Navier-Stokes equation: irreversibility, turbulence and ensembles equivalence

Giovanni Gallavotti
INFN and U. Roma "La Sapienza"
March 30, 2019

Abstract: *The NS equation is considered (in 2&3 dimensions) with a fixed forcing on large scale; the stationary states form a family of probability distributions on the fluid velocity fields depending on a parameter R (Reynolds number). It is proposed that other equations could lead to -exactly- the same distributions via a mechanism closely analogous to the coincidence of the canonical and microcanonical averages of local observables in the statistical mechanics thermodynamic limit (proposed, here, to correspond to the limit in which the UV cut-off N, regularizing the equations, is removed to infinity).*

The simplest geometry in which the Navier-Stokes equation can be studied is a periodic geometry; the fluid is supposed incompressible and enclosed in a periodic box, taken to be $\Omega = [0, 2\pi]^d$, $d = 2, 3$. The velocity field $\mathbf{u}(\mathbf{x})$, together with its evolution equation, will be represented via its Fourier's transform $\mathbf{u_k}$ by:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \sum_{\mathbf{0}\neq\mathbf{k}\in Z^d} \mathbf{u_k} e^{i\mathbf{k}\cdot\mathbf{x}}, && \mathbf{x}\in\Omega \\ \dot{\mathbf{u}}_\mathbf{k} &= \mathbf{Q}(\mathbf{u})_\mathbf{k} - \nu\mathbf{k}^2\mathbf{u_k} - \mathbf{f_k}, && \mathbf{k}\cdot\mathbf{u_k} = 0 \end{aligned} \tag{1}$$

where $\mathbf{Q}$ is quite simple in $d = 2$ because, setting $\|\mathbf{k}\|^2 \stackrel{def}{=} \sum_{\gamma=1}^d k_\gamma^2$ and $|\mathbf{k}| \stackrel{def}{=} \max_{1\le\gamma\le d}|k_\gamma|$, $\mathbf{u_k}$ is essentially a (complex) scalar $\mathbf{u_k} = i\frac{\mathbf{k}^\perp}{\|\mathbf{k}\|}u_\mathbf{k}$, $u_\mathbf{k} = \overline{u}_{-\mathbf{k}} \in C$, while if $d = 3$ $\mathbf{u_k} = \overline{\mathbf{u}}_{-\mathbf{k}}$ is orthogonal to $\mathbf{k}$; then $\mathbf{Q}(\mathbf{u})_\mathbf{k}$ is

$$\begin{aligned} d=2: &\quad \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \frac{(\mathbf{k}_1^\perp\cdot\mathbf{k}_2)(\mathbf{k}_2^2-\mathbf{k}_1^2)}{2\|\mathbf{k}_1\|\,\|\mathbf{k}_2\|\,\|\mathbf{k}\|} u_{\mathbf{k}_1}u_{\mathbf{k}_2}, \\ d=3: &\quad -\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} i\,(\mathbf{u}_{\mathbf{k}_1}\cdot\mathbf{k}_2)(\mathbf{u}_{\mathbf{k}_2} - \frac{\mathbf{k}\cdot\mathbf{u}_{\mathbf{k}_2}}{\|\mathbf{k}\|^2}\mathbf{k}). \end{aligned} \tag{2}$$

To avoid convergence and existence questions the $\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2$ and the summations are restricted to waves with $1 \le |\mathbf{k}|, |\mathbf{k}_1|, |\mathbf{k}_2| \le N$ so that the equations Eq.(1),(2) will actually define the NS equations regularized with "ultraviolet cut-off" ("UV") at N. But *attention will be devoted to properties that admit a limit as* $N \to \infty$.

Standing assumptions on the forcing $\mathbf{f}$ will be, aside from $\mathbf{k}\cdot\mathbf{f_k} = 0$, that $|\mathbf{f_k}| \equiv 0$ for $|\mathbf{k}| > k_f$ with k_f independent on the UV cut-off N, so that the scale of $\mathbf{f}$ will remain k_f^{-1}, *i.e.* "large", as the UV cut-off scale $N^{-1} \to 0$, furthermore $\mathbf{f}$ will be *fixed throughout the discussion* and supposed of size 1, *e.g.* $\|\mathbf{f}\|^2 = \sum_\mathbf{k}|\mathbf{f_k}|^2 = 1$, $\|\mathbf{f}|\mathbf{k}|\| \le k_f$.

The regularized equations have smooth solutions for all initial data if $\nu > 0$ and eventually satisfy N-independent *a priori* bounds[1] namely, if $\min_{\mathbf{k}\neq\mathbf{0}}\|\mathbf{k}\| = 1$:

$$\begin{aligned} \mathcal{E}(\mathbf{u}) &\stackrel{def}{=} \sum_\mathbf{k} |\mathbf{u_k}|^2 \le \nu^{-2}, \qquad d = 2, 3 \\ \mathcal{D}(\mathbf{u}) &\stackrel{def}{=} \sum_\mathbf{k} \mathbf{k}^2|u_\mathbf{k}|^2 \le k_f^2\nu^{-2}, \quad d = 2 \end{aligned} \tag{3}$$

bounding respectively energy $\mathcal{E}(\mathbf{u})$, enstrophy $\mathcal{D}(\mathbf{u})$.

Let $R \stackrel{def}{=} \nu^{-1}$; then, for each R, the evolution will lead, as time $t \to \infty$, a given intial datum $\mathbf{u}$ randomly chosen (with respect to a distribution with a smooth density) in M_N=space of the velocity fields[2] to a stationary state μ_R^N, which depends on the UV cut-off. The state μ_R^N may be not unique and, particularly if $d = 3$, reaching it may require selecting a sequence $t \to \infty$ of times.

Such stationary states form a set $\mathcal{E}_R^N$. In regimes of strong turbulence (*i.e.* R large) it can be expected that μ_R^N is just unique and reached without attention to the considered sequence of times $t \to \infty$.

The aim of this paper, pursuing the analogies between turbulence and statistical mechanics, [1–3], is to study whether properties of the stationary states in $\mathcal{E}_R^N$, which become *independent* of the UV cut-off N as $N \to \infty$, are shared by properties of stationay states of *other equations of motion* on the same phase space M_N, [4, 5].

Here an *exact equivalence* is proposed between the equation obtained by replacing $\nu\mathbf{k}^2u_\mathbf{k}$ by $\alpha(\mathbf{u})\mathbf{k}^2\mathbf{u_k}$ with the multiplier $\alpha(\mathbf{u})$ chosen so that $\mathcal{D}(\mathbf{u}) = \sum_\mathbf{k}\mathbf{k}^2|\mathbf{u_k}|^2$, has a value *En exactly constant* in time; see Eq.(8) below for an explicit expression of $\alpha(\mathbf{u})$, [6].

The stationary states of the two equations form two collections $\mathcal{E}^{i,N}, \mathcal{E}^{r,N}$ (labels i, r for irreversible,reversible) whose distributions $\mu_R^{i,N}, \mu_{En}^{r,N}$ are parameterized respectively by the Reynolds (or Grashof) number R and by the initial data enstrophy En (equal to the constant value of $\mathcal{D}(\mathbf{u})$) and depend on the UV cut-off N. A pair of such states will be called *corresponding* if the parameters R, En are such that

$$\mu_R^{i,N}(\mathcal{D}) = En \tag{4}$$

which will acquire, see below, the physical meaning that the work done per unit time by the forcing is the same in corresponding stationary distributions. Proposal:

Let O be any observable localized[3] on waves $\mathbf{k}, |\mathbf{k}| < K$, and a pair of corresponding states $\mu_R^{i,N}$ and $\mu_{En}^{r,N}$ for the two equations above, then:

$$\lim_{N\to\infty}\mu_R^{i,N}(O) = \lim_{N\to\infty}\mu_{En}^{r,N}(O) \qquad \text{for all } K < \infty. \tag{5}$$

[1] *i.e.* any initial data $\mathbf{u}$ for time large enough satisfy a bound larger then the one called here *a priori* average. Also if the time average of $\mathcal{D}_4(\mathbf{u}) \stackrel{def}{=} \sum_\mathbf{k}\mathbf{k}^4|u_\mathbf{k}|^2$ exists it is $\le k_f^4\nu^{-2}$.

[2] M_N dimension is $\mathcal{N} = 4N(N+1)$ if $d = 2$, $\mathcal{N} = 8N^3 + 12N^2 + 6N$ if $d = 3$.

[3] *i.e.* O is a function of the Fourier's components $\mathbf{u_k}$ with $|\mathbf{k}| < K$.

This is inspired by the existence in equilibrium statistical mechanics of "equivalent ensembles": where suitably corresponding stationary distributions which are very different nevertheless attribute the same average values to large classes of observables "in the thermodynamic limit". For instance the microcanonical and the canonical distributions at fixed energy density and, respectively, at fixed average kinetic energy density can be considered stationary states for suitable Hamiltonian equations in which the total energy E is conserved or, respectively, the average kinetic energy is conserved, [3, 7].

In these cases it is possible to establish a correspondence between stationary distributions, also called "states", in such a way that observables O which are *local*, *i.e.* which depend on particles located in a region Λ of space, small compared to the total volume V containing the system, will have the same average value in corresponding stationary states. *Exactly* the same in the limit in which $V \to \infty$ with Λ remaining fixed: *i.e.* in the *thermodynamic limit*.

The analogy proposed here is to consider the UV cutoff N as playing the role of the total volume in statistical mechanics, with the role of the energy being played by the enstrophy $\mathcal{D}(\mathbf{u})$ and the role of the inverse temperature being plaid by the Reynolds number R.

The correspondence between stationary states $\mu_R^{i,N}$ of the Eq.(1) and states $\mu_{En}^{r,N}$ of the equation with the "variable viscosity" $\alpha(\mathbf{u})$ replacing ν, as above, is established by Eq.(4) and $N \to \infty$ is analogous to the thermodynamic limit.

The intuition behind the proposal is that, at least for large R, in the variable viscosity equation $\alpha(\mathbf{u})$ fluctuates strongly generating a *homgeneization* phenomenon implying that the local observables, being variable only at large scale, "see" $\alpha(\mathbf{u})$ as a constant equal to its time average. This argument fails in the cases in which the stationary states are "laminar", *i.e.* non chaotic: yet it will be argued that the equivalence holds even in such cases.

Further motivation is obtained remarking that the NS equations follow, via suitable scaling limits, from the intermolecular microscopic equations. The latter are reversible while the Eq.(1) are irreversible because the time reversal $I\mathbf{u} = -\mathbf{u}$ does not anticommute with the evolution flow, denoted $\mathbf{u} \to S_t\mathbf{u}$: *i.e.* $S_tI \neq IS_{-t}$.

Since the time reversal symmetry is a fundamental one, it should be possible to describe the fluid phenomena on their macroscopic scale also via reversible equations: irreversibility is conceptually due to the chaotic motions and only phenomenologically represented by the viscosity coefficient. The equations defined above by replacing ν by $\alpha(\mathbf{u})$, so built that the enstrophy $\mathcal{D}(\mathbf{u})$ is exactly constant, are reversible (because $\alpha(-\mathbf{u}) \equiv -\alpha(\mathbf{u})$) and provide a natural test of the idea.

Looking for equations equivalent to the NS equations on the local observables the above choice is also somewhat privileged.

Because imposing that the averages of the observable $W(\mathbf{u}) = \sum_{\mathbf{k}} \overline{\mathbf{f}}_k \cdot \mathbf{u}_{\mathbf{k}}$ (*i.e.* the power injected per unit time by the forcing, which is a local observable by the standing assumption on $\mathbf{f}$) be the same in corresponding states, is implied by[4] $\frac{1}{R}\mu_R^{i,N}(D) = \mu_{En}^{r,N}(\alpha)En$. Hence the property $\mu_{En}^{r,N}(\alpha) = \frac{1}{R}$, on which the homogeneization interpretation is based, is the same as the $\mu_R^{i,N}(\mathcal{D}) = En$ if the conjecture holds. Hence checking that:

$$\lim_{N\to\infty} R\mu_{En}^{r,N}(\alpha) = 1, \tag{6}$$

is a first check, with a simple physical interpretation, of the proposed claims.

In statistical mechanics, however, there are many other ensembles equivalent to the canonical or microcanonical. So the conjecture can possibly be extended to other equations, in which other observables are imposed to be exactly constant, by replacing the $\frac{1}{R} = \nu$ in NS by a suitable multiplier.

For instance α could be chosen so that the energy $\mathcal{E}(\mathbf{u}) = \sum_{\mathbf{k}} |\mathbf{u}_{\mathbf{k}}|^2$ is exactly constant and local observables have equal averages in corresponding stationary states $\mu_R^{i,N}, \widehat{\mu}_E^{r,N}$ of the irreversible equation Eq.(1) and, respectively, of the new equation: with the correspondence established by the condition $\mu_R^{i,N}(\mathcal{E}) = E$, analogous to Eq.(4).

Then corresponding states $\mu_R^{i,N}, \widehat{\mu}_E^{r,N}$ would be determined by $\mu_R^{i,N}(\mathcal{E}) = E$; and "equality of averages of local observables in corresponding distributions" should imply that the averages of the local observable $W(\mathbf{u}) \overset{def}{=} \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}}$, power injected by the forcing, *i.e.* $\frac{1}{R}\mu_R^{i,N}(D) \equiv \mu_R^{i,N}(W)$ and $\widehat{\mu}_E^{r,N}(\alpha D) \equiv \widehat{\mu}_E^{r}(W)$[5] *are equal*, leading to the test:

$$\lim_{N\to\infty} \frac{1}{R}\mu_R^{i,N}(\mathcal{D}) = \lim_{N\to\infty} \widehat{\mu}_E^{r,N}(\alpha\mathcal{D}) \tag{7}$$

which has the same interpretation as Eq.(6), *i.e.* the average energy dissipated per unit time is the same in two equivalent evolutions.

Of course if this is correct also other equations can be imagined which are equivalent to the NS equations, [5].

Returning to the conjecture, consider the equations $\dot{\mathbf{u}}_{\mathbf{k}} = Q_{\mathbf{k}}(\mathbf{u}) - \alpha\mathbf{k}^2\mathbf{u}_{\mathbf{k}} - \mathbf{f}_{\mathbf{k}}$ with $\alpha = \frac{1}{R}$: these are irreversible equations. If, instead, $\alpha(\mathbf{u}) = \alpha_d(\mathbf{u})$ for $d = 2, 3$, respectively, is defined so that the enstrophy $\mathcal{D}(\mathbf{u}) = \sum_{\mathbf{k}} \mathbf{k}^2|\mathbf{u}_{\mathbf{k}}|^2$ is constant, *i.e.* (with $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$)

$$\begin{aligned} \alpha_2(\mathbf{u}) =& \frac{\sum_{\mathbf{k}} \mathbf{k}^2 \overline{\mathbf{f}}_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}}}{\sum_{\mathbf{k}} \mathbf{k}^4 |\mathbf{u}_{\mathbf{k}}|^2} \\ \alpha_3(\mathbf{u}) =& \alpha_2(\mathbf{u}) - i\frac{\sum_{\mathbf{k}_1,\mathbf{k}_2} \mathbf{k}^2 (\mathbf{u}_{\mathbf{k}_1} \cdot \mathbf{k}_2)(\mathbf{u}_{\mathbf{k}_2} \cdot \mathbf{u}_{-\mathbf{k}})}{\sum_{\mathbf{k}} \mathbf{k}^4 |\mathbf{u}_{\mathbf{k}}|^2} \end{aligned} \tag{8}$$

[4] This follows by mutiplying both sides of the equations by $\overline{\mathbf{u}}_{\mathbf{k}}$ and summing over $\mathbf{k}$.

[5] Obtained as in the above footnote[4].

and evolution is *reversible*

The equivalence condition $En = \mu_R^{i,N}(\mathcal{D})$ yields, if $W(\mathbf{u}) = \sum_{\mathbf{k}} \overline{\mathbf{f}}_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}}$, that the identities[6] $\frac{1}{R}\mu^{i,N}(\mathcal{D}) \equiv \mu_R^{i,N}(W)$ and $\mu_{En}^{r,N}(W) \equiv \mu_{En}^{r,N}(\alpha)En$ should imply, as mentioned before Eq.(6), equality of the averages of the injected average power W (as W is a local observable). Hence for $\alpha = \alpha_i, i = 2, 3$ the mentioned test of the equivalence in Eq.(6) should hold reflecting the heuristic idea of equivalence as a homogeneization phenomenon.

Eq.(6) takes a particularly simple form if $d = 2$:

$$\lim_{N\to\infty} R\,\mu_{En}^{r,N}\Big(\frac{\sum_{\mathbf{k}} \mathbf{k}^2 \overline{\mathbf{f}}_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}}}{\sum_{\mathbf{k}} \mathbf{k}^4 |\mathbf{u}_{\mathbf{k}}|^2}\Big) = 1 \tag{9}$$

while if $d = 3$ it takes the form:

$$\lim_{N\to\infty} R\,\mu_{En}^{r,N}\Big(\alpha_2(\mathbf{u}) - i\frac{\sum_{\mathbf{k}_1,\mathbf{k}_2} \mathbf{k}^2 (\mathbf{u}_{\mathbf{k}_1} \cdot \mathbf{k}_2)(\mathbf{u}_{\mathbf{k}_2} \cdot \mathbf{u}_{-\mathbf{k}})}{\sum_{\mathbf{k}} \mathbf{k}^4 |\mathbf{u}_{\mathbf{k}}|^2}\Big) = 1 \tag{10}$$

The above relations can be seen as *sum rules* which test quite critically the OK41-scaling laws, [8, Ch.5]: naive application of the OK scaling in $d = 2$ would give a $\log R$ divergence of the limit in Eq.(9), while in $d = 3$ the α_2 contribution to Eq.(10) would tend to 0 as $N \to \infty$ (as $R^{-\frac{1}{2}}$). Hence the first relation is a sum rule that depends strongly on the details of the OK laws and their corrections, while in $d = 3$ the second contribution, after α_2, involves the whole inertial range and is more delicate.

A few simulations can be invoked to support the above ideas in $d = 2$; in all of them the initial data rare random fields with enstrophy prefixed at its average computed in a preliminary run; a first among them is illustrated below:

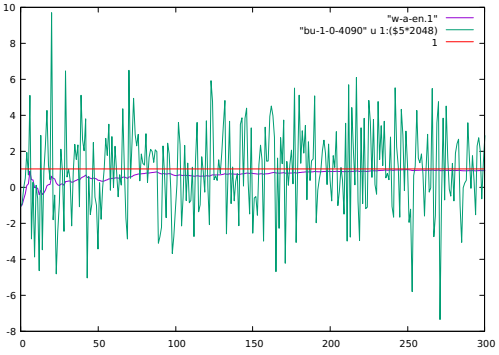

Fig.1: The running average of the reversible friction $R\alpha(u) \equiv R\frac{2Re(f_{-\mathbf{k}_0} u_{\mathbf{k}_0})\mathbf{k}_0^2}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$ *(blue), superposed to the conjectured value* 1 *(red) and to the fluctuating values* $R\alpha(u)$ *(green)reported every 20: reversible NS,* R*=2048,* $\mathcal{N} = 224$ *modes, integration step* $h = 2^{-13}$*, Lyapunov exponent* $\lambda \simeq 1.5$*, axis unit* $4h$*, constant (randomly chosen) forcing on wave* $\mathbf{k}_0 = \pm(2, -1)$.

Other considerations will next be presented with suggestions of several tests, related to the above discussion and to ideas summarized in [9].

[6] Obtained as in the above footnote[4].

The Fig.1 shows large fluctuations of the variable α for the *reversible* $d = 2$ *NS evolution.* This variable is closely related to the divergence $\sigma(\mathbf{u}) = -\mathrm{Tr}\frac{\partial \dot{\mathbf{u}}}{\partial \mathbf{u}}$ of the Jacobian matrix (*i.e.* the phase space contraction): which generates a very similar graph. Since the evolution is reversible it is tempting to check whether the *fluctuation relation* (FR, [10]) is satisfied by the probability $\pi_\tau(p)dp$ that $p \overset{def}{=} \frac{1}{\tau}\int_0^\tau \frac{\sigma(S_t\mathbf{u})}{\sigma_+} dt \in dp$ with $\sigma_+ \overset{def}{=} \mu_{En}^{r,N}(\sigma)$, *i.e.* whether for all UV cut-off N and asymptotically as $\tau \to \infty$ it is:

$$\frac{\pi_\tau(p)}{\pi_\tau(-p)} = e^{p\sigma_+\tau + o(\tau)},\ |p| < 1 \tag{11}$$

The FR is a consequence, for reversible evolutions, of the *chaotic hypothesis*, [9, 10], which states that a chaotic evolution has an attracting smooth manifold $\mathcal{A}$ on which the evolution is strongly unstable (as in an Anosov flow). But FR is a property of the phase space contraction $\sigma_{\mathcal{A}}(\mathbf{u})$ *measured on the surface of the attracting set* $\mathcal{A}$.

The latter is of course very difficult to describe analytically as $\mathcal{A}$ is in general only a (often small) subset of the surface of constant enstrophy on which the flow takes place. Except when
(a) $\mathcal{A}$ coincides with the entire surface of constant enstrophy $D(\mathbf{u}) = En$, or
(b) when, by virtue of some symmetry, the $\sigma_{\mathcal{A}}(\mathbf{u})$ on the surface of constant enstrophy is proportional to the $\sigma(\mathbf{u})$ on the full phase space, which is a quantity that can be computed directly from the equations of motion.

The question suggests studying the local Lyapunov exponents,[7] which can be quite easily measured, and the results of a simulation can be helpful to comment on the above items (a),(b), see Fig.2.

In the case (a) the number of positive and negative Lyapunov exponents has to be the same. In the simulations similar to the one leading to Fig.1, but *with low* UV *regularization* N, this seems to be the case, at least if the local Lyapunov exponents are studied: therefore it would be interesting to try to check the CH in this case, assuming that the local Lyapunov exponents reflect at least the count of the positive and negative ones, as it appears to be for $R = 2048$ and less than $\mathcal{N} = 224$ modes with the exponents recorded roughly every $4\lambda^{-1}$ units of time ($\lambda^{-1} \sim$ maximal Lyapunov exponent) and averaged. The graph of the exponents is not reproduced here.

The case (b) arises, at the same R, if the UV regularization is increased, *e.g.* at $\mathcal{N} = 960$ the fraction of positive local exponents is measurably $< \frac{1}{2}\mathcal{N}$. This is illustrated in Fig.2 which gives the local exponents for both the reversible and irreversible flow in an equivalence condition.

[7] Local exponents are defined in terms of the OR decomposition of the linearization of the flow over a time step h, *i.e.* of the Jacobian matrix $1 + h\frac{\partial \dot{\mathbf{u}}}{\partial \mathbf{u}}$, averaged over the evolution of $\mathbf{u} \to S_t\mathbf{u}$, [11], or in terms of the eigenvalues of the symmeric part of the Jacobian, [12, 13].

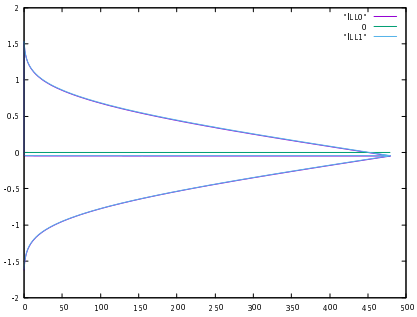

Fig.2: $R = 2048$, $\mathcal{N} = 960$ modes, **local** *exponents ordered by decreasing values λ_k, $0 \le k < N/2$, and increasing λ_{N-k}, $1 \le k \le N/2$ and the lines $\frac{1}{2}(\lambda_k + \lambda_{N-1-k})$, computed for the reversible and irreversible equations (superposed), as well as the line $\equiv 0$ (it is hard to distinguish the latter two lines).*

The first interesting feature is that the local exponents of the reversible and irreversible cases are almost overlapping: this cannot be attributed to the conjecture because the exponents *are not local observables* and suggests that equivalence might extend, for selected observables, beyond the local observations.

The second aspect is difficult to see on Fig.2: in it the line 0 is also drawn and it almost overlaps with the line $k \to \frac{1}{2}(\lambda_k + \lambda_{N-1-k})$. However it is *definitely distinct* as it could be seen magnifying Fig.2.

The positive and negative Lyapunov exponents appear "paired" to a line which is not horizontal, but close it.

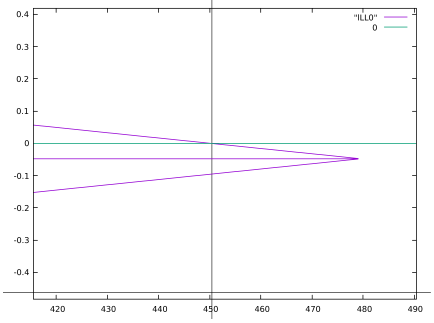

Fig.3: Detail of Fig.2 showing the irreversible exponents (only) and the line $\equiv 0$ which illustrates the dimensional loss of $\simeq \frac{450}{480}$. $R = 2048$, $\mathcal{N} = 960$ modes.

Nevertheless under the, appealing but daring, hypothesis that the pairs which consist of two negative exponents pertain to the attraction to the attracting set $\mathcal{A}$ proportionality between the total phase space contraction and the contraction of the surface elements on $\mathcal{A}$ can be established, [9, Sec.4.4]. The proportionality factor turns out to be about the ratio φ between the number of pairs of exponents of opposite sign and the total number of pairs (*i.e.* the total dimension of M_N). In the case of Fig.2,3 this would be $\varphi \simeq \frac{450}{480}$ and, in testing the FR, the σ_+ should be replaced by $\varphi\sigma_+$: which could be observed, in principle. However the graph draws local exponents and to find φ it would be necessary to use the Lyapunov exponents: hence the above value is, st best, an indication of dimensional loss.

Finally the following is a plot in a low number of modes of the local observable $(\mathrm{Re}\, u_{1,1})^4/\langle(\mathrm{Re}\, u_{1,1})^4\rangle$

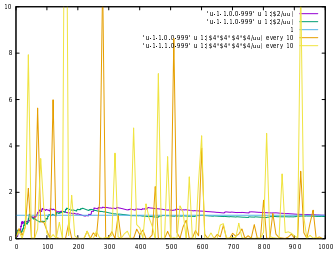

Fig.4: Graphs of the local observable $x^4/\langle x^4\rangle$ with $x = \mathrm{Re}\, u_{1,1}$, irreversible (yellow) and running averages of $x^4/\langle x^4\rangle$ superposing reversible and irreversible evolutions (blue and red) and the line 1 (reported every 20), both graphs converge to the conjectured 1 with the irreversible running average converging more slowly. $R = 2048$ and $\mathcal{N} = 48$ modes ($N = 3$) integration step $h = 2^{-13}$ data saved every $4h$ but reported in the graph every 20. Check of agreement with the conjectured value 1 (straight line).

The idea behind the conjecture is that it is due to the chaotic motion: however the NS equations are derived from the microscopic motions which undoubtedly are chaotic: therefore the conjecture is formulated for *all* Reynolds' number R. In particular at small R, where the motion may be laminar (*i.e.* periodic) or several attractors may coexist, the equivalence will mean that the stationary states of the reversible and irreversible equations can be put in one-to-one correspondence in which local observables have equal average valus in the UV limit; in close analogy with phases coexistence in statistical mechanics.

Hence equivalence may not hold for equations which are not derived from microscopic motions, like the fluid equations with linear viscosity $-\nu\mathbf{u_k}$ or NS equations with *fixed* UV cut-off or other models, [9], for which reversible equations have been considered. In the latter cases some equivalence might still remain: which should hold quite generally if at *fixed equations* the limit $R \to \infty$ is considered, [14].

[1] D. Ruelle, *Chaotic motions and strange attractors* (Accademia Nazionale dei Lincei, Cambridge University Press, Cambridge, 1989).

[2] D. Ruelle, Proceedings of the National Academy of Science **109**, 20344 (2012).

[3] D. Ruelle, Journal of Statistical Physics **157**, 205 (2014).

[4] Z. She and E. Jackson, Physical Review Letters **70**, 1255 (1993).

[5] G. Gallavotti, Physica D **105**, 163 (1997).

[6] G. Gallavotti, European Physics Journal Special Topics **227**, 217 (2018).

[7] D. Ruelle, *Statistical Mechanics* (Benjamin, New York,

1969, 1974).
[8] G. Gallavotti, *Foundations of Fluid Dynamics* ((second printing) Springer Verlag, Berlin, 2005).
[9] G. Gallavotti, *Nonequilibrium and irreversibility*, Theoretical and Mathematical Physics (Springer-Verlag and http://ipparco.roma1.infn.it & arXiv 1311.6448, Heidelberg, 2014).
[10] G. Gallavotti and D. Cohen, Physical Review Letters **74**, 2694 (1995).
[11] G. Benettin, L. Galgani, A. Giorgilli, and J. Strelcyn, Meccanica **15**, 9 (1980), ISSN 1572-9648.
[12] D. Ruelle, Communications in Mathematical Physics **87**, 287 (1982).
[13] E. Lieb, Communications in Mathematical Physics **92**, 473 (1984).
[14] L. Biferale, M. Cencini, M. DePietro, G. Gallavotti, and V. Lucarini, Physical Review E **98**, 012201 (2018).

Also: http://ipparco.roma1.infn.it
Acknowledgement: I am indebted for computer support from Roma 2, INFN and NYU; I am grateful to a referee for remarking several typos.
e-mail: giovanni.gallavotti@roma1.infn.it