary to be able to justify that the measurements of W_1 really evaluate the work W done on the system: in concrete cases it may be not easy to be sure that all forces are taken into account, in particular the ones that change H_0 to H_t. It may also be difficult to make sure that the protocol used is always exactly the same.

C. Ruelle-Lieb bounds

There are remarkable *rigorous* bounds on the averages, with respect to the stationary distributions, of the eigenvalues of the symmetric part of the Jacobian matrix J for the NS equations, symbolically given by $J_{i,j} = \nu\delta_{ji}\Delta - \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, acting on the incompressible velocity fields.[45] The averages over time of such eigenvalues are a kind of "local exponents". The estimates give an upper estimate $\widetilde{N}_J$ to the maximum number of exponents which add up, ordered by decreasing size, to a non negative value, hence also an upper bound on the number of non negative Lyapunov exponents.

The numbers $\widetilde{N}_J$ are, [82, 83], bounded in dimension 2 and 3 (and more); and *in dimension* 2 the bounds can be expressed, [83, Eq(34)], in terms of the average $D = \langle\sum_{\mathbf{k}} \mathbf{k}^2|u_{\mathbf{k}}|^2\rangle_{irr}$ of the enstrophy $D(u)$:

$$\widetilde{N}_J \leq A(2\pi)^2\sqrt{R^2 D}, \qquad A = 0.55... \tag{19.3}$$

As seen in Sec.XVII $\frac{1}{R}D = \langle W\rangle_{irr}$ with W the power spent by the external force, see comment on Eq.(17.7).

The 2-dimensional estimates are in [83, Eq/(43)]: there are also found similar estimates, in higher dimension, extending earlier ones in [82]. In ≥ 3 dimensions the $\widetilde{N}_J$ are not bounded in terms of $\mu^a_\cdot(\int dx(\boldsymbol{\partial}\mathbf{u}(x))^2)$, $a = rev, irr$, but involve powers of $\boldsymbol{\partial}\mathbf{u}$ higher than 2.

The estimates apply to the irreversible NS equations, truncated at arbitrary ultraviolet cut-off, and involve only the eigenvalues of $\frac{1}{2}(J+J^*)$ averaged in time: which can be evaluated in simulations. Being rigorous they can be important in checks of the accuracy of simulations. In the reversible equations R^2D should be replaced by $\langle\alpha(u)^{-2}D(u)\chi(\alpha(u))\rangle_{rev}$ where $\chi(z) = 1$ if $z \geq 0$ and 0 otherwise.

D. Wishes

Several tests of the FR have been performed in the literature. Unfortunately the FR is confused with the similar relation called above the "transient fluctuation theorem" (possibly even omitting the qualification of transient).

It is quite interesting that in most *simulations* the tests performed really deal with the FR; hence it would be very interesting to mount experiments to test the FR (in nonequilibrium situations).

Many experimental works, very delicate and difficult, that claim to have tested the FR have, unfortunately, instead only tested the above transient relation; and could be, *perhaps even easily*, devoted to a real test of FR. Or the tests have been devoted to check the linearity in the symmetry relation Eq.(6.3) for the fluctuations of a quantity identified with p but neither attempting to check the relation with the phase space contraction nor examining the validity of the assumption that the attracting set has dimension equal to the of phase space.

Beautiful laboratory experiments on nano materials, proteins, granular materials, ... have been performed and are, very often, remarkable and innovative from the technical view point but in all cases, that I am aware of, at best they test the transient theorem. The study of the FR has not yet attracted sufficient interest, with the notable exception of the many numerical simulations.

The check of the fluctuation relation in stationary states of systems in nonequilibrium is a test of the Chaotic Hypothesis, which is a *physical assumption*, unlike the checks of the transient fluctuation relation (which *per se* does not test any physical assumption, because testing an identity or a theorem does not provide new information).

Also some experimental works limit the analysis to studying the PDF of the work done or of the heat arising in the experiment but, in my view, not always enough attention is dedicated to check that all forces acting are taken into account. This leads, sometimes, to claim (more or less openly) that the PDF of the work or heat generated at temperature T in a process cannot be normalized with k_BT, as in Eq.(4.3): therefore, it is concluded, the stationary FR is false.

The problem seems to be a certain *resilience* to invest time to follow the ideas behind the chaotic hypothesis (*i.e.* Anosov systems) and the general Axiom A attractors: an example of the attitude towards these ideas is in the quoted statements in [8]; see also Appendix A below. In spite of all the above remarks I still hope that the FR relation will be tested in "real" experimental contexts.

Appendix A: About certain comments on CH

In the above sections quotes from [8] have been reproduced without really commenting them. The reason is that the quotes were written when the Chaotic Hypothesis had been just developed and many had not yet had the time to really study the subject.

But the same comments have appeared in a second edition of the just quoted book, [87], which I have seen only very recently (after completion of the present text). Since the comments have had some resonance the next few lines try to clarify some of the issues.

The Author of [87] criticizes the use of the Anosov systems as paradigm of chaotic motions. The full section from p.344 to p.347 discusses the merits and demerits of Anosov systems. On p. 344 begins

... has discussed the possibility that the useful properties exhibited by certain oversimplified and quite rare dynamical systems, termed "Anosov systems", have counterparts in the more usual thermostatted systems studied with nonequilibrium simulation methods. Anosov systems are oversimplifications, like square clouds or spherical chickens...

This seems to refer to the proposal that the "Axiom A" systems should be the right paradigm for generic chaotic systems, [19, 88]: a proposal which however is not centered on Anosov systems. The Axiom A systems are systems which have an the attracting set $\mathcal{A}$ on which motion has strong chaotic properties (is essentially hyperbolic).

And the CH just proposes, in its final formulation, (1996), that for many purposes the axiom A paradigm can be strengthened and simplified by requiring *in addition* that $\mathcal{A}$ is an attracting surface, possibly of dimension lower than that of phase space, on which the motion is an Anosov system. Even in time reversible cases $\mathcal{A}$ *can be different from its time reversal image*. This is explicitly stated with related problems and examples in [20, 21] and in several successive publications.

The underlying idea being that it is not possible to distinguish, in a system of physical interest, a fractal of Hausdorff dimension $= 10^6 + 3.1415...$ from a surface of exactly 10^6 dimensions.

In summary the Chaotic Hypothesis only assumes that the dynamics under consideration behaves (in some respects) like Anosov dynamics. This is after all not too astonishing if the most relevant degrees of freedom are chaotic like those of Anosov systems.

Most of the subsequent criticism in [87] is anchored on keeping the identification between the CH and the proposal that the whole dynamical system is an Anosov system.

On p.346 the fluctuation theorem is called a "retrospective result" and identified with the true Fluctuation Theorem, Sec.VI above, claiming:

"*These same "results" were actually given earlier by Denis Evans and several of his coworkers, for more general circumstances and through more elementary arguments.*"

but no reference is made here to the applicability of the "earlier retrospective result" to stationary nonequilibria to which the Fluctuation Theorem applies, see[46].

Then on p. 347 the view is found that:

"*Theoretical constructs such as "measures", should be viewed with a healthy suspicion until algorithms for evaluating them are supplied. The chaos inherent in interesting differential equations guarantees that our only access to the "strange sets" which constitute attractors and repellers will be representative time series from dynamical simulations. In no way can we construct, or even conceive of constructing, a Sinai-Ruelle-Bowen measure for an interesting system.*"

However for most purposes *by the CH Hamiltonian systems should be considered Anosov systems* (literally, except of course the integrable ones). Hence the assumption that the attracting set is the full phase space is not always unreasonable.

Furthermore it is useful to stress that there are easy examples of systems satisfying *the CH, with equal or disjoint attracting and repelling surfaces, time reversible, with as many degrees of freedom and negative Lyapunov exponents as wished (unrelated to the number of positive ones) and whose SRB measure is explicitly and completely constructed*, [11, Sec.10.2].

Appendix B: Local Fluctuations. An example.

The phase space contraction in the evolution of a macroscopic system is typically a macroscopic quantity: whether it is the amount of heat ceded to the thermostats or the amount of work performed by the systems.

Therefore the average phase space contraction σ_+ which controls the large fluctuations, Sec.VI and the occurrence of "anomalous" patterns, Sec.XII, cannot be really observed in measurements on macroscopic systems.

Avoiding comments on the many experimental fluctuations observations which claim to check the FR,[47] the question asked here is whether a kind of fluctuation relation could be defined, and constrain quantities depending on events that can be observed in very small parts of the system.

In other words is it possible to give a meaning to a *local fluctuation relation*? [2, Ch4.9].

The following relies on Sec.XV: it is inserted as it provides a quite interesting example on how to make use of the symbolic dynamics representation of the Anosov systems.

[47] Sometimes claiming to have checked it and sometimes claiming the opposite, while very often dealing with unrelated transient phenomena.

A simple example, in a system with time reversal symmetry, will be discussed in which a *local entropy production rate* can be defined and checked to satisfy a local version of FR. A general view on the matter can be found in [89, 90].

The analysis deals again with maps rather than flows.[48]

Consider a system with a translation invariant spatial structure, *e.g.* a periodic chain, or a d-dimensional square lattice $[-L, L]^d$ with periodic boundary, of $(2L)^d$ weakly interacting Anosov maps.

The phase space of the system is $M = \{\mathbf{x} = (\dots x_1, x_0, x_1, x_2, \dots)\} = \mathcal{M}_0^{(2L)^d}$, where $x_i \in \mathcal{M}_0$ are points in a manifold $\mathcal{M}_0$: to fix ideas we *take* $\mathcal{M}_0$ *to be a torus*, on which an Anosov map $\overline{S}_0$ acts; then define the "coupled map":

$$\overline{S}_\varepsilon(\mathbf{x})_i = \overline{S}_0 x_i + \varepsilon g(x_{i-1}, x_i, x_{i+1}),\ i = ..., 0, 1, \dots \quad \text{(B.1)}$$

where g is a smooth perturbation, *i.e.* a smooth periodic function on $\mathcal{M}_0^3$.

If ε is small and the perturbation has short range it is proved in [35, 93, 94] that, defining the map $\overline{S}_\varepsilon$ as in Eq.(B.1) with periodic boundary condition (*i.e.* identifying the site $-L$ with L), the map $\overline{S}_\varepsilon$ remains, if ε is small enough, still Anosov. It is conjugated to $\overline{S}_0$, via a Hölder continuous correspondence $\mathbf{\Theta}_\varepsilon$, see[34], by associating points x and x' with the same history under $\overline{S}_0$ and $\overline{S}_\varepsilon$. Furthermore there is $\varepsilon_0 > 0$ such that the above holds for $|\varepsilon| < \varepsilon_0$ *uniformly* in the system size L. [50]

Here the purpose is to study whether a local version of the FR can hold at least in an example derived from $\overline{S}_\varepsilon$: but the $\overline{S}_\varepsilon$ is, in general, not reversible. A related reversible map S_ε^{rev} can be easily constructed on the "doubled" phase space $\mathcal{M} = \mathcal{M}_0 \times \mathcal{M}_0$ by setting:

$$S_\varepsilon^{rev}(\mathbf{x}, \mathbf{y}) = (\overline{S}_\varepsilon(\mathbf{x}), (\overline{S}_\varepsilon)^{-1}(\mathbf{y})) \quad \text{(B.2)}$$

which is reversible for the time reversal map $I : (\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$. In the rest of this section this system will be considered in more detail.

A Markovian partition $\mathcal{P}_0^L$ for $\overline{S}_0^L \times (\overline{S}_0^L)^{-1}$ will be chosen to be the product of partitions $\mathcal{P}_{-\frac{L}{2}}, \dots, \mathcal{P}_{\frac{L}{2}-1}$ for the single site maps $\overline{S}_0$ and $\overline{S}_0^{-1}$; and $\mathcal{P}_\varepsilon^L$ will be the partition $\mathbf{\Theta}_\varepsilon \mathcal{P}_0^L$ existing and defined by the structural stability map $\mathbf{\Theta}_\varepsilon$, conjugating $\overline{S}_0^{rev}$ to $\overline{S}_\varepsilon^{rev}$, [11, Sec.10.2].

Hence the history of a point x will be a sequence of labels $\sigma_{i;j}$ with $i \in M$ and $j \in (-\infty, \infty)$: naturally i can be called a "space label" while j a "time label". The superscript rev will be omitted in what follows, to simplify notations.

The analysis in Sec.XV applied to the Anosov map S_ε, will give a representation of the volume distribution μ_0 and of the SRB distribution $\mu_{srb,\varepsilon}$ for S_ε via, respectively, suitable potentials $\mathbf{\Phi}_\varepsilon, \mathbf{\Phi}_\varepsilon^\pm$.

Let $[0, \tau]$ be a time interval and $\Lambda = [-\frac{1}{2}L, \frac{1}{2}L]^d = M$, $|\Lambda| = L^d$. Via the Jacobian matrix $J_\Lambda(x) = \partial_x(S_\varepsilon x)$ define the *phase space contraction* and the *time averaged contraction* per site as, respectively:

$$\begin{aligned} \eta_{\Lambda,\varepsilon}(x) &= -\frac{1}{|\Lambda|} \log |\det(\partial_x(S_\varepsilon x))| \\ \eta_{\Lambda,\varepsilon,\tau} &= \frac{1}{|\Lambda|} \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{j=0}^{\tau} \eta_{\Lambda,\varepsilon}(S_\varepsilon^j x) \end{aligned} \quad \text{(B.3)}$$

The limit of $\eta_{L,\varepsilon,+}$ in Eq.(B.3) as $\tau \to \infty$ exists with probability 1 with respect to the volume μ_{vol}, as well to the SRB distribution $\mu_{srb,\varepsilon}$, and is x-independent aside x's in a set of 0 volume: because the statistical properties of the volume distribution are those of the Gibbs distribution with potential $\mathbf{\Phi}_\varepsilon^+$, hence enjoy strong ergodicity properties, as any SRB distribution, with respect to time translations. [51]

The phase space contraction $\sum_{t=0}^{\tau} \eta_{L,\varepsilon}(S_\varepsilon^t x)$ can be expressed, see Sec.XV, via the potentials $\mathbf{\Phi}_\varepsilon^+, \mathbf{\Phi}_\varepsilon^-, \mathbf{\Phi}_\varepsilon^\varphi$, where $\mathbf{\Phi}_\varepsilon^\varphi$ is a potential that describes the interpolation between $\mathbf{\Phi}_\varepsilon^-$ to $\mathbf{\Phi}_\varepsilon^+$ and which is therefore "localized" (see comment to Eq.((15.1))) in the sense that $\mathbf{\Phi}_\varepsilon^\varphi(\boldsymbol{\sigma}_I) \neq 0$ only if I contains the sites 0 or L and $|\mathbf{\Phi}_{\varepsilon,I}^\varphi(\boldsymbol{\sigma}_I)| \leq Ce^{-\kappa|I|}$ for some $C, \kappa > 0$). Given the symbolic history $\boldsymbol{\sigma}$ of x, the Eq.(15.6) can be expressed as:

$$\frac{1}{\tau L^d} \sum_{K \subset M \times [0,\tau]} (\mathbf{\Phi}_{\varepsilon,K}^+(\boldsymbol{\sigma}_K) - \mathbf{\Phi}_{\varepsilon,K}^-(\boldsymbol{\sigma}_K)) + \dots \quad \text{(B.4)}$$

where $K = I \times [a, b]$ is a *parallelepiped* in $\Lambda \times [0, \tau]$, and $\mathbf{\Phi}_{\varepsilon,K}^z \overset{def}{=} \sum_{t\in[a,b]} \mathbf{\Phi}_{I+t}^z$ for $z = \pm, \varphi$, and the $\dots$ indicate a correction $\sum_K \mathbf{\Phi}_{\varepsilon,K}^\varphi(\boldsymbol{\sigma}_K)$,[52]. t A natural mathematical definition of the "local average phase space contraction" could be the $-\frac{1}{\tau\Lambda_0} \sum_{t=0}^{\tau} \log J_{\Lambda_0}(S^x)$ where

[48] This time *the reason is not "for simplicity"*, but because in the case of *coupled flows*, even if the coupling has short range and is weak, there seems to be no detailed and constructive general theory of the SRB distributions, because no simple conditions are known that, via perturbation techniques, yield hyperbolicity of the flow and allow studying its properties.[49] Instead, at least in the case of coupled maps, the theory is quite well understood, [11, 35, 93, 94], as in the example in Eq.(B.1) below, at small coupling ε.

[50] The SRB distribution for the evolution $\overline{S}_\varepsilon$ with $L = \infty$ could also be defined via the SRB distribution $\mu_{srb,\varepsilon}^L$ for the system $\overline{S}_\varepsilon$ consisting of the sites labeled from $-L$ and L and then taking the limit $\mu_{srb} = \lim_{L\to\infty} \mu_{srb,L}$, [93].
This is possible because of the uniformity in $\varepsilon < \varepsilon_0$: below, however, $L < \infty$ will be fixed, keeping in mind that the results will hold for all L if $|\varepsilon| < \varepsilon_0$.

[51] Likewise the space-time limit $\eta_{\varepsilon,+}$ exists because of the space-time ergodicity of the short range Gibbs processes describing the volume as well as the SRB distributions.

[52] Relatively vanishing as L^{-1} uniformly in x.

$J_{\Lambda_0}(x) = |\det(\partial_i (S_\varepsilon x)_{i'})|$. But this is a quantity difficult to express in a useful way.

However it is also possible to propose a *different* definition of local average phase space contraction based on the representation Eq.(B.4) of the average of the logarithms of the full Jacobian. The latter can be expressed as Eq.(B.4) up to a quantity uniformly bounded in L: and the contribution to Eq.(B.4) from the parallelepipeds K's *entirely contained* in $\Lambda_0 \times [0,\tau]$ is:

$$\eta^{loc}_{\Lambda_0,\varepsilon,\tau} \stackrel{def}{=} \frac{1}{\tau L_0^d} \sum_{K \subset \Lambda_0 \times [0,\tau]} \Big(\mathbf{\Phi}_{+,K}(\boldsymbol{\sigma}) - \mathbf{\Phi}_{-,K}(\boldsymbol{\sigma}) \Big) \quad \text{(B.5)}$$

see Eq.(15.1),(15.4); the $\eta^{loc}_{\Lambda_0,\varepsilon,\tau}$ can be, heuristically, called the "local contraction rate". It can be uniformly bounded (in τ, L).

Given Λ_0 let $\eta^{loc}_{\Lambda_0,\varepsilon,+}$ be the time average $\eta^{loc}_{\Lambda_0,\varepsilon,\tau}$ define:

$$p' = \frac{1}{\tau} \frac{\eta_{\Lambda,\varepsilon,\tau}(x)}{\eta_{\Lambda,\varepsilon,+}}, \quad p = \frac{1}{\tau} \frac{\eta^{loc}_{\Lambda_0,\varepsilon,\tau}(x)}{\eta^{loc}_{\Lambda_0,\varepsilon,+}} \quad \text{(B.6)}$$

and remark that $\eta^{loc}_{\Lambda_0,\varepsilon,+} = \eta_{\Lambda,\varepsilon,+} + o(L_0^{-1})$ (because of the SRB distribution representation of a Gibbs process).

It can also be shown that *to leading order* as $L_0, L, \tau \to \infty$ the large deviation rates for p', p in Eq.(B.6) have the form $\tau L^d \zeta_\infty(p')$, $\tau L_0^d \zeta^0_\infty(p)$, with $\zeta_\infty = \zeta^0_\infty$ because ζ_∞ is obtained as a thermodynamic limit of a kind of partition function: for a proof see [95, (5.14)].

Therefore by the FT applied to S_ε it is $L^d \zeta_\infty(p') - L^d \zeta_\infty(-p') = L^d p' \eta_{\Lambda,\varepsilon,+}$ and, since $\eta^{loc}_{\Lambda_0,\varepsilon,+} = \eta_{\Lambda,\varepsilon,+} + o(L_0^d)$, the large deviations rate for p in Eq.(B.6) satisfies a FR of the form

$$\begin{aligned} & L_0^d(\zeta_\infty(p) - \zeta_\infty(-p)) = p\, L_0^d \eta^{loc}_{\Lambda_0,\varepsilon,+} \\ & = p\, r\, (L^d \eta_{\Lambda,\varepsilon,+}) \end{aligned} \quad \text{(B.7)}$$

with $r = \frac{L_0^d}{L^d}$ and $|p| \le p^*, p^* \ge 1$, up to corrections of $O(L_0^{d-1})$: which means that the global and local large fluctuations rates are proportional and trivially related by a rescaling which equals $r = (\frac{L_0}{L})^d$ up to a correction bounded $\kappa^{-1} L_0^{d-1}$ with κ bounding the range of the SRB potential, as in Eq.(15.1).

The universal slope 1 in the global FR is modified into $r = (\frac{L_0}{L})^d$ in the local FR. The Eq.(B.7) *can be proved for the system in* Eq.(B.2).

However p in Eq.(B.6) *is not related to a measurable quantity*, as it cannot be hoped to be able to measure directly the local phase space contraction .defined as in Eq.(B.5).

Still the phase space contraction is often related to the amount of heat ceded or the work done on the surroundings by a system in a stationary state, as exemplified in the case of Eq.(4.4): hence it is tempting to test, in cases in which the latter quantities are accessible to local measurements, whether Eq.(B.7) holds. This is attempted in some simulations, [44].

The interest of the above special example lies in the statements independence on the total size of the systems: they also mean that the fluctuation theorems *may lead to observable consequences* if one looks at the far more probable microscopic fluctuations of the local entropy production rate,[35, 93, 94]. For more details see [33].

Appendix C: Reversible heating

Imagine a rarefied gas enclosed in a cubic container of side L described by a canonical distribution at inverse temperature β^{-1}. The potential energy is $\sum_{i=1}^N mgz_i + \sum_{i,j} v(x_i - x_j) \stackrel{def}{=} M\, g\, H + V$, with M=total mass and H the height of the center of mass. The initial free energy if $F(\beta, g) = -\beta^{-1} \log \int e^{-\beta(V+gP)} d^{3N}p d^{3N}q$. The entropy can be computed via Gibbs' formula $S_0 = -\int \rho(p,q) \log \rho(p,q) d^{3N}p d^{3N}q$.

The gas is set out of equilibrium by changing the gravity g to a new value g' for instance suddenly at time $t = 0$ or following a given prescription $t \to g(t), t \in [0,\tau]$ with $g(\tau) = g', \tau < \infty$. Then it is let to evolve.

Since the evolution is Hamiltonian (although not autonomous) $\rho(p,q)$ evolves in $\rho(p,q;t)$ and the latter tends, as $t \to \infty$, to a new equilibrium state in the gravity potential $mg'z$; but $-\int \rho(p,q;t) \log \rho(p,q;t) d^{3N}p d^{3N}q$ remains equal to S_0. Therefore at the end of the evolution the new distribution $\rho(p,q,\infty)$ will be an equilibrium state of the system in the modified gravity field.

It will not be, however, any more a canonical Gibbs state at temperature β^{-1} in a gravity field with acceleration g'; if the system is ergodic on the energy surface then the final distribution reached at infinite time after suddenly increasing the gravity g to a new value g' will be (integration over p', q' only)"

$$\mu^\infty(dpdq) = \int \mu_\beta(dp'dq') \mu^{mc}_{E(p',q')}(dpdq) \quad \text{(C.1)}$$

where $\mu^{mc}_E(dpdq)$ is the microcanonical distribution with energy E and $E'(p',q') = K(p') + V(q') + Mg'H(q')$ is the sum of the kinetic energy, internal potential energy and energy of the center of mass in a gravity acceleration g'.

The distribution Eq.(C.1) will be equivalent to a canonical Gibbs distribution (with temperature different from β^{-1}) only in the thermodynamic limit: in the finite system that we are considering it will be different by corrections vanishing in the thermodynamic limit. Yet the new state will be a stationary state close to a canonical (or any other equivalent) equilibrium state,

To estimate, actually to define, the temperature β'^{-1} of the new state imagine to identify the above μ^∞ with a canonical distribution $\mu_{\beta'}$, *i.e. neglect the finite volume corrections.* Computing the Gibbs entropy S_∞ of the new equilibrium (reached after infinite time) and make use of the identity between the Gibbs entropies of the initial

and final states:

$$
\begin{aligned}
&F = -\frac{1}{\beta}\log\int e^{-\beta(V+gP)}d^{3N}pd^{3N}q = -T\log Z_0\\
&S = -\partial_T F = -\log Z_0 - \beta\langle V+gP\rangle\\
&\partial_g S|_\beta = \beta^2(\langle PV\rangle - \langle P\rangle\langle V\rangle) + g(\langle P^2\rangle - \langle P\rangle^2)
\end{aligned}
\tag{C.2}
$$

with $T=\beta^{-1}$, $P=MgH$ and $\partial_g S|_\beta \neq 0$ at $g=0$: thus the new equilibrium *cannot* have the same entropy as the initial state if the temperature remained the same unless $\beta' \neq \beta$ (because in general $\partial_g S|_\beta \neq 0$, *e.g.* if $V \simeq 0$ it is $\langle P^2\rangle > \langle P\rangle^2$). If the final state has to become a canonical distribution at some temperature (*e.g.* the above estimated β'^{-1}) then the system will have to be attached to a thermostat and some heat exchange will take place and the entire transformation will be irreversible: in any event, if the system container was really adiabatic and at any finite (or infinite) time the gravity acceleration was dropped back to the initial value, then the system should in the same time return to the initial canonical state. See also [96] for the analysis of equally interesting cases.

Appendix D: Arnold-Euler geodesics

The analysis of [46] applies to the geodesic flows: which are Hamiltonian flows with Hamiltonian $H(\mathbf{p},\mathbf{q}) = \frac{1}{2}g(\mathbf{q})^{-1}\mathbf{pp}$. Hence the Lyapunov exponents of $\dot{\mathbf{q}} = g(\mathbf{q})\mathbf{p}, \dot{\mathbf{p}} = -\frac{1}{2}(\boldsymbol{\partial}_{\mathbf{q}}g(\mathbf{q})^{-1})\mathbf{p}\,\mathbf{p} - \nu\mathbf{p} + \mathbf{f}$ with $\mathbf{f}$ independent of $\mathbf{p},\mathbf{q}$ are paired to $-\frac{1}{2}\nu$, see footnote[18], with a general $H(\mathbf{p},\mathbf{q})$.

The Euler flow is a geodesic flow, [97, 98], and the canonical coordinates are $\mathbf{u},\boldsymbol{\delta}$ where $\boldsymbol{\delta}$ is the diffeomorphism bringing the reference state O of the fluid into the actual state A: if a force $\mathbf{f}$ is added which does not change the Hamiltonian nature of the motion (*e.g.* if, as in [46], $\mathbf{f}(\mathbf{q})$ is locally conservative) and if, furthermore, a viscosity force of the form $-\nu\mathbf{u}$ is also added, then the above shows that pairing takes place (formally) to $-\frac{\nu}{2}$.

A formal proof that the Euler equations for a fluid can be written as Arnold's geodesic flow is summarized as follows (correcting also typos in the somewhat obscure argument in the appendix theof [51]). Let $\boldsymbol{\xi}\to\mathbf{x}=\boldsymbol{\delta}(\boldsymbol{\xi})$ be the diffeomorphism in $Sdiff(T)\subset diff(T)$ [53] mapping the reference fluid configuration O into the actual one A. Here and below the differential operators are denoted $\partial,\dots$ but are intended to be $\partial_{\boldsymbol{\delta}(\boldsymbol{\xi})},\dots$ when operating on functions of $\boldsymbol{\delta}(\boldsymbol{\xi})$.

[53] Notation: $diff(T)$ denotes the set of the general torus diffeomorphisms and $Sdiff(T)$ is the subset of diffeomorphisms with Jacobian $J(\boldsymbol{\delta})(\boldsymbol{\xi}) \equiv \det(\frac{\boldsymbol{\partial}\boldsymbol{\delta}(\boldsymbol{\xi})}{\delta\boldsymbol{\xi}}) = 1$ *e.g.* , if $d=3$, $J(\boldsymbol{\delta})(\boldsymbol{\xi}) = \boldsymbol{\partial}\delta_1(\boldsymbol{\xi})\wedge\boldsymbol{\partial}\delta_2(\boldsymbol{\xi})\cdot\boldsymbol{\partial}\delta_3(\boldsymbol{\xi})\equiv 1$.

A map from O to a configuration $A'\in diff(T)$ *infinitesimally close* to one in $Sdiff(T)$ can be parameterized by $\boldsymbol{\delta},\boldsymbol{\zeta}$ with $\boldsymbol{\delta}\in Sdiff(T)$ and $\boldsymbol{\zeta}$ an infinitesimal variation of the form $\boldsymbol{\partial}z(\boldsymbol{\delta}(\boldsymbol{\xi}))$ for some scalar z.

The coordinates $\boldsymbol{\delta},\boldsymbol{\zeta}$ can be checked to form a system of coordinates "*orthogonal and well adapted*", in the sense of definition 12 in [49], to the surface $J(\boldsymbol{\delta}(\boldsymbol{\xi}))=1$ in the space $diff(T)$ of torus diffeomorphisms in the metric $g(\boldsymbol{\delta},\boldsymbol{\zeta})$ attributing to an infinitesimal variation $(\mathbf{w}+\mathbf{z})$ of $(\boldsymbol{\delta},\boldsymbol{\zeta})$ the square length:

$$
\int d\boldsymbol{\xi}(\mathbf{w}(\boldsymbol{\delta}(\boldsymbol{\xi}))^2 + \boldsymbol{\partial}z(\boldsymbol{\delta}(\boldsymbol{\xi}))^2)d\boldsymbol{\xi} \tag{D.1}
$$

for the *infinitesimal* variations $(\mathbf{w},\boldsymbol{\partial}z)$ of $(\boldsymbol{\delta},\boldsymbol{\zeta})$ with $\mathbf{w}$ a divergence free field and $\boldsymbol{\partial}z$ a gradient field. Furthermore if $\boldsymbol{\xi}\to\boldsymbol{\delta}(\boldsymbol{\xi})$ is an incompressible configuration in $Sdiff(T)$ and $\boldsymbol{\delta}'(\boldsymbol{\xi}) = \boldsymbol{\delta}(\boldsymbol{\xi}) + \mathbf{u}(\boldsymbol{\delta}(\boldsymbol{\xi})) + \boldsymbol{\partial}z(\boldsymbol{\delta}(\boldsymbol{\xi}))$ an *infinitesimally close* one in $diff(T)$ then

$$
\begin{aligned}
&J(\boldsymbol{\delta}'(\boldsymbol{\xi})) - 1 = \Delta z(\boldsymbol{\delta}(\boldsymbol{\xi})),\ \text{("Liouville's theorem")}\\
&\int d\boldsymbol{\xi}(J(\boldsymbol{\delta}'(\boldsymbol{\xi}))-1)^2 = \int d\boldsymbol{\xi}(\Delta z(\boldsymbol{\delta}(\boldsymbol{\xi})))^2\\
&\qquad = \int d\mathbf{x}(\Delta z(\mathbf{x}))^2
\end{aligned}
\tag{D.2}
$$

depending only on the "violation of the constraint" $J-1$ in the above orthogonal and well adapted coordinates.

Therefore a general theorem on constrained motions, (for the constraint $J(\boldsymbol{\delta}(\boldsymbol{\xi}))=1$) can be applied: because the *perfection criteria* for the constraint are met as a consequence of Eq.(D.1) , (D.2), see definition 13, Sec. 3.7, and proposition 13, Sec. 3.8 in [49]. Hence the motions obey Euler-Lagrange equations for the Lagrangian $\mathcal{L}(\dot{\boldsymbol{\delta}},\boldsymbol{\delta}) = \int\frac{1}{2}\dot{\boldsymbol{\delta}}(\boldsymbol{\xi})^2 d\boldsymbol{\xi}$ with the *ideal holonomic constraint* $J(\boldsymbol{\delta}(\boldsymbol{\xi}))\equiv 1$.

In other words the motions driven by the Lagrangian $\mathcal{L}_\Lambda(\dot{\boldsymbol{\delta}},\boldsymbol{\delta}) = \int d\boldsymbol{\xi}\Big(\frac{1}{2}\dot{\boldsymbol{\delta}}(\boldsymbol{\xi})^2 + \Lambda(J(\boldsymbol{\delta}(\xi))-1)^2\Big)$, in the space $diff(T)\supset Sdiff(T)$, are motions which depend on the auxiliary parameter Λ and, if the initial data are an incompressible configuration $\boldsymbol{\delta}\in Sdiff(T)$, in the limit as $\Lambda\to+\infty$ converge, at any prefixed time, to motions driven by the constrained Lagrangian $\mathcal{L}$.

Explicitly, let $\mathbf{u}(\mathbf{x}) = \dot{\boldsymbol{\delta}}(\boldsymbol{\xi})$: since at time $\varepsilon>0$ $\boldsymbol{\delta},\mathbf{u}$ become $\boldsymbol{\delta}_\varepsilon(\boldsymbol{\xi})$ and $\mathbf{u}_\varepsilon(\mathbf{x}) = \boldsymbol{\delta}_\varepsilon(\mathbf{x}+\varepsilon\mathbf{u}(\mathbf{x}))$, it follows that $\underset{\sim}{\ddot{\boldsymbol{\delta}}}(\boldsymbol{\xi}) = \partial_t\underset{\sim}{\mathbf{u}}(\mathbf{x}) + (\mathbf{u}(\mathbf{x})\cdot\boldsymbol{\partial})\underset{\sim}{\mathbf{u}}(\mathbf{x})$ and the Euler-Lagrange equations for the constrained $\mathcal{L}$ become the Euler equations for a perfect incompressible fluid, obtained as a geodesic flow for the Hamiltonian $H(\mathbf{u},\boldsymbol{\delta})$ defined as

$$
\int d\boldsymbol{\xi}\big(\frac{1}{2}\mathbf{u}(\boldsymbol{\delta}(\boldsymbol{\xi}))^2 + (\Delta^{-1}((\boldsymbol{\partial}\underset{\sim}{\mathbf{u}})\cdot(\underset{\sim}{\partial}\mathbf{u})))(\boldsymbol{\delta}(\boldsymbol{\xi}))\big) \tag{D.3}
$$

with $p(\mathbf{x}) = -\Delta^{-1}((\underset{\sim}{\boldsymbol{\partial}}\mathbf{u})\cdot(\boldsymbol{\partial}\underset{\sim}{\mathbf{u}}))(\mathbf{x})$ interpreted as "pressure".

If a force $\mathbf{f}(\mathbf{x}) = (-\partial_2\Phi(\mathbf{x}),\partial_1\Phi(\mathbf{x}))$, with Φ a given scalar independent of the fluid configuration,

acts on the fluid then to Eq.(D.3) a term $\boldsymbol{\delta}(\boldsymbol{\xi}) \cdot (-\partial_2\Phi(\boldsymbol{\delta}(\boldsymbol{\xi})), \partial_1\Phi(\boldsymbol{\delta}(\boldsymbol{\xi})))$ has to be added.

Recalling that $\mathbf{x} = \boldsymbol{\delta}(\boldsymbol{\xi})$, the equations of motion are the pair:

$$\begin{aligned} &\partial_t \boldsymbol{\delta}(\boldsymbol{\xi}) = \mathbf{u}(\boldsymbol{\delta}(\boldsymbol{\xi})) \\ &\partial_t \underset{\sim}{\mathbf{u}}(\mathbf{x}) + (\mathbf{u}(\mathbf{x}) \cdot \boldsymbol{\partial})\underset{\sim}{\mathbf{u}}(\mathbf{x}) = -\nu\underset{\sim}{\mathbf{u}}(\mathbf{x}) - \underset{\sim}{\boldsymbol{\partial}} p(\mathbf{x}) + \underset{\sim}{\mathbf{f}}(\mathbf{x}) \end{aligned} \tag{D.4}$$

and by the just mentioned result in [46], display a set of Lyapunov exponents, local and global, paired to $-\nu/2$.

In the Euler-Arnold case the exponents are naturally divided in two classes: the exponents relative to the $\mathbf{u}$ coordinates, 'fluid exponents', which do not depend (explicitly) from the evolution of the $\boldsymbol{\delta}$ coordinates and the others.

It would be natural to think that the other exponents are simply a copy of the first: if so the fluid exponents, alone, would be paired. However if true this cannot be a general property of geodesic flows to which friction and forcing are added (as it fails in the flow with Hamiltonian $H = \frac{1}{2}p^2$, *i.e.* for the equations $\dot{q} = p, \dot{p} = -\nu p + f$). I neither succeded in proving the just mentioned double degeneracy of the Lyapunov spectrum for the Arnold-Euler equations nor in convincing myself that it is a reasonable hypothesis.

The results just discussed *do not apply to the NS flow* because the viscosity is $\nu\Delta\mathbf{u}$: however a pairing to a line which is not constant but which depends on the scale in which the motion is studied, compatible with the above simulations, was proposed in [51].

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