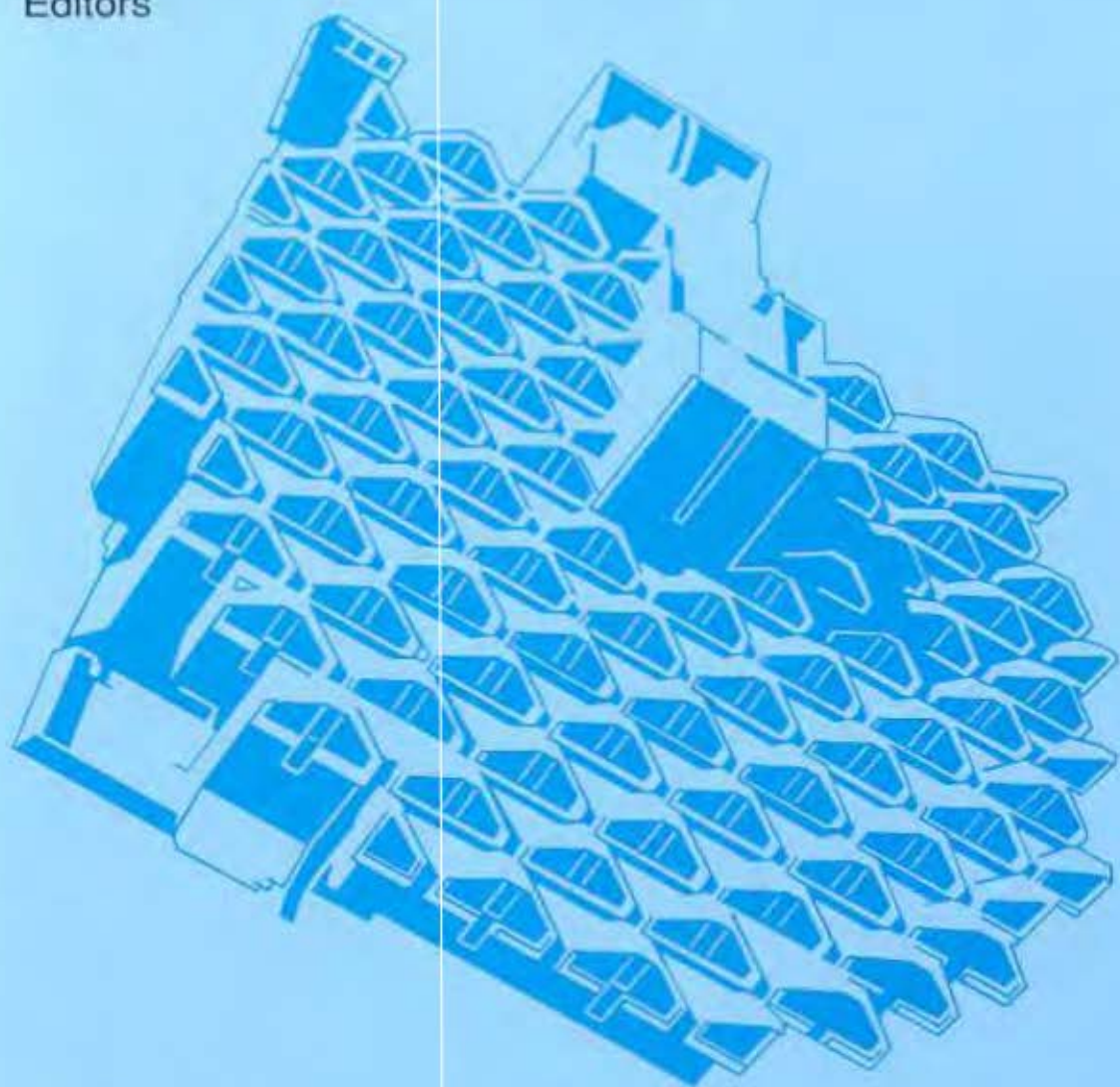


MATHEMATICAL **PHYSICS** **TOWARDS** **THE 21st CENTURY**

R. N. Sen and A. Gersten
Editors



Ben-Gurion University of the Negev Press

Mathematical Physics Towards the 21st Century

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Mathematical Physics Towards the 21st Century
(Based on the Beer-Sheva Workshop, 14-19 March, 1993)

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Editors' Foreword

The subject we call *mathematical physics* today began when, shortly after the end of the Second World War, a few physicists, Arthur Wightman and Rudolph Haag among them, realised that the successes of quantum electrodynamics really meant that

Nel mezzo del cammin di nostra vita
mi ritrovai per una selva oscura
che la diritta via era smarrita.¹

The Beer-Sheva Workshop, which was held between 14 and 19 March, 1993, was a modest celebration of what those dark woods have revealed to luminous eyes. The gathering was small, intimate and happy; but much that was discussed, during lectures and away from them, would have interested a wider audience. However, had the gathering been larger, the easy informality which had prevailed would invariably have given way, and the accent would have shifted from the insecurity of *unsolved* problems to the firm but conventional ground of newly-solved ones.

The present volume should be regarded as a continuation of the Workshop, but not its “proceedings”. Proceedings of conferences tend, sometimes, to have a sepulchral quality which makes readers—and writers—wary of them. Our aim has not been to record an event, but rather to produce an intellectual companion and sparring-partner equally for those who were not present at that event. This is what we requested of our authors, and were more than pleasantly surprised at the response. Indeed, one of our toughest problems has been to decide what to call this volume. We could not use the term “Proceedings”, not just because it may put people off, but also because it would have been hugely inaccurate. Several of the speakers have not contributed to this volume – no pressure was put on them to contribute². We asked for, and received, contributions from participants who were not among the speakers. The written contributions of some speakers are very different from their talks. Larry Biedenharn, Heide Narnhofer and Walter Thirring, who could not come at the last moment, provided us with their manuscripts. In the end, the editors settled for “A Report”—a phrase which, through bureaucratic zeal, has become sufficiently ill-defined. It might have suited our purpose, but was, mercifully, shot down by the publisher. So this volume comes into the world without a sub-title!

About the contents of the volume, we cannot do better than letting the authors speak for themselves. One-line summaries are provided, by definition, in the table of contents, and some of the authors have provided abstracts as well. In the main, the editors have had to attend only to “cosmetic” matters. However, even this has required

¹Dante Alighieri, *Inferno*, Canto I:1-3

²Minor exception: R. N. Sen bet Barry Simon 10 shekels that he (Barry) would eventually become so enthused that he *would* make a written contribution—and lost!

the exercise of a surprising amount of editorial tyranny. We acknowledge, with thanks, the gracious responses of the authors to our many demands, big, small and maybe even trivial.

One of these demands, which was purely technical, deserves special mention. This book has been typeset in $\text{\textit{AMS-LATeX}}$, and we requested, and received (with one exception), $\text{\textit{TeX}}$ or $\text{\textit{LATeX}}$ source files from the authors. Complaints, however, were voiced that publishers were trying to get the authors to do more and more of their work! We believe that this should be justified *by reducing the cost of the book to the buyer* (the American Mathematical Society has estimated that typesetting accounts for about half of its publication costs). The use of $\text{\textit{TeX}}$ or $\text{\textit{LATeX}}$ source files should reduce unit prices of books to end-users by something like 30 to 50%, which clearly has not happened. *The pricing of this volume reflects the fact that much of the technical work required was made available to the publisher, free of cost, by the authors and the editors.*

We had originally intended to print the articles in the order in which they were received, but could not always adhere to this. Once or twice, formatting considerations (a diagram *had to be* on an odd or an even page) supervened; in a few cases, we encountered technical problems which held us up. Advice from the Technical Support Group of the American Mathematical Society is gratefully acknowledged. The article by Mark Steiner was not the last we received, but we chose it for the (arbitrary) closure of this volume to symbolise that, in our opinion, at least one new chapter needs to be opened.

The reader will find that large areas of mathematical physics are not represented at all in these pages. This volume is not a comprehensive one. Neither was the Workshop itself – a small and intimate meeting, which is not a topical conference, could hardly be otherwise. It is surely possible to produce companion volumes to fill the lacunae. Perhaps they will be, in the fullness of time. We would like to use this as an excuse to express some personal opinions.

Mathematical physics is already a difficult subject, but there may be a case for making it even more so. In addition to mathematics and physics, it may be useful to require (of some!) a background in metaphysics, or the no-mans's land between physics and philosophy. A case for this can be made by looking at (fairly recent) history. Between the two wars, Sir James Jeans and Sir Arthur Eddington wrote a number of semi-popular books³ on the impact of quantum mechanics and the theory of relativity on physical reality and our perception of it. Written in elegant English, these books held more than a certain fascination, but philosophically they were wholly untenable. The authors were quite properly “trashed” by Susan Stebbing in a little book entitled *Philosophy and the Physicists*⁴. It is only to be expected that any change in our fundamental perceptions will cause perplexity, wonderment, and, eventually, “philosophising”. The enthusiastic but poorly-informed philosophising which followed the birth of the theory of relativity or of quantum mechanics was largely ignored by physicists who concerned themselves with understanding *phenomena*, of which there was both a large backlog and a new cascade. However, endeavours such as string theory or non-commutative geometry would not, from the empirical viewpoint, provide more

³For example, *The Mysterious Universe*, by J. H. Jeans, CUP, Cambridge, 1931; *The Nature of the Physical World*, by A. S. Eddington, CUP, Cambridge, 1932

⁴Pelican Books, Harmondsworth, England, 1944

than “marginal” gains. They *would* (if successful) shake the conceptual foundations of the subject. This is a situation which is tailor-made for metaphysics, both good and bad, and we should make some effort, if not to encourage the good, then at least to avoid the bad.

Other questions which may gain from careful excursions into metaphysics are those associated with certain limiting processes. One of the distinguishing features of mathematical physics after the Second World War has been the care with which it has handled limits. Despite this, some limits, from Boltzmann to Aizenman and Fröhlich, have caused much perplexity (and, in the latter case, given rise to the sobriquet *destructive field theory*)⁵. The same could be said, more generally, of the whole relationship between microphysics and macrophysics, and, to a lesser extent, of the theory of the renormalization group. The problem of *controlling* the left-hand side of $\lim_{n \rightarrow \infty} A^{(n)} = B$ can only be posed when *both* $A^{(n)}$ and B are known. The questions which arise in the specification of B may *sometimes* border on the metaphysical (for example, in the hydrodynamic and the thermodynamic limits).

Indeed, the frontiers between physics and metaphysics have been changing, through the ages, due both to advances in methods of experimentation and observation, and to the formulation of new concepts which become necessary or desirable. It is the second which is of immediate interest to the mathematical physicist, because concepts are seldom used in physics unless they are expressible in mathematical terms. If one wants to conquer new territory from metaphysics for mathematical physics, one has to have an idea of where the territory lies and what it looks like; that is, one has to have an acquaintance with metaphysics.

To this day, nobody really understands what Wigner called *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*⁶. In fact, the question has become rather like Englishmen and the Weather: “Everybody talks about it, but nobody *does* anything about it”; the only exception known to the editors being H.-J. Borchers⁷. However, this question deserves more attention than it has received, if only for a practical reason. The success of mathematics in physics has inspired many attempts to use it *in areas where present-day mathematics may not be applicable*, for example, in parts of theoretical biology. If one had a clearer perception of why mathematics succeeds so well in physics, one could perhaps form clearer ideas of its limitations, as well as of its potential power, in other fields in which it is sought to be applied.

These are formidable problems, but then Mathematical Physics captivates precisely because it makes such great demands. We can, perhaps, tell the young reader of this book that this is a part of our heritage which should never be surrendered. But, amid the “blood, sweat and tears”, this volume tries to continue the avowed purpose of the Workshop, which was more “to inspire, rather than to inform”. It was designed to be enjoyed, rather than endured. Perhaps that is why it will endure.

To all those who made this possible, our grateful thanks!

R. N. Sen and A. Gersten
Beer-Sheva, April 1994

⁵See, however, the contribution of John Klauder to this volume

⁶Comm. Pure and Appl. Math. 13, 1 (1960)

⁷The theory of ordered spaces, Commun. Math. Phys. 132, 593-611 (1990)

Note on The Divine Comedy

The literal translation of the three lines quoted earlier is something like "In the middle of the journey of our lives, I found myself in a dark wood, Where the straight path was lost". In the present context, the quotation should be taken more in the literal than in the allegorical sense. The Italian text, a free prose translation, and a very good commentary on *La Divina Commedia* (The Divine Comedy) has been provided by Charles S. Singleton, Bollingen Series, Princeton University Press, Princeton, 1977 (etc.). For a poetic rendition, we would recommend the version by Dorothy Sayers, *The Comedy of Dante Alighieri, the Florentine*, Penguin Press, Harmondsworth, England, 1949 (etc.). She renders the first verse as

Midway this way of life we're bound upon
I woke to find myself in a dark wood,
Where the right road was wholly lost and gone.

The interested reader may also consult: Dorothy L. Sayers, *Introductory Papers on Dante* (with a preface by Barbara Reynolds), Methuen, London, 1954. The reason why we have used the quotation may be found in Russell's comments on Dante: Bertrand Russell, *History of Western Philosophy*, George Allen and Unwin, London, 1946.

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EINSTEIN'S PRINCIPLE OF MAXIMAL SPEED IN CLASSICAL AND QUANTUM PHYSICS

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Abstract

Einstein's principle of maximal speed has dramatic consequences in several branches of physics. Among them, we shall discuss the following: (a) The linearity of a bijection of $\mathbb{R}^d$ which, together with its inverse, maps light rays onto light rays. (b) Reasons why Einstein's principle holds in dispersive media. (c) Einstein's principle as a fundamental property in quantum field theory: locality and spectrum conditions. (d) The structure of quantum field theories with broken Lorentz symmetry. (e) Particles when Wigner's definition of particles does not apply. (f) Perspectives and open problems.

1 Historical Remarks

In 1906 A. Einstein [20] formulated the principle of maximal speed as a fundamental principle of physics. With this principle he ended a long discussion which was led by Poincaré, Lorentz, Minkowski, and many others. He realized that two observers (moving with respect to each other) have to agree on some parameters concerning the transfer of information between them. Einstein decided to try the electromagnetic field as the carrier of information and in addition to make the velocity of light into a universal constant, which was the same for every observer. By this procedure he introduced the Lorentz group (of light) as the transformation group to be used for changing the (inertial) reference frame. This group is at the same time the invariance group of Maxwell's equations, which makes many considerations and calculations easier. But it is not necessary for the two to coincide, as the example of acoustics shows. The generally-believed principle of simplicity in physics requires that for light, which has a double function in the theory of relativity, the symmetry and the transformation groups must coincide. By accepting this, Einstein made a choice in favour of the invariance group of Maxwell's equations as opposed to the Galilei group. It was clear, at that time, that there was a conflict between these two groups. Einstein was not the first to realize the importance of the invariance group of a branch of physics. In this connection one should mention Woldemar Voigt [42] who, in 1887, used the invariance group of the wave equation – the Lorentz group – in order to describe the Doppler effect, discovered in 1842 [18].

In view of our discussion it might be interesting to look again at the interpretation of the experiments of Michelson in Berlin [29] and of Michelson and Morley in Chicago

[30]. From my point of view the outcome of the experiment does not say much about the invariance group of nature. It deals, rather, with the problem of whether or not optics and mechanics transform according to the same group. Optics deals with light rays and mechanics with the mechanical support of the semi-transparent plates and mirrors. The experiment showed that they transform according to the same group. This discussion means that one has to be careful to distinguish between the invariance group of a branch of physics, and the transformation group of the observers. Comparison of the Doppler effect in acoustics and in optics gives a good example of the pitfalls present in this field. There is no doubt about the Lorentz group as symmetry group of physics; however, not only the experimental, but also the conceptual problems should be discussed carefully.

In order to be more precise we have to fix some notations:

Minkowski space is a four-dimensional (more generally n -dimensional) space furnished with the metric

$$x^2 := (x_0)^2 - (\vec{x})^2.$$

Two points x, y are called *timelike* if $(x - y)^2 > 0$. They are called *spacelike* if $(x - y)^2 < 0$. They are called *lightlike* if the Minkowski distance between them is zero. A **light ray** is a straight line such that all pairs of points on it are lightlike.

A **light cone** with vertex (or origin) y is the set $\{x; (x - y)^2 \geq 0\}$. This light cone consists of two parts, the forward and the backward light cone. The forward cone consists of those elements for which $x_0 - y_0 \geq 0$ holds.

The **Lorentz group** consists of those transformations which leave the Minkowski scalar product unchanged. The inhomogeneous Lorentz group is often called the **Poincaré group**.

2 Mathematical Problems concerning the Lorentz Group

Most textbooks dealing with special relativity pass immediately from the invariance of the velocity of light to the invariance of the scalar product in Minkowski space. This procedure yields the Lorentz group rather easily. Einstein also started from the Lorentz group, but he had a different motivation. For him the invariance group was the starting point, and this group had to be continuous and linear. His point of view can best be seen in his Princeton lectures on general relativity in 1921 [21].

Since the modern tendency is to replace mathematical (e.g., continuity and linearity, as above) by physical principles (e.g., causality, as in the following) the derivation of the Lorentz group became a problem, at least for mathematically minded people. The first to consider this question was Christopher Zeeman [45]. He found the following

Theorem (Zeeman) *Let L be any one to one mapping of the Minkowski space onto itself. Assume L and L^{-1} map light rays onto light rays and leave the origin fixed. If the dimension of the Minkowski space is larger than two, then it is a Lorentz transformation followed by a contraction.*

It might be instructive to look at a proof of this result which is different from the original one. It is taken from a paper by Borchers and Hegerfeldt [10]. The main

problem consists of showing that the map is linear and hence also continuous. A result from elementary mathematics (see, e.g., van der Waerden [40]) shows that this is the case if L and L^{-1} map every straight line onto a straight line. One can construct a family of hyperplanes which are mapped onto hyperplanes again. Take a fixed light ray ℓ and consider the union of all light rays passing through the fixed one. It is the same as taking the union of the boundaries of all light cones having their origins on the fixed light ray ℓ . The union gives the whole space except the hyperplane tangent to the light cone containing ℓ . Hence the union is

$$\{\mathbb{R}^n \setminus H(\ell)\} \cup \ell.$$

The last term is the union of the origins of the light cones. $H(\ell)$ is the unique hyperplane containing ℓ and no timelike direction. Now L maps $\ell \rightarrow \ell'$ and every light ray through ℓ onto a light ray through ℓ' . Since L and L^{-1} are one-to-one they must map the complement onto the complement, i.e., $H(\ell) \setminus \ell \rightarrow H(\ell') \setminus \ell'$. Hence we obtain

$$LH(\ell) = H(L\ell).$$

Since we can represent every spacelike straight line as the intersection of these hyperplanes, L also maps every spacelike straight line onto a spacelike straight line. Having so many straight lines at hand it is easy to construct two-planes out of them containing timelike directions. Therefore L maps all straight lines onto straight lines. Hence the map is linear. The rest is simple.

Of course an experimentalist cannot check an infinity of directions. Therefore, one should ask for how many directions has linearity to be checked. The answer is somewhat disappointing. One needs a continuum of directions in order to guarantee the full linearity of the map. (This question is related to the extension theory of fields. Because of a result of Artin and Schreier [1] one needs a non-countable infinity of extensions in order to get from the field of rational numbers to the field of real numbers.) If, however, one is willing to assume continuity – as one always does in physics – then the situation becomes much simpler. In this case it is sufficient to check linearity for $(n+1)$ different directions, provided all n subdirections are linearly independent. These results can be found in a paper by Borchers and Hegerfeldt [11].

3 Einstein's Principle and Dispersive Media

The principle of maximal velocity for signals poses some problems for dispersive media. In such media there are always some wavelengths for which the phase-velocity is higher than the speed of light. It is clear that no information can be transported by plane waves. However, it remains to be understood how this fact accords with Einstein's principle. This question was first studied by Sommerfeld and Brillouin [34], [13]. They first considered the problem of defining the signal velocity, but could not give a satisfactory answer. Therefore they investigated the case of a half-space filled with the dispersive medium, the other half-space being *in vacuo*. One lets a wave packet with a sharp front come from the vacuum side and strike the surface of the dispersive medium. Then one looks at a distance d from the surface (inside the matter) and obtains the following:

Result of Brillouin and Sommerfeld Assume that the wave packet has a sharp beginning and a box-like shape. Assume also that it strikes the surface at $t = 0$.

(a) For $t < \frac{d}{c}$ there is no field.

(b) If c_1 is the group velocity in the dispersive medium, then for

$$\frac{d}{c_1} > t > \frac{d}{c}$$

the field is small (compared with the intensity of the wave packet).

(c) The wave packet is essentially only visible for $t > \frac{d}{c_1}$.

Remarks:

1) The results of Brillouin and Sommerfeld have been confirmed experimentally by Palósz and Pleshko in 1969 [31].

2) Sommerfeld gave a physical interpretation of the existence of the "precursor". This can be found in his textbook on electrodynamics [35].

Brillouin and Sommerfeld found that there was no field for times $t < \frac{d}{c}$. A mathematical understanding of this phenomenon came only with the introduction of response functions and the discovery of dispersion relations by Kramers [27] and Kronig [28]. Causality implies that the dielectric constant $\epsilon(\omega)$ as a function of the frequency has an analytic continuation into the upper complex half-plane. If one solves Maxwell's equation by Fourier analysis, an integral which gives zero for $t < \frac{d}{c}$ but not zero for $t > \frac{d}{c}$ is obtained. The quantity $\frac{d}{c}$ appears because the integrand changes its character at this point. (In the imaginary direction it is exponentially decreasing for $t < \frac{d}{c}$ and exponentially increasing for $t > \frac{d}{c}$.) Therefore below this value one can deform the contour in order to obtain zero. Beyond this value the method does not work anymore. The detailed results of Sommerfeld and Brillouin cannot be obtained by this technique.

Since I was dissatisfied with treating such a fundamental problem only in the one-dimensional setting I considered the question again and found a solution for another idealized situation: The dispersive medium fills the whole space and there is a light-flash at the origin at $t = 0$. Here one has a source term which is a delta-function in space and in time for which Maxwell's equations can be solved exactly. In this situation one finds a spherical wave, the wavefront of which travels with the speed of light. The reason for this is again the response-behaviour of the dielectric function. However, this investigation leads to something more. Response-functions have an analytic continuation into the upper complex frequency half-plane. In the case of optics the imaginary part of the dielectric function is the absorptive part and hence positive. But the investigation of the signal problem shows that the imaginary part of this function too has to be positive. That means $\epsilon(\omega)$ has to be a Herglotz function. If one chooses $\mu = 1$ — as one mostly does in optics — then one obtains for the index of refraction the formula

$$n(\omega) = 1 + \frac{2}{\pi} \int_0^{\infty} \frac{t}{t^2 + \omega^2} a(t) dt$$

where $a(\omega)$ is the absorptive part of the refractive index. These results can be found in [7].

The problem of inhomogeneous media, in which the refractive index depends on spatial position and on the frequency, still remains.

4 Einstein's Principle in the Theory of Finitely Many Particles

In quantum mechanics we shall encounter some new aspects of the principle of maximal speed. The first one is concerned with the fact that not all observables commute.

The standard way of describing non-commuting quantities is by operators on a Hilbert space $\mathcal{H}$. Physical states are represented by quadratic forms generated by the vectors of this Hilbert space, better known as expectation values. The observables are realized as operators acting on this Hilbert space. It is generally accepted that two observables which do not influence each other will commute. But it is clear that two observations at distant places and at the same time cannot influence each other. Therefore the operators describing these experiments must commute. For a systematic description of this situation a principle of localization for observables is needed.

Every measurement is performed in a laboratory. Moreover, it is done by a device which has to be switched on in order to prepare for a measurement, and switched off after the measurement is over. Therefore one should associate with every observable x a domain $O(x)$ in space-time, where $O(x)$ is the smallest domain which is influenced by x and in which it can be measured. Then it is clear that x can be measured in every larger domain. We also assume that if x can be measured in O_1 and y in O_2 then $x + y, xy + yx$ and $i(xy - yx)$ can be measured in $O_1 \cup O_2$. We associate with every observable a domain in Minkowski space, and denote by $\mathcal{A}(O)$ all those observables which can be measured in the domain O . As is usual in quantum mechanics one passes from the set of observables to the algebra they generate. If we do this we get:

A C^* -algebra $\mathcal{A}(O)$ is associated with every bounded open region O in Minkowski space such that $O_1 \subset O_2$ implies

$$\mathcal{A}(O_1) \subset \mathcal{A}(O_2).$$

Certain pairs of domains have the property of having a spacelike separation. In this case, by Einstein's principle, no information can pass from one domain to the other. Therefore the observables located in the different domains must commute:

If two bounded regions O_1 and O_2 have a spacelike separation (this means $a \in O_1$ and $b \in O_2$ implies always $(a - b)^2 < 0$), then every operator in $\mathcal{A}(O_1)$ commutes with every operator in $\mathcal{A}(O_2)$.

This is the famous locality condition of quantum field theory, often called Einstein causality.

The other aspect of Einstein's principle is concerned with the description of translation symmetry. In quantum physics symmetries are no longer described by symplectic but rather by unitary transformations. This is a consequence of the probability interpretation of wave functions. Space and time translations should be generated by the momentum and the energy respectively. Since we deal with particle physics the energy and momentum operators have to describe the energy and momentum of particles. Therefore the expectation values of these operators have to obey Einstein's principle. This implies that the Minkowski squares of these quantities should be positive and bounded from above by the velocity of light. This can only hold for arbitrary vectors

in the Hilbert space $\mathcal{H}$ if the common spectrum of the four operators is confined to the forward light cone.

The common spectrum of the energy and momentum operators has to be confined to the forward light cone. The energy and momentum operators are the generators of the time and space translations.

This consequence of Einstein's principle is called the *spectrum condition*. For further discussion of the "axioms" of quantum field theory in the setting of local observables I refer to the paper of Haag and Schroer [26] and the paper of Haag and Kastler [25].

After having derived two consequences from Einstein's principle we have to enquire about their consistency. In order to formulate the first condition we had to introduce the concept of localization of an observable. This implies, although we did not mention it, that essentially all observables are localized. The only quantities which are not localized are limits of localized observables. Non-localized quantities are energy and momentum. The experimentalist measures these quantities in a laboratory at least approximately. The experimental physicist regards them as limits of local observables. Will this be confirmed by the theory? A positive answer was given by Borchers [4, 5], who found the following result:

Theorem *Assume that we have a representation π of the local observables $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Assume also that there exists on $\mathcal{H}$ a continuous unitary representation $U'(a)$ of the translation group fulfilling the spectrum condition and implementing the space-time-translations on the observables. Then there exists a continuous unitary representation $U(a)$ of the translation group with*

- (i) $U(a) \in \pi(\mathcal{A})''$.
- (ii) *The spectrum of $U(a)$ is also contained in the forward light cone.*
- (iii) $U(a)$ *implements the same transformation on the observables as $U'(a)$.*

This result is not completely satisfactory because it says nothing about the uniqueness of the representation belonging to $\pi(\mathcal{A})''$. If we are dealing with a Hilbert space containing the vacuum vector then there is no problem. If, however, we deal with a Hilbert space describing charged particles then we do not have a ground state. How shall we define the representation describing the physical energy and momentum? If we look again for orientation at the vacuum sector, we realize that the representation here is characterized by a minimality condition, namely, that the energy of the ground state has the value zero. Trying to search for a similar condition for sectors different from the vacuum one has to ask whether or not the minimality condition has a unique solution. The answer is given by Borchers and Buchholz [12].

Theorem *Let $\pi(\mathcal{A})$ be a representation of the local observables on a Hilbert space $\mathcal{H}$ and let $U'(a)$ be a continuous representation of the translation group, satisfying the spectrum condition and implementing the translations. Then there exists a unique representation $U(a)$ with the properties*

- (i) $U(a)$ *belongs to $\pi(\mathcal{A})''$,*
- (ii) $U(a)$ *fulfills the spectrum condition,*
- (iii) $U(a)$ *implements the translations, and*
- (iv) $U(a)$ *fulfills a minimality requirement (which we did not explicitly define).*

5 The Lorentz Group

Up to now we spoke only about the translations, but what about the Lorentz transformations? In a quantum field theory which fulfills Einstein's principle – as defined in the last section – do we have also a representation of the Lorentz group? The answer is no, at least not for a representation describing charged particles. It is Gauß' law which prevents things from being so well-behaved. Recall that Gauß' law gives rise to singularities already in classical electrodynamics. It also causes difficulties in quantum electrodynamics – the so-called infrared problem. Before going into a detailed discussion let us look at the vacuum sector. Here no isolated charged particles are present and therefore one might have a representation of the Lorentz group. It is often assumed that in the vacuum representation there is a representation of the Lorentz group. We have investigated whether or not there exists a representation, and have found that it is possible to prove existence up to a point, that is in every situation which depends only on one space and one time coordinate. Though we can show that one has representations of the pure Lorentz boosts, we do not know how these different transformations fit together.

We need some notation to describe the result:

A **wedge** is defined by means of two vectors t, s with $t^2 = 1, s^2 = -1$ and $(t, s) = 0$. We define

$$W(t, s) = \{x; |(t, x)| \leq -(s, x)\}.$$

Usually the translates of a wedge are also called wedges. To a wedge W one can associate the algebra generated by $\cup\{\mathcal{A}(O); O \subset W\}$. This algebra will be denoted by $\mathcal{A}(W)$.

Duality: Let π be a representation of the local observables. Then we say π fulfills the duality condition if the commutant of $\pi(\mathcal{A}(W))$ coincides with the algebra of the opposite wedge:

$$\pi(\mathcal{A}(W))' = \pi(\mathcal{A}(W'))''.$$

If $W = W(t, s)$, then the opposite wedge is $W' = W(t, -s)$. W' consists of those points which are spacelike to all interior points of W .

Now we can formulate the result regarding the Lorentz boosts, which is due to Borchers [8]:

Theorem *Let π be a vacuum representation of the local observables on a Hilbert space $\mathcal{H}$. Vacuum representation means that we have a unitary representation $U(a)$ of the translations fulfilling the spectrum condition. They have the vacuum vector Ω as fixed point. If we choose a timelike and a spacelike direction (t, s) , as in the definition of a wedge, then there is a continuous unitary representation $U(\Lambda)$ of the Lorentz group of the two-plane spanned by the two vectors (t, s) . This transformation has the correct property for all objects depending only on the variables of the two-plane:*

$$U(\Lambda)U(a)U^*(\Lambda) = U(\Lambda a)$$

for all a in the two-plane spanned by (t, s) . Moreover, one has

$$U(\Lambda)\pi(\mathcal{A}(O))U^*(\Lambda) = \pi(\mathcal{A}(\Lambda O))$$

for all domains which are cylindrical in all directions perpendicular to the two-plane.

In addition one finds an antiunitary reflection J in this two-plane with

$$\begin{aligned} (1) \quad & JU(a)J = U(-a) \\ (2) \quad & J\pi(\mathcal{A}(O))J = \pi(\mathcal{A}(-O)) \end{aligned}$$

again for vectors in that two-plane and for cylindrical sets.

The construction of the operators $U(\Lambda)$ and J uses Tomita's theory of modular Hilbert algebras, the so-called Tomita-Takesaki theory [39], [36]. The existence of these operators is a consequence of the mathematical theory. Therefore we do not understand whether or not the transformations belonging to different two-planes match to give a representation of the whole Lorentz group. The problem is the behaviour of the transformations $U(\Lambda)$ in the directions perpendicular to this two-plane. Later on we shall discuss the infra-red problems appearing in theories with electromagnetic interactions. There we shall see that in a sector carrying electric charge one cannot have a representation of the Lorentz group. For such a theory it is very likely that in the corresponding vacuum sector as well there is no representation of the full Lorentz group.

Nevertheless in a recent article Buchholz and Summers [16] have put forward a set of assumptions which might give rise to a representation of the Lorentz group. We do not want to discuss it here because it would lead us too far into the Tomita-Takesaki theory. Nevertheless, it should be mentioned that their primary aim was not the construction of the Lorentz group but a characterization of the vacuum state.

Recently Narnhofer and Thirring [38] have suggested a covariant QED by using non-continuous representations of the canonical commutation relations and of the Lorentz group. In their theory the Hilbert space has a positive metric but it is not separable.

The situation in other particle representations is completely different. It has been observed by Fröhlich, Morchio and Strocchi [23] and also by Gervais and Zwanziger [24] that in a sector carrying electric charge there is no representation of the Lorentz group. This is one consequence of the infrared problems. It is related to the fact that a charged particle is surrounded by a cloud of soft photons. A nice proof has been given by D. Buchholz [14].

In a theory describing, among others, electromagnetism the electric field strength should be an observable quantity. Let us define

$$\Phi(\vec{n}) = \lim_{r \rightarrow \infty} r^2(\vec{n}, \vec{E}(r\vec{n})).$$

In this formula $\vec{E}$ is a sufficiently regularized electric field operator. Obviously this is a quantity which commutes with every observable. Hence it is a multiple of the identity (if we restrict ourselves to factor representations) $\Phi(\vec{n}) = \varphi(\vec{n})\mathbb{1}$. The function $\vec{n} \rightarrow \varphi(\vec{n})$ describes the asymptotic flux. Gauß' law implies that this quantity is not identically zero because $\int \varphi(\vec{n})d\omega$ is the total charge when integrated over the unit sphere.

6 The Particle Structure

Let us recall Wigner's identification of particles with irreducible representations of the Poincaré group [44]. We have just learned that there is no representation of the Poincaré

group in sectors carrying electric charge. Does it mean that we have lost the concept of particles? Before answering this question let us look at the experimental situation. From the experimental evidence (see Faragó and Jánosy [22]) we know that the relation $E^2 - (\vec{p})^2 = m^2$ holds with very high accuracy for the electron as well. How can we understand this experimental result in view of the fact that we have broken Lorentz symmetry? From the latter one should expect that the spectrum of the translations can be located in an arbitrary set in the forward light cone. That this is not the case is one of the mysteries of the interplay between the locality and the spectrum condition. The first partial result has been obtained by Borchers and Buchholz [12] and has been generalized by Borchers [5]:

Theorem *Let π be a representation of the algebra of local observables and assume that $U(a)$ is the unique minimal representation of the translations described in section 4. Then the spectrum $U(a)$ is a set which is invariant under Lorentz transformations.*

From this result we learn that an isolated part of the spectrum cannot have arbitrary shape, but must be a hyperboloid. In the case of the vacuum sector we know more: If we decompose the spectrum into the part which is regular and into the part which is singular with respect to Lebesgue measure, then each of these parts is located on a Lorentz-invariant set. This can be found in [3]. If we have a theory dealing with infra-particles introduced by Schroer [33] then the particle is characterized by a singularity in the spectral function, but up to now it has not been proved that this singularity is located on a hyperboloid, although there are several indications that this is true.

One of the fascinating consequences of relativistic invariance in quantum physics was the prediction of antiparticles by Dirac [17]. These have been found and we now know that to every particle there exists an antiparticle. (Sometimes a particle is its own antiparticle.) What can we say about antiparticles in the theory of local observables? There are two aspects to this problem. The first is concerned with the consequences, even when one stays in a single sector. The second deals with the construction of charged fields. Let me start with the first aspect.

Assume that we have a representation of the local observables in a sector describing a charged particle of mass m . If we do not know any more about our theory then we do not know whether or not there are other particles in the theory. We do not know, in particular, whether or not there are particles with a smaller mass. Can we say something more about the spectrum in the sector we are dealing with? In order to get from the one-particle state to another state in this sector we have to add a neutral family of particles. The simplest of such families is a particle-antiparticle pair. Since such a pair has a mass at least $2m$ we conclude that the continuous part of the spectrum cannot begin before $3m$. If we are in the vacuum sector then the particle itself is neutral and the continuous part cannot begin before $2m$. In the vacuum sector the theoretical prediction and the physical intuition coincide. In charged sectors it cannot be proved in full generality but only under (reasonable) additional assumptions. The result concerning the vacuum sector has been obtained by Wightman [43]:

Theorem *The spectrum of the translations in the vacuum sector is an additive set. This means that if p_1, p_2 belong to the spectrum then so does $p_1 + p_2$.*

The result about the spectrum of the translations in a charged sector can be found in [6].

Theorem *Let π be a factor representation of a theory of local observables fulfilling the spectrum condition, and assume that $U(a)$ is the unique minimal representation of the translations. Assume furthermore that the spectrum of $U(a)$ consists of two parts:*

- (a) *Isolated hyperboloids with the masses $m_0 < m_1 \cdots m_i < \cdots$.*
- (b) *The rest starting at $m_c > m_i$ where m_c denotes the beginning of the continuous spectrum or the first accumulation point of the m_i 's.*

Then

$$3m_0 \geq m_c.$$

This is not the best result one could hope for. I think that one day it will be possible to show that there is no gap in the spectrum beyond $3m$. For proving this conjecture one has to compute the envelope of holomorphy of an edge of the wedge problem where the coincidence domain is not of the Dyson type.

We finally want to discuss the possibility of constructing charged fields. It should be remarked that charged fields are not observable quantities. They do not appear in the setting of the theory. But note that if we have a charged field $\psi(x)$ and smear this with a test function with support in a bounded domain O then the operator $\psi(f)$ has two properties which are essential for what follows:

- 1) $\psi(f)$ maps from the vacuum sector into the sector carrying the charge of the field.
- 2) $\psi(f)$ commutes with all observables which are localized in the spacelike complement of O .

These properties are the fundamentals of the so-called theory of superselection sectors. One has to start with two representations, a vacuum and a charged representation. Let us assume unitary equivalence for the two representations of the algebra belonging to the spacelike complement of a bounded region. This condition is to hold for every bounded region. Under this condition one is able to construct a whole sequence of representations and a charged field connecting all of them. As usual in quantum field theory these charges are connected with a gauge group of the first kind. If, in addition, one assumes the duality condition then these fields can be localized. Now one can investigate their commutators or anticommutators. It turns out that these fields either commute or anticommute. If one finds another sector related to the vacuum in the same manner, then it is possible to construct another charged field which commutes or anticommutes with the first field. This method has been found by Borchers [3].

Some years later this construction was generalized by Doplicher, Haag and Roberts [19]. They allowed mappings the images of which are no longer the whole Hilbert space. This generalization allows one to treat the case of nonabelian gauge groups and to deal with statistics other than the Bose and the Fermi.

But what about a justification of the assumption made by Borchers, Doplicher, Haag and Roberts? A breakthrough in this question was made by Buchholz and Fredenhagen [15]. They assumed an isolated mass-hyperboloid in the charged sector. From this assumption they constructed a representation of the vacuum sector. For these two representations Buchholz and Fredenhagen found the following properties:

Theorem *In the situation just described there exists a charged field which can be localized not in bounded regions but in arbitrary small spacelike cones. If the dimension of the space is higher than two then for the statistics one has the same possibilities as in the theory of Borchers, Doplicher, Haag and Roberts.*

Let me end our discussion with some remarks:

- 1) In the theorem of Buchholz and Fredenhagen there appears the condition of local normality. This is a condition often used in physics. It says that in a fixed region one should not be able to notice in which representation one is. Locally one should always find the same physics. If one assumes that the local algebras are von Neumann algebras then a theorem of Takesaki and Winnink [37] shows that the representations must be locally normal. There are certain problems in the theory of local observables where one would like to have smooth behaviour under translations. But this is not possible with local von Neumann algebras.
- 2) We always assumed that there are two sectors. One of them is the vacuum sector. In situations like that of Buchholz and Fredenhagen the existence of such a sector can be proved. But unfortunately there exists no method which shows that every theory of local observables must have vacuum states. One might take this as an indication that there exists theories without a vacuum. At least it does not contradict Einstein's principle.
- 3) We did not succeed in constructing a representation of the Lorentz group in the vacuum sector although we know that there exist representations of Lorentz boosts. It seems to me that in case of interaction with the electromagnetic field the Lorentz symmetry should be broken in the vacuum sector as well. At least there is a conflict between Lorentz symmetry and the existence of charged fields. This has nothing to do with the possible localization of the charged field. As long as this field commutes asymptotically with the observables it is impossible to have Lorentz symmetry in the vacuum sector which is broken in the charged sector.

7 Problems

- 1) The principle of maximal speed has been well understood for homogeneous, dispersive media. But what about inhomogeneous media? This question is not a purely academic one, because the interesting phenomena seem to be in regions with drastic change of the refractive index. This is well known from nonlinear optics and also from the tunnelling microscope.
- 2) There exist several good arguments indicating that the Lorentz symmetry is broken in every sector carrying electric charge [23], [24], [14]. But the proofs are not free from defects. One has to bound the fluctuations for the charge operator. The fact that these fluctuations are bounded in the vacuum sector is no solution. This is due to the fact that one cannot interchange the integration over a sphere of radius R with taking the limit $R \rightarrow \infty$.

It is instructive to look at the free electrodynamic field in a coherent state as introduced by Roepstorff [Ro]. Here the fluctuation remains finite only for states in the vacuum sector. The question is whether there are better arguments.

3) We are still missing a proof for the existence of a vacuum state. Could it be that there exists none in field theories describing gauge fields? Could it be that the old arguments in favour of a vacuum state [9] are too optimistic?

If the Lorentz symmetry is broken only in charged sectors, then there is a contradiction between the existence of a Lorentz invariant vacuum, the broken symmetry in the charged sector, and the existence of charged fields commuting asymptotically with the observables. It is doubtful that the Lorentz symmetry is broken in the vacuum sector, because of the connection between Lorentz transformations and the modular group of the algebras belonging to wedge domains [2], [8].

4) Wightman [43] proved that in every irreducible vacuum representation the spectrum of the translation is an additive set, i.e.,

$$p_1, p_2 \in \text{spec}\{U(a)\} \longrightarrow p_1 + p_2 \in \text{spec}\{U(a)\}.$$

If a charged sector is generated by charged fields then one has the relation

$$p_1, p_2, p_3 \in \text{spec}\{U(a)\} \longrightarrow p_1 + p_2 + p_3 \in \text{spec}\{U(a)\}.$$

Try to prove this relation without using charged fields. There are results indicating that this is true [6].

5) It is generally believed that there is no discrete mass in the middle of the continuous spectrum, unless this mass is separated from the continuum by a quantum number. Also a particle interacting with a massless neutral field (photon) is most likely transformed by this interaction into an infraparticle [33], [14]. If this is true then discrete masses exist only if they are isolated. Can one prove this for interacting theories? The difficulty seems to be the lack of a good axiomatic definition of interaction.

6) We know, as a result of Borchers and Buchholz [12], [5], that the support of the spectrum of the minimal translations is invariant under Lorentz transformations. In the vacuum sector the invariance property holds for both the Lebesgue-continuous and the Lebesgue-singular part of the spectral measure [3]. Most likely the same is true for arbitrary sectors. But how could one prove this?

7) Recently, some remarkable results have been obtained concerning the connection between the translation group and the modular group of wedge domains [8], [41]. So far these investigations have been only one- or two-dimensional because the modular group commutes with the symmetry group of the wedge. Is there any way of including the extra dimensions into the above consideration?

8) So far the relations between the translations and the modular group have been established for the vacuum sector. For a charged sector the translations as well as the modular group are connected with the corresponding groups in the vacuum sector by cocycle relations. One should find relations between modular groups and translations in other sectors as well. What are these relations?

9) In quantum field theory there are two aspects of Einstein's principle, namely, causality in configuration space and the spectrum condition in momentum space. The first principle can easily be generalized to hyperbolic space-time manifolds. But what about the generalization of the spectrum condition? We know from the classical interpretation

what it means that a particle does not travel faster than light. There are two problems, namely the definition of the *local* speed, and furthermore the separation of the kinetic energy from the gravitational energy.

In order to define a family of reasonable states one has to answer the above questions.

10) The relations between the translations and the modular group are only concerned with wedges and translations along light rays. These objects are also present in general relativity, provided the space, together with its order structure, is homeomorphic to the Minkowski space. Here a chance of introducing the concept of "positive energy" into quantum theory on curved space-times seems to be present.

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ERGODIC PROPERTIES OF QUANTUM FIELD THEORIES

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1 Introduction

Boltzmann's ideas about the time evolution of large systems have developed into two mathematical disciplines, topological dynamics and measure-theoretic ergodic theory. In the former one considers a dynamical system consisting of a C^* -algebra $\mathcal{A}$ (the "observables") and an automorphism Θ of $\mathcal{A}$ (the "time evolution" for unit time), whereas in the latter one studies Θ -invariant states over $\mathcal{A}$. In both cases one is interested in the long time behaviour ($\lim_{n \rightarrow \infty} \Theta^n$). This framework can be taken over directly to quantum mechanics, where $\mathcal{A}$ is not commutative. (We add as standing assumption $\Theta \neq id$ and $1 \in \mathcal{A}$.) However, before generalizing the various classical theorems we have to discuss to which physical situation this framework applies. The system we are interested in is bulk matter which consists of electrons and nuclei. Electrons are fermions, and since the difference between various isotopes are of no relevance for us we might as well consider odd mass number isotopes. Then all particles are fermions, which makes the mathematics much easier. Since we are interested in large systems which are spatially unbounded but have a finite particle density the appropriate description is a quantized fermi field. Thus a minimal algebra $\mathcal{A}$ will be the CAR algebra generated by the annihilation operators $a_f = \int d^3x \bar{f}(x) \alpha(x)$, $f \in L^2(\mathbb{R}^3)$, i.e., the norm closure of the polynomials

$$(1.1) \quad a = c + \sum_{n,m} a_{f_1}^* \dots a_{f_n}^* a_{g_1} \dots a_{g_m}, \quad c \in \mathbb{C}.$$

From the CAR relations

$$(1.2) \quad [a_f, a_g^*]_+ = \langle f|g \rangle$$

one infers that $\|a_f\| = \|f\|_2$. The observable algebra $\mathcal{A}_0$ will be the subalgebra where $n = m$. $\mathcal{A}$ is a pleasant C^* -algebra, it is simple (i.e., it has no proper two-sided ideal) and it is UHF (i.e., the norm closure of an increasing sequence of matrix algebras). $\mathcal{A}_0$ has the two-sided ideal consisting of the a 's for which $n = m$ and $c = 0$ in (1.1).

States are positive linear functionals ω over $\mathcal{A}$ and by the GNS construction they give a representation Π_ω of $\mathcal{A}$ in a Hilbert space $\mathcal{H}_\omega$. Popular states are the equilibrium states Φ_ω for a quasifree time evolution $\tilde{f}(k) \rightarrow e^{ie(k)t} \tilde{f}(k)$. They are defined by

$$(1.3) \quad \Phi_\beta(a_f^* a_g) = \int \frac{d^3k}{(2\pi)^3} \frac{\tilde{f}(k) \tilde{g}(k)}{1 + e^{\beta(\varepsilon(k) - \mu)}}$$

and vanishing reduced ($n \geq 3$)– point functions. They have the following features:

1. Π_{Φ_β} is a factor (i.e., if $\Pi_\omega(\mathcal{A})'$ is the commutant of $\Pi_\omega(\mathcal{A})$, $\Pi_\omega(\mathcal{A})''$ the commutant of $\Pi_\omega(\mathcal{A})'$, then the center $\Pi_\omega(\mathcal{A})' \cap \Pi_\omega(\mathcal{A})''$ is trivial).
2. For $\beta \neq \infty$ Φ_β is faithful (i.e., $\Phi_\beta(a^*a) = 0 \leftrightarrow a = 0, a \in \mathcal{A}$).
3. Φ_0 is tracial (i.e., $\Phi_0(ab) = \Phi_0(ba)$).
4. Φ_∞ (the Fock state) is pure (i.e. Π_{Φ_∞} is irreducible).

To describe the dynamics of particles interacting via a potential v one usually writes down a Hamiltonian

$$(1.4) \quad H = \frac{1}{2m} \int dx \nabla \alpha^*(x) \nabla \alpha(x) + \int dx dx' \alpha^*(x) \alpha^*(x') v(x - x') \alpha(x') \alpha(x) \\ =: T + V$$

and defines a time evolution by

$$(1.5) \quad \Theta^t(a_f) = e^{iHt} a_f e^{-iHt}.$$

Indeed in the Fock representation Π_{Φ_∞} , where one has sectors with finite particle number $N = \int dx \alpha^*(x) \alpha(x)$, there is no doubt that for reasonable v 's the restriction H_N will define a time evolution in these sectors. However, in general this will not be a one-parameter automorphism group of $\mathcal{A}$ but only of $\Pi_{\Phi_\infty}(\mathcal{A})''$. Having an automorphism of $\mathcal{A}$ means a dynamics independent of the state and this, in general, cannot be expected. Generically potentials are not stable in the sense that there exists $c \in \mathbb{R}^+$ such that

$$(1.6) \quad H_N > -cN \quad \forall N.$$

This condition ensures that the energy per particle stays finite for $N \rightarrow \infty$. If it does not hold the motion gets faster and faster as N increases [1] and a limiting dynamics exists only after rescaling the time. Thus one can hope for an automorphism of $\mathcal{A}$ only for stable potentials. Unfortunately, the situation is worse because of the following argument. $K(\alpha_f) = \alpha_f^*$ is an automorphism of $\mathcal{A}$, $K^2 = 1$. $K\Theta^t K$ would again be an automorphism of $\mathcal{A}$ generated by KHK . Since $\alpha(x)$ is not bounded we should express H by the a_f 's, otherwise the time derivative $i[H, a]$ will lead out of $\mathcal{A}$. But regularizing H somehow in the form:

$$(1.7) \quad T = \sum_f a_{\nabla f}^* a_{\nabla f}, \quad V = \sum_{f,g} a_f^* a_f v(f, g) a_g^* a_g$$

we see $KTK = -T + c_1$, $KVK = V + c_2 N + c_3$, $c_i \in \mathbb{R}$. Removing the regularization the constants c_i become infinite but they have no influence on the time development of gauge invariant quantities. For the existence of the N -independent dynamics we need either $H_N > -cN$ or $H_N < cN$. Since $0 < T \not\leq cN$ we need therefore

$$(1.8) \quad T + V > -cN \quad \text{and} \quad T - V > -cN, \quad \text{that is} \quad -cN - T < V < cN + T.$$

In general, there is no such operator inequality since $V \sim N^2$. For fermions there is a way out by observing that the particle density in configuration space $\alpha^*(x) \alpha(x)$ is

unbounded in spite of the exclusion principle because by going to high momenta we can pack arbitrarily many fermions in any volume. However, the fermion density in phase space $\rho(z)$ is bounded by 1 (in units $\hbar = 1$) if

$$z = (x, p) \in \mathbb{R}^6, \quad \rho(z) = a_z^* a_z, \quad a_z = \int \frac{d^3 y}{\pi^{3/4}} e^{i p y} e^{-(x-y)^2/2} \alpha(y).$$

We have $0 \leq \rho(z) \leq 1$ and if we use a velocity dependent potential which cuts off high momenta, then (1.8) should be satisfied. Indeed we have

Proposition (1.9)

The potential $V = \int d^6 z d^6 z' a_z^* a_z' v(z - z') a_z' a_z$ satisfies $-\|v\|_1 N \leq V \leq \|v\|_1 N$, $\|v\|_1 = \int d^6 z |v(z)|$.

Proof: We have the operator inequality

$$a_z^* a_z' a_z' a_z \leq a_z^* a_z \|a_z' a_z'\| = a_z^* a_z$$

and thus

$$V \leq \int dz dz' |v(z - z')| a_z^* a_z = \|v\|_1 N,$$

and similarly the left inequality.

Once the obstacle of instability is removed we get indeed an automorphism of the CAR algebra. This is stated by the next theorem, the proof of which is based on expansion in the coupling constant and is clearly too involved to be given here [2].

Theorem (1.10)

$H = T + V$, v from (1.9) with $\|v\|_1 < \infty$ defines by $\Theta^t a_f = e^{iHt} a_f e^{-iHt}$ a one parameter group of automorphisms of the CAR algebra which is norm continuous and Galilei invariant.

Remarks (1.11)

1. Norm continuous means $\lim_{t \rightarrow 0} \|\Theta^t a - a\| = 0 \ \forall a \in \mathcal{A}$. There is no uniformity, $\lim_{t \rightarrow 0} \sup_{a \in \mathcal{A}, \|a\|=1} \|\Theta^t a - a\| = 0$ would imply bounded H .
2. Galilei invariant means that together with the shift $\sigma: \sigma^x a_f = a_{fx}$, $f_x(y) = f(x + y)$, the boost $\gamma: \gamma^p a_f = a_{e^{ipx}f}$, and the gauge transformation (first kind) $\nu: \nu^\alpha a_f = e^{i\alpha} a_f$, the time evolution Θ satisfies

$$\begin{aligned} \sigma^x \circ \nu^\alpha &= \nu^\alpha \circ \sigma^x, \quad \gamma^p \circ \nu^\alpha = \nu^\alpha \circ \gamma^p, \quad \gamma^p \circ \sigma^x = \sigma^x \circ \gamma^p \circ \nu^{-px} \\ \Theta^t \circ \nu^\alpha &= \nu^\alpha \circ \Theta^t, \quad \Theta^t \circ \sigma^x = \sigma^x \circ \Theta^t, \quad \Theta^t \circ \gamma^p = \Theta^t \circ \sigma^{-pt} \circ \nu^{-p^2 t/2}. \end{aligned}$$

This expresses the fact that $\Theta, \sigma, \gamma, \nu$ provide a realization of the central extension of the Galilei group by norm continuous automorphisms of the CAR algebra.

3. A local potential means v is p -independent in which case $\|v\|_1 = \infty$ and (1.10) does not apply. However, there is no limit on the momentum cut-off and one can hope that for stable systems where high momenta do not occur in a reasonable subset of states, then (1.10) gives a physically acceptable description of the time evolution.

When we develop the general theory in the next section we shall always have this model in mind since it exists mathematically and is relevant for physics.

2 Topological Dynamics

The knowledge an observer has about a system is contained in the state. Thus the properties of $(\mathcal{A}, \Theta)$ are objective in the sense that they do not depend on any observer. On the other hand, $(\mathcal{A}, \Theta, \omega)$ reflects how the situation appears to a particular observer. Similarly, elements from $\Pi_\omega(\mathcal{A})'' \setminus \Pi_\omega(\mathcal{A})$ are extrapolations from some observer and are different for different ω [3]. Thus we shall first study properties of $(\mathcal{A}, \Theta)$ alone and then, in the next section, the additional information contained in invariant states. It turns out that there is a close interplay between these points of view. From some states we can draw conclusions about $(\mathcal{A}, \Theta)$ and sufficient algebraic structure of $(\mathcal{A}, \Theta)$ implies some features of all invariant states.

Proposition (2.1)

Among the following properties of a dynamical system $(\mathcal{A}, \Theta)$

- (i) all Θ -invariant elements of $\mathcal{A}$ are $\sim \mathbb{1}$,
- (ii) all Θ -quasiperiodic elements of $\mathcal{A}$ are $\sim \mathbb{1}$,
- (iii) $(\mathcal{A}, \Theta)$ is a K -system,
- (iv) $\mathcal{A}$ has only trivial Θ -invariant closed subalgebras,

there are the implications

$$\begin{array}{ccc} \text{(iv)} & \implies & \text{(i)} \\ & & \uparrow \\ \text{(iii)} & \implies & \text{(ii)} \end{array}$$

Explanations (2.2)

ad (i) This is usually called ergodicity and is too weak a property since it does not imply mixing. Nevertheless it excludes systems where Θ is an inner automorphism (i.e., $\Theta(a) = U^{-1}aU$, $U \in \mathcal{A}$) and in particular finite quantum systems (where $U = e^{iH}$). It also excludes finite Θ -invariant subalgebras $\mathcal{A}_0$ since Θ restricted to $\mathcal{A}_0$ would be an automorphism of $\mathcal{A}_0$ and they always have invariant elements.

ad (ii) Quasiperiodicity for an element $a \in \mathcal{A}$ means $\forall \varepsilon > 0, N \in \mathbb{Z}^+ \exists |n| > N$ with $\|\Theta^n a - a\| < \varepsilon$. This property is of interest in connection with Poincaré's recurrence theorem which says that classically almost all orbits keep coming arbitrarily close to their origin. Nevertheless classical systems may have property (ii) since the observables are smooth functions on phase space and they may never regain their original form.

ad (iii) K -system [4] means that $\mathcal{A}$ has a C^* -subalgebra $\mathcal{A}_0$ with the properties

- (a) $\Theta^n \mathcal{A}_0 \supset \mathcal{A}_0 \forall n \in \mathbb{Z}^+$
- (b) $\bigvee_{n=0}^{\infty} \Theta^n \mathcal{A}_0 = \mathcal{A}$ ($\bigvee$ means norm completion of the algebra generated by the $\Theta^n \mathcal{A}_0$)
- (c) $\bigwedge_{n=0}^{\infty} \Theta^{-n} \mathcal{A}_0 = c \cdot \mathbb{1}$ ($\bigwedge$ is the intersection).

Rephrased it means that Θ^{-1} gives an isomorphism between $\mathcal{A}_1 := \Theta\mathcal{A}_0$ and its proper subalgebra $\mathcal{A}_0$ but this isomorphism has no proper invariant subalgebra (this would remain in the "tail" $\bigwedge_{n=0}^{\infty} \Theta^{-n}\mathcal{A}_0$). (b) might be considered as a definition of $\mathcal{A}$ such that the isomorphism $\Theta : \mathcal{A}_n \rightarrow \mathcal{A}_{n+1}$ extends to an automorphism $\mathcal{A} \rightarrow \mathcal{A}$.

ad (iv) This is equivalent to requiring that $\forall z \cdot \mathbb{1} \neq a \in \mathcal{A}$ the C^* -algebra generated by $\{\Theta^n a, n \in \mathbb{Z}\}$ is all of $\mathcal{A}$. For our purposes it is too strong a condition since the particle number conserving time evolutions (1.10) of the CAR algebra leave the subalgebras

$$\mathcal{A}_{n_0} = \{c + \sum_{n > n_0} a_{f_1}^* a^*(f_2) \dots a_{f_n}^* a_{g_1} \dots a_{g_n}\}$$

$\forall n_0 \in \mathbb{Z}^+$ invariant. (iv) is not related to (iii) since particle-conserving time evolutions may nevertheless generate K -systems. On the other hand, the tensor product of two K -systems $(\mathcal{A}, \Theta_1)$ and $(\mathcal{A}, \Theta_2)$ can be endowed naturally with a K -structure; but $(\mathcal{A}_1 \otimes \mathcal{A}_2, \Theta_1 \otimes \Theta_2)$ has $(\mathcal{A}_1 \otimes \mathbb{1})$ and $(\mathbb{1} \otimes \mathcal{A}_2)$ as invariant subalgebras, thus (iii) $\not\Rightarrow$ (iv). Furthermore (iv) is too weak to imply (ii) since there are trivial examples of periodic systems with only trivial subalgebras.

Proof of (2.1)

(ii) $\Rightarrow$ (i). Invariant elements are quasiperiodic.

(iv) $\Rightarrow$ (i). The invariant elements form a closed invariant subalgebra.

(iii) $\Rightarrow$ (ii). Let b be a quasiperiodic element. Because of property (b) $\forall \varepsilon > 0$ we can find an $n(\varepsilon) \in \mathbb{Z}^+$ with $\inf_{a \in \mathcal{A}_{n(\varepsilon)}} \|b - a\| < \varepsilon$, $\mathcal{A}_n = \Theta^n \mathcal{A}_0$. As the norm is invariant under Θ we have $\forall N \in \mathbb{Z}^+ \inf_{a \in \mathcal{A}_{n(\varepsilon)-N}} \|\Theta^{-N} b - a\| < \varepsilon$, and because b is quasiperiodic $\inf_{a \in \mathcal{A}_{n(\varepsilon)-m}} \|b - a\| \leq 2\varepsilon$ for some $m > N$ and thus $\forall N \in \mathbb{Z}^+ \inf_{a \in \mathcal{A}_{n(\varepsilon)-N}} \|b - a\| \leq 2\varepsilon$. This means that $\forall \varepsilon > 0, n \in \mathbb{Z}, \inf_{a \in \mathcal{A}_n} \|b - a\| < \varepsilon$ and since the $\mathcal{A}_n$ are norm-closed, $b \in \mathcal{A}_n \forall n \in \mathbb{Z}$. Thus the property (c) requires $b = c \cdot \mathbb{1}, c \in C$.

In classical topological dynamics there is the important notion of topologically mixing [5] which we generalize to the noncommutative case as follows [6, 7, 8, 9]:

Definition (2.3)

A dynamical system $(\mathcal{A}, \Theta)$ is called

- (i) mixing if $\forall a, b \in \mathcal{A} \exists N, \varepsilon > 0$ such that $\|a\Theta^n b\| > \varepsilon \|a\| \|b\| \forall n > N$,
- (ii) strongly mixing if $\lim_{n \rightarrow \infty} \|a\Theta^n b\| = \|a\| \|b\|$.

Remarks (2.4)

1. The intuitive meaning of mixing is visualized best in the classical case where a and b are continuous functions on phase space. Even if they have disjoint supports such that $ab = 0$ if one is time translated their supports will eventually overlap. (i) says that b is dispersed so finely that eventually its support keeps overlapping with the one of a forever. (ii) says that even the part close to the maximum of b will eventually meet the maximum of a and keep overlapping it.

2. Classically (i) and (ii) are equivalent since (i) must hold for all functions supported near the maxima of a and b . Obviously (ii) $\Rightarrow$ (i). In the non-abelian case (i) is actually weaker than (ii). This is shown by the shift on the spin chain $\vec{\sigma}_i \rightarrow \vec{\sigma}_{i+1}$, $i \in \mathbb{Z}$, $[\vec{\sigma}_i, \vec{\sigma}_j] = 0$ for $i \neq j$, and the fermion chain $a_i \rightarrow a_{i+1}$, $a_i^* a_j + a_j a_i^* = \delta_{ij}$. The former dynamical system is strongly mixing, but the latter is not. To see this, consider the projections $p_i = \frac{1}{2}(1 + a_i + a_i^*)$. We calculate $p_i p_j p_i = \frac{1}{2} p_i \forall i \neq j$. Thus $\|p_0(\Theta^n p_0)p_0\| = \|p_0 p_n\| = \frac{1}{2} < 1 \forall n \in \mathbb{N}$. Nevertheless both systems are K -systems and the fermion chain is mixing as well (see below).
3. Since $\|a\Theta^n b\| = \|b^* \Theta^{-n} a^*\|$ there is no distinction between $n \rightarrow \infty$ and $n \rightarrow -\infty$.

Proposition (2.5)

(2.3, i) $\Rightarrow$ (2.1, ii).

Proof: Suppose b is a quasiperiodic element $\neq c \cdot \mathbb{1}$. Then either $B = b + b^*$ or $B = i(b - b^*)$ has at least two different spectral values and is hermitian and also quasiperiodic. Suppose $f_{\pm}$ are two smooth functions with disjoint support having their maxima at the two spectral values respectively. Then $f_{\pm}(B)$ are $\neq 0$ and $f_+(B)f_-(B) = 0$. Suppose there were some ε with $\|f_+(B)\Theta^n f_-(B)\| > \varepsilon \|f_+\| \|f_-\| \forall n > N$. Since $f_- f_-^*$ is quasiperiodic there is some $m > N$ with $\|\Theta^m f_- f_-^* - f_- f_-^*\| \leq \varepsilon^2 \|f_-\|^2 / 4$ and we have

$$\|f_+ \Theta^m f_-\| = \|f_+(\Theta^m(f_- f_-^*) - f_- f_-^*)f_+^*\|^{1/2} \leq \frac{\varepsilon}{2} \|f_+\| \|f_-\|$$

contradicting our assumption.

3 Representations by Invariant States

Each invariant state ω gives a GNS representation Π_{ω} of $(\mathcal{A}, \Theta)$ in a Hilbert space $\mathcal{H}_{\omega} = \overline{\Pi_{\omega}(\mathcal{A})|\Omega\rangle}$ where Θ is unitarily implemented by U , $U\Pi_{\omega}(a)|\Omega\rangle = \Pi_{\omega}(\Theta a)|\Omega\rangle$ [10,11]. The ergodic properties of $(\mathcal{A}, \Theta)$ have their counterpart in properties of ω . The following proposition shows that in a pure quantum situation one automatically gets mixing.

Proposition (3.1)

Let ω be a Θ -invariant state of $(\mathcal{A}, \Theta)$ (i.e., $\omega \circ \Theta = \omega$) such that ω is faithful on $\Pi_{\omega}(\mathcal{A})''$ (i.e. $\langle \Omega | a^* a | \Omega \rangle = 0 \Leftrightarrow a = 0 \forall a \in \Pi_{\omega}(\mathcal{A})''$) and $\mathcal{Z}_{\omega} = \Pi_{\omega}(\mathcal{A}) \cap \Pi_{\omega}(\mathcal{A})' = c \cdot \mathbb{1}$.

Then the following properties are equivalent:

- (i) ω is mixing,
- (ii) $w\text{-}\lim_{n \rightarrow \infty} U^n = |\Omega\rangle\langle\Omega|$.
- (iii) $w\text{-}\lim_{n \rightarrow \infty} \Pi_{\omega}(\Theta^n a) = \omega(a) \cdot \mathbb{1} \forall a \in \Pi_{\omega}(\mathcal{A})$,
- (iv) $(\mathcal{A}, \Theta, \omega)$ is weakly asymptotically abelian.

Remarks (3.2)

1. Faithful states are dense and so this requirement does not leave us with exceptions only. There are examples of states faithful on $\Pi_\omega(\mathcal{A})$ and not on $\Pi_\omega(\mathcal{A})''$ but they do not seem to be important in physics. The advantage of faithfulness is that they supply the modular automorphism σ_t^ω which satisfies the KMS condition [10,11] $\langle \Omega | ab | \Omega \rangle = \langle \Omega | b \sigma_i^\omega(a) | \Omega \rangle$. If ω is Θ -invariant σ^ω and Θ commute where σ^ω is the modular automorphism to ω .
2. The center $\mathcal{Z}_\omega$ is the classical part of the system and may be nontrivial even if $\mathcal{A}$ is simple and therefore its center is trivial. Our requirement means that Π_ω is a factor representation; there the classical observables have fixed values.
3. For systems where one can show asymptotic abelianness in the norm $(\lim_{n \rightarrow \infty} \|[a, \Theta^n(b)]\| = 0 \forall a, b \in \mathcal{A})$, (3.1) implies that all factor states are mixing. For the observable algebra $\mathcal{A}$ it holds for space translations but for time translations it is an unproven conjecture.

ad (i) Mixing means $\lim_{n \rightarrow \infty} \omega(a\Theta^n(b)c) = \omega(ac)\omega(b)$. Since ω is Θ -invariant the limits $n \rightarrow \infty$ and $n \rightarrow -\infty$ are equivalent:

$$\omega(a\Theta^n(b)c) = \omega(b\Theta^{-n}(c\sigma_i^\omega(a))) = \omega(d^{-1}b\Theta^{-n}(c\sigma_i^\omega(a))\sigma_i^\omega d).$$

ad (ii) w-lim means weak limit in the Hilbert space $\overline{\Pi_\omega(\mathcal{A})|\Omega}$. Strong limit is impossible since strong limits of unitaries must be isometries.

ad (iii) Again strong limits are impossible:

$$\begin{aligned} 0 &= \text{s-lim } \Pi_\omega(\Theta^n(a^* - \omega(a^*))) \cdot \text{s-lim } \Pi_\omega(\Theta^n(a - \omega(a))) \\ &= \text{s-lim } \Pi_\omega(\Theta^n(a^* - \omega(a^*))(a - \omega(a))) = \omega((a^* - \omega(a))(a - \omega(a))). \end{aligned}$$

Faithfulness requires $a = \omega(a)$ and thus strong convergence holds only for multiples of $1\mathbb{1}$ (which are invariant under Θ).

ad (iv) Weak asymptotic abelianness means

$$\lim_{n \rightarrow \infty} \omega(a[b, \Theta^n c]d) = 0 \quad \forall a, b, c, d \in \mathcal{A}.$$

It is not possible for inner automorphisms $\Theta(a) = U^{-1}aU$ since $\Theta(U) = U$ and thus the condition would make $[b, U] = 0 \forall b \Leftrightarrow \Theta = id$.

Proof of (3.1)

(i) $\Leftrightarrow$ (ii) Both require

$$\omega(a\Theta^n(b)c) = \langle \Omega | \sigma_{-i}^\omega(c) a U^n b | \Omega \rangle \rightarrow \langle \Omega | ac | \Omega \rangle \langle \Omega | b | \Omega \rangle.$$

(i) $\Leftrightarrow$ (iii) Both require

$$\langle \Omega | a\Theta^n(b)c | \Omega \rangle \rightarrow \langle \Omega | ac | \Omega \rangle \cdot \langle \Omega | b | \Omega \rangle.$$

(iv) $\Rightarrow$ (iii) Since in the von Neumann algebra $\Pi_\omega(\mathcal{A})''$ bounded sets are weakly compact, so $\Pi_\omega(\Theta^n a)$ will have weak cluster points $\Pi_\omega(\Theta^{n_i} a) \rightarrow a'' \in \Pi_\omega(\mathcal{A})''$. Asymptotic abelianness requires $a'' \in \Pi_\omega(\mathcal{A})'$ and thus $a'' \in \mathcal{Z}_\omega = c \cdot \mathbb{1}$. Weak continuity of ω implies $\omega(a) = \langle \Omega | \Pi_\omega(\Theta^{n_i} a) | \Omega \rangle \rightarrow \langle \Omega | a'' | \Omega \rangle$ and thus $a'' = \omega(a) \cdot \mathbb{1}$.

(iii) $\Rightarrow$ (iv)

$$\lim_{n \rightarrow \infty} \omega(ab\Theta^n(c)d) = \lim_{n \rightarrow \infty} \omega(a\Theta^n(c)bd) = \omega(abd)\omega(c).$$

Remarks (3.3)

1. Note that (i), (iii) and (iv) hold even for all elements from $\Pi_\omega(\mathcal{A})''$. For this to happen faithfulness is crucial. In the Fock state $\Pi_\omega(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$ and the unitary generators of rotations are time invariant but do not commute. Thus there $\Pi_\omega(\mathcal{A})$ is asymptotic abelian but $\Pi_\omega(\mathcal{A})''$ is not.
2. The properties (3.1) imply that ω is extremal invariant (i.e., $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, $\lambda \in (0, 1)$, $\omega_{1,2}$ invariant, implies $\omega_1 = \omega_2 = \omega$). The reason is that in such a decomposition ω_i can be written $\omega_i(a) = \langle \Omega | \Pi_\omega(\mathcal{A})a' | \Omega \rangle$, a' an invariant positive element from $\Pi_\omega(\mathcal{A})'$. For faithful states $\Pi_\omega(\mathcal{A})'$ and $\Pi_\omega(\mathcal{A})''$ are anti-isomorphic, and invariance of ω implies this anti-isomorphism for their invariant elements. Since the only invariant element from $\Pi_\omega(\mathcal{A})''$ is $c \cdot \mathbb{1}$ the same holds for $\Pi_\omega(\mathcal{A})'$ and thus $a' = \mathbb{1}$. Extremal invariance implies in turn that (i), (iii), and (iv) hold in the mean (ergodicity).

Proposition (3.4)

If (3.1) holds $(\mathcal{A}, \Theta)$ is mixing.

Proof:

$$\begin{aligned} \|a\Theta^n(b)\| &= \|a\Theta^n(bb^*)a^*\|^{1/2} \geq (\omega(a\Theta^n(bb^*)a^*))^{1/2} \\ &\rightarrow \omega(aa^*)^{1/2}\omega(bb^*)^{1/2} > \varepsilon > 0 \end{aligned}$$

since ω is faithful.

In some cases pure states are more convenient and even if they are not faithful $\Pi_\omega(\mathcal{A})$ will be if $\mathcal{A}$ is simple. In this case a strengthening of (3.1, i) gives an even stronger result.

Proposition (3.5)

If $\mathcal{A}$ is simple and Π_ω is hyperclustering, $(\mathcal{A}, \Theta)$ is strongly mixing for ω .

Explanations (3.6)

1. Simplicity implies that the center of $\Pi_\omega(\mathcal{A})$ is trivial because a nontrivial two-sided ideal can be constructed with nontrivial elements of the center. This does not yet mean that $\Pi_\omega(\mathcal{A})''$ has trivial center but that all representations $\Pi(\mathcal{A})$ have to be faithful.
2. Hyperclustering means

$$\omega(a\Theta^n(b)d\Theta^n(c)) \rightarrow \omega(ad)\omega(bc).$$

It implies $\omega(c[a, \Theta^n b][a^*, \Theta^n b^*]d) \rightarrow 0$ and thus strengthens (3.1, iii) to

$$\text{s-}\lim_{n \rightarrow \infty} [\Pi_\omega(a), \Pi_\omega(\Theta^n b)] = 0$$

(strong asymptotic abelianness). Together with mixing this implies conversely hyperclustering.

Proof of (3.5): For simplicity we write $\Theta^n a = a_n$, etc. Consider the operator inequality

$$cd_n ab_n b_n^* a^* d_n^* c^* \leq \|ab_n\|^2 cd_n d_n^* c^*$$

which gives us

$$\frac{\omega(cd_n ab_n b_n^* a^* d_n^* c^*)}{\omega(cd_n d_n^* c^*)} \leq \|ab_n\|^2.$$

Let n tend to infinity and remember that commutators go strongly to zero. Thus the left hand side tends to

$$\frac{\omega(caa^* c^*)}{\omega(cc^*)} \frac{\omega(dbb^* d^*)}{\omega(dd^*)}.$$

Now take the sup over d and c and take into account that faithfulness of Π_ω means

$$\sup_{c \in \mathcal{A}} \frac{\omega(caa^* c^*)}{\omega(cc^*)} = \|a\|^2.$$

Thus $\lim_{n \rightarrow \infty} \|ab_n\| \geq \|a\|\|b\|$, but generally $\|ab_n\| \leq \|a\|\|b\|$, and thus

$$\lim_{n \rightarrow \infty} \|a\Theta^n b\| = \|a\|\|b\|.$$

Corollary (3.7)

Let $\mathcal{A}$ be an UHF algebra and $(\mathcal{A}, \Theta)$ norm asymptotic abelian, then $(\mathcal{A}, \Theta)$ is strongly mixing.

Remark (3.8)

Norm asymptotic abelianness is satisfied, in particular, when Θ is a quasi-free automorphism of the CAR algebra, $\Theta(a_f) = a_{Uf}$, where the unitary operator U has (apart from the eigenvector $|\Omega\rangle$) an absolutely continuous spectrum and $\mathcal{A}$ is the even part of this algebra. In this case $\|[a_f, \Theta^n a_g^*]_+\| = \langle f|U^n g \rangle$ which goes to zero.

Proof of (3.7): The unique tracial state τ is invariant under all automorphisms α ($\tau \circ \alpha$ would be another tracial state and thus equals τ). It is known to be faithful over $\Pi_\tau(\mathcal{A})''$ which is a type II_1 factor and has trivial center. Thus by (3.1) τ is mixing and even hyperclustering since it is norm asymptotic abelian. Thus (3.5) applies.

Corollary (3.9)

Let Θ of the CAR algebra $\mathcal{A}$ be Galilei invariant. Then $(\mathcal{A}, \Theta)$ is mixing and $(\mathcal{A}_0, \Theta)$ is strongly mixing.

Proof: The tracial state τ qualifies for (3.1). From the representation theory of the Galilei group one knows that, apart from the invariant vector, U_t which generates the time translations has an absolutely continuous spectrum [12]. One knows that $|\Omega\rangle$ is the

only translation invariant vector in $\mathcal{H}_\tau$ and therefore the only Galilei invariant vector. Thus $U_n \rightarrow |\Omega\rangle\langle\Omega|$ and the conditions of (3.1) are satisfied.

For the strong mixing of $\mathcal{A}_0$ we have to use the strong graded asymptotic abelianness discussed in the Appendix. If we take a and b to be even in the proof of (3.5) and decompose c and d into even and odd parts (c_g, d_g and c_u, d_u resp.), we find that

$$\begin{aligned} \omega(cd_t ab_t b_t^* a^* d_t^* c^*) &= \\ &= \omega((c_g + c_u)(d_{gt} + d_{ut}) ab_t b_t^* a^* (d_{gt}^* + d_{ut}^*)(c_g^* + c_u^*)) \\ &\rightarrow \omega((c_g + c_u) a a^* c_g^*) \omega((d_{gt} + d_{ut}) b_t b_t^* (d_{gt}^* + d_{ut}^*)) \\ &\quad + \omega((c_g + c_u) a a^* c_u^*) \omega((d_{gt} - d_{ut}) b_t b_t^* (d_{gt}^* - d_{ut}^*)) \\ &= \omega(c_g a a^* c_g^* + c_u a a^* c_u^*) \omega(d_g b b^* d_g^* + d_u b b^* d_u^*) \\ &= \omega((c_g + c_u) a a^* (c_g^* + c_u^*)) \omega((d_g + d_u) b b^* (d_g^* + d_u^*)). \end{aligned}$$

Thus the proof of strong mixing goes through in the same way as in (3.5).

Remarks (3.10)

1. Even without appeal to the representation theory of the Galilei group one can show that τ is mixing using only its multiplication law [13]. One uses the fact that in Galilei frames which are moving faster and faster a time translation looks more and more like a space translation, and the latter is mixing.
2. For mixtures of even and odd elements of $\mathcal{A}$ we cannot expect strong mixing because of the fermionic chain example (2.4.2).

Proposition (3.11)

If $(\mathcal{A}, \Theta)$ is a K -system, then Φ_0 is mixing.

Proof: Denote by $P_n \in \Pi_{\Phi_0}(\mathcal{A}_n)'$ the orthogonal projector projecting onto $\overline{\mathcal{A}_n|\Omega}$. Since weak convergence of projections to projections implies strong convergence we can deduce the following relations:

$$(a) \quad U^{-n'} P_n U^{n'} = P_{n+n'} \supset P_n \quad \forall n' \in \mathbb{N}, n \in \mathbb{Z},$$

$$(b) \quad s\text{-}\lim_{n \rightarrow \infty} P_n = \mathbb{1},$$

$$(c) \quad s\text{-}\lim_{n \rightarrow \infty} P_n = |\Omega\rangle\langle\Omega|.$$

(a) is obvious, and since we are dealing with II_1 factors there are conditional expectations $E_n : \mathcal{A} \rightarrow \mathcal{A}_n$ which are related to P_n by $E_n(a)P_n = P_n a P_n \quad \forall a \in \mathcal{A}$. Furthermore $P_n|\Omega\rangle = |\Omega\rangle \quad \forall n$ such that $P_n a |\Omega\rangle = E_n(a)|\Omega\rangle$. The sequence $E_n(a)$ will, for $n \rightarrow -\infty$, have weak cluster points $a_{-\infty}$ such that $a_{-\infty} \in \bigvee_n \mathcal{A}_n = z\mathbb{1}$ and thus $a_{-\infty} = \Phi_0(a)$ since $\Phi_0(E_n(a)) = \Phi_0(a)$. For $n \rightarrow \infty$ P_n converges to $\mathbb{1}$ on the dense set of states of the form $a_m|\Omega\rangle$, $a_m \in \mathcal{A}_m$ for some m . These two observations then demonstrate (b) and (c). As for mixing we need only consider $\Phi_0(a\Theta^n b)$ since Φ_0 is faithful. Thus, if $b \in \mathcal{A}_0$ and using $U^n|\Omega\rangle = |\Omega\rangle$, we have

$$\begin{aligned} |\Phi_0(a\Theta^n b) - \Phi_0(a)\Phi_0(b)| &= |\langle\Omega|aU^{-n}(P_0 - |\Omega\rangle\langle\Omega|)b|\Omega\rangle| \\ &= |\langle\Omega|a(P_n - |\Omega\rangle\langle\Omega|)U_{-n}b|\Omega\rangle| \\ &\leq \|\langle\Omega|a(P_n - |\Omega\rangle\langle\Omega|)\| \cdot \|b|\Omega\rangle\| \end{aligned}$$

and consequently c) tells us that $\forall \varepsilon \exists N$ such that

$$|\Phi_0(a\Theta^n b) - \Phi_0(a)\Phi_0(b)| < \varepsilon \Phi_0(b^*b)^{1/2} \quad \forall n < -N, b \in \mathcal{A}_0.$$

Clearly the same holds for b from any $\mathcal{A}_n$ and since the $\mathcal{A}_n$ are norm dense in $\mathcal{A}$ we have $\forall a, b \in \mathcal{A}, \varepsilon \exists N$ such that

$$|\Phi_0(a\Theta^n b) - \Phi_0(a)\Phi_0(b)| < \varepsilon \quad \forall n < -N.$$

Corollary (3.12)

If $\mathcal{A}$ is UHF and $(\mathcal{A}, \Theta)$ a K -system, then for the tracial state all conditions (3.1) are satisfied and $(\mathcal{A}, \Theta)$ is mixing.

Corollary (3.13)

If $\mathcal{A}$ is UHF, then (2.1) can be sharpened to: $(\mathcal{A}, \Theta)$ is a K -system $\Rightarrow (\mathcal{A}, \Theta)$ is mixing $\Rightarrow$ all quasiperiodic elements of $\mathcal{A}$ are $\sim \mathbb{1}$, $\Rightarrow$ all invariant elements of $\mathcal{A}$ are $\sim \mathbb{1}$.

Remark (3.14)

K -systems are not only mixing but even K -mixing which is the maximally possible uniformity in mixing. Whereas complete uniformity in clustering in the sense that $\forall a \in \mathcal{A}, \|a\| = 1, \varepsilon > 0 \exists N$ such that $|\omega(a\Theta^n b) - \omega(a)\omega(b)| < \varepsilon \|b\| \quad \forall b \in \mathcal{A}, n > N$ is impossible for faithful ω (take $b = \Theta^{-n}a^*$), we found in the proof of (3.11) that $\forall a \in \mathcal{A}, m \in \mathbb{Z}, \varepsilon > 0 \exists N$ such that

$$|\omega(a\Theta^n b) - \omega(a)\omega(b)| < \varepsilon \|b\| \quad \forall b \in \mathcal{A}_m, n < -N.$$

Since the $\mathcal{A}_m$ are dense that is the best one can hope for. It is related to the strong convergence of the P_n and is equivalent to saying that all states when restricted to one of the “strictly local” algebras $\mathcal{A}_m$ converge for $n \rightarrow \infty$ strongly to the equilibrium ω . Such states φ can be written $\varphi(b) = \omega(ab), a \in \mathcal{A}, \omega(a) = \mathbb{1}$, and we have

Proposition (3.15)

In the Schrödinger representation $\forall \varepsilon > 0 \exists N \in \mathbb{Z}^+$ such that

$$\|\Phi_0 \circ \Theta^n|_{\mathcal{A}_m} - \Phi_0|_{\mathcal{A}_m}\| = \sup_{b \in \mathcal{A}_m, \|b\|=1} |\Phi_0(a\Theta^n b) - \Phi_0(b)| < \varepsilon \quad \forall n < -N.$$

Remark (3.16)

In a forthcoming paper we will generalize the results for Φ_0 to arbitrary faithful invariant factor states.

4 The Increase of Entropy with Time

One of the key formulas of quantum statistical mechanics is von Neumann’s expression for the entropy of a state over $\mathcal{B}(\mathcal{H})$ given by a density matrix ρ

$$(4.1) \quad S = - \operatorname{Tr} \rho \ln \rho$$

This relation has been generalized for states over arbitrary von Neumann algebras $\mathcal{A}$ [17] but one always meets with the same difficulties when one discusses the question of entropy increase.

1. All these expressions have to be invariant under automorphisms and therefore cannot change with time. Furthermore, for infinite system S is infinite.
2. The entropy of the state restricted to a subalgebra $\mathcal{A}_0$, $\Theta\mathcal{A}_0 \neq \mathcal{A}_0$ may change with time but it may increase or decrease. Furthermore, if the state is time invariant – $\omega \circ \Theta = \omega$ – even for a subalgebra the entropy does not change:

$$(4.2) \quad S(\omega|_{\Theta\mathcal{A}_0}) = S(\omega \circ \Theta|_{\mathcal{A}_0}) = S(\omega|_{\mathcal{A}_0}).$$

The following two ways have been proposed to get around these difficulties:

1. Dynamical Entropy

For some infinite algebras $\mathcal{A}_0$ one can give a well defined meaning for the entropy increase per unit time

$$S(\omega|_{\Theta\mathcal{A}_0}) - S(\omega|_{\mathcal{A}_0}) = \infty - \infty.$$

If $\mathcal{A}_0 = \bigvee_{n=-\infty}^0 \Theta^n \mathcal{B}_0$, $\mathcal{B}_0 = \text{finite}$, then classically

$$(4.3) \quad h_\omega(\mathcal{B}_0, \Theta) \equiv S(\omega|_{\Theta\mathcal{A}_0}) - S(\omega|_{\mathcal{A}_0}) = \lim_{m \rightarrow \infty} (S(\omega|_{\bigvee_{n=-m}^1 \Theta^n \mathcal{B}_0}) - S(\omega|_{\bigvee_{n=-m}^0 \Theta^n \mathcal{B}_0}))$$

is well defined. The dynamical (or Kolmogorov-Sinai [18]) entropy is the maximal increase (per unit time) of the entropy of these algebras

$$(4.4) \quad h_\omega(\Theta) = \sup_{\mathcal{B}_0 = \text{finite}} h_\omega(\mathcal{B}_0, \Theta).$$

In quantum theory the union $\bigvee_{n=-m}^0 \Theta^n \mathcal{B}_0$ may be too big and S is not monotonic so the definition (4.3) is no good. Here one needs a more refined theory which has been elaborated in the past years [19]. One now disposes of a dynamical entropy of an automorphism of an arbitrary C^* -algebra [20] for which almost all the desired properties have been demonstrated. The theory is too extensive to be given here [21].

2. Relative Entropy

We all have learned in thermodynamics that for part of a system which exchanges energy with the rest, it is not the entropy which tends to a maximum but the free energy which tends to a minimum. Again the free energy of an infinite system may be infinite but the difference between the free energies of a locally disturbed equilibrium ν and the equilibrium state ω may be finite.

The relative entropy $S(\nu|\omega)$ is precisely this difference. For finite quantum systems where states can be described by density matrices it is

$$(4.5) \quad S(\nu|\omega) = -\text{Tr } \nu(\ln \nu - \ln \omega)$$

and if ω is a canonical state (for $\beta = 1$) $\omega = e^{-H+F_\omega}$, $F_\omega = \text{equilibrium free energy}$, we find

$$S(\nu|\mu) = -\text{Tr } \nu(\ln \nu + H - F_\omega) = \langle H \rangle_\nu - S(\nu) - F_\omega \equiv F_\nu - F_\omega.$$

There is a general definition of $S(\nu|\omega)$ for arbitrary states of C^* -algebras [19, 22]. For infinite systems $S(\nu|\mu)$ is finite if $\nu < c\omega$ and ν is a local perturbation of ω . Again the definition is invariant under automorphisms so that $S(\nu|\omega)$ does not change with time. For subsystems it may change but not necessarily monotonically. We shall see that this monotonicity is a special feature of K -systems.

Theorem (4.6)

Let $(\mathcal{A}, \mathcal{A}_0, \Theta)$ be a K -system, ω an invariant state satisfying (3.11) and φ a normal state for $\Pi_\omega(\mathcal{A})''$ with $\lambda_1\omega \leq \varphi \leq \lambda_2\omega$, $\lambda_i \in \mathbb{R}^+$. Then for any $m \in \mathbb{Z}$, $\mathcal{A}_m = \Theta^m \mathcal{A}_0$,

$$\Delta F(n) = S(\varphi \circ \Theta^n_{|\mathcal{A}_m} | \omega_{|\mathcal{A}_m})$$

converges monotonically for $-\infty < n < \infty$ from 0 to $S(\varphi|\omega)$.

Remarks (4.7)

1. Even in finite dimensions $S(\varphi|\omega)$ is not continuous and becomes infinite where ω is zero but φ greater than zero. Thus we need a condition on the states.
2. We have seen that for mixing systems any normal state converges weakly towards equilibrium. However, $\varphi \rightarrow \omega$ does not imply $S(\varphi|\omega) \rightarrow 0$ since $S(\varphi|\omega) > \frac{1}{2}\|\varphi - \omega\|^2$. Thus it is only the strong convergence (3.12) of the restriction of φ which ensures the convergence of the free energy of the subsystems $\mathcal{A}_n$.
3. In general, even if the free energy converges towards its equilibrium value it will not be monotonic because of thermal fluctuations. It is a special feature of K -systems that there are subalgebras where this does not happen.
4. The thoughtful reader will be perplexed by the fact that ΔF converges away from its equilibrium value 0 to its maximal value $S(\varphi|\omega)$. However, it is not yet said whether Θ or Θ^{-1} is the real time development. The physical K -systems are actually time reversal invariant in the sense that there is an anti-automorphism K , $K^2 = 1$, with $K\Theta K = \Theta^{-1}$. Then for $K(\mathcal{A}_0)$ the free energy decreases if it increases for $\mathcal{A}_0$. For time reversal invariant systems there are necessarily as many subsystems with increasing free energy as there are with decreasing free energy.

Proof of (4.6)

(i) Monotonicity

$$\varphi \circ \Theta^n_{|\mathcal{A}_m} = \varphi_{|\mathcal{A}_{m+n}}$$

and for $n < n'$ $\mathcal{A}_{m+n} \subset \mathcal{A}_{m+n'}$. The relative entropy has the monotonicity [19]

$$S(\varphi_{|\mathcal{A}} | \omega_{|\mathcal{A}}) \leq S(\varphi_{|\mathcal{B}} | \omega_{|\mathcal{B}}) \quad \text{if } \mathcal{A} \subset \mathcal{B}.$$

(ii) The limit $n \rightarrow -\infty$

We have seen in (3.12) that

$$\lim_{n \rightarrow -\infty} \|\varphi_{|\mathcal{A}_n} - \omega_{|\mathcal{A}_n}\| \rightarrow 0$$

and for states satisfying the hypothesis [16] of (4.6) this implies

$$S(\varphi_{|\mathcal{A}_n} | \omega_{|\mathcal{A}_n}) \rightarrow 0 \quad \text{for } n \rightarrow -\infty.$$

- (iii) S can be written as sup over an expression linear and therefore weak*-continuous in φ and ω [19]. Thus it is weak*-lower semicontinuous and this implies

$$\lim_{n \rightarrow \infty} S(\varphi|_{\mathcal{A}_n}|\omega|_{\mathcal{A}_n}) \geq S(\varphi|\omega).$$

However, monotonicity insures the opposite inequality.

Appendix:

Strong Asymptotic Graded Abelianness

In this appendix we shall give some details of the proof that the anticommutator of the field operators in the Fock representation go strongly to zero for large time separations. We shall work now only in Π_∞ and write for simplicity a_f for $\Pi_\infty(a_f)$. We start with the

Proposition (A.1)

$$\text{st-} \lim_{t \rightarrow \pm\infty} \tau_t a_f = 0.$$

Proof: The intuition is that $\tau_t a_f$ absorbs a particle which for $t \rightarrow \pm\infty$ is far away. However, in the vectors of the Fock representation far away there are scarcely any particles, thus strong convergence to zero. For the mathematical details see [13].

Corollary (A.2)

- a) $\text{w-} \lim_{t \rightarrow \pm\infty} \tau_t a_f^* = 0,$
- b) $\text{st-} \lim_{t \rightarrow \pm\infty} [a_f^*, \tau_t a_g]_+ = 0,$
- c) $\text{st-} \lim_{t \rightarrow \pm\infty} [a_f, \tau_t a_g]_+ = 0.$

Proof: $a_t \rightarrow \infty$ implies $a_t^* \rightarrow 0$. Strong convergence does not hold for a_f^* since $\|\tau_t a_f^*|0\rangle\| = \|f\|_2 \forall t$. b), c) are immediate, actually $[\]_-$ would converge strongly just as well.

Proposition (A.3)

$$\text{st-} \lim_{t \rightarrow \pm\infty} [a_g^*, \tau_t a_f^*] = 0.$$

Proof: A total set of vectors in $\mathcal{H}_\infty$ can be written as $A_n^*|0\rangle$ where $A_n^* = \int dx_1 \dots dx_n \psi(x_1, \dots, x_n) a^*(x_1) \dots a^*(x_n)$. From the identity

$$\begin{aligned} (a_g^* \tau_t a_f^* + (\tau_t a_f^*) a_g^*) A_n^* &= \\ &= (\tau_t a_f^*) a_g^* A_n^* + (-)^n a_g^* A_n^* \tau_t a_f^* + a_g^* [(\tau_t a_f^*) A_n^* + (-)^{n-1} A_n^* \tau_t a_f^*] \end{aligned}$$

we conclude that (A.3) is proved if we show that $[(\tau_t a_f^*) A_n^* + (-)^{n-1} A_n^* \tau_t a_f^*]|0\rangle$ goes strongly to zero.

If we have in the n -particle sector of $\mathcal{H}_\infty$ a Hamiltonian

$$H_{1,\dots,n} = \sum_{i=1}^n \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i,j=1}^n v_{ij}$$

then this strong convergence is equivalent to

$$e^{iH_1, \dots, nt} e^{-iH_2, \dots, nt} - e^{iH_1 t} \rightarrow 0$$

or

$$e^{-iH_1 t} e^{iH_1, \dots, nt} e^{-iH_2, \dots, nt} \rightarrow \mathbb{1}$$

or

$$e^{it(H_1 + H_2, \dots, n + \sum_{j=2}^n v_{ij}(t))} e^{-it(H_1 + H_2, \dots, n)} \rightarrow \mathbb{1}$$

with $v_{ij}(t) = e^{-iH_1 t} v_{ij} e^{iH_1 t}$.

This condition says that the Møller operators between particle 1 and the rest become trivial if particle 1 is far away. This property has been verified in [14] for those interactions which lead to a Galilei invariant automorphism of $\mathcal{A}$.

Finally we have to show strong convergence of $[a_g, \tau_t a_f^*]_+$. Weak convergence follows from (A.2a) but strong convergence requires a special consideration.

Proposition (A.4)

$$\text{st-} \lim_{t \rightarrow \pm\infty} [a_g, \tau_t a_f^*]_+ = 0.$$

Proof: As in the proof of (A.3) we use the decomposition

$$\begin{aligned} (a_g \tau_t a_f^* + (\tau_t a_f^*) a_g) A_n^* &= \\ &= (\tau_t a_f^*) a_g A_n^* + (-)^n a_g A_n^* \tau_t a_f^* + a_g [(\tau_t a_f^*) A_n^* + (-)^{n-1} A_n^* \tau_t a_f^*]. \end{aligned}$$

The last $[\]$ is the same as before. In the first two terms we have to commute a_g to the right since the whole expression acts on $|0\rangle$. Now $a_g A_n^* = A_{n-1}^* + (-)^n A_n^* a_g$ with some A_{n-1}^* and there remains only $a_g \tau_t a_f^* |0\rangle$ but $\tau_t a_f^* |0\rangle = e^{iH_1 t} a_f^* |0\rangle = a_{f_t}^* |0\rangle$ and thus $a_g \tau_t a_f^* |0\rangle = \langle g | f_t \rangle |0\rangle$ and since $\langle g | f_t \rangle \rightarrow 0$ this part goes even in norm to zero.

These propositions finally prove

Theorem (A.5)

If $a_{1,2} \in \mathcal{A}$ have grading $g_{1,2}$, then

$$\text{st-} \lim_{t \rightarrow \pm\infty} (a_1 \tau_t a_2 + (-)^s (\tau_t a_2) a_1) = 0$$

where $s = 0$ if g_1 and g_2 are odd, otherwise $s = 1$.

(A.5) may not be optimal and it could be that it even holds as norm limit but we were not able to prove that. In this case (3.1) would hold for all extremal invariant states.

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QUANTIZATION OF FIELD THEORY WITH NONTRIVIAL GEOMETRY

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Abstract

Outside of perturbation theory, the canonical quantization procedure for gauge fields turns out to be incomplete. This fact, discovered by Gribov in 1976, is a consequence of the geometrical nontriviality of space underlying the fields for non-Abelian gauge groups. In the canonical nonrelativistic approach, after imposing a gauge-fixing condition one finds that the Hamiltonian does not exist for fields which are not weak enough. Moreover, the conventional path integral can no longer be used for quantization when the underlying space is topologically nontrivial. Simple arguments show that the path integral cannot be a substitute for canonical quantization. The method of constrained dynamics, where the system is embedded in an Euclidean space (or super-space), also turns out to be inconsistent. A way to reinforce the foundations of quantum field theory would be to replace the traditional commutation relations for coordinate and momentum operators by a more sophisticated principle, making use of group-theoretical arguments, which would take into account the geometry of the underlying space. The required construction may be based upon ideas of geometric quantization. Examples show the consistency and the power of the geometrical method. The Berezin quantization of symmetric Kähler manifolds is described in some detail. Hopefully, it will be used as a pattern for a future theory.

1 Introduction

Everybody believes now that the theory of non-Abelian gauge fields is the basis of fundamental physics. Besides the inherent mathematical beauty, revealed in the geometric interpretation of the fields as connections in fiber bundles, gauge fields have opened new horizons in physics, since they solved the long-standing problem of ultraviolet divergencies. At small distances, the fields are suppressed, and one can make use of perturbative expansions, the only tool available at present for working with quantized fields. Yet can we go beyond perturbation theory? Gauge fields increase at large distances from coloured sources, and nobody really knows how to treat them. As a result, at least two crucial physical problems remain unsolved: i) What is the structure of the physical vacuum? ii) What is the reason for colour confinement?

Even in the perturbative treatment, a consistent quantization of non-Abelian field theory was a sophisticated business. It took more than a decade after the discovery

by Yang and Mills [1] until an approach leading to an appropriate Feynman-graph technique was suggested by Faddeev and Popov [2]. It took another decade until a careful analysis by Gribov [3] proved an apparent inconsistency in the quantization procedure. In spite of a lot of effort, and more than three hundred publications, no satisfactory way out of the difficulty has yet been proposed. Actually, we do not know what is the system which is perturbed by gauge interactions. One may hope still that the perturbative technique is reliable, but non-perturbative arguments, like instantons and their physical consequences, deserve a more solid foundation.

Many people have understood, of course, that the problem exists and is serious. For instance, discussing the functional integral definition of the effective action in 1981, Bryce DeWitt [4] noted: "Although this integral is widely believed to have a meaning that transcends perturbation theory, every definition of it to date (in the context of field theory) ultimately makes use of a regularized asymptotic expansion around one or more chosen backgrounds, which are usually stationary points of $S[\varphi]$." (The latter is the classical action functional.) In his talk to this conference, B. Simon remarked: "Gauge theory is not understood well by mathematical physics...". A few hours later, referring to the path integral with constraints, J. Klauder noted here: "Faddeev's approach does not always work".

The purpose of this talk is twofold. First, to stress that conventional quantization methods do not work if the geometry of the manifold describing a dynamical system is nontrivial. This is the case, in particular, for non-Abelian gauge fields. Second, to show that sometimes the concept of quantization can be generalized properly, leading to consistently quantized dynamical systems.

A. CRISIS IN STANDARD QUANTIZATION PROCEDURES

2 Two Approaches to Field Quantization

In order to remind ourselves of the essence of the trouble, let us consider a field theory with a simple compact non-Abelian Lie group G . The vector gauge potential is represented by four anti-Hermitian matrices $A_\mu(x)$, $\mu = 0, 1, 2, 3$, in a group representation. (Boldface letter will be used for the three-vector parts of the gauge fields $\mathbf{A} = (A_1, A_2, A_3)$.) In matrix form, the gauge transformation is

$$(1) \quad A_\mu(x) \rightarrow U^\dagger A_\mu U + U^\dagger \partial_\mu U,$$

where the unitary matrix $U(x)$ represents an element of G at the point x of space-time. For any group representation, the covariant derivative is defined by

$$(2) \quad D_\mu(A) \equiv I \partial_\mu + A_\mu(x), \quad \partial_\mu \equiv \partial/\partial x^\mu,$$

where I is the unit matrix. The field strengths are given by matrices

$$(3) \quad F_{\mu\nu}(x) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = [D_\mu(A), D_\nu(A)].$$

The gauge-field Lagrangian density is

$$(4) \quad \mathcal{L} = -\frac{1}{4g^2} \text{tr}(F^{\mu\nu} F_{\mu\nu}) + \text{tr}(J^\mu A_\mu),$$

where J^μ is a conserved external current density, also an anti-Hermitian matrix in the representation space, $\partial_\mu J^\mu = 0$, and g is the coupling constant.

There are, essentially, two ways to quantize a field theory.

The first (traditional) way of the field quantization is, in principle, as follows. One starts from a relativistic Lagrangian, and declares some of the fields to be the generalized coordinates. The corresponding generalized momenta are obtained immediately from the Lagrangian. After constructing the phase space at every point of the Euclidean space, one postulates the canonical Heisenberg commutation relations, and looks for their realization in the Hilbert space of state vectors. The problem specific to any gauge field theory (including electrodynamics, where the solution is well known) is the choice of the *basic* coordinate fields. Even though every field has four components, there are only two “physical” photon (or gluon) states. The timelike component appears in the Lagrangian *without* its time derivative, so that the corresponding generalized momentum vanishes identically. This reduces the number of local degrees of freedom to three. Furthermore, the longitudinal component of A_μ appears *only* in its time derivative, so that its motion is “free”, unlike the dynamics of the transverse components, which are quantized as oscillators (anharmonic for non-Abelian fields). As a result, longitudinal photons do not appear in the Fock space construction and are excluded from dynamics. They are responsible for the instantaneous Coulomb interaction, which does not change the character of the radiation field, and they do not contribute to radiative corrections in electrodynamics. On the other hand, the longitudinal gluons induce an “anti-screening” in non-Abelian theories, but they cannot be eliminated consistently, as we shall see below.

The second (modern) way of quantization is to *postulate* the Feynman path integral for the generating functional in its original (Lagrangian) form. The inherent obstacle is that, because of the gauge freedom, the generating functional is infinite, since the path integral does not exist. The conventional procedure is to separate an infinite factor, deemed irrelevant, by means of a gauge-fixing condition. This can be done consistently in electrodynamics, and the result is in agreement with the canonical approach, but this is not the case in non-Abelian theories owing to the interaction between the longitudinal and transversal gluons.

A provocative question is whether an alternative quantization exists. The light-cone quantization method, proposed by Dirac [5] (see also more recent works [6]), is a nice example which apparently deserves further elaboration. This aspect is, however, beyond our present concerns.

3 Gauge Fields in the Coulomb Gauge

There are a number of ways of introducing canonical variables for the gauge fields. Let us consider one of them, which uses the Coulomb gauge condition and sets A_0 to zero by an appropriate gauge transformation. Actually, for any given field configuration $\vec{A}_0(x)$, one can find a solution of the linear first-order differential equation for the transformation U_C ,

$$(5) \quad \partial U_C / \partial x^0 = -\vec{A}_0 U_C,$$

and, introducing the resulting U_C into Eq. (1), perform the transformation $\tilde{A} \rightarrow A$. Now the Lagrangian density is reduced to its three-dimensional form,

$$(6) \quad \mathcal{L} = -\frac{1}{2g^2} \text{tr}(\dot{\mathbf{A}}^2 - \mathbf{B}^2) - \text{tr}(\mathbf{J}\mathbf{A}),$$

where $\mathbf{B}$ is the (chromo)-magnetic field vector,

$$(7) \quad \mathbf{B}(x) = \text{rot } \mathbf{A} + (\mathbf{A} \times \mathbf{A}) \equiv \text{Rot } \mathbf{A}.$$

There still remains a gauge freedom owing to transformations independent of time x^0 , since $\mathbf{B}$ is independent of the longitudinal component of $\mathbf{A}$, which is present in the kinetic energy, i.e. in the first term in Eq. (6). In order to separate the free motion in that direction, one can change the variables, $\mathbf{A} \rightarrow (\mathbf{Q}, X)$, where $\nabla \mathbf{Q} = 0$, and X is a unitary matrix. One can set $X = \exp(i\Phi)$, with a self-adjoint Φ , which may be treated as a generalized coordinate representing the longitudinal gauge field. This field does not describe any propagating wave, but mediates interactions, such as the electrostatic force mediated by the longitudinal photon. Unlike the electrostatic field, Φ also mediates charge (“colour”) in the non-Abelian case, leading to asymptotic freedom. As to the transverse vector field $\mathbf{Q}$, it represents the usual gluons, which are supposed never to be released into “empty” space (the effect of colour confinement), but which behave like massless spin-1 particles at small distances. The change of variables is

$$(8) \quad \mathbf{A} = X^\dagger \mathbf{Q} X + X^\dagger \nabla X.$$

Given a field $\mathbf{A}$, one can apply ∇ to the above equation and get a differential equation for X ,

$$(9) \quad [D_l(\mathbf{A}), X^\dagger \nabla_l X] = \nabla \mathbf{A}, \quad l = 1, 2, 3.$$

For the Abelian theory (or for a field configuration for which $[A_l, A_k] = 0 \forall l, k$) this is just the Poisson equation for Φ , and the solution exists and is unique under the standard boundary conditions at infinity. In the non-Abelian theory, more analysis is required, but as soon as X is obtained, one can recover $\mathbf{Q}$ from Eq. (8), completing the change of variables. We shall return to this equation later.

Substituting the result into the Lagrangian, one can see that X appears only in the form

$$(10) \quad V \equiv \dot{X} X^\dagger = i \int_0^1 e^{i\tau\Phi} \dot{\Phi} e^{-i\tau\Phi} d\tau.$$

Now the part of the Lagrangian density which depends on the longitudinal fields takes the form

$$(11) \quad \mathcal{L}_V = \frac{1}{2g^2} \text{tr}(V \Delta(\mathbf{Q}) V + 2V R),$$

where

$$(12) \quad \Delta(\mathbf{Q}) V \equiv \nabla[\mathbf{D}(\mathbf{Q}), V] = \Delta V + [\mathbf{Q}, \nabla V], \quad R \equiv g^2 J^0 + [\mathbf{Q}, \dot{\mathbf{Q}}].$$

Actually, the matrix R plays the role of the total colour charge density, including external sources *and* the gluon field itself.

There is no “potential” acting in the direction of the longitudinal field. Therefore, trying to calculate the partition function or a generating functional, one would get a formal infinity. Take, for example, a free particle; the corresponding evolution kernel depends only on the differences of coordinates, its diagonal elements are constant, so that the trace, which is the integral in coordinates, diverges as the total space volume. This is also evident in the path-integral formulation; the functional integral does not exist, since the operator of the quadratic form in the exponent is singular. Free motion in some degrees of freedom does not make the problem more difficult in principle, but it must be separated before the standard methods of perturbation theory are applied. Essentially, the same issue appears in few-body problems where the center-of-mass motion should be separated.

The analogy with few-body problems suggests a solution: One should consider the composite system in the center-of-inertia frame, setting the velocity to zero. Our problem is somewhat less simple; the Lagrangian in Eq. (11) contains both quadratic and linear terms in V . Therefore one has to fix V at the minimum of the Lagrangian. Then V would satisfy a Poisson-type equation

$$(13) \quad \Delta(\mathbf{Q})V = -R.$$

Introducing its formal solution into Eq. (11), one obtains an additional term in the Lagrangian for the gluon field:

$$(14) \quad \mathcal{L}_C = -\frac{1}{2g^2} \text{tr} \left(R \Delta(\mathbf{Q})^{-1} R \right).$$

This expression has a straightforward interpretation: it is the chromostatic energy of coloured fields. Unlike Maxwell electrodynamics, the radiation field $\mathbf{Q}$ is present here; first, it contributes to the colour density R , second, the Coulomb law is modified by its presence in the Laplacian defined in Eq. (12).

4 Quantization and the Gribov Ambiguity

For quantizing the gluon field we may try to follow, in essence, the procedure explained by Bjorken and Drell [7] for the electromagnetic field. The resulting Lagrangian density, which depends on the transverse gluon fields, is as follows:

$$(15) \quad \mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}}) = -\frac{1}{2g^2} \text{tr} \left[\dot{\mathbf{Q}}^2 - (\text{rot } \mathbf{Q})^2 - \tilde{\mathbf{J}}\mathbf{Q} - R(\Delta(\mathbf{Q}))^{-1} R \right],$$

where $\tilde{\mathbf{J}}$ is a modified source current. The Lagrangian looks non-degenerate, since $\nabla \mathbf{Q} = 0$, and we are ready to define the Hamiltonian. The canonical momentum conjugate to the gluon field is obtained as usual from the Lagrangian density,

$$(16) \quad \mathbf{P} = \partial \mathcal{L} / \partial \dot{\mathbf{Q}}, \quad \nabla \mathbf{P} = 0.$$

In order to carry out the canonical quantization, one has to *postulate* the canonical commutation relations for the transverse gluon fields, namely (The commutation relations are written down for the fundamental representation of $SU(N)$, as an example)

$$(17) \quad \hat{P}_{l',\beta'}^{\alpha'}(\mathbf{x}') \hat{Q}_{l,\alpha}^{\beta}(\mathbf{x}) - \hat{Q}_{l,\alpha}^{\beta}(\mathbf{x}) \hat{P}_{l',\beta'}^{\alpha'}(\mathbf{x}') = -i\hbar(\delta_{\alpha}^{\alpha'}\delta_{\beta'}^{\beta} - \frac{1}{N}\delta_{\alpha}^{\beta}\delta_{\beta'}^{\alpha'})\delta_{ll'}^{\text{tr}}(\mathbf{x} - \mathbf{x}'),$$

where $l, l' = 1, 2, 3$, and $\alpha, \beta = 1, \dots, N$ - the dimensionality of the representation. The transverse delta-function [7] is defined by

$$(18) \quad \delta_{ll'}^{\text{tr}}(\mathbf{x} - \mathbf{x}') \equiv \int \frac{d^3k}{(2\pi)^3} e^{ik(\mathbf{x}-\mathbf{x}')} \left(\delta_{ll'} - \frac{k_l k_{l'}}{k^2} \right).$$

The quantization procedure requires construction of a Hamiltonian operator, which can be found unambiguously provided that it is a self-adjoint operator in the Fock space of transverse gluon states.

The Lagrangian (15) is quadratic in $\dot{\mathbf{Q}}$, and its form is typical for motion in a Riemannian space,

$$(19) \quad L(q, \dot{q}) = \frac{1}{2} (\dot{q}, M(q) \dot{q}) + (\dot{q} \cdot v(q)) + U(q),$$

where $M(q)$ is a metric matrix and $v(q)$ a vector field. In our case, the coordinate dependence in M and in the vector potential appears because R depends linearly on $\dot{\mathbf{Q}}$, as seen from Eq. (12). The introduction of the Hamiltonian formalism is straightforward; one has to evaluate M^{-1} . In principle, it is known how to treat systems like that. If one works within the q -representation, the Schrödinger equation is based upon the Laplace operator. An equivalent way is the path integral, which has specific features for Riemannian spaces [8] (see also reviews [9, 10] and references therein). Apparently, the Hamiltonian obtained is a reasonable starting point for perturbation theory, in spite of the fact that it is an involved (nonlinear and nonlocal) functional of the gluon field $\mathbf{Q}$. Like any Hamiltonian we are used to working with, it is quadratic in the generalized momentum $\mathbf{P}$ and one can use the Feynman path integral and Feynman graph techniques. The resulting theory is like quantum electrodynamics, yet the colour charge density $[\mathbf{Q}, \mathbf{P}]$ is added to external sources. Asymptotic freedom appears already at this stage because polarization of the gluon vacuum produces an anti-screening effect. Higher corrections in $\mathbf{Q}$ lead to an infinite number of interaction terms, which make the Feynman graph technique a complicated business [11], unless the Faddeev-Popov ghost fields are introduced.

The problem is, however, that the Lagrangian does not exist because the operator $\Delta(\mathbf{Q})$ has no inverse unless the field $\mathbf{Q}$ is weak enough. Actually, Gribov showed that this operator is singular for some field configurations. The set of such configurations in the functional space is called the *Gribov horizon*, which can be formally described by means of the functional equation

$$(20) \quad \text{Det}(\Delta(\mathbf{Q})) = 0, \quad \mathbf{Q} \in \mathcal{G},$$

where Det stands for the Fredholm determinant. The Gribov horizon $\mathcal{G}$ is the boundary of the domain in the functional space in which one can use the perturbation series. The

structure of $\mathcal{G}$ is not known yet; apparently, it is a great challenge to mathematical physics.

As we know from quantum mechanics, for coordinate spaces having a nontrivial topology, i.e., bounded or closed in some direction, the problem of representation of the canonical commutation relations needs a special investigation. That is the reason why for non-Abelian gauge theory, even after postulating Eq. (17), we do not arrive at a consistent canonical quantization. The boundary conditions on the horizon should be taken into account.

5 Path Integral Does Not Help

5.1 Gauge theory with ghost fields

Technically, the method of Faddeev and Popov is very advantageous. An apparently invariant Lagrangian density is written in terms of anticommuting ghost fields $\zeta, \bar{\zeta}$, which are matrices like A_μ ,

$$(21) \quad \mathcal{L} = \text{tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\beta} (\partial^\mu A_\mu)^2 + \partial^\mu \bar{\zeta} D_\mu(A) \zeta + J^\mu A_\mu + \bar{\eta} \zeta + \eta \bar{\zeta} \right].$$

Here the second term is introduced for gauge fixing, β is a constant, and anticommuting fields $\eta, \bar{\eta}$ are the sources of the ghost fields. The functional integral has to be evaluated in $A_\mu, \zeta, \bar{\zeta}$. The integral in the ghost fields leads to the Fredholm determinant of the linear operator in the space of matrix fields Φ ,

$$(22) \quad \square(A)\Phi \equiv \partial^\mu (\partial_\mu \Phi + [A_\mu, \Phi]).$$

Excluding the irrelevant zero factor, one gets the following functional in A_μ ,

$$(23) \quad \text{Det} \left(\frac{\square(A)}{\square(0)} \right) = \text{Det}(\hat{I} + \Omega(A)),$$

where Ω is a (non-local) linear operator. The problem is, however, that $\square(A)$ may have zero eigenvalues for sufficiently strong fields A . This is a relativistic form of the Gribov ambiguity. One can also say that a relativistic linear condition $\partial^\mu A_\mu = F$, where F is a given function, does not fix the gauge. There may be two gauge-equivalent fields A and A' satisfying this condition

$$(24) \quad A'_\mu = U^\dagger A_\mu U + U^\dagger \partial_\mu U, \quad \partial^\mu A'_\mu = \partial^\mu A_\mu,$$

which are related by a nontrivial unitary transformation U , if the latter satisfies the equation

$$(25) \quad \partial^\mu (U^\dagger \partial_\mu U + U^\dagger A_\mu U) = 0.$$

For sufficiently large A_μ , this equation may have nontrivial solutions. Moreover, it was shown by Singer [12] (see also a review in Ref. [13]) that the Gribov ambiguity is a universal property of the fibre bundle given by the non-Abelian gauge fields and is

independent of the particular choice of the linear gauge condition. The problem has been under investigation till now (see, e.g., Refs. [14], [15]), but a satisfactory solution has not yet been found.

Thus we see that the functional space of gauge fields has a complicated geometry. The structure of the Gribov horizon is not known, and even if it were, canonical quantization would have to be performed with proper boundary conditions. It is not enough, of course, to write the formal path integral for a theory of the field φ ,

$$(26) \quad \mathcal{Z} = \int D\varphi \exp \left[\frac{i}{\hbar} \int \mathcal{L}(\varphi, \partial\varphi) d^4x \right],$$

restricting the functional integral to the domain where the field φ takes its values. It is known that this trick does not work in quantum mechanics, since the path integral does not enable one to incorporate the boundary conditions, which carry important information about the dynamics of the system. (The simplest example is, probably, the comparison between a particle in a box with infinite walls and zero boundary conditions and the particle on a circle, where one has to impose periodic boundary conditions on the wave functions.)

5.2 Embedding by Lagrangian constraints

In classical mechanics one can embed the system into an Euclidean (or another suitable) space, describing the manifold by means of constraint conditions which are introduced into the Lagrangian via the corresponding Lagrange multipliers λ :

$$(27) \quad L(q, \dot{q}) = L_0(q, \dot{q}) + \sum_j \lambda_j \chi_j(q, \dot{q}),$$

where $\chi_j = 0$ are the constraints. This method was even extended to system with anticommuting degrees of freedom. Some people claimed that this approach enables a path integral quantization: one has just to put the Lagrangian into the path integral, and integrate in $q(t)$ and $\lambda(t)$. The λ 's appear linearly in the exponent, so the integrals produce δ -functions of χ 's. Simple examples show that this is not the way, except for the trivial case where the constraints are linear in q .

Take, for instance, a free particle on a ν -dimensional sphere of radius r , so that the only constraint is $\mathbf{q}^2 = r^2$. An explicit calculation [16] shows that the path integral leads to the correct result only if an additional term appears in the effective Lagrangian

$$(28) \quad L_{\text{eff}} = \frac{1}{2} \dot{\mathbf{q}}^2 + \frac{1}{2} \lambda (\mathbf{q}^2 - r^2) + \frac{\hbar^2}{r^2} \left[\frac{(\nu - 1)^2}{8} + \frac{1}{12} \right].$$

The quantum correction term is just a constant for a sphere, but it must be quite sophisticated for a general constraint function. The physical reason is clear; imposing a constraint upon a coordinate, one gets large quantum fluctuations in the corresponding momentum, which contribute to the energy of the system. Only in special cases like linear constraints is this an additive constant which can be just discarded.

5.3 Hamiltonian constraints

Another way of describing classical motion on a manifold is to start from an Euclidean phase space and a Hamiltonian $H(\mathbf{q}, \mathbf{p})$ which preserves the constraints $\chi_j(\mathbf{q}, \mathbf{p})$. In that case the motion is confined to the manifold $\chi_j = C_j$, where the constants C_j are given by initial conditions and not by the external constraints. Now the phase-space path integral could be a way to quantization, if everything is defined properly.

For example, let us assume that a ν -dimensional configuration manifold M_0 is specified by a constraint $\chi(\mathbf{q}) = C_0$, where $\mathbf{q}$ is a vector in the Euclidean space $\mathbb{R}^{\nu+1}$. The problem is to extend the dynamics from the manifold to the whole space $\mathbb{R}^{\nu+1}$, and to find a Hamiltonian $H(\mathbf{q}, \mathbf{p})$ in the phase space $\mathbb{R}^{2\nu+2}$ which would preserve the constraint χ . A reasonable way of doing that is to include χ together with two other functions $\varphi(\mathbf{q}, \mathbf{p})$ and $\kappa(\mathbf{q}, \mathbf{p})$ into a closed Lie algebra of Poisson brackets and to look for a Hamiltonian preserving all three functions:

$$\begin{aligned} \{\chi, \kappa\} &= 2\varphi, \quad \{\chi, \varphi\} = \chi, \quad \{\kappa, \varphi\} = -\kappa; \\ (29) \quad \{\chi, H\} &= \{\varphi, H\} = \{\kappa, H\} = 0. \end{aligned}$$

This is the simplest rank-1 noncompact Lie algebra, that of $SU(1, 1)$, and the Hamiltonian may be taken as its Casimir invariant

$$(30) \quad H(\mathbf{q}, \mathbf{p}) = \chi\kappa - \varphi^2.$$

Let us look for a function H containing a quadratic form in $\mathbf{p}$, which can be considered as a kinetic energy; then φ is a linear function of $\mathbf{p}$. We set

$$(31) \quad H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}(\mathbf{p} \cdot \mathbf{K}\mathbf{p}) + (\mathbf{p} \cdot \mathbf{a}) + u, \quad \varphi(\mathbf{q}, \mathbf{p}) = (\mathbf{p} \cdot \mathbf{v}) + w,$$

and $\kappa = (H + \varphi^2)/\chi$. Here $\mathbf{K}(\mathbf{q})$ is a linear operator in $\mathbb{R}^{\nu+1}$ (a matrix field), $\mathbf{a}(\mathbf{q})$ and $\mathbf{v}(\mathbf{q})$ are vector fields, and $u(\mathbf{q})$, $w(\mathbf{q})$ are scalar functions of $\mathbf{q}$.

Now Eqs. (29) hold under the following conditions:

$$(32) \quad (\mathbf{v}\nabla)\mathbf{K} = \mathbf{V}\mathbf{K} + \mathbf{K}\mathbf{V}, \quad (\mathbf{v} \cdot \mathbf{n}) = \chi, \quad \mathbf{K}\mathbf{n} = 0,$$

$$(33) \quad (\mathbf{v}\nabla)\mathbf{a} - \mathbf{V}\mathbf{a} = \mathbf{K}\nabla w, \quad (\mathbf{v}\nabla)u = (\mathbf{a}\nabla)w, \quad (\mathbf{a} \cdot \mathbf{n}) = 0,$$

where we have used the notations

$$(34) \quad \nabla^\alpha = \partial/\partial q_\alpha, \quad n^\alpha = \nabla^\alpha \chi, \quad V_\beta^\alpha(\mathbf{q}) = \nabla^\alpha v_\beta.$$

The problem is straightforward, in principle. First of all, take any matrix field $\mathbf{K}_0(\mathbf{q})$ in the ν -dimensional tangent bundle to the original manifold M_0 and, setting the components of $\mathbf{K}_0$ normal to M_0 equal to zero, reconstruct $\mathbf{K}(\mathbf{q})$ outside M_0 by means of the differential equation (32). The standard way to do this is to solve the ordinary differential equations

$$(35) \quad dQ_\alpha/d\sigma = v_\alpha(\mathbf{Q}),$$

for characteristics $Q_\alpha(\sigma)$, with the initial conditions $Q_\alpha(0) = q_\alpha^0$ on the original manifold, where $q^0 \in M_0$. As soon as the solution is found, the problem is reduced to an ordinary linear differential matrix equation,

$$(36) \quad d\mathbf{K}/d\sigma = \mathbf{V}_\sigma \mathbf{K} + \mathbf{K} \mathbf{V}_\sigma,$$

with $\mathbf{V}_\sigma \equiv \mathbf{V}(\mathbf{Q}(\sigma))$, and $\mathbf{K}(0) = \mathbf{K}_0$. Actually, the fields which determine two elements of the Lie algebra, φ and χ , are subject to a single condition, $(\mathbf{v}\nabla)\chi = \chi$. In principle, given any vector field $\mathbf{v}$ satisfying the latter, one can try to construct the desired Lie algebra. Now the classical motion is always confined to a surface given by $\chi = \text{const}$, determined by the initial conditions, since χ is an integral of the motion, even though the constraint does not appear in the equations of motion explicitly.

When the system exists in Euclidean phase space, the quantization procedure is defined by the Heisenberg commutation relations. The operator ordering is determined by the requirements that the operators corresponding to φ and κ be self-adjoint. One should note that they do not commute with χ , the constraint operator, which is just natural: the normal component of the momentum and the corresponding kinetic energy are undefined after the constraint is imposed. However, there is no problem with the energy operator as given by H , in which the contribution from the motion along the normal to the manifold has been excluded. The method described here was applied to quadratic constraints (e.g., motion on a sphere) $\chi(\mathbf{q}) = (\mathbf{q}, \mathbf{C}\mathbf{q})$, for arbitrary constant matrix $\mathbf{C}$, and the results are reasonable [17]. However, this method is not universal, as in general there is only a restricted domain around M_0 in which equations (35) can be solved (unless they are linear, as in the example mentioned above, in which case one can take $\mathbf{v} = \mathbf{q}/2$). The formal reason is the appearance of caustics on the rays $\mathbf{Q}(\sigma)$.

Thus embedding the system by constraints into an Euclidean space does not provide a general solution to the problem of quantization.

B. QUANTIZATION AS REPRESENTATION THEORY

6 Recapitulation of the Quantization Principle

The basic postulates of quantum theory were the Heisenberg commutation relations. According to Weyl [20], standard quantum mechanics in ν degrees of freedom can be considered as the representation theory of the Lie algebra generated by the $(2\nu + 1)$ elements P_α, Q_β, I (where $\alpha, \beta = 1, \dots, \nu$), which consists of the canonical commutation relations

$$(37) \quad [P_\alpha, Q_\beta] = -i\hbar\delta_{\alpha\beta}I, \quad [P_\alpha, I] = 0 = [Q_\beta, I].$$

The representation was given (in Schrödinger's theory) by linear operators in the Hilbert space of square integrable functions. Using P_α, Q_β, I as generators of the associative algebra of observables, and selecting various Lie algebras (as given by commutators of the linear operators) embedded into the universal enveloping algebra defined by generators satisfying the relations (37), one gets quantum dynamics, symmetries and everything else relevant to physics. The Lie algebra of the classical Poisson brackets appears as the

classical limit of the commutators. It looked as if the quantization principle could be extended to any system of coordinate-momentum pairs in classical Hamilton mechanics. In his "Principles" [18], Dirac called Eqs. (37) the "*fundamental quantum conditions*", which "... provide us with a basis for calculating commutation relations between other dynamical variables". That was the foundation of his "transformation theory".

Yet, some 20 years after those pioneering works, Dirac [5] felt compelled to present a different formulation of the quantization principle. For any relativistic theory there are ten "fundamental quantities" $P_\mu, M_{\mu\nu}$ (generators of the Poincaré group), which are characteristic of the dynamical system. From geometrical arguments, one gets the relations between the fundamental quantities

$$(38) \quad \begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, P_\rho] &= -g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -g_{\mu\rho}M_{\nu\sigma} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\rho\nu} + g_{\nu\sigma}M_{\rho\mu}. \end{aligned}$$

Dirac wrote [5]: "The problem of finding a new dynamical system reduces to the problem of finding a new solution of these equations". An elementary solution is, of course, that for a single relativistic particle

$$(39) \quad P_\mu = p_\mu, \quad M_{\mu\nu} = q_\mu p_\nu - q_\nu p_\mu,$$

where q_μ, p_ν are four-component operators which satisfy the canonical commutation relations (37).

Now the principle is: *start from a Lie algebra*, say, given by a basis $\{\hat{T}_A\}$,

$$(40) \quad [\hat{T}_A, \hat{T}_B] = f_{AB}^C \hat{T}_C,$$

where f_{AB}^C are structure *constants*. One has to construct a representation of the Lie algebra, and its (associative) enveloping algebra which would be an algebra of observables. Previously, the standard problem of quantization was formulated as a mapping of a given classical Poisson-bracket Lie algebra into an associative algebra of operators. Now we are looking for an inverse procedure: given a Lie algebra, let us find its representations in terms of generalized Poisson brackets in a phase-space manifold M. Actually, we are doing a *de-quantization* of the Lie algebra,

$$(41) \quad \hat{T}_A \rightarrow T_A(x), \quad x \in M.$$

Here $T_A(x)$ are functions on M, which are called *symbols* of $\hat{T}_A$. It is assumed that the *canonical commutation relations* (40) are represented by the Poisson brackets which are first-order differential operators,

$$(42) \quad \{T_A(x), T_B(x)\}_{\text{P.B.}} \equiv \varpi_{ab} \partial^a T_A \partial^b T_B,$$

where $\partial^a \equiv \partial/\partial x_a$, and $\varpi_{ab}(x)$ is a (skew-symmetric) tensor field on M. The Jacobi identities impose conditions upon ϖ ,

$$(43) \quad \varpi_{ad} \partial^d \varpi_{bc} + \varpi_{bd} \partial^d \varpi_{ca} + \varpi_{cd} \partial^d \varpi_{ab} = 0.$$

The next *assumption* is that the matrix ϖ is not singular everywhere on M , so that there exists an inverse matrix field $\omega^{ab}(x)$. Now the Jacobi identities are satisfied if ω is a closed two-form on M ,

$$(44) \quad \partial^c \omega^{ab} + \partial^b \omega^{ca} + \partial^a \omega^{bc} = 0, \quad \varpi_{ac} \omega^{bc} = \delta_a^b,$$

so that it has the following *local* representation

$$(45) \quad \omega^{ab} = \partial^a \lambda^b - \partial^b \lambda^a.$$

Here $\lambda^a(x)$ is a connection one-form; its local transformation law is like that of a gauge field,

$$(46) \quad x \rightarrow \tilde{x} \rightsquigarrow \lambda^a \rightarrow \tilde{\lambda}^a = J_b^a \lambda^b + \partial^a \Lambda, \quad J_b^a \equiv \partial \tilde{x}_b / \partial x_a,$$

where Λ is a function defined (locally) on M . If the connection form λ and the Hamiltonian $H(x)$ are given, one can define an *action functional*

$$(47) \quad \mathcal{A} = \int [\lambda^a(x) dx_a - H(x) dt].$$

Applying the variational principle to this action, one obtains the Hamilton equations of motion

$$(48) \quad \delta \mathcal{A} / \delta x = 0 \rightsquigarrow \dot{x}_a = \{H, x_a\}_{\text{P.B.}},$$

where the Poisson brackets are those defined by Eqs. (42) and (45).

The representation of the Lie algebra given by symbols of its elements in the phase space M enables one to construct the algebra of observables. This algebra, noncommutative in general, must be associative and have a number of other properties (as discussed in a classical work by Jordan, von Neumann and Wigner [19]). First of all, using $T_A(x)$ one has to generate symbols of the corresponding Lie group elements,

$$(49) \quad \hat{G}_\xi \equiv \exp(\xi^A \hat{T}_A) \rightarrow G_\xi(x),$$

where ξ^A are the Lie group parameters. The group multiplication law defines the symbol of a product of group elements

$$(50) \quad \hat{G}_\xi \hat{G}_\eta = \hat{G}_\zeta \rightarrow G_\zeta(x),$$

where ζ is a known function of ξ and η . Now one considers *functions* on M given by integrals on the group manifold, which are symbols of the corresponding operators in the representation space:

$$(51) \quad \hat{F} = \int \hat{G}_\xi f(\xi) d\mu(\xi) \rightarrow F(x) = \int G_\xi(x) f(\xi) d\mu(\xi).$$

The integration is over the group with the invariant measure, and $f(\xi)$ is a function called the *co-symbol* of operator $\hat{F}$. When the symbols and the co-symbols are in one-to-one correspondence, the multiplication law (50) determines a multiplication law for the symbols which is associative but not commutative. The symbol of a commutator for any

two operators is not given, in general, by their Poisson bracket, unless the operators are elements of the original Lie algebra. The correspondence between the Poisson brackets and the commutators appears, however, in the semi-classical approximation (see below).

The Lie structure induces an action of the Lie group on M . One may assume that M is homogeneous. Then it can be considered as a coset space, $M = G/H$, where G is the Lie group and H is the subgroup which is called the gauge group of the dynamical theory.

Of course, the main problem is to find a symplectic form λ which would be consistent with the original Lie algebra. A general solution will probably be found some day; meanwhile, let us look at particular cases which have been elaborated previously.

1) The Heisenberg - Weyl group, $G \equiv W_\nu$, generated by the Lie algebra of (37), $H \equiv U(1)$, $M = \mathbb{R}^{2\nu}$. This is standard quantum theory in the Weyl picture [20]. Here ω , as well as ϖ , is a constant non-singular skew-symmetric matrix. The symplectic one-form is linear in x , and can be written as $\mathbf{p} d\mathbf{q}$, or $-\mathbf{q} d\mathbf{p}$, or $-i(zd\bar{z} - \bar{z}dz)$, where $z = \frac{1}{2}(\mathbf{p} - i\mathbf{q})$ (the last is used for coherent states), depending on the representation one wants to work with. The differences between the above one-forms are total differentials of quadratic functions.

2) Instead of (37), one can start from the Clifford algebra with anticommutators,

$$(52) \quad [\Gamma_a, \Gamma_b]_+ = \delta_{ab}I, \quad [\Gamma_a, I] = 0, \quad a = 1, \dots, n.$$

The (super-)group parameters ξ^a are supposed to be anticommuting, i.e., they are generators of the Grassmann algebra. Elements of G are written as

$$(53) \quad G_\xi = \exp(\xi^a \Gamma_a + \xi^0 I), \quad \xi^a \xi^b = -\xi^b \xi^a, \quad \xi^0 \xi^a = \xi^a \xi^0.$$

Thus G may be called the Weyl *super-group*. Again $H \equiv U(1)$, and the symbols ("functions" on M) belong to the Grassmann algebra. This scheme was actually developed by Schwinger [21] in 1953 (also in Ref. [22]) in order to introduce the fermion (electron) fields into the quantum action principle for quantum electrodynamics. Schwinger called the anticommuting coordinates in M the variables of the *second kind*. Symbols of the operators are elements of the Grassmann algebra, dual to G , and its generators are anticommuting: $x_a x_b = -x_b x_a$. The interpretation, properties of symbols and the path integral (both for even and odd n) may be found in Ref. [23], where this formalism was used for the description of spin variables. In general, the total number of states is given by the dimensionality of the representation of the Clifford algebra (52), $N = 2^{\lfloor n/2 \rfloor}$. There is only one irreducible representation. One can combine both the systems, Bose-type of Eq. (37) and Fermi-type of Eq. (52), in a supersymmetric way.

3) Let $G \equiv W_\nu$, as in the first example, and $H \equiv U(1) \times L^{2\nu}$, where the latter is a 2ν -dimensional lattice. Now the phase space is a torus, $M = \mathbb{T}^{2\nu}$, and the symbols $F(x)$ must be periodic functions. This is another extension of standard quantum theory invented by Schwinger [24]. The coordinates and momenta satisfy Eqs. (37), but their spectra are discrete, as the support of the symbols is compact. Moreover, the total number of states N is *finite* in this case, like in the previous example. For instance, for one degree of freedom, if the periods in (q, p) are a and b , respectively, one has

$$(54) \quad F(q+a, p) = F(q, p) = F(q, p+b) \rightsquigarrow N = ab/2\pi\hbar.$$

4) Another dynamical system with one degree of freedom: $G \equiv SU(2)$, $H \equiv U(1)$, $M \equiv S^2$, i.e., the phase space is a two-dimensional sphere. This example was described by Souriau [25] in 1970. The number of states is finite, and the radius of the sphere R is quantized,

$$(55) \quad N = 2j + 1, \quad R^2 = j\hbar.$$

Here $2j$ is any positive integer, and the algebra of observables is determined by an irreducible unitary representation of $SU(2)$, given by the angular momentum j .

5) For any Lie group, the co-adjoint orbit can be considered as a phase space. This is the *geometric quantization* of Kirillov and Kostant. (For reviews, see [26], [27]).

6) The Berezin quantization on Kähler manifolds [28], [29]. Berezin gave a general approach, and worked out the particular cases of a sphere (the example 4 above) and a hyperbolic (Lobachevsky) plane, where $G \equiv SU(1, 1)$ and $H \equiv U(1)$. Berezin's approach is described in the next section.

7 Berezin Quantization

Given a Kähler manifold M^m and the Kähler potential $K(z, \bar{z})$ (where $z \equiv \{z^\alpha\}$, $\alpha = 1, \dots, m$ are local coordinates) one has a closed two-form ω and the corresponding Poisson brackets,

$$(56) \quad \omega \equiv \omega^{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta, \quad \omega^{\alpha\bar{\beta}} \equiv \partial^\alpha \partial^{\bar{\beta}} K(z, \bar{z}), \quad \partial^\alpha \equiv \partial / \partial z_\alpha;$$

$$(57) \quad \{F_1, F_2\}_{\text{P.B.}} \equiv \varpi_{\alpha\bar{\beta}} \left(\partial^\alpha F_1 \partial^{\bar{\beta}} F_2 - \partial^\alpha F_2 \partial^{\bar{\beta}} F_1 \right).$$

The potential defines a Riemannian structure on M ,

$$(58) \quad ds^2 \equiv \omega^{\alpha\bar{\beta}} dz_\alpha d\bar{z}_\beta.$$

The corresponding Riemannian measure on M is the same as that obtained by the wedge product of the two-forms defined in (56),

$$(59) \quad d\mu(z, \bar{z}) \equiv \det \omega \prod_{\alpha=1}^m dz_\alpha d\bar{z}_\alpha = \omega \wedge \dots \wedge \omega.$$

Berezin [28] introduced the Hilbert space $\mathcal{H}$, which is the holomorphic line bundle on M , locally defined as a set of holomorphic functions $\psi(z)$, with the inner product

$$(60) \quad (\psi_2, \psi_1) \equiv C \int_M \overline{\psi_2(z)} \psi_1(z) e^{-K(z, \bar{z})} d\mu(z, \bar{z}),$$

where C is a normalization constant, see Eq. (68) below. Under holomorphic coordinate transformations, the potential and the wave functions transform as follows:

$$(61) \quad z \rightarrow \tilde{z} = f(z) \rightsquigarrow K(z, \bar{z}) \rightarrow K(z, \bar{z}) + \Phi(z) + \overline{\Phi(z)}, \quad \psi(z) \rightarrow e^{\Phi(z)} \psi(z),$$

where $f(z)$ and $\Phi(z)$ are holomorphic functions. Thus one has $\Lambda(z, \bar{z}) = \Phi(z) + \overline{\Phi(z)}$ in Eq. (46).

The Kähler potential determines the reproducing kernel L in $\mathcal{H}$,

$$(62) \quad C \int_{\mathbf{M}} L(\zeta, \bar{z}) \psi(z) e^{-K(z, \bar{z})} d\mu(z, \bar{z}) = \psi(\zeta), \quad \forall \psi \in \mathcal{H},$$

$$(63) \quad L(\zeta, \bar{z}) = e^{K(\zeta, \bar{z})} = \sum_n \varphi_n(\zeta) \overline{\varphi_n(z)},$$

where $\{\varphi_n(z)\}$ is an orthonormal basis in $\mathcal{H}$. Linear operators in $\mathcal{H}$ are represented by their symbols, $\hat{A} \rightarrow A(z, \bar{z})$:

$$(64) \quad (\hat{A}\psi)(\zeta) = C \int_{\mathbf{M}} A(\zeta, \bar{z}) L(\zeta, \bar{z}) \psi(z) e^{-K(z, \bar{z})} d\mu(z, \bar{z}).$$

The multiplication law for the symbols (the *-product) is given by

$$(65) \quad \hat{A}\hat{B} = \hat{P} \rightarrow P(\zeta, \bar{\eta}) = C \int_{\mathbf{M}} A(\zeta, \bar{z}) B(z, \bar{\eta}) \times \\ \exp [K(\zeta, \bar{z}) - K(z, \bar{z}) + K(z, \bar{\eta}) - K(\zeta, \bar{\eta})] d\mu(z, \bar{z}).$$

The product is clearly associative. In the case of Euclidean phase space one has $K(z, \bar{z}) = z_\alpha \bar{z}_\alpha$, and $\varphi_n(z)$ describe the usual coherent states. Any function which has a Fourier transform corresponds to an operator. A similar requirement is imposed upon the functions on $\mathbf{M}$: operators correspond to functions for which the integral in Eq. (64) would not take ψ out of $\mathcal{H}$; i.e., the integral in Eq. (60) for the norm of $\hat{A}\psi$ must be convergent.

Now the trace of any operator is given by the integral of its symbol,

$$(66) \quad \text{tr } \hat{A} = C \int_{\mathbf{M}} A(z, \bar{z}) d\mu(z, \bar{z}).$$

The symbol of the unit operator is a constant, $\hat{I} \rightarrow 1$; the number of states N is finite if the manifold is compact, and (like its volume V) is given by an integral

$$(67) \quad V = \int_{\mathbf{M}} d\mu(z, \bar{z}), \quad \text{tr } \hat{I} = N,$$

$$(68) \quad C^{-1} = V/N = \int_{\mathbf{M}} e^{-K(z, \bar{z})} d\mu(z, \bar{z}).$$

The normalization condition follows from the requirement that the norm of the function $\varphi_0(z) \equiv 1$, which belongs to $\mathcal{H}$ if $\mathbf{M}$ is compact, be unity.

There exists a class of homogeneous Kähler manifolds where G is a compact simple Lie group and the potentials can be constructed explicitly. This is the subject of the next section.

8 Homogeneous Kähler Manifolds

Borel's method provides a complete description of the Kähler manifolds of the type

$$(69) \quad M = G/T^k \otimes H',$$

where G is a simple compact Lie group, H' is a semi-simple subgroup, and $T^k = U(1) \otimes \cdots \otimes U(1)$ (k times) – a torus. The number k is called the rank of M , and $1 \leq k \leq r \equiv \text{rank } G$. We shall use the standard notations: $\Delta_+(G)$ is the set of positive roots α of the Lie algebra of G , $\gamma_{(j)} \in \Delta_+(G)$ are the primitive roots ($j = 1, \dots, r$), $\mathbf{w}^{(j)}$ are the fundamental weights, and

$$(70) \quad \rho \equiv \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \sum_{j=1}^r \mathbf{w}^{(j)}.$$

Any unitary irreducible representation is specified with its highest weight vector,

$$(71) \quad \mathbf{l} = \sum_{j=1}^r l_j \mathbf{w}^{(j)},$$

where l_j are nonnegative integers. For the representation of G specified by $\mathbf{l}$, the Kähler potential is given by the sum

$$(72) \quad K_{\mathbf{l}}(z, \bar{z}) = \sum_{j=1}^r l_j K^{(j)}(z, \bar{z}),$$

where $K^{(j)}$ are the fundamental potentials. If all l_j are nonzero, the gauge group is the maximal torus of G , $H = T^r$ and H' is trivial. The manifold M is called the flag manifold; its dimensionality is maximal, $m = \frac{1}{2}(\dim(G) - r)$ – the number of positive roots of G . The Kähler structure is based upon Borel's nilpotent group $B \subset G^c$ (the latter is the complexification of G), which has elements of the form

$$(73) \quad b(z) = \exp \left(\sum_{\alpha \in \Delta_+} \hat{E}_{\alpha} z_{\alpha} \right) \in B.$$

Here $\hat{E}_{\alpha}$ are elements of the canonical basis in the Lie algebra, i.e., $\hat{T}_A \equiv \{\hat{E}_{\alpha}, \hat{E}_{-\alpha}, \hat{H}_j\}$ and $\hat{H}_j$ are the basis elements of the Cartan subalgebra. In any representation the matrix elements of $b(z)$ are polynomials in z . The fundamental Kähler potentials have the following general form

$$(74) \quad K^{(j)}(z, \bar{z}) = \log \det_{(j)} b^{\dagger}(z) b(z),$$

where $\det_{(j)} M$ stands for a principal minor of the matrix M in the fundamental representation of G^c . The symbols of $\hat{T}_A$ are also expressed in terms of the fundamental representation of $b(z)$. (Technical details and references may be found in Ref. 30.)

If a number of components of $\mathbf{l}$ are zero, i.e., $(\mathbf{l} \cdot \gamma^{(j)}) = 0$ for $\gamma^{(j)} \in \Delta'_+ \subset \Delta_+$, the dimensionality of M is lowered, since the gauge group is enlarged and acquires a non-Abelian factor, so that $H = T^k \otimes H'$, where H' is semisimple. Moreover, $\text{rank } M =$

$k < r = \text{rank } G$. The Lie algebra of H' is generated by $\hat{E}_\beta$, where $\beta \in \Delta'_+$, and one has to set $z_\beta = 0$ in Eq. (73). Thus M is a subspace of the flag manifold for $k < r$.

The integrals in Eqs. (67) and (68) can be calculated explicitly, and the result is

$$(75) \quad N = \prod_{\alpha \in \Delta_+ \setminus \Delta'_+} \frac{(\alpha \cdot (1 + \rho))}{(\alpha \cdot \rho)}, \quad V = (2\pi)^m \prod_{\alpha \in \Delta_+ \setminus \Delta'_+} \frac{(\alpha \cdot 1)}{(\alpha \cdot \rho)}.$$

The expression for N is, of course, the famous Weyl formula for the dimensionality of the representation (the product can be extended to all $\alpha \in \Delta_+$). The expression for the volume of M stems from Berezin's formula for the invariant measure

$$(76) \quad d\mu(z, \bar{z}) = \text{const} \cdot e^{-K_\rho(z, \bar{z})} \prod dz d\bar{z}.$$

The homogeneous Kähler manifolds can be described by means of Dynkin diagrams. Any irreducible group representation is specified by nonnegative integers l_j attached to each node of the diagram. The union of all the nodes with $l_j = 0$ is just the Dynkin diagram for H' in the coset representation of M , Eq. (69). For example, the graph below

$$(77) \quad \odot \rightarrow \bigcirc \xrightarrow{l_3} \odot \begin{matrix} \nearrow \bigcirc^{l_1} \\ \searrow \odot \end{matrix}$$

where the nodes with $l_j = 0$ are marked with a dot, represents a manifold of rank 2, $M = SO(10)/U(1) \otimes U(1) \otimes SU(2) \otimes SU(3)$. Its dimensionality is $m = \frac{1}{2}(45 - 1 - 1 - 3 - 8) = 16$.

Two classes of rank-1 manifolds for unitary groups $G = SU(r+1)$ are well known. Here we have $l_j = 0$ for all $j \neq q$ for a given q . If $q = 1$ or $q = r$, the node with nonzero l is at the edge of the Dynkin chain graph, and the manifold is the complex projective space, i.e., $M = U(r+1)/U(1) \otimes U(r) \equiv \mathbb{CP}^r$. If $q \neq 1, r$, that node is in the middle, and one has the Grassmann manifold, $M = SU(p+q)/U(1) \otimes SU(q) \otimes SU(p) \equiv \text{Gr}(p, q)$, where $m = pq$, $p = r + 1 - q$.

The fundamental Kähler potentials can be written down explicitly for any unitary group. The primitive roots for $SU(n)$ are given in terms of orthonormal vectors in n -dimensional Euclidean space, $\gamma^{(j)} = e_j - e_{j+1}$ for $j = 1, \dots, n-1$. All the positive roots are $\alpha_{ij} = e_i - e_j$, and the complex parameters in Eq. (73) are z_{ij} , for $1 \leq i < j \leq n$. The fundamental representation of $b(z)$ is a $n \times n$ matrix having the elements $b_{jj} = 1$, $b_{ji} = 0$ for $i < j$, and $b_{ij} \equiv \zeta_{ij}$ which are simple polynomials in $z_{i'j'}$, where $i \leq i' < j' \leq j$,

$$(78) \quad \zeta_{ij} = z_{ij} + \frac{1}{2} \sum_{k=i+1}^{j-1} z_{ik} z_{kj} + \dots + \frac{1}{(j-i)!} z_{i,i+1} z_{i+1,i+2} \dots z_{j-1,j}.$$

Now let us construct a Grassmann tensor of rank j , substituting the first j columns in the matrix representation of $b(z)$ for the basis vectors and evaluating the determinant

of the resulting $n \times n$ matrix

$$(79) \quad \mathbf{T}_{(j)}(z) \equiv \det \begin{pmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_1 & \zeta_{1,j+1} & \dots & \zeta_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{e}_{j+1} & \dots & \mathbf{e}_{j+1} & 1 & \dots & \zeta_{j+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{e}_n & \dots & \mathbf{e}_n & 0 & \dots & 1 \end{pmatrix}.$$

The fundamental Kähler potentials are given by

$$(80) \quad K^{(j)}(z, \bar{z}) = \log \left(\overline{\mathbf{T}_{(j)}(z)} \cdot \mathbf{T}_{(j)}(z) \right), \quad j = 1, \dots, n-1$$

(the dot stands for the scalar product in the tensor space). Moreover,

$$(81) \quad \exp \left(K^{(j)}(z, \bar{z}) \right) = \sum_{k=0}^{\nu} |P_k^{(j)}(\zeta)|^2, \quad \nu = \frac{n!}{j!(n-j)!},$$

where $P_k^{(j)}(\zeta)$ are polynomials which are $(n-j) \times (n-j)$ minors of the $n \times n$ matrix in Eq. (73). Taking the polynomials for all j , one gets a basis in $\mathcal{H}$ for the representation with $\mathbf{l} = \rho$.

The fundamental potentials for manifolds of a lower rank are obtained from Eq. (80) by setting constraints on the coordinates. For example, to get the potential for $\mathbb{CP}^{n-1}$ one can set $\zeta_{in} = z_i$ for $i = 1, \dots, n-1$ and $\zeta_{ij} = 0$ for $j < n$. The resulting Kähler potential is

$$(82) \quad K_l = l \log \left(1 + \sum_{i=1}^{n-1} |z_i|^2 \right),$$

and the corresponding metric given by Eq. (58) has the familiar Fubini-Study form. For a number of other Kähler manifolds, the potentials have been calculated explicitly [30].

9 Semiclassical Approximation

As soon as one understands the quantum kinematics, including the operator-symbol correspondence, the dynamics is determined by a Hamiltonian operator $\hat{H}$. The time evolution is given by the operator $\exp(-it\hat{H})$. In order to describe the time evolution, one can represent the symbol of the evolution operator by means of an analogue of the path integral. We shall use the identity

$$(83) \quad \hat{U}_t \equiv e^{-i\hat{H}t} = \left(e^{-i\hat{H}t/\mathcal{N}} \right)^{\mathcal{N}}, \quad \forall \mathcal{N},$$

and write an approximate expression for the symbol of the evolution operator at small times, $\tau = t/\mathcal{N}$, where $\mathcal{N} \rightarrow \infty$,

$$(84) \quad e^{-i\hat{H}\tau} \rightarrow e^{-i\tau H(z, \bar{z})} + O(\tau^2).$$

The multiplication is given by Eq. (65), and the result is the following representation for the symbol of the evolution operator

$$(85) \quad U_i(z_0, \bar{z}) = \lim_{N \rightarrow \infty} \int \cdots \int \exp [K(z_0, \bar{z}_1) - K(z_0, \bar{z})] \times \\ \exp \left\{ \sum_{n=1}^{N-1} (K(z_n, \bar{z}_{n+1}) - K(z_n, \bar{z}_n) - i\tau H(z_n, \bar{z}_n)) \right\} \times \\ \prod_{n=1}^{N-1} C d\mu(z_n, \bar{z}_n).$$

The semiclassical approximation follows from the assumption that the highest vector of the representation is large. In this case the integrals can be calculated by the method of steepest descent. The extremal trajectories in M are determined by the set of equations for $n = 1, \dots, N-1$,

$$(86) \quad \begin{aligned} \kappa(z_n, \bar{z}_{n+1}) - \kappa(z_n, \bar{z}_n) - i\tau \partial H(z_n, \bar{z}_n) &= 0, & \kappa^\alpha(\zeta, \bar{z}) &\equiv \partial K(\zeta, \bar{z}) / \partial \zeta_\alpha, \\ \bar{\kappa}(z_{n-1}, \bar{z}_n) - \bar{\kappa}(z_n, \bar{z}_n) - i\tau \bar{\partial} H(z_n, \bar{z}_n) &= 0, & \bar{\kappa}^\alpha(\zeta, \bar{z}) &\equiv \partial K(\zeta, \bar{z}) / \partial \bar{z}_\alpha. \end{aligned}$$

Under standard conditions on the Hamiltonian, in the limit $\tau \rightarrow 0$ the region $z_{n+1} \rightarrow z_n$ contributes most to the integral, so that

$$\kappa^\alpha(z_n, \bar{z}_{n+1}) - \kappa^\alpha(z_n, \bar{z}_n) \approx \omega^{\alpha\bar{\beta}}(z_n, \bar{z}_n)(\bar{z}_{n+1} - \bar{z}_n)_\beta,$$

and equations (86) are reduced to the usual differential equations of motion, which can be obtained from the variational principle applied to the action of Eq. (47) with the symplectic one-form corresponding to the Kähler potential

$$(87) \quad \mathcal{A} = - \int_0^t [i\bar{\kappa}^\alpha(z, \bar{z}) d\bar{z}_\alpha + H(z, \bar{z}) d\tau].$$

The boundary conditions for z and $\bar{z}$ are given at the opposite ends of the time interval: $z(0) = z_0$, $\bar{z}(t) = \bar{z}$. Thus one recovers classical mechanics on M with Poisson brackets as given by Eqs. (56) and (57). Equation (85) looks like a way to the path integral representation for the quantal dynamics on M , but actually this is true only in the semiclassical limit, since the accuracy of the substitution in Eq. (84) needs a further analysis.

The semiclassical approximation is relevant if K is proportional to a large parameter. This parameter must also be present in the Hamiltonian H , otherwise the classical dynamics would be trivial. For homogeneous Kähler manifolds the large parameter appears if all nonzero components of l are large. It is remarkable that for large l the volume is proportional to the number of states, $V \approx (2\pi)^m N$, which is an analogue of the Schwinger quantization of the phase space volume, cf. Eq. (54). In order to understand the role of the Planck constant, let us consider the complex projective space, where l has only one nonzero component l . If the essential domain in $M = \mathbb{CP}^m$ is restricted to $|z|^2 < l^{-1}$ and $l \rightarrow \infty$, we set $l = R^2/\hbar$ and get from Eq. (82)

$$(88) \quad K_l(z, \bar{z}) \approx l|z|^2 = |Rz|^2/\hbar.$$

Thus $x_\alpha \equiv Rz_\alpha$ may be considered as complex coordinates in the Euclidean phase space E^{2m} , which is the neighbourhood of the origin in $\mathbb{CP}^m$. For more sophisticated Kähler manifolds, one can also reproduce the canonical commutation relations, equivalent to Eqs. (37), in a restricted domain of the phase space, but with an anisotropy which cannot be eliminated because of boundary conditions.

10 Conclusion

Despite the triumphs of the standard model, we cannot be sure at present that we understand the underlying field theory beyond perturbative schemes. Apparently, the perturbative approach is already exhausted and any further progress will be due to a development of new methods of quantization. However, embedding the system by constraints into an Euclidean space and using the Heisenberg commutation relations does not provide a general solution to the problem of quantization.

Geometric quantization on homogeneous Kähler manifolds shows that the principles of quantum theory may be extended considerably. The conventional way from a Lagrangian for coordinates taking their values in a manifold to a phase space, which is the cotangent bundle, and a Hamiltonian specified in that space, is probably not universal enough. The guideline to a deeper insight is given by group-theoretical arguments. The theory is elaborated reasonably for compact semi-simple Lie groups acting upon homogeneous phase-space manifolds. An extension of the formalism to non-compact (locally compact) groups and super-groups will probably lead to new and interesting developments.

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EXTENDING QUANTUM GROUP STRUCTURE TO POINCARÉ SYMMETRY

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Abstract

The quantum group extension of Poincaré symmetry is discussed briefly and the recently introduced κ -Dirac equation is gauged to include the Coulomb field. Precision quantum optics data is used to estimate the dimensional (deformation) parameter κ .

1 Introduction

One of the more interesting recent innovations in quantum physics is an algebraic structure [1] called a “quantum group”. There are essentially two new ideas – for a physicist – in the concept of a quantum group: (a) *symmetry deformation*: the structure constants of a Lie symmetry algebra become functions of a parameter, q , such that for $q \rightarrow 1$, the usual Lie algebra is obtained and (b) *non-commutative co-multiplication*. Ad (a) let us remark that many fundamental generalizations in physics have the nature of a deformation: quantum mechanics, for example, may be considered as a deformation of classical mechanics (with $\hbar$ as the deformation parameter). To understand (b) intuitively, it is helpful to recognize that the addition of angular momenta is formally a co-multiplication, which is commutative. In the q -group generalization, this addition depends on order and becomes non-commutative. In consequence of the non-commutative co-multiplication the coordinates of the carrier space become non-commutative [2].

For symmetries defined by a semi-simple Lie algebra (for example, the quantal rotation group, the Lorentz group, or $SU_{3\text{color}}$) there now exist structure theorems [1-3] categorizing the quantum group extension. The associated representation theory [1-6] – so important for physical applications – is now well developed, though still of current interest. The situation is very different for non-semi-simple Lie symmetry groups, for which the quantum group deformation is not unique [7]. Some of the most fundamental and important symmetries in physics – for example the Poincaré group – are in this category.

What is the motivation underlying this extension to quantum group symmetry? Deformations can be interpreted, in a sense, as a new kind of symmetry breaking. Conceivably this mechanism might, for example, give mass to vector bosons without need of the Higgs mechanism (this idea is being pursued by Connes using non-commutative differential geometry). It has been suggested (by Majid) that the deformation parameter q could serve as some kind of a regulator for finitizing divergences. The q -deformation of a direct product group need not be a direct product of q -groups, which suggests that the quantum group structure might serve, say, to unify the $SU(2) \times U(1)$ electroweak group.

More concretely, one motivation, especially for field theory, stems from an old result of Gel'fand that the essence of a compact topological group, G , is captured by the algebra of continuous functions on the group. Let $f, h \in C(G)$, $g \in G$. Then we have a commutative Hopf algebra:

$$(1.1) \quad \begin{array}{lll} \text{multiplication:} & (f \cdot h)(g) & = f(g)h(g), \\ \text{co-multiplication:} & \Delta(f)(g_1 \otimes g_2) & = f(g_1 g_2), \text{ and} \\ \text{antipode:} & \gamma(f)(g) & = f(g^{-1}). \end{array}$$

A quantum group (in Drinfeld's definition) is an extension of this structure to non-commutative, non-co-commutative Hopf algebras.

In the present paper we will survey briefly the extension of q -group structure to the Poincaré group, based primarily on the work of refs. [8] and [9], and then turn to the κ -Dirac equation recently developed by Nowicki, Tarlini and Sorace [9]. By gauging this equation, we obtain the κ -Dirac-Coulomb equation, and then seek to confront this new structure with the very accurately known quantum optics data on the hydrogen-atom spectrum to assess the validity of the quantum group generalization.

2 The κ -Poincaré Quantum Group

There are two distinct approaches to a q -extension of the Poincaré group being pursued at present. One procedure [10] is based on inducing a q -Poincaré group from the symmetries of non-commutative Minkowski space (which, as Dirac pointed out long ago, has its own difficulties of physical interpretation). The second, more conservative, procedure [8, 9] uses the familiar embedding of the Poincaré group in the de Sitter group $SO(3, 2)$, which is semi-simple and thus has a unique q -group extension. Using a contraction procedure, one can then obtain a quantum version of the Poincaré group.

This latter approach has been developed by Lukierski, Nowicki and Ruegg [8] and extended by Nowicki, Sorace and Tarlini [9], obtaining a deformed Poincaré algebra by contracting the standard q -deformation of the (anti-) de Sitter algebra $so(3, 2)$. Taking the limit of the de Sitter radius $R \rightarrow \infty$ with an accompanying limit of the deformation parameter q , such that $\lim(R \ln q) = \kappa^{-1}$, one obtains the κ -Poincaré quantum group.

The algebra structure is:

$$\begin{aligned}
 (2.1) \quad & [P_i, P_j] = 0, \quad [P_i, P_0] = 0, \\
 & [M_i, P_j] = i\epsilon_{ijk}P_k, \quad [M_i, P_0] = 0, \\
 & [L_i, P_0] = iP_i, \quad [L_i, P_j] = i\kappa\delta_{ij}\sinh\left(\frac{P_0}{\kappa}\right), \\
 & [M_i, M_j] = i\epsilon_{ijk}M_k, \quad [M_i, L_j] = i\epsilon_{ijk}L_k \\
 & [L_i, L_j] = -i\epsilon_{ijk}\left(M_k \cosh\left(\frac{P_0}{\kappa}\right) - \frac{1}{4\kappa}P_k P_l M_l\right).
 \end{aligned}$$

Here $P_\mu \equiv \{P_0, P_i\}$ are the deformed generators for energy and momenta, the M_i are the spatial rotation generators (they close on an undeformed Hopf co-commutative subalgebra), and the L_i are the deformed boost generators. (One sees that only the boost structure is deformed.)

We refer to refs. [8–9] for the co-algebra and antipode structure.

The two deformed invariant operators of the κ -Poincaré group are:

$$\begin{aligned}
 (2.2) \quad & C_1 = \left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right)\right)^2 - P_i P_i, \\
 & C_2 = \left(\cosh\left(\frac{P_0}{\kappa}\right) - \frac{P_i P_i}{4\kappa^2}\right) W_0^2 - W_i W_i
 \end{aligned}$$

where

$$W_0 = P_i M_i \quad \text{and} \quad W_i = \kappa \sinh\left(\frac{P_0}{\kappa}\right) M_i + \epsilon_{ijk} P_j L_k.$$

The Lorentz Lie algebra representation using the Dirac matrices is also a representation of the Poincaré Lie algebra with the four-momentum generators represented by zero. This representation also fulfills the commutation relations (2.1). We can then “add” a spinless representation and the Dirac matrix representation (by using the co-multiplication) to obtain the desired spinorial representation for $s = \frac{1}{2}$. We obtain:

$$\begin{aligned}
 (2.3) \quad & \mathcal{P}_\mu^{\text{total}} = P_\mu, \quad \mathcal{M}_i^{\text{total}} = M_i + m_i, \\
 & \mathcal{L}_i^{\text{total}} = L_i + \exp\left(-\frac{P_0}{2\kappa}\right) l_i - \frac{1}{2\kappa} \epsilon_{ijk} m_j P_k.
 \end{aligned}$$

The deformed Dirac operator is required to be invariant under the generators in (2.3). From the commutation relations (and the properties of the Dirac matrices) one can verify that this deformed operator:

$$(2.4) \quad \mathcal{D} \equiv -\exp\left(-\frac{P_0}{2\kappa}\right) \gamma_i P_i + \gamma_4 \kappa \sinh\left(\frac{P_0}{\kappa}\right) - \frac{i}{2\kappa} \gamma_4 P_i P_i,$$

obeys the required relations: $[\mathcal{D}, \mathcal{L}_i^{\text{total}}] = [\mathcal{D}, \mathcal{M}_i^{\text{total}}] = [\mathcal{D}, \mathcal{P}_\mu^{\text{total}}] = 0$.

Moreover, the square of $\mathcal{D}$ can be found to be:

$$(2.5) \quad \mathcal{D}^2 = C_1 \left(1 + \frac{1}{4\kappa^2} C_1\right) = -\frac{4}{3} C_2.$$

Thus the deformed Dirac equation may be written in the explicit form:

$$(2.6) \quad \left(-\exp\left(-\frac{P_0}{2\kappa}\right) \gamma_i P_i + \gamma_4 \kappa \sinh\left(\frac{P_0}{\kappa}\right) - \frac{i}{2\kappa} \gamma_4 P_i P_i\right) \psi = m \left(1 + \frac{m^2}{4\kappa^2}\right)^{\frac{1}{2}} \psi,$$

where $m^2 = C_1$. (Note that in the limit $\kappa^{-1} \rightarrow 0$, one recovers the usual Dirac equation.)

Let us expand the κ -Dirac equation, (2.6), in powers of κ^{-1} . The resulting equation, to an error $\approx \kappa^{-2}$, is:

$$(2.7) \quad \left[(\gamma_4 P_0 - \gamma_i P_i) + \frac{1}{2\kappa} (\gamma_4 (P_0^2 - P_i P_i) - m P_0) \right] \psi = m \psi.$$

3 The κ -Dirac-Coulomb Equation

In order to obtain the κ -Dirac-Coulomb equation, it is necessary now to gauge eq. (2.7), thereby introducing the Coulomb potential. In general, gauging of quantum groups is a current research problem [11], but for the $U(1)$ group of electromagnetism and a one-body equation (no co-multiplication) the straightforward gauging $P_\mu \rightarrow P_\mu - e A_\mu$ is quite satisfactory [11]. (Note that in (2.7), unlike eq. (2.6), there are no operator ordering problems after the gauging.)

Accordingly, we gauge (2.7), $P_0 \rightarrow H + \frac{\alpha Z}{r}$, introducing the (attractive) Coulomb potential. We obtain the first order κ -Dirac Coulomb equation:

$$(3.1) \quad H\psi = \left\{ \left(\gamma_4 \gamma_i P_i + m \gamma_4 - \frac{\alpha Z}{r} \right) + \frac{1}{2\kappa} \left(\left(H + \frac{\alpha Z}{r} \right)^2 - P_i P_i - \gamma_4 m \left(H + \frac{\alpha Z}{r} \right) \right) \right\} \psi.$$

To an error of order κ^{-2} , we can identify H in the perturbation terms with the Dirac Hamiltonian H_D ((3.1), with $\kappa^{-1} = 0$). The perturbation conserves angular momentum and parity.

Since the unperturbed (bound-state) Dirac-Coulomb problem has solutions degenerate in the sign of Dirac's quantum number, one would normally apply 2×2 degenerate perturbation theory for an order κ^{-1} calculation of the shifted degenerate energy levels. However, parity is conserved by the perturbation, so that the 2×2 matrix becomes diagonal. The diagonal matrix elements of the perturbation in (3.1) are to be taken between eigenstates of the unperturbed Dirac-Coulomb Hamiltonian. Using the quadratic Dirac Hamiltonian for $P_i P_i$ we can put the perturbation in a more convenient form. Denoting the perturbation in (3.1) (the term multiplied by $1/2\kappa$) by H_{pert} we find: (Dirac's notation)

$$(3.2) \quad H_{\text{pert}} = m^2 - \frac{i\alpha Z}{r^2} \rho_1 \sigma \cdot \hat{r} - E_D m \rho_3 - m \rho_3 \frac{\alpha Z}{r}.$$

To our surprise, matrix elements of (3.2) were found to be *zero*. In fact, the perturbation is *identically zero*. It is easy to show this when (3.2) has been put in the form,

$$(3.3) \quad H_{\text{pert}} = m^2 - \{H_D, m \rho_3\} - m \rho_3 \frac{\alpha Z}{r}.$$

Using the Dirac Hamiltonian (3.3) can be seen to vanish. *The first order correction to the κ -Poincaré-Dirac-Coulomb problem is identically zero.* This is quite remarkable and indicates that the κ -Poincaré quantum group tends to make minimal changes in standard structures.

4 Further Remarks and Conclusion

In an attempt to determine some experimental limit on the length scale κ^{-1} it would be natural now to proceed to the second-order corrections in the κ -Poincaré-Dirac-Coulomb problem. It is easily seen from (2.6) that this procedure must fail, *since the second-order perturbation is singular*. This singularity comes from the term $\frac{1}{6}(\alpha Z/r)^3$ introduced via $\kappa \sinh(P_0/\kappa)$ in (2.6), and cannot be eliminated.

Thus we are at an impasse in this attempt to bound the κ -Poincaré length scale experimentally. There is, however, a rather speculative way to proceed further. One of the motivations underlying the quantum group approach is that the quantum parameter may possibly serve as a convergence factor. The singular terms in the κ -Poincaré-Dirac-Coulomb problem enter through exponentials of the P_0 operator, which suggests that finite (time) displacements occur. Thus it is not unreasonable to hope that the κ -Poincaré length scale may somehow cut-off the singular terms in the Coulomb problem. For the singular second order perturbation the cut-off enters logarithmically, and is thus relatively insensitive to the cut-off. In fact, the ground state second-order (cut-off) energy shift determines a length-scale that is self-consistent (the cut-off scale in the singular integral is of the same order of magnitude as that scale determined by the cut-off energy shifts).

The ground-state second-order (cut-off) energy shift is found to have the order of magnitude:

$$(4.1) \quad \Delta E(1s_{\frac{1}{2}}) \approx \left(\frac{m}{\kappa}\right)^2 \cdot (m\alpha^6).$$

Let us now compare this estimated shift, $\Delta E(1s_{\frac{1}{2}})$, with the observed experimental limits [12] on the accuracy with which the $1s_{\frac{1}{2}}$ eigen-energy is known. A reasonable estimate from the results in ref. 12 is that any deviation from the eigen-energy of the $1s_{\frac{1}{2}}$ state is less than approximately $10^{-3} MHz$. Since the Rydberg, $\frac{1}{2}m\alpha^2$, is $3289 THz$ this implies:

$$(4.2) \quad \left(\frac{m}{\kappa}\right)^2 (m\alpha^6) \approx (m\alpha^2) \times 10^{-12}, \text{ or } \frac{m}{\kappa} \approx 10^{-2}.$$

This implies that the κ -Poincaré length scale is: $\kappa^{-1} \approx 10^{-13}$ cm or smaller.

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THE GENERALIZED ERLANGEN PROGRAM AND A GEOMETRY FOR FOUR-DIMENSIONAL CONFORMAL FIELDS

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Abstract

In the first part we attempt to summarize XXth Century Physics, in the light of Kelvin's 1900 speech "Dark Clouds over XIXth Century Physics". Contrary to what is usually said, Kelvin predicted that the "clouds" (relativity and quantum mechanics) would revolutionize physics and that one hundred years might be needed to harmonize them with classical physics. Quantum Gravity can be considered as a leftover from Kelvin's program – so are the problems with the interpretation of quantum mechanics. At the end of the XXth Century, the Standard Model is the new panoramic synthesis, drawn in gauge-geometric lines – realizing the Erlangen program beyond F. Klein's expectations. The hierarchy problem and the smallness of the cosmological constant are our "clouds", generations and the Higgs sector are to us what radioactivity was in 1900. In the second part we describe Metric-Affine spacetimes. We construct the Noether machinery and provide expressions for the conserved energy and hypermomentum. Superimposing conformal invariance over the affine structure induces the Virasoro-like infinite constraining algebra of diffeomorphisms, applied with constant parameters and opening the possibility of a 4DCFT, similar to 2DCFT.

1 Physics of the 20th Century: A Personal Evaluation

The new results presented in this paper represent joint work by three authors, as listed above; but as the first three sections were mostly a personal reaction of the first-named

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author, who attended the Beersheva conference and is grateful for the hospitality extended by Ben Gurion University and was drawn into making these remarks by the challenging questions posed by R. Sen, the organizer, in his convocation document – that author (YN) should be solely blamed for any errors, bad judgement or pretentiousness in their presentation. This is also the reason for the compound title.

In this *Fin de Siècle* spirit, the mood may well border on the apocalyptic. As you will see, we tend to steer away from such millenary fever. Whatever the approach, however, it is instructive to go back to April 1900 and read the famous (and much maligned) lecture delivered by William Thomson, first Baron Kelvin (and indeed better known by the peerage title he chose, i.e., Lord Kelvin) at the meeting of the Royal Institution of Great Britain (an organisation similar to the AAAS in the USA) [1]. The title was “**Nineteenth Century Clouds over the Dynamical Theory of Heat and Light**”; notice the emphasis on the clouds rather than on the harmony of the whole, which these clouds were marring (Kelvin is generally presented as having taken the opposite view and of pooh-poohing the “clouds”):

...The beauty and clearness of the dynamical theory, which asserts heat and light to be modes of motion, is at present obscured by two clouds. The first came into existence with the undulatory theory of light... it involved the question ‘How could the Earth move through an elastic solid, such as essentially is the luminiferous ether?’ [It was supposed to carry transverse oscillations – YN]. The second is the Maxwell-Boltzmann current doctrine regarding the partition of energy... a cloud that has obscured the brilliance of the molecular theory of heat and light during the last quarter of the Nineteenth Century.

Kelvin is referring to the Michelson-Morley experiment (behind the null results of which the Special Theory of Relativity was lurking) and to the specific heat of the solids, experimentally contradicting the equipartition theorem (an issue to be resolved by Quantum Theory). Kelvin, a great believer in the ether, had used much imagination in inventing for it structures that could be reconciled with the Michelson-Morley experiment. He had worked very hard and yet had failed to find a way of making the classical theory compatible with these two paradoxical results. He continues:

And here, I am afraid, I must end by saying that the difficulties are so great in forming anything like a comprehensive theory, that we cannot even imagine a finger-post pointing a way that can lead us towards an explanation. I only say we cannot **now** imagine it. But this time next year – this time ten years – **this time one hundred years** – probably it will be just as easy as we think it is to understand that glass of water, which now seems so plain and simple. I cannot doubt but that... **we shall learn to look on things in a different way** – when that which is now a difficulty will be the only commonsense and intelligible way of looking at the subject.

The usual rendering of this talk describes Kelvin as claiming that the entirety of physics was now described by the grand synthesis of Maxwell’s electrodynamics and of thermodynamics with the kinetic theory of gases, with only two minor difficulties still marring

the view, but soon to be resolved and integrated in that grand scheme. Instead, we find Kelvin convinced of the impossibility of integrating these two persistent difficulties within the 1900 picture, estimating that harmony will require “learning to look on things in a different way” and that it could take anything between one... to one hundred years to achieve that harmony!

Indeed, within a year, there was Planck’s quantization as an ad-hoc hypothesis; within five years there was Einstein creating Special Relativity and inventing the photon. Note that four years earlier, Becquerel had first hit upon the unknown world of the nucleus – the facts apparently still appearing vague enough, to the point of escaping Kelvin’s attention; but within eleven years after the speech, Rutherford was discovering the nucleus and presenting the explicit challenge of a new layer of matter with its interactions – to be included in the next grand synthesis!

The story has indeed unfurled at the rate Kelvin estimated. Within fifteen years Einstein ushered in General Relativity, needed for the harmonization of Special Relativity with Newtonian physics. Within twenty-five years, Heisenberg, Born and Schrödinger had replaced the Planck-Einstein ad-hoc theory by Quantum Mechanics; so much for the clarification of Kelvin’s two “clouds” and for the understanding of their relationship to the classical picture of the 19th Century. Note, though, that it is only in the last few years that one has been able to go beyond the residual ad-hoc elements in the “Copenhagen interpretation” of Quantum Mechanics, with a first glimpse of how the classical picture could emerge as a decoherence limit in a fundamentally “quantum” world (see the work of Zeh, Joos, Zureck, Hartle, etc.). Note, however, that this program is just as incomplete as statistical irreversibility: only by having the number of decohering scatterings “reach” infinity, do we arrive at a classical apparatus – just as the Second Law does not generate true irreversibility without having the number of particles “reach” infinity. This is a mystery that we inherited from Boltzmann and appear to hand over “doubled” to the next generations.

But what about marrying the two new theories, in order to attain the new grand synthesis as expected by Kelvin? Forty-eight years after the speech, Tomonaga, Schwinger, Feynman and Dyson indeed managed to reconcile Quantum Mechanics with Special Relativity, but only for the case of QED. There were now new forces that had also to be taken care of. Together with Gell-Mann and Zweig, YN had had the privilege in 1960-63 of identifying a deeper layer in the underlying structure of nuclear matter (quarks), the level at which the phenomena ushered in by Becquerel and Rutherford seem to originate. Yang and Mills provided the foundations for the new mechanism in 1953, Veltman and ’t Hooft welded it to Special Relativity in 1970-71 and the way was open for a new synthesis. Seventy-five years from that speech, the work of ’t Hooft, Gross, Wilzcek and Politzer, of Weinberg, Fritzsche and Gell-Mann (with previous installments in the sixties by Nambu and YN himself) now led to the formulation of QCD for the description of 90% of what had been defined as the Strong nuclear interaction; Weinberg, Salam and Glashow had constructed the Electroweak Hamiltonian, experimentally corroborated in 1978-82. The “**Standard Model**” was born, the first presentation of a new grand synthesis.

2 Quantum Gravity as the Missing Piece: Kelvin's Program for the 20th Century

Yet twenty years later, very close to the 21st Century, even the task defined by Kelvin has not yet been accomplished: **Quantum Gravity** still eludes us! The marriage of Quantum Mechanics with Relativity, the final merger and integration of the two "dark clouds", has not been accomplished. Note that it is precisely a conceptual gap which is lurking, as long as we have no theory of Quantum Gravity: Leon Rosenfeld had suggested in the fifties that General Relativity might be left as a classical theory, thus just providing the "arena" for quantum physics (a view which would have given the Copenhagen interpretation a "fundamental" status). However, it was immediately shown that one could then have easily violated and neutralized Heisenberg's Uncertainty Relations, by using measurements based on Gravity. **This is thus the unresolved leftover from Kelvin's 1900 "project", even after almost one hundred years!** If we do not manage to formulate the correct theory of Quantum Gravity by the year 2000, we shall have failed in clearing up the "Nineteenth Century Clouds" of Kelvin's title, humbly passing on the task to the 21st...

What is the situation with Quantum Gravity? First, we know for sure that Einstein's Lagrangian is not (perturbatively) renormalizable. 't Hooft and Veltman had evaluated in 1974 [2] the gravitational one-loop amplitude in the vacuum and had found that it is finite. There was therefore a suspicion (or should we say a hope?) that even though dimensional arguments made it most probable that the theory would diverge, some miraculous hidden symmetry might yet cancel the infinities, as had happened for the one-loop case. However, the calculation of the two-loop diagram was very complicated. It was thus only ten years later, when Goroff and Sagnotti [3] had access to a supercomputer and evaluate the two-loop diagram, that we could have an answer: There is no miracle and the end result is plain infinity.

There are two possible paths one might follow:

- a) Keep Einstein's Lagrangian, but use a non-perturbative approach (and prove that the theory has finite quantum solutions); this is the approach followed by Ashtekar [4], for example, who will speak about it at this meeting; as a matter of fact, his method is the great hope for this approach at this moment and the candidate solutions [5] appear to be very interesting – but there is as yet no indication with respect to finiteness. As to these "loop space" solutions for the quantum Hamiltonian constraints equations for both Maxwell and Einstein theories, they appear to vindicate Kelvin's vortex atoms, made of "knots tied in the vortex lines of the aether". An important advantage of Ashtekar's Hamiltonian formulation is that with his variables, General Relativity acquires the structure of a complex Yang-Mills theory [6] of gauged (or local) translations.
- b) Modify the Lagrangian, adding new short range contributions so that Einstein's Lagrangian would reemerge as the long range (i.e., larger than Planck length) effective contribution. Just adding terms quadratic in the curvature already makes the theory renormalizable [7], but it is then not unitary. The results we are presenting in the next sections of this paper are partly motivated by an alternative modification, involving Metric-Affine Gravity [8]. Spacetime is assumed to

be non-Riemannian underneath Planck dimensions (or above Planck energies); a Goldstone-Higgs mechanism [9] breaks the $SL(4, \mathbb{R})$ local gauge symmetry, yielding Einstein's Riemannian theory as the low-energy effective theory. We have shown that this model is also renormalizable [9], but we have not yet clarified whether or not it is unitary. Supergravity [10], especially the $N = 8$ version, was another such possible route; the result however has only provided an improvement in the renormalizability features (less divergences) but not a cure. At the end, we may have to start with both a non-Riemannian (perhaps Metric-Affine) geometry and supersymmetry (which would then be far more complicated) or alternatively conformal symmetry (this relates to the ideas presented in the next section).

Another massive attempt to resolve the riddle of Quantum Gravity has been the Quantum Superstring (QSS) [11]. It does provide finite gravitational amplitudes – although the perturbation series yields a divergent sum, most probably just indicating the existence of non-perturbative solutions. Physics (i.e., the true, unlinearized theory) requires the embedding manifold (“target space” is the professional appellation) to be curved; it can then be regarded as the equivalent of the Einstein Lagrangian with all higher order counterterms, plus a dilaton and an axial-vector interaction, etc. Above Planck mass, it goes over to a symmetry limit corresponding to having the diffeomorphisms in the 2-dimensional string world sheet turning into an active (global) symmetry [12]. However, it requires 10 (or 26) dimensions. Although the vanishing of anomalies severely constrains the internal (gauged) symmetry to being either $E(8) \times E(8)$ or $\text{Spin}(32)$ – at the level of the 10 or 26 dimensions – the further reduction to 4-dimensional spacetime spoils it all – in the sense of introducing almost total freedom. Since Gravity does not stand by itself here – this was the additional appeal of the QSS, the “Theory of Everything” (TOE) – we have to know what precisely happens to the “folded up” dimensions. Since “anything goes”, the TOE loses its predictive power – including a certain arbitrariness in the related Quantum Gravity sector.

Physicists who persist in working on the theory (as a TOE) cherish the hope of developing a String Field Theory. This goes presumably beyond the concept of locality – or generalizes it in a peculiar way – so dear to most participants in this meeting, which emphasizes the achievements of Axiomatic Quantum Field Theory. In String Quantum Field Theory, one would write an equation for the Superstring Functional $\Psi[g_{\alpha\beta}(\zeta^\gamma), X^\mu(\zeta^\gamma)]$ in which the argument X^μ is the embedding coordinate, which is itself a local Quantum Field over the ζ^γ coordinates of the string's world sheet (or of the world hypersheet for a higher dimensional “extendon” – see next paragraph). The lowest energy solution for Ψ should select the true vacuum [13]. This, however, still appears as a distant dream. Meanwhile, another spin-off relates to **non-critical strings**, i.e., not in the dimensions removing the conformal anomalies; as a result, the 2-dimensional sheet metric $g_{\alpha\beta}(\zeta^\gamma)$ preserves one field component, namely 2-dimensional gravity. The embedding manifold is taken care of by collapsing it to 0 or 1 dimension, treating it as a set of internal degrees of freedom. Non-critical strings have thus been used as a theoretical laboratory for quantum gravity, providing us for the first time with the opportunity to look at “almost physical” (only 2 dimensions!) black hole-dilaton [14] quantum solutions of a gravitational theory [15].

There is also another departure stemming from the work on the string. Strings were the first extended structures to be studied as elementary quantum objects, i.e.,

one-dimensional “extendons”. They contain this spin 2 massless state and therefore caused all this excitement. However, maybe this is a red herring; should we go on to higher-dimensional extendons [16]? This route has begun to be explored; one model happens to represent a quantum membrane whose low energy truncation is 11-dimensional Supergravity [10] (or $N = 8$ in $D = 4$) which had generated such expectations in 1976-82.

Note that I have dealt here with the operational issues and have not touched upon “technical” problems of Quantum Field Theory, such as the adaptation of the local formalism constructed for Minkowski space – to either just curved spacetime or even to non-Riemannian manifolds. Again, should the answer come from the string, we shall have to deal with third quantization, etc., and also ensure that the truncation underneath Planck energy yields a local (massless) field theory. However, we are not yet there.

3 New Physics for the 21st Century

Quantum Gravity represents unfinished business from the legacy of the two Nineteenth Century clouds. What about new, Twentieth Century, fresh ones? Here are two such “clouds”, plus what is perhaps the tip of an iceberg, somewhat like radioactivity was in 1900.

The two clouds may, as a matter of fact, also relate to Quantum Gravity; if they end up being resolved by it, it will mean that they too are still part of cleaning up the Nineteenth Century sky. The (perhaps new) clouds are known as the **Hierarchy Problem** and the **Mystery of the Cosmological Constant**. The “iceberg” is what is known as the **Higgs Sector** and which YN called, back in 1964, the **Fifth Interaction** [17]; it may also relate to the problem of the “Generations” and of the hypothetical constituents of quarks and leptons, as we shall see.

Note *en passant* that we still have the unfinished business of GUTs (Grand Unified Theories), after the experimental failure of (at least the direct application of) $SU(5)$ – no proton decay at the 10^{31} years’ lifetime level. There is no doubt that we need an algebraically constraining structure, to explain the cancellation of the chiral anomalies between electrically charged leptons and quarks. The candidates are other forms of $SU(5)$, or instead, $SO(10)$ or $E(8)$ – the latter fits with a superstring TOE. YN and collaborators – Thierry-Mieg and Sternberg, and independently Fairlie, have suggested a gauged internal superalgebra [18] as a possible alternative to a GUT. This would be $SU(2)_L \times U(1)_W \subset SU(2/1) \subset SU(7/1)$. The geometric features are interesting, coincide with Quillen’s theory of the “superconnection” and are related to Connes’ Non-commutative Geometry [19].

The hierarchy issue derives from the renormalization features of scalar fields. These enter into present theories as the spontaneous symmetry breakdown (Higgs) fields. Their interaction with Gravity – or alternatively their interaction with the gauge fields in GUTs – all seem to indicate unstable masses, or masses tending to inflate via radiative corrections [20]. This is catastrophic in a theory that has more than one spontaneous symmetry breakdown, e.g., one at 10^{16} GeV to separate QCD from $SU(2)_L \times U(1)_W$ and one at 100 GeV to break the Electroweak gauge. The lower energy Higgs fields tend to acquire masses at the 10^{16} GeV level too, within a GUT, and as a matter of fact, all

Higgs fields appear to climb to Planck energies because of gravity. The answers that have been suggested are either overall Minkowski **Supersymmetry** (with the partner particles in the 500-1000 GeV region), or **Technicolor**, i.e. a new type of matter, with “**techniquarks**” making up the Higgs mesons as “technipions”, glued by “technigluons”. Supersymmetry would simply extend to bosons the zero-mass condition imposed by Chiral Symmetry on fermions, and 100-1000 GeV represent a good approximation to zero, on a 10^{16} GeV scale. Both solutions would exhibit some samples of new particles around 500-1000 GeV. The answer thus awaits the results of future experiments at the SSC or LHC. Let me express the hope that these machines will indeed be built in this last decade of the century.

Let us now turn to the **cosmological constant**. The present experimental bounds put its density ρ_Λ [21] at $-2\rho_{\text{obs}} < \rho_\Lambda < 4\rho_{\text{obs}}$, where $\rho_{\text{obs}} \simeq 10^{-29} \text{ g/cm}^3$ is the observed mean energy density of the universe. On the other hand, as the Planck values represent the only numerical input in the interactions with gravity, it is not surprising that the calculation of radiative corrections predict for the density of the cosmological constant ($\hat{=}$ the vacuum energy) a value of the order of Planck density,

$$\rho_\Lambda \simeq \rho_{\text{Planck}}, \quad \rho_{\text{Planck}} = \frac{10^{-5} \text{ g}}{(10^{-33} \text{ cm})^3} \simeq 10^{94} \text{ g/cm}^{-3}.$$

There is thus a discrepancy of 123 orders of magnitude between theoretical and experimental values. . . Coleman and others have conceived of various mechanisms [22] that could “make it easier” to generate a ‘baby universe’ than carry a large Λ . (This is in the spirit of color confinement, where emitting a meson serves as an emergency drainage for extra energy in a hadron, thus evading the emission of a quark or other color-carrying particles). Note that as far as the phenomenology goes, it is possible that the experimental value of Λ will end up being around the present observational upper bound and that this vacuum energy will thus resolve the missing-mass problem in cosmology (to which we have as yet not given the status of a “Kelvin cloud” – perhaps we should have. . .). This would still allow the energy density in the first stages of the Big Bang to be very much larger than the cosmological vacuum energy density (which is constant in time, whereas the energy density decreases very fast with time). Returning to the theoretical versus experimental Λ dilemma, there are problems with the “baby universe” solution and the situation is at best confusing, at worst highly paradoxical – a perfect Kelvin cloud.

We now deal with the “iceberg”. Several of the participants at this meeting are friends who were already “in the field” thirty years ago. They will remember that when Gell-Mann and YN identified $SU(3)$ (the “Eightfold Way”) as the group classifying the hadrons, with highly successful perturbatively derived mass formulae for masses (including both Gell-Mann/Okubo and Coleman/Glashow, etc.), magnetic moments, weak amplitudes, etc. – and roughly symmetric (only slightly broken) strong-interactions amplitudes, Oakes and Yang [23] raised a pertinent question: How can a strong-interaction symmetry be broken and yet yield successful perturbative results? With unitarity allowing us to cut anywhere and insert a complete set of intermediate states, we can have the strong breaking multiplying itself and propagating all over the group manifold. A year later, YN [17] offered an answer, which is the one that was adopted twenty years later, with the advent of QCD. The answer was that the traditional “**Strong Interaction**” should be split into two interactions, a non-perturbative one (QCD

in the 1973 Standard Model), generating the main characteristics (Regge trajectories, etc. . .) and a somewhat weaker force, admitting a perturbative description, which YN called “The Fifth Interaction”¹, responsible for $SU(3)$ -breaking (the mass of the s quark), for the muon mass, etc. It thus couples to “flavors” such as strangeness, whereas QCD couples to color. The Fifth has not yet been identified at all, and it could well be that it should couple at the level of the hypothetical constituents of quarks and leptons – for instance through strangeness and muon number representing the presence of a specific charge at that level, with zero value for the first generation fields (u, d, e, ν_e), one for the second (c, s, μ, ν_μ) and two for the third (t, b, τ, ν_τ).

A segregation of $SU(3)$ breaking has thus indeed been adopted, which is why YN described QCD as taking care of 90% of the Strong Interactions only. In the present picture, all quark and lepton masses are inputs – at most they are “blamed” on Yukawa interactions with scalar fields, generally one interaction for each mass. Clearly, this is just a way of hiding our ignorance. In addition, there are the masses of the Higgs-Goldstone fields involved in breaking $SU(2)_L \times U(1)_W$ and perhaps of yet another Higgs set, at 10^{16} GeV. Together with the Cabibbo angle and other parameters of the Kobayashi-Maskawa mixing matrix, the Fifth has to explain some 15–20 parameters.

Another unfinished business is represented by the mystery of the three (or four, the last one with a very massive neutrino) **generations** – perhaps in itself an indication of the existence of a further structure layer with **rishons, quinks, preons**, . . . as constituents. There would then also have to be new forces operating at the new level. This is on top of the “GUT” group or supergroup containing the Standard Model symmetries in an irreducible constraining structure, which we mentioned earlier.

As you see, the 21st Century physicists have quite a job in store. Whether or not in the future (do not expect this to happen before another fifty years) there will be a “Final Theory” [24] is, in our opinion an open question. Chew thought we had reached it in the Sixties (the Bootstrap) and there was nothing that could be said against the idea from first principles, it had to be tried out . . . Hawking’s inaugural lecture at Cambridge in 1979 was entitled “The End of Physics” – he thought Supergravity and the GUT would provide all the answers. Weinberg’s recent book “Dreams of a Final Theory” is based on the hope that the TeV accelerators will do it, but he stresses that this is only a possibility, with the alternatives involving a Pandora’s box at 0.5–1 TeV. This would be just as interesting may be more so, in our view.

So much for the 21st Century . . . Let us now return to our immediate worries.

4 Metric-affine Spacetime: Potentials and Field Strengths

We now move on to the gravitational theory I mentioned, in explaining that Quantum Gravity program in which one starts with a Post-Riemannian Theory which then reduces to Einstein’s for distances much larger than the Planck length (10^{-33} cm) and

¹The appellation “Fifth Interaction” was plagiarized in the Seventies, when for a while it looked as if there is an additional medium range component to Gravity. This appears to have been yet another red herring.

for energies below Planck energy (10^{19} GeV). We have hypothesized [8] a Metric-Affine spacetime, i.e., one in which the homogeneous local symmetry is $\overline{GL}(4, \mathbb{R})$, instead of the Lorentz $\overline{SO}(1, 3)$, the inhomogeneous local symmetry thus being given by the Affine Group $\overline{A}(4, \mathbb{R})$, instead of the Poincaré group (the overline denotes [25] double-coverings, $\overline{SO}(1, 3) = SL(2, \mathbb{C})$, etc.). Should we find that scale invariance is not essential at the primordial stage, we would stay with the unimodular versions $\overline{SL}(4, \mathbb{R})$ and $\overline{SA}(4, \mathbb{R})$.

We start with differential forms [8] in which the α, β indices represent anholonomic ($\hat{=}$ tangent space) variables, carrying the action of anholonomic $\overline{GL}(4, \mathbb{R})$:

(4.1)

Potential	deg. p	p - form	$A(4, \mathbb{R})$ variation	Field strength	$(p+1)$ - form	Bianchi identity
Metric	0	$g_{\alpha\beta}$	$2\epsilon_{(\alpha\beta)}$	nonmetricity	$Q_{\alpha\beta} = -Dg_{\alpha\beta}$	$DQ_{\alpha\beta} = 2R_{(\alpha\beta)}$
Coframe	1	ϑ^α	$-\epsilon_\beta^\alpha \vartheta^\beta$	torsion	$T^\alpha = D\vartheta^\alpha$	$DT^\alpha = R_\gamma^\alpha \wedge \vartheta^\gamma$
Connection	1	Γ_α^β	$D\epsilon_\alpha^\beta$	generalized curvature $\rightarrow$ both $(\alpha\beta) \& [\alpha\beta]$	$R_\alpha^\beta = d\Gamma_\alpha^\beta + \Gamma_\gamma^\beta \wedge \Gamma_\alpha^\gamma$	$DR_\alpha^\beta = 0$

The matter fields are [25] spinorial manifolds Ψ , with given density character w . In the anholonomic presentation, they can be multiplicity-free, e.g., based on $[(\frac{1}{2}, 0) + (0, \frac{1}{2})]$ and with a highly restrictive field equation [26]. Upon symmetry breakdown to a Riemannian manifold, in the flat tangent space, they reduce to an infinite sum of Lorentz spinors $\frac{1}{2} + \frac{5}{2} + \frac{9}{2} + \dots$. Going over to a holonomic picture [27] they cannot be multiplicity free any longer [26, 28, 29].

The action of the $\overline{GL}(4, \mathbb{R})$ gauge-covariant derivative can be split in two irreducible pieces: that of the traceless $\overline{SL}(4, \mathbb{R})$ (4-volume preserving) with connection ${}^\dagger\Gamma$ and a rescaling piece involving the trace, represented by the Weyl one-form $Q := (1/4)Q^\alpha_\alpha$:

$$(4.2) \quad D\Psi = d\Psi + {}^\dagger\Gamma_\alpha^\beta \wedge \rho(L^\alpha_\beta)\Psi + \frac{1}{2}wQ \wedge \Psi.$$

For a given Ψ , the generators L_α^β of $\overline{SL}(4, \mathbb{R})$ act on the fields via their representation $\rho(L_\alpha^\beta)$ and $w = \rho(L_\gamma^\gamma)$. Note that although $Q_{\alpha\beta}$ is listed in (4.1) as a field strength, it plays the role of a Yang-Mills like potential here, because of an identity relating $\Gamma_{\alpha\beta}$ to metric (the Riemannian piece, given by the Christoffel formula), torsion and nonmetricity [30].

5 Framework of Metric-Affine Dynamics: Noether Currents and Identities

In a first order formalism we assume that the material Lagrangian 4-form depends generally on Ψ , its gradient $d\Psi$, on the potentials of (4.1) and on their gradients or the relevant field strengths. Minimal coupling would not require the last two, but Pauli-type terms might arise through some additional constraint or at the symmetry breaking stage; also, we may encounter such terms as a result of the redundancy in “potential” p -forms in our list (4.1): $10 + 4 + 16 = 30$, as against e.g., $16 + 4 = 20$ parameters of $\overline{A}(4, \mathbb{R})$. As a matter of fact we know a priori that the status of the metric in a Metric-Affine theory is somewhat contrived. We thus write

$$(5.1) \quad L = L(g_{\alpha\beta} dg_{\alpha\beta}, \vartheta^\alpha, d\vartheta^\alpha, \Gamma_\alpha{}^\beta, d\Gamma_\alpha{}^\beta, \Psi, d\Psi).$$

We define the variational derivative with respect to a p -form — which is identical with a covariant variational derivative,

$$(5.2) \quad \frac{\delta L}{\delta \Psi} := \frac{\partial L}{\partial \Psi} - (-1)^p d \frac{\partial L}{\partial (d\Psi)} = \frac{\partial L}{\partial \Psi} - (-1)^p D \frac{\partial L}{\partial (D\Psi)}.$$

The matter currents are the metrical stress-energy $\sigma^{\alpha\beta}$, the canonical energy-momentum Σ_α , and the hypermomentum Δ_β^α , which is asymmetric in α and β . They are defined by

$$(5.3a) \quad \sigma^{\alpha\beta} := 2 \frac{\delta L}{\delta g_{\alpha\beta}} = 2 \frac{\partial L}{\partial g_{\alpha\beta}} + 2 D \frac{\partial L}{\partial Q_{\alpha\beta}},$$

$$(5.3b) \quad \Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha} = \frac{\partial L}{\partial \vartheta^\alpha} + D \frac{\partial L}{\partial T^\alpha}$$

$$(5.4) \quad \Delta^\alpha{}_\beta := \frac{\delta L}{\delta \Gamma_\alpha{}^\beta} = \rho(L^\alpha{}_\beta) \Psi \wedge \frac{\partial L}{\partial (D\Psi)} + 2g_{\beta\gamma} \frac{\partial L}{\partial Q_{\alpha\gamma}} + \vartheta^\alpha \wedge \frac{\partial L}{\partial T^\beta} + D \frac{\partial L}{\partial R_\alpha{}^\beta}$$

What is meant by gauging the translations is somewhat ambivalent [31]. We have introduced the gauged $\overline{A}(4, \mathbb{R})$ and shall return to it; but we can also explicitly evaluate the variation of the Lagrangian under L_ξ , a Lie-derivative, i.e., a local anholonomized analytical diffeomorphism, also yielding a “translation current” — since local diffeomorphisms are identical to a local gauging of translations, except for the non Lie algebra aspects of Lie derivatives. We obtain the canonical energy-momentum current [32].

$$(5.3c) \quad \Sigma_\alpha = e_\alpha \rfloor L - (e_\alpha \rfloor D\Psi) \wedge \frac{\partial L}{\partial D\Psi} - (e_\alpha \rfloor \Psi) \wedge \frac{\partial L}{\partial \Psi} \\ - (e_\alpha \rfloor Q_{\beta\gamma}) \frac{\partial L}{\partial Q_{\beta\gamma}} - (e_\alpha \rfloor T^\beta) \wedge \frac{\partial L}{\partial T^\beta} + D \frac{\partial L}{\partial T^\alpha} - (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \frac{\partial L}{\partial R_\beta{}^\gamma}$$

where $\rfloor$ implies “interior” product ($\hat{=}$ contracting a 1-form with a “vector-field”, i.e., an anti-derivative operator). The expression in the first line of (5.3c) survives the

transition to Special Relativity and, quite generally, loses its third contribution for a 0-form Ψ . This transition to Special Relativity can be accomplished by successively enforcing $Q_{\beta\gamma} = 0$, $T^\beta = 0$, $R_\beta{}^\gamma = 0$ via Lagrange multipliers.

Obtaining Noether identities for curved manifolds corresponds to evaluating the effect of local gauging. Here we gauge $\overline{A}(4, \mathbb{R}) = \mathbb{R}^4 \ltimes \overline{\text{GL}}(\mathbb{R})$, where gauging $\mathbb{R}^4$ is equivalent to applying $\text{diff}(4, \mathbb{R})$ as explained. We find

$$(5.5) \quad \begin{aligned} D\Sigma_\alpha &\equiv (e_\alpha]T^\beta) \wedge \Sigma_\beta + (e_\alpha]R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma - \frac{1}{2}(e_\alpha]Q_{\beta\gamma})\sigma^{\beta\gamma} + \\ &+ (e_\alpha]D\Psi)\frac{\delta L}{\delta\Psi} + (-1)^p(e_\alpha]\Psi) \wedge D\frac{\delta L}{\delta\Psi} + \\ &\simeq (e_\alpha]T^\beta) \wedge \Sigma_\beta + (e_\alpha]R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma - \frac{1}{2}(e_\alpha]Q_{\beta\gamma})\sigma^{\beta\gamma} \end{aligned}$$

$$(5.6a) \quad D\Delta^\alpha{}_\beta + \vartheta^\alpha \wedge \Sigma_\beta - g_{\beta\gamma} \sigma^{\alpha\gamma} \equiv -\rho(L^\alpha{}_\beta)\Psi \wedge \frac{\delta L}{\delta\Psi} \simeq 0.$$

where $\equiv$ and $\simeq$ respectively denote “strong” and “weak” [$\hat{=}$ on mass shell, here requiring satisfaction of the matter field equations $\frac{\delta L}{\delta\Psi} = 0$]. Taking the trace in (5.6a) yields the scale (dilation) identity,

$$(5.6b) \quad D\Delta + \vartheta^\alpha \wedge \Sigma_\alpha - \sigma^\alpha{}_\alpha \equiv -\rho(L^\gamma{}_\gamma)\Psi \wedge \frac{\delta L}{\delta\Psi} \simeq 0.$$

we observe the role of the traces in $\vartheta^\alpha \wedge \Sigma_\alpha$ and $\sigma^\alpha{}_\alpha$, which becomes important when, e.g., for renormalizability purposes, one has to construct “improved” expressions [33]. Moreover, if we define a metrical energy-momentum 3-form

$$(5.7a) \quad \sigma_\alpha := e_\beta] \sigma_\alpha{}^\beta,$$

the 2nd Noether identity (5.6a) allows us to rewrite it quite generally as

$$(5.7b) \quad \sigma_\beta \equiv \Sigma_\beta + e_\alpha] D\Delta^\alpha{}_\beta + \rho(L^\alpha{}_\beta) e_\alpha](\Psi \wedge \frac{\delta L}{\delta\Psi}).$$

Imposing invariance of metric and connection under Lie derivatives we use the generalized Killing equation for vector fields,

$$(5.7c) \quad \mathcal{L}_\xi g = 0, \quad \mathcal{L}_\xi \Gamma_\alpha{}^\beta = 0$$

We define a new current 3-form

$$(5.8a) \quad \epsilon_{MA} := \xi^\alpha \Sigma_\alpha + (e_\beta] \widehat{D}\xi^\gamma) \Delta^\beta{}_\gamma,$$

where

$$(5.8b) \quad \widehat{D} = d + \widehat{\Gamma}, \quad \widehat{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha] T^\beta$$

which is “weakly” conserved,

$$(5.8c) \quad d\epsilon_{MA} \simeq 0.$$

6 Superimposing Conformal Invariance

We now go beyond the general Metric-Affine spacetimes and impose in addition conformal symmetry. We introduce a conformal Killing vector ζ , so that (5.7) is replaced by

$$(6.1a) \quad \mathcal{L}_\zeta g = \omega g, \quad \mathcal{L}_\zeta \Gamma_\alpha^\beta = \frac{1}{2} \delta_\alpha^\beta d\omega$$

with the scaling transformations now involving the metrical weight λ ,

$$g \rightarrow g' = \exp(\lambda\omega) g = \Omega^\lambda g$$

where the conformal function $\Omega = \exp\{\omega(x)\}$ is determined by

$$(6.1b) \quad \omega = \frac{1}{2} e_\gamma \rfloor \widehat{D}\zeta^\gamma - \zeta \rfloor Q$$

which is obtained [31] from (6.1a). This modifies (5.8a) which is replaced by

$$(6.2) \quad d\left[\epsilon_{MA} - \frac{1}{2}\omega\Delta\right] \simeq \frac{1}{2}\omega(\vartheta^\alpha \wedge \Sigma_\alpha)$$

which becomes an exact conservation law in the case of traceless $\vartheta^\alpha \wedge \dagger \Sigma_\alpha = 0$.

We can also rewrite the expression in the square bracket in (6.2) while separating the traceless pieces $\dagger \Sigma_\alpha$ and $\dagger \Delta^\alpha_\beta$ from the traces in Σ_α and Δ^α_β . Denoting it by ϵ_C we obtain [32],

$$(6.3) \quad \epsilon_C = \epsilon_{MA} - \frac{1}{2}\omega\Delta = \zeta^\alpha \dagger \Sigma_\alpha + (e_\beta \rfloor \widehat{D}\zeta^\gamma) \dagger \Delta^\beta_\gamma + \frac{1}{4}\zeta \rfloor (\vartheta^\alpha \wedge \Sigma_\alpha) + \frac{1}{2}(\zeta \rfloor Q)\Delta.$$

Alternately, using the “strong” form (5.6b) of the local-gauge Noether identities [34], we have been able [8] to derive, for scalar fields, a “strongly improved” version, this time for the metrical energy-momentum tensor current, for scalar fields,

$$(6.4) \quad \vartheta^\alpha \wedge \dagger \sigma_\alpha = \vartheta^\alpha \wedge \sigma_\alpha + 3 D\Delta = \left[4 U(|\sigma|) - \sigma \frac{\partial U(|\sigma|)}{\partial \sigma} \right] \eta,$$

where we use the “improved”

$$(6.4a) \quad \dagger \sigma_\alpha := \sigma_\alpha + \frac{3}{4} e_\alpha \rfloor D\Delta$$

in terms of (5.3a) and (5.7a). The potential $U(|\sigma|)$ is a polynomial type, and η is the volume 4-form.

Imposing conformal invariance on a Metric-Affine spacetime is, however, an extremely strong condition. To realize why this is so, we turn to the flat spacetime “orbital” representation of the relevant Noether currents and will use a component notation. For the shears in $sl(4, \mathbb{R})$, we have the currents (underlining denotes the flat case),

$$(6.5a) \quad \underline{\dagger \Delta}^{\alpha(\beta\gamma)} = x^{(\beta} \underline{\tilde{t}}^{\gamma)\alpha} - \frac{1}{4} g^{\beta\gamma} x^\lambda \underline{\tilde{t}}_\lambda^\alpha$$

for the dilations

$$(6.5b) \quad \underline{\Delta}^\alpha = x^\gamma \tilde{t}_\gamma^\alpha,$$

and for the proper conformal currents

$$(6.5c) \quad \underline{K}^{\alpha\beta} = (2 x^\beta x^\gamma - g^{\beta\gamma} x^2) \tilde{t}_\gamma^\alpha.$$

Here the “improved” $\tilde{t}_\gamma^\alpha$ turns out to be the Hodge dual of the improved version of the metric energy-momentum 4-form (5.3a),

$$(6.5d) \quad \tilde{t}_\alpha^\gamma := {}^*\dagger\sigma_\alpha^\gamma, \quad \text{with} \quad \dagger\sigma_\alpha^\gamma := \vartheta^\gamma \wedge \dagger\sigma_\alpha.$$

Using this geometrical defining representation, Ogievetsky [35] proved for the Lie algebraic generators (i.e., the $\int d^3x$ integrals) of (6.5), under recurring commutation,

$$(6.6) \quad \left[\int \eta_\alpha \dagger \underline{\Delta}^{\alpha(\beta\gamma)}, \int \eta_\mu \underline{K}^{\mu\nu} \right] \dots = \text{diff}(4, \mathbb{R})$$

(η_α is the hypersurface 3-form) i.e., the commutation relations never close and generate instead the infinite algebra of analytical diffeomorphisms, with generators,

$$(6.7) \quad L_\beta \langle p^0 \dots p^3 \rangle_{\mu\nu\rho\dots\kappa} = \int \prod_{i=0}^3 (x^i)^{p_i} \tilde{t}_\beta^\alpha \eta_\alpha$$

The algebraic relations are preserved anholonomically. Thus, a metric-affine space-time which admits a conformal symmetry will have its frames locally invariant (in the **active** operational sense) under the group of analytical diffeomorphisms. This result overlaps with the fact that we have included in our affine gauge approach local translations, i.e., active diffeomorphisms, except that whereas the latter are only infinitesimal (their generators do not form a Lie algebra anyhow), the Ogievetsky transformations can be integrated to finite diffeomorphisms, without involving an infinite-component connection, since the algebraic generators are multiplied by constant parameters solely: the local dependence has already been taken care of through the generating Taylor expansion itself.

The emergence of an explicit infinite-dimensional Lie algebra may make it possible to treat conformal fields in 4 dimensions similarly to what is done in the special case of two dimensions. In $n = 2$, there is an infinite-dimensional conformal algebra which is isomorphic to the algebra of analytic two-dimensional diffeomorphisms [36]. In two dimensions, this feature constrains the fields and leads to the highly restrictive “fusion” rules [36], which have recently put 2-dimensional conformal field theory into the focus of interest of statistical mechanics and string theory. Note that the Ogievetsky algebra in 4 dimensions should possess a quantum extension with central charges as in the case of its 2-dimensional analog, the Virasoro algebra. Neither this extension nor the representation theory have been investigated to date.

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THE EMERGENCE OF GAUGE THEORY

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Introduction

Although future historians of Physics will remember the first part of the twentieth for the emergence of Relativity and Quantum Mechanics they may well remember the rest of the century for the emergence of gauge theory. Indeed they may argue that the emergence of gauge-theory as the principle behind the fundamental forces was the more fundamental discovery of the century on the grounds that it completely changed our conception of *dynamics* whereas Quantum Mechanics and Special Relativity changed only our ideas about *kinematics*. Furthermore, it showed that Geometry was not only the stage on which physics took place, but was part of the physical drama. The first appreciation of intimate relationship between geometry and physical force came, of course, with Einstein's theory of gravitation. In that case the geometry was strictly metrical. The appreciation of the intimate relationship between geometry and the other fundamental forces emerged much more slowly, partly because in those cases the geometry was non-metrical, and non-metrical geometry was itself an innovation. The process took about fifty years altogether, thirty to discover the basic structure of the fundamental forces (a structure which we now know as non-abelian or Yang-Mills gauge theory) and twenty to discover how such a theory should be applied. It is about the first thirty years of this evolution that I wish to speak here.

Strictly speaking, of course, the roots of present-day gauge theory lie in the early theory of magnetism, when it was realized that the magnetic field should be expressed, not as the gradient of a scalar potential, but as the curl of a vector potential, $\vec{B} = \vec{\nabla} \times \vec{A}$. In contrast to the scalar potential, which was thereby defined up to a constant, the vector potential $\vec{A}$ was thereby defined only up to the gradient of a scalar, and thus was born the concept of gauge invariance. The idea of the vector potential $\vec{A}$, and the fact that it was defined only up to the gradient of a scalar, was well-known by the middle of the 19th century (motivating Stokes' theorem for example) but the name vector potential came much later (though not later than 1892 [1]). Maxwell, for example, called the vector potential the electromagnetic momentum, presumably because of the analogy between the kinematical momentum $\vec{p}$ and the canonical momentum $\vec{p} + e\vec{A}$. The vector potential $\vec{A}$ was used extensively, of course, in the work of Lorentz and even more extensively after the advent of special relativity, when it could then be extended to the four-vector $\mathbf{A}$ which included the electric scalar potential ϕ . Even then, however, the significance of gauge invariance does not seem to have been fully appreciated. For example, Mie's 1912 theory of electromagnetism [2] was not gauge invariant, but was

not criticized on those grounds, and Einstein's development of his gravitational theory was delayed because, by failing to take the gauge freedom (in this context coordinate-freedom) fully into account, he thought for some time that his field equations were underdetermined.

Einstein's Gravitational Theory and its Repercussions

Einstein soon realized the true situation, of course, and proceeded to construct his famous 1916 theory of Gravitation. This theory constituted not only the enormous advance within the context of gravitation for which it is justly celebrated, but also provided the starting point for modern gauge theory. Indeed, once the role of Riemannian geometry as the origin of gravity was appreciated the idea that the other forces should have a geometrical origin became quite natural. However, this idea in its simplest form foundered on the fact that four-dimensional Riemannian geometry was too restrictive to permit a generalization from gravitation to the other interactions, in particular to electromagnetism and no other geometries were known to the physicists. Accordingly it is not surprising that the first attempt to generalize Einstein's gravitational theory stayed within the context of Riemannian geometry and changed only the dimensions of the Riemannian space. This was the 5-dimensional Kaluza-Klein (KK) theory in which one extra component of the metric tensor was set equal to unity and the other four identified with the electromagnetic potential. The main difficulty with this theory was that one had to make the unwarranted assumption that the dependence of the metric on the fifth dimension was negligible (in the same way that the dependence of a quasi-static Einstein metric on the time dimension is negligible). It might be mentioned that today, in the light of spontaneous symmetry breaking and the consequent possibility of dimensional reduction, the assumption would not appear to be so unwarranted. But because of this difficulty and some other more technical problems the proposal was temporarily abandoned. Nevertheless, as we shall see later, the KK theory was to play an important role in motivating present-day gauge theory, and in recent times it has been revived in its own right in the context of string theory, for which a higher number of dimensions (ten and twenty-six) is natural.

The major development for modern gauge theory took place, however, not in physics but in pure mathematics, where in 1917 Levi-Civita [5], motivated by some earlier work of Christoffel, introduced the idea of parallel transport. In doing so he replaced the metric by an affine connection and so liberated differential geometry from its traditional metric-dependence. The importance of this development, for both mathematics and physics, can hardly be over-estimated. In mathematics it started the series of developments in both differential geometry and topology which we associate with names like Cartan (who immediately generalized Levi-Civita's symmetric connection to a general one), Whitney, de Rham, Hodge, Chern and Steenrod, which culminated in the theory of fibre bundles [6] about 1950. Although Einstein kept himself abreast of the early mathematical developments through Cartan, who even wrote a review paper [7] at Einstein's specific request, almost nothing of this mathematical progress was known to physicists in general, and nothing at all was known by those physicists

who were to invent gauge theory. Indeed most of them made the connection with fibre bundles only at a much later date. As we shall see, for some of them the discovery of gauge theory came from KK theory and for the others it came from an element of Levi-Civita’s original theory which had been introduced into physics by Weyl [8] in 1918, almost immediately after Levi-Civita’s original paper. What Weyl had proposed in the 1918 paper was that the general coordinate invariance of Einstein’s theory should be extended to include invariance with respect to parallel-transport transformations of the form $g_{\mu\nu}(x) \rightarrow g_{\mu\nu}(x) \exp(\int dx \cdot A(x))$, where $g_{\mu\nu}$ is the metric tensor. Incidentally, it was in this 1918 paper of Weyl that the word gauge(*Eich*)-transformation first made its appearance and had the everyday meaning of a change of length or a change in calibration. Einstein (in a postscript at the end of Weyl’s paper) was quick to point out that Weyl’s proposal was not physically acceptable, because the non-integrability of the scale-factor implied that the times of clocks would depend on their history. Nevertheless, the idea was launched, and it was soon realized that it was not the idea itself that was wrong, but only its sphere of application. What was realized, in fact, was that the non-integrable phase factor should be attached not to the gravitational metric but to the quantum-mechanical wave function, where it should also pick up a factor i and become a phase factor. In other words, $g_{\mu\nu}(x) \rightarrow g_{\mu\nu}(x) \exp(\int dx \cdot A(x))$ should be replaced by $\psi(x) \rightarrow \psi(x) \exp(i \int dx \cdot A(x))$. Surprisingly, this idea seems to have been first suggested [9] in 1922 before the wave function was invented but, not so surprisingly, the suggestion came from none other than Schrödinger, who formulated it (rather obscurely) in terms of Bohr quantization rules. Indeed it is claimed in [10] that this 1922 paper of Schrödinger played an important role in his later invention of the wave function (although he did not refer to it in the 1926 papers). After the invention of the wave function the idea could be formulated much more clearly, namely, as the statement that in wave mechanics the change $\vec{p} \rightarrow \vec{p} + e\vec{A}$ of classical electrodynamics becomes $\vec{\partial} \rightarrow \vec{\partial} + (ie/\hbar)\vec{A}$ and hence a gauge transformation of $\vec{A}$ could be viewed as a phase transformation $\psi(x) \rightarrow \exp(i e \alpha(x)/\hbar) \psi(x)$ of the wave function. This observation was made independently by Fock [11] and London [12] in 1927, and an interesting letter from London to Schrödinger linking his observation to the 1922 paper can be read in [13]. The transference of his idea to the wave function was endorsed enthusiastically by Weyl, and in his now-famous 1929 paper [14] he went even further and pointed out that the standard theory of electromagnetism was not only compatible with the gauge principle but could actually be derived from it. Unfortunately, Weyl continued to call the transformations $\psi(x) \rightarrow \exp(i e \alpha(x)/\hbar) \psi(x)$ gauge transformations, although they no longer had anything to do with length or calibration. But, apart from this minor flaw, the 1929 paper was fundamental in so many ways that it must rank among the most important contributions of the century.

Weyl’s 1929 Paper

In this paper [14], which was neglected for many years but has now become famous, Weyl laid the foundation not only for modern gauge theory but for many other later developments. Among these were:

- The use of parallel transport in General Relativity

- The use of the Vierbein (which he called Achsenkreuz) for gravitation itself and for gravitating spinors
- The derivation of Energy, Momentum and Angular Momentum conservation from Euclidean invariance
- The introduction of the 2-component Weyl spinor
- The first discussion of the possibility of P and T violation
- The first mention of a possible relation between the P, T and charge-conjugation
- The use of covariant differentiation for electromagnetism
- The derivation of charge conservation from gauge invariance
- The introduction of the non-integrable phase factor $\exp\left((ie/h) \int A_\mu dx^\mu\right)$
- Most important of all, the introduction of the electromagnetic gauge principle

Not all of these developments were completely new. Some had been introduced by Weyl himself in earlier work, the derivation of conservation laws from symmetry principles was originally due to Noether [15], and the use of the Vierbein for gravitating spinors had been proposed by Wigner [16] shortly before. But in the 1929 paper these and the new concepts were presented in a manner that was consistent and comprehensive. Einstein's comment on Weyl's earlier book 'Raum-Zeit-Materie' [17], namely that it was *a master-symphony, in which each detail had its part to play in the whole, and the total was grandiose*, was equally applicable here. Weyl himself laid particular stress on the derivation of conservation laws from symmetry principles (Noether's theorem). Indeed he claimed that it was the derivation of charge conservation from gauge invariance that convinced him of the correctness of the gauge principle.

It is amusing to contrast Einstein's enthusiastic welcome for the paper with the reaction of Pauli. The latter, on first seeing a preliminary version in U. S. National Academy Proceedings, made some quite scathing remarks and, on reading the final version, only partially withdrew them. An amusing account of the two letters which he wrote to Weyl can be found in ref [18]. Pauli seems to have been particularly upset by the two-component spinor, on the grounds of parity violation. But, behind the facade, he really appreciated Weyl's approach, as was shown by subsequent events (see below) and by his Handbuch article on relativity [2].

As is now well-known, Weyl's non-integrable phase factor plays a crucial role in the Aharonov-Bohm effect [19] and in the Yang-Wu [20] formulation of electrodynamics, and has been generalized to non-abelian theories in the form of the Wilson loop [21]. An interesting comment, due to Yang [22], is that from the non-integrable phase factor point of view Einstein's objection to Weyl's gravitational theory is just the objection that there is no Aharonov-Bohm effect for clocks in a gravitational field!

In spite of the depth and originality of Weyl's 1929 paper it appears to have had little impact during the following two decades, except for the fact that the concepts of Vierbein and parallel transport came into common use in General Relativity theory

and that it had some influence on the attempts by Einstein and Schrödinger [23] to construct a unified theory of Gravitation and Electromagnetism. It was only toward the end of the forties that the ideas were again taken up, this time by Yang. Today it is well-known that Yang’s considerations of Weyl’s gauge principle led to what is now called Yang-Mills, or non-abelian, gauge theory in 1954. What is perhaps not so well-known is that Yang had already begun to consider the problem when he was a graduate student and that similar ideas were being considered independently by other people, notably Klein, Pauli, Shaw and Utiyama. What I wish to do here is sketch both the Yang-Mills and the other approaches. I should perhaps hasten to add that am not raising any question of priority since Yang and Mills were the first to formulate the theory in a clear and logical manner and to publish in a regular journal. The other approaches were more speculative (except possibly for Utiyama, whose results were obtained slightly later, though not as late as 1956, as is generally thought). It should also be pointed out, perhaps, that the fact that others were on the same track in no way detracts from the YM result. Rather does it emphasize the fundamentality of the YM idea by showing how it could emerge from a number of quite different considerations.

Klein’s Theory

The very first Yang-Mills-type theory was produced by Oskar Klein in 1938. Up to recently Klein’s contribution was almost unknown, partly because it was produced just before the outbreak of World War II, partly because it was produced only as a conference talk [24] and partly because the notation and physical application were rather obscure. Indeed the paper seems to have been appreciated by only one member, Møller, of the illustrious audience present at the talk, with many of the greats such as Bohr, Gamow, von Neumann and Wigner, and it was not even quoted, much less developed, in any subsequent paper. What Klein did was to make the natural generalization of the existing Kaluza-Klein theory, which had *one* extra dimension (a circle) to the corresponding theory with *two* extra dimensions (a 2-sphere).

Since $SO(3)$ is the symmetry group of the 2-sphere and Klein included the nucleons as a multiplet, it is not surprising that he arrived at an $SU(2)$ gauge theory. Because of the Kaluza-Klein derivation the field strengths came out automatically and he wrote them in the form

$$A_{rs} = \partial_r A_s - \partial_s A_r + \frac{ie}{hc}(B_r \tilde{B}_s - B_s \tilde{B}_r) \quad (30)$$

$$B_{rs} = \delta_r A_s - \delta_s A_r, \quad (31)$$

and similarly for $\tilde{B}_{rs}$, where δ_r is the EM covariant derivative. He then postulated a Lagrangian and derived the field equations

$$\epsilon_r \partial_r - \frac{ie}{2hc} \epsilon_r \begin{pmatrix} 0 & \tilde{B}_r \\ B_r & 2A_r \end{pmatrix} \begin{pmatrix} n \\ p \end{pmatrix} = \frac{Mc}{h} \begin{pmatrix} n \\ p \end{pmatrix} \quad (22)$$

for the nucleons, where ϵ_r are the Dirac matrices, together with the field equations

$$\delta_l B_{kl} - \frac{2ie}{hc} A_{kl} B_l + \frac{\mu^2 c^2}{h^2} B_k = j_l, \quad (40)$$

(and similarly for $\tilde{B}_l$) for the gauge-fields, where j_l is the matter-current. These are essentially the Yang-Mills equations, but in an unconventional notation and with ad hoc masses for the gauge-fields. I have used Klein's equation numbering in order to give some idea of the sequence and the number of intermediate steps.

Klein's problem was how to identify the gauge fields physically and how to give them masses. As one can see from (40) he solved the mass problem by simply inserting masses for the B -fields in the Lagrangian (in gross violation of gauge invariance). He then identified the A -field with the electromagnetic field and suggested that the B -fields should mediate the nuclear forces. But it was not clear which of the nuclear forces they should mediate, which is not surprising if we recall that the same problem was still around twenty years later. Recall that in 1938 Yukawa's meson had only recently been proposed, and it was not clear whether it should be scalar or vector! Indeed, the only property of the nuclear forces that had been established beyond doubt was the charge independence of the strong interactions. This property prompted Møller to ask the single question that followed the talk, namely, how the fact that Klein's two new mesons would have to be electrically charged could be reconciled with charge-independence. Astonishingly, Klein responded by pointing out that this difficulty could be overcome by making the following simple change in his array of vector mesons

$$\begin{pmatrix} A_k & \tilde{B}_k \\ B_k & A_k \end{pmatrix} \rightarrow \begin{pmatrix} A_k - C_k & \tilde{B}_k \\ B_k & A_k + C_k \end{pmatrix} \quad (2)$$

If one takes into account that the array (2) is just $\text{diag}(1, -1)$ times the usual present-day array one sees that Klein made the generalization $SU(2) \rightarrow SU(2) \times U(1)$ on the spot! This is just the generalization that Glashow made twenty years later for the *weak* interactions when he generalized Schwinger's 1957 $SU(2)$ model to the standard model. Finally, Klein proceeded to second-quantize the system in the manner used at the time for QED. The full Klein paper, the Møller discussion and a commentary by David Gross can be found in ref. [23].

Pauli's Theory

A second Yang-Mills-type theory was produced by Pauli in 1953, as a sequel to a discussion following a talk by Pais at the Lorentz Conference [25] in Leiden in 1953. During the discussion Pauli had said:

I am very much in favour of the general principle to bring empirical conservation laws and invariance properties in connection with the mathematical groups of transformations of the laws of Nature. If besides the conservation of energy-momentum and of charge the conservation of the property defined as number of nucleons and charge-independence of the nuclear forces are well established they have indeed, as Pais tried now to express mathematically, also to be connected with group theoretical properties of the laws of nature. . . I would like to ask in this connection whether the transformation group (*isospin*) with constant phases can be amplified in a way analogous to the gauge group for electromagnetic potentials in such a way that the meson-nucleon interaction is connected with the amplified group. The main

problem here seems to be the proper incorporation of the coupling constant into the group.

To which Pais replied that

...an amplified group is very much on my mind as a means of specifying the coupling in more detail. I have no results so far.

Pauli and Pais were clearly unaware of Klein’s contribution, and, stimulated by the discussion, Pauli independently generalized Kaluza-Klein theory from the circle to the 2-sphere. The results, similar to Klein’s, were sent to Pais in letters dated July and December 1953, in which Pauli says he wrote the theory down *just to see how it looks*. He considered his most important result to be the expression for the gauge field, which, having been induced from the Kaluza-Klein theory, was automatically the correct one. However, Pauli did not attempt to abstract his theory from the Kaluza-Klein context or make any explicit application to the nuclear interactions, possibly because he realized that the classical gauge-field masses would have to be zero.

1 Yang-Mills Theory

Both the Klein and Pauli theories were formulated within the context of Kaluza-Klein theory and neither author seems to have envisaged the possibility of constructing a non-abelian gauge theory in its own right. This was left to Yang, who, as mentioned above, had begun to think about this possibility while still a graduate student. He writes [26]:

While a graduate student in Kunming and in Chicago, I had thoroughly studied Pauli’s review articles on field theory. I was very much impressed with the idea that charge conservation was related to the invariance of the theory under phase changes, an idea, I later found out, due originally to H. Weyl. I was even more impressed with the fact that gauge invariance *determined* all the electromagnetic interactions. While in Chicago I tried to generalize this to isotopic spin interactions by the procedure later written up in [27], equations (1) and (2) [*covariant derivative*]. Starting from these it was easy to get equation (3) [*transformation law for the gauge-potential*]. Then I tried to define the field-strengths $F_{\mu\nu}$ by $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ which was a ‘natural’ generalization of electromagnetism. This led to a mess, and I had to give up. But the basic idea remained attractive, and I came back to it several times in the next few years, always getting stuck at the same point... As more and more mesons were discovered and all kinds of interactions were being considered, the necessity to have a *principle* for writing down interactions became more obvious to me. So while at Brookhaven [*in the summer of 1953*] I returned once more to the idea of generalizing gauge invariance. My office mate was R. L. Mills who was about to finish his Ph. D. at Columbia with N. Kroll. We worked on the problem and eventually produced [27]. We also wrote an Abstract for the April 1954 meeting of the AMS in Washington, which became [28]. Different motivations were

emphasized in the two papers. The formal aspect of our work did not take long and was essentially finished by February 1954. But we found that we were unable to conclude what the mass of the gauge-particle should be. We toyed with the dimensional argument that, for a pure gauge theory, there is no quantity with the dimension of a mass to start with, and therefore the gauge particle must be massless. But we quickly rejected this line of reasoning.

The first public presentation of their results seems to have been during a seminar given at the Institute in Princeton at the end of February. In this seminar Pauli (who, as mentioned above, had produced a Kaluza-Klein version and was aware of the problem with the gauge-field masses) caused some disruption by refusing to accept Yang's statement that he was not sure whether these masses should be zero. (In the light of our present knowledge concerning spontaneous breakdown and confinement, Pauli's demand for immediate information was quite a tall order!). In any case Yang and Mills soon realized that the mass and renormalization problems for this theory were not going to be solved overnight, and sent the manuscript for publication in June 1954.

Shaw's Theory

One of the most remarkable and least known approaches to non-abelian gauge theory was that of Ronald Shaw, who in 1954 was studying for his doctorate [29] under Professor Salam at Cambridge. Quite unaware of the work of Yang and Mills, Shaw introduced $SU(2)$ Yang-Mills theory in his thesis, starting from a completely different angle, namely from a formulation of electromagnetism in which the usual $U(1)$ group was replaced by $SO(2)$. This formulation appeared in a preprint of Schwinger's and, according to Shaw, on reading it he saw *in a flash* that the $SO(2)$ formulation could be generalized to $SU(2)$. Even the construction of the kinetic terms does not seem to have given him any difficulty. It may be of interest reproduce the relevant two pages of his thesis (in slightly abbreviated form and with the original equation-numbering):

Now under a general transformation (13) ($\psi' = (1 + \frac{1}{2}ic_i\tau_i)\psi$) with $\vec{c}$ a function of position, L_0 is no longer invariant but becomes

$$L'_0 = L_0 + \frac{1}{2}\partial^\alpha \vec{c} \cdot \bar{\psi} \gamma_\alpha \vec{\tau} \psi. \quad (16)$$

The additional term can be compensated by introducing an interaction

$$L_1 = -\frac{iq}{2}\vec{B}^\alpha \cdot \bar{\psi} \gamma_\alpha \vec{\tau} \psi, \quad (17)$$

where

$$q\vec{B}^\alpha = q(\vec{B}^\alpha - \vec{c} \times \vec{B}^\alpha) + \partial^\alpha \vec{c}. \quad (18)$$

In order to obtain a theory invariant under general isotopic spin transformations it only remains to construct an invariant free field Lagrangian for the $\vec{B}$ -field. It is natural in analogy with with electromagnetic case, to define

$$\vec{f}^{\alpha\beta} = \partial^\alpha \vec{v}^\beta - \partial^\beta \vec{v}^\alpha \quad (19)$$

$$L_2 = -\frac{1}{4} \vec{f}^{\alpha\beta} \dot{\vec{f}}_{\alpha\beta}. \quad (20)$$

However (20) is not invariant under the transformation (18)... A properly invariant Lagrangian can be obtained by defining

$$\vec{F}^{\alpha\beta} = \vec{f}^{\alpha\beta} - q(\vec{B}^\alpha \times \vec{B}^\beta - \vec{B}^\beta \times \vec{B}^\alpha) \quad (22)$$

$$L_2 = -\frac{1}{4} \vec{F}^{\alpha\beta} \cdot \vec{F}_{\alpha\beta} \quad (23)$$

L_2 is now invariant under (18) as $\vec{F}^{\alpha\beta}$ transforms solely as a vector in E_3 .

This work was done in the Spring of 1954, around the time that Yang was giving his seminar in Princeton, and thus it came a little later than the work of Yang and Mills. But, of course, it was completely independent. The thesis was not submitted until 1955 because it had to be completed by further work on the Poincaré and Lorentz groups. (The latter work appeared finally as Part I and the above-quoted extract in Part II, which adds to the confusion in dates!). As Professor Salam was Shaw’s thesis adviser it is sometimes maintained that he should have urged Shaw to publish his result at once. But, in Shaw’s own words:

Sometime later [*still early in 1954?*] I showed it to Salam, but I do not think he appreciated it at the time. I do not at all blame him for this—I probably told him about it in a very dismissive way, since the relevant particles, surely $m = 0$, I thought, did not exist. Much later [*end of 1954?*] he told me that Yang and Mills had had the same idea and told me to write mine up (which I did not).

Utiyama’s Theory

Utiyama’s published contribution [30] is quite well-known. But because it was first published in 1956, quoted Yang and Mills and dealt with all simple compact groups and with Gravitation, it is usually regarded as a generalization of the Yang-Mills paper which was written as a sequel to that paper. This is not the case. As we now know from his published statements, Utiyama derived the YM equations slightly later than Yang and Mills but quite independently. These statements appear in two books which, as far as I can ascertain, are available only in Japanese and Japanese/Russian respectively but two of my colleagues have been kind enough to translate the relevant parts of the two books (the translations agree in detail where they overlap). So I can let Utiyama speak for himself.

Book 1 [31] Brief summary of Chapter 10, entitled Memoir of Remorse

1954, January: In the end of January, I (R. Utiyama) received a letter from the Institute for Advanced Study (IAS), Princeton, inviting me to stay at the IAS from September, 1954. I decided to prepare some completed work prior to going to the IAS so that I could publish papers even if I could

not work successfully there. In fact, I had already the idea of the *general gauge theory*, including the basic structure of the theory, in my mind before I received the letter from the IAS. So what I should do was simply to spell it out in the form of a paper. I completed the work at the end of March 1954.

1954, May or June: I gave a talk on the work at a small workshop held at the then Research Institute for Fundamental Physics (*now Yukawa Institute for Theoretical Physics*), Kyoto University. I got a negative response.

1954, September: I arrived at Princeton. I met Prof. Minoru Kobayashi (then Professor at Kyoto University) who was my supervisor at Osaka University and had been staying at the IAS from one year earlier as a visiting fellow. When I explained to him my idea of general gauge theory, he told me that recently Prof. Yang announced a theory which is quite similar to mine. Some days later, I found a preprint by Yang in my mailbox. I took a look of it and found many formulae already familiar to me – I immediately realised that he had found the same theory that I had developed. I was shocked too deeply to examine Yang's paper closely and to compare it with my work with care.

1955, March: As my appointment was prolonged for one more year, I felt that I should work on a subject which was more important than those I had engaged so far at the IAS. So I decided to return to the general gauge theory and took a closer look on Yang's paper which had been published in 1954. At this moment I realised for the first time that there was a significant difference between Yang's theory and mine. The difference was that Yang merely found an example of non-Abelian gauge theory whereas I had developed a general idea of gauge theory which could contain gravity as well as electromagnetic theory. Thus I decided to publish my work by translating it into English, and adding an extra section where Yang's theory is discussed as a example of my general gauge theory. I did so partly because his paper had already been published, and partly because I thought that it would not be fair if I did not refer to his paper after I read it, even if I developed the idea of the general gauge theory before knowing his work. My paper was accepted for publication in *Phys. Rev.* in July 1955 and published early in 1956.

Remark: I admit that it was Yang and Mills who first found a new type of (non-Abelian) gauge theory (though not gravity). However, it is very very disappointing for me to see that non-Abelian gauge theory is called by the name 'Yang-Mills theory' only, without the slightest mention of me who developed the general framework of the gauge theory. I very much regret not having submitted my paper to some Japanese journal in March, 1954 when I completed the work.

The most interesting aspect of Utiyama's approach is that it was motivated by gravitational theory and that (partly as a result of this) it was formulated for arbitrary simple Lie groups, not just $U(1)$ and $SU(2)$ or even compact simple groups. In fact, Utiyama

had hoped to use this approach to unify the interactions. As we now know (partly thanks to Utiyama himself), strict unification of gravitation with the other interactions along these lines is impossible since gravitation cannot be formulated as a set of local field equations for the gauge-potentials, but only for the metric (or the Vierbein). Nevertheless, Utiyama was the first to suggest that the broader analogy between gravitation and electromagnetism which had so fascinated Weyl, Einstein and Schrödinger [22] could be extended to include the nuclear interactions. As he says himself in the second book [32]:

If I am allowed to refer to my work on the general gauge theory, I would like to stress that it is my paper which first showed clearly that the theory of gravity could fall into the framework of the gauge theory, which would be a first step toward the the grand unified theory in the modern viewpoint. Even more importantly, my paper pointed out that fields carrying a fundamental force – either gravity or electromagnetism – must in fact be those termed as ‘connections’ in mathematics, which are now called ‘gauge fields’. That the notion of connections is indispensable in establishing a theory of interactions was the basic assertion that I wanted to make in my paper.

Summary

As everybody knows the Yang-Mills paper did not immediately herald a new dawn for the nuclear interactions. Indeed the hurdles still to be overcome, the infra-red, unitarity and renormalization problems on the theoretical side and the zero-mass, neutral current and gauge-field identifications on the experimental side, were so formidable that for more than a decade the theory languished in a backwater. Only with the advent of the concept of spontaneous symmetry breaking and suitable renormalization techniques did the theory become feasible, and only with the the experimental discovery of neutral currents and the experimental/theoretical discovery of asymptotic freedom did it become completely acceptable. Of course, even when fully accepted gauge theory provides us only with a geometrical framework for the fundamental interactions but does not yet combine the strong and electroweak interactions in an intrinsic way, much less combine these two with gravitation and there is no shortage of open questions (are there Higgs bosons? what is the origin of the fermion masses? where does CP-violation fit in? and so on). We are still far from the end-station.

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SELF-INTERACTING SCALAR FIELDS AND (NON-) TRIVIALITY

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Abstract

For five or more space-time dimensions, and probably four as well, the quartic self-coupled relativistic scalar field defined by way of a conventional lattice limit becomes trivial (Gaussian). The same fate befalls the quartic self-coupled ultralocal scalar field when defined by a conventional lattice limit. However, ultralocal models admit alternative, nontrivial results when approached nonconventionally. A reexpression of these nontrivial results as they would arise from the continuum limit of a lattice-space formulation reveals the need for an additional, nonclassical local potential. This additional potential does not vanish when the quartic coupling constant goes to zero, which suggests that it represents a hard-core interaction the existence of which is predicted on formal grounds. It is proposed that the use of a similar kind of additional, nonclassical local potential may actually result in nontrivial continuum limits for $(\phi^4)_n$ relativistic models in space-time dimensions $n \geq 4$.

1 Introduction

1.1 Background

Nonrenormalizable quantum field theories have generally received a very bad press and carry a very bad reputation. However, it is not entirely clear that such theories are as badly behaved as many researchers would have us believe. Consider the case of $(\phi^4)_5$, for example. To be sure when such a theory is approached perturbatively an infinite number of distinct counterterms need to be specified requiring, as the story goes, an infinite number of experiments to define the theory thus rendering the theory powerless for predictive purposes [1]; this result is, of course, unsatisfactory. Viewed nonperturbatively, for example, through the mechanism of a standard lattice formulation (see below), the continuum limit results in a (possibly generalized) free field [2]; this is a wholly unsatisfactory answer of another variety. Why do such disparate answers emerge? Is there a way to rationalize these results and, more importantly, can one see how finite and nontrivial results could possibly emerge?

Negative preconceptions against nonrenormalizable quantum field theories are so strong that many would ask why one might even expect finite and nontrivial results.

One possible answer to such questions lies in the fact that the classical theory of a typical nonrenormalizable model, *e.g.*, $(\phi^4)_5$, exists and is nontrivial [3]. How can one reconcile a trivial quantum theory limiting, in the case of large quantum numbers, to a *nontrivial* classical theory? Such a paradox cries out for an *alternative* quantization procedure as compared to any of the standard ones, a procedure that will yield a nontrivial quantum theory that can indeed limit to the appropriate classical theory as $\hbar \rightarrow 0$.

1.2 Hard-Core Interactions

Some years ago the basic philosophy of an apparently alternative quantization scheme was put forward [4], a scheme that we believe still has considerable merit although in practice it has not been possible to implement it as yet for a covariant theory such as $(\phi^4)_5$. Let us sketch in a formal and heuristic fashion the basic argument. For our formulation we choose a Euclidean-space functional integral with an imaginary source term, *i.e.*, we study

$$C\{h\} = \mathcal{N} \int e^{i \int h \phi d^n x - \int \left\{ \frac{1}{2} [(\nabla \phi)^2 + m^2 \phi^2] + g \phi^4 \right\} d^n x} \mathcal{D}\phi$$

where $d^n x$ refers to an n -dimensional space time and $\hbar = 1$. We recall that

$$\left(\int \phi^4(x) d^n x \right)^{\frac{1}{2}} \leq K \int [(\nabla \phi(x))^2 + m^2 \phi^2(x)] d^n x$$

holds for finite K for smooth classical functions $\phi(x)$ only if $n \leq 4$, while for $n \geq 5$ no finite K exists [5]. Of course, the support of the functional measure is on distributions rather than smooth functions. Nevertheless, any distribution can be approached as a limit of smooth functions, in which case, for any $n \leq 4$,

$$\lim \frac{\left(\int \phi^4 d^n x \right)^{\frac{1}{2}}}{1 + K \int [(\nabla \phi)^2 + m^2 \phi^2] d^n x} \leq 1$$

holds as well for the distributional fields. From this fact the inference is drawn that the free (Euclidean) action controls the interaction action, and in particular, as $g \rightarrow 0$ the free theory emerges,

$$\begin{aligned} \lim_{g \rightarrow 0} C\{h\} &= \mathcal{N} \int e^{i \int h \phi d^n x - \frac{1}{2} \int [(\nabla \phi)^2 + m^2 \phi^2] d^n x} \mathcal{D}\phi \\ &= \exp \left\{ -\frac{1}{2} \frac{1}{(2\pi)^n} \int \frac{|\tilde{h}(k)|^2}{k^2 + m^2} d^n k \right\} \\ &\equiv C_0\{h\}. \end{aligned}$$

In short, the interacting theory is continuously connected to the standard free theory. This fact may seem obvious, but it is worth reemphasizing.

On the other hand, for $n \geq 5$ the stated Sobolev inequality holds for no finite K . This situation is interpreted [4] as effectively introducing a *hard core* within the functional integral thereby projecting out, for any $g > 0$, certain fields that would otherwise be allowed in case $g \equiv 0$. We crudely denote such a projection operator by $X\{\phi\}$ and observe, for $n \geq 5$ and $g > 0$, that formally

$$C\{h\} = \mathcal{N} \int e^{i \int h \phi d^n x - \int \left\{ \frac{1}{2} [(\nabla \phi)^2 + m^2 \phi^2] + g \phi^4 \right\} d^n x} X\{\phi\} \mathcal{D}\phi.$$

As a consequence it follows that

$$\begin{aligned} \lim_{g \rightarrow 0} C\{h\} &= \mathcal{N} \int e^{i \int h \phi d^n x - \frac{1}{2} \int [(\nabla \phi)^2 + m^2 \phi^2] d^n x} X\{\phi\} \mathcal{D}\phi \\ &\equiv \overline{C}_0\{h\} \neq C_0\{h\}. \end{aligned}$$

Thus—and *this is a principal fact*—the interacting theory is *not* continuously connected to the free theory, but, rather, it is continuously connected to an alternative theory termed the pseudo-free theory [4, 6].

1.3 Lattice Approach

Another argument that suggests a radical reformulation of the problem may also be given. Consider the standard lattice-space formulation of the problem at hand given basically by

$$\begin{aligned} C\{h\} &= \lim_{a \rightarrow 0} M \int \cdots \int \exp \left\{ i \sum h_k \phi_k a^n - \frac{1}{2} \sum (\phi_{k^*} - \phi_k)^2 a^{n-2} \right. \\ &\quad \left. - \frac{1}{2} m_0^2(a) \sum \phi_k^2 a^n - g_0(a) \sum \phi_k^4 a^n \right\} \prod d\phi_k. \end{aligned}$$

Here a denotes the lattice spacing, $k = (k_1, \dots, k_n)$, $k_j \in \mathbb{Z}$, denotes a lattice site, and k^* denotes one of n nonequivalent nearest neighbor sites to k . Allowance is made for the bare constants m_0^2 and g_0 ($g_0 \geq 0$) to depend on the cutoff; a similar a -dependent factor may be included for the gradient term as well. The goal is to adjust such factors so that in the continuum limit, $a \rightarrow 0$, a nonzero, nontrivial (*i.e.*, non-Gaussian) result emerges. The continuum limit may be taken for a finite volume, and an infinite volume limit may be taken at a later stage. In rough outline this procedure defines the lattice-space formulation of the quartic (or any other lower-bounded polynomial) self-interacting scalar field.

This procedure has been admirably successful for $(\phi^4)_2$ and $(\phi^4)_3$, but leads to trivial results for $(\phi^4)_n$, $n \geq 5$, and most probably for $(\phi^4)_4$ as well [hereafter we assume triviality for $(\phi^4)_4$]. One possible way to understand these results is to observe that the given lattice formulation has made one hidden but basic assumption, namely that of *asymptotic freedom*. In short, the high-momentum degrees of freedom ($|p| \gtrsim \hbar/a$) implicitly integrated out to arrive at the lattice expression at finite a have been treated in just the same way as when $g_0 \equiv 0$. For $n = 2$ and 3 , $(\phi^4)_n$ is indeed asymptotically free and its implicit assumption is consistent. For $n \geq 4$, $(\phi^4)_n$ is *not* asymptotically free for $g_0 > 0$ and its implicit assumption is not consistent unless $g_0 = 0$. In brief, the only $(\phi^4)_n$, $n \geq 4$, theory consistent with the implicitly assumed asymptotic freedom is the free theory.

On the strength of this discussion we seek to replace the standard lattice formulation by a nonstandard one that has implicitly treated its high-momentum degrees of freedom differently than in the free theory. This different treatment takes the form of an additional interaction, and should formally correspond to the appearance of the hard-core projection in the continuum limit. In particular, the contributions from the additional interaction should not vanish as $g \rightarrow 0$ giving rise to the pseudo-free theory.

If the scenario outlined above was actually true it would resolve several issues. On the one hand, blame for divergences in the perturbation series would be laid primarily

on the expansion of a hard-core interaction. [As a rough analogy, imagine expanding in a hard-core internuclear potential designed to keep nuclei rigidly apart.] By incorporating the vestiges of the hard-core interaction in the pseudo-free theory a meaningful perturbation expansion should exist when taken about the pseudo-free theory. On the other hand, there is also the possibility that a nonperturbative and nontrivial exact solution emerges. Perhaps even the $(\phi^4)_4$ model may receive an alternative treatment along the foregoing lines.

Unfortunately, we are not in a position to offer one shred of concrete evidence that the proposal outlined above applies to any quartic, self-coupled relativistic scalar field. However, we can offer instead an account of another model that is intrinsically more singular than the relativistic model. The virtue of this alternative (ultralocal) model is that: (i) it has all of the important difficulties appearing in the relativistic case; (ii) it is completely soluble outside of perturbation theory; (iii) the solution has all the features alluded to with regard to hard-core behavior; and (iv) the model has a lattice-space path-integral formulation that makes clear how the nontraditional lattice-space action resolves all difficulties leading to a well-defined, thoroughly satisfactory, and explicit solution.

2 Ultralocal Models

2.1 Classical Theory

The alternative model we shall examine is an ultralocal quantum field theory with a classical action given by

$$\int \left\{ \frac{1}{2} [\dot{\phi}^2(x, t) + m^2 \phi^2(x, t)] + g \phi^4(x, t) \right\} d^s x dt.$$

This model differs from a relativistic model by the absence of the spatial gradient term, $[\nabla \phi(x, t)]^2$, and thus the temporal evolution of the field at any one spatial point is completely independent of the temporal evolution of the field at any other spatial point. The nonlinearity is not limited to $\phi^4(x, t)$ but could be replaced by other local powers, *e.g.*, $\phi^{40}(x, t)$. Ultralocal models have been rather thoroughly studied previously [7–9] and we shall extract from this earlier work only what is needed here. However, it should be remarked that the lattice-space formulation of these models presented in this paper has not appeared in the literature previously. In spite of its appearance here within a specialized model, we conjecture that the general form of the lattice-space formulation may possibly be relevant in the relativistic case as well. The ultralocal model is not substantially different whether the number of space dimensions $s = 1$ or $s \geq 2$, and for purposes of illustration in this article attention is confined to just one spatial dimension, $s = 1$.

2.2 Standard Approaches

Before presenting the proper solution to these models, let us very briefly examine them from two conventional approaches, namely, by standard perturbation theory and by a standard lattice-space formulation. The (Euclidean) momentum-space propagator reads

$(p_0^2 + m^2)^{-1}$ [rather than $(p_0^2 + p_1^2 + m^2)^{-1}$ as in the relativistic case]. Hence integration over the spatial momentum p_1 (or summation if the space is compact) always gives a divergence to every loop. Let the cutoff value of this divergent integral (sum) be called Λ . Observe that there is a one-loop divergence proportional to Λ for any even number ℓ , $\ell \geq 4$, of external legs that needs a counterterm which is of order ℓ in the fields for all even ℓ ; this fact already classifies these models as nonrenormalizable. On the other hand, if we include, for example, only the subset of iterated simple bubble graphs for the connected four-point function, the resultant geometric series leads to a contribution to the renormalized coupling of the form $g_0/(1 + g_0 f \Lambda)$ for some positive factor f . As Λ diverges this contribution vanishes. Likewise more complicated graphs with four external legs can be repeatedly iterated and summed, and as Λ diverges, their contribution to the four-point coupling constant also vanishes. The latter argument suggests that a free theory should emerge, a result that is supported by a standard lattice-space approach as we now discuss. The lattice-space expression of interest is just that given earlier in the relativistic case save that the term $\sum (\phi_{k^*} - \phi_k)^2$ now includes a sum over only *one* nearest neighbor (Euclidean time) rather than n inequivalent nearest neighbors. The result is a product over the lattice points in the spatial direction of independent, similar expressions, a product, which in the continuum limit goes to a Gaussian result simply on the basis of the central limit theorem independently of the choices made for the a -dependent model parameters. [In like fashion, one may assert in the relativistic case (for $n \geq 4$) that the correlations between fields at different lattice sites are simply too weak to counteract the strong Gaussian tendencies of the central limit theorem in the continuum limit.] As will become evident later, our nonstandard lattice-space formulation of the ultralocal models will, in effect, capitalize on standard ways to beat the central limit theorem leading to a Poisson rather than a Gaussian process.

2.3 Operator Analysis

We now briefly summarize the results of an operator analysis of ultralocal models [7–9]. Assume the local quantum field operator $\phi(x)$ becomes selfadjoint after smearing with a test function $f(x)$ at sharp time. Focus on the ground state $(|0\rangle)$ expectation value

$$E(f) \equiv \langle 0 | e^{i \int \phi(x) f(x) dx} | 0 \rangle = e^{- \int L[f(x)] dx},$$

where the expression on the far right hand side is a direct consequence of the translation invariance of the ground state and the assumed independent dynamical behavior for each point in space. Let $f(x) = p \chi_\Delta(x)$ where $\chi_\Delta(x) = 1(0)$ if $x \in (\mathcal{Q})\Delta \subset \mathbb{R}$; set $\Delta = \int \chi_\Delta(x) dx$. Then since E is a characteristic function it follows that $\exp\{-\Delta L[p]\} = \int e^{ip\lambda} d\rho_\Delta(\lambda)$, $\lambda \in \mathbb{R}$, holds for some probability measure ρ_Δ for each $\Delta > 0$. Choosing $L[-p] = L[p]$ on symmetry grounds, the form of $L[p]$ is given by

$$L[p] = \lim_{\Delta \rightarrow 0} \int [1 - \cos(p\lambda)] \Delta^{-1} d\rho_\Delta(\lambda) = ap^2 + \int_{|\lambda|>0} [1 - \cos(p\lambda)] d\sigma(\lambda)$$

where $a \geq 0$ and σ is a nonnegative measure such that $\int_{|\lambda|>0} [\lambda^2/(1 + \lambda^2)] d\sigma(\lambda) < \infty$; here, in the last line, we have given the most general form for the indicated limit.

We concentrate on $a = 0$ and an absolutely continuous choice for σ , $d\sigma(\lambda) = c^2(\lambda)d\lambda$, $c(-\lambda) = c(\lambda) \in \mathbb{R}$, to arrive at

$$L[p] = \int [1 - \cos(p\lambda)] c^2(\lambda) d\lambda,$$

which expresses $E(f)$ in terms of the "model function" $c(\lambda)$.

An operator realization for $\phi(x)$ is now straightforward to obtain. Let $A(x, \lambda)$ and $A^\dagger(x, \lambda)$ denote conventional irreducible Fock representation operators for which $|0\rangle$ is the unique no particle state, $A(x, \lambda)|0\rangle = 0$, for all $(x, \lambda) \in \mathbb{R}^2$. The only nonvanishing commutator is

$$[A(x, \lambda), A^\dagger(x', \lambda')] = \delta(x - x')\delta(\lambda - \lambda').$$

Then

$$\phi(x) = \int A^\dagger(x, \lambda)\lambda A(x, \lambda)d\lambda + \int A^\dagger(x, \lambda)\lambda c(\lambda)d\lambda + \int c(\lambda)\lambda A(x, \lambda)d\lambda.$$

This expression may be usefully simplified if we introduce $B(x, \lambda) \equiv A(x, \lambda) + c(\lambda)$, and $B^\dagger(x, \lambda) \equiv A^\dagger(x, \lambda) + c(\lambda)$, which satisfy $[B(x, \lambda), B^\dagger(x', \lambda')] = \delta(x - x')\delta(\lambda - \lambda')$, and if we use $\int \lambda c^2(\lambda)d\lambda = 0$, as a principal value integral, if necessary. Then it follows that

$$\phi(x) = \int B^\dagger(x, \lambda)\lambda B(x, \lambda)d\lambda.$$

Since

$$\exp i \int f(x)dx \int B^\dagger(x, \lambda)\lambda B(x, \lambda)d\lambda =: \exp \int dx \int B^\dagger(x, \lambda)[e^{if(x)\lambda} - 1]B(x, \lambda)d\lambda :,$$

where $::$ denotes normal ordering, it follows that

$$E(f) = \langle 0 | \exp i \int f(x)\phi(x)dx | 0 \rangle = \exp - \int dx \int c(\lambda)\{1 - \cos[f(x)\lambda]\}c(\lambda)d\lambda$$

as required. We desire that $\int f(x)\phi(x)dx$ have absolutely continuous spectrum for all smooth f , say with compact support. Then it follows from the Riemann-Lebesgue lemma that $\int c^2(\lambda)d\lambda = \infty$, i.e., $c(\lambda) \notin L^2(\mathbb{R})$, and we set $c(\lambda) = b^{1-\gamma}|\lambda|^{-\gamma}e^{-\gamma|\lambda|}$, where $\gamma, \frac{1}{2} \leq \gamma < \frac{3}{2}$, is called the "singularity parameter". (The physical meaning of γ as well as cases where $\gamma \geq \frac{3}{2}$ are discussed in [9]; these topics are not important here.) With $[\cdot] \equiv \text{dimension}(\cdot)$, observe that $[\lambda] = [x] = L$ (length), and that $[c] = L^{-1}$. In addition, to maintain dimensions, we are forced to introduce an auxiliary constant b with $[b] = L^{-1}$; in prior work on this model units were implicitly chosen so that $b = 1$.

Renormalized products follow directly from an operator product expansion. In particular, since

$$\phi(x)\phi(y) = \delta(x - y) \int B^\dagger(x, \lambda)\lambda^2 B(x, \lambda)d\lambda +: \phi(x)\phi(y) :,$$

it is reasonable to choose

$$\phi_R^2(x) = b \int B^\dagger(x, \lambda)\lambda^2 B(x, \lambda)d\lambda$$

as the renormalized square. The factor b has been inserted to preserve dimensions. Formally, $\phi_R^2(x) = Z\phi^2(x)$ where $Z = b/\delta(0)$ is a dimensionless “zero” multiplier. In like fashion one defines an arbitrary renormalized local power,

$$\phi_R^q(x) = b^{q-1} \int B^\dagger(x, \lambda) \lambda^q B(x, \lambda) d\lambda \equiv Z^{q-1} \phi^q(x),$$

for $q = 1, 2, 3, \dots$. If $V(u) = \sum_{q=1}^Q v_{2q} u^{2q}$, $v_{2Q} > 0$, is an even, lower-bounded polynomial, then

$$Z^{-1}V(Z\phi(x)) \equiv b^{-1} \int B^\dagger(x, \lambda) V(b\lambda) B(x, \lambda) d\lambda$$

is a natural candidate for the density of the potential energy expressed in renormalized field powers. The full Hamiltonian is then expressed as $\mathcal{H} = \int dx \int B^\dagger(x, \lambda) \mathfrak{h}(\partial/\partial\lambda, \lambda) B(x, \lambda) d\lambda$, where by analogy with usual models we choose $\mathfrak{h}(\partial/\partial\lambda, \lambda) = -\frac{1}{2}b^{-1}\partial^2/\partial\lambda^2 + v(\lambda)$ (*N.B.*, $\hbar = 1$). General arguments also imply that $\mathcal{H} = \int dx \int A^\dagger(x, \lambda) \mathfrak{h}(\partial/\partial\lambda, \lambda) A(x, \lambda) d\lambda$ from which it follows that $v(\lambda) = c''(\lambda)/2bc(\lambda)$. This relation connects the kinematics (*i.e.*, c) to the dynamics (*i.e.*, $\mathfrak{h}$) and shows that

$$\mathfrak{h} = -\frac{1}{2b} \frac{\partial^2}{\partial\lambda^2} + \frac{\gamma(\gamma+1)}{2b\lambda^2} + \frac{\gamma\eta'(\lambda)}{b\lambda} - \frac{1}{2b} \eta''(\lambda) + \frac{1}{2b} [\eta'(\lambda)]^2$$

Special interest focusses on $y = \bar{y} \equiv bm\lambda^2/2$ for which

$$\mathfrak{h} = \bar{\mathfrak{h}} \equiv -\frac{1}{2b} \frac{\partial^2}{\partial\lambda^2} + \frac{\gamma(\gamma+1)}{2b\lambda^2} + \frac{1}{2}bm^2\lambda^2 + (\gamma - \frac{1}{2})m,$$

an expression that will be relevant for the pseudo-free theory. Assume that $y(\lambda) \geq y_1 + y_2|\lambda|^{1+\epsilon}$, $\epsilon > 0$ and $y_2 > 0$, in which case the spectrum of $\mathfrak{h}$, as determined from $\mathfrak{h}u_\ell(\lambda) = \mu_\ell u_\ell(\lambda)$, is purely discrete and doubly degenerate with $0 < \mu_0 = \mu_1 < \mu_2 = \mu_3 < \dots$. For $\bar{\mathfrak{h}}$ the eigenvalues are $\mu_{2\ell} = \mu_{2\ell+1} = (2\ell + 2\gamma + 1)m$, $\ell = 0, 1, \dots$, independent of b . Generally, $\mathcal{H} = \sum \mu_\ell N_\ell$, where $N_\ell = \int A_\ell^\dagger(x) A_\ell(x) dx$ are number operators relevant to the modes $A_\ell(x) = \int u_\ell^*(\lambda) A(x, \lambda) d\lambda$. Consequently, $|0\rangle$ is the unique ground state of $\mathcal{H}$, $\mathcal{H}|0\rangle = 0$. [Observe that if $c(\lambda) \in L^2(\mathbb{R})$, then the ground state of $\mathcal{H}$ would be degenerate; this is the real reason for requiring that $c(\lambda) \notin L^2(\mathbb{R})$.] Lastly we observe that the Heisenberg field operator reads

$$\begin{aligned} \phi(x, t) &\equiv e^{i\mathcal{H}t} \phi(x) e^{-i\mathcal{H}t} \\ &= \int A^\dagger(x, \lambda) e^{i\mathfrak{h}t} \lambda e^{-i\mathfrak{h}t} A(x, \lambda) d\lambda + \int c(\lambda) \lambda e^{-i\mathfrak{h}t} A(x, \lambda) d\lambda + \int A^\dagger(x, \lambda) e^{i\mathfrak{h}t} \lambda c(\lambda) d\lambda \end{aligned}$$

from which various multi-point functions may be defined. For example, the two-point function reads

$$\begin{aligned} \langle 0 | \phi(x_2, t_2) \phi(x_1, t_1) | 0 \rangle &= \delta(x_2 - x_1) w_2(t_2, t_1), \\ w_2(t_2, t_1) &\equiv \int c(\lambda) \lambda e^{-i\mathfrak{h}(t_2 - t_1)} \lambda c(\lambda) d\lambda, \end{aligned}$$

while the connected (truncated) four-point function reads

$$\begin{aligned} \langle 0 | \phi(x_4, t_4) \phi(x_3, t_3) \phi(x_2, t_2) \phi(x_1, t_1) | 0 \rangle_c \\ = \delta(x_4 - x_3) \delta(x_3 - x_2) \delta(x_2 - x_1) w_4(t_4, t_3, t_2, t_1), \end{aligned}$$

$$w_4(t_4, t_3, t_2, t_1) \equiv \int c(\lambda) \lambda e^{-i\hbar(t_4-t_3)} \lambda e^{-i\hbar(t_3-t_2)} \lambda e^{-i\hbar(t_2-t_1)} \lambda c(\lambda) d\lambda.$$

Higher-order generalizations are evident. Thus the ultimate theory is well defined and nontrivial (non-Gaussian) as claimed. Moreover, it should be noted that the solution we have found is *noncanonical* in the sense that

$$\pi(x) \equiv \dot{\phi}(x) = i[\mathcal{H}, \phi(x)] = \int B^\dagger(x, \lambda) (-ib^{-1} \partial / \partial \lambda) B(x, \lambda) d\lambda$$

is only a local form, and even more clearly, that

$$\begin{aligned} [\phi(x), \pi(y)] &= i\delta(x-y) b^{-1} \int B^\dagger(x, \lambda) B(x, \lambda) d\lambda \\ &= i\delta(x-y) [b^{-1} \int c^2(\lambda) d\lambda + \dots] \end{aligned}$$

contains a divergent c -number contribution. Nevertheless, it is also worth remarking that the classical limit ($\hbar \rightarrow 0$) of the quantum solution we have presented does indeed coincide with the classical solution of the original motivating theory as found from the scale-covariant form of the classical equations of motion [10, 11].

3 Lattice-Space Formulation for Ultralocal Models

We next turn our attention to reexpressing the solution just outlined within a lattice-space formulation. First we present the answer and then proceed to justify it. We assert that the (Euclidean space) characteristic functional $C\{h\}$ previously represented by the formal functional integral may be correctly calculated by the following lattice-space limit:

$$\begin{aligned} C\{h\} &= \lim_{a \rightarrow 0} \lim_{\epsilon \rightarrow 0} M \int \cdots \int \exp\{i\Sigma h_k \phi_k a \epsilon \\ &\quad - \frac{1}{2}(ba)\Sigma(\phi_{k^*} - \phi_k)^2 \epsilon^{-2} a \epsilon - \frac{1}{2}(ba)m^2 \Sigma \phi_k^2 a \epsilon - (ba)^3 g \Sigma \phi_k^4 a \epsilon \\ &\quad - \frac{1}{2}b^2(ba)^{-3} \Sigma[\gamma(\gamma+1)\phi_k^2 - \gamma a^{-2} \delta^2](\phi_k^2 + a^{-2} \delta^2)^{-2} a \epsilon\} \Pi d\phi_k. \end{aligned}$$

Here a is the spatial lattice spacing, $[a] = L$, ϵ is the Euclidean time lattice spacing, $[\epsilon] = L$, b denotes the dimensional parameter ($[b] = L^{-1}$) introduced previously, and $\delta = \delta(a)$, $[\delta] = L$, will be defined shortly. The dimensionless combination ba is the regularized form of Z introduced in the previous section, and $Z \rightarrow 0$ in the continuum limit. Note that the traditional terms in the action involve Z to positive powers, while the nontraditional term (proportional to γ) involves Z^{-3} .

To proceed further we first change integration variables from ϕ_k to $\lambda_k \equiv \phi_k a$ which gives

$$\begin{aligned} C\{h\} &= \lim_{a \rightarrow 0} \lim_{\epsilon \rightarrow 0} \tilde{N} \int \cdots \int \exp\{i\Sigma h_k \lambda_k \epsilon \\ &\quad - \frac{1}{2}b\Sigma(\lambda_{k^*} - \lambda_k)^2 \epsilon^{-1} - \frac{1}{2}bm^2 \Sigma \lambda_k^2 \epsilon - b^3 g \Sigma \lambda_k^4 \epsilon \\ &\quad - \frac{1}{2}b^{-1} \Sigma[\gamma(\gamma+1)\lambda_k^2 - \gamma \delta^2](\lambda_k^2 + \delta^2)^{-2} \epsilon\} \Pi d\lambda_k. \end{aligned}$$

Let $k = (k_1, k_2) \equiv (p, q) \in \mathbb{Z}^2$, where p refers to the temporal direction and q refers to the spatial direction of the lattice. Note that the spatial label on λ may be dropped as it appears only as a dummy integration variable. Thus we have

$$\begin{aligned} C\{h\} &= \lim_{a \rightarrow 0} \lim_{\epsilon \rightarrow 0} \Pi_q N \int \cdots \int \exp\{i \Sigma h_{p,q} \lambda_p \epsilon \\ &\quad - \frac{1}{2} b \Sigma (\lambda_{p+1} - \lambda_p)^2 \epsilon^{-1} - \frac{1}{2} b m^2 \Sigma \lambda_p^2 \epsilon - b^3 g \Sigma \lambda_p^4 \epsilon \\ &\quad - \frac{1}{2} b^{-1} \Sigma [\gamma(\gamma+1) \lambda_p^2 - \gamma \delta^2] (\lambda_p^2 + \delta^2)^{-2} \epsilon\} \Pi d\lambda_p. \end{aligned}$$

Observe, for each q , that the indicated expression is the lattice form of a one-dimensional quantum mechanical path integral. The limit $\epsilon \rightarrow 0$ may now be taken and the result written in two different but equivalent forms. On the one hand,

$$\begin{aligned} C\{h\} &= \lim_{a \rightarrow 0} \Pi_q \mathcal{N} \int \exp\{i \int h_q(t) \lambda(t) dt \\ &\quad - \int [\frac{1}{2} b \dot{\lambda}^2(t) + \frac{1}{2} b m^2 \lambda^2(t) + b^3 g \lambda^4(t) \\ &\quad + \frac{1}{2} b^{-1} \{\gamma(\gamma+1) \lambda^2(t) - \gamma \delta^2\} [\lambda^2(t) + \delta^2]^{-2} dt\} \mathcal{D}\lambda \end{aligned}$$

while, on the other hand,

$$C\{h\} = \lim_{a \rightarrow 0} \Pi_q \int \psi_\delta^*(\lambda) T e^{\int [i h_q(t) \lambda - \mathfrak{h}_\delta(\partial/\partial \lambda, \lambda)] \mathfrak{D}t} \psi_\delta(\lambda) d\lambda$$

where T denotes time ordering, and $\psi_\delta(\lambda)$ denotes the normalized ground state of $\mathfrak{h}_\delta$. According to the usual association it follows that

$$\begin{aligned} \mathfrak{h}_\delta(\partial/\partial \lambda, \lambda) &= -\frac{1}{2} b^{-1} \frac{\partial^2}{\partial \lambda^2} + \frac{1}{2} b m^2 \lambda^2 + b^3 g \lambda^4 \\ &\quad + \frac{1}{2} b^{-1} [\gamma(\gamma+1) \lambda^2 - \gamma \delta^2] (\lambda^2 + \delta^2)^{-2} + \text{const.} \end{aligned}$$

where the const. is chosen so that $\mathfrak{h}_\delta \psi_\delta = 0$ (in the present case const. = 0). With $\delta > 0$ and very small it follows that

$$\psi_\delta(\lambda) = B (\lambda^2 + \delta^2)^{-\gamma/2} e^{-y(\lambda, \delta)},$$

where $y(\lambda, \delta) \simeq y(\lambda)$, and B denotes a (real) normalization constant chosen so that

$$1 = \int |\psi_\delta(\lambda)|^2 d\lambda = B^2 \int (\lambda^2 + \delta^2)^{-\gamma} e^{-2y(\lambda, \delta)} d\lambda.$$

First let $\gamma > \frac{1}{2}$ in which case an adequate estimate of B when δ is very small is given by

$$1 = B^2 \int_{-\infty}^{\infty} (\lambda^2 + \delta^2)^{-\gamma} d\lambda = B^2 G / \delta^{2\gamma-1}$$

where $G = \int_{-\infty}^{\infty} (x^2 + 1)^{-\gamma} dx = \sqrt{\pi} \Gamma(\gamma - \frac{1}{2}) / \Gamma(\gamma)$. We now demand that $B^2 \equiv b^{2-2\gamma} a$, which finally determines $\delta = \delta(a)$ to be

$$\delta = (G b^{2-2\gamma} a)^{\frac{1}{2\gamma-1}}.$$

[If $\gamma = \frac{1}{2}$ then $\delta \propto b^{-1} \exp(-1/2ba)$.] Evidently $[\delta] = L$, and $\delta \rightarrow 0$ as $a \rightarrow 0$. Let $\psi_\delta(\lambda) = a^{\frac{1}{2}} c_\delta(\lambda)$, in which case $c_\delta(\lambda) = b^{1-\gamma} (\lambda^2 + \delta^2)^{-\gamma/2} e^{-\gamma(\lambda, \delta)}$. Evidently $c_\delta(\lambda) \rightarrow c(\lambda)$ as $\delta \rightarrow 0$. In the new notation

$$\begin{aligned} C\{h\} &= \lim_{a \rightarrow 0} \Pi_q a \int c_\delta(\lambda) T e^{\int [i h_q(t) \lambda - \bar{h}_\delta] \bar{\partial} t} c_\delta(\lambda) d\lambda \\ &= \lim_{a \rightarrow 0} \Pi_q \{1 - a \int c_\delta(\lambda) T (1 - e^{i \int h_q(t) \lambda dt}) e^{-\int \bar{h}_\delta \bar{\partial} t} c_\delta(\lambda) d\lambda\}. \end{aligned}$$

At this point the singularity at $\lambda = 0$ of $c(\lambda)$ is rendered harmless [by symmetry and by the factor $(1 - e^{i \int h_q(t) \lambda dt})$] so that within the integral, and to the order needed, the limit $\delta \rightarrow 0$ can be taken giving

$$C\{h\} = \lim_{a \rightarrow 0} \Pi_q \{1 - a \int c(\lambda) T (1 - e^{i \int h_q(t) \lambda dt}) e^{-\int \bar{h} \bar{\partial} t} c(\lambda) d\lambda\},$$

or finally

$$C\{h\} = \exp\left\{-\int dx \int c(\lambda) T (1 - e^{i \int h(x,t) \lambda dt}) e^{-\int \bar{h} \bar{\partial} t} c(\lambda) d\lambda\right\}.$$

Although the answer obtained from the nontraditional lattice-space formulation is expressed in Euclidean time it is clear that this result corresponds to the one derived by the operator method (which, of course, it was designed to do). Observe that the additional, nontraditional term in the lattice-space formulation was absolutely essential to obtain the correct, non-Gaussian result. Note as well the fine tuning needed of the δ -parameter which carries a dependence on the spatial lattice (a) even within the purely temporal lattice path integrals.

Next, we observe that

$$\begin{aligned} \lim_{g \rightarrow 0} C\{h\} &= \bar{C}_0\{h\} \equiv \lim_{a \rightarrow 0} \lim_{\epsilon \rightarrow 0} M \int \cdots \int \exp\{i \Sigma h_k \phi_k a \epsilon \\ &\quad - \frac{1}{2} (ba) \Sigma (\phi_k - \phi_k)^2 a \epsilon - \frac{1}{2} (ba) m^2 \Sigma \phi_k^2 a \epsilon \\ &\quad - \frac{1}{2} b^2 (ba)^{-3} \Sigma [\gamma(\gamma+1) \phi_k^2 - \gamma a^{-2} \delta^2] (\phi_k^2 + a^{-2} \delta^2)^{-2} a \epsilon\} \Pi d\phi_k \\ &= \exp\left\{-\int dx \int \bar{c}(\lambda) T (1 - e^{i \int h(x,t) \lambda dt}) e^{-\int \bar{h} \bar{\partial} t} \bar{c}(\lambda) d\lambda\right\} \end{aligned}$$

where

$$\begin{aligned} \bar{c}(\lambda) &\equiv b^{1-\gamma} |\lambda|^{-\gamma} e^{-\frac{1}{2} b m \lambda^2}, \\ \bar{h} &\equiv -\frac{1}{2b} \frac{\partial^2}{\partial \lambda^2} + \frac{\gamma(\gamma+1)}{2b \lambda^2} + \frac{1}{2} b m^2 \lambda^2 + (\gamma - \frac{1}{2}) m. \end{aligned}$$

This is the expression that defines the pseudo-free theory and it clearly is non-Gaussian.

It is easy to see that instead of ϕ^4 we could have studied, for example, ϕ^{40} just as well [replace $(ba)^3 g \phi_k^4$ by $(ba)^{39} g \phi_k^{40}$ in the nontraditional lattice-space form], but as the coupling constant $g \rightarrow 0$ the very same pseudo-free theory emerges. Observe, from the form of the answer for $C\{h\}$, that a perturbation expansion in powers of g about the pseudo-free theory makes perfectly good sense. Finally, we note that if a symmetry breaking potential is used, e.g., $g_1 \phi^4 + g_2 \phi^3$, then the model function is not even, $c(\lambda) \neq c(-\lambda)$, $L[p]$ is correspondingly more general, and the degeneracy of $\bar{h}$ is lifted so that $0 < \mu_0 < \mu_1 < \mu_2 < \mu_3 < \dots$.

4 Discussion

Although the ultralocal model has never been (and probably never will be) promoted for its physical applications, it nevertheless does serve as a model quantum field theory with the features of solubility and nontriviality. When approached by standard procedures of solution, the calculations are of such a nature that they lead either to a nonrenormalizable theory or to a free theory, both unacceptable results from a physical point of view. The unconventional solution arrived at in Sec. 2.3 on the basis of general principles and a few reasonable assumptions exhibits the consequences of a hard-core interaction in so far as the limit of the generating functional when the coupling constant is reduced to zero is not that of the free theory but (by definition) the pseudo-free theory. Given the full solution in the form of the characteristic functional $C\{h\}$, it is not too difficult to specialize h to being piecewise constant on lattice sites, and by an inverse Fourier transformation uncover a suitable effective lattice action that yields the desired result in the continuum limit. The resultant lattice action, as discussed in Sec. 3, contains the expected terms plus an additional local potential the net effect of which is to heavily weigh very small values of the field. This major shift of weighting, *e.g.*, for the λ variables, from moderate to very small values is the principal ingredient involved in changing the resultant process from Gaussian to Poissonian. As with any continuum limit, the various coefficients involved require fine tuning in order for a satisfactory limit to emerge. Moreover, the result can reasonably be considered to correspond to a quantization of the original system since the additional terms in the effective lattice action are implicitly proportional to $\hbar^2$.

It is natural to ask whether a similar additional local potential in the lattice action, perhaps one that also favors small field values and corresponds formally to a hard-core interaction, may be relevant in constructing nontrivial theories for various relativistic models such as $(\phi^4)_n$, $n \geq 4$. If the additional local potential is $O(\hbar)$ then it would not be expected to influence the classical limit. One may even speculate that the additional local potential in the relativistic case, just as in the ultralocal case, should be a properly regularized form of “const./ $\phi^2(x)$ ” since (i) it should be local to preserve relativistic covariance, (ii) a power law wave function modification, which formally gives rise to such a local potential, ensures that the additional local potential is indeed $O(\hbar)$, and (iii) the proposed additional potential formally scales in the same manner as $\pi^2(x)$ and thus may be considered as a renormalization correction to π^2 . Whether or not this speculation is close to the mark is just part of the larger question of how to find such additional local potentials should they exist, or, alternatively, to show that no such modification applies in the relativistic case. These clearly represent significant questions for the future.

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DEFINING LOCAL THERMODYNAMIC EQUILIBRIUM

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1 Introduction

The title of this paper may evoke two responses: “Everybody knows what local thermodynamic equilibrium means”, and: “Nobody has given a really satisfactory definition of local thermodynamic equilibrium”. I can defend both under suitable conditions; and I shall do so in the following, concentrating mainly on the second one. This article will not give answers, let alone a precise definition of local thermodynamic equilibrium; my aim is to bring the problems involved into sharp focus.

The generic case of a macroscopic system is one which requires thermodynamics in its description, but which is not in global equilibrium; any viscous fluid, any elastic medium with internal friction exhibit generation and dissipation of heat. That is, the concepts of heat — and thus thermodynamics — enter necessarily; but global thermodynamic equilibrium can be expected only under special circumstances. There is another line of arguments based on grounds of principle: Starting from Galilei group theoretic considerations in defining the concept of continuous media one finds that one cannot avoid the notion of internal energy and thus thermodynamics [RSS]. It is to be stressed that we are speaking of *thermodynamics*, not merely of *thermostatistics*.

Equilibrium relations are well established; and there are relations for non-equilibrium as well, e.g., Fick’s law, Fourier’s law, Onsager’s reciprocity relations. These are successful phenomenological relations akin to the spirit of linear response theory; but they presuppose tacitly that the medium under consideration can be considered as a thermodynamic equilibrium system *locally*, i.e., quantities used to describe systems in thermal equilibrium are assumed to exist at every point, that is, they are now continuous functions of space and time. This is how one would define local thermodynamic equilibrium phenomenologically; and this is not a deep insight, hence my first statement: “Everybody knows what local thermodynamic equilibrium means.”

Of course, not every macroscopic system can be considered as a continuous system of that sort; there are many cases where a microscopic view is necessary which takes the particle structure into account. Take as an example the solar photo- and chromosphere above relatively cool solar spots, which are governed by strong magnetic fields where particle motions are rather turbulent; there are good arguments that this is not a local thermodynamic equilibrium situation [St].

The question of systems treated microscopically brings us to the following problem: Under which circumstances can we extract a continuous media picture from the particle

point of view, i.e., under which circumstances can we define, and how can we define, the notion of local thermodynamic equilibrium from the viewpoint of Statistical Mechanics?

Equilibrium statistical mechanics is by now a well established, mathematically rigorous theory. The essential idealization involved is the passage to the thermodynamic limit, i.e., to a system of infinitely many particles in an infinite volume but with finite mean density; this gets rid of the problems of boundary effects, and enables one to describe phase transitions in a clear-cut way. It is worth recalling the basic axiom of equilibrium statistical mechanics: *Global equilibrium states are defined by the (classical or quantum) KMS condition*. For finite systems these are just the Gibbs states; for infinite systems the axiom is validated, firstly, by its success in reproducing thermodynamics (or better: thermostatics), and, secondly, by the theorems of Haag, Kastler and Trych-Pohlmeyer [HKT] and of Pusz and Woronowicz [PW]. Stationarity and dynamical stability are minimal requirements for a state to be called an equilibrium state; the first theorem is a stability theorem stating in essence that these two conditions — apart from technical assumptions — are necessary and sufficient to imply the KMS condition. The second one is a passivity theorem which can be rephrased by saying that it is precisely the KMS states which are distinguished by the second law of thermodynamics in Kelvin's formulation: There are no cyclic processes converting heat into mechanical work if the state of the systems obeys the KMS condition.

Physical intuition tells us that in a closed system of very many particles "most" initial states evolve towards equilibrium under the microdynamical time evolution. This can hold only if the states are not measured down to the last atom, that is, if not all (compatible) microscopic observables are measured. In order to convert the intuition into a theorem one has to specify what equilibrium means, which are the relevant observables, and, moreover, to define the notion of "most states". This then could replace the above axiom, albeit subject to another axiom that the exceptional states are irrelevant; but no such theorem is known to me.

For non-equilibrium systems the situation is far less clear. In order to arrive at a formulation of similar mathematical rigor we would like to perform the thermodynamic limit, too; but now it has to be done "pointwise", i.e., every macroscopic point, which is actually a macroscopically small volume, has to be considered as an infinite system. This involves a scaling limit. Problems connected to this difficulty have been tackled, but up to now no rigorous definition of local thermodynamic equilibrium has been given; and this is the justification of my second statement in the first paragraph. I shall elaborate on that.

The intuitive idea behind the definition of global equilibrium by the Gibbs states or, in the microcanonical ensemble, via equipartition of probability among the microstates, is that inevitably there are small perturbations not taken into account explicitly in the Hamiltonian, which, in the course of time, lead to a maximally mixed state, the (mean) energy being the only conserved quantity (apart from the particle density and external parameters). These perturbations — "a grain of dust in the system" — act as thermalizing agents. I want to stress that the definition of equilibrium is not a natural outcome of the dynamical laws of mechanics or quantum mechanics; actually it is an additional assumption, a first "hypothesis on thermalization", which goes beyond the dynamical laws. My thesis is that we need another thermalizing hypothesis singling out states exhibiting thermal equilibrium *locally*.

2 Phenomenological Considerations

Let us start with a definition:

A system is in local thermodynamic equilibrium at a given time if there is a well-defined temperature $T(x)$ at any macroscopic point x .

The term “well-defined” is meant not only mathematically but physically: the temperature is defined as the parameter governing thermal equilibrium between two systems, i.e., coupling a heat reservoir at temperature $T(x)$ to the system at point x would not change the state of the system. As an additional requirement we demand continuity of $T(x)$ in the interior of the volume under consideration. Furthermore, in order that the concept is of any use, local thermodynamic equilibrium should prevail for at least a finite time interval.

It is not hard to invent necessary conditions for local thermodynamic equilibrium with a microscopic point of view in mind. Consider a continuous system which is to be described hydrothermodynamically. In order to speak of local thermodynamic equilibrium we should be able to exhibit a “macroscopically small” volume V at the “macroscopic point x ”, i.e., small compared to length scales L on which typical macroscopic quantities change markedly. Furthermore, it is certainly necessary to have “many” particles in V . Finally, the time needed to reach local equilibrium should be small compared to time scales on which macroscopic changes occur.

To make this a bit more precise in the case of a fluid consider a small volume V flowing along with mean velocity $\langle v \rangle$. Define $l := V^{1/3}$ and $L := (\langle \nabla v \rangle / \langle v \rangle)^{-1}$. L denotes the length scale on which macroscopic velocity changes occur. Our first requirement then reads $l \ll L$ or, equivalently,

$$l \langle \nabla v \rangle \ll \langle v \rangle.$$

Let ρ be the mass density, m the particle mass; then the second requirement is

$$l \gg \left(\frac{m}{\rho} \right)^{1/3}.$$

A lower bound for the relaxation time is given by the mean time t_1 needed for a particle to traverse the volume V which is flowing along with velocity $\langle v \rangle$: if Δv is the mean velocity fluctuation, then $t_1 = l / \Delta v$. The time scale for macroscopic changes can be estimated by the time needed for a particle to traverse the length L : $t_2 := L / \langle v \rangle \approx \langle \nabla v \rangle^{-1}$, which has to be large compared to t_1 , consequently,

$$l \langle \nabla v \rangle \ll \Delta v.$$

This is what I have to say about the definition of local thermodynamic equilibrium on a phenomenological level.

3 The Boltzmann Equation

As a step intermediate between the macroscopic description and the full microscopic picture let us consider the example of a system described by the Boltzmann equation.

This is an intermediate — mesoscopic — description because it already involves an idealization, i.e., the Boltzmann-Grad limit [Gr] which leads from the full (classical) dynamical equations containing the exact particle interaction to an equation for the one-particle distribution function $f(\vec{x}, \vec{v}, t)$. The Boltzmann-Grad limit can be interpreted as letting the particle number N go to infinity and simultaneously shrinking the mass m and diameter σ (or the range of the interaction) of the particles such that the mass density and the cross section remain finite. The resulting system is to be considered as an ideal gas with no interaction left apart from an (idealized) hard core interaction.

Let us note that only in this case of vanishing interaction is the description by the one-particle distribution function sufficient: Even for thermodynamics, where we do not need the full information about the microscopic state, we would in general get wrong results when calculating the density of the potential energy with the help of $f(\vec{x}_1, \vec{v}_1, t)f(\vec{x}_2, \vec{v}_2, t)$ instead of the correct two-particle distribution function which need not factorize.

There is a special class of solutions: Hilbert's "normal" solutions [Hi] which are defined via their method of construction by a series expansion. Their first moments give the mass density ρ , the mean velocity $\vec{u}$ and the density of the kinetic energy ϵ_{kin} :

$$\begin{aligned}\rho(\vec{x}, t) &= \rho_0 \int f(\vec{x}, \vec{v}, t) d^3v, \\ u_k(\vec{x}, t) &= \rho_0 \int v_k f(\vec{x}, \vec{v}, t) d^3v, \quad k = 1, 2, 3, \\ \epsilon_{\text{kin}}(\vec{x}, t) &= \rho_0 \int |\vec{v}|^2 f(\vec{x}, \vec{v}, t) d^3v,\end{aligned}$$

the constant ρ_0 is the global mass density. Conversely, any normal solution is completely characterized by its first moments at a fixed initial time. That is, the above quantities provide a complete description of our system, they are determined by their initial values alone, and are sufficient to recover $f(\vec{x}, \vec{v}, t)$. This property is not shared by non-normal solutions.

Let us define the internal energy density

$$\epsilon_{\text{int}}(\vec{x}, t) := \epsilon_{\text{kin}}(\vec{x}, t) - |\vec{u}(\vec{x}, t)|^2 / 2\rho(\vec{x}, t);$$

it is then in the spirit of our idealization — the Boltzmann gas is an ideal gas — to put $\epsilon_{\text{int}}(\vec{x}, t)$ proportional to the temperature T at the point $\vec{x}$ at time t . We may replace $\epsilon_{\text{kin}}(\vec{x}, t)$ by $T(\vec{x}, t)$ thus arriving at a hydrothermal description of our system with a well-defined temperature at any macroscopic point, i.e., exhibiting local thermodynamic equilibrium.

Some remarks are in order.

1. The above characterization of normal solutions as local thermodynamic equilibrium states hinges on the proportionality of ϵ_{int} and T . But this can be considered a rather arbitrary definition of T taken over from thermostatics. Remember that the temperature is originally defined as parameter governing thermal equilibrium between two systems; there is no such justification like that here. An ideal gas in global thermal equilibrium exhibits a Maxwellian velocity distribution; but this is not the case for Hilbert's solutions, it holds true only in lowest approximation.
2. What is the thermalizing hypothesis behind the above definition of local thermodynamic equilibrium? The Boltzmann equation implies Boltzmann's H -Theorem, i.e.,

evolution towards global equilibrium for all solutions (and not only for the normal ones). Thus time reversal invariance is broken. The relevant assumption for that is hidden in the derivation of the Boltzmann equation: the assumption of “molecular chaos”, i.e., the assumption that the velocities of colliding particles are statistically independent before the collision. Of course, this is not the case immediately after the collision; but it is assumed that statistical independence is restored by the “grain of dust” before the next collision occurs. There is a second part of the thermalizing hypothesis: the assumption that we have to consider normal solutions, and that $\varepsilon_{\text{int}} \propto T$.

3. The Boltzmann equation is of course not the most general case of a classical mesoscopic equation (or set of equations). In general, one has a hierarchy of equations for the n -particle distribution functions, $n = 1, \dots, \infty$. But in that case, as in the case of general Boltzmann solutions, the connection to thermohydrodynamics is by no means obvious. Furthermore, one should start from quantum systems on the microscopic level, as is done in a paper of Narnhofer and Sewell [NS]; cf. the comments on that paper in Section 5.

4 The Scaling Limit

Let us now start from a general microscopic point of view. We have to keep in mind that defining local thermodynamic equilibrium means defining a macroscopic time evolution, too, starting from the microscopic interaction. Global thermal equilibrium states are time invariant ones, and the KMS condition is defined relative to the microscopic time evolution. But states exhibiting local thermodynamic equilibrium will, in general, not be stationary on a macroscopic scale although we want them to be locally KMS and hence time invariant microscopically.

The following gives an indication of the difficulties we have to face: Defining a macroscopic time evolution may come close to solving the problem of evolution towards equilibrium of fairly arbitrary initial states. Up to now only the *return* to equilibrium has been treated satisfactorily, where the initial states are locally disturbed equilibrium states.

Starting from a microscopic picture requires a twofold idealization: taking the thermodynamic limit *locally* (i.e., “many” particles in a small volume dV), and at the same time shrinking the (macroscopically) small volume to a point in order to define the local fields of velocities, pressure, temperature etc. This can actually be achieved by performing a scaling limit; a possible scheme which, at the same time, yields a macroscopic time evolution by means of a simultaneous time scaling will be described in the following.

Let the macroscopic system be contained in a region $\Gamma \subset \mathbb{R}^3$; for brevity we shall write $x \in \Gamma$ instead of $\vec{x} \in \Gamma$. We start — using the language of quantum theory — with a quasi-local algebra of observables $\mathcal{A}$ generated by a net $\{\mathcal{A}(\Lambda)\}_{\Lambda \subset \mathbb{R}^3}$, $\mathcal{A} = \bigcup \mathcal{A}(\Lambda)$, a net $\{H_\Lambda\}$ of corresponding Hamiltonians, and a net $\{\omega_\Lambda\}$ of states over $\mathcal{A}(\Lambda)$. We denote the states over $\mathcal{A}$ by $\mathcal{S}(\mathcal{A})$. Our aims are: 1) to “attach” a copy of $\mathcal{A}$ to every $x \in \Gamma$ (corresponding to the local infinite system); 2) to single out suitable nets $\{\omega_\Lambda\}$ defining “macroscopic” states, where a macroscopic state is a set $\{\omega_x \in \mathcal{S}(\mathcal{A})\}_{x \in \Gamma}$ of states over $\mathcal{A}$ attached to the points of Γ , describing the local properties; 3) to define the dynamics of the macroscopic states based on the microscopic interaction given by

a net of Hamiltonians $\{H_\Lambda\}$. The region Γ will be considered finite in order to leave open the possibility of introducing macroscopic boundary conditions.

Let us introduce a scaling parameter κ which will tend to infinity, and scale Γ by a factor of κ in order to "gain a microscopic point of view": $\Lambda_\kappa := \kappa\Gamma$; we assume that the origin of the macroscopic coordinate system is in Γ . The "microscopic" coordinates will be denoted by $\xi = \kappa x$. Now let us consider an arbitrary strictly local observable $A \in \mathcal{A}_{\Lambda_0}$, Λ_0 fixed. For sufficiently large κ , and x in the interior of Γ , Λ_0 as well as the translated region $\Lambda_0 + \kappa x$ are contained in Λ_κ ; the set of space translated observables $\{\sigma_{\kappa x}(A) \in \mathcal{A}(\Lambda_\kappa); \kappa \in \mathbb{R}^+\}$ is considered as representing the copy of A attached to $x \in \Gamma$.

We now require that the sequence $\omega_\kappa \equiv \omega_{\Lambda_\kappa}$ be chosen such that

$$(1) \quad \lim_{\kappa \rightarrow \infty} \omega_\kappa(\sigma_{\kappa x}(A)) =: \omega_x(A)$$

exists for all strictly local A and all $x \in \Gamma$. By continuity, ω_x can be extended to a state over $\mathcal{A}$. The set $\{\omega_x\}_{x \in \Gamma}$ is said to be a macroscopic state; and ω_y is the local state corresponding to the macroscopic point $y \in \Gamma$.

It is not hard to construct fairly arbitrary macroscopic states. The construction is roughly as follows. Start from a given set $\{\omega_x\}$ of local states depending continuously on x . Divide Λ_κ into disjoint subsets $\Lambda_{\kappa i}$ with centers $\xi_{\kappa i} = \kappa x_{\kappa i}$ making sure that the $x_{\kappa i}$ form a dense set of Γ as $\kappa \rightarrow \infty$ while the subsets $\Lambda_{\kappa i}$ increase to $\mathbb{R}^3$. Define ω_κ by gluing together the states $\omega_{x_{\kappa i}}$ belonging to the centers of the subsets of Λ_κ . Finally, take the limit $\kappa \rightarrow \infty$. We omit the subtleties of the construction; but we claim that the resulting macroscopic state has the set $\{\omega_x\}$ as local states. In this way we may construct a macroscopic state of continuously varying local inverse temperature $\beta(x)$ by choosing the states ω_x as $\beta(x)$ -KMS states.

Of course, in order to do so we need a well-defined microscopic time evolution. For that end let us assume that the automorphisms α_t^Λ of $\mathcal{A}_\Lambda$, given by

$$\alpha_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda},$$

define a limiting automorphism α_t of $\mathcal{A}$ or at least a map α_t^* of a suitable subset of $S(\mathcal{A})$ describing the time evolution of the microscopic system in the Heisenberg or in the Schrödinger picture. The concept of a KMS state is defined in the Schrödinger picture as well [Ro], but let us stick to the Heisenberg picture for simplicity.

The crucial point in this scheme is the definition of the time evolution of the *macroscopic* states. If a disturbance of the microscopic system spreads with a finite velocity one cannot expect any change of ω_x in time induced by the properties of the system at neighbouring points unless time is scaled, too. For the sake of generality let us introduce a time scaling factor $\gamma = \gamma(\kappa)$, $\lim_{\kappa \rightarrow \infty} \gamma(\kappa) = \infty$ and write $\tau = \gamma t$, where τ is now the microscopic time. In the spirit of equation (1) we would like to define $\omega_{t,x}$ by

$$(2) \quad \omega_{t,x}(A) := \lim_{\kappa \rightarrow \infty} \omega_\kappa \circ \alpha_{\gamma t}^{\Lambda_\kappa}(\sigma_{\kappa x}(A)),$$

provided the limit exists. Its existence is to be considered as an additional requirement on a net $\{\omega_\kappa\}$ defining a macroscopic state.

There is a case for taking $\gamma(\kappa) = \kappa^2$ (van Hove limit) [vH, Oj]; this choice corresponds to the fact that in Brownian motion the mean distance travelled in time t

is proportional to $\sqrt{t}$. On the other hand, if there is a finite propagation velocity of a microscopic perturbation (which is the case for lattice systems under fairly general assumptions [BR, Thm. 6.2.11]) one has to choose $\gamma(\kappa) \propto \kappa$ in order to arrive at finite macroscopic propagation velocities. This choice can be justified in still another way:

The requirement of Galilei invariance on the macroscopic as well as on the microscopic scale forces $\gamma = \kappa$.

This follows rather easily by consideration of the transformation properties of macroscopic quantities $f(x, t)$ given by limits of microscopic quantities which might depend on micro-states ω_κ and one or more observables A_κ :

$$f(x, t) = \lim_{\kappa} F_{\kappa}(\omega_{\kappa}, A_{\kappa}; \kappa x, \gamma(\kappa)t)$$

(the dependence of ω_κ and A_κ on κ is assumed to be only through the size of the micro-system; they should be the “same” for all κ , $\{\omega_\kappa\}$ forming a macro-state, and $A_\kappa \rightarrow A$). Applying Galilei boosts to the macroscopic variables (x, t) on the one hand, and to the microscopic variables $(\xi = \kappa x, \tau = \gamma(\kappa)t)$ on the other, one finds that necessarily $\gamma(\kappa) \propto \kappa$. However, as all *laboratory* frames are related by *macroscopic* Galilei transformations, the case for assuming microscopic invariance is rather weak.

The hard task is not to invent a scheme as the one sketched above, but to prove the existence of the macroscopic time evolution. I know of only two extreme cases where it has been done: the first one is the trivial case of a free Fermi gas [FR], the second one is an inhomogeneous mean field model [DRW]. But let us assume for a moment that this task has been performed. Then the scaling limit allows the definition of local thermal quantities, and of local thermodynamic equilibrium: for the microscopic system attached to a point x we have the microscopic time evolution given by α_t , and we can ask whether ω_x is a $\beta(x)$ -KMS state, i.e., whether the temperature at point x is given by $1/k\beta(x)$, try to compute its entropy density, etc.; any such macroscopic state $\{\omega_x\}$ is then to be called a local thermodynamic equilibrium state.

But this is too naïve a view!

Thesis: *Thermodynamics necessitates two “hypotheses on thermalization”: a first one to describe global equilibrium, i.e., to single out a set of states which are to be interpreted as equilibrium states; and a second one in order to enforce local equilibrium.*

Note that this goes beyond the microscopic picture.

To see that another hypothesis is needed for local thermodynamic equilibrium one may take a look at those two models for which the macroscopic time evolution is known to exist. The mean field model is definitely not trivial, but it suffers from the same defect as the first one: starting from an initial macroscopic state with well-defined local temperature $\beta(x)$, which can be constructed, one ends up — after arbitrarily short macroscopic times — with a state which is no longer locally KMS. This is to be expected in the first model because there is no interaction; it is to be expected of the second model, too: here we have to take $\gamma = 1$, i.e., time is not scaled at all because “mean field interaction” implies infinitely long range interaction, the velocity of spreading of a disturbance is infinite; and this means an immediate destruction of local equilibrium due to the effects of different temperatures at different points.

Similar results are to be expected in more realistic models (for which the above scheme is very hard to implement), hence the requirement of a “thermalizing agent”, cf. the molecular chaos hypothesis in the case of the Boltzmann equation.

For the general case one might think of introducing an averaging over microscopic times together with the limiting procedure; this is suggested by a paper of Ojima [Oj] on entropy production. A second possibility would be to adopt the ideas of information thermodynamics of finite volume systems [Rob] to the infinite case. But at the moment these are only vague ideas which will not be followed here.

5 Related Papers

There are many papers related to our topic; but this is not the place to review them all. I want to mention only a few dealing with nonequilibrium thermodynamics, in order to point out where or how a definition of local thermodynamic equilibrium has been circumvented. I hasten to stress that this is a highly subjective random selection.

The first paper I want to cite is the work of Davies [Da] treating a model for heat conduction. Here heat baths are coupled to every point of the system, their temperatures are to be adjusted according to the interaction. But this means introducing local thermodynamic equilibrium by hand.

As a representative of papers deriving classical macroscopic, or mesoscopic, equations — possibly showing irreversibility — from microscopic ones let me cite a paper by Narnhofer and Sewell [NS] on “Vlasov hydrodynamics of a quantum mechanical model”. They derive the Vlasov hierarchy from the microscopic quantum dynamics by means of a “hydrolocal” limit. But in that stage the concept of local thermodynamic equilibrium does not enter. The set of Vlasov equations describe classical correlation functions on a mesoscopic level comparable to the Boltzmann equation, but in greater generality. Further effort is needed to derive macroscopic hydrothermal equations from these functions. One important result of this paper is a theorem stating that the Vlasov dynamics respects “molecular chaos”: once the correlation functions factorize as time $t = 0$, they do so at all times. So it seems that this hypothesis is an important part of a thermalizing hypothesis in this case.

Sewell wrote a series of papers [Se1, Se2, Se3] determining restrictions imposed on macroscopic phenomenological laws by microphysics. A lot is achieved by that method; but due to the assumption of the very existence of the macroscopic laws the question of the definition of local thermodynamic equilibrium could be avoided. The last paper contains a nice model, a heavy quantum particle coupled to an infinite chain of small particles, which, in an appropriate limit, yields a classical equation of motion with a frictional force for the massive particle; i.e., the particle tends to the (trivial) equilibrium state of being at rest. But, of course, *local* thermodynamic equilibrium is trivial here.

Verbeure and collaborators have succeeded in deriving quantum central limit theorems [GVV1, GVV2, GVV3] which can be applied in the study of fluctuations. This has been done in a paper by Verbeure and Zagrebnov [VZ]. They showed the existence of a macroscopic limiting algebra of fluctuations. But this is not a macroscopic dynamics in the sense sketched above. First of all, it refers to an equilibrium situation, and furthermore, the resulting macroscopic algebra of fluctuations is dependent on the state underlying the construction.

6 Conclusion

Our conclusion can be kept very short: Whatever may have been done, a second thermalizing hypothesis is still missing.

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THINGS TO THINK ABOUT

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The way I interpret the challenge posed by the title of this conference is:

1. What developments in fundamental physical theory have we witnessed in the second half of this century?
2. What has been the contribution of tough and patient application of mathematical analysis or of heavy mathematical artillery or of recent mathematical concepts to the elucidation of the consequences of old or newer physical theories?
3. Do we have any guesses about the emergence of a new, deeper fundamental theory starting from radically different basic concepts. Any ideas what these concepts should be?
4. From which sources do we expect the key to such a development?
 - i) Experiments?
 - ii) Conceptual analysis of the incongruities in existing theories?
 - iii) Inspiration from studies in pure mathematics?

Let me give some personal opinions on some of these questions.

It appears fair to say that in the period in question the frontiers of fundamental theory meant essentially high energy physics, elementary particles, and relativistic quantum physics. The available framework for this was Quantum Field Theory. The oldest of the listed speakers at this conference started their scientific endeavours just around the major breakthrough in QED due to the development of covariant perturbation expansions and renormalization. Before this, say in 1940, it was widely felt that Quantum Field Theory was incurably sick and radically new ideas were needed. Many proposals were advanced then, some of which recurred in later decades, e.g., the existence of a smallest length, the replacement of space-time by a pure S-matrix theory, indefinite metric in Hilbert space. The success of renormalized QED showed that there was much more true physical information in QED than expected. On the other hand it did not transform QED into a well defined theory but only provided a set of rules which seemed to work miraculously. Dirac called the scheme ugly. Heisenberg considered it as a distraction of attention from the really important questions.

The most spectacular progress in the following forty years was the path from QED to the standard model of elementary particle physics. The essential ingredients are the principle of locality based on a given (classical) 4-dimensional space-time continuum with given causal structure and metric, together with the local gauge principle which generalizes QED by saying that we should consider a fibre bundle over space-time whose structure group relates to the internal symmetries and whose sections and connections provide the basic variables of the theory. These principles are used to construct a Lagrangian, defining a "classical" field theory, to which the recipes of "quantization" are applied (either by the canonical formalism or in the form of Feynman's sum over histories). Still, at this stage, the standard model is not a well defined theory.

Another development during these decades has been a substantial clarification of the conceptual and mathematical structure of the framework of the theory, beginning in the fifties with Wightman's axioms, the study of analyticity properties of vacuum expectation values of products of field operators, and other functions which can be considered as describing the solution of the theory; the discussion of inequivalent representations of algebras involving infinitely many degrees of freedom, and the understanding of particle aspects and collision theory in terms of local fields. Then came the formulation of the framework and its physical interpretation in terms of algebras of local observables without recourse to pointlike fields, the study of qualitative consequences independent of specific models, most notably the occurrence of global gauge groups, charge supers-election and their relation to statistics; the emergence of structural properties needed to ensure the existence of thermal states and "asymptotic completeness" (description of all states in terms of asymptotic particle configurations), prominent among them the so-called nuclearity condition. There were the very important results on scaling properties, critical behaviour, conformal invariance and "asymptotic freedom".

A third line of endeavour was the work on constructive field theory which aimed at showing that specific field theoretical models yield mathematically well-defined theories whose solutions satisfy the principles mentioned above. In the course of this effort many techniques of estimates were developed which found application in other areas such as nonrelativistic quantum statistical mechanics. Similar in spirit is lattice-gauge theory which has the even more ambitious aim of obtaining quantitative physical consequences in realistic models by the use of powerful computers.

Underlying all the work mentioned so far is the assumption of an a priori given space-time with Poincaré symmetry and strict locality. Concerning its status it is probably fair to say that quite a number of results of permanent value have been obtained, that there is reasonable hope that a viable theory can be developed along these lines without a fundamental change in the concepts; a theory which is adequate for the range of energies available now or becoming available in the next half century in the laboratories. It will contain probably a few dozen unexplained empirical constants, among them possibly an ultraviolet cut-off. Much has to be done and it will need more than one decade to reach this level. On the side of the general structural analysis, the area closest to my competence, the two most important tasks seem to me the understanding of the local gauge principle in quantum physics and the classification or characterization of specific models. I believe that these tasks are within reach if they are vigorously attacked in the coming years.

Coming now to point 3) it is evident that there is a deep conceptual problem in the

synthesis of general relativity with quantum physics. Due to this, but not only for that reason, there is a widespread feeling (which I share) that we stand on the verge of a revolution of fundamental theory and that the new theory will not start from a space-time continuum as a primary input. The views about the road to this goal are widely divergent. If we are modest and just want to achieve the abovementioned synthesis by going over from Maxwell-Dirac theory (which is QED) to Einstein-Maxwell-Dirac theory then we are invited to look at germs in a fiber bundle over a 4-dimensional base space where the fiber has the noncompact structure group $SL(2, \mathbb{C}) \times U(1)$ or perhaps $GL(2, \mathbb{C})$. The relation between two neighboring germs is described by a $(1,1)$ -spinor in the fiber. Then we have to build up the theory in the large by patching together these germs in a manner consistent with local quantum physics. The first part suggests the use of Ashtekar variables. The second part needs new ideas.

If we are more radical and discard the 4-dimensional space-time continuum even locally then the game is open to ingenious imagination. Superstrings? Non-commutative geometry? There is, however one aspect which may even be amenable to experimental study. This concerns the standard interpretation of quantum physics and the division problem. To do physics we must subdivide the universe. This is traditionally done by making first of all a distinction between the observer with his instruments and the observed system. The space-time coordinates are attributed to the side of the observer and considered as classical variables; these are then used to subdivide the observed system. Now the bodies needed to establish a reference system are necessarily macroscopic and they are also physical systems subject to quantum mechanical laws both with regard to their internal structure and their center of mass position. The meaningful accuracy in determining the position of an event with respect to a reference system can certainly be no better than 10^{-8} cm. Speaking of "physics at the scale of 10^{-16} cm" means that we study processes with energy-momentum transfer of 10^{16} cm $^{-1}$ (in units $\hbar = 1, c = 1$). Such a high-energy event may be rendered visible by a star of tracks which defines a collision center with the abovementioned poor accuracy. We may make the hypothesis that the event marks a point in space-time with an accuracy of 10^{-16} cm even if we cannot substantiate this claim by observations. But if we take this hypothesis more seriously than just an allegorical manner of speaking it means first of all that we accept the existence of real events, not synonymous with observation results, but as irreversible transitions from a possibility to a fact, spontaneously happening and unpredictable in the individual case. Moreover those real events form the operational basis for the space-time concept beyond the classical level. Thus the arrow of time must enter on the basic level of the theory. There is the problem of understanding the conditions for subdivision of complex phenomena into distinct events. One may expect that this needs idealizations analogous to those met in the subdivision of matter into elementary constituents. It is unlikely that a 4-dimensional continuum can serve as an adequate receptacle for patterns of events.

So much for speculations which partly are my answer to 4). I do believe in the harmony of good mathematics and good physics. I do not believe, however, in the hopes expressed in 4(iii) unless they go hand in hand with (ii) and as much as possible with (i).

NON-COMMUTATIVE GEOMETRY

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1 Introduction

Among the problems that naturally belong to the realm of mathematical physics, the question of the ultimate nature of space and time is one of the oldest, the most fundamental, and perhaps the most difficult. It touches our very basic mechanisms of perception of the Universe, and must be dealt with sooner or later if we want to understand the relationship that exists between the elementary constituents of matter, the forces acting between them, and their mutual interactions.

There are two important questions, raised already by ancient Greeks, to which we still don't have clear answers: First, do the notions of space and time reflect the existence of something that is present independently of any material object, and prior to it; second, are space and time continuous and infinitely divisible, or have they rather a discrete nature? Then, does the notion of a point in space and time possess any physical meaning? Parmenides and Zeno [1] answered the last question negatively, pretending that an infinitely precise localisation in space and time leads to logical contradictions. Newton [2], and later Kant [3], felt that space and time are prior to any physical phenomena and can be compared to an arena without which physical events could not occur (or "take place" – notice how even our language itself acknowledges such a point of view). Curiously enough, after so many centuries of the development of science, we still seem to be very far from the definite answer to these questions.

2 The Two Pillars of Modern Physics

The advent of the two most important new physical theories of this century, General Relativity and Quantum Mechanics, did not bring any better understanding of the nature of space-time at the microscopic level. General Relativity developed our knowledge about global properties of space and time at very large distances, and raised questions concerning the global topology of the Universe; in parallel, the methods of Differential Geometry, which were the best suited as the mathematical language of this theory, were very different from the methods of Quantum Physics, in which since its beginning physicists studied the properties of the algebra of the observables, considering the state vectors, as well as geometric points and trajectories, as artefacts and secondary notions. This approach has been inspired by the works of John von Neumann [4], and – as we

shall see in the following sections – has much in common with the non-commutative geometry, where the very notion of a point loses its meaning.

P. A. M. Dirac was already aware of the possibility of a radical modification of geometrical notions, and in his fundamental papers written in 1926 [5] he evokes the possibility of describing the phase space physics in terms of a non-commutative analogue of the algebra of functions, which he referred to as the “quantum algebra”, together with its derivations, which he called “quantum differentiations”.

Of course, this kind of geometry seemed strange and even useless from the point of view of General Relativity. Einstein thought that further problems of physics should be solved by subsequent development of geometrical ideas, and it seemed to him that to have $a \times b$ not equal to $b \times a$ was something that does not fit very well with geometry as he understood it [6].

During several decades, mostly in the sixties and the seventies, a lot of effort has been made in order to find a unifying approach to both these great theories. In doing so, people either tried to generalize one of the two theories so that the other one would follow, or tried to merge them together via embedding into some more general unified theory. Most of the activity in this field belonged essentially to the first category. The Hamiltonian formulation of General Relativity by R. Arnowitt, S. Deser and C. Misner [7], and later the Wheeler-DeWitt equation which generalizes the Schrödinger equation for quantum wave functions describing the state of a 3-dimensional geometry of the Universe [8] can be considered as the attempts to quantize General Relativity. Simultaneously, the geometric quantization developed by J. M. Souriau, D. Simms, A. Lichnerowicz, B. Kostant and S. Sternberg ([9], [10], [11]) has been an attempt to derive the rules of quantum mechanics by interpreting the observables and the state vectors as elements of algebras of operators and functions defined on classical manifolds with sufficiently rich and subtle geometry (e.g., symplectic manifolds, fibre bundles, jet spaces, and the like).

Unfortunately, it seems that these developments and generalizations of two great theories of twentieth century have never met in the middle of the road; on the contrary, the dominant impression is that they have led systematically in opposite and diverging directions. Nevertheless, they brought more insight into the understanding of the most important differences between these theories, and have thrown much light on possibilities of extensions and deformations of both domains of theoretical physics.

3 Deformations of Geometry

Both General Relativity and Quantum Theory have been tested by their creators against the classical limit, in which they tend in some sense to their classical counterparts, which are Newtonian mechanics and gravitation. The mathematical expression of this fact is formulated in terms of the *deformations* of the respective structures. The notion of deformation plays the central rôle in modern attempts that try to generalize the geometric structure of physical realm. To be more precise, we can cite the example of the relation existing between the Lorentz and the Galilei groups: The Lorentz group can be considered as a deformation of the Galilei group, with the characteristic parameter c^{-1} ; when this parameter tends to zero, the Lorentz group is said to undergo a *contraction* into the Galilei group. Similarly, the quantization procedure proposed by J. E. Moyal

[12] is a deformation of the usual Poisson algebra which is contracted back to it when the characteristic parameter of deformation which is here the Planck constant $\hbar$ tends to zero. Finally, Special Relativity may be considered as a contraction of General Relativity when the characteristic parameter G tends to zero (although some space-times different from the Minkowskian one can appear when the Ricci tensor is set to zero). Now, with three fundamental constants of Nature, $\hbar$, G and c^{-1} serving as deformation parameters, one can imagine seven different contractions of the hypothetical unified theory that would deserve the name of "Relativistic Quantum Gravity", and which is yet to be invented. The seven contractions correspond to the vanishing of:

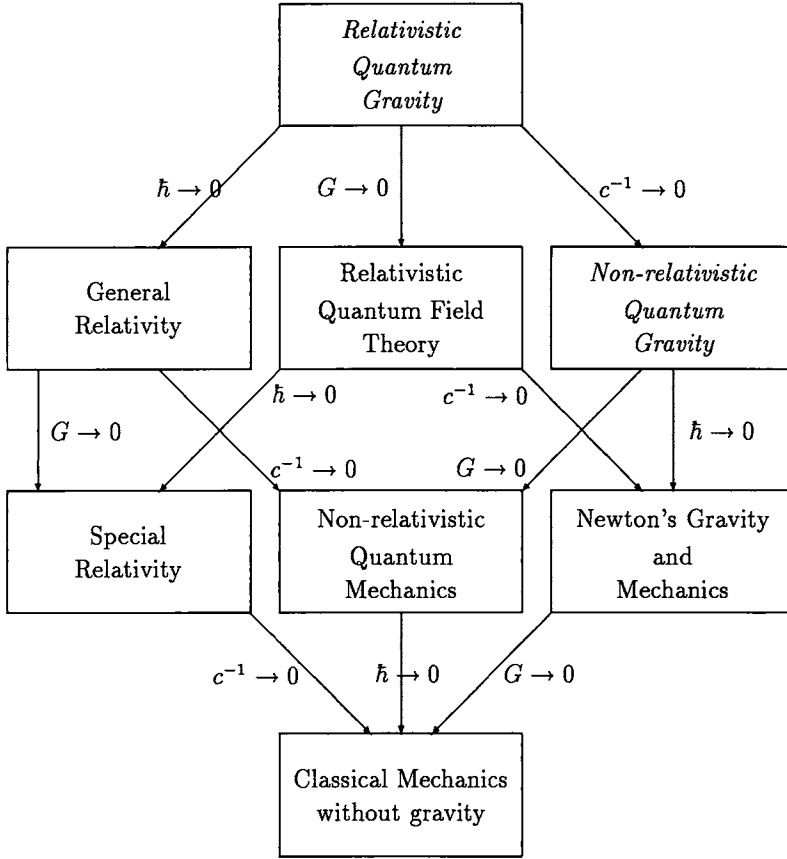
- a) one of the three parameters, i.e., $\hbar$, G , or c^{-1} only;
- b) two out of three parameters at once, i.e., $(\hbar \text{ and } G)$, $(\hbar \text{ and } c^{-1})$ and $(G \text{ and } c^{-1})$;
- c) all three parameters at once, i.e., $(\hbar, G \text{ and } c^{-1})$.

The diagram on the facing page shows the relations between the corresponding theories, as well as their usual denominations (when we know them ...). We did not take into account the fact that taking the double limits might be non-commutative, which cannot be excluded *a priori* and would have complicated our diagram even more.

Two of the theories displayed here have not found their realization yet: "Relativistic Quantum Gravity" and "Non-Relativistic Quantum Gravity". It is not at all clear whether these hypothetical theories can be realized without introducing some new deformation parameter depending on a new physical constant, and whether this constant should be independent or related to the three fundamental constants $\hbar$, c and G or not.

Of course, one can see that all known physics is based on observations concerning matter whose elementary constituents have masses negligible as compared with the Planck mass $m_P = \sqrt{\hbar c/G}$. This suggests that a new deformation parameter is nothing else but the ratio of the characteristic mass scale of the masses of observed elementary particles to the Planck mass. In such a case the new deformation would place the hypothetical theory inside the 3-dimensional simplex spanned by the directions $(\hbar, c^{-1}, G)$ in contrast to the known theories which are found on the edges or on the faces of the simplex. But it might happen that an entirely new and unknown parameter is needed in order to describe the new physical reality that will emerge at very high energies and at very small space-time scales.

The deformations envisaged by this new extension of physical theories are supposed to modify the intimate structure of the space-time, and in recently proposed theories, that can be labeled as non-commutative extensions of geometry and group theory, they follow the spirit of the most successful modification of this kind, which is the Born-Heisenberg matrix algebra of observables in which the functions describing the position and momentum of an elementary particle do not commute anymore. In past few years, the non-commutativity of algebras of functions describing the structure of differential manifolds has been explored; the resulting mathematical models can be referred to as general non-commutative geometries. There exist many variations and realizations of such models. In what follows, we shall present a short review of the most important deformations of usual structures used for the description of various manifolds serving as models of space and time.



4 Examples of Non-Commutative Geometries

4.1 The Grassmann Algebra

The Grassmann algebra of exterior forms in mathematics and the supersymmetry theory in physics that mixes bosonic and fermionic fields inspired the geometry of supermanifolds, which can be seen as a particular and very rigid model of non-commutative geometry. A supermanifold is described by the set of local commuting coordinates $\{x^i\}$, $i = 1, 2, \dots, N$, and the set of anticommuting generators of Grassmann algebra, $\{\theta^\alpha\}$, $\alpha = 1, 2, \dots, M$, such that $\theta^\alpha \theta^\beta + \theta^\beta \theta^\alpha = 0$. The generalized functions on such a manifold can be conceived as Grassmann algebra-valued functions on the ordinary manifold V_N described by the commuting variables x^i :

$$(1) \quad F = F_0(x) + W_\alpha(x)\theta^\alpha + \omega_{\alpha\beta}(x)\theta^\alpha\theta^\beta + \dots + \eta_{\alpha_1\alpha_2\dots\alpha_M}(x)\theta^{\alpha_1}\theta^{\alpha_2}\dots\theta^{\alpha_M}$$

Such an algebra is naturally Z_2 -graded, i.e., it splits into two linear subspaces, $\mathcal{A}_0$

and $\mathcal{A}_1$, whose elements are said to be of grade 0 or 1 respectively, and which satisfy the graded commutation relations:

$$(2) \quad FG + (-1)^{\text{grad}(F) \cdot \text{grad}(G)} GF = 0.$$

The grade is 0 for even powers of θ , and is 1 for odd powers of θ . In physics, when the variables $\{x^i\}$ describe the Minkowskian space-time, the variables θ^α and $\bar{\theta}^\beta$ transform as spinors under the representations $D^{(0, \frac{1}{2})}$ and $D^{(\frac{1}{2}, 0)}$ of the Lorentz group. This enables one to introduce the graded generalization of the Poincaré algebra, whose generators contain the combinations of ordinary (grade 0) and anticommuting (grade 1) derivations, e.g.,

$$(3) \quad \mathcal{D}_\alpha = \partial_\alpha + \sigma_{\alpha\dot{\beta}}^j \bar{\theta}^{\dot{\beta}} \partial_j, \quad \mathcal{D}_{\dot{\beta}} = \partial_{\dot{\beta}} + \sigma_{\alpha\dot{\beta}}^j \theta^\alpha \partial_j$$

which anticommute to give a combination of ordinary partial derivatives. The grade 1 “partial derivatives” ∂_α satisfy the graded Leibniz rule

$$(4) \quad \partial_\alpha \theta^\beta \theta^\gamma = \delta_\alpha^\beta \theta^\gamma - \delta_\alpha^\gamma \theta^\beta$$

from which it follows that

$$(5) \quad \partial_\alpha \partial_\beta + \partial_\beta \partial_\alpha = 0.$$

The theory of supersymmetry has been introduced by F. A. Berezin, A. Volkov and N. Akulov, J. Wess and B. Zumino, and later developed by B. Kostant, S. Sternberg and Y. Ne’eman, A. Rogers and M. Batchelor, K. Gawedzki, U. Bruzzo, A. Perez-Rendón, and many others, mostly during the seventies ([13] to [16]). The most important physical consequence of this theory is the possibility of elementary processes in which fermions can be transformed into bosons, and vice versa. Different hypotheses are made concerning the energy scale at which such processes could be observed; up to now, there is no conclusive evidence that supersymmetry might play any rôle at the energies accessible to our experimentation.

4.2 Differential Calculus on Lattices

The position and momentum operators of a quantum object generate the Heisenberg algebra of observables. It is defined by $2n$ generators satisfying the commutation relations of the corresponding Poisson algebra:

$$(6) \quad x^i x^j - x^j x^i = 0, \quad p^i p^j - p^j p^i = 0, \quad p^i x^j - x^j p^i = i\hbar \delta^{ij}$$

Such quantities are realized mathematically as *unbounded* operators in Hilbert spaces. Another idea may stem from these relations, if we recall that in the classical mechanics of a point-particle, in the absence of interactions one has $p^k = m(dx^k/dt)$; therefore, assuming that the time and its differential behave as commuting variables, we can write the last commutation relation as

$$(7) \quad dx^j x^k - x^k dx^j = \frac{i\hbar}{m} dt \delta^{jk}.$$

This suggests the possibility of a general deformation of usual differential geometry by introducing the following commutation relations:

$$(8) \quad x^i x^j - x^j x^i = 0, \quad x^i dx^j - dx^j x^i = \sum_k a_k^{ij} dx^k.$$

If we require the usual behaviour for the exterior differential d , i.e., $d^2 = 0$, then, applying d to the relations above, one gets the following consistency conditions:

$$(9) \quad a_k^{ij} = a_k^{ji}, \quad \sum_m a_m^{ik} a_n^{jm} = \sum_m a_m^{jk} a_n^{im}$$

i.e., the matrices $a^{(i)}$ with entries $a_k^{(i)j}$ should commute. The anticommutation of the differentials becomes also the consistency condition. In one dimension, the matrices $a_k^{(i)j}$ reduce to a number, and we have $x dx - dx x = a dx$. It can be shown that such a differential calculus remains consistent even if we replace the number a by a function of x . In two dimensions, there exist five different (up to a linear equivalence) realizations of such a calculus, besides the standard one with $a_k^{(i)j} = 0$. These non-commutative geometries have been studied by F. Müller-Hoissen [17] in order to describe gauge theories on lattices. This can be understood quite easily when we consider the direct consequence of $[x, dx] = a dx$, namely that one has

$$(10) \quad x^n dx = dx(x + a)^n,$$

and for any function of x ,

$$(11) \quad f(x) dx = dx f(x + a),$$

that is, commuting the differential dx through $f(x)$ has the effect of a translation of the argument by a . This non-commutative differential calculus simulates the action of a discrete translation group. One can consistently define *right* and *left* derivatives with respect to x :

$$(12) \quad (\partial_x^R f)(x) = \frac{1}{a} [f(x + a) - f(x)]$$

and

$$(13) \quad (\partial_x^L f)(x) = \frac{1}{a} [f(x) - f(x - a)],$$

satisfying the “twisted” Leibniz rule:

$$(14) \quad \partial_x^R(fg)(x) = (\partial_x^R f)g(x + a) + f(x)(\partial_x^R g)(x),$$

and

$$(15) \quad \partial_x^L(fg)(x) = (\partial_x^L f)g(x - a) + f(x)(\partial_x^L g)(x).$$

There are many interesting applications of this non-commutative differential calculus to the gauge field theory on lattices [18].

4.3 Quantum Spaces and Quantum Groups

A more radical deformation of usual behaviour of functions describing the coordinates and their differentials consists in modifying the commutation relations not only between the coordinates and their differentials, but also between the coordinates themselves, and between the differentials as well, which would represent a very profound modification of the space-time structure. Moreover, if we look for the transformations that would keep these new relations invariant, we discover that such transformations *cannot* be described by means of ordinary groups; which therefore need to be generalized. Such new generalizations have been introduced by V. Drinfeld, L. Faddeev and S. L. Woronowicz, ([19, 20, 21]), and they are known under the name of "Quantum Groups". The literature on this subject is very abundant; we shall cite the papers by S. L. Woronowicz [21] as well as those by L. C. Biedenharn [22], J. Wess and B. Zumino [23], L. A. Takhtajan [24], and V. G. Kac [25]; the list is far from being exhaustive, so that we shall limit ourselves to an outline of the main idea illustrated by a simple example. In conformity with the spirit of quantum field theories, the most important mathematical object to be studied is the algebra of observables, which are usually functions of few fundamental ones. This approach can be extended to the mathematical study of Lie groups: Indeed, we can learn almost everything concerning the group's structure from the algebraic structure of functions (real or complex) defined on it. Consider a compact manifold G which is also a Lie group; let e denote its unit element. The algebra A of functions defined on G has a very particular structure, which is implemented by the following three mappings:

i) for each $f \in A$, there is an element of $A \otimes A$, denoted by Δf , such that $\Delta f(x, y) = f(xy)$; The mapping Δ :

$$(16) \quad A \longrightarrow A \otimes A$$

is called the *coproduct*.

ii) There exists a natural mapping from A into $\mathbb{C}$ (or $\mathbb{R}$) defined by

$$(17) \quad \epsilon : f \longrightarrow f(e) \in \mathbb{C}$$

which is called the *co-unit*.

iii) There exists a natural mapping of A into itself:

$$(18) \quad (Sf)(x) = f(x^{-1})$$

which is called the *antipode*.

It is easy to see that in the case of the algebra of functions defined on a Lie group the co-product is non-commutative if the Lie group is non-commutative; however, the multiplication law in the algebra A itself remains commutative as long as we consider the functions taking their values in $\mathbb{C}$ or $\mathbb{R}$. This particular structure of an associative commutative algebra A with the three operations defined above, the *co-product*, the *co-unit* and the *antipode* is called a *Hopf algebra*. Now, the natural extension that comes to mind is to abandon the postulate of the commutativity of the product in A ; in this case, the structure is named a *Quantum Group*. It should be stressed that a quantum group

is not a group, but a general algebra which behaves as the algebra of functions defined on a Lie group only in the commutative case. One of the most interesting aspects of this theory is the fact that quantum groups arise quite naturally as the transformations of non-commutative geometries known under the name of *quantum spaces* introduced by Yu. Manin, J. Wess and B. Zumino, and others. We shall illustrate how a quantum group can be constructed by a simple example in two dimensions called the *Manin plane*.

Consider two “coordinates” x and y spanning a linear space and satisfying

$$(19) \quad xy = q yx$$

with a complex parameter q different from 1. Consider a transformation

$$(20) \quad x' = a x + b y, \quad y' = c x + d y$$

which preserves the relation $xy = q yx$, i.e., such that

$$(21) \quad x'y' = q y'x'.$$

We shall suppose that the quantities a, b, c, d commute with the “coordinates” x, y ; the simplest realization of this requirement is achieved by assuming (disregarding the nature of the entries of the matrix) that the multiplication of x by a, b , etc. is tensorial, i.e., when we set by definition

$$(22) \quad x' = a \otimes x + b \otimes y.$$

Then the conservation of the q -commutation relations between x and y leads to the following rules for a, b, c and d :

$$(23) \quad ac = q ca, \quad bd = q db, \quad ad = da + q cb - \left(\frac{1}{q}\right)bc$$

In order to fix all possible binary relations between the coefficients a, b, c and d we need three extra relations, which would define bc, ab and cd . Such relations can be obtained if we define the “differentials”

$$(24) \quad \xi = dx, \quad \eta = dy, \quad \xi^2 = 0, \quad \eta^2 = 0,$$

satisfying the twisted p -commutation relations

$$(25) \quad \xi\eta + \left(\frac{1}{p}\right)\eta\xi = 0$$

with a new complex parameter p . Assuming that the exterior differentiation commutes with the transformation matrix and requiring the same relations for ξ' and η' , we get

$$(26) \quad bc = \left(\frac{q}{p}\right)cb, \quad ab = p ba, \quad cd = p dc.$$

With these relations the matrix algebra defined above becomes associative and can be given the structure of a Hopf algebra as follows:

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}$$

and the co-unit as

$$\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The antipode S of a quantum matrix should be defined as its inverse. In order to make such a definition operational, we need a non-commutative generalization of the determinant of a matrix. Such a “ (q, p) -determinant” should be defined as the combination of parameters appearing in the transformation law for the “elementary area element”, i.e., the exterior product of the differentials ξ and η :

$$(27) \quad \xi' \eta' = D_q \xi \eta,$$

which yields immediately

$$(28) \quad D_q = ad - pbc = da - \left(\frac{1}{q}\right)bc.$$

The determinant D_q commutes with a and d , but has non-trivial commutation relations with the off-diagonal elements b and c (In what follows, we shall omit the subscript q for the sake of simplicity):

$$(29) \quad Db = \left(\frac{p}{q}\right)bD, \quad Dc = \left(\frac{q}{p}\right)cD.$$

It should also possess an inverse D^{-1} , which in fact is a new element extending the algebra, and satisfying

$$(30) \quad D^{-1}D = \mathbb{1} = DD^{-1}.$$

Applying these identities to the commutation relations of D , one finds easily that D^{-1} commutes with a and b , and satisfies

$$(31) \quad bD^{-1} = \left(\frac{p}{q}\right)D^{-1}b, \quad cD^{-1} = \left(\frac{q}{p}\right)D^{-1}c.$$

It is easy to see that

$$(32) \quad \Delta(D) = D \otimes D, \quad \Delta(D)\Delta(D^{-1}) = \Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$$

and

$$(33) \quad \Delta(D^{-1}) = D^{-1} \otimes D^{-1}.$$

The antipode of any matrix can be determined now as follows:

$$(34) \quad S \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = D^{-1} \begin{pmatrix} d & \left(\frac{-1}{q}\right)b \\ -qc & a \end{pmatrix} = \begin{pmatrix} d & \left(\frac{1}{p}\right)b \\ -pc & a \end{pmatrix} D^{-1}.$$

Also

$$(35) \quad S(D) = D^{-1}, \quad S(D^{-1}) = S(D),$$

but $S^2 \neq \mathbb{1}$. The inverse of the antipode mapping can also be defined as

$$(36) \quad S \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = D \begin{pmatrix} a & pqb \\ (\frac{1}{pq})c & d \end{pmatrix}.$$

The algebra generated by the matrices defined above, whose entries a, b, c and d satisfy the (q, p) -commutation relations is a Hopf algebra; it is denoted by $GL_{p,q}(2, \mathbb{C})$.

A differential calculus on such algebras has been developed by S. L. Woronowicz; the notion of covariant differentiation, if it can be introduced properly, may lead to new and rich extensions of the ideas of connections, curvatures and gauge fields.

4.4 Non-commutative Differential Geometry

In the examples of non-commutative generalizations of spaces of states or of algebras of observables, we have looked up to now only at the linear cases. A more general non-commutative geometry should imitate the situations encountered in the ordinary differential geometry of manifolds. Therefore we should replace the algebra of smooth functions on a manifold (the maximal ideals of this algebra can be identified as points of the corresponding manifold) by a more general associative algebra, which can be non-commutative. The *derivations* of this algebra will naturally generalize the notion of *vector fields*; their dual space will generalize the fields of *1-forms*, and one can continue as far as possible, trying to construct the analogues of a metric, integration, volume element, Hodge duality, Lie derivatives, connection and curvature, and so forth. It is amazing how almost all of these objects known from the classical version of differential geometry find their counterparts in the non-commutative case. The differential algebras of this type have been studied by A. Quillen [27], A. Connes [28] and M. Dubois-Violette [29]; their application to the new mathematical models of the gauge theories, including the standard model of electroweak interactions of Weinberg and Salam, has been worked out by M. Dubois-Violette *et al.* [30], by A. Connes and J. Lott [31], R. Coquereaux *et al.* [32], and other authors. Here we shall give the simplest example of a realization of the non-commutative geometry proposed in [30], realized with the algebra of complex $n \times n$ matrices, $M_n(\mathbb{C})$. Any element of $M_n(\mathbb{C})$ can be represented as a linear combination of the unit $n \times n$ matrix $\mathbb{1}$ and $(n^2 - 1)$ hermitian traceless matrices E_k , $k = 1, 2, \dots, (n^2 - 1)$:

$$(37) \quad B = \beta \mathbb{1} + \sum \alpha^k E_k.$$

One can choose a basis in which the following relations hold:

$$(38) \quad E_k E_m = \left(\frac{1}{n}\right) g_{km} \mathbb{1} + S_{km}^j E_j - \left(\frac{i}{2}\right) C_{km}^j E_j$$

with real coefficients satisfying $S_{km}^j = S_{mk}^j$, $S_{km}^k = 0$, $C_{km}^j = -C_{mk}^j$, $C_{km}^k = 0$, and $g_{km} = C_{kl}^p C_{pm}^l$. Then C_{kl}^m are the structure constants of the Lie group $SL(n, \mathbb{C})$, and g_{kl} its Killing-Cartan metric tensor. All the derivations of the algebra $M_n(\mathbb{C})$ are *interior*, i.e., the basis of the derivations is given by the operators ∂_k such that

$$(39) \quad \partial_k E_m = \text{ad}(iE_k) E_m = i[E_k, E_m] = C_{km}^l E_l.$$

By virtue of the Jacobi identity, we have

$$(40) \quad \partial_k \partial_m - \partial_m \partial_k = C_{km}^l \partial_l.$$

The linear space of derivations of $M_n(\mathbb{C})$, denoted by $\text{Der}(M_n(\mathbb{C}))$, is *not* a left module over the algebra $M_n(\mathbb{C})$ — this is the first important difference with respect to the usual differential geometry, in which a vector field can be multiplied on the left by a function, producing a new vector field. The canonical basis of 1-forms dual to the derivations ∂_k is defined formally by the relations

$$(41) \quad \theta^k(\partial_m) = \delta_m^k \mathbb{1}.$$

These 1-forms span a left module over $M_n(\mathbb{C})$, i.e., $E_l \theta^k$ is also a well-defined 1-form; indeed, $E_l \theta^k(\partial_m) = E_l \delta_m^k \mathbb{1} = E_l \delta_m^k$.

The exterior differential d is defined as usual, first on the 0-forms (“functions”) by the identity

$$(42) \quad df(X) = Xf,$$

with f a function, X an arbitrary vector field. Here we have

$$(43) \quad (d\mathbb{1})(\partial_m) = \partial_m \mathbb{1} = \text{ad}(iE_m)\mathbb{1} = i[E_m, \mathbb{1}] = 0,$$

so that $d\mathbb{1} = 0$, and

$$(44) \quad dE_k(\partial_m) = \partial_m(E_k) = i[E_k, E_m] = C_{mk}^l E_l.$$

Because the Lie algebra $SL(n, \mathbb{C})$ is semi-simple, the matrices C_{km}^l are non-degenerate, and the above relation can be solved in θ^k 's giving the explicit expression

$$(45) \quad dE_k = C_{km}^l E_l \theta^m.$$

The fact that $d^2 = 0$ follows then directly from the Jacobi identity. The Grassmann algebra of p -forms is defined as usual, with the wedge product

$$(46) \quad \theta^k \wedge \theta^m = \frac{1}{2}(\theta^k \otimes \theta^m - \theta^m \otimes \theta^k).$$

We have then

$$(47) \quad d\theta^k + \frac{1}{2}C_{mi}^k \theta^m \theta^i = 0.$$

If we define the *canonical 1-form* $\theta = \sum E_k \theta^k$, we can easily prove that it is coordinate-independent. Moreover, it satisfies the important relation

$$(48) \quad d\theta + \theta \wedge \theta = 0.$$

Let ω be a p -form. The anti-derivation i_X with respect to a vector field X can be defined as usual,

$$(49) \quad (i_X \omega)(X_1, X_2, \dots, X_{p-1}) = \omega(X, X_1, X_2, \dots, X_{p-1}).$$

The Lie derivative of a p -form ω with respect to a vector field X is defined as

$$(50) \quad L_X \omega = (i_X d + d i_X) \omega.$$

It is easy to check now that the 2-form $\Omega = d\theta$ is invariant with respect to the derivations of A , i.e., that

$$(51) \quad L_X \Omega = 0$$

for any vector field X belonging to $\text{Der}(M_n(\mathbb{C}))$. The 2-form Ω is also non-degenerate, and it is a closed 2-form by definition, because

$$(52) \quad d\Omega = d^2\theta = 0.$$

The 2-form Ω defines a Hamiltonian structure in the algebra $M_n(\mathbb{C})$ in the following sense: Let $f \in M_n(\mathbb{C})$ be an element of our algebra; then Ham_f is the *Hamiltonian vector field of f* defined by the equality

$$(53) \quad \Omega(\text{Ham}_f, X) = X f$$

for any X belonging to $\text{Der}(M_n(\mathbb{C}))$. The *Poisson bracket* of two “functions” (observables) f and g is then defined as

$$(54) \quad \{f, g\} = \Omega(\text{Ham}_f, \text{Ham}_g).$$

A simple computation shows then that

$$(55) \quad \{E_k, E_m\} = \Omega(\partial_k, \partial_m) = i [E_k, E_m].$$

Therefore, in our simple version of non-commutative geometry, classical and quantum mechanics merge into one and the same structure: The Poisson bracket of any two matrix “functions” (observables) is equal, up to a factor that can be chosen as the Planck constant $\hbar$, to their *commutator*.

This simple and beautiful picture is of course somewhat perturbed in the case of *infinite-dimensional algebras* for which not all the derivations are interior and might have other representations than the commutator with an observable.

The volume element induced by the canonical Cartan-Killing metric and the corresponding Hodge duality $\star$ can be also introduced in a classical manner. The volume element is given by

$$(56) \quad \eta = \frac{1}{(n^2 - 1)!} \epsilon_{i_1 i_2 \dots i_{n^2-1}} \theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_{n^2-1}}.$$

Any $(n^2 - 1)$ -form is proportional to the volume element η ; the integral of such a form will be defined as the *trace* of the matrix coefficient in front of η . Then the scalar product is readily introduced for any pair of p -forms α and β as follows:

$$(57) \quad (\alpha, \beta) = \int (\alpha \wedge \star \beta).$$

At this stage we can introduce a non-commutative analogue of Kaluza-Klein type theory, which will lead to a generalization of gauge field theories. In ordinary differential

geometry the fact of using a Cartesian product of two differential manifolds, or a fibre bundle locally diffeomorphic with such a product, can be translated into the language of the corresponding function algebras; as a matter of fact, in the case of the Cartesian product of two manifolds, the algebra of functions defined on it is the tensor product of algebras of functions defined on each of the manifolds separately.

Consider the space-time manifold V_4 with its algebra of smooth functions $C^\infty(V_4)$. Let us define the tensor product

$$(58) \quad A = C^\infty(V_4) \otimes M_n(\mathbb{C}).$$

It can be shown (cf. [30]) that

$$(59) \quad \text{Der}(A) = [\text{Der}(C^\infty(V_4)) \otimes \mathbb{1}] \oplus [C^\infty(V_4) \otimes \text{Der}(M_n(\mathbb{C}))].$$

In other words, a general derivation in our tensor product algebra replacing the algebra of smooth functions on a fibre bundle space, can be written as the following vector field

$$(60) \quad X = X^\mu(x)\partial_\mu + \xi^k(x)\partial_k$$

with $\mu, \nu = 0, 1, 2, 3$; $k, l = 1, 2, \dots, (n^2 - 1)$. A general 1-form defined on such vector fields splits naturally into *four* different components:

$$(61) \quad A = A_\mu^0(x)\mathbb{1}dx^\mu + A_\mu^k(x)E_kdx^\mu + B_m^0(x)\mathbb{1}\theta^m + B_m^k(x)E_k\theta^m.$$

The exterior differential of a 1-form A takes into account the two kinds of differentiation; e.g., for a general matrix-valued 0-form ("function") $\Phi = \Phi^0\mathbb{1}(x) + \Phi^m(x)E_m$ we have

$$(62) \quad d(\Phi) = (\partial_\mu\Phi^0)dx^\mu + (\partial_\mu\Phi^m)E_mdx^\mu + \Phi^m C_{km}^l E_l\theta^k.$$

The notion of *covariant derivation* can be introduced quite naturally by considering a free (right) hermitian module H over the algebra A . If we choose a *unitary element* e in H , then any element of H can be represented as $\Phi = eB$, with $B \in A$. Then the *covariant derivative* must have the following basic property:

$$(63) \quad \nabla(\Phi D) = (\nabla\Phi)D + \Phi \otimes dD$$

for arbitrary $\Phi \in H$, $D \in A$. Now, if $\Phi = eB$, we shall have

$$(64) \quad \nabla\Phi = (\nabla e)B + e \otimes dB,$$

and there exists a unique element $\alpha \in \Lambda^1(M_n(\mathbb{C}))$ such that

$$(65) \quad \nabla e = e \otimes d\alpha$$

satisfying the hermiticity condition $\bar{\alpha} = -\alpha$. The elements B and α are called the *components* of the field Φ and the connection ∇ in the gauge e .

Let U be a unitary matrix from the algebra A . Under a change of gauge

$$(66) \quad e \longrightarrow eU$$

the components B and α transform as follows:

$$(67) \quad B \longrightarrow U^{-1}B, \quad \alpha \longrightarrow U^{-1}\alpha U + U^{-1}dU.$$

This is the exact analogue of the gauge theory in the non-commutative case. When applied to the connection 1-form (denoted now by A instead of α), these principles lead to the following expression of the gauge field tensor $F = dA + A \wedge A$:

$$(68) \quad F = (F_{\mu\nu}^0 \mathbb{1} + G_{\mu\nu}^k E_k) dx^\mu \wedge dx^\nu + [(D_\mu B_l^0) \mathbb{1} + (D_\mu B_l^m E_m)] dx^\mu \wedge \theta^l + G_{kl}^m E_m \theta^k \wedge \theta^l$$

where

$$(69) \quad F_{\mu\nu}^0 = \partial_\mu A_\nu^0 - \partial_\nu A_\mu^0$$

represents the abelian $U(1)$ -gauge field;

$$(70) \quad G_{\mu\nu}^k = \partial_\mu A_\nu^k - \partial_\nu A_\mu^k + C_{lm}^k A_\mu^l A_\nu^m$$

represents the $SU(2)$ -gauge field;

$$(71) \quad D_{mu} B_k^0 = \left(\frac{1}{m}\right)(\partial_\mu B_k^0)$$

is the derivative of the scalar triplet B_k^0 ;

$$(72) \quad D_\mu B_k^m = \left(\frac{1}{m}\right)(\partial_\mu B_k^m + C_{sr}^m A_\mu^s B_k^r)$$

is the covariant derivative of the scalar (Higgs-type) multiplet B_k^m ; finally,

$$(73) \quad G_{kl}^m = \left(\frac{1}{m^2}\right)(C_{kl}^p B_p^m - C_{sr}^m B_k^s B_l^r)$$

represents the potential contribution of the Higgs multiplet. Here m is the dimensional parameter ($\dim[m] = \text{cm}^{-1}$) introduced in order to give the proper dimension to the 1-forms θ^k . The parameter m can be later related to the characteristic mass scale of the theory. The generalized action integral is equal – in conformity with the definition of integration on the algebra of p-forms in the non-commutative case – to the *trace* of the integral over space-time V_4 of the expression $F \wedge \star F$:

$$(74) \quad \text{Tr} \int (F \wedge \star F) d^4x$$

The multiplet of scalar fields B_l^m plays here the rôle of the symmetry-breaking Higgs-Kibble field, whose quartic potential appearing in the last part of the action integrand possesses multiple local minima or maxima. In this example, when all other fields are set equal to 0, there exist several configurations of B_l^m corresponding to vacuum states representing different gauge orbits. Indeed, it is easy to see that $G_{kl}^m = 0$ not only when $B_l^m = 0$, but also for $B_l^m = \delta_l^m$. These two vacua *cannot* be transformed one into another by means of a gauge transformation, which is a novel feature when compared with the known classical versions of gauge theory coupled with Higgs fields.

Although this simple generalization of gauge theory including a non-commutative sector of differential geometry contains naturally the gauge group $SU(2) \times U(1)$, the Higgs multiplet arising here does not have the generally-required properties, i.e., *it is not* a doublet of complex scalar fields that can be coupled in a different way to the left- and right-handed fermions; we have instead a tensor multiplet B_l^m which can admit 16 different vacuum configurations, most of them degenerate saddle points in the parameter space. Also the mass spectrum of bosons appearing in the theory is not satisfactory. Developing the bosonic fields present in the model, A_μ^0 , A_μ^k and B_k^0 , and linearizing the equations around the vacua given by $B_l^m = 0$ or $B_l^m = \delta_l^m$ respectively, we obtain on the gauge orbit $B_l^m = 0$:

- masses of A_μ^0 and A_μ^k equal zero;
- masses of B_l^0 and B_l^m all equal $\sqrt{nm}$;
- whereas on the gauge orbit $B_l^m = \delta_l^m$,

the $U(1)$ gauge field $A^0\mu$ remains massless while the $SU(2)$ -gauge field acquires the mass $\sqrt{2nm}$; the scalar multiplet B_m^0 acquires the mass $\sqrt{2}m$, and the Higgs multiplet itself, B_l^m , develops a mass spectrum with values 0, $\sqrt{2}m$ and $2\sqrt{2}m$, which makes this version of unified $SU(2) \times U(1)$ theory unrealistic.

More realistic versions of non-commutative gauge models, reproducing quite well all the properties of the electroweak interactions required by the experiment, have been proposed by R. Coquereaux *et al.* [32], by A. Connes and M. Lott [31], by M. Dubois-Violette *et al.*, [30], and by J. Fröhlich *et al.*, [33]. In all these models the non-commutative algebra of complex matrices is tensorized with a Z_2 -graded algebra, which in the simplest realization can be conceived of as an algebra of 2×2 matrices that splits into two linear subspaces called “*even*” (corresponding to diagonal matrices) and “*odd*” (corresponding to the off-diagonal matrices), with respective *grades* being 0 and 1, which under matrix multiplication add up *modulo* 2. The exterior derivations change the grade of an element by 1, and satisfy the *graded Leibniz rule*

$$(75) \quad d(AB) = (dA)B + (-1)^{\text{grad}(A)\text{grad}(B)} A dB.$$

This enables one to represent the connection form (interpreted as the gauge-field potential) in the following form:

$$(76) \quad \begin{pmatrix} A & W^+ \\ W^- & Z \end{pmatrix}$$

where the gauge fields A and Z belong to the even part of the algebra, while the fields W^+ and W^- belong to the odd part; moreover, all these fields are themselves 2×2 matrix-valued 1-forms. Developing this theory around the appropriately chosen vacuum configuration one can quite correctly reproduce the mass spectrum, with the mass of neutral Z -boson $\frac{2}{\sqrt{3}}$ times bigger than the mass of the charged W^- boson, which corresponds to the Weinberg angle of 30° . More details can be found in the papers cited above. There exists a tentative generalization of graded non-commutative geometry to the case of the Z_3 -grading, exposed in a recent article by the author (R. Kerner, [34])

5 Conclusion

In this short overview we have tried to show, mostly by means of concrete and simple examples, how the ideas of non-commutative geometry penetrate various domains of contemporary theoretical physics, and above all, the theory of quantized fields whose very existence requires the clarifications concerning the basic notions of space and time.

At present, i.e., by the last decade of the Twentieth century, there is a very intense and diversified activity in this field, which is being pursued by a lot of physicists and mathematicians. The number of publications devoted to this subject nowadays may be compared with the number of papers devoted to string theory in the second half of the eighties, which has somewhat diminished since then. There is, though, a big difference between these two trends, which consists of the following:

Strings have been conceived by physicists first as a new kind of field theory, in which *one-dimensional* entities were treated as primary geometrical objects instead of points. Only after the relativistic quantum physics of such objects had been investigated and developed, it suggested the necessity of introducing new mathematical structures, essentially the non-compact infinite Lie algebras such as the Virasoro or Kac-Moody algebras. By contrast, non-commutative geometry and differential associative algebras developed first as mathematical theories in their own rights, to be then tested against the applications in already-existing physical theories.

Until now, the non-commutative geometries have been adapted to the description of unified theories of elementary interactions, especially the electroweak interaction model of Weinberg and Salam, but it has not led to any new predictions yet. It seems clear that in order to perform a real breakthrough, the non-commutative geometry of quantum spaces of some kind should be raised to the rank of independent physical theories, modifying our concepts of coordinates, points, trajectories and fields, as it happened when Einstein applied the Riemannian geometry to the description of physical realm. This is yet to come; the solution does not seem to be close at hand.

In our everlasting quest for the ultimate understanding of the nature of space and time, after the big leap forward which has occurred at the very beginning of this century with the advent of General Relativity, and which enabled us to enter a new, great and beautiful room of The Lord's edifice, it seems that we are stopped, at the end of the same century, by a tightly closed door that is supposed to lead to another room, in which the *quantized space-time* extends its charms. We are trying to use non-commutative geometry, quantum groups and quantum spaces as the keys that should open this door. Maybe we shall succeed this time, maybe not. It cannot be excluded, however, that there is a largely opened door just on another side, only we are not looking in that direction – and which will be shown to us soon!

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TOWARDS A MACROSTATISTICAL MECHANICS*

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Abstract

I discuss the general question of the derivation of the statistical mechanics of macroscopic variables from the quantum structures of many-particle systems, as represented in the thermodynamic limit. I then present a number of strands of such a macrostatistical mechanics, including (a) a derivation of the electrodynamics of superconductors from their order structure and gauge covariance; (b) a hydrodynamics, with transition from deterministic to stochastic flow, of a quantum plasma model; and (c) a non-linear generalisation of Onsager's irreversible thermodynamics.

1 Introduction

The 'miracle' of statistical physics is that the microscopically chaotic dynamics of many-particle systems conspires to generate macroscopic laws of relatively simple structure, such as those of thermodynamics and hydrodynamics. In view of the complexity of the microscopic picture and the simplicity of the macroscopic one, it is natural to seek an approach to the subject, that is centred on macro-observables, with microscopic input limited to very general principles, e.g., conservation laws, ergodicity, etc. Such an approach would evidently be at the opposite pole from the conventional microscopically based 'many-body theory'.

In fact, there are already important macroscopically-based theories in statistical physics, most notably Onsager's [On] linear irreversible thermodynamics and Landau's [LL, Ch. 17] fluctuating hydrodynamics. However, these theories are essentially heuristic because, apart from questions of rigour, they lack the structures needed for a precise characterisation of their purportedly key ingredient of macroscopicity. Moreover, the same thing can be said about all works devised within the traditional framework of the statistical mechanics of strictly finite systems.

On the other hand, the 'revolution' in statistical mechanics, based on the formulation of many-particle systems in the thermodynamic limit [Ru, HHW, He], has provided us with just the framework required for a sharp characterisation of macroscopicity, and even of different levels thereof [Se1, GVV]. My objective here is to discuss the project of

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extracting a ‘macrostatistical mechanics’ (MSM) from the quantum structures of many-particle systems, within this framework. In fact, we already have some strands of such a discipline in the works of Hepp and Lieb [HL] on the derivation of the macroscopic dynamics, with non-equilibrium phase transition, of a laser mode; of myself [Se1, Ch. 4] on the formulation of an extended classical thermodynamics with phase structure; and of the Leuven school [GVV] on macroscopic fluctuation theory.

In this article, I shall bring together further contributions, from three different areas, towards an MSM. The first of these (Sec. 2) consists of a derivation of the electromagnetic properties of superconductors from their order structure and gauge covariance [Se2,3]. The second (Sec. 3) is an extraction of the hydrodynamics, with non-equilibrium phase transition, of a quantum plasma model [Se4]; and the third (Sec. 4) consists of a quantum-statistical derivation of a non-linear generalisation of Onsager’s irreversible thermodynamics [Se5]. I shall conclude, in Sec. 5, with some further brief comments about the project of a macroscopically-centred statistical mechanics.

2 Macroscopic Quantum Theory of Superconductivity

At the *phenomenological* level, the principal electrodynamic properties of superconductors are their capacity to support persistent electric currents (superconductivity) and their perfect diamagnetism (Meissner effect). These two properties are intimately related, since the Meissner effect is the mechanism whereby the supercurrents screen the magnetic field they generate from the interior of the body [Lo]. Thus, superconductivity arises from the combination of the Meissner effect with the thermodynamic metastability of the supercurrents and their magnetic fields.

Although it appears to be widely accepted that the microscopic theory of Bardeen-Cooper-Schrieffer [BCS] leads to the electrodynamics of metallic superconductors, the arguments employed both there and in related works [An, Ri] are radically flawed in that (a) they are based on totally uncontrolled approximations, and (b) they violate (exact) gauge covariance of the second kind *at the Hamiltonian level*, and thus do not even admit precise definition of a local current density. As regards ceramic, i.e., high T_c , superconductivity, the microscopic theory is less developed than in the metallic case, and has not yet led to an electrodynamics.

On the other hand, the BCS characterisation of the structure of the superconductive phase by electron pairing, first proposed by Schafroth [Sc], has been amply substantiated by experiments on the Josephson effect [Jo] and the quantisation of trapped magnetic flux in multiply-connected superconductors [DF]. Moreover, Yang [Ya], generalising ideas of O. Penrose [Pe, PO], pointed out that this characterisation is captured by the hypothesis of *off-diagonal long range order* (ODLRO). This is a macroscopic quantum property, representing a well-defined order structure in a gauge covariant way. Furthermore, it is a property also possessed by certain Ansätze, e.g., [ZA], for the high T_c superconductive phase of ceramics.

I shall now sketch an approach [Se2, 3] I have made to the electrodynamics of superconductors, based on the assumption of ODLRO. This is designed to relate the electromagnetic properties to the order structure of these systems in purely macroscopic

quantum terms (cf. eqn. (2.8) below). For brevity, I shall confine myself here to the derivation of the Meissner effect from ODLRO.

The Model. We take the quantum model, Σ , to be an infinitely extended system of electrons, and possibly also of ions or phonons, in a Euclidean space X : lattice systems may be formulated analogously. Points in X will generally be denoted by x , (sometimes by y, a or b) and the Lebesgue measure by dx . It will be assumed that the model enjoys the properties of gauge covariance of the second kind, and that its interactions are translationally invariant.

The electronic part of Σ is formulated in terms of a quantised field $\psi = (\psi_\uparrow, \psi_\downarrow)$, satisfying the canonical anticommutation relations. Thus, in a standard way, the C^* -algebra $\mathcal{F}_{el}$ of the CAR over the Hilbert space $\mathcal{H} := L^2(X, dx)$ is defined by the specifications that

1. there are linear maps $\psi_\uparrow, \psi_\downarrow$, from $\mathcal{H}$ into $\mathcal{F}_{el}$ satisfying the CAR

$$[\psi_\alpha(f), \psi_\beta(g)^*]_+ = \delta_{\alpha\beta} \langle g, f \rangle_{\mathcal{H}}; [\psi_\alpha(f), \psi_\beta(g)]_+ = 0. \quad (2.1)$$

2. $\mathcal{F}_{el}$ is generated by $\{\psi_\alpha(f), \psi_\alpha(f)^* | f \in \mathcal{H}; \alpha = \uparrow, \downarrow\}$.

The algebra $\mathcal{F}_{el}$ is then the *field algebra* of the electrons. Space translations and gauge transformations are represented by the homomorphisms σ_0, γ of the additive groups $X, C_R^\infty(X)$, respectively, into $\text{Aut}(\mathcal{F}_{el})$, defined by the formulae

$$\sigma_0(a)\psi(f) = \psi(f_a) \quad \forall a \in X, \text{ with } f_a(x) := f(x - a) \quad (2.2)$$

and

$$\gamma(\chi)\psi(\phi) = \psi(\phi \exp(i\chi)) \quad \forall \chi \in C_R^\infty(X). \quad (2.3)$$

The *global* gauge automorphisms are those for which χ is constant.

Let $\mathcal{K} := \{f_1 \otimes f_2 | f_1, f_2 \in \mathcal{H}\}$. We define the *pair field* Ψ to be the mapping of $\mathcal{K}$ into $\mathcal{F}_{el}$ given by

$$\Psi(F) = \psi_\uparrow(f_1)\psi_\downarrow(f_2) \text{ for } F = f_1 \otimes f_2 \quad (2.4)$$

Hence, by (2.2)-(2.4),

$$\sigma_0(a)\Psi(F) = \Psi(F_a), \text{ with } F_a(x_1, x_2) = F(x_1 - a, x_2 - a) \quad (2.5)$$

and

$$\gamma(\chi)\Psi(F) = \Psi(g(\chi)F); \quad (2.6)$$

with

$$(g(\chi)F)(x_1, x_2) := F(x_1, x_2) \exp i(\chi(x_1) + \chi(x_2)).$$

In a standard way, we define the *observable algebra* of the electrons to be the subalgebra $\mathcal{A}_{el}$ of $\mathcal{F}_{el}$ that is elementwise invariant under the global gauge automorphisms $\psi \rightarrow \psi \exp(i\alpha)$, with α constant. Space translations and gauge transformations of the electronic observables are then given by the restrictions to $\mathcal{A}_{el}$ of the automorphism groups $\sigma_0(X)$ and $\gamma(C_R^\infty(X))$, respectively.

The construction of the C^* -algebra $\mathcal{A}_0$ of the observables of the other species of particles of Σ is effected in a similar way. We take the C^* -algebra of observables of the system to be $\mathcal{A} := \mathcal{A}_{el} \otimes \mathcal{A}_0$ and canonically identify $\mathcal{A}_{el}$ with its subalgebra $\mathcal{A}_{el} \otimes I$.

It is assumed that the automorphism groups $\sigma_0(X)$ and $\gamma(C_R^\infty(X))$ extend from $\mathcal{A}_\text{el}$ to $\mathcal{A}$.

We assume, for simplicity,¹ that the dynamics of Σ , as given by a canonical limiting form of that of finite versions of the system, corresponds to a one-parameter group $\{\alpha(t)|t \in \mathbb{R}\}$ of automorphisms of $\mathcal{A}$.

An equilibrium state ω at inverse temperature β may then be characterised by the Kubo-Martin-Schwinger (KMS) conditions [HHW], namely that, for arbitrary $A, B \in \mathcal{A}$, there is a function F on $\mathbb{C}$ that is analytic in the interior of the strip $\text{Im}(z) \in [0, \hbar\beta]$ and continuous on its boundaries, and satisfies the relations

$$F(t) = \omega(A\alpha(t)B); \quad F(t + i\hbar\beta) = \omega((\alpha(t)A)B). \quad (2.7)$$

Off-Diagonal Long Range Order. A state ω is said to possess the property of off-diagonal long-range order (ODLRO) if there is a mapping $\Phi : \mathcal{K} \rightarrow \mathbb{C}$, such that

$$\lim_{|y| \rightarrow \infty} [\omega(\Psi(F)^* \Psi(G_y)) - \overline{\Phi(F)} \Phi(G_y)] = 0 \quad \forall F, G \in \mathcal{K} \quad (2.8)$$

where the bar denotes complex conjugation; and $\Phi(G_y)$ does not tend to zero, for all $G \in \mathcal{K}$ as $y \rightarrow \infty$.

Φ is then termed the *macroscopic wave function* for the state ω .

Note that, although $\Psi(F)$ does not belong to the observable algebra $\mathcal{A}$, the argument of ω in (2.8) does.

Lemma 2.1. *The ODLRO conditions define the macroscopic wave function up to a constant phase factor; i.e., if Φ_1, Φ_2 are two such functions satisfying these conditions, for the same state ω , then $\Phi_2 = \Phi_1 \exp(i\eta)$, where η is a real-valued constant.*

Proof. Assuming that Φ_1, Φ_2 both satisfy the ODLRO conditions with respect to the same state ω , it follows from (2.8) that

$$\lim_{|y| \rightarrow \infty} [\overline{\Phi_1(F)} \Phi_1(G_y) - \overline{\Phi_2(F)} \Phi_2(G_y)] = 0.$$

Since this is valid for all $F, G \in \mathcal{K}$, we may replace F by $F' \in \mathcal{K}$, thereby obtaining

$$\lim_{|y| \rightarrow \infty} [\overline{\Phi_1(F')} \Phi_1(G_y) - \overline{\Phi_2(F')} \Phi_2(G_y)] = 0.$$

On multiplying the complex conjugate of the first equation by $\Phi_2(F')$ and that of the second one by $\Phi_1(F')$, and then taking the difference, we see that

$$\lim_{|y| \rightarrow \infty} \overline{\Phi_1(G_y)} [\Phi_1(F) \Phi_2(F') - \Phi_1(F') \Phi_2(F)] = 0. \quad (2.9)$$

Since, by the above definition of ODLRO, there are elements G of $\mathcal{K}$ for which $\Phi(G_y)$ does not tend to zero, as $|y| \rightarrow \infty$, it follows that the quantity in the square brackets of (2.9) vanishes. Consequently, as $\Phi_{1,2}$ are non-zero, by the same stipulation,

$$\Phi_2(F) = c \Phi_1(F) \quad \forall F \in \mathcal{K}$$

where c is a complex-valued constant, and since Φ_1, Φ_2 both satisfy (2.8), it follows immediately that this is just a constant phase factor $\exp(i\eta)$.

¹The more generally valid assumption of a W^* -dynamical system [Se6], employed in Sec. 4 of this article, leads to precisely the same results as the present one.

Gauge and Space Translational Covariance. Assume now that the system is subjected to a classical, static magnetic field, of induction

$$B = \text{curl } A. \quad (2.10)$$

The time-translational automorphisms then become A -dependent, and we denote them by $\alpha_A(\mathbb{R})$. We assume that they are covariant w.r.t. space translations and gauge transformations, i.e., that

$$\sigma_0(a)\alpha_A(t)\sigma_0(-a) = \alpha_{A_a}(t); \text{ with } A_a(x) := A(x-a) \quad (2.11)$$

and

$$\gamma\left(\frac{e\chi}{\hbar c}\right)\alpha_A(t)\gamma\left(\frac{-e\chi}{\hbar c}\right) = \alpha_{A+\nabla\chi}(t). \quad (2.12)$$

We note now that, in the particular case where B is uniform, it may be represented by the vector potential

$$A = \frac{1}{2}B \times x. \quad (2.13)$$

Thus, choosing

$$\chi(x) = -\frac{1}{2}(B \times a) \cdot x, \quad (2.14)$$

we see that

$$A + \nabla\chi = A_a. \quad (2.15)$$

Hence, defining $\sigma : X \rightarrow \text{Aut}(\mathcal{A})$ by the formula

$$\sigma(a) = \gamma\left(\frac{e\chi}{\hbar c}\right)\sigma_0(a), \quad (2.16)$$

it follows from (2.11), (2.12) and (2.16) and the definition of $\mathcal{A}$ that σ is a representation of X in $\text{Aut } \mathcal{A}$ which commutes with the time translations $\alpha_A(\mathbb{R})$, with the gauge fixed by eqn. (2.13). In view of these properties, we take σ to be the space translational automorphisms in the presence of the uniform magnetic field B .

Note. By (2.5), (2.6) and (2.16), the canonical extension of σ to $\mathcal{F}_{\text{el}}$, and in particular to the pair field Ψ , yields

$$\sigma(a)\Psi(F) = \Psi(s(a)F) \quad (2.17)$$

where

$$(s(a)F)(x_1, x_2) = F(x_1 - a, x_2 - a) \exp\left(\frac{ie(B \times a) \cdot (x_1 + x_2)}{2\hbar c}\right). \quad (2.18)$$

Hence,

$$[s(a), s(b)]_- = 2i \sin\left(\frac{eB \cdot (a \times b)}{\hbar c}\right) s(a+b) \text{ and } s(a)s(-a) = I \quad \forall a, b \in X \quad (2.19)$$

where $[\cdot, \cdot]_-$ denotes commutator. Thus, as $\sigma(a)$ and $\sigma(b)$ do not, in general intercommute, it follows from (2.17) that the extension of $\sigma(X)$ to the field algebra $\mathcal{F}_{\text{el}}$ is non-abelian. This is crucial for our derivation of the Meissner effect below.

ODLRO and the Meissner Effect. The essential distinction between normal diamagnetism and the Meissner effect is that the former can support a uniform, static, non-zero magnetic induction and the latter can not. Thus, we base our derivation of the Meissner effect on considerations of the response of an state with ODLRO to the action of a uniform magnetic field.

Proposition 2.2. *The system cannot support uniform, non-zero magnetic induction in translationally invariant states with ODLRO.*

The following Corollary follows immediately from this Propostion and elementary thermodynamics.

Corollary 2.3. *Assuming that translational symmetry is preserved in the equilibrium state ω_A , then either*

(1) ω_A possesses the property of ODLRO and $B = 0$,

or

(2) ω_A does not possess ODLRO and is normally diamagnetic.

Further, assuming that, in the absence of a magnetic field, the free energy density of the ODLRO phase is lower, by Δ , than that of the normal one, the former phase will prevail, and thus the system will exhibit the Meissner effect, provided that the applied field, H , satisfies the condition that $|H| < H_c := (8\pi\Delta)^{1/2}$.

Comment. This result signifies that the Meissner effect arises from a rigidity of the macroscopic wave-function in the face of an applied magnetic field. This corresponds precisely to London's [Lo] idea of rigidity, transferred to the macroscopic level.

Proof of Proposition 2.2. By the translational invariance of ω_A and the formula (2.17) for σ ,

$$\omega_A(\Psi(s(a)F)^*\Psi(s(a)G)) \equiv \omega_A(\Psi(F)^*\Psi(G)). \quad (2.20)$$

Further, it follows from the ODLRO condition (2.8) and the definition (2.18) of s that

$$\lim_{y \rightarrow \infty} [\omega_A(\Psi(s(a)F)^*\Psi(s(a)G_y)) - \overline{\Phi(s(a)F)}\Phi(s(a)G_y)] = 0$$

and hence, by (2.20), that

$$\lim_{y \rightarrow \infty} [\omega_A(\Psi(F)^*\Psi(G_y)) - \overline{\Phi_a(F)}\Phi_a(G_y)] = 0 \quad (2.21)$$

where

$$\Phi_a = \Phi \circ s(a). \quad (2.22)$$

Equation (2.21) implies that Φ_a , as well as Φ , serves as a macroscopic wave function for the state ω_A , and therefore, by Lemma 2.1 and eqn. (2.22), that

$$\Phi \circ s(a) = c(a)\Phi$$

where $c(a)$ is a complex number of unit modulus. Similarly, if b is another element of X , then

$$\Phi \circ s(b) = c(b)\Phi$$

where $c(b)$ is also a complex number of unit modulus. Thus, the last two equations imply that

$$\Phi \circ [s(a), s(b)]_- = 0.$$

On taking the composition product of this equation with $s(-a - b)$ and using (2.19), we see that,

$$\sin\left(\frac{eB \cdot (a \times b)}{\hbar c}\right) \Phi = 0 \quad \forall a, b \in X$$

from which it follows that *either* $B = 0$ *or* $\Phi = 0$, which is the required result.

3 Hydrodynamical Phase Transition in a Quantum Plasma

The quantum Jellium model is a system of electrons, interacting via Coulomb forces both with one another and with a uniform, positively charged, neutralising background. It is thus a model of a many-particle system with realistic interactions. Its thermodynamic properties have been established at the level of mathematical physics [LN]. I shall now sketch a treatment [Se4, 7] leading from its many-particle Schrödinger equation to an Eulerian hydrodynamics with a non-equilibrium phase transition. The three principal stages of the derivation of this result are the following. Firstly, we derive a classical Vlasov equation, governing a *large scale* description of its dynamics, from the microscopic Schrödinger equation, together with certain initial and regularity conditions. Secondly, we observe that the Vlasov dynamics corresponds precisely to the Liouville equation for a *Lagrangian* hydrodynamics, governing the evolution of the time (t)-dependent position, $X_t(x)$, of a ‘fluid particle’ located initially at the point x . Then, finally, we show that this Lagrangian hydrodynamics reduces to a deterministic *Eulerian* form, provided that a certain parameter, governing the non-uniformity in the initial density profile, lies below a certain critical value, and that otherwise it corresponds to a *stochastic* flow.

The Jellium Model, $\Sigma^{(N,L)}$, consists of N electrons in a cube, $K^{(L)}$, of side L , with uniform neutralising positive charge background. We assume periodic boundary conditions. Our objective is to obtain a quantum theoretical derivation of the hydrodynamics of $\Sigma^{(N,L)}$ in a limit where N and L tend to infinity and the mean particle density,

$$\bar{n} = N/L^3 \tag{3.1}$$

remains fixed and finite.

Denoting the gradient operator in $K^{(L)}$ by $\nabla^{(L)}$, we represent the position vectors and momenta of the electrons by the standard multiplicative and differential operators $\{X_j, P_j = -i\hbar \nabla_j^{(L)} | j = 1, \dots, N\}$ in the Hilbert space $\mathcal{H}^{(N,L)}$ of square integrable anti-symmetric functions on $(K^{(L)})^N$. The Hamiltonian is then the self-adjoint operator in $\mathcal{H}^{(N,L)}$ given by

$$H^{(N,L)} = \frac{-\hbar^2}{2m} \sum_{j=1}^N \Delta_j^{(L)} + e^2 \sum_{j,k(>j)=1}^N U^{(L)}(X_j - X_k) \tag{3.2}$$

where $-e, m$ are the electronic charge and mass, respectively, $\Delta^{(L)}$ is the Laplacian for $K^{(L)}$, and $U^{(L)}(X)$ is the difference between $|X|^{-1}$, periodicised w.r.t. $K^{(L)}$, and its space average over that cube, i.e.,

$$U^{(L)}(X) = \frac{4\pi}{L^3} \sum^{(L)} \frac{\exp(iQ \cdot X)}{Q^2} \quad (3.3)$$

the superscript (L) over Σ signifying that summation is taken over the non-zero vectors $Q = (2\pi/L)(n_1, n_2, n_3)$, with the n 's integers. The pure states of $\Sigma^{(N,L)}$ are represented by the normalised vectors, $\Psi^{(N)}$ in $\mathcal{H}^{(N,L)}$ and their evolution is governed by the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi_T^{(N)}}{\partial T} = H^{(N,L)} \Psi_T^{(N)} \quad (3.4)$$

with T the time variable. We shall assume the following initial kinetic and potential energy bounds for $\Sigma^{(N,L)}$ -more precisely, for the family of systems $\{\Sigma^{(N,L)}\}$, with N, L satisfying (3.1).

(I.1)^(L) The expectation value of the total kinetic energy per particle, for the initial state $\Psi_0^{(N)}$, is less than some finite N -independent constant $B/2m$, i.e.,

$$(\Psi_0^{(N)}, P_1^2 \Psi_0^{(N)}) < B. \quad (3.5)$$

(I.2)^(L) The expectation value of the total potential energy, for the initial state $\Psi_0^{(N)}$, is less than some finite N -independent constant, $e^2 C/2$, times $N^{5/3}$. This bound corresponds to the electrostatic energy of a continuous distribution of charge, whose density is a smooth function of X/L , and signifies that

$$(\Psi_0^{(N)}, U^{(L)}(X_1 - X_2) \Psi_0^{(N)}) < C N^{2/3}. \quad (3.6)$$

We base our macroscopic description of the model on scales of length, time and particle momentum given by L , ω^{-1} and $mL\omega$, where ω is the classical plasma frequency, i.e.,

$$\omega = (4\pi\bar{n}e^2/m)^{1/2}. \quad (3.7)$$

For this description, we employ the rescaled space and time coordinates, $x = X/L$, $t = \omega T$, respectively. Under this rescaling, $\Sigma^{(N,L)}$ is mapped onto a system $\Sigma^{(N)}$ of particles in a unit cube, K , with periodic boundaries. This rescaling induces a unitary mapping of $\mathcal{H}^{(N,L)}$ onto the Hilbert space $\mathcal{H}^{(N)}$ of antisymmetric square integrable functions on K^N , under which $\Psi_T^{(N)}$ is transformed to the state of $\Sigma^{(N)}$ given by

$$\psi_t^{(N)}(x_1, \dots, x_N) = L^{3N/2} \Psi_{\omega^{-1}t}^{(N)}(Lx_1, \dots, Lx_N). \quad (3.8)$$

The Schrödinger equation (3.4) thus transforms to

$$i\hbar_N \frac{\partial \psi_t^{(N)}}{\partial t} = H^{(N)} \psi_t^{(N)} \quad (3.9)$$

where

$$H^{(N)} = \frac{1}{2} \sum_{j=1}^N p_j^2 + N^{-1} \sum_{j,k(>j)=1}^N U(x_j - x_k) \quad (3.10)$$

$$p_j = -i\hbar_N \nabla_j \quad (3.11)$$

$$\hbar_N = \frac{\hbar}{mL^2\omega} = \frac{\hbar}{m\omega} \left(\frac{\bar{n}}{N}\right)^{2/3} \quad (3.12)$$

is a dimensionless effective 'Planck constant', ∇ is the gradient operator for K , and

$$U(x) = \sum_q^{(1)} \exp(iq \cdot x)/q^2 \quad (3.13)$$

the superscript (1) over Σ signifying that summation is taken over the non-zero vectors $2\pi(n_1, n_2, n_3)$, with the n 's integers. It follows from (2.13) that

$$\Delta U(x) = 1 - \delta(x) \quad (3.14)$$

where Δ is the Laplacian and δ the Dirac distribution for K .

We formulate the dynamics of $\Sigma^{(N)}$ in terms of its characteristic functions

$$\mu_t^{(N,n)}(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n) := \quad (3.15)$$

$$(\psi_t^{(N)}, \Pi_{j=1}^n (\exp(i\xi_j \cdot j/2) \exp(i\eta_j \cdot x_j) \exp(i\xi_j \cdot p_j/2)) \psi_t^{(N)})$$

where the ξ 's and η 's run over the ranges $\mathbb{R}^3$ and $(2\pi\mathbb{Z})^3$, respectively.

The initial condition for $\Sigma^{(N)}$, corresponding to (I.1)^(L) for $\Sigma^{(N,L)}$, is

$$(I.1) \quad (\psi_0^{(N)}, p_1^2 \psi_0^{(N)}) < N^{-2/3} b \rightarrow 0 \text{ as } N \rightarrow \infty \quad (3.16)$$

where b is a finite constant.

The initial condition (I.2)^(L) transforms to the following form for $\Sigma^{(N)}$, which is unaffected by the above rephasing of the initial state.

$$(I.2) \quad (\psi_0^{(N)}, U(x_1 - x_2) \psi_0^{(N)}) < c \quad (3.17)$$

where c is a finite constant. We further assume that

$$(I.3) \quad \lim_{N \rightarrow \infty} \rho_0^{(N,1)}(x) = \sigma_0(x) \quad \forall x \in K \quad (3.18)$$

where the function σ_0 is continuous; and that

$$(I.4) \quad \lim_{N \rightarrow \infty} (\mu_0^{(N,n)}(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n) - \Pi_1^n \mu_0^{(N,1)}(\xi_j, \eta_j)) \equiv 0. \quad (3.19)$$

This last condition signifies that, in the limit $N \rightarrow \infty$, the initial correlations of $\Sigma^{(N)}$ are of zero range: the assumption behind this is that the initial state of the microscopic model $\Sigma^{(N,L)}$, carries only short range correlations, as in a pure phase [Ru, Sel].

We note here that it was shown by explicit construction, in the Appendix of Ref. [Se7], that the initial conditions (I.1-4) are perfectly viable.

The macroscopic dynamics of the Jellium model, then, is represented by the time-dependence of the characteristic functions $\mu_t^{(N,n)}$, in the limit $N \rightarrow \infty$, subject to the initial conditions (I.1-4).

The Vlasov Dynamics. We note now that the dynamics of the model $\Sigma^{(N)}$, as given by (3.9) and (3.10), has the simplifying features that (a) the effective, dimensionless Planck constant, $\hbar_N$, governing its quantum behaviour, tends to zero as $N \rightarrow \infty$; and (b) the pair interaction potential scales as N^{-1} . The properties (a) and (b) are generally the hallmarks of a classical and of a mean field theory, respectively, in the limit $N \rightarrow \infty$. In fact, were it not for the Coulomb singularity in U , one could infer immediately from [NS, Sp] that the Schrödinger equation (3.9), subject to the initial conditions (I.1-4), led to a classical (mean field theoretic) Vlasov dynamics in the limit $N \rightarrow \infty$. Specifically, if the two-body potential U in (3.10) were a twice continuously differentiable function, then the following results would prevail.

(A) [NS, Sp] Pointwise convergence of $\mu_t^{(N,1)}$, as $N \rightarrow \infty$, to the characteristic function, μ_t , of a classical probability measure m_t on $(K \times \mathbb{R}^3)$, that satisfies the weak Vlasov equation

$$\begin{aligned} \frac{d}{dt} \int f(x, v) dm_t(x, v) &= \int v \frac{\partial}{\partial x} f(x, v) dm_t(x, v). \\ - \int \nabla U(x - y) \cdot \frac{\partial}{\partial v} f(x, v) dm_t(x, v) dm_t(y, w) &\quad \forall f \in C_0^2(K \times \mathbb{R}^3) \end{aligned} \quad (3.20)$$

(B) [NS, Sp] Pointwise convergence of $\mu_t^{(N,n)}$ to the characteristic function of the classical probability measure $m_t \otimes m_t \dots \otimes m_t$ (n factors) on $(K \times \mathbb{R}^3)^n$.

(C) [Se7] The initial condition for m , ensuing from (I.1) and (I.3), that

$$\int dm_0(x, v) f(x, v) = \int dx \sigma_0(x) f(x, 0). \quad (3.21)$$

(D) [Ne, Se7] The unique determination of the solution of the Vlasov equation (3.20), subject to the initial condition (3.21), by that of the Newtonian mean field theoretic problem

$$\frac{dX_t(x)}{dt} = V_t(x); \quad \frac{dV_t(x)}{dt} = - \int dy \sigma_0(y) \nabla U'(X_t(x) - X_t(y)); \quad (3.22)$$

with

$$X_0(x) = x; \quad V_0(x) = 0$$

according to the formula

$$\int dm_t(x, v) f(x, v) = \int dx \sigma_0(x) f(X_t(x), V_t(x)). \quad (3.23)$$

In order to extend these results to the Jellium model, $\Sigma^{(N)}$, we require certain regularity conditions on the dynamics of the system, that serve to tame the Coulomb singularity. Accordingly, we invoke certain supplementary assumptions (R), specified in [Se4], whose essential import is that the Coulomb repulsion keeps the electrons apart, sufficiently to ensure that the magnitude of the internal electric field remains bounded. Under these assumptions, we have the following result.

Proposition 3.1. [Se4] *The Schrödinger equation (3.10) for $\Sigma^{(N)}$, together with the initial conditions (I.1-4) and the regularity conditions R, leads to the Vlasov dynamics specified by (A)-(D).*

Eulerian Versus Stochastic Hydrodynamics. The Newtonian mean field theory, given by (3.22), corresponds to a *Lagrangian* hydrodynamics, in which $X_t(x)$ and

$V_t(x)$ are the position and velocity, respectively, of a 'fluid particle'; and the Vlasov equation (3.20) is just the Liouville equation representing its probabilistic description. Our aim now is to show that the reducibility of the Vlasov dynamics to a deterministic Eulerian hydrodynamics depends on the invertibility of X_t .

Case (a): X_t Invertible. In this case, the Jacobian

$$J_t(x) = \frac{\partial(X_t^{(1)}, X_t^{(2)}, X_t^{(3)})}{\partial(x^{(1)}, x^{(2)}, x^{(3)})} \quad (3.24)$$

with $x^{(j)}$ (resp. $X_t^{(j)}$) the j -th component of x (resp. X_t), is strictly positive, and so we can define

$$\sigma_t(x) = \sigma_0(X_t^{-1}(x))/J_t(x); \text{ and } u_t(x) = V_t(X_t^{-1}(x)). \quad (3.25)$$

Thus, since, by (3.23) and (3.25),

$$\int dm_t(x, v) f(x, v) = \int dx \sigma_t(x) f(x, u_t(x)) \quad \forall f \in C(K) \quad (3.26)$$

i.e., formally,

$$dm_t(x, v) = \sigma_t(x) \delta(v - u_t(x)) dx dv$$

it follows that $u_t(x)$ and $\sigma_t(x)$ are the drift velocity and normalised particle density, respectively, at position x and time t . Moreover, by (3.14), (3.22) and (3.25), u_t, σ_t evolve according to the Euler-cum-Maxwell hydrodynamical equations

$$\frac{\partial \sigma_t}{\partial t} + \nabla(\sigma_t u_t) = 0; \quad \frac{\partial u_t}{\partial t} + (u_t \cdot \nabla) u_t = E_t; \quad \nabla \cdot \dot{E}_t = (\sigma_t - 1). \quad (3.27)$$

Case (b): X_t Non-Invertible. In this case, we cannot employ the formulae (3.25) to define the time-dependent density and drift velocity. Instead, we have to consider the situation where the equation

$$X_t(y) = x$$

has several solutions, labelled by an index set S , for y as a function of x and t , i.e.,

$$y = \{Y_t^{(s)}(x) | s \in S\}.$$

In this case, equation (3.26) implies that

$$\int dm_t(x, v) f(x, v) = \sum_{s \in S} \int dx \sigma_t^{(s)}(x) f(x, u_t^{(s)}(x)) \quad (3.28)$$

where

$$\sigma_t^{(s)}(x) = \sigma_0(Y_t^{(s)}(x)) |K_t^{(s)}(x)|; \quad u_t^{(s)}(x) = V_t(Y_t^{(s)}(x)) \quad (3.29)$$

and

$$K_t^{(s)}(x) = \frac{\partial(Y_t^{(s,1)}, Y_t^{(s,2)}, Y_t^{(s,3)})}{\partial(x^{(1)}, x^{(2)}, x^{(3)})}. \quad (3.30)$$

Thus, (3.28) signifies that the macroscopic state of the system at time t corresponds to a statistical mixture of different streams, indexed by S , with densities $\sigma_t^{(s)}$ and drift

velocities $u_i^{(s)}$. This implies that the local density and drift velocity of the system are now *stochastic* variables of a hydrodynamics still governed by the Vlasov equation.

We may summarise the above observations in the following form.

Proposition 3.2. *If X_t is invertible, then the macroscopic dynamics of the system corresponds to a deterministic hydrodynamics given by the Euler-cum-Maxwell equations (3.27). Otherwise, it is a stochastic flow, still governed by the Vlasov equation.*

Example of Hydrodynamic Phase Transition. Let us now specialise to the situation where σ_0 is a function of just a single coordinate, $x^{(1)}$. In this case, as the regularity conditions (R) imply uniqueness of the solution of (3.22) [Se4], the flow becomes effectively one-dimensional, with $X_{1,t}$ dependent on the coordinate $x^{(1)}$ only, and $X_t^{(2)}, X_t^{(3)}$ remaining fixed at $x^{(2)}, x^{(3)}$, respectively. For notational convenience, we shall henceforth drop the superscript (1) from $X_t^{(1)}$ and $x^{(1)}$, and denote by $X_t(x)$ the first component of the vector hitherto represented by this symbol. Thus, by equations (3.22),

$$X_t(x) = x + \int_0^t ds(t-s) \int_0^1 dy \sigma_0(y) F(X_s(x) - X_s(y)) \quad (3.31)$$

where

$$F(x) = - \int_0^1 dx^{(2)} \int_0^1 dx^{(3)} \frac{\partial U}{\partial x}(x, x^{(2)}, x^{(3)}). \quad (3.32)$$

Let

$$J_t(x) = \frac{\partial X_t(x)}{\partial x}. \quad (3.33)$$

Then, by Prop. 3.2 and the implicit function theorem, the necessary and sufficient condition for deterministic hydrodynamics is that J_t has no zeroes. We note also that the definition (3.33) permits us to re-express (3.31) in the form

$$X_t(x) = x + \int_0^t ds(t-s) \int_0^1 dy \sigma_0(y) F\left(\int_y^x dz J_s(z)\right). \quad (3.31)'$$

Proposition 3.3. *Under the specified assumptions, the hydrodynamics of the model is deterministic and given by the Euler-cum-Maxwell equations (3.27), provided that*

$$\sigma_0(x) > \frac{1}{2} \forall x \in [0, 1]. \quad (3.34)$$

Otherwise there is a transition to a stochastic flow at a certain time τ , given by the least positive value of t for which

$$\sigma_0(x) + (1 - \sigma_0(x)) \cos(t) = 0 \quad (3.35)$$

for some $x \in [0, 1]$.

Proof. By (3.13), (3.14) and (3.31)-(3.33),

$$J_t(x) = 1 + \int_0^t ds(t-s) J_s(x) \left(-1 + \int_0^1 dy \sigma_0(y) \delta(X_s(x) - X_s(y))\right) \quad (3.36)$$

where now δ is the Dirac distribution on $[0, 1]$, subject to periodic boundary conditions.

Let us first suppose that X_t is invertible, i.e., that J_t is strictly positive, over a time interval $0 \leq t < \tau_0$, for some positive τ_0 . In this case,

$$J_s(x)\delta(X_s(x) - X_s(y)) \equiv \delta(x - y) \quad \forall s \in [0, \tau_0)$$

and therefore (3.36) reduces to

$$J_t(x) = 1 + \int_0^t ds(t-s)(\sigma_0(x) - J_s(x)),$$

i.e.,

$$\left(\frac{d^2}{dt^2} + 1\right)J_t(x) = \sigma_0(x); \quad \text{with} \quad J_0(x) = 1; \quad \dot{J}_0(x) = 0$$

where $\dot{J}_t = dJ_t/dt$. Hence,

$$J_t(x) = \sigma_0(x) + (1 - \sigma_0(x))\cos(t).$$

In view of the non-negativity of σ_0 , this equation implies that J_t is strictly positive for all $t \geq 0$ if and only if the condition (3.34) is fulfilled. Otherwise, J_t changes sign at some point $x \in [0, 1]$ when t reaches the value τ specified in the statement of the Proposition. Hence, our assumption of the invertibility of X_t is untenable if (3.34) is violated; and therefore, by Prop. 3.2, the flow becomes stochastic in this case.

The proof (cf. [Se4]) of the converse, i.e., that (3.34) implies the invertibility of X_t and thus the deterministic Euler-cum-Maxwell flow, stems from the explicit form of our regularity condition (R).

4 Macrostatistics and Non-Equilibrium Thermodynamics

Here, I present an approach to the general problem of formulating the structures imposed on non-equilibrium thermodynamics by the underlying quantum mechanics of many-particle systems. This is based on a combination of macroscopic and microscopic treatments of a generic system, Σ , of particles occupying a Euclidean space $X = \mathbb{R}^d$. The argument consists of four parts. The first (Sec. 4.1) is a formulation of a mathematical framework for non-equilibrium thermodynamics, based on a macroscopic, continuum mechanical model, M , of Σ . The second (Sec. 4.2) is an algebraic quantum statistical formulation of a microscopic model, $\mathcal{Q}$, of the same system. The third (Sec. 4.3) consists of a treatment of the connection between M and $\mathcal{Q}$, leading to a *classical macrostatistical mechanics* (CMSM) of the observables of $\mathcal{Q}$ that correspond to the hydrodynamical variables of M . This admits a clear formulation of *local equilibrium* conditions and of a generalised version of Onsager's regression hypothesis [On], namely that the macroscopic fluctuations about the flow given by M is governed by the same dynamics as the externally induced 'weak' perturbations to that flow. In Sec. 4.4, I derive a generalised form of the Onsager reciprocity relations for M from the structure of CMSM. The key elements of this derivation are the symmetry properties of the model, stemming from the microscopic reversibility of $\mathcal{Q}$, and the assumption of local equilibrium.

4.1 The Macroscopic Model, M

In the classical thermodynamical description, the *equilibrium states* of Σ are represented by a certain set $q = (q_1, \dots, q_n)$ of intensive variables, corresponding to global densities of extensive conserved quantities, which we shall characterise in Sec. 4.2. Its entropy density, s , is then a function of these variables, and its equilibrium properties of governed by the form of s . The thermodynamic variables $\theta = (\theta_1, \dots, \theta_n)$ conjugate to q are defined by the formula

$$\theta = \frac{\partial s(q)}{\partial q}, \quad \text{i.e.,} \quad \theta_k = \frac{\partial s(q)}{\partial q_k}, \quad k = 1, 2, \dots, n \quad (4.1)$$

and the pressure, p , is the function of θ given by

$$p(\theta) = \sup_q (s(q) - q \cdot \theta) \quad (4.2)$$

the dot representing the $\mathbb{R}^n$ inner product. Thus, the inverse of the formula (4.1) (in the pure phase region) is

$$q = -\frac{\partial p(\theta)}{\partial \theta} \quad (4.3)$$

In non-equilibrium thermodynamics, the state of the system is generally not translationally invariant, and the macroscopic variables q become functions of position, x , and of time, t . In other words, the macrostate is an n -component *classical field*

$$q(x, t) \equiv q_t(x) \equiv (q_{1,t}(x), \dots, q_{n,t}(x)).$$

The laws of continuum mechanics, such as those of hydrodynamics or heat conduction, are then of the *deterministic* form

$$\dot{q}_t = \mathcal{F}(q_t) \quad (4.4)$$

where $\mathcal{F}$ is some functional of the fields q_t . Equivalently, defining the local thermodynamic conjugates θ_t of the fields q_t by

$$\theta_t(x) \equiv \theta(x, t) := \frac{\partial s(q(x, t))}{\partial q}, \quad (4.5)$$

it follows from (4.3) that the equation of motion (4.4) may be re-expressed as

$$\dot{q}_t \equiv \frac{dq_t}{dt} = \Phi(\theta_t); \quad \Phi := \mathcal{F} \circ \left(-\frac{\partial p}{\partial \theta}\right). \quad (4.6)$$

Note. The definition (4.5) of θ_t does not require the microstate of Σ to simulate one of equilibrium at a local level. On the other hand, some assumption of local equilibrium will be needed for the determination of properties of the functional $\mathcal{F}$.

Example. An example of a law of the above form is the non-linear, n -component diffusion given by

$$\dot{q}_{k,t} = \sum_{i=1}^n \nabla \cdot (L_{k,i}(\theta_t) \nabla \theta_{i,t}) \quad (4.6a)$$

Returning to the general structure, we impose the following assumptions on the dynamical law (4.6).

(M.1) *The system is confined to a single phase region, i.e., the values of $q_t(x)$, $\theta_t(x)$ always lie in domains where the functions s and p are $C^{(\infty)}$.*

(M.2) *$\mathcal{F}(q) = 0$ if $q(x)$ is constant, i.e., the system is in equilibrium if its local thermodynamic variables $q(x)$ (or equivalently $\theta(x)$) are spatially uniform.*

(M.3) *The equation of motion (4.6) is covariant w.r.t. space-time translations and scale transformations $x \rightarrow \lambda x$, $t \rightarrow \lambda^r t$, with r a positive integer ($=2$ in the case of (4.6a)).²*

Let y_t , η_t be 'small' perturbations of q_t , θ_t , respectively, and let

$$[\Lambda(\theta)\eta_t](x) := \frac{d}{dz}\Phi(\theta_t(x) + z\eta_t(x))|_{z=0} \quad (4.7)$$

in the pointwise sense, for the class of functions η_t for which the r.h.s. of this formula is well-defined. Then it follows from (4.6) and (4.7) that the *linearised* equation of motion for y_t is

$$\dot{y}_t = \mathcal{L}(\theta_t)y_t \quad (4.8)$$

where

$$\mathcal{L}(\theta_t) = \Lambda(\theta_t)B(\theta_t) \quad (4.9)$$

and $B(\theta)$ is the Hessian $s''(q)$, i.e.,

$$(B(\theta))y_k := \sum_{i=1}^n \frac{\partial^2 s(q)}{\partial q_k \partial q_i} y_i. \quad (4.10)$$

The next assumption permits us to formulate the perturbed dynamics as the evolution of a tempered distribution. Thus, defining $\mathcal{S}^{(n)}(X)$ to be the space of the Schwartz $\mathcal{S}$ -class $\mathbb{R}^n$ -valued functions on X , and $\mathcal{S}^{(n)'}(X)$ to be its dual space, we assume that

(M.4) *the linear operator $\mathcal{L}(\theta_t)$ extends by continuity to a transformation of $\mathcal{S}^{(n)'}(X)$, and the formula (4.8), considered now as an equation of motion in that space, has a unique solution*

$$y_t = T(\theta|t, t')y_{t'} \quad \forall t \geq t' \geq 0 \quad (4.11)$$

with

$$T(\theta|t, t_0) \equiv T(\theta|t, t_1)T(\theta|t_1, t_0) \text{ and } T(\theta|t, t) = 1 \quad \forall t \geq t_1 \geq t_0 \geq 0.$$

Comments. In the equilibrium case, where $\theta_t = \bar{\theta}$, a constant, the two-parameter family $T(\theta|.,.)$ of transformations of $\mathcal{S}^{(n)'}(X)$ reduces to a one-parameter semi-group $\{\tilde{T}(\bar{\theta}|t)|t \in \mathbb{R}_+\}$, where

$$\tilde{T}(\bar{\theta}|(t - t_0)) \equiv T(\bar{\theta}|(t, t_0) = \exp(\mathcal{L}(\bar{\theta})(t - t_0)). \quad (4.12)$$

Further, in the case of the non-linear diffusion (4.6a), $\mathcal{L}(\theta^{(0)})$ reduces to the form

$$\mathcal{L}(\bar{\theta}) = L(\bar{\theta})B(\bar{\theta})\Delta; \quad L(\bar{\theta}) = [L_{kl}(\bar{\theta})]; \quad B = [B_{kl}(\bar{\theta})] \quad (4.12a)$$

where Δ is the Laplacian.

The following Proposition follows immediately from (M.1) and (M.4).

²Note that this assumption is not always fulfilled: for example, the Navier-Stokes equation is not scale invariant.

Proposition 4.1. *The process y is covariant w.r.t. space translations and space-time scale transformations, i.e., defining*

$$\phi_t^{(a,b,\lambda)}(x) \equiv \phi^{(a,b,\lambda)}(x, t) := \phi(a + \lambda x, b + \lambda^\tau t), \text{ for } \phi = \theta, q, y \quad (4.13)$$

and for arbitrary $(a, b, \lambda) \in X \times \mathbb{R} \times \mathbb{R}_+$,

$$(T(\theta|t, t')y_{t'})^{(a,b,\lambda)} \equiv T(\theta^{(a,b,\lambda)}|t, t')y_{t'}^{(a,b,\lambda)} \quad (4.14)$$

4.2 The Quantum Model, $\mathcal{Q}$.

In order to accommodate the dynamics of both the microscopic and macroscopic observables of Σ , we construct its quantum model, $\mathcal{Q}$, as a W^* -dynamical system [Se6] $(\mathcal{A}, \alpha, \mathcal{N}(\mathcal{A}))$, where $\mathcal{A}$ is a W^* -algebra of observables, $t \rightarrow \alpha_t$ is a representation of $\mathbb{R}$ in $\text{Aut}(\mathcal{A})$, corresponding to the dynamics of the system, and $\mathcal{N}(\mathcal{A})$ is the folium of normal states on $\mathcal{A}$. In fact, this algebra is constructed as the weak closure of the largest locally normal representation, π , of the standard [HHW] quasi-local C^* -algebra, $\tilde{\mathcal{A}}$, that can support the dynamical and thermodynamical structures we require. Thus, we characterise $\mathcal{A}$, α and π according to the following prescription.

(1) Let L be the set of bounded open regions of X . Then for each $\Lambda \in L$, there is a type-I factor, $\mathcal{A}(\Lambda) (\subset \mathcal{A})$, representing the observables of that region and satisfying the conditions of isotony and local commutativity; and $\mathcal{A}$ is the weak closure of $\mathcal{A}_L := \bigcup_{\Lambda \in L} \mathcal{A}(\Lambda)$. We extend automorphisms γ of $\mathcal{A}$ to unbounded observables, Q , affiliated³ to this algebra, according to the formula

$$\exp(i\lambda\gamma(Q)) = \gamma(\exp(i\lambda Q)) \quad \forall \lambda \in \mathbb{R}. \quad (4.15)$$

We similarly extend anti-automorphisms of $\mathcal{A}$ to its unbounded affiliates.

(2) The local energy observable (Hamiltonian) $H(\Lambda)$ for the region $\Lambda (\in L)$ is, in general, an unbounded one, affiliated to $\mathcal{A}(\Lambda)$. We assume that $\mathcal{A}$ is equipped with a Wigner time-reversal anti-automorphism, ρ , which leaves this observable invariant, i.e.,

$$\rho H(\Lambda) = H(\Lambda). \quad (4.16)$$

(3) Space translations are represented by a homomorphism σ of X into $\text{Aut}(\mathcal{A})$, such that $\sigma(x)\mathcal{A}_0(\Lambda) \equiv \mathcal{A}_0(\Lambda + x)$ and $\sigma(x)H(\Lambda) = H(\Lambda + x)$, this last condition representing an assumption of translationally invariant interactions.

(4) The dynamical automorphisms α are given by infinite volume limits of those of finite versions of Σ , according to the formula

$$\alpha_t A = s - \lim_{\Lambda \uparrow} (\exp(iH(\Lambda)t/\hbar) A \exp(-iH(\Lambda)t/\hbar)) \quad \forall t \in \mathbb{R}, A \in \mathcal{A}_L. \quad (4.17)$$

Thus, by the translational covariance of $H(\Lambda)$, α_t commutes with the space translational automorphisms, σ , and, by (4.16), satisfies the *microscopic reversibility* condition

$$\rho \alpha_t \rho = \alpha_{-t} \quad \forall t \in \mathbb{R}. \quad (4.18)$$

³Recall that, if $\mathcal{M}$ is a W^* -algebra of operators in a Hilbert space $\mathcal{H}$, then the unbounded operators, in $\mathcal{H}$, affiliated to $\mathcal{M}$ are the densely defined ones that commute with $\mathcal{M}'$.

(5) We formulate the thermodynamics of the system in terms a Hermitian quantum field $\hat{q} = (\hat{q}_1, \dots, \hat{q}_n)$, the $\hat{q}_k$'s being densities of locally conserved quantities. We assume that $\hat{q}$ is a tempered distribution, affiliated to $\mathcal{A}$, that transforms covariantly w.r.t space translations. Thus, $\hat{q}$ is a mapping of $\mathcal{S}^{(n)}(X)$ into the self-adjoint affiliates of $\mathcal{A}$. We assume that the function $\exp(i\hat{q}(\cdot))$ is strongly continuous, and that $\hat{q}$ transforms covariantly w.r.t. space translations, i.e.,

$$\sigma(x)[\hat{q}(f)] = \hat{q}(f_x), \text{ where } f_x(y) = f(x - y) \forall x, y \in X, f \in \mathcal{S}^{(n)}(X). \quad (4.19)$$

We assume, for simplicity, that $\hat{q}$ is invariant under time reversals, i.e.,

$$\rho\hat{q}(f) \equiv \hat{q}(f) \quad (4.20)$$

and that its components satisfy the following commutation relations, which signify that their space integrals over finite volumes intercommute, up to 'surface effects',

$$[\hat{q}_k(g), q_l(h)]_- = i\hbar j_{k,l}(\nabla(gh)) \quad \forall g, h \in \mathcal{S}(X), \quad (4.21)$$

where $(gh)(x) := g(x)h(x)$ and $j_{k,l}$ a tempered distribution. Further, denoting $\alpha_t[\hat{q}(f)]$ by $\hat{q}_t(f)$, we assume a local conservation law of the form

$$\frac{\partial \hat{q}_t(f)}{\partial t} = j_t(\nabla f) \quad (4.22)$$

where j_t is a tempered distribution.

(6) We take the equilibrium thermodynamic variables of $\mathcal{Q}$ to be the 'observables at infinity' [LR], given by the global spatial average $\hat{q}^\infty$ of $\hat{q}$ over X . Further, denoting by $\hat{s}$ the standard [Ru] entropy density functional on the translationally invariant states on $\mathcal{A}_0$, we assume that these observables form a complete thermodynamic set, in the sense that [Sel, Ch. 4]

(a) for each expectation value, q , of $\hat{q}$, there is precisely one translationally invariant state ($\in \mathcal{N}(\mathcal{A})$) that maximises $\hat{s}$; and

(b) no proper subset of $\hat{q}$ possesses this property.

The equilibrium thermodynamics of the system is thus given by the form of the resultant entropy density, $s(q)$. We identify q, s with the objects denoted by these symbols in the macroscopic model M ; and, defining θ according to (4.1), we denote by ω_θ the maximising state of condition (a). We assume that this state is stationary,⁴ i.e., α -invariant, in view of the fact that $\hat{q}_\infty$ is a globally conserved quantity; and we designate it as the equilibrium state corresponding to θ , i.e., to q . We note that, by (4.18) and the thermodynamic completeness condition (a),

$$\omega_\theta \circ \rho = \omega_\theta \quad (4.23)$$

(7) We define π to be the largest locally normal representation of $\tilde{\mathcal{A}}$ that supports the dynamical group α and the quantum field $\hat{q}$, as defined above.

Thus, the quantum model, $\mathcal{Q}$, is given by $(\mathcal{A}, \alpha, \sigma, \hat{q}, \mathcal{N}(\mathcal{A}))$, as specified by the conditions (1)-(7).

⁴The proof of this is straightforward for lattice systems, since one can show within the framework of [Sel, Ch. 4], that, for these, ω_θ satisfies the KMS conditions, and is therefore stationary.

4.3 Relationship between M and $\mathcal{Q}$

Our formulation of this relationship is based on the idea that the phenomenological law (4.6) corresponds to the dynamics of the quantum field $\hat{q}$ in a large-scale limit. Thus, in view of the scale-invariance assumption (M.1) for M , we introduce a length parameter L and reformulate $\mathcal{Q}$ on a length scale L and a time scale L^r , defining the quantum field $\hat{q}^{(L)}$ on these scales by the formula

$$\hat{q}_t^{(L)}(f) \equiv \hat{q}^{(L)}(f, t) := \hat{q}(f^{(L)}, L^r t) \quad (4.24)$$

where

$$f^{(L)}(x) := L^{-d} f(x/L). \quad (4.25)$$

We assume that L is also the length scale of spatial variations of the initial state, $\omega^{(L)}$, of $\mathcal{Q}$, i.e., that there is a map $A \rightarrow \bar{A}$ of $\mathcal{A}$ into $C(X)$ and a tempered distribution $q_0(\in \mathcal{S}^{(n)})$, such that

$$\lim_{L \rightarrow \infty} \omega^{(L)}(\sigma(Lx)[A]) = \bar{A}(x) \quad \forall x \in X \quad (4.26)$$

and

$$\lim_{L \rightarrow \infty} \omega^{(L)}(\hat{q}_0^{(L)}(f)) = q_0(f) \quad \forall f \in \mathcal{S}^{(n)}. \quad (4.27)$$

We define the fluctuation field $\hat{\xi}^{(L)}$ by the formula

$$\hat{\xi}_t^{(L)}(f) := L^{d/2} (\hat{q}_t^{(L)}(f) - \omega^{(L)}(\hat{q}_t^{(L)}(f))), \quad \forall f \in \mathcal{S}^{(n)}(X), \quad t \in \mathbb{R}. \quad (4.28)$$

Our basic assumptions for the large-scale dynamics of the model are the following.

(I) *For each M -process q , there is an equivalence class of initial states, $\{\omega^{(L)}\}$, parametrised by L , such that*

(a) *the mapping*

$$f^{(1)}, \dots, f^{(m)}; t^{(1)}, \dots, t^{(m)} \rightarrow \omega^{(L)}(\hat{\xi}_{t^{(1)}}^{(L)}(f^{(1)}) \dots (\hat{\xi}_{t^{(m)}}^{(L)}(f^{(m)}))$$

of $(\mathcal{S}^{(n)})^m \times \mathbb{R}^m$ into $\mathbb{C}$ is continuous, for all positive integers m .

(b) *The expectation value of the quantum field $\hat{q}_t^{(L)}$ reduces to that of the classical one, q_t , of M in the limit $L \rightarrow \infty$, i.e.,*

$$\lim_{L \rightarrow \infty} \omega^{(L)}(\hat{q}_t^{(L)}(f)) = q_t(f) \quad \forall f \in \mathcal{S}^{(n)}(X) \quad (4.29)$$

where q_t is the solution of (4.4), with initial value given by (4.27).

(c) *The quantum stochastic process $\hat{\xi}^{(L)}$ converges to a classical one, ξ , as $L \rightarrow \infty$, i.e.,*

$$\lim_{L \rightarrow \infty} \omega^{(L)}(\hat{\xi}_{t^{(1)}}^{(L)}(f^{(1)}) \dots (\hat{\xi}_{t^{(m)}}^{(L)}(f^{(m)})) = \mathbf{E}[\theta | (\xi_{t^{(1)}}(f^{(1)}) \dots \xi_{t^{(m)}}(f^{(m)}))] \quad (4.30)$$

$$\forall t^{(1)}, \dots, t^{(m)} \in \mathbb{R}_+, \quad f^{(1)}, \dots, f^{(m)} \in \mathcal{S}^{(n)}(X), \quad r \in \mathbb{N}$$

where the expectation functional $\mathbf{E}[\theta | \cdot]$ is governed by the restriction of θ to the closed interval between the minimum and maximum of $\{t^{(1)}, \dots, t^{(m)}\}$.

Comments.

1) The classicality of the limits of $q^{(L)}$ and $\xi^{(L)}$ here are assumed to arise from the commutation rules (4.21), together with asymptotic abelian properties of $\mathcal{Q}$ with respect to time.

2) Since (c) implies that the dispersion in $\hat{q}_t^{(L)}$, for the state $\omega^{(L)}$, tends to zero as $L \rightarrow \infty$, it follows from (b) that the quantum process $\hat{q}_t^{(L)}$ reduces to the classical one, q_t , in this limit.

3) It follows from (b) and (c) that ξ_t is an $\mathcal{S}^{(n)'} -$ valued random variable.

Let

$$\omega_\tau^{(L)} := \omega^{(L)} \circ \alpha(L^\tau \tau) \quad \forall \tau \in \mathbb{R}_+. \quad (4.31)$$

Then $\{\omega_t^{(L)}\}$ satisfies the conditions of (I), and the replacement of $\omega^{(L)}$ by $\omega_\tau^{(L)}$ corresponds to that of θ by $\theta^{(\tau)}$ in (4.30), where

$$\theta_t^{(\tau)} = \theta_{t+\tau}, \quad \forall t, \tau \in \mathbb{R}_+ \quad (4.32)$$

i.e.,

$$\mathbf{E}[\theta_t | (\xi_{t(1)+\tau}(f^{(1)}) \dots \xi_{t(m)+\tau}(f^{(m)}))] \equiv \mathbf{E}[\theta_t^{(\tau)} | (\xi_{t(1)}(f^{(1)}) \dots \xi_{t(m)}(f^{(m)}))]. \quad (4.33)$$

In the particular case where $t^{(1)} = \dots = t^{(m)} = 0$, the r.h.s. of this equation depends on $\theta^{(\tau)}$ only through the value of $\theta_0^{(\tau)} \equiv \theta_\tau$, by (4.32). Thus, by (4.30), the equal time correlation functions for the process ξ are of the form

$$\mathbf{E}[\theta_t | \xi_\tau(f^{(1)}) \dots \xi_\tau(f^{(m)})] \equiv \mathbf{E}[\theta_\tau | \xi_0(f^{(1)}) \dots \xi_0(f^{(m)})] \quad (4.34)$$

Our next assumption is that the space-time clustering properties of $\mathcal{Q}$ render the process ξ Gaussian (cf. [GVV]), and that the infinite separation of the relevant relaxation time-scales of the models M and $\mathcal{Q}$ ensure that it is Markovian.

(II) *The process ξ is Gaussian and temporally Markovian.*

It follows immediately from this assumption that the process ξ is completely determined by its two-point function. Our next assumption is the following generalisation of Onsager's regression hypothesis [On].

(III) *The fluctuation process ξ is governed by precisely the same dynamics as the perturbation, y , to the deterministic process q , i.e., by (4.11),*

$$\mathbf{E}[\theta_t | (\xi_{t+\tau}(f) \xi_t(g))] = \mathbf{E}[\theta_t | (\xi_t(T(\theta_t | t + \tau, t)^* f) \xi_t(g))].$$

Hence, by (4.34)

$$\mathbf{E}[\theta_t | (\xi_{t+\tau}(f) \xi_t(g))] = \mathbf{E}[\theta_t | (\xi_0(T(\theta_t | t + \tau, t)^* f) \xi_0(g))] \quad (4.35)$$

$$\forall f \in \mathcal{S}^{(n)}(X), t \in \mathbb{R}, \tau \in \mathbb{R}_+.$$

Thus, the process is determined by the form of T and of the expectation functional $\mathbf{E}[\theta_t | \cdot]$ on the algebra generated by ξ_0 . In order to formulate the action of space translations and scale transformations on the process, we define

$$f^{(a, \lambda)}(x) := \lambda^{-d/2} f(\lambda^{-1}(x - a)) \quad \forall a \in X, \lambda \in \mathbb{R}_+ \quad (4.36)$$

and

$$\xi_0^{(a,\lambda)}(f) := \xi_0(f^{(a,\lambda)}) \quad \forall a \in X, \lambda \in \mathbb{R}_+. \quad (4.37)$$

Proposition 4.2 *Under the above assumptions and definitions,*

$$\begin{aligned} \mathbf{E}[\theta_b | (\xi_0^{(a,\lambda)}(T(\theta_b | b + \tau, b)^* f) \xi_0^{(a,\lambda)}(g))] = \\ \mathbf{E}[\theta_0^{(a,b,\lambda)} | \xi_0(T(\theta_0^{(a,b,\lambda)} | \tau, 0)^* f) \xi_0(g)] \quad \forall a \in X, b, \tau \in \mathbb{R}_+, \lambda \in \mathbb{R}_+ \quad ((4.38)) \end{aligned}$$

with $\theta^{(a,b,\lambda)}$ as defined by (4.13).

Proof. The result is obtained by replacing each f by $f^{a,\lambda}$, in (4.30), and using eqns. (4.13), (4.14), (4.19), (4.27), (4.28), and (4.35)–(4.37).

We note now that the local properties of the process ξ , in the neighbourhood of a space-time point (a, b) , is given by the form of the l.h.s. of (4.38), in the limit $\lambda \rightarrow 0$. Moreover, by (4.13), the function $\theta^{(a,b,\lambda)}$, which occurs there, tends pointwise to a constant, $\theta(a, b)$, in this limit. These observations leads us to the following *local equilibrium* assumption.

$$\begin{aligned} (V) \quad \lim_{\lambda \rightarrow 0} \mathbf{E}[\theta_0^{(a,b,\lambda)} | \xi_0(T(\theta_0^{(a,b,\lambda)} | \tau, 0)^* f) \xi_0(g)] = \\ \mathbf{E}[\theta(a, b) | (\xi_0(T(\theta(a, b) | \tau, 0)^* f) \xi_0(g))] \quad \forall f, g \in \mathcal{S}^{(n)}(X), \tau \geq 0 \end{aligned} \quad (4.39)$$

and further, the r.h.s. of this formula is precisely the same as for the fluctuations of the field $\hat{q}$, about an equilibrium state $\omega_{\theta(a,b)}$, as defined in item (6) of §4.2, i.e.,

$$\begin{aligned} \mathbf{E}[\theta(a, b) | \xi_0(T(\theta(a, b) | \tau, 0)^* f) \xi_0(g)] \equiv \\ \lim_{L \rightarrow \infty} \omega_{\theta(a,b)}([\alpha(L^* \tau) \hat{\xi}_0(f)] \hat{\xi}_0(g)) \end{aligned} \quad (4.40)$$

Hence, by (4.12) and (4.40),

$$\begin{aligned} \mathbf{E}[\theta(a, b) | \xi_0(\exp(\mathcal{L}(\theta(a, b))\tau)^* f) \xi_0(g)] = \\ \lim_{L \rightarrow \infty} \omega_{\theta(a,b)}([\alpha(L^* \tau) \hat{\xi}_0(f)] \hat{\xi}_0(g)) \end{aligned} \quad (4.41)$$

4.4 Consequences of (I)-(V): Generalised Onsager Relations

Let $\mathcal{R}(\theta)$ be the range of the function $\theta \equiv \{\theta(a, b) | a \in X, b \in \mathbb{R}_+\}$. We shall employ the above theory to obtain properties of $\mathbf{E}[\bar{\theta}]$ and $\mathcal{L}(\bar{\theta})$ for arbitrary $\bar{\theta} \in \mathcal{R}(\theta)$.

(a) **Symmetry Property of Time Correlations Functions.** In view of the microscopic reversibility conditions (4.18), (4.20) and (4.23), together with the stationarity of $\omega_{\bar{\theta}}$, it follows from (4.30), with $\omega^{(L)} = \omega_{\bar{\theta}}$, that

$$\omega_{\bar{\theta}}(\xi_t(f) \xi_0(g)) \equiv \omega_{\bar{\theta}}(\xi_0(f) \xi_{-t}(g)) \equiv \omega_{\bar{\theta}}(\xi_t(g) \xi_0(f)) \quad \forall \bar{\theta} \in \mathcal{R}(\theta).$$

Hence, by (4.41), we have the symmetry property

$$\mathbf{E}[\bar{\theta} | \xi_0(\exp(\mathcal{L}(\bar{\theta})^* \tau) f) \xi_0(g)] \equiv \mathbf{E}[\bar{\theta} | \xi_0(\exp(\mathcal{L}(\bar{\theta})^* \tau) g) \xi_0(f)] \quad \forall \bar{\theta} \in \mathcal{R}(\theta). \quad (4.42)$$

(b) **The Static Two-point Function.** It follows immediately from (4.41) that $\mathbf{E}[\bar{\theta}]$ inherits the translational invariance of $\omega_{\bar{\theta}}$. Hence, in view of the tempered distribution property of ξ_0 , the generalised function $(x, y) \in X^2 \rightarrow \mathbf{E}[\bar{\theta}](\xi_0(x)\xi_0(y))$ is an $\mathcal{S}^{(n)'}(X)$ -class distribution $F(x-y)$; and, by Prop. 4.2, $F(\lambda x) \equiv \lambda^{-d}F(x)$, and is therefore of the form $C\delta(x)$, where C is an n -by- n matrix. Thus,

$$\mathbf{E}[\bar{\theta}]\xi_0(f)\xi_0(g) = \langle Cf, g \rangle \quad \forall \bar{\theta} \in \mathcal{R}(\theta) \quad (4.43)$$

where the angular brackets denote the inner product for the Hilbert space $\mathcal{H}^{(n)}$ of square integrable functions from X into $\mathbb{R}^n$, as defined by the formula

$$\langle f, g \rangle = \int f(x) \cdot g(x) dx \quad (4.44)$$

the dot denoting the $\mathbb{R}^n$ scalar product. Moreover, it follows [Se5] from a treatment of the linear response of $\omega_{\bar{\theta}}$ to local Hamiltonian perturbations $\hat{q}(f)$ that, under mild technical assumptions, $C = B(\bar{\theta})^{-1}$, where B is specified in (4.10). Hence, by (4.43),

$$\mathbf{E}[\bar{\theta}]\xi_0(f)\xi_0(g) = \langle B(\bar{\theta})^{-1}f, g \rangle. \quad (4.45)$$

(c) **Generalised Onsager Relations.** It follows immediately from (4.42) and (4.45) that

$$\langle B(\bar{\theta})^{-1} \exp(\mathcal{L}(\bar{\theta})^* \tau) f, g \rangle \equiv \langle B(\bar{\theta})^{-1} \exp(\mathcal{L}(\bar{\theta})^* \tau) g, f \rangle. \quad (4.46)$$

Hence, by (4.9), we have the following result:

Proposition 4.3. *Under the above assumptions, Λ satisfies the generalised Onsager relation*

$$\langle \Lambda(\bar{\theta})^* f, g \rangle \equiv \langle \Lambda(\bar{\theta})^* g, f \rangle \quad \forall \bar{\theta} \in \mathcal{R}(\theta), \quad f, g \in \mathcal{S}^{(n)'}(X) \quad (4.47)$$

i.e., $\Lambda(\bar{\theta})$, considered as an operator in $\mathcal{H}^{(n)}$, with domain $\mathcal{S}^{(n)}$, is symmetric.

Comment. In the case of the non-linear diffusion given by (4.6a), it follows from (4.12a) that (4.47) reduces to the form

$$L_{kl}(\theta)(x, t) = L_{lk}(\theta)(x, t) \quad \forall x \in X, \quad t \in \mathbb{R}_+.$$

5 Concluding Remarks

I have endeavoured to show here how, at least in certain domains, a macroscopically-based approach to statistical mechanics can serve to determine the form imposed by quantum mechanics on the structure of phenomenological laws. By contrast with the standard many-body theory, the microscopic input here is limited to very general principles; and this serves to pare down the conceptual structure of the theory to its essentials.

Of course, the relative simplicity gained by this approach is dependent on a number of assumptions, specified in the previous Sections, that are very difficult to verify constructively. Furthermore, the dynamical systems treated in Sections 3 and 4 have the simplifying, and rather particular, property of scale covariance. In the case of the plasma model, this stems from the fact that the Coulomb potential is given by a power

law: in the case of the non-equilibrium thermodynamics of Sec. 4, it is an assumed property of the macroscopic dynamics.

Thus, it is clear that the formulation of a coherent, general formulation of the statistical mechanics of macroscopic variables poses deep problems, concerning both its underlying assumptions and its potential scope. I would hope that inroads into these problems may be achieved through the study both of suitable models and of the relevant general structures.

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QUANTUM FIELD THEORY ON NONTRIVIAL SPACETIMES

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Classical field theory is intimately connected with differential geometry. Notions like the exterior derivative and the Hodge dual are crucial for electromagnetism, general relativity is related to Riemannian geometry, and the formalism of fibre bundles and connections is tailor-made for the formulation of Yang-Mills theories. One would like to have a similar geometrization for quantum field theory as well.

In the path integral formalism of quantum field theory this does not seem to be a principal problem. The basic quantity in this approach is the partition function

$$Z = \int \mathcal{D}A e^{iS(A)/\hbar}.$$

Here S is a functional on the classical field configuration A (the action) and $\hbar$ is Planck's constant divided by 2π . $\mathcal{D}A$ is a (formal) integral over all field configurations. In the limit $\hbar \rightarrow \infty$ only the stationary (critical) points of S contribute to the integral, so one recovers classical field theory. While the path integral is of great heuristic value, and in some cases has led to verifiable conjectures, its mathematical status is still unsatisfactory.

An alternative way is the direct formulation of quantum field theory on spacetimes more general than Minkowski space. Two kinds of problems arise. One is connected with the possible absence of a timelike Killing vector field. In such a case the notion of a ground state is not meaningful. Another problem is the existence of global obstructions like the absence of a spin structure on certain spacetimes.

Let me first describe the ground-state problem. Existence of a ground state means physically a stability property. Such a property seems to be crucial for every realistic physical theory. Mathematically, in Minkowski space one has the following structure: Let $\mathcal{A}(\mathcal{O})$ be the algebra of observables generated by a field $A(x), x \in \mathcal{O}$. We may choose $\mathcal{A}(\mathcal{O})$ to be the smallest von Neumann algebra in the vacuum representation to which all smeared fields $A(f)$, $f \in \mathcal{D}(\mathcal{O})$ are affiliated. Now using Takesaki's theorem on singular representations one can show [1, 2] that all representations in which the translations are implemented and satisfy the spectrum condition are normal on $\mathcal{A}(\mathcal{O})$. It was a proposal made by Haag, Narnhofer and Stein [3], and independently by Kay [4], that the ground-state problem be formulated as the problem of choosing the correct local von Neumann algebras $\mathcal{A}(\mathcal{O})$.

In the case of linear field theories satisfying the covariant Klein-Gordon equation

$$(D^\mu D_\mu + m^2)A = 0$$

a C^* -algebra can be given directly. Let (on a globally hyperbolic spacetime) $E_{\pm}$ be the retarded and the advanced propagator respectively:

$$(D^{\mu}D_{\mu} + m^2)E_{\pm}(x, y) = \delta(x, y)$$

$$\text{supp} E_{\pm} = \{(x, y), x \text{ later/earlier than } y\}.$$

Then we define a symplectic form σ on $\mathcal{D}(M, \mathbb{R})$,

$$\sigma(f, g) = \int dx dy f(x)(E_+ - E_-)(x, y)g(y)$$

and the C^* -algebra of canonical commutation relations is generated by operators $W(f)$ satisfying the following relations:

$$W(f)W(g) = e^{-i\sigma(f, g)/2}W(f + g),$$

$$W(f)^* = W(-f).$$

$\mathcal{A}(\mathcal{O})$ is the von Neumann algebra generated by $W(f)$, $\text{supp} f \subset \mathcal{O}$, with respect to a suitable class of states. The best choice seems to be the class of Hadamard states,

$$\omega(W(f)) = e^{-(f, Kf)/2}$$

where the kernel $K(x, y)$ of the positive semidefinite quadratic form K has the form of Hadamard's fundamental solution of the wave equation,

$$K(x, y) = s^{-2} + \log(s)v(x, y) + w(x, y),$$

with w smooth, s denoting the geodesic distance between x and y . It was recently shown by Verch [5] that different Hadamard states lead to quasi-equivalent factorial representations of $\mathcal{A}(\mathcal{O})$.

A generalization to interacting field theories is not yet available. For the general case, Haag, Narnhofer and Stein formulated the scaling limit condition, and Radzikowski [6] generalized this to a condition on the wavefront set.

An interesting idea was put forward by Buchholz and Summers [7]. These authors observed that the vacuum in Minkowski space can be uniquely characterized by the geometric action of modular operators on wedges (Bisognano-Wichmann theorem). Hence they proposed that one should look for states where the modular operators for some regions act geometrically.

As an example, let us consider de Sitter space. De Sitter space may be realized as a spacelike hyperboloid in a 5-dimensional Minkowski space:

$$M = \{x \in \mathbb{R}^5, x_0^2 - \sum_{i=1}^4 x_i^2 = -1\}.$$

We start from a field theory on the 5-dimensional Minkowski space. This leads to a net of von Neumann algebras $\mathcal{A}_0(\mathcal{O})$. We then define a net on M . As local regions $\mathcal{O}$ we use double cones, wedges and their spacelike complements. We set

$$\mathcal{A}(\mathcal{O}) := \bigvee_{\lambda > 0} \mathcal{A}_0(\lambda \mathcal{O})$$

This net is covariant under $SO(4,1)$. Moreover, from the spectrum condition on Minkowski space together with the Bisognano-Wichmann Theorem we obtain that for wedges W the modular operators act geometrically on M . The stability requirement may then be replaced by the KMS condition for Δ_W^H .

At the moment it is not clear to which spacetimes the method of Buchholz and Summers can be generalized. There is an interesting remark on this problem by Wightman. Wightman suggests that there might be a lesson to be drawn from “old quantum theory” where classical physics was enriched by certain rules allowing only very specific solutions. So, in quantum gravity, one might, in a first attempt, not treat all possible spacetimes on the same footing, but should look for a principle selecting only a few of them. Such a principle could very well be related to the geometric action of modular operators.

I now come to the problems connected to global properties of spacetimes. I do not have a systematic approach at the moment, but my impression is that my methods are generally applicable, and the interesting open questions are related to examples and their detailed properties.

Let me start from a simple example, occurring in conformal field theory. Let ϕ be the massless scalar field in 2d, and let $u = t - x, v = t + x$ be the usual light cone coordinates. The wave equation reads

$$\partial_v \partial_u \phi = 0,$$

hence $j = \partial_v \phi$ does not depend on v . We obtain a field theory on $\mathbb{R}$. Now j is covariant under Moebius transformations

$$u \mapsto \frac{au + b}{cu + d}, \quad ad - bc = 1.$$

The theory can then be canonically extended to a bigger manifold which is S^1 in this case. We have for each interval I on S^1 the local algebra $\mathcal{A}(I)$. The family $\mathcal{A}(I)$ has the structure of a pre-cosheaf but since the set of intervals is not directed there is no inductive limit. Instead one can define the global algebra $\mathcal{A}(S^1)$ as the free amalgamated product of the local algebras $\mathcal{A}(I)$. It can be characterized by the following universality condition [8]: There are homomorphisms ι_I of $\mathcal{A}(I)$ into $\mathcal{A}(S^1)$ which restrict, for $J \subset I$ on $\mathcal{A}(J)$ to ι_J such that, for any family π_I of representations on some Hilbert space $\mathcal{H}$ which has restrictions compatible with the pre-cosheaf structure, there is a unique representation π of $\mathcal{A}(S^1)$ such that $\pi \circ \iota_I = \pi_I$.

There is one representation, the vacuum representation, which is already faithful on $\mathcal{A}(I)$, hence $\mathcal{A}(I)$ is isometrically embedded into $\mathcal{A}(S^1)$.

Let me give two simple examples which show that the algebra which is characterized by this universality condition may be explicitly computed.

The first is the case of the $U(1)$ -current j , which has already been treated. We define $W_I(f) = e^{ij(f)}$, $f \in \mathcal{D}(I)$ with the algebraic relation

$$W_I(f)W_J(g) = e^{-i \int f g'} W_I(f + g).$$

The local algebras $\mathcal{A}(I)$ are generated by $W_I(f)$ with $f \in \mathcal{D}(I)$. The global algebra is generated by all operators $W_I(f)$ for all nontrivial intervals I with the additional relation

$$W_I(f) = W_J(f)$$

for $I \subset J$. The resulting algebra is the algebra generated by operators $W(f)$ with C^∞ -functions f on S^1 and the Weyl relation as before, together with the natural embeddings of local algebras. The new feature is that the global algebra is no longer simple; it has a nontrivial center which is actually big enough to label all positive energy representations [2].

The second example is somewhat more interesting. It is motivated by the treatment of Mack and Schomerus of the chiral Ising model [10] and is due to a collaboration with Brunetti and Szlachanyi [11]. One considers as local observables all bilinear expressions in a Majorana field ψ . The Majorana field satisfies the commutation relation

$$\{\psi(x), \psi(y)\} = 2\delta(x - y)$$

so that the smeared fields $\psi(f) = \int dx \psi(x) f(x)$, $f \in L^2(I, \mathbb{R})$ satisfy

$$\psi(f)^2 = \|f\|^2$$

and hence generate a Clifford algebra over $L^2(I, \mathbb{R})$. A Clifford algebra has a natural $\mathbb{Z}_2$ grading, and we define $\mathcal{A}(I)$ as the even part of this Clifford algebra. $\mathcal{A}(I)$ is generated by the operators $B_I(f, g) = \psi(f)\psi(g)$, $f, g \in L^2(I, \mathbb{R})$ which are $\mathbb{R}$ -bilinear in f and g and satisfy the relations

$$\begin{aligned} B_I(f, f) &= \|f\|^2 \\ B_I(f, g)B_I(g, h) &= \|g\|^2 B_I(f, h) \\ B_I(f, g)^* &= B_I(g, f). \end{aligned}$$

For the construction of the global algebra again one additional relation occurs, namely

$$B_I(f, g) = B_J(f, g)$$

for $I \subset J$ with the natural embeddings of the L^2 spaces. Let now I_1 and I_2 be disjoint intervals and let $J_\pm$ be intervals containing both of them and together covering S^1 . Let $f \in L^2(I_1)$ and $g \in L^2(I_2)$ with $\|f\| = \|g\| = 1$. We define a unitary Y by

$$Y = B_{J_+}(f, g)B_{J_-}(g, f)$$

and find that Y is independent of the choice of f, g, I_1, I_2, J_+, J_- within the restrictions listed above. Hence Y is in the center of $\mathcal{A}(S^1)$ and $Y^2 = 1$. The spectral values of Y distinguish the Neveu-Schwartz sector ($Y = -1$) from the Ramond sector ($Y = +1$).

If one replaces the local Clifford algebras by the von Neumann algebras obtained by weak closure in the vacuum representation, then the center of the global algebra is even larger. In general, provided there are no degenerate sectors, i.e., sectors with trivial monodromy with all other sectors, it was shown [12] that there are central elements C_α associated with each sector α , such that

$$\begin{aligned} C_\alpha C_\beta &= N_{\alpha, \beta}^\gamma C_\gamma \\ C_{\bar{\alpha}} &= C_\alpha^* \end{aligned}$$

where the coefficients $N_{\alpha, \beta}^\gamma$ are the fusion coefficients describing how often the sector γ occurs in the product of the sectors α and β . The value of C_α in the sector β is (up to a normalization) the $(\alpha\beta)$ entry of Verlinde's matrix S .

The construction of the global algebra may be done in an analogous way also for other spacetimes, for instance for conformal completions of Minkowski space in higher dimensions (cf. [13], [14], [15]).

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FUNCTIONAL INTEGRATION: COMMENTS ON AN OPEN DISCUSSION

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The strength of functional integration is its scope. Its weakness is the lack of a simple and robust mathematical framework. For the last couple of years, Pierre Cartier and I have examined a number of heuristic results obtained by functional integrals in order to identify their “*substantifique moelle*,”¹ and a number of rigorous results which can serve as pillars for an axiomatic basis of functional integration. I proposed to organize this open discussion by the various fields which have benefitted from, and enriched functional integration:

Analysis

- Functional analysis (integration by parts, change of variable of integration, iterated integrals, etc.)
- Calculus of variations
- Operator determinants
- Supermanifolds, etc.

Stochastic calculus

- Wiener processes, parabolic equations
- Poisson processes, wave equations
- First exit time, Dirichlet problems; bound states

Quantum physics

- Operator calculus (time ordered products of operators)
- Quantum Noether’s theorems, Takahashi-Ward identities, “anomalies”
- Effective actions, their low energy limits, wavelets
- Renormalization (with and without regularization)
- Gauge fields, etc.

¹François Rabelais.

Chemistry

Polymers, etc.

Group theory

Integration on symmetric spaces

Equivariant cohomology

Localization (generalization of Duistermaat-Heckman theorem)

Topology

Stochastic processes on fibre bundles (multiply connected spaces, etc.)

Topological invariants (Chern-Simons actions and Vasiliev invariants;

Supersymmetric actions and Euler-Poincaré characteristic)

Numerical analysis

Non-standard analysis

Computational techniques

Et cetera

The interesting and challenging comments made by the participants being difficult to incorporate in a well structured report, the participants were invited to submit papers² to a special issue of the *Journal of Mathematical Physics* on functional integration scheduled for 1994 or 1995.

²Proposals should be sent to Cécile DeWitt-Morette who coordinates the project.

WHAT ARE WE LEARNING FROM 2-DIMENSIONAL CONFORMAL MODELS?

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Abstract

Rational conformal field theories are reviewed from an (informal) axiomatic standpoint. This approach sheds new light on the classification of conformal current algebra models. On the other hand, the study of 2D CFT gives rise to new questions in the general theory. We display as an example embeddings of local chiral algebras which are not of the Doplicher-Haag-Roberts type.

The interrelation between this branch of mathematical physics and some problems of pure mathematics is emphasized.

1 General Remarks

If then I find myself writing, not mathematics but “about” mathematics, it is a confession of weakness, for which I may rightly be scorned . . .

G. H. Hardy, *A Mathematician’s Apology*

Special dates like the turn of a century give us both an opportunity and an excuse to distance ourselves for a moment from the fad of the day and current worries, to have a side look at the route we are following. Paying tribute to the title of our workshop, I shall begin by a general overview of 2-dimensional (2D) conformal field theory (CFT), a subject that has been particularly popular among mathematical physicists during the past decade. We shall then proceed to a more down-to-earth discussion of special examples designed to illustrate the interrelations between CFT models and the axiomatic approach to local quantum physics (to borrow the expression from the title of a recent book [21] by one of its founders).

Quantum field theory has been the most challenging endeavour of XXth century theoretical physics. Facing such a formidable problem it appears legitimate to spend a few years, even a couple of decades, trying to first master simpler lower dimensional examples. Such a justification was not “bought” by the physics community, though. One had to relate 2D CFT to surface critical phenomena before a breakthrough in the

field took place [2], describing an infinite family of universality classes of 2D critical behaviour. But it was not until the second coming of superstrings [20] that the subject became really hot. This gives me an occasion for another digression.

String theory strives to understand space-time coordinates as 2D quantum fields. The concept of a space-time point (not just of a point particle) is disappearing, giving room to little explored structures that belong to the general realm of non-commutative geometry (understood ultimately as a quantum field theory—see Connes [12]). This is an ambitious and romantic project. It led to a bizarre particle theory fashion that has no basis in (and no implication for) today's (or tomorrow's) experiment. Phenomenologists view such a development as unhealthy. Even mathematicians with an independent taste for physics, like Sergei Novikov, refer to repeated claims that superstrings solve the problem of quantum gravity as science fiction.

I would not venture to predict whether the fantastic leap of nearly twenty orders of magnitude in the energy scale, professed by string theorists, will be ever justified. Prediction, Niels Bohr's Danish proverb says, is always difficult, especially of the future. One should admit in any case that it is dangerous to engage in a long term project with no intermediate point of reference. It is customary to look for phenomenological implications as tests that we are on the right track: Confrontation with experiment or at least with another, more down-to-earth branch of particle theory (that in current parlance also includes elements of the theory of phase transitions). The development of mathematical physics, particularly in the last couple of decades, provides examples of applications in the opposite direction: Work in (low dimensional) quantum field theory and statistical mechanics leads to discoveries in pure mathematics. A prominent example is provided by recent progress in constructing knot invariants (the major steps can be traced back from [40], [27], [11]). From this (and not only from this) point of view string-theory-inspired studies can be favourably contrasted with the preceding fashion in "theoretical theory", (extended) supergravity. To be sure, the test of modern mathematics is not a monopoly of 2D CFT and related topological theories. An earlier example is provided by the Haag-Hugenholtz-Winnink form [22] of the Kubo-Martin-Schwinger boundary condition for equilibrium states in statistical mechanics which served as a ferment for completing the Tomita-Takesaki modular theory of von Neumann algebras (for an exposition with emphasis on ideas rather than techniques see Haag [21]).

Notably, as string theory fashion is fading away, problems of 2D CFT continue to attract attention of physicists and mathematicians alike. Let me, therefore, turn back to my original motivation and to the question in the title of this talk. More precisely: What are we learning as "axiomatic" quantum field theorists from 2D CFT models?

Leaving the discussion of the problem of local extensions of a chiral current algebra (in which I have been involved) to the subsequent more technical sections, let me mention here two other recent developments in the "algebraic" (Haag-Kastler) approach to local quantum theory, based on or inspired by CFT.

The first is the proof of the Haag duality property for a CFT in any number of space-time dimensions [6] (see also independent treatments for 2D CFT in [17] and [23]). A readable overview of the required concepts of duality, modular automorphisms and modular conjugation is now available in the book [21] (Secs. III.4, V.2 and V.4), and I will not recapitulate them here. One finds there (in V.4.2), in particular, a review

of the 1982 paper of Hislop and Longo that gives a geometric meaning to modular automorphisms in a CFT. A new ingredient and a motivation for the above-cited work has been a recent result of Borchers [3] (the modular operator for a wedge region having the expected commutation relations with translations and energy positivity yields, essentially, its geometric interpretation). It should be noted that global conformal invariance (implying that one is dealing with compactified space) is essential for the validity of the duality theorem—see [8].

Another development, motivated by CFT, is the introduction of the concept of a global algebra of observables associated with a topologically non-trivial space-time (see [15], [14] as well as Fredenhagen's article in this book). It has been inspired, in particular, by the work [7] on the $U(1)$ conformal current algebra on the circle. Such global algebras have a non-trivial centre (containing bounded functions of the $U(1)$ charge in the above examples as well as the monodromy-type operator $e^{2\pi i L_0}$, where L_0 is the conformal energy, for any CFT on the circle). Such an observable algebra is not represented faithfully in the vacuum (or any other single) superselection sector.

The discussion of chiral CFT corresponding to a given current algebra, undertaken in the following sections, has a double aim. Firstly, it is concerned with a beautiful achievement of 2D CFT which deserves to be remembered. Secondly, it provides examples of fields (of integer conformal dimension) which are not uniquely determined by their relation to a conformal current algebra that plays the role of a chiral observable algebra $\mathfrak{A}_k$. They yield local extensions of $\mathfrak{A}_k$ of a new type that is only recently being integrated within the algebraic approach to local quantum theory.

One of our objectives being to bridge the gap between different schools of thought in this branch of mathematical physics, we emphasize in Secs. 2 and 3 the point of view of “algebraic quantum field theorists” and add some remarks on its relation to more traditional 2D CFT in Sec. 4.

2 Axiomatic Rational CFT. Chiral Algebras and their Extensions by Simple Currents

The Belavin-Polyakov-Zamolodchikov paper [2] that started it all fits nicely in an axiomatic framework. The informal language used—with some variations—by the majority of theorists (the version I have in mind being reflected in [16] and [39]) can be viewed as a mixture of Haag's and Wightman's approaches. It starts, much like Doplicher, Haag and Roberts (DHR) do with a local observable algebra and proceeds to constructing different superselection sectors in terms of its positive energy irreducible representations (for a recent review of the DHR et al. work and a bibliography—see [25]). However, one considers both the observable and the field algebra as generated by *Wightman fields* (that are unbounded-operator valued distributions) rather than by local nets of C^* (or von Neumann) algebras. One can also take a more purist point of view by substituting smeared hermitian local currents $J(f)$ by some bounded functions of J , like $\exp(iJ(f))$, and stick closer to the canons of the Hamburg school—cf. [7] and [8], but this does not seem always practical.

What makes the division of fields in a 2D CFT into local observables and “charged primary fields” useful is the simple structure of the algebra of local observables in this

case. It consists of the tensor product $\mathfrak{A} \otimes \bar{\mathfrak{A}}$ of two (usually isomorphic) chiral algebras: the “right movers” algebra $\mathfrak{A}$ of fields like the “ $-$ ” component T of the stress-energy tensor, which depend only on a single light-cone variable $t - x$ and its left movers’ counterpart of fields which depend on $t + x$. As discussed in [28] it is natural to use also for $D = 2$ the cylindrical space-time

$$(2.1) \quad \tilde{M} = S^{D-1} \times \mathbb{R}^1$$

that appears for $D \geq 3$ as the universal cover of the conformal compactification of Minkowski space. Single-valued (in particular, observable) fields Φ on $\tilde{M}$ are periodic functions of x :

$$(2.2) \quad \Phi(x + 2\pi, t) = \Phi(x, t).$$

It follows that chiral observable fields are periodic functions of $t - x$ and can hence be viewed as functions of the complex variable $z = \exp\{i(t - x)\}$ that takes physical values on the unit circle. In this “analytic compact picture” (cf. [16]) the Fourier expansion coincides with the Laurent series in z ; in particular,

$$(2.3) \quad T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

where L_n generate the Virasoro algebra Vir ,

$$(2.4) \quad [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m}$$

($\delta_l = \delta_{l0}$). Hermiticity of T and energy positivity are expressed by

$$(2.5) \quad L_n^* = L_{-n}, \quad L_0 \geq 0.$$

The three operators $L_0, L_{\pm 1}$ span the $su(1,1)$ (Möbius) Lie subalgebra of Vir which gives rise to (infinitesimal) S^1 -preserving fractional linear transformations of z .

In a field theoretic context the algebra of observables is assumed to be unitarily represented in the (positive metric) Hilbert space of physical states. (By contrast, in statistical mechanics, positivity of the inner product that respects (2.5) is no longer motivated by the probabilistic interpretation.) Postulating the existence of a unique zero energy state, the vacuum, one proves that

$$(2.6) \quad L_n|0\rangle = 0 \quad \text{for } n \geq -1,$$

and hence the vacuum is Möbius invariant. In a unitary theory with a unique vacuum state the *central charge* c of Vir is a positive number (this is part of the 1976 Lüscher-Mack theorem—see, e.g., [16] Theorem 3.1; a result of this type was established even earlier, in little-known 1974 Brazilian lecture notes by B. Schroer cited there). The Virasoro central charge is an important characteristic of the vacuum of a CFT. The fact that it is a fixed number implies that products of fields do not give rise to tensor products of representations of Vir (but to the notion of “fusion rules”). It is instructive to have a fresh look at all local extensions of a given chiral current algebra $\mathfrak{A}_k$ with a fixed value $c = c(k)$ of the central charge of Vir . (A similar, more fundamental, problem

of classifying the local extensions of the chiral algebra generated by the stress-energy tensor is not yet solved.)

Let $\mathfrak{G}$ be a simple Lie algebra with (skew) product realized as a commutator, $[v_1, v_2]$, and a Killing form

$$(2.7) \quad (v_1, v_2) = \frac{1}{2h} \text{tr } v_1 v_2$$

where the matrices v_i in the right hand side are taken in the ($\dim \mathfrak{G}$ -dimensional) adjoint representation of $\mathfrak{G}$ and h is its *dual Coxeter number* — $\frac{1}{2}$ the eigenvalue of the Casimir operator C_2 in the adjoint representation (for $\mathfrak{G} = su_n$, $h = n$). The algebra $\mathfrak{A}_k(\mathfrak{G})$ is generated by currents $J(z, v)(z \in \mathbb{C}, v \in \mathfrak{G})$ satisfying the commutation relations:

$$(2.8) \quad [J(z_1, v_1), J(z_2, v_2)] = J(z_2, [v_1, v_2])\delta(z_{12}) - k(v_1, v_2)\delta'(z_{12}).$$

Here the Dirac δ -function on S^1 has the formal power series expansion of $z_{12}^{-1} + z_{21}^{-1}$:

$$(2.9) \quad \delta(z_{12}) = \sum_{n \in \mathbb{Z}} z_1^{-n-1} z_2^n, \quad z_{12} = z_1 - z_2$$

and the *level* k takes non-negative integer values for unitary integrable representations of the Kac-Moody algebra $\mathfrak{G}$ (see [24]). The algebra $\mathfrak{A}_k(\mathfrak{G})$ contains the (Sugawara) stress-energy tensor (see [16] and references therein) with central charge proportional to the dimension of the Lie algebra:

$$(2.10) \quad c(k) = \frac{k}{k+h} \dim \mathfrak{G}.$$

Restricting our attention to the case $\mathfrak{G} = su_n$ we can write simple explicit formulas, to begin with

$$(2.11) \quad \mathfrak{A}_{kn} \equiv \mathfrak{A}_k(su_n) \Rightarrow c(k, n) = \frac{k}{k+n}(n^2 - 1).$$

Let α_i be the *simple roots* of su_n normalized by $\alpha_i^2 = 2$, so that $\alpha_i \alpha_j$ is the su_n Cartan matrix. Every unitary irreducible representation of su_n is given by its *highest weight*

$$(2.12a) \quad \Lambda = \sum_{i=1}^{n-1} \lambda_i \Lambda^{(i)}, \quad \lambda_i \in \mathbb{Z}_+ (= \{0, 1, 2, \dots\}),$$

where $\Lambda^{(i)}$ are the *fundamental weights* satisfying

$$(2.12b) \quad (\Lambda^{(i)}, \alpha_j) = \delta_j^i, \quad i, j = 1, \dots, n-1 \quad (= \text{rank } su_n).$$

There is a finite set of unitary integrable positive energy (highest weight) representations of $\mathfrak{A}_{kn}$ for which the non-negative integers λ_i are restricted by

$$(2.13) \quad \lambda_1 + \dots + \lambda_{n-1} \leq k.$$

If $\mathfrak{A}_{kn}$ is the (maximal) chiral observable algebra of the CFT under consideration then each Λ satisfying (2.13) defines a *superselection sector* $\mathcal{H}_\Lambda$ with *conformal energy spectrum* $\{\Delta(\Lambda) + l, l \in \mathbb{Z}_+\}$ and *highest weight vector* $|\Lambda\rangle$ of minimal energy

$$(2.14) \quad (L_0 - \Delta(\Lambda))|\Lambda\rangle = 0, \quad 2(k+n)\Delta(\Lambda) = C_2(\Lambda).$$

Here $C_2(\Lambda)$ is the second order Casimir operator of su_n :

$$(2.15) \quad \begin{aligned} C_2(\Lambda) &= \sum_{i,j=1}^{n-1} \lambda_i a_{ij} (\lambda_j + 2) \\ nC_2(\Lambda) &= \sum_{i=1}^{n-1} i(n-i) \lambda_i (\lambda_i + n) + \sum_{i < j} i(n-j) \lambda_i \lambda_j, \end{aligned}$$

$a_{ij} = a_{ji}$, $na_{ij} = i(n-j)$ for $i \leq j$, (a_{ij}) being the inverse Cartan matrix, while the vector

$$(2, \dots, 2) \equiv 2 \sum_{i=1}^{n-1} \Lambda^{(i)}$$

appearing implicitly in (2.15) is the sum of all positive roots of su_n .

Each admissible Λ defines a finite set of chiral vertex operators (CVO)

$$(2.16) \quad V_\Lambda(z; \Lambda_f, \Lambda_i) : \mathcal{F}(\Lambda_i) \rightarrow \mathcal{F}'(\Lambda_f).$$

Here $\mathcal{F}(\Lambda)$ is a dense subspace of $\mathcal{H}_\Lambda$ of sl_2 analytic vectors (in the sense of [32]), complete with respect to a suitable locally convex topology, and $\mathcal{F}'(\Lambda)$ is an extension of $\mathcal{H}_\Lambda$, that is the topological dual to $\mathcal{F}(\bar{\Lambda})$, $\bar{\Lambda}$ being the conjugate of Λ . $\mathcal{F}(\Lambda)$ contains all finite energy vectors of $\mathcal{H}_\Lambda$ as well as infinite series of the form $\exp(zL_{-1})|\Lambda\rangle$ with $|z| < 1$. The pairs (Λ_i, Λ_f) are restricted by the $\mathfrak{A}_{kn}$ fusion rules derived in [19]. In particular, if $\Lambda_i = 0$ is the weight of the trivial representation of su_n (corresponding to the vacuum vector) then $\Lambda_f = \Lambda$, so that V_Λ intertwines between the vacuum sector and the sector of highest weight Λ .

The CVOs are, in general, multivalued functions of z with monodromy, depending on the conformal weights:

$$(2.17) \quad V_\Lambda(e^{2\pi i} z; \Lambda_f, \Lambda_i) = \exp\{2\pi i(\Delta(\Lambda_f) - \Delta(\Lambda) - \Delta(\Lambda_i))\} V_\Lambda(z; \Lambda_f, \Lambda_i).$$

For each Λ (2.12) there exist pairs (Λ_i, Λ_f) , allowed by the fusion rules, for which the monodromy (i.e., the exponential factor in (2.17)) is non-trivial. If all admissible superselection sectors (whose Kac-Moody weights satisfy (2.12)) are actually present in the theory then the $\mathfrak{A}_{kn}$ fusion rules will be saturated: None of the allowed CVOs will vanish identically. If, however, one only allows for a subset $\mathcal{R}$ of superselection sectors that is closed under composition then it is possible to find pairs (k, Λ) for which all remaining CVOs V_Λ are single valued. For $\mathfrak{A}_{k2}$ this is the case for

$$(2.18) \quad \lambda_k = k = 4n, \mathcal{R} = \{\lambda = 2m, m = 0, 1, \dots, 2n\}.$$

There is a single-valued hermitian Bose primary field $V_k(z)$ (of even isospin) which has the characteristic intertwining property for a *simple current* (see [36]):

$$(2.19) \quad V_k : \mathcal{F}_\lambda \rightarrow \mathcal{F}'_{k-\lambda}$$

(the relevant feature being that there is a single term in the right hand side).

Extensions by simple currents also appear for higher values of n . Their structure depends on the decomposition of n into primes. For a prime n the *extended vacuum space* $\tilde{\mathcal{H}}_0$ is the direct sum of n "coherent spaces"

$$(2.20) \quad \tilde{\mathcal{H}}_0 = \bigoplus_{l=0}^{n-1} \mathcal{H}_l$$

where $\mathcal{H}_l$ is an $\mathfrak{A}_{kn}$ module of highest weight $k\Lambda^{(l)}$; $\Lambda^{(j)}$, $j = 1, \dots, n-1$ are the fundamental weights (2.12 a); $\Lambda^{(0)} = 0$ is the vacuum weight. The corresponding conformal dimensions

$$(2.21) \quad \Delta(k\Lambda^{(l)}) = \frac{k}{2n}l(n-l)$$

are integer for k , a multiple of n , n odd. If we use the triplets of topological vector spaces

$$(2.22) \quad \mathcal{F}_l \subset \mathcal{H}_l \subset \mathcal{F}'_l$$

for arbitrary integer l , setting

$$(2.23) \quad \mathcal{F}_{l+n} \equiv \mathcal{F}_l \quad \text{etc.}$$

then the intertwining properties of simple currents (and hence, their fusion rules) are given by

$$(2.24) \quad V_{l_1} : \mathcal{F}_{l_2} \rightarrow \mathcal{F}'_{l_1+l_2}.$$

The observation that product of simple currents is again a multiple of a simple current or of the unit operator, together with the fact that all the above simple currents have integer dimensions, implies that the fields V_l , $l = 1, \dots, n-1$ give rise to a local extension of $\mathfrak{A}_{kn}$.

If n is not a prime, then there are extensions of $\mathfrak{A}_{kn}$ involving a subset of the V_l , but for a wider range of k . For instance, $\mathfrak{A}_{k9}$ admits a local extension by a pair of conjugate primary fields of weights $k\Lambda^{(3)}$ and $k\Lambda^{(6)}$ and conformal dimension k for all $k \in \mathbb{N}$. For $n = 2m$, even, we have again a set of $n-1$ simple currents of weights $k\Lambda^{(l)}$ with k a multiple of $2n$, but we also have at least two more infinite sets:

1. $k\Lambda^{(l)}$ for l even and k a multiple of m ;
2. $k\Lambda^{(m)}$ for k a multiple of 4 if m is odd and for all k if m is even.

Extensions by simple currents are not strangers in algebraic quantum field theory. They are of the DHR type: There exists a (finite) gauge group (of the first kind) implemented unitarily on $\mathcal{H}_0$ and acting by automorphisms on $\mathfrak{A}_{kn}$ which leaves each element of the unextended algebra $\mathfrak{A}_{kn}$ invariant. For $n \geq 3$ prime this is the $2n$ element group D_n (the symmetry group of a regular polygon of n sites); it is generated by two elements U_1 and C whose action on $\mathcal{H}_0$ is given by

$$(2.25) \quad \begin{aligned} U_1 \mathcal{H}_l &= \exp\left(2\pi i \frac{l}{n}\right) \mathcal{H}_l, \quad l = 0, 1, \dots, n-1 \\ C|k\Lambda^{(l)}\rangle &= |k\Lambda^{(n-l)}\rangle, \quad [C, \mathfrak{A}_{kn}] = 0 \end{aligned}$$

($\Lambda^{(n)} = \Lambda^{(0)} = 0$). Eq. (2.25) implies the properties

$$(2.26) \quad U_1^n = 1 = C^2, \quad CU_1C = U_1^* = U_1^{-1}.$$

The simple currents V_l transforming according to the law

$$(2.27) \quad U_1 V_l(z) U_1^{-1} = \exp\left(2\pi i \frac{l}{n}\right) V_l(z), \quad C V_l(z) C = V_{n-l}(z)$$

give rise to a reducible representation of D_n which splits into $\frac{1}{2}(n-1)$ 2-dimensional irreducible components $\mathcal{T}_l$ acting on the pairs of conjugate currents (V_l, V_{n-l}) . The fusion rules for such pairs follow the tensor product expansion law

$$(2.28) \quad \mathcal{T}_i \otimes \mathcal{T}_j = \mathcal{T}_{|i-j|} \oplus \mathcal{T}_{(i+j)} \quad (\mathcal{T}_l = \mathcal{T}_{n-l}, \mathcal{T}_0 = 2.1).$$

(Note that Eq. (2.28) remains valid if we replace each $\mathcal{T}_i$ by its dimension, 2.) The predictability of the resulting extension is also displayed by the fact that the operator expansions of products of simple currents do not change in the passage from $\mathfrak{A}_{kn}$ to $\tilde{\mathfrak{A}}_{kn}$. The structure of superselection sectors for the extended chiral algebra, although it does change, is also well understood given the additional generators of $\tilde{\mathfrak{A}}_{kn}$ and the associated gauge group.

As we shall see in Sec. 3 below – there are surprises for some exceptional extensions of $\mathfrak{A}_{kn}$ by primary fields which are not simple currents.

3 Non DHR Extensions of $\mathfrak{A}_{k2}$

There has been an unspelled belief that a CVO (and the corresponding 2D field) is essentially determined by its properties relative to the original chiral observable algebra; in a mathematical language, by its behaviour as an element of an $\mathfrak{A}_{kn}$ module. Indeed, given the chiral algebra and the normalization of 2-point functions of primary CVOs all conformal blocks characterizing the chiral theory appear to be determined by the polynomial (pentagon and hexagon) identities of [31]. In effect, this is only true whenever the full set of $\mathfrak{A}_{kn}$ superselection sectors is present. For special values of the level k and for appropriate subsets of primary weights these identities admit more solutions. An efficient method addressing this problem, at least for small values of n , is provided by the Knizhnik-Zamolodchikov (KZ) equation [26].

The basic ingredients in deriving the KZ equation are the 1968 Sommerfield-Sugawara formula for the stress energy tensor T as a quadratic function of the current J and the T - and J -Ward identities for primary fields (a derivation along these lines parallel to the KZ paper was given in [38]; for a pedagogical review and further references see [16]).

We are looking for pairs of conjugate mutually local, single valued, primary chiral fields $V_\Lambda, V_{\bar{\Lambda}}$ where

$$(3.1) \quad \Lambda = (\lambda_1, \dots, \lambda_{n-1}) \leftrightarrow \bar{\Lambda} = (\lambda_{n-1}, \dots, \lambda_1)$$

(an example being the pairs of simple currents V_l, V_{n-l} of the preceding section). Their 4-point function

$$(3.2) \quad \langle 0 | V_\Lambda(z_1) V_{\bar{\Lambda}}(z_2) V_\Lambda(z_3) V_{\bar{\Lambda}}(z_4) | 0 \rangle = \left(\frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{14}} \right)^{2\Delta} f(\eta),$$

where $\Delta = \Delta(\Lambda) = \Delta(\bar{\Lambda})$ is given by (2.13) and η is the Möbius-invariant cross ratio,

$$(3.3) \quad z_{ij} = z_i - z_j, \quad \eta = \frac{z_{12}z_{34}}{z_{13}z_{24}},$$

should be a rational function of its arguments with (possible) singularities at coinciding points. The prefactor is chosen to carry the dominating singularity of the 2-point function of the pair $V_\Lambda V_{\bar{\Lambda}}$ for neighbouring arguments. A straightforward analysis shows that if $V_\Lambda, V_{\bar{\Lambda}}$ are to be local Bose fields then the dimension Δ should be integer and f should be a polynomial solution of the KZ equation

$$(3.4) \quad \left\{ (k+n) \frac{d}{d\eta} - \frac{C_{12}}{\eta} + \frac{C_{23}}{1-\eta} \right\} f(\eta) = 0.$$

Here C_{ij} is the su_n Casimir operator acting in the corresponding tensor product of irreducible representations of weights Λ and $\bar{\Lambda}$. We have omitted throughout the internal symmetry indices in V and f as well as the invariant tensor prefactor. It is instructive to display them in the simplest su_2 case for which the above problem was first solved – see [29].

We use the classical realization of an irreducible su_2 representation of weight (twice isospin) λ in the space of polynomials of degree λ in a formal variable ζ (see [41] for an early use of this realization in the present context). The su_2 properties of ζ are reflected, in particular, in the current-field Ward identity which can be written in the form

$$(3.5) \quad V_\lambda(z_1, \zeta_1) J(z_2, \zeta_2) |0\rangle = \frac{1}{z_{12}} (\zeta_{12}^2 \partial_{\zeta_1} - \lambda \zeta_{12}) V_\lambda(z_1, \zeta_1) |0\rangle.$$

The advantage of this relation is the similarity between the Möbius transformation properties of the world sheet variable z and the SU_2 properties of ζ . This similarity is displayed in writing down the invariant amplitude: f in (3.2) should be substituted by

$$(3.6) \quad \zeta_{13}^\lambda \zeta_{24}^\lambda f(\eta, \xi), \quad \xi = \frac{\zeta_{12} \zeta_{34}}{\zeta_{13} \zeta_{24}} \quad (\zeta_{ij} = \zeta_i - \zeta_j).$$

C_{ij} then appear in Eq. (3.4) as second order differential operators in ξ :

$$(3.7) \quad \begin{aligned} C_{12} &= \lambda(\lambda+1) - \lambda^2 \xi - [2\lambda(1-\xi) + \xi] \xi \frac{\partial}{\partial \xi} + \xi^2 (1-\xi) \frac{\partial^2}{\partial \xi^2}, \\ C_{23} &= \lambda(\lambda+1) - \lambda^2 (1-\xi) + (2\lambda\xi + 1 - \xi)(1-\xi) \frac{\partial}{\partial \xi} + \xi(1-\xi)^2 \frac{\partial^2}{\partial \xi^2}. \end{aligned}$$

The requirement that V_λ is a local Bose field implies the crossing symmetric boundary conditions

$$(3.8) \quad f(0, \xi) = \xi^\lambda, \quad f(1, \xi) = (\xi - 1)^\lambda.$$

The result of [29] is summarized in the following

Theorem *The polynomial solution of Eq. (3.4), (3.7) satisfying the boundary conditions (3.8) is either the one with maximal weight, $\lambda = k$,*

$$(3.9) \quad f_k(\eta, \xi) = (\xi - \eta)^k,$$

that is the 4-point amplitude of the (hermitian) simple current V_k , or it corresponds to one of the following exceptional values of k, λ (and Δ)

$$(3.10) \quad k = 10, \quad \lambda = 4 \left(\Delta = \frac{1}{2} \right), \quad \lambda = 6 \left(\Delta = 1 \right);$$

$$(3.11) \quad k = 28, \quad \lambda = 10 \left(\Delta = 1 \right), \quad \lambda = 18 \left(\Delta = 3 \right).$$

An important intermediate step in the proof of the Theorem is provided by the crucial

Lemma *If a local chiral field V is not of the type V_k then the operator product expansion $V \times V$ involves a primary current of dimension 1.*

A similar assertion is valid for $\mathfrak{A}_{k3}$ where the role of V_k is taken by the pair V_{k0}, V_{0k} (see [37]). It is however not generally true as demonstrated by a counterexample for $\mathfrak{A}_6(F_4)$ – see [36].

Remarkably, both (3.10) and (3.11) as well as the first bosonic simple current (with $k = 4, \Delta = 1$) correspond to *conformal embeddings* ([34], [1]) of su_2 (of the corresponding level k) into a (rank 2) simple Lie current algebra of level 1 – see [5]. This means, in particular, that the stress energy tensors and the Virasoro central charge of the two CFT coincide:

$$(3.12) \quad c(4, 2) = \frac{3 \times 4}{4 + 2} = 2 = c(1, 3), \quad \tilde{\mathfrak{A}}_{4,2} = \mathfrak{A}_{13}$$

(in the notation (2.11));

$$(3.13) \quad c(10, 2) = \frac{3 \times 10}{10 + 2} = \frac{5}{2} = c_1(sp_4), \quad \tilde{\mathfrak{A}}_{10,2} = \mathfrak{A}_1(sp_4);$$

$$(3.14) \quad c(28, 2) = \frac{3 \times 28}{28 + 2} = \frac{14}{5} = c_1(G_2), \quad \tilde{\mathfrak{A}}_{28,2} = \mathfrak{A}_1(G_2).$$

This observation determines completely the superselection structure of the extended theory (and proves its existence without the necessity of going beyond the 4-point functions in the above analysis). For both su_3 and sp_4 the level 1 condition,

$$(3.15) \quad \lambda_1 + \lambda_2 \leq 1 \quad \text{for} \quad \mathfrak{A}_{13} \text{ and } \mathfrak{A}_1(sp_4),$$

implies the presence of two “charged sectors” (besides the vacuum sector). They are labeled by the fundamental weights, (1,0) and (0,1). The corresponding Casimir operators and conformal dimensions are

$$(3.16) \quad C_2(su_3) = \frac{2}{3} \{ \lambda_1(\lambda_1 + 3) + \lambda_2(\lambda_2 + 3) + \lambda_1\lambda_2 \}, \quad \Delta_{10}(su_3) = \frac{1}{3} = \Delta_{01}(su_3);$$

$$(3.17) \quad C_2(sp_4) = \lambda_1 \left(\frac{1}{2} \lambda_1 + \lambda_2 + 2 \right) + \lambda_2(\lambda_2 + 3), \quad \Delta_{10}(sp_4) = \frac{5}{16}, \quad \Delta_{01}(sp_4) = \frac{1}{2}.$$

The level 1 condition for G_2 gives

$$(3.18) \quad 2\lambda_1 + \lambda_2 \leq 1 \text{ for } \mathfrak{A}_1(G_2).$$

Thus, there is a single charged sector in this case, of G_2 weight $(0, 1)$; furthermore,

$$(3.19) \quad C_2(G_2) = 2\{\lambda_1(\lambda_1 + \lambda_2 + 3) + \frac{\lambda_2}{3}(\lambda_2 + 5)\},$$

hence,

$$\Delta_{01}(G_2) = \frac{2}{5} = \Delta_6(su_2, k=28) \left(= \frac{3.4}{30} \right).$$

All this has been known since the 1987 work of Nahm and Bouwknegt. What seems to have remained unnoticed either by these authors or by anyone else after the appearance of [5] is the drastic change of properties of the $\mathfrak{A}_{k2}$ primary fields of dimension 1 in the passage $\mathfrak{A}_{k2} \rightarrow \tilde{\mathfrak{A}}_{k2}$ for the two exceptional cases ($k=10$ and $k=28$). The 4-point conformal blocks of the $\Delta=1$ $\mathfrak{A}_{k2}$ CVO for both $k=10, \lambda=6$ and $k=28, \lambda=10$ are multivalued transcendental functions that give rise to 5-, respectively 11-, dimensional representations of the braid group $\mathfrak{B}_4$ on 4 strands (or, more precisely, of the mapping class group of the 4-punctured sphere – see [39] Appendix A). They are solutions of the KZ equation consistent with the $\mathfrak{A}_{k2}$ fusion rules

$$(3.20) \quad \begin{aligned} \underline{6} \times \underline{6} &= \sum_{I=0}^4 \underline{2I} \quad \text{for } k=10 \left(\Delta_{2I} = \frac{I(I+1)}{12} \right) \\ \underline{10} \times \underline{10} &= \sum_{I=0}^{10} \underline{2I} \quad \text{for } k=28 \left(\Delta_{2I} = \frac{I(I+1)}{30} \right). \end{aligned}$$

By contrast, the 4-point function of the $\Delta=1$ current in the $\tilde{\mathfrak{A}}_{k2}$ theory is a single valued meromorphic function invariant under the permutation group

$$\mathfrak{S}_4 = \mathfrak{B}_4 / \mathfrak{M}_4.$$

Here $\mathfrak{M}_4$ is the (normal) monodromy subgroup of $\mathfrak{B}_4$. (In fact, only the factor group

$$\mathfrak{D}_3 = \mathfrak{S}_4 / \mathbb{Z}_2 \times \mathbb{Z}_2$$

acts faithfully on functions of η and ξ – see [29].) The 4-point amplitude $f_{k\lambda}(\eta, \xi)$ for $(k, \lambda) = (10, 6)$ and $(28, 10)$ is a polynomial solution of the same KZ equation corresponding to different fusion rules:

$$(3.21) \quad \begin{aligned} \underline{6} \times \underline{6} &= \underline{0} + \underline{6} \quad \text{for } \tilde{\mathfrak{A}}_{10,2}; \\ \underline{10} \times \underline{10} &= \underline{0} + \underline{10} + \underline{18}, \quad \underline{10} \times \underline{18} = \underline{10} + \underline{18} + \underline{28}; \\ \underline{18} \times \underline{18} &= \underline{0} + \underline{10} + \underline{18} \quad \text{for } \tilde{\mathfrak{A}}_{28,2}. \end{aligned}$$

As expected, only fields of integer conformal dimension appear in the right hand side. In contrast with the situation of extensions by simple currents (discussed in Sec. 2) the embedding $\mathfrak{A}_{k2} \subset \tilde{\mathfrak{A}}_{k2}$ for $k=10$ and 28 is not of a DHR type. Indeed, if the currents $\underline{6} = (10, 6), \underline{10} = (28, 10)$ etc. belonged to finite $(d_6-, d_{10}-) \dots$ -dimensional

representations of a (compact) gauge group, then the corresponding dimensions should satisfy algebraic relations copying (3.21):

$$(3.22) \quad d_6^2 = 1 + d_6; \quad d_{10}^2 = 1 + 2d_{10} \quad (d_{10} = d_{18}, \quad d_{28} = 1).$$

The solution d_6 of the first equation is the “golden ratio”

$$(3.23a) \quad d_6 = \frac{1 + \sqrt{5}}{2} = \frac{\sin\left(\frac{2\pi}{5}\right)}{\sin\frac{\pi}{5}},$$

and $d_{10}(=d_{18})$ is also irrational:

$$(3.23b) \quad d_{10} = 1 + \sqrt{2} = \frac{\sin\frac{3\pi}{8}}{\sin\frac{\pi}{8}}.$$

Such non-integer numbers clearly cannot be the dimensions of representations of an ordinary group. They can, of course, appear as “quantum dimensions” of quantum group representations. It is, in fact, customary to encounter such quantum dimensions for primary CVOs with non-trivial braid group statistics. It is unusual, however, to have a single-valued local Bose field associated with an “irrational dimension”.

In order to resolve the puzzle one should make a clear distinction between the CVOs and their, say, 4-point blocks on the one hand, and a local chiral field with its correlation functions, on the other hand.

Two local 4-point functions may be built out of the 4-point block $S_{2I}(I = 0, 1, 2, 3, 4)$ of the CVO V_6 (I labelling the isospin of the intermediate states in the s -channel). One is the standard (“diagonal”) 2-dimensional Green function of a spin zero field constructed out of products of V_6 and $\bar{V}_6$. Its Euclidean expression can be written as

$$(3.24) \quad \sum_{I=0}^4 N_{2I}^2 |S_{2I}|^2, \quad N_0 = 1, \quad N_{2I} > 0.$$

The second is a meromorphic 4-point function of a local chiral field $J_6(z)$ of the form

$$(3.25) \quad S_0(z_1, z_2, z_3, z_4) + a_6 S_6(z_1, z_2, z_3, z_4)$$

(corresponding to $J_6 = V_6(6, 0) + V_6(0, 6) + c_6 V_6(6, 6)$). A computation (based on the results of Section 3D of [39]) shows that the invariant ratio of structure constants is just equal to 2:

$$\frac{a_6^2}{N_6^2} = 2.$$

It would be interesting to understand this ratio in the line of the algebraic interpretation of the embedding of $\mathfrak{A}_{10,2}$ in $\mathfrak{A}_1(sp_4)$ given recently in [33]¹.

¹The author thanks K.-H. Rehren for correspondence on this point and Yassen Stanev for his part in computing the above ratio of structure constants.

4 Discussion and Outlook

We outlined in Secs. 2 and 3 an approach to chiral CFT designed to make contact with the algebraic framework for local quantum physics. One starts with a manageable algebra of observables, the su_n chiral current algebra $\mathfrak{A}_{kn}$ for a fixed level k . The standard (diagonal) CFT based on such an algebra involves all $\binom{k+n-1}{n-1}$ superselection sectors of highest weights satisfying (2.13). Other CFTs dealing with the same current algebra are constructed whenever $\mathfrak{A}_{kn}$ admits a local extension $\tilde{\mathfrak{A}}_{kn}$ by $\mathfrak{A}_{kn}$ primary fields. There is an infinite series of levels k admitting extensions by simple currents which are of conventional DHR type, and a finite number of exceptional cases for which the fusion properties of the additional local currents are substantially different from those of the CVOs with the same quantum numbers in the standard $\mathfrak{A}_{kn}$ theory.

For a newcomer to 2D CFT the above picture could be misleading, first, because it ignores the historical development of the field, and second, because it is incomplete. The following remarks aim to indicate the main omissions.

The classification of 2D conformal current algebra models and of minimal CFT followed a rather different path. It started with Cardy's study [10] of the effect of boundary conditions on the operator content of 2D CFT and culminated in the ADE classification of modular invariant partition functions (see [9] and references therein). This approach is attractive because of its interrelation with a beautiful branch of modern mathematics—the study of modular forms—and deserves at least a brief summary (especially, since it is much in the spirit of the general remarks of the introduction).

The energy distribution of a chiral superselection sector (λ) of $\mathfrak{A}_{k2}$ is coded in the character

$$(4.1) \quad \chi_\lambda(\tau) = \text{tr}_{\mathcal{H}_\lambda} \exp\{2\pi i \tau (L_0 - \frac{c}{24})\}, \quad \text{Im } \tau > 0,$$

$$\lambda = 0, 1, \dots, k.$$

The partition function of a 2D CFT is written as a sesquilinear form in the characters of the theory

$$(4.2) \quad Z(\tau) = \sum_{\lambda, \mu=0}^k n_{\lambda\mu} \chi_\lambda(\tau) \overline{\chi_\mu(\tau)}$$

where $n_{\lambda\mu}$ are non-negative integers and $n_{00} = 1$ (reflecting the uniqueness of the vacuum state). The characters $\{\chi_\lambda, \lambda = 0, 1, \dots, k\}$ transform under a $(k+1)$ -dimensional representation of the modular group

$$(4.3) \quad \tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

The existence of a path integral representation for Z that is globally reparametrization invariant implies its modular invariance.

The classification of modular invariant partition functions of the type (4.2) is related to the number theoretic properties of the height $h = k + 2$ and involves the Coxeter exponents of the Weyl group of simply-laced Lie groups. It is so beautiful that I cannot resist the temptation to state it again here.

The Weyl group of a rank r simple Lie algebra is generated by r reflections $s_i \alpha = \alpha - 2(\alpha, \alpha_i) \alpha_i \alpha_i^{-2}$, $i = 1, \dots, r$ in the r dimensional root space (α_i being the simple roots). The eigenvalue spectrum of the product $s_1 s_2 \dots s_r$ is independent of the order of factors. It consists of roots of 1:

$$(4.4) \quad \exp\left(2\pi i \frac{m_\nu}{h}\right), \quad \nu = 1, \dots, r$$

where m_ν are positive integers,

$$(4.5) \quad 1 \leq m_1 \leq m_2 \leq \dots \leq m_r < h,$$

called the *Coxeter exponents* while h is the Coxeter number introduced in Sec. 2 (see, e.g., [4]).

The classification theorem of [9] can be formulated as follows. To each height h there correspond as many modular invariant partition functions as there are simply-laced simple Lie algebras with Coxeter number equal to h . They can be split into three infinite series and two finite sets. The standard ("diagonal") CFT exist for any $h = n \geq 2$ and correspond to $su_n \approx A_{n-1}$. The partition function is given in this case by (4.2) with $n_{\lambda\mu} = \delta_{\lambda\mu}$. For $h = 4\rho + 2$ ($k = 4\rho$) we have, in addition, the $D_{2\rho+2}$ series of partition functions

$$(4.6) \quad Z_{2\rho-2} = \sum_{\nu=0}^{\rho-1} |\chi_{2\nu} + \chi_{4\rho-2\nu}|^2 + 2|\chi_{2\rho}|^2.$$

They correspond to the extension of the observable algebra by a simple current of weight $\lambda = 4\rho$. The two conformal embeddings into $C_2 \simeq sp_4$ and G_2 —see (3.13) and (3.14)—correspond to the exceptional Lie algebras E_6 and E_8 :

$$(4.7) \quad Z(E_6) = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2$$

$$(4.8) \quad \begin{aligned} Z(E_8) = & |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + \\ & + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2. \end{aligned}$$

In all cases the number of characters appearing in the formula for Z (each counted with its multiplicity) is equal to the rank of the corresponding simply laced Lie algebra while its Coxeter exponents are given by $\lambda + 1$ where λ is the su_2 weight of χ . The factor 2 in front of $|\chi_{2\rho}|^2$ is equal to the multiplicity of the Coxeter exponent $2\rho + 1$ in (4.4). The $D_{2\rho+3}$ series ($\rho = 1, 2, \dots$) associated with heights $h(= k + 2) = 4\rho + 4$ corresponds to

$$(4.9) \quad \begin{aligned} Z_{2\rho+3} = & \sum_{I=0}^{2\rho+1} |\chi_{2I}|^2 + \sum_{\nu=1}^{\rho} (\chi_{2\nu-1} \bar{\chi}_{2\rho-2\nu} + \\ & + \chi_{2\rho-2\nu} \bar{\chi}_{2\nu-1}) + |\chi_{2\rho+1}|^2. \end{aligned}$$

The exceptional E_7 CFT corresponds to

$$(4.10) \quad \begin{aligned} Z(E_7) = & |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 \\ & = |\chi_8|^2 + (\chi_2 + \chi_{14}) \bar{\chi}_8 + \chi_8 (\bar{\chi}_2 + \bar{\chi}_{14}), \quad h = 18. \end{aligned}$$

The $D_{2\rho+3}$ series does not refer to an extension of the chiral observable algebra, but to an automorphism of the fusion rules (in accord with the general analysis of [30]). The E_7 partition function is associated with an automorphism of the D_{10} fusion rules.

Thus, listing the local extension of $\mathfrak{A}_{k_2}$ takes care of roughly a half of the non-diagonal 2D CFT based on the su_2 current algebra: The D-even and E-even series. The remaining twisted series have to be understood in 2D terms, since they describe non-standard coupling of left and right blocks.

To summarize, classifying $\mathfrak{A}_{k_2}$ chiral current algebra CFT relates unexpectedly two at first sight different mathematical problems: finding all polynomial solutions of the su_2 KZ equation and singling out the modular invariant subspaces in the family of finite dimensional $SL(2, \mathbb{Z})$ -modules spanned by the characters (4.1). The solution of both problems involves the Coxeter exponents of the Weyl group of simply-laced simple Lie algebras. Branches of hard core mathematics, long regarded as the realm of pure art, immune to any kind of applications, are becoming integral parts of modern quantum field theory.

The success is still confined to special classes of rational CFT. Going from su_2 to su_3 , one had a good conjecture about the set of all modular invariant partition functions for some time (see, e.g., [13] and references therein) before an attempt to proving it was made in [18]. It is therefore, rewarding to see a complete study of the local extensions of $\mathfrak{A}_{k_3}$ —see [37]. The interpretation of the resulting set of integers (the relevant su_3 weights and the selected heights $k + 3$) in terms of arithmetic objects, such as the Coxeter exponents and number, are still missing in this case.

Physics and mathematics have gone some way towards each other from the “missed opportunities”, noted by Freeman Dyson a couple of decades ago to the recent manifestations of a concerted effort in unravelling common structures in the two sciences.

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ON THE MANIFESTATIONS OF PARTICLES

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Abstract

We discuss the mathematical description of stable particle-like systems appearing in relativistic quantum field theory (a) at large and small scales, and (b) at non-zero temperatures.

1 Introduction

Stable particle-like systems are met with in high energy physics in very different settings: Elementary particles, as well as their bound states, are observed at asymptotic times in collision processes. At the other end of the scale, i.e., at short distances, particle-like structures confined in hadronic matter (partons) appear in deep inelastic processes. In thermal states, they may also exist as individual constituents of a hot plasma (quarks, gluons). It is a somewhat embarrassing fact that for most of these fundamental elementary systems a fully satisfactory mathematical description is not at hand.

In the conventional treatment, originating from E. Wigner's pioneering analysis [1], particles are identified with vector states in irreducible representations of the Poincaré group, or its covering group. This characterization of particles has proved to be extremely useful: It has led to an understanding of the role of mass and spin, it is a key ingredient in the construction of field theoretic models, it enters in the renormalization procedure, and it is the basis for the physical interpretation of the theory in terms of collision states. These applications reveal the value of a sound *a priori* particle concept within the theoretical setting.

However, Wigner's particle concept applies only to a restricted class of elementary systems, a well known counterexample being particles carrying electric charge. The states of such particles always contain infinitely many low energy photons and therefore cannot be described by vectors in some Poincaré invariant superselection sector of the physical Hilbert space; in fact, neither mass nor spin can sharply be defined for these states [2, 3]. The inappropriate Wigner concept of the particle is at the origin of the infrared problems in the early treatments of quantum electrodynamics; it led B. Schroer to the notion of infraparticles [4]. Although one has learned in the meantime how to handle these difficulties, a clear-cut mathematical characterization of electrically charged particles has not yet emerged. Summarizing the folklore, these particles are regarded as certain limiting cases of Wigner-type particles, which are accompanied by

photon clouds (cf. [5, 6] for a rigorous formulation of this idea). This picture suffices, in many applications, for a consistent interpretation of theoretical calculations, e.g., of collision cross sections. But it somewhat obscures the true nature of the underlying physical systems.

In the case of partons the situation is even less satisfactory. These entities made their first appearance in an ingenious interpretation of experimental data [7], relying on Wigner's particle concept. Their theoretical description has later been formalized in the perturbative treatment of quantum chromodynamics, which also relies on that concept through the use of Feynman rules. In this way the existence of partons has now been firmly established, but it is also clear that the particle concept used has only a tentative meaning since partons are (expected to be) confined and physical states of these entities do not exist. Again, one pays for the inappropriate particle concept by (a) tantalizing infrared problems in the perturbative setting, and (b) the need to introduce in an *ad hoc* way phenomenological fragmentation functions which are tailored to convert the perturbative results into physically significant statements. These difficulties may in principle be avoided by considering only physical states, as e.g., in lattice gauge theory. But then one loses contact with the simple parton picture, and the problem of a description of the parton content of states reappears.

Problems of still another kind arise if one wants to describe the particle features of thermal states. Thinking of the deconfinement transition in quantum chromodynamics, for example, it would be of interest to characterize the constituents of such states. Of similar physical interest are the quasiparticles [8], i.e., collective excitations of the constituents which frequently exhibit particle properties as well. The constituents of a thermal state may be expected to have properties which do not deviate too much from those of an elementary particle, at least in the case of low densities. Nevertheless, their mathematical description causes problems because of the influence of the thermal background. In fact, neither the constituents, nor the quasiparticles can obey a sharp dispersion law if there is interaction [9]. This fact makes it difficult to distinguish them from other modes, and again one runs into inconsistencies if one imposes from the outset a wrong particle concept [10].

It is the aim of this article to report on attempts to overcome these conceptual difficulties. The results obtained so far are not complete, but they shed new light on the problems and seem to be a solid basis for further studies. Thus, in accord with the idea of this Workshop, this contribution intends to be both, a status report and a bit of science fiction.

The article is organized as follows: Sec. 2 contains an outline of algebraic quantum field theory, which is the natural framework for particle analysis. The particle structures which appear in this setting at asymptotic times are described in Sec. 3. These results are fairly complete and lead to a unified treatment of elementary particles and infraparticles in terms of *weights* [11]. Moreover, they provide a general method by which the asymptotic particle content can be extracted from any state.

The particle structures appearing in the short distance limit are the theme of Sec. 4. For their characterization the concept of a scaling algebra is introduced which provides an algebraic version of the renormalization group and allows a systematic discussion of the scaling limit of any quantum field theory [12]. It turns out that this limit can always be interpreted in terms of vacuum states on the scaling algebra. The particle content

of the theory in the short-distance limit can thus be determined by a particle analysis of the local excitations of these vacua, as indicated above, and one may therefore hope for a complete classification of the pertinent structures.

Section 5 is devoted to the particle aspects of thermal equilibrium states. For the investigation of the particle structures in the scaling limit one can apply the method of the scaling algebra, yielding the same results as before. But if one proceeds to larger scales, the analysis becomes more difficult since the specific dynamical properties of the thermal background come into play. Nevertheless it appears to be possible to distinguish the constituents of thermal states in precise mathematical terms [13]. A clear-cut mathematical characterization of quasiparticles remains, however, as an intriguing problem.

The article closes with some “concluding contemplations”.

2 The Framework

Before entering into details, it may be helpful to point to the similarities of our approach and the strategy of Wigner. Wigner took as the starting point of his analysis the principles of *special relativity*, which are encoded in the properties of the space-time symmetry group, the Poincaré group. Since symmetries act on physical states by automorphisms, it appeared to be natural to assume that particles, being the most elementary systems, give rise to irreducible representations of the Poincaré group or its covering group.¹ This idea of a particle then led naturally to the problem of classifying these representations [1].

In the present approach we start from the principle of *locality*, which is encoded in algebraic relations between observables. The notion of locality is of vital importance in our characterization of particles, where we appeal to the heuristic picture of a particle as a stable, singly localized entity [14]. This picture will lead us to a comprehensive description of particles in terms of specific states (or weights) on the algebra of observables, depending on the respective physical situation. Thus, like Wigner, we characterize particles by a condition of simplicity, and we are then led to the classification of the corresponding representations of the observables, in complete analogy to his approach.

As already indicated, we shall make use of the framework of algebraic quantum field theory [15] which allows us to discuss the particle aspects of relativistic quantum field theories in full generality. For the convenience of those readers who are not familiar with this setting, we briefly list the relevant assumptions and add a few comments.

1. (*Locality*) We suppose that the local observables of the underlying theory generate a net of local algebras over d -dimensional Minkowski space $\mathbb{R}^d$, i.e., an inclusion-preserving map

$$(1) \quad \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$$

from the set of open, bounded regions $\mathcal{O}$ in Minkowski space to unital C^* -algebras $\mathcal{A}(\mathcal{O})$. We will think of $\mathcal{A}(\mathcal{O})$ as the algebra generated by observables which can be

¹At the time of its invention this assumption must have seemed to be exceedingly general. But today we see its weak point: symmetries can be spontaneously broken so that their action on states gives rise to incoherent systems, belonging to different superselection sectors. It then makes no sense to classify the states according to the irreducible representations of the symmetry group.

measured in $\mathcal{O}$. The algebra generated by all local algebras $\mathcal{A}(\mathcal{O})$ (as a C^* -inductive limit) will be denoted by $\mathcal{A}$. The net is supposed to satisfy the principle of locality (Einstein causality), i.e.,

$$(2) \quad \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)' \text{ if } \mathcal{O}_1 \subset \mathcal{O}_2',$$

where $\mathcal{O}'$ denotes the spacelike complement of $\mathcal{O}$ and $\mathcal{A}(\mathcal{O})'$ the set of operators in $\mathcal{A}$ which commute with all operators in $\mathcal{A}(\mathcal{O})$. We note that this abstract algebraic setting is quite natural for our purposes since we are interested in physical systems whose Hilbert space description has yet to be determined.

2. (*Covariance*) The Poincaré group $\mathcal{P}_+^\uparrow$ is represented by automorphisms of the net. Thus for each $(\Lambda, x) \in \mathcal{P}_+^\uparrow$ there is an $\alpha_{\Lambda, x} \in \text{Aut } \mathcal{A}$ such that, in an obvious notation,

$$(3) \quad \alpha_{\Lambda, x}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O} + x)$$

for any region $\mathcal{O}$. We amend this fundamental postulate by a continuity condition, i.e., we assume that for any $A \in \mathcal{A}$ the function $(\Lambda, x) \rightarrow \alpha_{\Lambda, x}(A)$ is strongly continuous. This is no loss of generality since these observables suffice to separate the states of physical interest.

3. (*States*) Physical states are described by positive, linear and normalized functionals ω on $\mathcal{A}$. By the GNS-construction, any state ω gives rise to a representation π_ω of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$, and there exists a cyclic vector² $\Omega_\omega \in \mathcal{H}_\omega$ such that

$$(4) \quad \omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega), \quad A \in \mathcal{A}.$$

Since the dual space of $\mathcal{A}$ contains an abundance of states corresponding to very different physical situations (few and many body systems, equilibrium and non-equilibrium systems, etc.), one has to select the states of actual interest by additional conditions.

It is a characteristic feature of few body systems that they have finite total energy and momentum. This suggests to characterize the corresponding states ω by the condition [16] that the space-time translations x act on the GNS-space $\mathcal{H}_\omega$ by a continuous unitary representation $U_\omega(x)$ which satisfies the relativistic spectrum condition (i.e., the generators have joint spectrum in the forward lightcone $\bar{V}_+$) and implements the translations of observables,

$$(5) \quad U_\omega(x)\pi_\omega(A)U_\omega(x)^{-1} = \pi_\omega(\alpha_x(A)), \quad A \in \mathcal{A}.$$

As a matter of fact, under these conditions there exists a distinguished representation $U_\omega(x)$ with Lorentz invariant spectrum, as a consequence of locality [17]. States ω and representations π_ω with these properties are said to have positive energy. If a positive energy state ω is invariant under translations, i.e., $\omega(\alpha_x(A)) = \omega(A)$ for all x and $A \in \mathcal{A}$, its GNS-vector Ω_ω is invariant under the action of $U_\omega(x)$ and may be interpreted as a vacuum vector.

If also the Lorentz transformations Λ act on $\mathcal{H}_\omega$ by a continuous unitary representation $\Lambda \rightarrow U_\omega(\Lambda)$, and $U_\omega(\Lambda)\pi_\omega(A)U_\omega(\Lambda)^{-1} = \pi_\omega(\alpha_\Lambda(A))$, $A \in \mathcal{A}$, we speak of Poincaré-covariant representations. It is reasonable to assume that states ω , representing the vacuum, are invariant under Lorentz transformations, thereby giving rise to

²This means that $\pi_\omega(\mathcal{A})\Omega_\omega$ is dense in $\mathcal{H}_\omega$.

Poincaré-covariant representations. But, as was pointed out in the Introduction, the condition of full Poincaré covariance for non-vacuum states is generally too restrictive, as it excludes, from the outset, states carrying electric charge.

We will also consider states ω on $\mathcal{A}$ which describe systems in thermal equilibrium. These states are characterized by the KMS-condition, which is the proper description of Gibbs ensembles in the thermodynamic limit [18, 15]. In the formulation of this condition there enters a distinguished time axis, which is fixed by the rest frame of the heat bath. Denoting by t the translations along this axis, the KMS-condition says that ω is a thermal equilibrium state at inverse temperature $\beta > 0$ if the following holds: for any given $A, B \in \mathcal{A}$ the functions $t \rightarrow \omega(A\alpha_t(B))$ and $t \rightarrow \omega(\alpha_t(B)A)$ are continuous boundary values of some analytic function in the strip $\{z \in \mathbb{C} : 0 < \text{Im} z < \beta\}$, at its lower and upper edge, respectively.

3 Particles and Infraparticles

States ω of positive energy are expected to describe, at asymptotic times, configurations of a few stable particles, possibly accompanied by clouds of low energy massless particles. Following this idea and the conception of a particle as some well-localized entity, it seems natural to analyze the asymptotic particle aspects of ω by studying the properties of the expectation values $\omega(\alpha_x(A))$, $A \in \mathcal{A}$, for large timelike x . The dominant contributions should originate from a vacuum state ω_0 ,

$$(6) \quad \lim_x \omega(\alpha_x(A)) = \omega_0(A), \quad A \in \mathcal{A},$$

since the probability for finding some deviation from the vacuum in a finite space-time volume about x (fixed by the localization properties of the respective observable A) should tend to 0 in a state of finite energy because of dispersive effects ("spreading of wavepackets").³ Here we are primarily interested in the local deviations from the vacuum, however. Following Araki and Haag [20], we therefore proceed to a subalgebra $\mathcal{C} \subset \mathcal{A}$ of observables which are insensitive to the vacuum.

The algebra $\mathcal{C}$ is constructed as follows: we first consider the vector space $\mathcal{L}_0 \subset \mathcal{A}$ of almost local operators⁴ L_0 such that the functions $(\Lambda, x) \rightarrow \alpha_{\Lambda, x}(L_0)$ are C^∞ and have, with respect to x , Fourier transforms with support in some arbitrary compact subset Γ in the complement of the closed forward light-cone $\bar{V}_+$ (they "transfer" energy-momentum Γ). Such operators are easily obtained by regularization of (almost-) local operators $A \in \mathcal{A}$ with respect to the action of the Poincaré group,

$$(7) \quad L_0 = \int d\mu(\Lambda, x) f(\Lambda, x) \alpha_{\Lambda, x}(A).$$

Here μ is the Haar measure on $\mathcal{P}_+^1$ and f is any test function whose Fourier transform with respect to x has support in some Γ . By left multiplication of the elements of $\mathcal{L}_0$

³This argument is actually not quite correct since the vacuum may be "slowly varying" at infinity, cf. [19]. But one can show within the present general setting that the family of states $\omega \circ \alpha_x$ always has weak $*$ -limit points which are vacuum states.

⁴An operator $B \in \mathcal{A}$ is said to be almost local if there exists an approximating sequence $B_r \in \mathcal{A}(\mathcal{O}_r)$, $\mathcal{O}_r$ being a family of concentric balls of radius r , such that $\|B - B_r\|$ tends to 0 for large r faster than any inverse power of r .

with arbitrary elements from $\mathcal{A}$ and linear combinations we obtain a (non-closed) left ideal $\mathcal{L} \subset \mathcal{A}$ which is invariant under the action of the Poincaré group. We call $\mathcal{L}$ the left ideal of localizing operators, for reasons which will become clear later. Because of the specific energy-momentum transfer Γ of the operators in $\mathcal{L}_0$, any operator $L \in \mathcal{L}$ annihilates the vacuum state(s), and in fact all positive energy states of sufficiently small energy (depending on L).

We then define the algebra $\mathcal{C}$ as the linear span of all operators of the form L^*L , $L \in \mathcal{L}$, i.e., $\mathcal{C} = \mathcal{L}^*\mathcal{L}$ in shorthand. The algebra $\mathcal{C}$ is smaller than the “algebra of detectors” introduced in [20]. But it separates all positive energy states which are orthogonal to the vacuum and is therefore big enough for our purposes.

We can now proceed with our asymptotic analysis of positive energy states. Since the operators $C \in \mathcal{C}$ annihilate the vacuum, at asymptotic times the leading contributions to the expectation values $\omega(\alpha_x(C))$, $C \in \mathcal{C}$, should be due to single stable particles, triggering C . Contributions from multiple excitations of the vacuum may be expected to appear only in the next-to-leading order since the probability for finding more than one particle in a space-time region of fixed size should decay substantially faster with time. As far as the single-particle contributions in the functions $x \rightarrow \omega(\alpha_x(C))$ are concerned, they will tend to zero at large times because the particles can be anywhere, and one has no control over their localization. This picture suggests that these functions be integrated over spacelike surfaces, to compensate for the dispersion.

At this point there arises a problem: do the integrals $\int d\mathbf{x} \omega(\alpha_{t,\mathbf{x}}(C))$, $C \in \mathcal{C}$ (where we have introduced proper coordinates $x = (t, \mathbf{x})$) exist? An affirmative answer, which may be regarded as a first confirmation of our heuristic picture, follows from the subsequent lemma [21]. Its proof is based on a harmonic analysis of the space-time translations $\alpha_{\mathbb{R}^d}$.⁵

Lemma 3.1: *Let $E(\cdot)$ be the spectral resolution of the generators of space-time translations $U(x)$ in any positive energy representation π of $\mathcal{A}$. Then, for any $C \in \mathcal{C}$ and any compact subset Δ of the spectrum,*

$$\limsup_{r \rightarrow \infty} \left\| E(\Delta) \int_{|\mathbf{x}| < r} d\mathbf{x} \pi(\alpha_{t,\mathbf{x}}(C)) E(\Delta) \right\| < \infty.$$

Thus if ω is a positive energy state whose spectral support (with respect to the generators of the space-time translations in its GNS-representation) is contained in some compact set Δ , we see from the lemma that the integrals in question are well-defined and uniformly bounded in t (since the spectral support of ω is invariant under time translations). In fact, using the lemma, one can equip $\mathcal{C}$ with a directed set of seminorms, and the positive linear functionals ρ_t , $t \in \mathbb{R}$ on $\mathcal{C}$ given by

$$(8) \quad \rho_t(C) = \int d\mathbf{x} \omega(\alpha_{t,\mathbf{x}}(C)), \quad C \in \mathcal{C}$$

form an equicontinuous family with respect to the resulting topology. Thus there exist weak *-limit points if t approaches $\pm\infty$, respectively. Actually, one would expect

⁵The general harmonic analysis of automorphism groups [22] seems still in its initial stages. It would be useful to have a classification of their possible spectral properties, in analogy to the notion of measure classes in the case of unitary groups.

that in physically reasonable theories the functionals ρ_t converge since the asymptotic particle configurations should stabilize. This question is closely related to the problem of asymptotic completeness, cf. below. In order to deal with possible oscillations of asymptotic particle configurations, we proceed to time averages of the functionals ρ_t ,

$$(9) \quad \bar{\rho}_t = \frac{1}{\lambda(t)} \int_t^{t+\lambda(t)} dt' \rho_{t'},$$

where λ is some positive, slowly increasing function. According to the preceding discussion, we expect that the limit points of the functionals $\bar{\rho}_t$ contain the relevant information about the asymptotic particle content of ω .

Working hypothesis: The limit points $\{\sigma_{\pm}\}$ of the functionals $\bar{\rho}_t, t \rightarrow \pm\infty$ describe the outgoing, respectively incoming stable particle content of the state ω .

The problem of describing particles appearing at asymptotic times has thus turned into a problem of mathematical analysis and interpretation of the structure of the functionals $\sigma_{\pm}$. Such an analysis has been carried out in cooperation with M. Poppmann [23], cf. also [11], and we give here an account of some of the results. Since in this discussion the time direction plays no role, we shall omit the index $\pm$ in $\sigma_{\pm}$ in the following.

It is apparent from their definition that the functionals σ on $\mathcal{C}$ are positive and linear. (But they are not normalizable; note that $\mathcal{C}$ does not contain a unit.) Thus any σ induces a positive semi-definite scalar product on the left ideal $\mathcal{L}$. Proceeding as in the GNS-construction, one obtains, by taking quotients with respect to elements of zero norm and completion, a Hilbert space $\mathcal{H}$ and a linear map $|\cdot\rangle : \mathcal{L} \rightarrow \mathcal{H}$ with dense range, such that

$$(10) \quad \langle L_1 | L_2 \rangle = \sigma(L_1^* L_2) \quad L_1, L_2 \in \mathcal{L}.$$

Here $\langle \cdot |$ denotes of course the corresponding antilinear map of $\mathcal{L}$ into the dual space of $\mathcal{H}$. As suggested by the notation, it may be helpful to think of $\langle \cdot |$ and $|\cdot\rangle$ as improper Dirac bra and ket vectors, respectively, which become normalizable after application of the elements of $\mathcal{L}$. In special cases, they may also be identified with improper states in a rigged Hilbert space formalism. Some important properties of the maps $|\cdot\rangle$ are summarized in the

Proposition 3.2: *i) The operators $\pi(A), A \in \mathcal{A}$ given by*

$$\pi(A)|L\rangle \doteq |AL\rangle, \quad L \in \mathcal{L}$$

*are well-defined and bounded. The corresponding map $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ defines a *-representation of $\mathcal{A}$.*

ii) The functions $(\Lambda, x) \rightarrow |\alpha_{\Lambda, x}(L)\rangle, L \in \mathcal{L}_0$ are C^∞ . ($\mathcal{L}_0$ is the generating space of $\mathcal{L}$.)

iii) The operators $U(x)$ given by $U(x)|L\rangle \doteq |\alpha_x(L)\rangle, L \in \mathcal{L}$ define a continuous unitary representation $x \rightarrow U(x)$ of space-time translations on $\mathcal{H}$. The spectrum of the generators of $U(x)$ is contained in some (shifted) light cone $\bar{V}_+ + q$.

We note that the vector q in the latter statement depends on the spectral support of the underlying state ω .

Definition: A linear map $|\cdot\rangle$ of $\mathcal{L}$ into some Hilbert space $\mathcal{H}$ with dense range is called a *particle weight* if it has the properties given in Proposition 3.2.

The representations π of $\mathcal{A}$ induced by the particle weights constructed above will in general be highly reducible. (Note that in the construction of $|\cdot\rangle$ there appeared direct integrals of states.) Since elementary systems, such as particles, should give rise to irreducible representations of $\mathcal{A}$, it is natural to pose the question of whether $|\cdot\rangle$ can be decomposed into pure components. An affirmative answer is given in the following statement.

Proposition 3.3: *Let $|\cdot\rangle$ be a particle weight. There exists a measure space (Σ, μ) and a measurable family of pure particle weights $|\cdot\rangle_\lambda, \lambda \in \Sigma$ (inducing irreducible representations of $\mathcal{A}$) such that the following direct integral decomposition holds:*

$$(11) \quad |\cdot\rangle = \int d\mu(\lambda) |\cdot\rangle_\lambda.$$

Remark: In this form the statement holds actually only if $\mathcal{A}$ is separable. We refrain from discussing here the complications arising in the general case.

This result tells us that it is possible to resolve the asymptotic particle structure described by σ into components with definite superselection quantum numbers (charges). More can be said, but before we proceed we have to discuss some technicalities.

Definition: A particle weight $|\cdot\rangle$ is said to be *regular* if $\sum_{i=1}^n |L_i^* L_i\rangle = 0$ implies $|L_i\rangle = 0, L_i \in \mathcal{L}, i = 1 \dots n$. (Note that this implication would hold trivially if $\mathcal{L}$ were to contain a unit.)

It seems likely that pure particle weights of physical interest are regular. But we have not been able to establish this property in general.

Proposition 3.4: *Let $|\cdot\rangle_\lambda$ be a pure particle weight. There exists a unique vector $p_\lambda \in \mathbb{R}^d \setminus \bar{V}_-$ such that the continuous unitary representation $x \rightarrow U_\lambda(x)$ of space-time translations given by*

$$U_\lambda(x)|L\rangle_\lambda \doteq e^{ip_\lambda x} |\alpha_x(L)\rangle_\lambda, \quad L \in \mathcal{L}$$

has Lorentz invariant spectrum in $\bar{V}_+$. If $|\cdot\rangle_\lambda$ is regular, then $p_\lambda \in \bar{V}_+ \setminus \{0\}$.

The interpretation of this result is obvious: pure particle weights $|\cdot\rangle_\lambda$ have sharp energy-momentum p_λ . Appealing to the picture of a plane wave, the role of the operators $L \in \mathcal{L}$ becomes clear: they localize the wave and thereby produce a normalizable state. Incidentally, this explains the terminology “left ideal of localizing operators”. It is not excluded by our results that p_λ be spacelike. The corresponding “tachyonic” particle weights would be non-regular, but become positive energy states after localization.

Let us next discuss the behaviour of particle weights under Lorentz-transformations. Given a pure particle weight $|\cdot\rangle_\lambda$ and any Lorentz transformation Λ , we can define another particle weight $|\cdot\rangle_{\lambda, \Lambda}$, by

$$(12) \quad |L\rangle_{\lambda, \Lambda} \doteq |\alpha_\Lambda^{-1}(L)\rangle_\lambda, \quad L \in \mathcal{L}.$$

One easily verifies that $|\cdot\rangle_{\lambda, \Lambda}$ has energy-momentum Λp_λ . Restricting Λ to the little (stability) group $\mathcal{R}(p_\lambda)$ of p_λ , one can explore the internal (“spin”) degrees of freedom

of $|\cdot\rangle_\lambda$. Here we restrict attention to the physically relevant case in which the number of these degrees of freedom is finite (finite particle multiplets).

Definition: A pure particle weight $|\cdot\rangle_\lambda$ is said to have finite spin if the space of sesquilinear forms on $\mathcal{L} \odot \mathcal{L}$ generated by ${}_{\lambda,\Lambda}\langle\cdot|\cdot\rangle_{\lambda,\Lambda}$, $\Lambda \in \mathcal{R}(p_\lambda)$ is finite dimensional.

Proposition 3.5: *Let $|\cdot\rangle_\lambda$ be a pure particle weight with finite spin and let $(\pi_\lambda, \mathcal{H}_\lambda)$ be the corresponding GNS-representation of $\mathcal{A}$. There exists on $\mathcal{H}_\lambda$ a continuous unitary projective representation $\Lambda \rightarrow V_\lambda(\Lambda)$ of the little group $\mathcal{R}(p_\lambda)$ such that*

$$V_\lambda(\Lambda)\pi_\lambda(A)V_\lambda(\Lambda)^{-1} = \pi_\lambda(\alpha_\Lambda(A)), \quad A \in \mathcal{A}.$$

Moreover, the linear space of particle weights generated by $V_\lambda(\Lambda)|\cdot\rangle_{\lambda,\Lambda}$, $\Lambda \in \mathcal{R}(p_\lambda)$ is finite dimensional.

Remark: We recall that a representation of the full Lorentz group may not exist (infraparticle problem).

Let us discuss the consequences of this result for regular particle weights in physical space-time $d = 4$. There we have to distinguish two cases: If the mass-square p_λ^2 is positive, the little group of p_λ is isomorphic to $O(3)$. Hence the projective representation V_λ can be lifted to a true representation of its covering group $SU(2)$, and the particle weights can be decomposed into components of fixed (half) integer spin. So in this case we arrive at the same conclusions as Wigner. If $p_\lambda^2 = 0$ the situation is, however, different. In this case the little group of p_λ is isomorphic to $E(2)$, and it may well be that V_λ cannot be lifted to a true representation of its two-fold covering if Lorentz boosts are spontaneously broken in the representation $(\pi_\lambda, \mathcal{H}_\lambda)$. The spin (helicity) of particle weights can still be defined in these cases, but it does not have to be (half) integer. Thinking of the spin-statistics theorem, this leaves open the possibility of massless infraparticles in physical space-time with unusual statistics.

There are further results which substantiate the heuristic expectation that particle weights describe the timelike asymptotic particle content of states (such as specific localization properties of the systems described by particle weights, local normality properties etc. [23]). But the results quoted so far may suffice for the motivation of the following:

Postulate: Particles appearing at asymptotic times in few body systems are described by (pure) particle weights $|\cdot\rangle$, mapping the left ideal $\mathcal{L}$ of localizing operators into some Hilbert space $\mathcal{H}$. The ensembles which can experimentally be prepared by the process of localizing particles are described by the range of $|\cdot\rangle$.

What have we gained by this point of view? a) We have now a formalism which allows a unified treatment of particles, infraparticles, tachyons (should they exist), etc. The conceptual problem being solved, it is now a well-defined mathematical problem to determine the specific properties of these systems (Poincaré covariance versus spontaneous breaking of Lorentz boosts, negative mass squares, etc.). b) Our point of view is the basis for a simple and general formula by which the asymptotic particle content can be extracted from any state ω ,

$$(13) \quad \langle L|L \rangle = \lim_t \int d\mathbf{x} \omega(\alpha_{t,\mathbf{x}}(L^*L)), \quad L \in \mathcal{L}.$$

(Here the quotation marks indicate the possible need for time averages and the transition to subnets.) It is a striking fact, first stated implicitly in [20] in a more restricted setting, that in this way the (charged) particle content of a theory can be uncovered by using only the local observables in the vacuum sector.

We conclude this discussion with a list of open problems and remarks.

i) When does a theory have a non-trivial asymptotic particle structure, i.e., under which conditions are the limits $\langle \cdot | \cdot \rangle$ in (13) non-zero? It would be of great interest to formulate these conditions in general terms (using the underlying net structure) since it would clarify the question of how the particle aspects are encoded in the local properties of a theory. It seems likely that the following two conditions are the key ingredients: the condition of primitive causality [24], being the abstract expression of the existence of a dynamical law, and a condition of causal independence [25], now called the split-property [26]. A heuristic argument to this effect has been given in [27], but it has not yet matured into a proof.

ii) When do the asymptotic particle configurations stabilize, i.e., under what conditions can the quotation marks in (13) be removed (existence of limits)? As is clear from quantum mechanical scattering theory [28], a positive answer will depend on decent “phase space properties” of the theory. In the present setting, this requirement can be expressed in terms of compactness or nuclearity conditions (cf. [29] for a comprehensive account and references).

iii) Is there a general collision theory based on particle weights? Some interesting progress on this problem has been made in collaboration with U. Stein. We have developed a novel algorithm which seems suitable for the construction of collision states of (infra-) particles of prescribed charge, spin, mass and momentum (possibly accompanied by an unspecified cloud of massless particles with arbitrarily small total energy), cf. [11]. It is based on the algebra $\mathcal{C} = \mathcal{L}^* \mathcal{L}$ and does not require the knowledge of charge carrying fields. It thereby fills the remaining gap in [20] in the direct construction of collision cross sections from observables in the vacuum sector. A proof that the method works has so far only been given in massive, asymptotically complete theories [30]. But the method can be directly applied (and tested) in any model.

iv) What is the statistics of particle weights? An approach to this problem could be based on an extension of the DHR-analysis [31] to superselection sectors describing non-localizable charges (such as the electric charge), and there are some promising ideas to this effect [32].

Another approach could look as follows: It seems possible to construct, with the help of the operators $L \in \mathcal{L}$, loops in the space of collision states, describing the exchange of two (or more) particles of the same kind. In order to determine the statistics of the particles one has to lift these loops to the space of collision state vectors. The appropriate lift seems to be the Berry lift or its generalizations (cf. [33] and references quoted there), and the statistics would then be given by the corresponding monodromy operator (Berry phase). Some preliminary calculations indicate that this definition of statistics gives, in the case of localizable charges, the same answers as the definition of statistics in [31]. Another question, closely related to this problem is: When do antiparticle weights exist?

v) When does a theory have a complete particle (weight) interpretation? A definitive mathematical formulation of this question has not yet been accomplished. (Note

that an unambiguous scattering matrix does not exist in the case of infraparticles.) But there is some closely related problem which is of interest in its own right: In high energy collision experiments only some very limited aspect of completeness is really tested, which primarily relies on the existence of conservation laws. One prepares, for example, (initial) states ω with relatively sharp energy-momentum and checks whether the energy-momentum carried by the outgoing particles adds up to the same result. So there are two quantities involved: (a) The energy-momentum pumped into the state ω in the process of its preparation, which may be identified with $\omega(P)$, P being the energy-momentum operator, and (b) the energy-momentum carried by the asymptotic particles, which according to Proposition 3.4 can be identified with the values p_λ corresponding to the particle weights $|\cdot\rangle_\lambda$ appearing in the asymptotic decomposition (11) of $|\cdot\rangle$. Hence, in accord with the experimental facts, the following should hold:

$$(14) \quad \omega(P) = \int d\mu(\lambda) p_\lambda,$$

where μ is the measure appearing in the decomposition (11),⁶ and a similar relation should hold for the other conserved quantities (charge, spin, etc.) which can be attributed to particles. It seems that such a weak form of asymptotic completeness suffices for a consistent interpretation of most collision experiments. We conjecture that relation (14) holds in all theories admitting a local stress energy tensor.

4 Ultraparticles

First a word about the terminology. The term *infraparticle* was coined by B. Schroer [4] as a catch-phrase alluding to the infrared problems which arise in models with long range forces. In the preceding section we have seen how infraparticles (like particles) appear at large distances in states of moderate energy. Thus in this respect the term *infraparticle* also fits well. In the present section we are interested in the particles appearing at short distances in the limit of arbitrarily high energies. We call these particles *ultraparticles*. Some of them may be infraparticles as well, e.g., leptons, others do not show up at large scales, e.g., partons. The latter phenomenon (confinement) may be understood as an infrared problem. But the term *ultraparticle* seems more appropriate also in these cases since it specifies the domain where these particles are observed.

Within the perturbative treatment of field theory, a powerful tool for the analysis of the ultraviolet properties of models is based on scaling transformations of fields (renormalization group transformations [34]). They allow an interpretation of the theory at small scales and have led to fundamental concepts such as the notion of asymptotic freedom. Although well known, let us recall that the scaled fields are regarded as elements of another theory at the original scale with e.g., different (running) coupling constants. Phrased differently: The interpretation of the original theory at small scales is accomplished in terms of other theories, but at fixed scale. We emphasize this fact in order to make clear from the outset that the particle structures which are of interest

⁶An unambiguous definition of μ requires a normalization of the particle weights $|\cdot\rangle_\lambda$, a problem not touched upon here.

here will not (and cannot) be described by states or weights on the given local net of observables $\mathcal{A}$.

The utility of scaling transformations for the structural analysis in algebraic quantum field theory has first been realized by Roberts [35] in the special case of scale-invariant theories. The methods were later refined in [36] (cf. also [37] and [38]) to cover also the case of non-scale-invariant theories. These proposals contain, however, a conceptually unsatisfactory element since they depend on the existence of pointlike quantum fields. From the algebraic point of view such fields (if they exist) are usually interpreted as a kind of coordinatization of the local algebras. This raises then the questions: Does the “principle of local stability”, proposed in [36], depend on a specific choice of such coordinates? Do the results on the structure of local algebras in [37] depend on the existence of such coordinates? Moreover, such interesting questions as to when there exist quantum fields describing the ultraviolet properties of a net of observables, or what is the possible ultraviolet structure if there are no such fields, are set aside from the outset.

In joint work with R. Verch, we have set up a framework which allows the systematic discussion of these problems [12], and we are presently investigating the ultraviolet properties of local nets with emphasis on the ultraparticle aspects. In the following an outline of the basic ideas in our approach is given, and some pertinent results with regard to the ultraparticle problem are discussed.

What is required in order to express the ideas of the renormalization group in the algebraic setting? One first needs a method to proceed from the local algebras $\mathcal{A}(\mathcal{O})$ at scale $\lambda = 1$, say, to algebras $\mathcal{A}_\lambda(\mathcal{O})$ describing the observations made at smaller scales $\lambda > 0$ in the regions $\lambda\mathcal{O}$, i.e., $\mathcal{A}_\lambda(\mathcal{O}) = \mathcal{A}(\lambda\mathcal{O})$.⁷ There are many ways of introducing such maps: conditional expectations, canonical endomorphisms based on the Tomita-Takesaki theory, scaling transformations of underlying quantum fields, etc. Apart from an important physical constraint which will be discussed below, it should not really matter which map one takes since they yield the same nets (they merely reshuffle the operators in the local algebras in different ways). So why not consider them all?

A convenient way of expressing this idea is provided by the concept of a scaling algebra $\underline{\mathcal{A}}$ associated with any given local net $\mathcal{A}$. The elements $\underline{A}$ of $\underline{\mathcal{A}}$ are bounded functions of the scaling parameter $\lambda \in (0, \infty)$ with values in $\mathcal{A}$, i.e., $\underline{A}_\lambda \in \mathcal{A}$, $0 < \lambda < \infty$. The algebraic operations in $\underline{\mathcal{A}}$ are defined pointwise by the corresponding operations in $\mathcal{A}$, and there is a C^* -norm on $\underline{\mathcal{A}}$ given by

$$(15) \quad \|\underline{A}\| = \sup_{\lambda} \|\underline{A}_\lambda\|.$$

The local structure of $\mathcal{A}$ is lifted to $\underline{\mathcal{A}}$ by setting

$$(16) \quad \underline{\mathcal{A}}(\mathcal{O}) = \{\underline{A} \in \underline{\mathcal{A}} : \underline{A}_\lambda \in \mathcal{A}(\lambda\mathcal{O}), 0 < \lambda < \infty\}.$$

Hence $\mathcal{O} \rightarrow \underline{\mathcal{A}}(\mathcal{O})$ defines a net over Minkowski space, and it suffices for our purposes to assume that $\underline{\mathcal{A}}$ is the C^* -inductive limit of the local algebras $\underline{\mathcal{A}}(\mathcal{O})$. It is easily verified that the net is local, i.e., $\underline{\mathcal{A}}(\mathcal{O}_1) \subset \underline{\mathcal{A}}(\mathcal{O}_2)'$ if $\mathcal{O}_1 \subset \mathcal{O}_2'$. Moreover, the action of

⁷Note that the map $\mathcal{O} \rightarrow \mathcal{A}_\lambda(\mathcal{O})$ defines again a local net over Minkowski space with Poincaré-transformations given by $\alpha_x^{(\lambda)} \doteq \alpha_{\lambda x}$ and $\alpha_\Lambda^{(\lambda)} \doteq \alpha_\Lambda$.

the Poincaré group on $\mathcal{A}$ can be lifted to a covariant action on the net $\underline{\mathcal{A}}$, induced by automorphisms $\underline{\alpha}_x, \underline{\alpha}_\Lambda \in \text{Aut } \underline{\mathcal{A}}$ given by

$$(17) \quad (\underline{\alpha}_x(\underline{A}))_\lambda \doteq \alpha_{\lambda x}(\underline{A}_\lambda), \quad (\underline{\alpha}_\Lambda(\underline{A}))_\lambda \doteq \alpha_\Lambda(\underline{A}_\lambda).$$

Not quite unexpectedly, there appears an additional symmetry in this setting: dilations. This group acts on the net by automorphisms $\underline{\delta}_\mu, 0 < \mu < \infty$, the action being given by

$$(18) \quad (\underline{\delta}_\mu(\underline{A}))_\lambda \doteq \underline{A}_{\mu\lambda}.$$

This formalism nicely expresses the geometrical aspects of scaling transformations. But it is probably of little use as it stands since there is some important physical constraint missing: We want to change the length scale, but the other physical units should stay constant. The velocity of light, $c = 1$, has not been changed by our procedure since we have scaled space and time by the same factor λ . But what about Planck's constant? Many of the (only abstractly characterized) functions $\underline{A}$ will describe a situation in which, simultaneously with the length, Planck's constant is also scaled in some uncontrolled manner. Thus we are not yet in a setting corresponding to the framework of the renormalization group, unless we manage to get rid of these unwanted elements.

In a first attempt to solve this problem, we tried to isolate in $\underline{\mathcal{A}}$ a "Planck-ideal" which, by taking quotients, fixes the value of Planck's constant at each scale. But this idea did not work. Finally, the solution to this problem has turned out to be amazingly simple. It consists of a direct characterization of the quotient of $\underline{\mathcal{A}}$, based on the following idea: Since the product of the dimensions of length and momentum is equal to the dimension of action, we have to scale momentum by λ^{-1} if we scale length by λ and want to keep Planck's constant fixed. The energy-momentum scale can be set by determining the energy-momentum transferred by "standard operators" to physical states. Hence if $\underline{A}_1 \in \mathcal{A}$ is such an operator which at scale $\lambda = 1$ transfers energy-momentum Γ , where Γ is some compact subset of $\mathbb{R}^d$, then its counterpart $\underline{A}_\lambda \in \mathcal{A}_\lambda$ at scale λ should transfer energy-momentum $\lambda^{-1}\Gamma$. Now the local operators $A \in \mathcal{A}$ can transfer any energy-momentum. But they are "quasi-local in momentum space" as a consequence of their continuity properties with respect to space-time translations. In fact, given $A \in \mathcal{A}$ and $\varepsilon > 0$ one can find some operator $A_\Gamma \in \mathcal{A}$ with compact energy-momentum transfer Γ such that $\|A - A_\Gamma\| < \varepsilon$.

After these remarks it is straightforward to express the requirement that Planck's constant be held fixed under scaling transformations: We have to restrict attention to functions $\underline{A}$ which, at every scale λ , can be approximated with uniform accuracy $\varepsilon > 0$ by operators with energy-momentum transfer in $\lambda^{-1}\Gamma$, where Γ is some fixed compact set. Phrased in formula form: $\sup_\lambda \|\underline{A}_\lambda - \underline{A}_{\lambda, \lambda^{-1}\Gamma}\| < \varepsilon$. After a moment's reflection one sees that this condition is equivalent to the requirement that $\sup_\lambda \|\alpha_{\lambda x}(\underline{A}_\lambda) - \underline{A}_\lambda\| \rightarrow 0$ as $x \rightarrow 0$, which simply means that the function $x \rightarrow \underline{\alpha}_x(\underline{A})$ has to be strongly continuous. Thus the condition " $\hbar = 1$ at all scales" amounts, in the framework of the scaling algebra $\underline{\mathcal{A}}$, to the condition of strong continuity under space-time translations.

The same considerations, as indicated above for the case of energy-momentum, apply of course also to angular momentum and lead to the requirement that also the functions $\Lambda \rightarrow \underline{\alpha}_\Lambda(\underline{A})$ have to be strongly continuous. Hence we are led to the

Definition: Let $\mathcal{A}$ be any local net complying with the assumptions given in Sec. 2. The canonically associated *scaling algebra* $\underline{\mathcal{A}}$ is the C^* -inductive limit of the local net $\mathcal{O} \rightarrow \underline{\mathcal{A}}(\mathcal{O})$ of functions $\underline{A}$ which are strongly continuous with respect to the action of the Poincaré group, cf. relations (16) and (17).

(We note that the concept of scaling algebra can also be applied to local nets on curved space-time [12].)

Remark: Within this setting the significance of the automorphisms $\underline{\delta}_\mu, 0 < \mu < \infty$ becomes clear: they induce scaling (renormalization group) transformations on $\underline{\mathcal{A}}$. In view of this fact it may seem natural to assume that $\underline{\delta}_\mu$ acts strongly continuously on $\underline{\mathcal{A}}$ (which amounts to further restrictions on the elements of $\underline{\mathcal{A}}$). It would, however, be too much to require continuity of the functions $\mu \rightarrow \underline{\delta}_\mu(\underline{A})$ at $\mu = 0$, as this would imply that $\text{n-lim}_{\mu \rightarrow 0} \pi(\underline{A}_\mu) = \text{const} \cdot 1$ in all positive energy representations π of $\mathcal{A}$. The corresponding functions $\underline{A}$ would therefore not be suitable for exploring the ultraviolet properties of a theory. In the following discussion we do not anticipate any continuity properties of the scaling transformations.

We next discuss how the structure of the states ω at small scales can be analyzed with the help of the scaling algebra. To this end we have to lift these states to $\underline{\mathcal{A}}$.

Definition: Let ω be any state on $\mathcal{A}$. Its canonical lift $\underline{\omega}$ to the scaling algebra $\underline{\mathcal{A}}$ is given by

$$(19) \quad \underline{\omega}(\underline{A}) \doteq \omega(\underline{A}_{\lambda=1}), \quad \underline{A} \in \underline{\mathcal{A}}.$$

Before we proceed let us clarify the physical interpretation of the states $\underline{\omega}$ on $\underline{\mathcal{A}}$. In order to recover this interpretation, one has to regard $\underline{\omega}$ as a (projected) state on the quotient $\underline{\mathcal{A}}/\ker \underline{\pi}$, where $\underline{\pi}$ is the GNS-representation induced by $\underline{\omega}$ and $\ker \underline{\pi}$ the kernel of $\underline{\pi}$. To see this, note that if π is the GNS-representation induced by ω , the nets $\underline{\mathcal{A}}/\ker \underline{\pi}$ and $\mathcal{A}/\ker \pi$ ($= \mathcal{A}$ if π is faithful, which we assume henceforth) are isomorphic⁸, an isomorphism ϕ being given by $\phi(\underline{A}/\ker \underline{\pi}) = \underline{A}_{\lambda=1}, \underline{A} \in \underline{\mathcal{A}}$. This isomorphism also intertwines the actions of the Poincaré group on the respective nets. These properties characterize ϕ up to internal symmetries of the algebra of observables $\mathcal{A}$. Moreover, if $\text{proj} \underline{\omega}$ denotes the state obtained by projecting $\underline{\omega}$ to the quotient $\underline{\mathcal{A}}/\ker \underline{\pi}$, there holds $\text{proj} \underline{\omega} \circ \phi^{-1} = \omega$. Thus the state $\text{proj} \underline{\omega}$ on the net $\underline{\mathcal{A}}/\ker \underline{\pi}$ contains the same physical information as ω , provided there exist no internal symmetries. If there are such symmetries, one still knows that $\text{proj} \underline{\omega}$ describes some state on $\mathcal{A}$ which lies on the orbit of ω under the action of the internal symmetry group. Hence if ω is a vacuum state, then $\text{proj} \underline{\omega}$ also describes a vacuum state which coincides with ω , unless the internal symmetry group is spontaneously broken. This information suffices for our purposes.

In order to analyze the state ω at scale λ , we have to apply the scaling transformation $\underline{\delta}_\lambda$ to its canonical lift $\underline{\omega}$, giving the state $\underline{\omega} \circ \underline{\delta}_\lambda$ on $\underline{\mathcal{A}}$. If we proceed as above and consider the projected state $\text{proj} \underline{\omega} \circ \underline{\delta}_\lambda$ on the net $\underline{\mathcal{A}}/\ker \underline{\pi}_\lambda$, where $\underline{\pi}_\lambda$ is the GNS-representation induced by $\underline{\omega} \circ \underline{\delta}_\lambda$, we recover the state ω on the net $\mathcal{A}_\lambda$, up to internal symmetry transformations. (The internal symmetry group is of course the same at each scale $\lambda > 0$.)

⁸Two nets are said to be isomorphic if there exists an isomorphism between the respective global algebras of observables which maps the local algebras corresponding to the same regions $\mathcal{O}$ onto each other. Two such nets describe "the same physics".

We are interested here in the structure of the physical states ω at scales $\lambda \rightarrow 0$, so we have to consider the corresponding sequence of states $\underline{\omega} \circ \underline{\delta}_\lambda$ in this limit. Here some technical nuisance appears since these sequences do not converge. To explain the simple origin of this phenomenon, let us consider the following toy model: we take as our algebra of observables the complex numbers, $\mathcal{A} = \mathbb{C}$. The corresponding scaling algebra $\underline{\mathcal{A}}$ is then the algebra of all bounded c -number functions $\underline{A} : (0, \infty) \rightarrow \mathbb{C}$ with norm $\|\underline{A}\| = \sup_\lambda |\underline{A}_\lambda| < \infty$. The canonical lift of the (unique) state ω on $\mathcal{A}$ to the scaling algebra $\underline{\mathcal{A}}$ is given by $\omega(\underline{A}) = \underline{A}_1$, and $\omega(\underline{\delta}_\lambda(\underline{A})) = \underline{A}_\lambda$.

Since we may not assume from the outset that all functions $\underline{A}$ converge for $\lambda \rightarrow 0$ (cf. the previous remark) it is clear that the sequence $\underline{\omega} \circ \underline{\delta}_\lambda$ will not converge in this limit. The problem is that the scaling algebra provides us with an abundance of mappings from the algebra $\mathcal{A}$ at scale $\lambda = 1$ to the corresponding algebras $\mathcal{A}_\lambda (= \mathcal{A})$ at other scales λ . (This was, after all, our initial idea.) To proceed, we make use of the fact that the set of states $\underline{\omega} \circ \underline{\delta}_\lambda$ is relatively compact in the weak *-topology induced by $\underline{\mathcal{A}}$. Thus there exist (many) convergent subnets for $\lambda \rightarrow 0$. In a sense, each convergent subnet corresponds to some particular choice of the abovementioned mappings.

Let us denote the limits of these subnets by $\underline{\omega}_{0,\iota}$, $\iota \in \mathbb{I}$ ($\mathbb{I}$ being some index set). Recalling the physical interpretation of states ω at scale λ in terms of $\underline{\omega} \circ \underline{\delta}_\lambda$, we consider the projected states $\text{proj } \underline{\omega}_{0,\iota}$ on $\underline{\mathcal{A}}/\ker \pi_{0,\iota}$, $\pi_{0,\iota}$ being the GNS-representation induced by $\underline{\omega}_{0,\iota}$. It is evident that in our model $\underline{\mathcal{A}}/\ker \pi_{0,\iota}$ is isomorphic to $\mathcal{A}$. Thus the abundance of functionals $\underline{\omega}_{0,\iota}$, $\iota \in \mathbb{I}$ on $\underline{\mathcal{A}}$ describes a single “physical state”, the state ω from which we started. (This is of course no surprise.)

After these preliminary remarks the reader will guess how we proceed: Given a state ω on $\mathcal{A}$, we consider the corresponding family of states $\underline{\omega} \circ \underline{\delta}_\lambda$ on $\underline{\mathcal{A}}$ and its weak *-limit points for $\lambda \rightarrow 0$: $\underline{\omega}_{0,\iota}$, $\iota \in \mathbb{I}$. We then propose as

Working hypothesis: Let ω be a state on $\mathcal{A}$ and let $\underline{\omega} \circ \underline{\delta}_\lambda$ be the corresponding family of scaled states on the scaling algebra $\underline{\mathcal{A}}$. The limit points $\underline{\omega}_{0,\iota}$, $\iota \in \mathbb{I}$ of this family for $\lambda \rightarrow 0$, projected on the respective nets $\underline{\mathcal{A}}/\ker \pi_{0,\iota}$, where $\pi_{0,\iota}$ is the GNS representation induced by $\underline{\omega}_{0,\iota}$, describe the structure of the state ω in the scaling limit (possibly modulo internal symmetry transformations).

If the nets $\underline{\mathcal{A}}/\ker \pi_{0,\iota}$ in this statement are all isomorphic, one can interpret the asymptotic structure of ω in terms of a single theory. We then say that the original theory has a unique scaling limit. (A justification for this terminology will be given below.) But this does not mean that the states $\text{proj } \underline{\omega}_{0,\iota}$ all coincide, there may appear new “degrees of freedom” in the scaling limit. The second possibility is that not all of the nets $\underline{\mathcal{A}}/\ker \pi_{0,\iota}$ are isomorphic. It means that the theory has no fixed interpretation in the scaling limit; it looks different at different scales.

After this lengthy preparation of our tools, let us turn to the actual analysis of the scaling limits. To this end we first have to fix the states which we want to consider. Being interested in states in high-energy physics, we pick the set $\mathcal{S}_0$ of all states which are locally normal [39] with respect to a given pure Poincaré invariant vacuum state ω_0 on $\mathcal{A}$. This set of states is actually quite big; it not only contains all finite energy excitations of ω_0 , but in general also thermal states, etc. Our first result says that it does not matter which state $\omega \in \mathcal{S}_0$ we consider, they all describe the same structure in the scaling limit. In fact there holds a stronger result.

Lemma 4.1: *Let $\omega_1, \omega_2 \in \mathcal{S}_0$. Then*

$$\lim_{\lambda \rightarrow 0} (\omega_1 \circ \underline{\delta}_\lambda - \omega_2 \circ \underline{\delta}_\lambda)(A) = 0, \quad A \in \underline{\mathcal{A}}.$$

In simple terms: all states $\omega \in \mathcal{S}_0$ look alike at small scales. (This justifies our terminology concerning the uniqueness of the scaling limit of a theory.) But what do they look like? Here is the answer for $d > 2$ dimensions:

Proposition 4.2: *Let $\omega \in \mathcal{S}_0$. Then each weak $*$ -limit point $\underline{\omega}_{0,\iota}$ of $\omega \circ \underline{\delta}_\lambda$, $\lambda \rightarrow 0$, is a pure Poincaré invariant vacuum state on $\underline{\mathcal{A}}$. (Thus it is also a pure, Poincaré invariant vacuum state if projected to the net $\underline{\mathcal{A}}/\ker \underline{\pi}_{0,\iota}$, since the kernel of the GNS-representation $\underline{\pi}_{0,\iota}$, induced by $\underline{\omega}_{0,\iota}$, is Poincaré invariant.)*

This result is the key to a general understanding of the scaling limits of quantum field theories. Let us first indicate how it can be used for their classification. There are the following three possibilities.

1. The theory has a *classical* scaling limit, i.e., $\underline{\mathcal{A}}/\ker \underline{\pi}_{0,\iota} \simeq \mathbb{C}$ for $\iota \in \mathbb{I}$. A trivial example of this kind is obtained if one (re)defines the algebras $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}$ to be equal to $\mathbb{C} \cdot 1$ for small regions $\mathcal{O}$. This artificial possibility is not what we here have in mind (and it could be excluded by a condition of additivity on the net $\mathcal{A}$). But our structural result seems to indicate that there may well exist quantum field theories which become classical in the sense that all correlations between bounded observables disappear at small scales. We note that under these circumstances the “classical ideal” in $\underline{\mathcal{A}}$ (generated by all commutators) is annihilated in the scaling limit, i.e., it is a proper ideal of $\underline{\mathcal{A}}$.

2. The theory has a unique *quantum* scaling limit, i.e., the (isomorphic) nets $\underline{\mathcal{A}}/\ker \underline{\pi}_{0,\iota}$, $\iota \in \mathbb{I}$, are non-abelian. This case corresponds to theories which have (in the language of the theory of the renormalization group) an ultraviolet fixed point. Of particular interest are the asymptotically free theories. According to the folklore, in such theories the underlying fields have almost canonical dimensions (apart from logarithmic corrections) and become free massless fields in the scaling limit. We therefore propose to characterize such theories in the algebraic setting by the condition that (for even dimensions d) the “Huygens-ideal” in $\underline{\mathcal{A}}$ (generated by all commutators of pairs of operators which are localized in strictly timelike separated regions) is annihilated in this limit.

3. The theory has no scaling limit, i.e., not all of the nets $\underline{\mathcal{A}}/\ker \underline{\pi}_{0,\iota}$, $\iota \in \mathbb{I}$, are isomorphic.⁹

We mention as an aside that, as a consequence of Proposition 4.2 and a result in [37], the weak closures of the local algebras $\mathcal{A}(\mathcal{O})$ (with respect to the folium of states $\mathcal{S}_0$) are of type III₁ for all double cones $\mathcal{O}$ if the underlying theory does not have a classical scaling limit.

Let us turn now to the discussion of ultraparticles, the actual subject of this section. So where are they? Proposition 4.2 provides the tools to answer also this question: The

⁹It may very well be that “physics” behaves like this scenario. Then the hope that one may guess a “grand unified theory” which describes physics at all scales would probably be an illusion. But this is not a real problem. After all, high energy experiments can only be carried out up to the scale λ_{budget} , and it should therefore be possible to understand the prevailing data in terms of the simpler cases 2 or 1.

ultraparticles are the particles (in the sense of the previous section) of the theory in the scaling limit. This characterization is meaningful, since in the scaling limit we are still in the framework of positive energy (vacuum) representations of some local net, and the methods of the preceding section can be applied. But what is more important is that this characterization complies with the physical picture which led to the successful interpretation of experimental data. We emphasize that in our analysis the “ultraparticles” are not put in by hand, they appear naturally as a possible structure within the general framework of local quantum field theory. This fact leads us to propose as a

Postulate: The ultraparticle content of the set of all states which locally look like few body systems (states of finite energy) is described in terms of the particle content of the vacuum representations of the scaling limit of the underlying theory, cf. Proposition 4.2 and Sec. 3.

We refrain from giving here a survey of our partial results on the structure of ultraparticles. But we would like to point out one interesting aspect of our analysis: Although we started from local (gauge-invariant) observables, we expect to recover in the scaling limit the full gauge-group of the underlying theory, including the *local* gauge group. The structure of the full gauge group is expected to be encoded in the *global* gauge group of the scaling limit theory since there is no difference between global and local symmetry transformations at a fixed space-time point. Since the global gauge group of any theory can be reconstructed from the net of local observables in a systematic manner [42], we have thus found a promising method to identify gauge quantum field theories, starting from local observables.

We conclude this section with an extract from the long list of intriguing questions in this new setting of algebraic quantum field theory.

i) What is the ultraparticle content of a given *state*? So far we have only described the ultraparticle content of a theory (by looking at the particle content of all states in the vacuum sector of the scaling limit theory). For the analysis of the specific ultraparticle structure of a given state ω one can probably proceed as in Sec. 3. One first distinguishes some left ideal $\tilde{\mathcal{L}} \subset \mathcal{A}$ which annihilates all vacuum states on $\mathcal{A}$. But now the operators in $\tilde{\mathcal{L}}$ have to localize the ultraparticles in momentum space (they have to cut off large momenta). In complete analogy to the discussion in Sec. 3 one could then try to extract the ultraparticle content of ω from the leading contributions in $\omega \circ \delta_\lambda(\tilde{\mathcal{L}}^* \tilde{\mathcal{L}})$, $\tilde{\mathcal{L}} \in \tilde{\mathcal{L}}$, for small λ . A further step would be the formulation of a collision theory for ultraparticles.

ii) When is a theory asymptotically free? We have characterized asymptotically free theories in an ad hoc way by adopting the folklore and converting its algebraic aspects into a condition which fits naturally into our framework. But it would be desirable to establish this condition in an intrinsic way. A possible approach (in massive theories like quantum chromodynamics) could be based on a study of the assumption that the S-matrices S_λ of the theory at scale λ tend to unity in the scaling limit $\lambda \rightarrow 0$.

iii) What is the field content of a theory? The scaling algebra seems to be an adequate instrument for analyzing this question, as well as for exploring, for example, the interesting ideas outlined in [40].

iv) What is the structure of the theory at large scales $\lambda \rightarrow \infty$? The general concept

of scaling algebra may also be relevant for a study of other “scaling limits” such as $c \rightarrow \infty$ or $\hbar \rightarrow 0$.

5 Thermoparticles

Let us turn now to the analysis of the particle-like constituents of thermal equilibrium states. As will become clear, these entities have to be distinguished from quasiparticles, describing the collective excitations of thermal states, and also from the other particle structures analyzed so far. We therefore propose to call them *thermoparticles*.¹⁰

How does one determine the thermoparticle content of a given KMS-state ω on $\mathcal{A}$? A complete answer to this question is not yet known, but for the special case of spatially homogeneous KMS-states some promising results have been obtained in collaboration with J. Bros [13], cf. also [41]. In this approach we proceed again from the heuristic idea that thermoparticles, like other particles, are almost point-like objects. It should therefore be meaningful to consider the situation in which such a particle is placed in an equilibrium state into some region, about a space-time point x , whose size is small compared to the mean volume occupied by the other constituents. (Phrased differently: the de Broglie wavelength of the particle should be small compared to its mean free path.) This particle will either collide with constituents of the state and thereby disappear in the thermal background, or manage to propagate to another space-time point y without intermediate collisions. The latter events should give rise to particle-like correlations between operations performed in almost point-like regions due to the creation and annihilation of a single particle or hole. The probability for such events may be small because of the effects of the thermal background. Nevertheless, the correlation functions of point-like localized operators may be expected to contain the desired information about thermoparticles in a clear-cut form.

Let us translate now these heuristic ideas into our mathematical setting. In order to simplify the discussion we assume, for the time being, that the constituents of the underlying equilibrium state ω are “neutral” in the sense that they can be created by local operations described by certain specific operators $A \in \mathcal{A}$. From the preceding discussion it is clear that most operators in $\mathcal{A}$ are not suitable for our purposes, since they describe operations in finite space-time regions $\mathcal{O}$ which in general will contain many constituents. The resulting excitations of ω may have a quasiparticle structure, but it seems hopeless to extract from them some relevant information about the thermoparticles. Since we do not know from the outset how small $\mathcal{O}$ has to be chosen in order to avoid this problem, it seems appropriate to proceed to the idealization in which the localization region of the underlying operators $A \in \mathcal{A}$ is contracted to a point. Thus, as in the preceding section, we are led to consider sequences of local operators $A_\lambda \in \mathcal{A}(\lambda\mathcal{O})$, where λ tends to 0. But now we want to use these operators to create particle-like excitations of the equilibrium state which can be observed at the original scale $\lambda = 1$.

As explained before, we expect that information about the thermoparticles can be extracted from the correlation functions of the operators A_λ . Assuming that ω is

¹⁰Our terminology does not intend to suggest that for a description of these particles one has to rely on the framework of “thermofield theory”.

invariant under space-time translations, these functions can be represented in the form

$$(20) \quad \omega(\alpha_y(A_\lambda)^* \alpha_x(A_\lambda)) = \int d\mu_\lambda(p) e^{ip(x-y)},$$

where μ_λ is some positive measure. The dominant contribution to this measure will in general be a discrete part at $p = 0$, corresponding to those events where the operation described by A_λ has no effect on the state ω . This uninteresting part has to be removed from μ_λ (e.g., by subtracting from A_λ its expectation value $\omega(A_\lambda)$ if ω describes a pure phase), and we assume in the following that this has been done. The remaining part of the measure describes the correlations between well-localized excitations of ω .

In order to extract from μ_λ the contributions due to the propagation of a single thermoparticle, we have to proceed to the limit $\lambda \rightarrow 0$. Here some further preparation is necessary. On the one hand, the energy-momentum content of the excitations produced by A_λ will become infinite in the limit $\lambda \rightarrow 0$. As in the case of quantum fields, this effect can be controlled by taking suitable space-time averages of the operators $\alpha_x(A_\lambda)$, i.e., the appropriate limit has to be understood in the sense of distributions. On the other hand, the probability that the operators A_λ create some excitation of ω will tend to 0 for $\lambda \rightarrow 0$ if the norms of these operators are uniformly bounded. In order to compensate for the latter effect, the measure μ_λ has to be renormalized. This is accomplished by picking any $N \in \mathbb{N}$ and fixing a corresponding normalization constant

$$(21) \quad Z_\lambda \doteq \int d\mu_\lambda(p) (1 + |p|)^{-N}, \quad \lambda > 0.$$

The resulting renormalized correlation functions

$$(22) \quad W_\lambda(x) \doteq Z_\lambda^{-1} \int d\mu_\lambda(p) e^{ipx}, \quad \lambda > 0,$$

regarded as elements of the space of tempered distributions, form an equicontinuous family. Thus there exist subsequences $\lambda_i \rightarrow 0$ such that the corresponding functions W_{λ_i} converge in the sense of distributions,

$$(23) \quad \lim_i W_{\lambda_i}(x) = W(x).$$

(In order not to overburden the notation, we do not indicate the possible dependence of the limit W on the choice of the subsequence or the number N in the normalization constant.) Making use of the KMS-condition, one finds that the permuted functions $W_\lambda^\pi(-x) = Z_\lambda^{-1} \omega(\alpha_x(A_\lambda) A_\lambda^*)$ also converge for the same subsequences λ_i ,

$$(24) \quad \lim_i W_{\lambda_i}^\pi(-x) = W^\pi(-x).$$

By performing this procedure for all possible sequences of operators A_λ and numbers N one arrives at a collection of tempered distributions W, W^π describing the correlations between arbitrary point-like operations in the underlying KMS-state ω . These distributions can be shown to have the following general properties:

- i) *Locality*: $W(x) = W^\pi(-x)$ for spacelike x .
- ii) *KMS-condition*: Denoting by t the translations along the time axis fixed by the rest system of the underlying heat bath at temperature $\beta^{-1} > 0$, the distribution-valued

function $t \rightarrow W(x+t)$ has an analytic continuation into the strip $0 < \text{Im } t < \beta$, and its boundary value at the upper edge of this strip is given by $\lim_{t \nearrow \beta} W(x+it) = W^\pi(-x)$.
 iii) *Cluster property*: $\lim_{t \rightarrow \pm\infty} W(x+t) = 0$.

It turns out that there exists a universal integral representation for these distributions [41]. We give here the result for the case where the commutator function $C(x) = W(x) - W^\pi(-x)$ is antisymmetric in the time variable. (The general case can be reduced to this one.)

Proposition 5.1: *Let W be a tempered distribution with properties specified above. Then*

$$(25) \quad W(x) = \int_0^\infty dm D(x;m) W^{(0)}(x;m),$$

where $D(x;m)$ is a distribution with respect to x, m and $W^{(0)}(x;m)$ is the correlation function of a non-interacting particle of mass m in a thermal equilibrium state at temperature β^{-1} ,

$$W^{(0)}(x;m) = \text{const} \int dp \varepsilon(p_0) \delta(p^2 - m^2) (1 - e^{-\beta p_0})^{-1} e^{ipx}.$$

This result provides us with the desired tool to uncover the thermoparticle content of equilibrium states: the individual terms $D(x;m)W^{(0)}(x;m)$ in (25) describe the propagation of an excitation of mass m which has been created at the origin. Here the free correlation function $W^{(0)}(x;m)$ corresponds to the unperturbed motion; it is in general modified by a "damping factor" $D(x;m)$ which subsumes the loss of probability of finding the excitation at other space-time points due to intermediate collisions. In the case of the stable constituents of thermal equilibrium states, one would expect that such collisions are the only mechanism by which these probabilities are reduced, in contrast to excitations such as metastable bound states, pairs and other composite systems, which disintegrate also without external impact. We therefore conjecture that thermoparticles contribute to the discrete (atomic) part of the integral representations (25), whereas all other excitations contribute to the continuous part. The contributions due to thermoparticles can thus be uncovered by inverting relation (25), cf. [13].

If the underlying thermoparticles are not "neutral", i.e., cannot be created and annihilated by operators $A \in \mathcal{A}$, one has to proceed from the algebra of observables to a corresponding field algebra $\mathcal{F}$ of charge-carrying fields. In the case of localizable charges, this field-algebra is still generated by a net $\mathcal{O} \rightarrow \mathcal{F}(\mathcal{O})$ of local field algebras [42], and the thermal equilibrium states can be extended to this net [43]. One can then proceed in the same way as in the preceding discussion and construct a family of correlation functions W , describing the effects of point-like localized operations induced by fields in $\mathcal{F}$. Again, the distributions W possess integral representations of the kind given in relation (25), and the thermoparticles should manifest themselves by discrete contributions in these representations. This leads us to propose as a

Working hypothesis: Thermoparticles, appearing as constituents of a thermal equilibrium state and carrying a localizable charge, manifest themselves by discrete contributions in the integral representations of the respective correlation functions of point-like observables or fields, cf. relations (23) and (25).

In contrast to other proposals [10], our approach provides a characterization of thermoparticles in unambiguous mathematical terms. But there remain several open problems, and we conclude this section by listing some of them.

i) What are the properties of thermoparticles? We have identified thermoparticles by the discrete contributions appearing in the integral decomposition of the correlation functions W . In this way we were able to specify their mass m and the corresponding “damping factors” $D(\mathbf{x}; m)$ which describe the dissipative effects of the thermal background. Since these factors are responsible for the modified dispersion laws of thermoparticles, it would be desirable to gather further information about their specific properties. For example, if the correlation functions satisfy the relativistic KMS-condition proposed in [44], the damping factors $D(\mathbf{x}; m)$ are locally analytic in $\mathbf{x}$, which in turn implies that the energy-momentum content of the corresponding thermoparticles is concentrated about the mass shell $p^2 = m^2$. Another problem of interest is the determination of the internal degrees of freedom of thermoparticles.

ii) How does one identify thermoparticles carrying non-localizable charges? A solution of this problem does not seem possible along the lines indicated in this section, since such particles cannot be created by almost point-like operations in view of the long-range fields which they carry along. Instead one should try to localize the thermoparticles already present in the state with the help of suitable localizing operators, in analogy to the strategy outlined in Sec. 3. By proceeding to point-like localizing operators, one may then hope to extract the thermoparticle content from the resulting correlation functions.

iii) How does one determine the thermoparticle content of non-homogeneous states? The basic strategy should be the same as for homogeneous states, but there arise additional mathematical complications in the construction and analysis of the correlation functions W .

iv) When does a theory describe thermoparticles? A general analysis of this problem would require the following steps: First, one has to understand under which conditions the limits W of the correlation functions are non-trivial: This question is closely related to the existence problem of point-like localized quantum fields. Second, one would then have to clarify when there appear discrete contributions in the integral representations of the distributions W . Here the conditions of primitive causality and of causal independence, mentioned in Sec. 3 in a similar context, should again be of relevance.

6 Concluding Contemplations

The topic of the present article has been a conceptual problem in quantum field theory: the notion of particle. We have seen that, depending on the physical situation under consideration, the description of particles leads to specific idealizations which call for their own mathematical framework (particle weights, scaling algebras, etc.).

The clarification of physical concepts requires a model-independent setting, and algebraic quantum field theory has proved to be a very useful tool in this respect. As the motto of this Workshop is an invitation to contemplate about the achievements and perspectives of our subjects, we conclude this article with some further remarks and views about the value of model-independent investigations for the development and understanding of relativistic quantum field theory.

The idea that one may gain some insight into physics by studying a mathematical setting (in contrast to a specific model) is sometimes regarded with suspicion. It is argued that a general framework, such as that of algebraic quantum field theory outlined in Sec. 2, contains too little specific information and can therefore only yield "soft" results which are applicable to all possible theories, no matter how weird.

The point missed here is the fact that a general framework is the appropriate arena for the *classification* of theories, and detailed information can frequently be obtained for the resulting special cases. Let us mention as a prominent example the analysis of Doplicher, Haag and Roberts, which led to a complete classification of the possible statistics of states carrying a localizable charge and the clarification of the relation between the type of statistics and the structure of the underlying global gauge group.

The problem of classifying theories often leads also to new and powerful *concepts*, an example being the "split property", first conceived of by Borchers, which proved to be useful not only in the structural analysis but also in the investigation of concrete models.

A model independent analysis is furthermore the basis for the development of general *algorithms*, well known examples being the Haag-Ruelle collision theory and the Lehmann-Symanzik-Zimmermann reduction formulas for the calculation of S-matrix elements. It also leads to the formulation of *criteria* which characterize physically significant properties of a theory. For example, the analysis of the possible range of the local degrees of freedom of quantum field theories, initiated by Haag and Swieca, led to compactness and nuclearity conditions which turned out to be a key e.g., to the understanding of thermodynamical properties of quantum field theories.

Finally, the results of a model independent analysis may point to the existence of interesting *new structures* which are compatible with the general principles, but remained undiscovered in the realm of models. An example of this kind are theories of massive particles carrying a quantum topological charge. Their specific properties were uncovered by Fredenhagen and the present author in a general analysis, which subsequently triggered a (successful) search for models. (Cf. [15] for more detailed information on the various results mentioned.)

As may be seen from these remarks, it is one of the principal and most interesting aims of algebraic quantum field theory to amend the basic postulates by further conditions, resulting either from a structural analysis or from physical considerations. The strategy is thus to tighten the framework and thereby to obtain more detailed information about the properties of physically significant theories. One may even hope that this strategy leads to new methods for the actual construction of models. Some interesting results in this direction have recently been obtained by Wiesbrock [45], who proved that conformal theories on the light ray can be constructed from special inclusions of two von Neumann algebras.

Another objection raised against the general framework of (algebraic) quantum field theory is the view that it may be too restrictive: all attempts to construct a mathematically consistent and physically significant quantum field theory fitting into this framework have failed so far, and it may be necessary to include "gravitation" in order to cure the agonizing ultraviolet problems. (The reader may recognize this view in some of the contributions in this volume.)

It is difficult to argue against this kind of critique if one cannot exhibit a coun-

terexample. But we dare to recall that the gravitational constant is small and that the development of physics has shown that "orders of magnitude" provide a natural grading of phenomena. It would therefore be a surprise if the theoretical understanding of present day high energy physics (and may be that of the next century) could not be based in a consistent way on the idealizations of special relativity. At least, there is no indication that there exists some basic obstruction to this effect. The present lack of relevant examples seems to have a much simpler explanation: it is a difficult task to construct them.

In order to get an idea about the degree of difficulties it may be instructive to compare the progress in quantum field theory with the development of quantum mechanics. It took 30 years from Planck's first observations till von Neumann's final rigorous formulation of the theory. Vital intermediate steps were taken by Einstein, Bohr, de Broglie and, finally, by Heisenberg and Schrödinger. So, roughly speaking, the time scale for progress in quantum mechanics was 6 years.

The corresponding scale in quantum field theory seems to be about 20 years: quantum field theory was already conceived by the fathers of quantum mechanics. The next vital step (covariant perturbation theory and renormalization) was taken 20 years later by Tomonaga, Schwinger, Feynman and Dyson and led to the consolidation of quantum electrodynamics. It then took even more than 20 years until the systematic study of the ultraviolet properties of quantum field theories led Symanzik, Gross and Wilczek, Politzer and others to the fundamental concept of asymptotic freedom, which is now a basic ingredient in the treatment of gauge theories, such as quantum chromodynamics. According to this time scale some further important progress in quantum field theory seems to be due, but one should not expect that it will lead to its ultimate consolidation. It may take another 20 or 40 years until quantum field theory reaches a form which also pleases mathematical physicists. So many of us may not be able to appreciate it.

It is perhaps this unpleasant perspective which makes us susceptible to all kinds of unconventional ideas which offer an apparently quicker solution. But the solution of a difficult problem needs time, sometimes much time, and continuous efforts (compare mathematics). It would therefore be regrettable if the beauties of physics in 2 and 10 dimensions let us forget the inherent possibilities of the more conventional approach and distract courageous people from pushing this problem towards its solution.

But what if the Lord is malicious and has enticed us to explore for more than six decades some cul-de-sac? Probably the only way to find him out would be a structural analysis which either reveals the internal inconsistency of our present physical ideas or an unavoidable clash between experimental facts and theoretical predictions. Thus the model-independent analysis of the "basic principles" is a vital part of the development of the theory and one may therefore expect that algebraic quantum field theory will continue to play its part in the consolidation of relativistic quantum field theory in the 21st century as well.

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STRING THEORY AND INTEGRABLE SYSTEMS

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Abstract

In this brief review we try to outline the basic structures underlying *integrable* quantum field theory models with *infinite-dimensional* symmetry groups which display *quantum group* symmetries. Certain aspects are treated in some detail: integrable systems of Kadomtsev-Petviashvili type and their reductions appearing in matrix models of strings; Hamiltonian approach to Lie-Poisson symmetries; quantum field theory approach to two-dimensional relativistic integrable models with dynamically broken conformal invariance. All field-theoretic models in question are of relevance to diverse branches of physics ranging from nonlinear hydrodynamics to string theory of fundamental particle interactions at ultra-high energies.

1 Introduction

A current view in theoretical physics is that fundamental laws of Nature can be understood in terms of field-theoretic models in a lower dimensional space-time possessing *infinite-dimensional symmetry groups* and, thus, as a rule being *integrable*. These models are a part of the rich and rapidly developing branch of *string theory* [1]. It is believed that string theory is the most viable candidate for a unified theory of all fundamental interactions at ultra-short distances which, in particular, will unify General Relativity and Quantum Mechanics – one of the major challenges of this century's Physics.

We have in mind two large classes of *integrable models*: conformal field theory (CFT) [2, 3] and massive completely integrable models [4, 5, 6] in $D = 2$ space-time dimensions. Typical examples of CFT are the *rational* CFT's, such as the extensively-studied Wess-Zumino-Novikov-Witten (WZNW) models for various Lie groups G , and models obtained from them by the gauging of different subgroups H of G [7]. In the class of massive integrable models, the Sine-Gordon, (nonabelian) massive Thirring models, Toda models for various groups, and the Korteweg-de Vries (KdV) and Kadomtsev-Petviashvili (KP) integrable *soliton* evolution equations and their hierarchies have been thoroughly investigated. The *integrability* which is common to both classes stems from the infinite-dimensional Lie-algebraic structure they share: in the first class this being

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the Noether symmetry algebra, and in the second, the Hamiltonian structure. The underlying infinite-dimensional symmetries (which manifest themselves through the Virasoro (conformal) [2] and affine Kac-Moody algebras [8] and their generalizations such as the W -algebras [9]) play a crucial role in integrability.

The interrelation between CFT and completely integrable models became clear recently through the appearance of KdV and KP integrable hierarchies in the *matrix model* description [10] of (sub)critical strings (i.e., 2-dimensional gravity interacting with conformal “matter” fields). Thus, it is precisely the integrable field theories which provide the proper framework for incorporation of the huge symmetries of string theory models.

The property of *integrability* is studied both in the classical and the quantum cases. The principal problems which one should investigate are, for the classical case: (a) Classification; (b) variables of the action-angle type; and (c) exact integration of the equations of motion; in the quantum case: (a) Classification; (b) exact scattering amplitudes, and (c) exact correlation functions of local fields off the mass shell. The concepts and tools appropriate to these are found in the theory of infinite-dimensional Lie algebras and groups, and *symplectic geometry*, or equivalently, *Hamiltonian mechanics*. The aim here is to uncover the common *geometrical* foundation of the field theories. In the realm of string theory and the theory of completely integrable systems, the intertwining of Hamiltonian and Lie-group structures in field-theoretic models gives rise to new features, which are clearly seen in the main methods for solving the quantization problem in integrable models: *the quantum inverse scattering method* [11, 12], representation theory of infinite-dimensional Lie algebras [8], and *Quantum Groups* (non-commutative and non-cocommutative Hopf algebras) [13, 12].

The generic integrable models are massive field theories which can be regarded as integrable perturbations of conformal field theories [14]. The latter describe the renormalization group fixed points. Such models have the advantage of being relativistically invariant and classifiable by the conformal models of which they are perturbations. Their essential feature, which shows explicitly the connection with conformal models, is the existence of *multi-Hamiltonian* structures, i.e., the existence of at least a second Hamiltonian structure which is compatible with the canonical R -matrix Kirillov-Kostant structure [15]. The fundamental Poisson brackets corresponding to these Hamiltonian structures (linear R -matrix brackets and quadratic (Sklyanin) R -matrix brackets) arise naturally and are well understood within the classical inverse scattering method [5]. They can also be deduced in the semiclassical limit from the basic algebraic structures:

1. Yang-Baxter equation for the quantum version of the R -matrix;
2. fundamental commutation relations for the quantum transfer matrix, involving the quantum R -matrix as “structure constants”,

of the quantum inverse scattering method [11] – the first systematic method for quantization of completely integrable models.

In a related development, Drinfeld [16, 13] was the first to understand the deeper algebraic and geometric nature of classical and quantum completely integrable models. He showed that the algebraic structures 1) and 2) mentioned above were the basic

structural relations of the *non-commutative and non-cocommutative Hopf algebras* which were ultimately named *Quantum Groups*. Furthermore, it was realized that a quantum group is a *deformation* of a classical Lie group much in the same way quantum mechanics is a deformation of classical Hamiltonian (symplectic) mechanics [17]. In the semiclassical limit the basic quantum group algebraic structures 1) and 2) go over into a special Hamiltonian structure on the classical Lie group G , called the *Lie-Poisson structure*, which is compatible with the group multiplication. This is precisely the class of Hamiltonian structures given by the quadratic fundamental R -matrix Poisson brackets mentioned above in the context of classical completely integrable models.

The concept of *quantum group symmetries* in integrable quantum field and statistical mechanics models has led to several developments: from generalization (i.e., *q-deformation*) of the internal and space-time symmetries in quantum field theory and the connection between spin and statistics, to quantum magnetic chains and the dynamics of critical phenomena [18]. Furthermore, the concepts of integrability and perturbations around exactly solvable theories have found their place in models of elementary particles such as in quantum chromodynamics [19].

In what follows, some of the topics mentioned above will be discussed in greater detail.

2 Matrix Models of Non-Perturbative Strings and Integrability

2.1 Conventional Perturbative String Theory

The standard geometrical formulation of *perturbative* string theory [20] provides the following prescription for calculating physical observables (fermionic degrees of freedom are discarded for simplicity): to construct scattering amplitudes one considers functional integrals over (Euclidean) string world-sheets $\Sigma_{A,G}$ – smooth Riemann surfaces embedded in D -dimensional (Euclidean) space-time R^D of genus G and area A :

$$(1) \quad Z_{\text{string}} = \sum_{G=0}^{\infty} g^{2G} \int dA e^{-\Lambda A} Z_{A,G}$$

$$(2) \quad Z_{A,G} = \int [\mathcal{D}h] \mathcal{D}X \exp \{-S_{\text{string}}[X, h]\} \prod_{i=1}^n \int d^2\sigma_i \sqrt{h} V(X, h; k_i)$$

$$(3) \quad S_{\text{string}} = \frac{1}{2} \int d^2\sigma \left[\sqrt{h} \left(h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + \Phi(X) R^{(2)}(h) \right) + \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \right]$$

Here the following notations are used: g denotes the string coupling constant, Λ is the cosmological constant, the string action S_{string} represents a typical $D = 2$ conformally invariant field theory describing $D = 2$ gravity given by the world-sheet metric $h_{ab}(\sigma)$ and coupled to world-sheet matter fields $X^\mu(\sigma)$ describing the embedding of $\Sigma_{A,G}$ in R^D . The functionals $G_{\mu\nu}, \Phi, B_{\mu\nu}$ represent the space-time dilaton-gravity multiplet. String

interactions are given in (1), (2) by geometrical splitting and joining of individual string world-sheets thus creating handles on the total world-sheet, whereas the asymptotic incoming and outgoing string states are given by vertex operators $V(X, h; k)$ (k denoting the momentum of the asymptotic state).

Henceforth, for simplicity, the vertex operators V 's in (2) will be suppressed, i.e., we shall concentrate on the string partition function.

The most difficult part of calculating (2) is handling the functional measure $[Dh]$ over the space of all *gauge-inequivalent* classes of metrics h_{ab} on $\Sigma_{A,G}$ w.r.t. reparametrization and Weyl conformal invariance. In the conformal gauge, $h_{ab} = e^\phi \hat{h}_{ab}(\tau)$ where ϕ is the Weyl conformal factor and $\hat{h}_{ab}(\tau)$ is a reference metric with constant curvature $R^{(2)}(\hat{h}_{ab})$ and which depends in general on the moduli $\{\tau\}$ of the Riemann surface $\Sigma_{A,G}$, the standard Faddeev-Popov gauge-fixing procedure yields [20, 21]:

$$(4) \quad [Dh] = \delta(h_{ab} - e^\phi \hat{h}_{ab}(\tau)) \Delta_{\Phi\Pi} D\phi(d\tau)$$

with the Faddeev-Popov determinant $\Delta_{\Phi\Pi}$ giving rise to the well-known conformal anomaly.

An important result about the entropy of random surfaces with fixed area A , first obtained by Zamolodchikov [22] in the semi-classical approximation and subsequently strengthened in [23], states that for large A :

$$(5) \quad Z_{A,G} \simeq_{A \rightarrow \infty} \text{const}_G e^{\Lambda_c A} A^{-\chi(1-\gamma_0/2)-1}$$

where Λ_c denotes a critical value of the cosmological constant Λ , $\chi = 2(1 - G)$ is the Euler characteristic of the surfaces, and γ_0 denotes a critical exponent depending on the world-sheet matter fields. Relation (5) implies for the string partition function (1):

$$(6) \quad Z_{\text{string}} \simeq (\Lambda - \Lambda_c)^{2-\gamma_0} \sum_{G=0}^{\infty} \text{const}_G \left(\frac{g^2}{(\Lambda - \Lambda_c)^{2-\gamma_0}} \right)^G$$

which shows that one can obtain the complete nonperturbative result for Z_{string} by taking the *double scaling* limit:

$$(7) \quad \Lambda \longrightarrow \Lambda_c, \quad g^2 \longrightarrow 0 \quad \text{such that} \quad g_{\text{ren}}^2 \equiv \left(\frac{g^2}{(\Lambda - \Lambda_c)^{2-\gamma_0}} \right) = \text{fixed.}$$

2.2 Lattice Regularization of String Theory; Matrix Model Formulation

It has been found [24] that statistical mechanical models of random matrices ("matrix models" for short) provide an adequate apparatus for the nonperturbative description of lattice-regularized string theory based on the method of *random triangulation* (and, more generally, random polygonization) of the (Euclidean) string world-sheet. Furthermore, ways have been proposed in refs. [10] for the correct implementation of the continuum limit as a double scaling limit (7), which admits exact solutions in string theory.

Let us note that, whereas matrix model formulation of random surfaces is adequate for solving integrable lattice models of planar statistical mechanics, its application to genuine string theory is limited so far to the case of $D \leq 2$ dimensional embedding space. Nonetheless, the exact solvability of matrix models provides an important testing ground and qualitative hints for nonperturbative string theory solutions in realistic cases (for extensive reviews, see [26]).

Since our main aim is to clarify the emergence of integrability structures in matrix models of string theory, we shall consider for illustrative purpose the simplest one-matrix model with partition function given by:

$$(8) \quad Z = \int d^{N^2} M e^{-NV(M)}, \quad V(M) = \sum_{k \geq 0} t_k \left(\frac{N}{\beta} \right)^{k/2-1} \text{Tr } M^k$$

with $M = \|M_{ij}\|$ being a $N \times N$ hermitian matrix. In ordinary perturbation theory defined in terms of a free propagator $\langle M_{ij} M_{kl} \rangle_0 \sim N^{-1} \delta_{ik} \delta_{jl}$ and k -leg vertices with weights $t_k N \left(\frac{N}{\beta} \right)^{k/2-1}$, each diagram Γ gives a contribution of the form:

$$(9) \quad \prod_{k \geq 3} \left(t_k N \left(\frac{N}{\beta} \right)^{k/2-1} \right)^{V_k(\Gamma)} N^{L(\Gamma)-P(\Gamma)} = \mathcal{W}_\Gamma[\{t\}] N^{\chi(\Gamma)} \left(\frac{\beta}{N} \right)^{-L(\Gamma)}$$

On the l.h.s. of (9) $V_k(\Gamma)$, $P(\Gamma)$ and $L(\Gamma)$ denote the numbers of k -leg vertices, propagators (links) and closed loops (faces) of Γ , whereas on the r.h.s. $\chi(\Gamma) = L(\Gamma) - P(\Gamma) + \sum_{k \geq 3} V_k(\Gamma)$ denotes the Euler characteristic of the two-dimensional polygonized surface spanned by Γ , and $\mathcal{W}_\Gamma[\{t\}]$ the product of the vertex weights. Clearly, $L(\Gamma) \equiv A(\Gamma)$ can be understood as the area of Γ . Thus the partition function (8) can be written as:

$$(10) \quad Z = \exp \left\{ \sum_{\text{conn. surfaces } \Gamma} N^{\chi(\Gamma)} e^{-(\ln \beta / N) A(\Gamma) + \ln \mathcal{W}_\Gamma[\{t\}]} \right\}$$

i.e., the *free energy* $\ln Z$ of the matrix model (10) represents the discretized regularized partition function of random surfaces (more precisely, pure $D = 2$ gravity with action $A(\Gamma)$ interacting with matter with action $\ln \mathcal{W}_\Gamma$) upon making the following identifications (comparing with (1)–(2), (5)): $1/N \simeq g$ (bare string coupling constant), $N/\beta \simeq e^{-(\Lambda - \Lambda_c)}$ (Λ - the cosmological constant). The double scaling limit (7) takes the form of a special continuum limit:

$$(11) \quad \frac{N}{\beta} \longrightarrow 1, \quad N \longrightarrow \infty, \quad \text{such that } g_{\text{ren}}^2 \equiv \left[\beta^2 \left(\frac{\beta}{N} - 1 \right)^{2-\gamma_0} \right]^{-1} = \text{fixed}.$$

The explicit solution for the partition function (8) is obtained by the method of orthogonal polynomials [25]. Diagonalizing the matrix $M = U \text{diag}(\lambda_1, \dots, \lambda_N) U^{-1}$, which is hermitian, and integrating over angle variables in U , one gets (rescaling $M \longrightarrow (\beta/N)^{\frac{1}{2}} M$):

$$(12) \quad Z = \int \prod_{i=1}^N d\lambda_i \Delta(\lambda) \exp \left\{ -\beta \sum_{i=1}^N V(\lambda_i) \right\} \Delta(\lambda),$$

$$V(\lambda_i) = \sum_{k \geq 0} t_k \lambda_i^k, \quad \Delta(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j).$$

Introducing a complete set of orthogonal polynomials $P_n(\lambda) = \lambda^n + \text{lower order terms}$:

$$(13) \quad \int d\lambda P_n(\lambda) e^{-\beta V(\lambda)} P_m(\lambda) = h_n \delta_{nm}$$

and re-expressing the Vandermonde determinant as $\Delta(\lambda) = \det \|P_i(\lambda_j)\|$, the integrals in (12) factorize, and give, using (13):

$$(14) \quad Z[\{t\}] = \prod_{n=0}^N h_n(\{t\})$$

where the dependence on the parameters of the random matrix potential is indicated. Explicit solutions for $h_n(\{t\})$ can be found by solving the flow equations w.r.t. t_k which correspond to integrable lattice hierarchies, as briefly discussed in the next subsection.

2.3 Differential Integrable Hierarchies from Matrix Models

The appearance of integrable hierarchies in the continuum (double scaling) limit (11) has been extensively discussed in the literature (see [26] and refs. therein). It is very interesting, however, that the flow equations can be obtained directly from discrete matrix models even before taking the continuum limit [27, 28]. This reveals their close connection to topological field theories [29].

The above result is obtained most easily by the method of orthogonal polynomials (13). On the Hilbert space spanned by $\{P_n(\lambda)\}_{n \geq 0}$, one introduces two conjugate operators $\mathcal{Q}, \mathcal{P}$ with matrix elements defined by:

$$(15) \quad h_n \mathcal{Q}_{mn} = \int d\lambda P_n(\lambda) e^{-\beta V(\lambda)} \lambda P_m(\lambda)$$

$$(16) \quad h_n \mathcal{P}_{mn} = \int d\lambda P_n(\lambda) e^{-\beta V(\lambda)} \frac{d}{d\lambda} P_m(\lambda)$$

From (15), (16) one easily gets the matrix model string equation (the second one below):

$$(17) \quad \mathcal{P} = \beta (V'(\mathcal{Q}))_{(-)} \longrightarrow [\beta (V'(\mathcal{Q}))_{(-)}, \mathcal{Q}] = 1$$

where the subscript $(-)$ denotes the strictly lower-diagonal part of the corresponding matrix. The string equation yields recursion relations for the matrix elements (15) of $\mathcal{Q}$.

It is straightforward to deduce from (15) and (13) the following flow equations:

$$(18) \quad \frac{\partial P_n(\lambda)}{\partial t_r} = \mathcal{Q}_{(-)nm}^r P_m(\lambda), \quad \mathcal{Q}_{nm} P_m(\lambda) = \lambda P_n(\lambda)$$

$$(19) \quad \frac{\partial \mathcal{Q}}{\partial t_r} = [\mathcal{Q}_{(-)}^r, \mathcal{Q}]$$

Eq. (19) is the integrability condition for eqs. (18) and it is compatible with the string equation (17). One can identify (19) with the Lax form of the flow equations of the integrable Toda lattice hierarchy by inserting into (19) the explicit form of the $\mathcal{Q}$ matrix elements:

$$(20) \quad \mathcal{Q}_{n,n+1} = 1, \quad \mathcal{Q}_{n,n} \equiv \mathcal{S}_{n-1}, \quad \mathcal{Q}_{n+1,n} = \frac{h_{n+1}}{h_n} \equiv \mathcal{R}_n \equiv e^{\phi_n - \phi_{n-1}}$$

the rest being zero owing to the recurrence relations for orthogonal polynomials. Eqs. (19) are now Hamiltonian, and the lowest one (with $r = 1$) is generated by the Toda lattice Hamiltonian:

$$(21) \quad H_{\text{Toda}} = \frac{1}{2} \sum_n \mathcal{S}_n^2 + \sum_n (e^{\phi_{n+1} - \phi_n} - 1), \quad \{\mathcal{S}_n, \phi_m\} = \delta_{nm}.$$

Following [28], it is possible to replace the discrete lattice integrable hierarchy (19) by a differential hierarchy at each fixed lattice site n where the continuum variable is $x = t_1$. Indeed, from the ($r = 1$) flow eqs. (18) and (19),

$$(22) \quad \frac{\partial P_{n+1}}{\partial t_1} = \mathcal{R}_n P_n, \quad \frac{\partial \mathcal{R}_n}{\partial t_1} = \mathcal{R}_n (\mathcal{S}_{n-1} - \mathcal{S}_n), \quad \frac{\partial \mathcal{S}_n}{\partial t_1} = \mathcal{R}_n - \mathcal{R}_{n+1}$$

one obtains (up to gauge transformation $P_n \rightarrow \psi_n = \exp \{ \int dt_1' \mathcal{S}_{n-1} \} P_n = h_n^{-1} P_n$, and similarly $\mathcal{Q}_{nm} \rightarrow \hat{\mathcal{Q}}_{nm} = h_n^{-1} \mathcal{Q}_{nm} h_m$):

$$(23) \quad \begin{aligned} \lambda \psi_n = \hat{\mathcal{Q}}_{nm} \psi_n &= h_n^{-1} (P_{n+1} + \mathcal{S}_{n-1} P_n + \mathcal{R}_{n-1} P_{n-1}) \\ &= [\partial + \mathcal{R}_n (\partial - \mathcal{S}_n)^{-1}] \psi_n \end{aligned}$$

where $\partial \equiv \frac{\partial}{\partial t_1}$. Once again using (22), one can rewrite the discrete evolution eqs. (18), (19) in a differential Lax form for a fixed lattice site n :

$$(24) \quad \frac{\partial \psi}{\partial t_r} = (L^r)_+ \psi, \quad \frac{\partial L}{\partial t_r} = [(L^r)_+, L]$$

where $r \geq 2$, $\psi \equiv \psi_n(t_1)$ and the subscript $+$ indicates taking the purely differential part of the corresponding pseudo-differential Lax operator (cf. last equality in (23)):

$$(25) \quad L = \partial + A(\partial - B)^{-1}, \quad A \equiv \mathcal{R}_n(t_1), \quad B \equiv \mathcal{S}_n(t_1).$$

Eqs. (24), (25) are immediately recognized as the 2-boson reduction of KP integrable hierarchy (see subsections 3.2 and 3.3).

Using a generalization of the method of orthogonal polynomials, it is possible to derive flow equations for integrable hierarchies also in the general case of multi-matrix models (describing random surfaces interacting with q different types of "matter"):

$$(26) \quad Z = \int \prod_{i=1}^q d^{N^2} M_i \exp \left\{ - \sum_{i=1}^q \left(\text{Tr } M_i M_{i+1} + \sum_{k \geq 0} t_{i,k} \text{Tr } M_i^k \right) \right\}$$

without passing to the continuum limit [28]. The appropriate generalization of (25) now reads:

$$(27) \quad L_q = \partial + \sum_{l=1}^q A_l (\partial - B_l)^{-1} (\partial - B_{l+1})^{-1} \dots (\partial - B_q)^{-1}$$

which is a $2q$ -boson reduction of KP integrable hierarchy (see subsections 3.2 and 3.3 for more details).

Finally, returning to the string partition function (14), one can show that $Z[\{t\}] = \tau_N(t)$ – the Toda lattice τ -function [27] subject to the so-called Virasoro constraints:

$$(28) \quad \mathcal{L}_s Z[\{t\}] = 0, \quad s \geq -1; \quad [\mathcal{L}_r, \mathcal{L}_s] = (r-s)\mathcal{L}_{r+s}$$

$$\begin{aligned} \mathcal{L}_{s \geq 0} &= \sum_{k \geq 0} k t_k \frac{\partial}{\partial t_{k+s}} + \sum_{k=0}^s \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{s-k}} \\ \mathcal{L}_{s < 0} &= \sum_{k \geq 0} k t_k \frac{\partial}{\partial t_{k-s}} - \sum_{k=0}^{|s|-1} \frac{\partial}{\partial t_{-k}} \frac{\partial}{\partial t_{s+k}} \end{aligned}$$

which are equivalent to the constraints on the pertinent integrable hierarchies imposed by the "string" equation (17). They are in fact Ward identities corresponding to the symmetry $\delta_s M = \varepsilon_s M^{s+1}$, $s \geq -1$ of (9). Similar relations hold for multi-matrix models. In this case the continuum (double scaling) limit $Z[\{t\}]$ can be identified with τ -functions of reduced KP hierarchies subject to the so-called W -constraints (generalizations of (28) which span W -algebras; cf. (50) below).

3 Integrable Systems in Classical Physics: Geometric Formulation

This section is devoted to a brief review of some principal structures and properties of integrable systems in classical mechanics and field theory. We first recall the notion of integrability.

Complete integrability: Consider a Hamiltonian system with n degrees of freedom possessing a standard Hamiltonian structure with Hamiltonian $H(p, q)$ and Poisson bracket $\{\cdot, \cdot\}$. A Hamiltonian system is called *completely* (or Liouville) *integrable* if it has n independent integrals of motion I_k , $k = 1, \dots, n$, which are in involution: $\{I_i, I_j\} = 0$.

For such a system one can find the *action-angle* canonical variables and write explicitly the general solution to the equations of motion.

Lax formulation: For infinite-dimensional (field theory) integrable Hamiltonian systems, there exists a convenient Lax (or “zero-curvature”) formulation [5]. In the Lax formulation, the phase space of the Hamiltonian system is parametrized by elements L taking values in some Lie algebra $\mathcal{G}$ and the dynamical equations of motion can be written in terms of a Lax pair L, P , the latter also taking values in $\mathcal{G}$, as the Lax-type equation:

$$(29) \quad \frac{dL}{dt} = [L, P]$$

The Lax formulation leads straightforwardly to the construction of the involutive integrals of motion. Any Ad-invariant function $I(L)$ on $\mathcal{G}$ is a constant of motion. It can be shown that any completely integrable Hamiltonian system admits a Lax representation (at least locally) [30].

A very wide class of integrable models can be constructed in the approach developed by Adler, Kostant and Symes, and by Reyman and Semenov-Tyan-Shansky (AKS-RS scheme) [31, 15] having roots in the coadjoint orbit method [32].

3.1 AKS-RS Scheme

Let G denote a Lie group and $\mathcal{G}$ be its Lie algebra. G acts on $\mathcal{G}$ by the adjoint action: $Ad(g)X = gXg^{-1}$, with $g \in G$ and $X \in \mathcal{G}$. Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$ relative to a non-degenerate bilinear form $\langle \cdot | \cdot \rangle$ on $\mathcal{G}^* \times \mathcal{G}$. The corresponding coadjoint action of G on $\mathcal{G}^*$ is obtained from the duality of $\langle \cdot | \cdot \rangle$: $\langle Ad^*(g)U | X \rangle = \langle U | Ad(g^{-1})X \rangle$. The infinitesimal versions of the adjoint and coadjoint transformations (for $g = \exp Y$) will be denoted by $ad(Y)X = [Y, X]$ and $\langle ad^*(Y)U | X \rangle = -\langle U | [Y, X] \rangle$, respectively. When $\mathcal{G}$ is endowed with an ad-invariant bilinear form $(\cdot, \cdot)$ (Killing form) allowing to identify $\mathcal{G}^*$ with $\mathcal{G}$, one has $ad^*(Y)U = [Y, U]$.

There exists a natural Poisson structure on the space $C^\infty(\mathcal{G}^*, BbbR)$ of smooth, real-valued functions on $\mathcal{G}^*$ called Kirillov-Kostant (KK) bracket, given by:

$$(30) \quad \{F, H\}(U) = - \left\langle U \left| \left[\nabla F(U), \nabla H(U) \right] \right. \right\rangle$$

where $F, H \in C^\infty(\mathcal{G}^*, \mathbb{R})$, the gradient $\nabla F : \mathcal{G}^* \rightarrow \mathcal{G}$ is defined by the formula $\frac{d}{dt}F(U + tV)|_{t=0} = \langle V | \nabla F(U) \rangle$ and $[\cdot, \cdot]$ is the Lie commutator on $\mathcal{G}$. On each orbit of G in $\mathcal{G}^*$ the Poisson bracket (30) gives rise to a non-degenerate symplectic structure. For any Hamiltonian function H on such an orbit one can write a Hamiltonian equation of motion $dU/dt = ad^*(\nabla H(U))U = [\nabla H(U), U]$ when $\mathcal{G}$ admits a Killing form).

In ref. [15] the R -operator (generalized R -matrix) was introduced as a linear map from a Lie algebra $\mathcal{G}$ to itself such that the bracket

$$(31) \quad [X, Y]_R \equiv \frac{1}{2}[RX, Y] + \frac{1}{2}[X, RY]$$

defines a second Lie-bracket structure on $\mathcal{G}$, or equivalently, defines a second Lie algebra $\mathcal{G}_R$ isomorphic to $\mathcal{G}$ as a vector space. The Jacobi identity for the R -commutator (31) implies that the modified Yang-Baxter equation (YBE) for the R -matrix should hold (for arbitrary $X_{1,2,3} \in \mathcal{G}$):

$$(32) \quad \sum_{\text{cyclic}(1,2,3)} \left[X_1, \left[RX_2, RX_3 \right] - R \left(\left[RX_2, X_3 \right] + \left[X_2, RX_3 \right] \right) \right] = 0.$$

A sufficient condition for the fulfilment of (32) is:

$$(33) \quad \left[RX, RY \right] - R \left(\left[RX, Y \right] + \left[X, RY \right] \right) = -\alpha \left[X, Y \right]$$

where α is an arbitrary constant. Eq. (33) is usually written in terms of the "ordinary" r -matrix $r \in \mathcal{G} \otimes \mathcal{G}$ which is isomorphic to $R \in \mathcal{G} \otimes \mathcal{G}^*$ via the Killing form¹:

$$(34) \quad \left[\begin{smallmatrix} 12 \\ r \end{smallmatrix}, \begin{smallmatrix} 13 \\ r \end{smallmatrix} \right] + \left[\begin{smallmatrix} 13 \\ r \end{smallmatrix}, \begin{smallmatrix} 23 \\ r \end{smallmatrix} \right] + \left[\begin{smallmatrix} 12 \\ r \end{smallmatrix}, \begin{smallmatrix} 23 \\ r \end{smallmatrix} \right] = \text{ad-invariant}$$

$$\begin{smallmatrix} 12 \\ r \end{smallmatrix} \equiv r_{ij} T^i \otimes T^j \otimes 1 \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}), \quad \text{etc.}$$

where $\mathcal{U}(\mathcal{G})$ is the universal enveloping algebra. Classification of all one-parameter solutions to (34) for simple Lie algebras $\mathcal{G}$ is given in refs. [33].

A new KK-type Poisson bracket $\{\cdot, \cdot\}_R$, called the R -bracket, can be introduced on $\mathcal{G}_R^* \simeq \mathcal{G}^*$ with the help of (31) by substituting the usual Lie commutator $[\cdot, \cdot]$ for the R -Lie commutator $[\cdot, \cdot]_R$ in (30):

$$(35) \quad \{F, H\}_R(U) = - \left\langle U \left| \left[\nabla F(U), \nabla H(U) \right]_R \right. \right\rangle.$$

A function H on $\mathcal{G}^*$ is called Ad^* -invariant (Casimir) if $H[Ad^*(g)U] = H[U]$. Infinitesimally $ad^*(\nabla H(U))(U) = 0$ for each $U \in \mathcal{G}^*$. Then it can be shown that [15]:

1. All ad^* -invariant functions are in involution with respect to both brackets (30) and (35).
2. The Hamiltonian equation on $\mathcal{G}^* \simeq \mathcal{G}_R^*$ takes the following generalized Lax form:

$$(36) \quad \frac{dU}{dt} = \frac{1}{2} ad^* \left(R(\nabla H(U)) \right) U = \left[\frac{1}{2} R(\nabla H(U)), U \right]$$

where the second equality holds when $\mathcal{G}$ admits a Killing form. Eq. (36) can be obtained from a variational principle with the following geometric action:

$$(37) \quad \mathcal{W}[U] = - \int \left\langle U \left| \mathcal{Y}_R(U) \right. \right\rangle - \int dt H[U]$$

¹ $r = r_{ij} T^i \otimes T^j$ and $\frac{1}{2} RX = T^i r_{ij} (T^j, X)$, where $\{T^i\}$ denotes a basis in $\mathcal{G}$.

$$(38) \quad dU = ad_R^*(\mathcal{Y}_R(U))U \longrightarrow d\mathcal{Y}_R = \frac{1}{2} [\mathcal{Y}_R, \mathcal{Y}_R]_R$$

where the integrals in (37) are along a smooth curve in the phase space $\mathcal{G}^*$; $H[U]$ is a Casimir on $\mathcal{G}^*$; $\mathcal{Y}_R$ is the Maurer-Cartan one-form on $\mathcal{G}_R$. The dependence on $U \in \mathcal{G}^*$ of $\mathcal{Y}_R$ is determined from the first eq. (38) with the R -coadjoint action:

$$(39) \quad \begin{aligned} ad_R^*(X)U &= \frac{1}{2} \left(ad^*(RX)U + R^*(ad^*(X)U) \right) \\ &= \frac{1}{2} [RX, U] - \frac{1}{2} R([X, U]) \end{aligned}$$

where the second equality in (39) holds when $\mathcal{G}$ admits a Killing form. Hence the above AKS-RS technique leads to a construction of completely integrable systems in which the complete set of integrals of motion in involution coincides with the set of independent Casimir functions on $\mathcal{G}^*$.

A realization of this scheme arises when the Lie algebra $\mathcal{G}$ decomposes as a vector space into two subalgebras $\mathcal{G}_+$ and $\mathcal{G}_-$, i.e. $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$. Let $P_\pm$ be the corresponding projections on $\mathcal{G}_\pm$. Then $R = P_+ - P_-$ satisfies the modified YBE (32). In this case eqs. (31), (35), (36) and (39) take the following form:

$$(40) \quad \begin{aligned} [X, Y]_R &= [X_+, Y_+] - [X_-, Y_-] \\ (ad_R^*(X)U)_\pm &= \mp [X_\mp, U_\pm]_\pm \end{aligned}$$

$$(41) \quad \begin{aligned} \left\{ \left\langle U_\mp \middle| X_\pm \right\rangle, \left\langle U_\mp \middle| Y_\pm \right\rangle \right\} &= \pm \left\langle U_\mp \middle| [X_\pm, Y_\pm] \right\rangle \\ \frac{dU}{dt} + \left[\left(\frac{\delta H}{\delta U} \right)_+, U \right] &= 0 \end{aligned}$$

where the following notations have been used:

$$X_\pm = P_\pm X \in \mathcal{G}_\pm, \quad U_\pm = P_\mp^* U \in (\mathcal{G}_\mp)^*, \quad [X, U]_\pm = P_\mp^*([X, U]) \in (\mathcal{G}_\mp)^*.$$

3.2 Algebra of Pseudo-Differential Operators and Integrable Hierarchies of Kadomtsev-Petviashvili type

Here the AKS-RS construction will be illustrated for $\mathcal{G} = \Psi\mathcal{DO}$, the algebra of pseudo-differential operators on the circle. An arbitrary pseudo-differential operator $X(x, D_x) = \sum_{k \geq -\infty} X_k(x) D_x^k$ is conveniently represented by its symbol [34] which is a Laurent series in the variable ξ : $X(\xi, x) = \sum_{k \geq -\infty} X_k(x) \xi^k$. The operator multiplication corresponds to the following symbol multiplication:

$$(42) \quad X(\xi, x) \circ Y(\xi, x) = \sum_{N \geq 0} \frac{1}{N!} \frac{\partial^N X}{\partial \xi^N} \frac{\partial^N Y}{\partial x^N}$$

determining a Lie algebra structure given by: $[X, Y] \equiv (X \circ Y - Y \circ X)$.

An invariant, non-degenerate bilinear form can be introduced on $\Psi\mathcal{DO}$:

$$(43) \quad \langle L|X \rangle \equiv \text{Tr}_A(LX) = \int dx \text{Res}_\xi \left(L(\xi, x) \circ X(\xi, x) \right)$$

which allows identification of the dual space $\Psi\mathcal{DO}^*$ with $\Psi\mathcal{DO}$.

There exist three natural decompositions of $\mathcal{G} = \Psi\mathcal{DO}$ into a linear sum of two subalgebras $\mathcal{G} = \mathcal{G}_+^\ell \oplus \mathcal{G}_-^\ell$ labelled by the index ℓ taking values $\ell = 0, 1, 2$:

$$(44) \quad \begin{aligned} \mathcal{G}_+^\ell &= \left\{ X_+ \equiv X_{\geq \ell} = \sum_{i=\ell}^{\infty} X_i(x) D^i \right\}, \\ \mathcal{G}_-^\ell &= \left\{ X_- \equiv X_{< \ell} = \sum_{i=-\ell+1}^{\infty} X_{-i}(x) D^{-i} \right\}. \end{aligned}$$

Correspondingly the dual spaces to subalgebras $\mathcal{G}_\pm^\ell$ are given by:

$$(45) \quad \begin{aligned} \mathcal{G}_+^{\ell*} &= \left\{ L_- \equiv L_{< -\ell} = \sum_{i=\ell+1}^{\infty} D^{-i} \circ u_{-i}(x) \right\}, \\ \mathcal{G}_-^{\ell*} &= \left\{ L_+ \equiv L_{\geq -\ell} = \sum_{i=-\ell}^{\infty} D^i \circ u_i(x) \right\}. \end{aligned}$$

Note that in (45) the differential operators are to the left. Henceforth, we shall skip the sign $\circ$ in symbol products for brevity.

After defining $R_\ell = P_+ - P_-$ for each of the three cases, eqs. (40) take the form:

$$(46) \quad \begin{aligned} [X, Y]_{R_\ell} &= [X_{\geq \ell}, Y_{\geq \ell}] - [X_{< \ell}, Y_{< \ell}], \\ \text{ad}_{R_\ell}^*(X)L &= [X_{\geq \ell}, L_{< -\ell}]_{< -\ell} - [X_{< \ell}, L_{\geq -\ell}]_{\geq -\ell} \end{aligned}$$

Choosing the infinite set of independent Casimir functions in the form:

$$(47) \quad H_{m+1} = \frac{1}{m+1} \int dx \text{Res } L^{m+1}, \quad m = 0, 1, 2, \dots$$

the three decompositions (45) of $\Psi\mathcal{DO}$ labelled by $\ell = 0, 1, 2$ yield, according to the AKS-RS scheme, three different integrable hierarchies – the standard KP hierarchy ($\ell = 0$) and the first and second *modified* KP hierarchies ($\ell = 1, 2$):

$$(48) \quad \frac{\partial L}{\partial t_m} + \left[(L^m)_{\geq \ell}, L \right] = 0$$

The three KP hierarchies are related through symplectic gauge transformations [35]. Here we shall concentrate on various *Poisson reductions*² of the standard KP hierarchy which, in particular, appear in the context of matrix models of strings.

The phase space of the standard KP integrable system is:

$$(49) \quad \mathcal{M}_{KP} = \left\{ L = D + \sum_{k=1}^{\infty} u_k(x) D^{-k} \right\}.$$

It is a trivial Poisson reduction from $\mathcal{M} = \Psi\mathcal{DO}^* \simeq \Psi\mathcal{DO}$. The KK Poisson brackets (first eq. (41)) read on (49),

$$(50) \quad \begin{aligned} \{u_n(x), u_m(y)\} &= \Omega_{n-1, m-1}(u(x)) \delta(x-y), \\ \Omega_{nm}(u(x)) &= \sum_{k=0}^n (-1)^k \binom{n}{k} u_{n+m-k+1}(x) D_x^k - \sum_{k=0}^m \binom{m}{k} D_x^k u_{n+m-k+1}(x) \end{aligned}$$

and are recognized as the centerless $\mathbf{W}_{1+\infty}$ algebra [9], which is isomorphic to the algebra of differential operators on the circle $\mathcal{DO}\mathcal{P}(S^1) \subset \Psi\mathcal{DO}$ [37].

Let us go back to the flow eqs. in (multi)matrix models (24) with Lax operator (27). One can show [38] that the space of $2q$ -boson Lax operators

$$(51) \quad \mathcal{M}_q = \left\{ L_q = D + \sum_{l=1}^q A_l (D - B_l)^{-1} (D - B_{l+1})^{-1} \dots (D - B_q)^{-1} \right\}$$

is a *Poisson reduction* of the full KP hierarchy given by (49). Therefore, the flow eqs. (24) are Hamiltonian and completely integrable. In [38] the Darboux canonical coordinate pairs $(a_r(x), b_r(x))$, $r = 1, \dots, q$ were found to be

$$(52) \quad \{a_r(x), b_s(y)\}_{P'} = -\delta_{rs} \partial_x \delta(x-y)$$

$$(53) \quad B_l = b_l + b_{l+1} + \dots + b_q, \quad A_q = a_q$$

$$A_{q-r}(a, b) = \sum_{n_r=r}^{q-1} \dots \sum_{n_2=2}^{n_3-1} \sum_{n_1=1}^{n_2-1} (\partial + b_{n_r} + \dots + b_{n_r-r+1}) \dots (\partial + b_{n_1}) a_{n_1}$$

By expressing the $2q$ -boson Lax operator (51) as a power series in D^{-1} it could be rewritten in the form (49):

²The general notions could be found in [36]. Let $(\mathcal{M}, P)$ be a smooth Poisson manifold with Poisson structure $P : T^*(\mathcal{M}) \rightarrow T(\mathcal{M})$ and let S be a smooth submanifold of $\mathcal{M}$ with an embedding $\mu : S \rightarrow \mathcal{M}$. A Poisson structure $P' : T^*(S) \rightarrow T(S)$ on $S \subset \mathcal{M}$ is called *Poisson reduction* of P if for an arbitrary pair of functions on $\mathcal{M}$ the following property is satisfied: $\mu^* (\{f_1, f_2\}_P) = \{\mu^* f_1, \mu^* f_2\}_{P'}$.

$$(54) \quad L_q = D + \sum_{k=1}^{\infty} U_k[(a, b)](x) D^{-k}$$

$$(55) \quad U_k[(a, b)](x) = a_q P_{k-1}^{(1)}(b_q) + \sum_{r=1}^{\min(q-1, k-1)} A_{q-r}(a, b) P_{k-1-r}^{(r+1)}(b_q, b_q + b_{q-1}, \dots, \sum_{l=q-r}^q b_l)$$

where $A_{q-r}(a, b)$ are the same as in (53), and $P_n^{(N)}$ denote the multiple Faà di Bruno polynomials

$$(56) \quad P_n^{(N)}(B_N, \dots, B_1) = \sum_{m_1 + \dots + m_N = n} (-\partial + B_1)^{m_1} \dots (-\partial + B_N)^{m_N} \cdot 1$$

Thus, the Poisson reduction described above leads to the construction of a series of $2q$ -boson representations, $q = 1, 2, \dots$, of $\mathbf{W}_{1+\infty}$ algebra in terms of Poisson brackets for the functions (55), as follows from (50):

$$(57) \quad \left\{ U_k[(a, b)](x), U_l[(a, b)](y) \right\} = \Omega_{k-1, l-1} \left(U[(a, b)] \right) \delta(x - y).$$

Until now only the first Hamiltonian structure for integrable systems in the AKS-RS scheme has been discussed. It is given by the KK bracket (35) on $\mathcal{G}_R^*$.

Completely integrable systems possess another remarkable feature – that of *multi-Hamiltonian* structure, i.e., they always possess at least two independent *compatible* Poisson structures³.

The second Hamiltonian structure for general Lax operators (49) has the form⁴:

$$(58) \quad \begin{aligned} \{ \langle L | X \rangle, \langle L | Y \rangle \} &= \text{Tr}_A \left((LX)_+ LY - (XL)_+ YL \right) \\ &+ \int dx \text{Res}([L, X]) \partial^{-1} \text{Res}([L, Y]) \end{aligned}$$

Here Tr_A denotes the Adler trace (43) and the subscript $+$ indicates taking the purely differential part. For the coefficient fields $u_k(x)$ of L (49) the second KP Poisson algebra (58) yields the *nonlinear* (i.e., non-Lie) $\widehat{\mathbf{W}}_{\infty}$ algebra [42]. The latter appears as a unique (modulo certain homogeneity assumptions) nonlinear deformation of $\mathbf{W}_{1+\infty}$ algebra. The Darboux variables for (58) for the operators L_q (51) can also be constructed (see next subsection). The second KP Hamiltonian structure can be understood as a *Lie-Poisson* structure on the Lie-Poisson group of purely integral operators – the Volterra group.

³Two Poisson structures $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are called *compatible* if their linear combination $\{\cdot, \cdot\}_{\lambda} = \{\cdot, \cdot\}_1 + \lambda \{\cdot, \cdot\}_2$ is also a Poisson structure.

⁴The second term on the r.h.s. of (58) is a Dirac bracket term due to the second class constraint $u_0 = 0$ in L , eq. (49).

3.3 Lie-Poisson Groups and Lie Bi-Algebras: Hamiltonian Approach

The notion of a Lie-Poisson group was introduced by Drinfeld [16]. A Lie group G is called *Lie-Poisson* if there exists a Poisson structure on the algebra of smooth functions on it $Fun(G)$ which is compatible with the group multiplication, i.e., $\Delta(\{F_1 F_2\}) = \{\Delta(F_1) \Delta(F_2)\}$, where Δ denotes the coproduct in $Fun(G)$ ($\Delta F(g, h) = F(gh)$). A Poisson structure with this property is called Lie-Poisson and is given by:

$$(59) \quad \begin{aligned} \{F_1(g), F_2(g)\} &= \left\langle \nabla_L F_1(g) \otimes \nabla_L F_2(g) \middle| r(g) \right\rangle \\ &= - \left\langle \nabla_R F_1(g) \otimes \nabla_R F_2(g) \middle| r(g^{-1}) \right\rangle \end{aligned}$$

$$(60) \quad \left\langle \nabla_L F(g) \middle| X \right\rangle = \frac{d}{dt} \bigg|_{t=0} F(e^{tX} g), \quad \left\langle \nabla_R F(g) \middle| X \right\rangle = \frac{d}{dt} \bigg|_{t=0} F(g e^{tX})$$

where $\nabla_{L,R}$ denote left, right Lie-derivatives, and $r(g)$ is a *cocycle* on G with values in $\mathcal{G} \otimes \mathcal{G}$:

$$(61) \quad r(gh) = r(g) + Ad(g) \otimes Ad(g) r(h)$$

In the case when $r(g)$ is a *coboundary*:

$$(62) \quad r(g) = Ad(g) \otimes Ad(g) r_0 - r_0$$

with $r_0 \in \mathcal{G} \otimes \mathcal{G}$ being a constant element, one finds that the Jacobi identity for (59) reduces precisely to the YBE (34) for r_0 as a classical r -matrix.

For matrix groups eq. (59) can be written in a simpler form:

$$(63) \quad \{g \circledast g\} = r(g) g \otimes g \quad \left(\{g \circledast g\} = -[r_0, g \otimes g] \text{ in the case of (62)} \right)$$

The cocycle condition (61) implies the following exterior derivative equation:

$$(64) \quad dr(g) = [Y(g) \otimes 1 + 1 \otimes Y(g), r(g)] - \phi(Y(g)),$$

$$(65) \quad \phi(X) = - \frac{d}{dt} \bigg|_{t=0} r(e^{Xt}) \quad \text{for } \forall X \in \mathcal{G}$$

where $Y(g) = dg g^{-1}$ denotes the Maurer-Cartan one-form on $\mathcal{G}$, and ϕ defined in (65) is a $\mathcal{G} \otimes \mathcal{G}$ -valued cocycle on $\mathcal{G}$:

$$(66) \quad \phi([X, Y]) = [X \otimes 1 + 1 \otimes X, \phi(Y)] + [\phi(X), Y \otimes 1 + 1 \otimes Y]$$

The solution of (64) is a coboundary $r(g)$ (62) if and only if ϕ is a coboundary:

$$(67) \quad \phi(X) = [r_0, X \otimes 1 + 1 \otimes X]$$

The cocycle ϕ allows to introduce a Lie-commutator $[\cdot, \cdot]_*$ on the dual space $\mathcal{G}^*$ and, correspondingly, coadjoint action $ad_*^*(\cdot)$ of $\mathcal{G}^*$ on $\mathcal{G}$ as follows:

$$(68) \quad \left\langle [U, V]_* \middle| X \right\rangle \equiv - \left\langle V \middle| ad_*^*(U)X \right\rangle = \left\langle U \otimes V \middle| \phi(X) \right\rangle, \quad U, V \in \mathcal{G}^*, \quad X \in \mathcal{G}$$

whereas the mixed commutator between elements of $\mathcal{G}$ and $\mathcal{G}^*$ is defined as:

$$(69) \quad [X, U] = ad^*(X)U - ad_*^*(U)X$$

An important theorem by Drinfeld [16] states that the Lie algebra $\mathcal{G}$ of each Lie-Poisson group G is a Lie-bialgebra and *vice versa*. Thus, by means of (68) and (69) the direct sum (as vector space) $\mathcal{D} = \mathcal{G} \oplus \mathcal{G}^*$ becomes itself a Lie algebra called the *double*, such that $\mathcal{G}$ and $\mathcal{G}^*$ are isotropic subalgebras of $\mathcal{D}$ w.r.t. the Killing form on $\mathcal{D}$:

$$(70) \quad \left((X_1, U_1), (X_2, U_2) \right) = \langle U_1 | X_2 \rangle + \langle U_2 | X_1 \rangle \quad \forall (X_{1,2}, U_{1,2}) \in \mathcal{D}$$

The triple $(\mathcal{D}, \mathcal{G}, \mathcal{G}^*)$ is also called *Manin triple*. The *double group* $\widetilde{D} \simeq G \times G^*$, corresponding to $\mathcal{D}$, is a direct product (as a manifold) of its subgroups G and G^* corresponding to $\mathcal{G}$ and $\mathcal{G}^*$, respectively. The above construction is symmetric under the replacements $\mathcal{G} \longleftrightarrow \mathcal{G}^*$, $G \longleftrightarrow G^*$.

The explicit solution for the Lie-Poisson group cocycle $r(g)$ (61) reads:

$$(71) \quad \langle U \otimes V | r(g) \rangle = \left\langle \left(g^{-1} V g \right)_- \middle| \left(g^{-1} U g \right)_+ \right\rangle \quad \forall U, V \in \mathcal{G}^*$$

where the subscripts $(\pm)$ indicate projections in the double algebra $\mathcal{D}$ along $\mathcal{G}$, $\mathcal{G}^*$ respectively.

As in the AKS-RS Lie-algebraic scheme, one can construct integrable Hamiltonian systems on Lie-Poisson groups. Indeed, from (61) or (62) one can easily verify that all $Ad(\cdot)$ -invariant functions on G , $H[hgh^{-1}] = H[g]$, are in involution w.r.t. the Lie-Poisson structure (59): $\{H_k[g], H_l[g]\} = 0$. Similarly, the analogues of the Hamiltonian eqs. of motion (36) and the associated geometric action (37) take the form:

$$(72) \quad \frac{\partial g}{\partial t_k} g^{-1} = -\hat{r}_g(\nabla_L H_k[g])$$

$$(73) \quad \mathcal{W}[g] = -\frac{1}{2} \int d^{-1} \left(\left\langle \hat{s}_g(Y(g)) \middle| Y(g) \right\rangle \right) - \int dt H_k[g]$$

where the action of the operator $\hat{r}_g : \mathcal{G}^* \rightarrow \mathcal{G}$ is defined by $\langle U | \hat{r}_g(V) \rangle \equiv \langle U \otimes V | r(g) \rangle$, cf. (71), $\hat{s}_g = \hat{r}_g^{-1}$, and d^{-1} denotes the inverse of the exterior derivative defined on the closed forms⁵.

An example of this construction, worked out in [39], is provided by the extended Volterra algebra of purely pseudo-differential operators (c is an arbitrary constant):

$$(74) \quad \mathcal{G} \equiv \left(\widetilde{\Psi \mathcal{DO}} \right)_- = \left\{ \sum_{k \geq 1} u_k(x) D^{-k} + c \ln D \right\}$$

Its dual is the extended algebra of differential operators (α an arbitrary constant):

$$(75) \quad \mathcal{G}^* \equiv \widetilde{\mathcal{DO}} \simeq \mathbf{W}_{1+\infty} = \left\{ \sum_{l \geq 0} D^l v_l(x) + \alpha \hat{E} \right\},$$

and the Lie-double is the extended algebra of all pseudo-differential operators:

$$(76) \quad \mathcal{D} \equiv \widetilde{\Psi \mathcal{DO}} = \left(\widetilde{\Psi \mathcal{DO}} \right)_- \oplus \widetilde{\mathcal{DO}}$$

Here $\hat{E}$ indicates the central element of $\mathbf{W}_{1+\infty}$, as well as of the whole $\widetilde{\Psi \mathcal{DO}}$, which is dual to $\ln D$, cf. [37, 39]. The corresponding extended Volterra group (exponentiation of (74)):

$$(77) \quad G \equiv \left(\widetilde{\Psi \mathcal{DO}} \right)_- = \left\{ g \equiv L = \left(1 + \sum_{k \geq 1} \tilde{u}_k(x) D^{-k} \right) \circ D^c \right\}$$

can be viewed as a set of spaces (for each fixed c) of Lax operators of generalized KP hierarchies. (The KP hierarchy (48) is recovered for $c = 1$). In [39] it was shown that the second Hamiltonian structure (58) coincides with the Lie-Poisson structure (63) with the cocycle $r(g)$ (71) for the group $G = \left(\widetilde{\Psi \mathcal{DO}} \right)_-$ (77).

Let us go back to the example of $2q$ -boson KP Lax operators appearing in the multi-matrix string models (51). By analogy with eqs. (53)–(52) one can express [41] the coefficient fields $(A_l, B_l)_{l=1}^q$ of L_q (51) in terms of Darboux canonical pairs of fields $(c_r, e_r)_{r=1}^q$ w.r.t. the second KP Hamiltonian structure (58):

⁵Closure follows from the exterior derivative equation, $d\hat{s}_g = ad^*(Y(g))\hat{s}_g - \hat{s}_g ad(Y(g)) - \hat{s}_g \phi(Y(g))\hat{s}_g$, which is satisfied by the inverse cocycle operator $\hat{s}_g$ as a consequence of eq. (64).

$$(78) \quad \{c_k(x), e_l(y)\} = -\delta_{kl} \partial_x \delta(x-y), \quad k, l = 1, 2, \dots, q$$

$$(79) \quad B_k = e_k + \sum_{l=k}^q c_l, \quad 1 \leq k \leq q; \quad A_q = \sum_{r=1}^q (\partial + c_r) e_r$$

$$(80) \quad A_k = \sum_{n_k=1}^k \left(\partial + e_{n_k} - e_{n_k+q-k} + \sum_{l_k=n_k}^{n_k+q-k} c_{l_k} \right) \\ \times \sum_{n_{k-1}=1}^{n_k} \left(\partial + e_{n_{k-1}} - e_{n_{k-1}+q-1-k} + \sum_{l_{k-1}=n_{k-1}}^{n_{k-1}+q-1-k} c_{l_{k-1}} \right) \times \dots \\ \times \sum_{n_2=1}^{n_3} (\partial + e_{n_2} - e_{n_2+1} + c_{n_2} + c_{n_2+1}) \sum_{n_1=1}^{n_2} (\partial + c_{n_1}) e_{n_1}, \quad 1 \leq k \leq q-1.$$

These equations are equivalent to the following "dressing" form for the $2q$ -boson KP Lax operator (51):

$$(81) \quad L_q = \mathcal{U}_q \dots \mathcal{U}_1 D \mathcal{V}_1^{-1} \dots \mathcal{V}_q^{-1}$$

$$(82) \quad \mathcal{U}_k \equiv (D - e_k) e^{\int c_k}, \quad \mathcal{V}_k \equiv e^{\int c_k} (D - e_k), \quad k = 1, \dots, q$$

Eqs. (78)–(80) or, equivalently, eqs. (81)–(82) can be viewed as *generalized Miura transformations* for the $2q$ -boson KP hierarchy⁶. The Miura form of L_q (81) reads explicitly:

$$(83) \quad L_q = D + \sum_{k=1}^{\infty} U_k[(c, e)](x) D^{-k}$$

$$(84) \quad U_k[(c, e)](x) = P_{k-1}^{(1)}(e_q + c_q) \sum_{l=1}^q (\partial + c_l) e_l + \\ \sum_{r=1}^{\min(q-1, k-1)} A_{q-r}(c, e) P_{k-1-r}^{(r+1)}(e_q + c_q, e_{q-1} + c_{q-1} + c_q, \dots, e_{q-r} + \sum_{l=q-r}^q c_l)$$

where $A_{q-r}(c, e)$ are the same as in (80), and $P_n^{(N)}$ are the (multiple) Faà di Bruno polynomials (cf. (56)).

Now, in complete analogy with eqs. (54)–(57), which yield a series of realizations of the linear $\mathbf{W}_{1+\infty}$ algebra in terms of $2q$ bosons, one obtains, after substitution of (83)–(84) into (58), a series of explicit Poisson bracket realizations of the nonlinear $\widehat{\mathbf{W}}_{\infty}$ algebra in terms of $2q$ bosonic fields for any $q = 1, 2, \dots$. This algebra plays an important rôle as a "hidden" symmetry algebra in string-theory-inspired models with black hole solutions [42].

Concluding this section, let us note that in the general case Lie-Poisson groups provide a natural geometric description of the *dressing symmetries* in completely integrable models [43, 44]. The Lie-Poisson structures (63) also appear in the context of the classical inverse scattering method [5] as fundamental *Sklyanin brackets* for the monodromy matrix $g \simeq T(\lambda)$ of the auxiliary linear spectral problem.

⁶For discussion of the generalized Miura transformation and the associated Kuperschmidt-Wilson theorem, we refer to [40].

4 Quantum Integrable Models

4.1 Quantization of Lie-Poisson Groups: Quantum Groups

The quantization of completely integrable Lie-Poisson models with a Hamiltonian structure led historically to the first explicit construction, by the quantum scattering method, [11], of *quantum groups*. These were subsequently identified with *quasi-triangular Hopf algebras* [13].

Among the various ways of introducing quantum groups there exists one, [12], whose conceptual point of view underscores both the quantum mechanical as well as the Hopf algebraic aspects in the quantization of Lie-Poisson groups. On the one hand, $Fun(G)$ can be viewed as an Abelian associative algebra of “observables” of a classical Hamiltonian system $(\mathcal{M}, \mathcal{P})$ with a phase space $\mathcal{M} = G$ and Poisson structure which is Lie-Poisson, i.e., $\mathcal{P} = \mathcal{P}_{LP}$ given by (59):

$$(85) \quad \mathcal{P}_{LP}(F_1, F_2) \equiv \{F_1, F_2\} = \langle \nabla_L F_1 \otimes \nabla_L F_2 - \nabla_R F_1 \otimes \nabla_R F_2 | r_0 \rangle,$$

where the classical r -matrix r_0 satisfies the classical YBE (34). From now on we shall consider only coboundary Lie-Poisson groups (62). On the other hand, one can check that $Fun(G)$ is endowed with a structure of a commutative and non-cocommutative Hopf algebra $\mathcal{A}_0(m, \Delta, S, \varepsilon) \equiv Fun(G)$ with a product $m(F_1, F_2)(g) = F_1(g)F_2(g)$, coproduct $\Delta(F)(g_1, g_2) = F(g_1 g_2)$, antipode $SF(g) = F(g^{-1})$ and co-unit $\varepsilon(F) = F(e)$, and this Hopf structure is compatible with the Poisson structure (85), i.e., $\Delta \circ \mathcal{P} = \mathcal{P} \circ \Delta$.

Thus, the quantization of a Lie-Poisson group G may be viewed as a generalization of Weyl quantization: $Fun(G) \longrightarrow Fun_h(G)$ of a classical Hamiltonian system defined by $(\mathcal{M} \equiv G, \mathcal{P}_{LP})$, i.e., it is a *non-commutative* deformation of the product $m \longrightarrow m_h$ with a deformation parameter $\hbar$, which satisfies the additional condition that the deformed algebra $Fun_h(G) \equiv \mathcal{A}_h(m_h, \Delta, S, \varepsilon)$ is a non-commutative and non-cocommutative Hopf algebra.

Let us recall [17] that Weyl quantization for a Hamiltonian system defined on a Poisson manifold $(\mathcal{M}, \mathcal{P})$ with local coordinates (x^i) and constant Poisson tensor $\{x^i, x^j\} = P^{ij}$ is given by the associative and non-commutative *Moyal* product:

$$(86) \quad \begin{aligned} m_h(F_1, F_2) &\equiv F_1 \star_h F_2 = m \circ e^{\frac{\hbar}{2}\mathcal{P}}(F_1, F_2) \\ &= F_1 \cdot F_2 + \frac{\hbar}{2} \{F_1, F_2\} + O(\hbar^2) \end{aligned}$$

$$\mathcal{P} = P^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} : Fun(\mathcal{M}) \otimes Fun(\mathcal{M}) \longrightarrow Fun(\mathcal{M}) \otimes Fun(\mathcal{M})$$

Here $\mathcal{P}$ is the Poisson bi-vector field. Note, that the form of the first order term in the $\hbar$ -expansion (last eq. (86)) is dictated by the *semiclassical correspondence principle*.

For Lie-Poisson groups $(G, \mathcal{P}_{LP})$, the deformed product m_h which defines the deformed Hopf algebra structure and satisfies the semiclassical condition can be constructed as follows [12]. Let us choose a basis $\{X^i\}$ in $\mathcal{U}(\mathcal{G})$ – the universal enveloping algebra of the Lie algebra $\mathcal{G}$ of G , and let $\pi_{L,R}$ denote the representations of $\mathcal{U}(\mathcal{G})$ in terms of left, right Lie derivatives $\pi_{L,R}(X^i) = \nabla_{L,R}^i$ (see eq. (60)). Then:

$$(87) \quad m_h = m \circ \tilde{\Lambda}, \quad \tilde{\Lambda} = (\pi_L \otimes \pi_L) (\Lambda) \circ (\pi_R \otimes \pi_R) (\Lambda^{-1})$$

$$(88) \quad \begin{aligned} \Lambda(X, Y) &= \sum_{\{\alpha\}, \{\beta\}} c_{\{\alpha\}, \{\beta\}}(h) \prod_{i=1}^{\dim \mathcal{G}} (X^i)^{\alpha_i} \prod_{j=1}^{\dim \mathcal{G}} (Y^j)^{\beta_j} \\ &= 1 + \frac{h}{2} r_{ij} X^i Y^j + O(h^2) \end{aligned}$$

with the following notations. The coefficients in $\Lambda : \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \rightarrow \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})[[h]]$ are power series in h ; $\{X^i\}$ and $\{Y^j\}$ are generator bases in the first and second copy of $\mathcal{U}(\mathcal{G})$ respectively; $\|r_{ij}\| = r_0$ is just the classical r -matrix satisfying (34). The associativity condition for m_h (87) implies the following quadratic equation for Λ on $\mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})[[h]]$ (X, Y, Z below correspond to the first, second and third factor $\mathcal{U}(\mathcal{G})$ in the tensor product):

$$(89) \quad \begin{aligned} \Lambda(X + Y, Z) \Lambda(X, Y) &= \Lambda(X, Y + Z) \Lambda(Y, Z) \\ \Lambda(X, 0) &= \Lambda(0, Y) = 1 \end{aligned}$$

Defining $\bar{R}(X, Y) = \Lambda^{-1}(Y, X) \Lambda(X, Y) \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})[[h]]$, one obtains from (89):

$$(90) \quad \begin{aligned} \bar{R}(X, Y) \bar{R}(X, Z) \bar{R}(Y, Z) &= \bar{R}(Y, Z) \bar{R}(X, Z) \bar{R}(X, Y) \\ \bar{R}(X, Y) \bar{R}(Y, X) &= 1; \quad \bar{R}(X, Y) = 1 + h r_{ij} X^i Y^j + O(h^2). \end{aligned}$$

$\bar{R}$ is called *universal quantum R -matrix* associated with the classical r -matrix r_0 . If $\rho : \mathcal{G} \rightarrow \text{End}(\mathcal{V})$ is some representation of $\mathcal{G}$ in a finite-dimensional vector space $\mathcal{V}$, then the matrix $R = (\rho \otimes \rho)(\bar{R}) \in \text{End}(\mathcal{V} \otimes \mathcal{V})$ satisfies the *quantum Yang-Baxter equation* (QYBE) and the "unitarity" condition, ($P \in \text{End}(\mathcal{V} \otimes \mathcal{V})$ being the permutation operator $PX \otimes Y = Y \otimes X$):

$$(91) \quad \begin{matrix} 12 & 13 & 23 & 23 & 13 & 12 \\ R & R & R & R & R & R \end{matrix}; \quad RPRP = 1; \quad R = 1 + h r_0 + O(h^2)$$

The indices 12, 13, 23 indicate the various embeddings of $R \in \text{End}(\mathcal{V} \otimes \mathcal{V})$ in $\text{End}(\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})$.

In particular, when G is a matrix group and the functions F_i are chosen to be the matrix elements of $g \in G$, $F_1(g) = g_{ab}$, $F_2(g) = g_{cd}$, one obtains from the deformed product (87) and using matrix tensor notations:

$$(92) \quad R \overset{1}{g} \overset{2}{g} \overset{2}{g} \overset{1}{g} R, \quad \overset{1}{g} = (g \otimes 1), \quad \overset{2}{g} = (1 \otimes g).$$

The semiclassical limit of (92) is given precisely by the Lie-Poisson bracket (63). Eq. (92) is nothing but the fundamental commutation relations for the matrix elements of the quantum monodromy matrix (the dependence on the spectral parameter has been suppressed) in the quantum inverse scattering method [11].

Various treatments and numerous applications of QYBE could be found in ref. [45]. For an incomplete list of parallel developments in abstract Hopf algebra approach, the reader is referred to [46].

4.2 Soliton Scattering in Completely Integrable Models

Let us consider a completely integrable $D = 2$ relativistic field theory defined by the action $S[\phi] = \int d^2x \mathcal{L}(\phi, \partial\phi)$ which is assumed to be a local functional of the fundamental fields (collectively denoted by ϕ) and their derivatives. It is useful to introduce the light-cone coordinates: $x^\pm = \frac{1}{2}(x^1 \pm x^0)$. Complete integrability implies the existence of an infinite number of independent integrals of motion in involution $Q^{(s)}$, whose densities are local (as functionals of ϕ and its derivatives) conserved currents:

$$(93) \quad Q^{(s)} = \oint \left(T^{(s+1)} dx^- + \Theta^{(s-1)} dx^+ \right), \quad s = 1, 2, \dots;$$

$$\partial_+ T^{(s+1)} = \partial_- \Theta^{(s-1)}$$

here s indicates the $D = 2$ Lorentz weight. Thus, if the property of complete integrability survives after quantization, it should imply, in particular, that the quantum *renormalized* Ward identities for the *renormalized* quantum conserved currents $(\tilde{T}^{(s+1)}, \tilde{\Theta}^{(s-1)})$ should be satisfied:

$$(94) \quad \partial_+ \langle \tilde{T}^{(s+1)}(x) \phi(x_1) \dots \phi(x_n) \rangle - \partial_- \langle \tilde{\Theta}^{(s-1)}(x) \phi(x_1) \dots \phi(x_n) \rangle = \delta - \text{function terms}$$

Here $\langle \dots \rangle$ denote the time-ordered correlation functions. The infinite set of Ward identities leads to severe restrictions on the particle ("soliton") scattering processes – conservation of all (odd) powers of momenta of incoming and outgoing particles:

$$(95) \quad \sum_{l=1}^{N_{\text{in}}} p_{l(\text{in})}^{2n+1} = \sum_{l=1}^{N_{\text{out}}} p_{l(\text{out})}^{2n+1}$$

with $N_{\text{in}}, N_{\text{out}}$ denoting the number of incoming, outgoing particles which, in turn, implies [47]:

1. No multi-particle production, i.e., $N_{\text{in}} = N_{\text{out}}$.
2. Factorization of multi-particle scattering amplitudes.

The last property leads to the remarkable Zamolodchikov's factorization eqs. for the 3-particle amplitudes [6], meaning that any 3-particle scattering process is accomplished as a sequence of 2-particle scatterings and, moreover, the amplitude does not depend on the order in which these 2-particle scatterings occur:

$$(96) \quad S_{i_1 i_2}^{k_1 k_2}(\theta_{12}) S_{k_1 i_3}^{j_1 k_3}(\theta_{13}) S_{k_2 k_3}^{j_2 j_3}(\theta_{23}) = S_{i_2 i_3}^{k_2 k_3}(\theta_{23}) S_{i_1 k_3}^{k_1 j_3}(\theta_{13}) S_{k_1 k_2}^{j_1 j_2}(\theta_{12})$$

with $\theta_{ab} \equiv \theta_a - \theta_b$, $a, b = 1, 2, 3$. In (96) the following notations are used: $S_{ij}^{kl}(\theta_{12})$ denotes the 2-particle scattering amplitude of incoming particles of "type" labelled by i and j into outgoing particles of type k and l and (on-mass-shell) momenta $p_a = m_a (\cosh \theta_a, \sinh \theta_a)$, $a = 1, 2$ (θ_a is the relativistic rapidity).

Now, denoting by $\mathcal{V}$ the vector space of internal particle symmetry (particle "types"), one can regard the matrix of the 2-particle amplitude as:

$$(97) \quad \begin{aligned} S(\theta_{12}) &= \| S_{ij}^{kl}(\theta_{12}) \| \in \text{Mat}(\mathcal{V}) \otimes \text{Mat}(\mathcal{V}) \quad (\text{for fixed } \theta_{12}) \\ \overset{12}{S}(\theta_{12}) &\equiv S(\theta_{12}) \otimes 1 \in \text{Mat}(\mathcal{V}) \otimes \text{Mat}(\mathcal{V}) \otimes \text{Mat}(\mathcal{V}) \end{aligned}$$

and similarly for $\overset{13}{S}(\theta_{13})$ and $\overset{23}{S}(\theta_{23})$. Then it is straightforward to identify (96) with the QYBE (91) in the quantum group framework.

A realization of the matrix quantum group relation (92) is provided by the symmetries of the soliton scattering states [49]. Let $T_{ij}(\theta)$ be an internal symmetry operator acting on the 1-particle asymptotic space: $|(\theta, i)\rangle \longrightarrow T_{ij}(\theta)|(\theta, j)\rangle$, $|(\theta, i)\rangle \in \mathcal{V}$ and $\|T_{ij}(\theta)\| \in \text{Mat}(\mathcal{V})$.

As a result of the integrability the 2-particle S -matrix

$$(98) \quad |(\theta_1, i_1), (\theta_2, i_2)\rangle^{\text{in}} = S_{i_1 i_2}^{j_1 j_2}(\theta_{12}) |(\theta_2, j_2), (\theta_1, j_1)\rangle^{\text{out}}$$

can be viewed (for fixed "rapidities") as a mapping $\overset{12}{S}: \overset{1}{\mathcal{V}} \otimes \overset{2}{\mathcal{V}} \longrightarrow \overset{2}{\mathcal{V}} \otimes \overset{1}{\mathcal{V}}$. Its invariance under the asymptotic states' symmetry on the subspace of 2-particle asymptotic states gives (in the notation of eqn. (92)):

$$(99) \quad \overset{12}{S}(\theta_{12}) \overset{1}{T}(\theta_1) \overset{2}{T}(\theta_2) = \overset{2}{T}(\theta_2) \overset{1}{T}(\theta_1) \overset{12}{S}(\theta_{12}).$$

This is straightforwardly identified with the structural relations (92) for quantum groups.

As pointed out in [50] quantum group relations of the form (96) and (99) appear in exactly solvable lattice models of planar statistical mechanics, but in this case – with purely imaginary "rapidities" ($\theta = i\alpha$, α being angle characterizing the rectangular lattice), $S_{ij}^{kl}(\alpha)$ being the matrix of Boltzmann weights at each lattice vertex, and $T_{ij}(\alpha)$ denoting the row transfer matrix.

4.3 Quantum Field Theory Approach to Integrable Models with Dynamically Broken Conformal Invariance

Finally, let us briefly discuss the construction of higher local quantum conserved currents satisfying the Ward identities (94), which is the heart of the quantum field theory approach to quantization of completely integrable models. Recently, Zamolodchikov [14] proposed a beautiful general formalism based on treating integrable models as *mass perturbations* of conformal field theories: $S[\phi] = S_{\text{conf}}[\phi] + \sum_i m_i \int d^2x B_i(\phi, \partial\phi)$, where the coupling constants m_i have positive mass dimensions and $B_i(\phi, \partial\phi)$ are composite fields with conformal dimensions less than 2. Zamolodchikov's approach is purely algebraic since in general it is not possible to explicitly find expressions for $S_{\text{conf}}[\phi]$ and $B_i(\phi, \partial\phi)$ as local functionals of the local fundamental fields $\{\phi\}$. It uses exact results from the representation theory of the Virasoro algebra, in particular, information about the spectrum of conformal field dimensions.

There exist, however, interesting classes of $D = 2$ integrable field theories, which are conformally invariant on the classical level but, upon quantization, undergo *dimensional transmutation*, manifested through dynamical *mass generation*. This leads to anomalous conformal symmetry breaking. An example is the $O(N)$ nonlinear sigma-model and its supersymmetric generalization defined by:

$$(100) \quad \mathcal{L}_{NL\sigma} = \frac{1}{2} \partial_+ n^a \partial_- n_a, \quad \vec{n}^2 = N/g, \quad \vec{n} = (n^1, \dots, n^N)$$

$$(101) \quad \mathcal{L}_{NL\sigma}^{\text{susy}} = \frac{1}{2} \partial_+ n^a \partial_- n_a + i \bar{\psi}^a \gamma^\mu \partial_\mu \psi_a - \frac{g}{N} (\bar{\psi}^a \psi_a)^2, \quad n^a \psi_a = 0$$

The dimensional transmutation phenomenon precludes the use of conformal perturbation approach to (100), (101). However, there is an alternative nonperturbative⁷ treatment of quantum field theory models with $O(N)$ or $SU(N)$ internal symmetry – the $1/N$ expansion [51].

Let us briefly illustrate the construction [52] of higher local quantum conserved currents for (100) [53] within the $1/N$ expansion framework (the same techniques applies to other $1/N$ -expandable integrable models as well). The $1/N$ expansion is obtained from the generating functional of the time-ordered correlation functions:

$$(102) \quad Z[J] = \int \mathcal{D}\vec{n} \prod_x \delta(\vec{n}^2 - N/g) \exp \left\{ i \int d^2x \left[\frac{1}{2} (\partial \vec{n})^2 + (\vec{J}, \vec{n}) \right] \right\} \\ = \int \mathcal{D}\sigma \exp \left\{ -\frac{N}{2} S_1[\sigma] + \frac{i}{2} \int d^2x d^2y (\vec{J}(x), (-\partial^2 + \sigma)^{-1} \vec{J}(y)) \right\}$$

$$(103) \quad S_1[\sigma] \equiv \text{Tr} \ln(-\partial^2 + \sigma) + \frac{i}{g} \int d^2x \sigma$$

by expanding the effective σ -field action (103) around its stationary point $\sigma_c \equiv m^2 = \mu^2 e^{-4\pi/g}$ (dynamically generated mass of the “Goldstone” field $\vec{n}$, μ being the renormalization scale), i.e., $\sigma(x) = m^2 + \frac{1}{\sqrt{N}} \tilde{\sigma}(x)$. As a result, one arrives at the $1/N$ diagram technique with (free) propagators in momentum space:

$$(104) \quad \langle n^a n^b \rangle_{(0)} = -i (m^2 + p^2)^{-1} \delta^{ab}, \quad \langle \tilde{\sigma} \tilde{\sigma} \rangle_{(0)} = \left(\Sigma(p^2) \right)^{-1} \\ \Sigma(p^2) = \int \frac{d^2k}{(2\pi)^2} \frac{1}{(m^2 + k^2)(m^2 + (p - k)^2)}$$

and trilinear $\tilde{\sigma}nn$ -vertices.

The $1/N$ expansion can be renormalized [54] by adapting the $BPHZ$ ⁸ renormalization technique. A remarkable property of the $1/N$ expansion is that the nonlinearity

⁷The term “nonperturbative” refers to expansions different from (or, e.g., partial resummations of) the ordinary perturbation theory w.r.t. the coupling constant g in (100) and (101) which is plagued by infrared divergences in $D = 2$.

⁸Bogoliubov-Parasiuk-Hepp-Zimmermann [55]. As shown in [54], the (supersymmetric) nonlinear sigma-models (100) and (101) are renormalizable within the $1/N$ expansion also in $D = 3$ space-time dimensions in spite of their naive nonrenormalizability w.r.t. the ordinary coupling constant perturbation theory.

of the “Goldstone” field $\vec{n}(x)$ is preserved on the quantum level as an identity on the correlation functions, in spite of the manifest linear $O(N)$ symmetry of (104):

$$(105) \quad \left\langle \mathcal{N} \left[\vec{n}^2 P(\vec{n}, \partial \vec{n}) \right] (x) \dots \right\rangle = \text{const} \left\langle \mathcal{N} \left[P(\vec{n}, \partial \vec{n}) \right] (x) \dots \right\rangle$$

where $P(\vec{n}, \partial \vec{n})$ is arbitrary local polynomial of the fundamental fields and their derivatives, and $\mathcal{N}[\dots]$ indicates renormalized normal product of the corresponding composite fields.

The first higher quantum conserved current (for $s = 3$ in the notations of (93), (94)) takes the following form:

$$(106) \quad \tilde{T}^{(4)} = \mathcal{N} \left[(\partial_-^2 \vec{n})^2 \right] + a_1 \mathcal{N} \left[((\partial_- \vec{n})^2)^2 \right]$$

$$(107) \quad \tilde{\Theta}^{(2)} = \left(\frac{1}{2} + a_2 \right) \mathcal{N} \left[(\partial_- \vec{n})^2 \sigma \right] + a_3 \partial_-^2 \sigma$$

where all coefficients $a_{1,2,3} = O(1/N)$ are expressed in terms of one-particle irreducible correlation functions and their derivatives in momentum space at zero external momenta. Their explicit form can be found order by order in $1/N$ from the renormalized $1/N$ -diagram technique described above [52].

Let us stress that the higher quantum conserved currents (106) and those for $s = 5, 7, \dots$ do not have analogues in the classical conformally invariant theory [56].

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QUANTUM GRAVITY: A MATHEMATICAL PHYSICS PERSPECTIVE

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1 Introduction

The problem of quantum gravity is an old one and over the course of time several distinct lines of thought have evolved. However, for several decades, there was very little communication between the two main communities in this area: particle physicists and gravitation theorists. Indeed, there was a lack of agreement on even what the key problems are. By and large, particle physics approaches focused on perturbative techniques. The space-time metric was split into two parts: $g_{\mu\nu} = \eta_{\mu\nu} + Gh_{\mu\nu}$, $\eta_{\mu\nu}$ being regarded as a flat kinematic piece, $h_{\mu\nu}$ being assigned the role of the dynamical variable and Newton's constant G playing the role of the coupling constant. The field $h_{\mu\nu}$ was then quantized on the $\eta_{\mu\nu}$ -background and perturbative techniques that had been so successful in quantum electrodynamics were applied to the Einstein-Hilbert action. The key problems then were those of handling the infinities. The gravity community, on the other hand, felt that a central lesson of general relativity is that the space-time metric plays a dual role: it is important that one and the same mathematical object determine geometry *and* encode the physical gravitational field. From this perspective, an ad-hoc split of the metric goes against the very spirit of the theory and must be avoided. If one does not carry out the split, however, a theory of quantum gravity would be simultaneously a theory of quantum geometry and the notion of quantum geometry raises a variety of conceptual difficulties. If there is no background space-time geometry—but only a probability amplitude for various possibilities—how does one do physics? What does causality mean? What is time? What does dynamics mean? Gravity theorists focused on such conceptual issues. To simplify mathematics, they often truncated the theory by imposing various symmetry conditions and thus avoided the field theoretic difficulties. Technically, the emphasis was on geometry rather than functional analysis. It is not that each community was completely unaware of the work of the other (although, by and large, neither had fully absorbed what the other side was saying). Rather, each side had its list of central problems and believed that once these issues were resolved, the remaining ones could be handled without much difficulty. To high energy theorists, the conceptual problems of relativists were perhaps analogous to the issues in foundations of quantum mechanics which they considered to be “unimportant for real physical predictions.” To relativists, the field theoretic difficulties of high energy physicists were technicalities which could be sorted out after

the conceptual issues had been resolved. This is of course a simplified picture. My aim is only to provide an impression of the general state of affairs.

Over the last decade, however, there has been a certain rapprochement of ideas on quantum gravity. Each side has become increasingly aware of the difficulties that were emphasized by the other. By and large, high energy theorists now agree that the non-perturbative techniques are critical. They see that the underlying diffeomorphism invariance should be respected. Even if one introduces a background structure for, e.g., regularization of operators, the final result should not make reference to such structures (unless of course they are physically important). Relativists have come to recognize that the field theoretic divergences have to be faced squarely. There is now a general agreement that although the truncated theories are interesting and provide insights into the conceptual (and certain mathematical) problems faced by the full theory, they are essentially toy models whose value, in the final analysis, is quite limited since they have only a finite number of degrees of freedom. Thus, the sets of goals of the two communities have moved closer.

These recognitions do not imply, however, that there is a general consensus on *how* all these problems are to be resolved. Thus, there are again many approaches. But this diversity in the lines of attack is very healthy. In a problem like quantum gravity, where directly relevant experimental data is scarce, it would be an error if everyone followed the same path. As Feynman (1965) put it:

It is very important that we do not all follow the same fashion... It's necessary to increase the amount of variety... and the only way to do is to implore you few guys to take a risk with your lives that you will not be heard of again, and go off in the wild blue yonder to see if you can figure it out.

And quite a few groups have taken the spirit of this advice seriously and “gone off in the yonder to figure it out.” What is striking is that, in spite of the diversity of their methods, some their results are qualitatively similar. One message that seems to keep coming back is that not only can one not assume a flat Minkowskian geometry at the Planck scale but in fact even the more general notions from Riemannian geometry would fail. The continuum picture itself is likely to break down. The lesson comes from certain computer simulations of 4-dimensional Euclidean gravity, (see e.g., Agishtein & Migdal (1992)) from string theory (see e.g., Gross & Mende (1988), Amati et al. (1990), Aspinwall (1993)) and from canonical quantization of 4-dimensional general relativity (Ashtekar et al. (1992)). The detailed pictures of the micro-structure of space-time that arises in these approaches is quite different at least at first sight and it is not clear that these pictures can be reconciled with one another in detail. Nonetheless, there *are* certain similarities in the results and most of them are obtained by using genuinely non-perturbative techniques.

The purpose of this article is to give a flavor of these ideas and techniques to mathematical physicists. I should emphasize that this is not a systematic survey. In particular, I will concentrate just on one approach—non-perturbative, canonical treatment of 4-dimensional, Lorentzian general relativity. Even within this approach, there are over 350 papers and I cannot do justice to them in this limited space. (For detailed reviews, see, e.g., Ashtekar (1991, 1992) and Pullin (1993).) Rather, I will just present

a few results that may be of interest to this audience, indicating, wherever possible, the degree of precision and rigor of the underlying calculations.

However, since the goal of the conference is to look to future, “towards the 21st century,” and since the organizers asked us to try to “inspire rather than merely inform,” I will begin in section 2 with a few remarks that may perhaps seem provocative to some mathematical physicists. In doing so, however, I am following the lead of other speakers and my hope is the same as theirs: These remarks may lead to stimulating exchanges of ideas and perhaps a re-examination of some of the basic premises. In section 3, the main difficulties of quantum gravity are outlined from the perspective of mathematical physics. Section 4 summarizes some recent developments. While general relativity is normally regarded as a theory of metrics, it can be recast as a dynamical theory of connections (Ashtekar, 1987). This shift of emphasis has two important consequences. First, it brings general relativity closer to theories of other interactions and one can draw on the numerous techniques that have been developed to quantize these theories. Second, the shift simplifies the basic equations considerably making them low order polynomials in the basic variables. (They are non-polynomial in the metric variables.) Recently, a number of rigorous results have been obtained to analyze theories of connections, particularly the ones such as general relativity in which there is an underlying diffeomorphism invariance. The main idea here is to use algebraic methods to develop an integration theory on the space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations and to explicitly construct a measure which is diffeomorphism invariant. These results now provide the foundation for a non-perturbative approach to quantum gravity based on Hamiltonian methods. They may also have other applications in mathematical physics. In section 5, this framework is used to show the existence of states in the full, non-perturbative quantum gravity which approximate a classical metric when coarse-grained at scales much larger than the Planck length *and* which exhibit a specific discrete structure at the Planck scale. If such a state is used in place of Minkowski space-time, the ultra-violet difficulties of Minkowskian field theories may disappear altogether. This is a concrete illustration of the results in non-perturbative quantum gravity that may have an impact on quantum field theory.

2 The Ultraviolet Catastrophe: A Matter of Gravity?

As we all know, for over 40 years, quantum field theory has been in a somewhat peculiar situation as far as realistic models are considered. On the one hand, perturbative treatments are available for, say, the electro-weak interaction and the results are in excellent agreement with experiments. It is clear therefore that there is something “essentially right” about these theories. On the other hand, their mathematical status has continued to be dubious and it has not been possible to say precisely what is right with them. And this is not because of lack of effort. Already by early sixties, quantum field theory had become an intellectually coherent subject (recall that *PCT spin and statistics and all that* was published in 1964). A considerable amount of imaginative—and often heroic—effort has gone into the field since then. And yet none of the physically realistic quantum field theories has reached a sound status in 4 space-time dimensions.

I think it is fair to say that the general attitude in the mathematical physics community is that the difficulties are of a mathematical nature. Realistic 4-dimensional theories are extremely involved; as Professor Wightman put it at the conference, “handling them with the present techniques is hellishly complicated.” So, the overall feeling seems to be that there is no obstacle of principle to construct a non-Abelian gauge theory such as QCD in 4-dimensions, but new mathematical tools are needed to make the task practicable.

One might worry about the fact that in all these theories, one uses Minkowski space as the underlying space-time, thereby ignoring all the Planck scale effects. Could this be a source of the difficulty? The general belief in the community seems to be that this is not the case; no new physical input from the Planck scale is needed to construct rigorously the quantum theory that underlies the standard model of particle physics. (This view was, for example, expressed in Professor Buchholtz’s talk.) Sure, the theory would be an approximation in that one would not be able to trust its predictions below, say 10^{-17} cm. But it would be internally consistent and agree with Nature as far as laboratory physics is concerned. Sure, we are idealizing the space-time geometry. But one does this all the time. Indeed, such idealizations always work: our general experience in physics tells us that, somehow, the phenomena at different scales decouple approximately. Indeed, very little progress could have occurred in absence of this decoupling. After all, when engineers build bridges, they don’t have to worry about the fact that it is quantum mechanics that governs the atoms of the bridge; they just use classical, Newtonian physics. They succeed because there is a factor of 10^{12} between the scale of the bridge and the atomic scale. There is a factor of 10^{16} between (say) the weak-interaction scale and the Planck length! Surely, one says, the Planck-size effects are unimportant to the problem of constructing a consistent (and therefore in particular, finite) quantum theory underlying the standard model.

I would like to argue that this is not necessarily the case. Recall, first, that the key difficulties of quantum field theory are the ultra-violet divergences. These arise *precisely* because we allow virtual processes involving arbitrary number of loops, each carrying an integral over arbitrarily large momenta. And it is not *our* choice to allow or disallow them: we are forced into it by the general principles of quantum field theory which include Poincaré invariance. Surely, as the energy involved becomes bigger, the approximation that one can ignore the gravitational effects becomes worse. If the microstructure of space-time is qualitatively different from that given by the continuum picture, the whole procedure is flawed; we shouldn’t—and couldn’t—integrate to arbitrarily high momenta since that is equivalent to integrating to arbitrarily small distances. Let us consider an analogy with atomic physics where we can successfully use non-relativistic quantum mechanics. Suppose for a moment that there was a general requirement that arose from the quantum principles which forced us, in the calculation of, say, the ground state energy of the hydrogen atom, to consider electrons with velocities arbitrarily close to the speed of light. *Then*, had we ignored special relativity altogether, we *would* probably have got an inconsistent theory: The hypothetical quantum principle would have *forced* us to bring special relativity into the treatment. The standard treatment of atomic physics in non-relativistic quantum mechanics is internally consistent *because* the calculation scheme does not involve steps which violate the basic premise and limitation of non-relativistic quantum mechanics. (It agrees well

with experiments because, in addition, these approximations are met in Nature.) Let us return to quantum field theory. In spite of the fact that we are interested here in processes in which physical energies (and masses of particles) are small compared to the Planck mass, we are forced to allow *virtual* processes involving arbitrarily high energies and these do probe the Planck-scale structure of the space-time geometry. But we insist on ignoring this structure altogether. *That*, it may well be, is the *physical* source of the ultraviolet catastrophe; it may be a matter of gravity.

Indeed, there are other instances in physics where mathematical difficulties signalled the need for changing the basic physical premise. Consider for instance the action at a distance models in classical, relativistic physics¹. The system of integro-differential equations one obtains is hard to manage mathematically and, if taken seriously, raises questions of predictability in relativistic physics. The “origin” of these difficulties lies in the physical inadequacy of the basic assumptions. Once we bring in the field degrees of freedom, these mathematical problems go away. Now, classical physics is described by a system of hyperbolic differential equations and causality is manifest. Another example is the birth of quantum mechanics itself. The mathematical description of the black-body radiation broke down in the framework of classical physics and pointed to the necessity of a radical revision of that framework. There are a number of other examples. Indeed, most radical changes of the conceptual framework are inspired, at least in part, by the fact that the older framework ran into serious mathematical difficulties. This does not of course imply that the same *must* happen with the ultra-violet divergences. These are only analogies. And there is no irrefutable evidence that more sophisticated mathematical techniques will not suffice to construct a consistent quantum field theory for the standard model. However, the examples suggest that we should be open to such a possibility.

I will conclude with a related but somewhat different point. Let me again use an analogy. Suppose, for a moment that special relativity had not been discovered. One might have learnt painfully that the predictions of Newtonian mechanics are not quite correct and have to be supplemented by powers of v/c where v is the velocity of the object under consideration. If we had been sufficiently clever, we would have discovered that to compute any physical effect, one can use a perturbation series in the powers of v/c . One could get sophisticated and worry about whether such series actually converge. This is perhaps similar to the present situation with the perturbation theory for the electro-weak interaction. In the hypothetical case of “special relativity”, proving convergence and other mathematical properties of the resulting series would be instructive. Like Lorentz and Poincaré, one could have even discovered the Lorentz transformations and found all sorts of equations which are “true.” However, without the *physical* shift of scenario that was provided to us by Einstein—that there is no absolute simultaneity—we would still be missing key insights. In a real sense, we would not really “understand” what the equations were telling us. The situation could be similar with the standard model. Suppose we do succeed in giving a mathematical meaning to the perturbative results. We will have a nice, consistent theory with convergence proofs. But it is possible that we may still be missing some key insights because we ignored the Planck scale physics. Indeed, it is not obvious that *all* effects of the quantum nature of geometry will be confined to the Planck scale. Let us take special relativity. It is true

¹This example was suggested to me by Jose Mourão.

that most effects are corrections in powers of v/c to the predictions of non-relativistic physics. But now and again, there are also *qualitative* predictions that have nothing to do with how large the velocity v of the particles involved is. Conversion of mass into enormous amount of energy happens in nuclear processes where all velocities are small. There is the prediction that associated with every particle there is an anti-particle. There is the CPT theorem. These are all qualitative effects completely unrelated to how fast the particles in question are moving. They come about because special relativity shifts the very paradigm within which one operates. The entire mathematical framework of quantum physics changes abruptly. The problems change. The tools change. The concepts change. There is a possibility that quantum field theory may undergo a similar radical change once the concepts from quantum gravity are brought in. Indeed, we will see in the subsequent sections that quantum gravity does make strong demands on how one should formulate and analyze problems. It insists on diffeomorphism invariance whence there is no background metric or causal structure; it trivializes the Hamiltonian and puts the burden of dynamics on the constraints of the theory; it introduces an essential non-locality through physical observables. The ground rules therefore seem quite different from those we are used to in Minkowskian quantum field theories.

Mathematical physicists can raise an immediate objection against the all these possibilities. After all, there do exist well-defined, consistent quantum field theories in 2 and 3 space-time dimensions. Why don't the Planck scale problems raise their annoying head there?² It turns out that there is a dramatic change in the properties of the gravitational field starting precisely at 4 dimensions! It acquires its *local* degrees of freedom only in 4 and higher dimensions. In dimension 3 or lower, there are no gravity waves and no gravitons. Therefore, the ultraviolet problems of field theories are, in a sense, decoupled from the quantum gravitational field. Indeed, one can turn the argument around and ask if it is a pure coincidence that the number of dimensions for which the ultra-violet problems of field theories seem so difficult to handle happens to be precisely the one at which gravity comes on its own. Or, is there a lesson lurking here that we have ignored?

I want to emphasize again that I do not regard the arguments given in this section as conclusive. It is a viable, logical possibility that a consistent quantum field theory incorporating the standard model will exist in 4-dimensions and will contain *all* the physics that is relevant to the scale of these interactions. Quantum gravity may have no effect whatsoever on this physics. However, I feel that this "mainstream" viewpoint in mathematical physics is also not watertight. There *are* differences between the current situation in field theory and previous examples in the history of physics where a clean decoupling occurred between physics at one scale and that at another. And, in the interest of variety, I believe it is important not to ignore altogether the possibility that the ultra-violet catastrophe may, in the end, be a matter of gravity.

3 Difficulties of Quantum Gravity

The importance of the problem of unification of general relativity and quantum theory was recognized as early as 1930's and a great deal of effort has been devoted to it in

²Indeed, Professor Chayes did ask this question during my talk!

the last three decades. Yet, we seem to be far from seeing the light at the end of the tunnel. Why is the problem so hard? What are the principal difficulties? Why can we not apply the quantization techniques that have been successful in theories of other interactions? In this section, I will address these questions *from the perspective of mathematical physics*.

The main difficulty, of course, is the lack of experimental data with a direct bearing on quantum gravity. One can argue that this need not be an unsurmountable obstacle. After all, one hardly had any experimental data with a direct bearing on general relativity when the theory was invented. Furthermore, the main motivation came from the incompatibility of Newtonian gravity with special relativity. We face a similar situation; we too are driven by what appears to be a fundamental tension between general relativity and quantum theory. However, it is also clear that the situation with discovery of general relativity is an anomaly rather than a rule. Most new physical theories—including quantum mechanics—arose and were continually guided and shaped by experimental input. In quantum gravity, we are trying to make a jump by some twenty orders of magnitude—from a Fermi to a Planck length. The hope that there is no dramatically new physics in the intermediate range is probably just that—a hope.

The experimental status, however, makes the situation even more puzzling. If there is hardly any experimental data, theorists should have a ball; without these “external, bothersome constraints”, they should be able to churn out a theory a week. Why then do we not have a *single* theory in spite of all this work? The brief answer, I think, is that it is *very* difficult to do quantum physics in absence of a background space-time. We have *very* little experience in constructing physically realistic, diffeomorphism invariant field theories. Indeed, until recently, there were just a handful of examples, obtained by truncating general relativity in various ways. It is only in the last three years or so that a significant number of diffeomorphism invariant models with an infinite number of degrees of freedom has become available, still, however, in low space-time dimensions.

As mentioned in the Introduction, one way out of this quandry—tried by the high energy physicists—was to simply break the diffeomorphism invariance to start with and introduce a flat background metric. As is well-known, however, the resulting perturbative quantum field theory is non-renormalizable. In the high energy community, this was considered a fatal flaw. At first, it was thought that the problem is with the starting point—general relativity. Therefore, attempts were made to modify Einstein’s theory. Perhaps the most notable of these modifications were the higher derivative theories and supergravity. However, these attempts at defining a local quantum field theory for gravity (with matter) which is consistent order by order in the perturbation expansion failed. Finally, these developments led, in the mid-eighties, to string theory. Since there were several talks on this subject in the conference, I will restrict myself just to a one line summary here: The perturbation series in string theory is believed to be finite order by order (in the string tension) but the series is believed not to be even Borel-summable. As a result, a great deal of effort is being devoted to constructing the theory non-perturbatively.

Returning to quantum general relativity, the failure of perturbation theory would, presumably, not upset the mathematical physicists a great deal. After all, they know that in spite of perturbative non-renormalizability, a quantum field theory *can* exist

non-perturbatively. This point was discussed in some detail in Professor Klauder's talk and Professor Wightman commented on it in the context of the Gross-Neveu model in 3-dimensions. Indeed, in the case of $(\text{GN})_3$, there appears to be no fundamental difference from renormalizable models. In particular, we learnt at this conference that the conjecture that such models should be based on distributions which are worse behaved than tempered distributions has been shown to be false. So, at a basic mathematical level, non-renormalizability seems not to be a fundamental consideration. Returning to gravity, as I will indicate below, there is some evidence from numerical simulations that quantum general relativity itself may exist non-perturbatively. One might therefore wonder: why have the standard methods developed by mathematical physicists not been applied to the problem of quantum gravity? What are the obstacles? Let me therefore consider some natural strategies that one may be tempted to try and indicate the type of difficulties one encounters.

First, one might imagine *defining* the goal properly by writing down a set of axioms. In Minkowskian field theories the Wightman and the Haag-Kastler axioms serve this purpose. Can we write down an analogous system for quantum gravity, thereby spelling out the goals in a clean fashion? Problems arise right away because both the system of axioms are rooted in the geometry of Minkowski space and in the associated Poincaré group. Let me consider the Wightman system (Streater & Wightman 1964) for concreteness. The zeroth axiom asks that the Hilbert space of states carry a unitary representation of the Poincaré group and that the 4-momentum operator have a spectrum in the future cone; the second axiom states how the field operators should transform; the third axiom introduces micro-causality, i.e., the condition that field operators should commute at space-like separations; and, the fourth and the last axiom requires asymptotic completeness, i.e., that the Hilbert spaces $\mathcal{H}^\pm$ of asymptotic states be isomorphic with the total Hilbert space. Thus, four of the five axioms derive their meaning directly from Minkowskian structure. It seems quite difficult to extend the zeroth and the fourth axioms already to quantum field theory in topologically non-trivial space-times, leave alone to the context in which there is no background metric whatsoever. And if we just drop these axioms, we are of course left with a framework that is too loose to be useful.

The situation is similar with the Osterwalder-Schrader system. One might imagine foregoing the use of specific axioms and just using techniques from Euclidean quantum field theory to construct a suitable mathematical framework. This is the view recently adapted by some groups using computer simulations. These methods have had a great deal of success in certain exactly soluble 2-dimensional models. The techniques involve dynamical triangulations and have been extended to the Einstein theory in 4 dimensions (see, e.g., Agishtein & Migdal (1992)). Furthermore, there is some numerical evidence that there is a critical point in the 2-dimensional parameter space spanned by Newton's constant and the cosmological constant, suggesting that the continuum limit of the theory may well exist. This is an exciting development and interesting results have now been obtained by several groups. Let us be optimistic and suppose that a well-defined Euclidean quantum theory of gravity can actually be constructed. This would be a major achievement. Unfortunately, it wouldn't quite solve the problem at hand. The main obstacle is that, as of now, there is no obvious way to pass from the Euclidean to the Lorentzian regime! The standard strategy of performing a Wick-rotation simply

does not work. First, we don't know which time coordinate to Wick-rotate. Second, even if we just choose one and perform the rotation, generically, the resulting metric will not be Lorentzian but *complex*. The overall situation is the following. Given an analytic Lorentzian metric, one can complexify the manifold and extend the metric analytically. However, the resulting complex manifold need not admit *any* Euclidean section. (Conversely, we may analytically continue an Euclidean metric and the resulting complex space-time need not have any Lorentzian section.) This is not just an esoteric, technical problem. Even the Lorentzian Kerr metric, which is *stationary*, does not admit an Euclidean section. Thus, even if one did manage to solve the highly non-trivial problem of actually constructing an Euclidean theory using the hints provided by the computer work, without a brand new idea we would still not be able to answer physical questions that refer to the Lorentzian world. More generally—i.e., going beyond computational physics—one can hope to use the Euclidean techniques for some specific calculations tailored to suitable approximations. However, for the *construction* of a consistent quantum field theory, as of now, the Euclidean techniques seem to be of little help.

Why not then try canonical quantization? This method lacks manifest covariance. Nonetheless, as we will see, one can construct a Hamiltonian framework without having to introduce any background fields, thereby respecting the diffeomorphism invariance of the theory. However, from the perspective of mathematical physicists, the structure of the resulting framework has an unusual feature which one has never seen in any of the familiar field theories: it is a dynamically constrained system. That is, most of the non-trivial content of the theory lies in its constraints. To see how this comes about, let us first consider Yang-Mills theory in Minkowski space. As is well-known, the Hamiltonian description of this theory has a constraint—the Gauss law. It tells us that, only those points $(A_a^i(\vec{x}), E_i^a(\vec{x}))$ of the phase space represent *physical* states of the classical theory for which the electric field E_i^a has zero covariant divergence, i.e., (A_a^i, E_i^a) satisfies the constraint $D_a E_i^a = 0$. The canonical transformations generated by this functional corresponds precisely to gauge transformations on (A_a^i, E_i^a) . In quantum theory, the corresponding operator equation is imposed on wave functionals $\Psi(A)$ to select the *physical* states: A state is physical if and only if $\hat{D}_a \hat{E}^a \circ \Psi(A) = 0$. This operator equation tells us simply that the physical states are gauge invariant: $\Psi(A) = \Psi(A^g)$, where A^g is the transform of A under a local gauge transformation $g(\vec{x})$. The dynamics on these states is generated by the Hamiltonian operator which, being gauge invariant, maps physical states to physical states. Let us return to general relativity. Now, the group of space-time diffeomorphisms is the “gauge group” of the theory. Hence, in the Hamiltonian framework, there are four constraints, three corresponding to “spatial” diffeomorphism on the 3-manifold fixed in the construction of the phase space and one corresponding to diffeomorphisms in “time-like directions” transverse to this surface. The canonical transformation generated by this last constraint functional defines the dynamics of the theory; it is therefore called the *Hamiltonian constraint*. Thus, we have a peculiar situation: on physical classical states, the generator of dynamics—the Hamiltonian—vanishes identically!³ Suddenly, then, mathematical physicists find themselves in an

³For simplicity I am assuming that the spatial slices are compact. In the asymptotically flat case, the Hamiltonian generating dynamics does not vanish identically; it equals a surface term. The total energy of the theory is thus analogous to the charge integral in QED. This situation is generic to

unfamiliar territory and all the experience gained from canonical quantization of field theories in 2 or 3 dimensions begins to look not so relevant. Normally, the key problem is that of finding a suitable representation of the CCRs—or, an appropriate measure on the space of states—which lets the Hamiltonian be self-adjoint. Now, the Hamiltonian seems trivial but all the difficulty seems concentrated in the quantum constraints. Furthermore, while the representation of the CCRs is to be chosen *prior* to the imposition of constraints, the scalar product on the space of states has physical meaning only *after* the constraints are solved. One thus needs to develop new strategies and modify the familiar quantization program appropriately.

Finally, one may imagine using techniques from geometrical quantization (see, e.g., Woodhouse 1980). However, since the key difficulty lies in the constraints of the theory, a “correct” polarization would be the one which is preserved, in an appropriate sense, by the Hamiltonian flows generated by the constraint functions on the phase space (Ashtekar & Stillermann, 1986). The problem of finding such a polarization is closely related to that of obtaining a general solution to Einstein’s equation and therefore seems hopelessly difficult in 4 dimensions. (Incidentally, in 3 dimensions, the strategy does work but only because there are no local degrees of freedom; every solution to the field equations is flat.) One may imagine using instead the Dirac (1964) approach to quantization of constrained systems: as in Yang-Mills theory, use the operator constraints to select physical states. This is in fact the procedure one uses in familiar examples such as a free relativistic particle: the classical constraint, $P^\alpha P_\alpha + \mu^2 = 0$ provides us, via the Dirac strategy, the Klein-Gordon equation $\eta^{\alpha\beta} \partial_\alpha \partial_\beta \Phi - \mu^2 \Phi = 0$ which incorporates quantum dynamics. Indeed, in the next two sections, we will use this strategy. We will see however that the representation best suited for solving the quantum constraints in this framework does not arise from any polarization whatsoever on the phase space. Thus, unfortunately, geometric quantization techniques do not seem to be well suited to this problem. Finally, one might imagine group theoretic method of quantization. An appropriate canonical group was in fact found (Isham & Kakas, 1984a, b) and it does “interact” well with the constraints of the theory which generate spatial diffeomorphisms (and triad rotations). However, the problem of incorporating the Hamiltonian constraint which generates dynamics again seems hopelessly difficult.

As the discussion suggests, one needs a new quantization program that can handle the peculiarities of general relativity. Such a program does exist. The basic ideas were introduced by Dirac already in the sixties and have been refined over the years by many authors. The approach to quantum general relativity that I will discuss in the next two sections is based on the framework developed in Ashtekar (1991) (chapter 10; see also Ashtekar & Tate 1993).

4 A Mathematical Framework for Quantum General Relativity

In this section, I will present a number of recent results that provide a mathematical framework for non-perturbative, canonical quantization of general relativity in 3 and 4 space-time dimensions. This is *not* an exhaustive treatment of canonical quantum

diffeomorphism invariant theories; it is not restricted to the Einstein-Hilbert action.

gravity. I will focus only on a few topics and omit most of the details. I will, however, provide references where these can be found. My aim is to illustrate the type of results that have been obtained and provide a feel of the current status of the field from the perspective of mathematical physics.

This section is divided into three parts. In the first, I will collect some basic results that relate general relativity to theories of connections. In the second—and the main part—I will report on some recent work on calculus on the space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations. In the third part, I will indicate how these results are now being used in quantum gravity.

4.1 Preliminaries

As I mentioned in the Introduction, it is possible to regard general relativity as a dynamical theory of connections. In 3 space-time dimensions, the connection in question is simply the spin-connection which enables one to transport the $SU(1,1)$ spinors along curves. In 4 dimensions, the appropriate connection turns out to be *chiral*; it enables one to transport (say) left-handed spinors along curves. In both cases, one can construct a Hamiltonian formulation in which all the equations are low order polynomials in the connection and its “canonically conjugate variable”—the analog of the electric field of Yang-Mills theory. However, unlike in the Yang-Mills theory, this field, E_i^a , has a dual interpretation: it can be regarded as a square root of the spatial metric. (In the 4-dimensional theory, this is just a spatial triad.) This dual interpretation enables one to pass back and forth between Yang-Mills theory and general relativity. In particular, the effect of Yang-Mills gauge transformations on the electric field can be re-interpreted as a triad rotation, which leaves the metric invariant—it is thus a gauge transformation also from the perspective of general relativity. Consequently, the field E_i^a is constrained to be divergence-free also in general relativity. However, as noted in Section 3, general relativity has additional constraints, a vectorial constraint that implies that physical states should be invariant under spatial diffeomorphisms and a scalar (or Hamiltonian) constraint which encodes dynamics of the theory.

Thus, in this connection-dynamics formulation of general relativity, the phase space, to begin with, is the same as in the Yang-Mills theory. Furthermore, the constraint sub-manifold of the phase space in general relativity is embedded in the constraint sub-manifold of the Yang-Mills theory. The difference lies, of course, in dynamics. In the Yang-Mills theory, it is generated by the Hamiltonian,

$$(4.1) \quad H(A, E) = \int d^3x (E_i^a E_a^i + B_i^a B_a^i),$$

which is (gauge invariant but otherwise) unrelated to the constraint (i.e., the Gauss law). In general relativity, there are additional constraints and time-evolution is coded in one of them. Hence, if one can solve the quantum constraints, one has essentially tackled the issue of dynamics. We will not need the explicit form of the constraints in what follows.

Let us conclude the preliminaries by noting that a key simplification arises in 3 dimensions: the general relativity constraints imply that the connection must be flat. Thus, in this case, the essence of the entire theory is contained in just two constraints: the first is the Gauss law which ensures gauge invariance and the second is flatness which

implies that the $SU(1, 1)$ connection—which serves as the configuration variable—must be flat. In the classical theory, the two constraints tell us that all dynamical trajectories correspond to flat metrics; there are no local degrees of freedom, no gravity waves. In the quantum theory, the imposition of these constraints implies that the physical states $\Psi(A)$ are gauge invariant functions of *flat* connections—i.e., are functions on the moduli space of flat connections on the given spatial, 2-manifolds. Since these spaces are *finite* dimensional, we have quantum *mechanics* rather than quantum field theory. In 4 dimensions, of course, no such simplification occurs. We again have the general relativity constraints in addition to the Gauss law. However, these do *not* imply that the connections must be flat. In the classical theory, there are local degrees of freedom, gravity waves and rich, albeit complicated, dynamics. In the quantum theory, we have a genuine field theory with infinite number of degrees of freedom.

For details, see, e.g., Ashtekar (1987, 1991) and Romano (1993).

4.2 Integration on the space $\mathcal{A}/\mathcal{G}$

As we saw in Sections 1 and 3, any attempt at non-perturbative quantization of gravity faces a host of conceptual problems. 3-dimensional general relativity is an excellent toy model to see how these problems can be faced since it has been solved exactly (see, e.g., Witten (1988), Ashtekar et al. (1989), Nelson & Regge (1989).) *However, one needs a new mathematical framework and a variety of new techniques.* Indeed, even the finished theory contains a number of unfamiliar notions. The basic observables are essentially non-local; the familiar operator-valued distributions are notably absent. To begin with, there is no Hamiltonian; indeed, no time to evolve anything in. There is no underlying space-time and hence no microcausality. Yet, the Hilbert space is well-defined and the observables are represented by self-adjoint observables. One can, if one wishes, regard a suitable dynamical variable as time and see how states “evolve” relative to this “internal clock.” One can show that this evolution is unitary. There are physical predictions. The theory is in a from that is unfamiliar from, say, constructive quantum field theory but has as much physical content as one can hope for. Therefore, in my talk I discussed this case in some detail. In this report, however, I will forego that discussion since there are a number of reviews on the subject (see, e.g., Carlip (1990, 1993) and Ashtekar (1991), chapter 17) Instead, I will focus here on recent developments which encompass the 4-dimensional theory.

A key mathematical problem is to develop integration theory on the space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations since, heuristically, this is the domain space of quantum states. In the 3-dimensional case, this problem is easy to solve because the (appropriate components of the) moduli space of *flat* $SU(1, 1)$ connections can be given naturally the structure of a finite dimensional symplectic manifold (see e.g., Ashtekar(1991), chapter 17); one can simply use the Liouville volume element to perform integration. Thus, although the domain space of quantum states is non-linear, the integration theory is simple because the space is *finite* dimensional. In 4 space-time dimensions, the situation is again simple for the case of *linearized gravity*—the theory of free gravitons in Minkowski space. This theory can be cast in the language of connections (see, e.g., Ashtekar (1991), chapter 11). Integration theory is again well-developed; the domain space is now linear and one can simply use the Gaussian measure as in free

field theories in Minkowski space. Thus, in this case, in spite of the presence of an infinite number of degrees of freedom, the integration theory is straightforward because of the underlying *linearity*. In the case of full, non-linear general relativity—and also for Yang-Mills theory—in 4-dimensions the problem is significantly more difficult because $\mathcal{A}/\mathcal{G}$ is *both* non-linear *and* infinite dimensional. Fortunately, a rigorous approach has now become available to tackle this problem. In particular, a diffeomorphism invariant measure has been found. I will now outline these developments and indicate in the next sub-section how they can be used in quantum gravity.

Fix an analytic 3-manifold Σ which will represent a Cauchy surface in space-times to be considered. We will consider $SU(2)$ connections on Σ . Since any $SU(2)$ bundle over a 3-manifold is trivial, we can represent any connection by a Lie-algebra valued 1-form A_a^i on Σ , where a is the spatial index and i , the internal⁴. Denote by $\mathcal{A}$ the space of smooth (say C^2) $SU(2)$ -connections equipped with one of the standard (Sobolev) topologies (see, e.g., Mitter & Viallet (1981)). $\mathcal{A}$ has the structure of an affine space. However, what is of direct interest to us is the space $\mathcal{A}/\mathcal{G}$ obtained by taking the quotient of $\mathcal{A}$ by (C^3) local gauge transformations. In this projection, the affine structure is lost; $\mathcal{A}/\mathcal{G}$ is a genuinely non-linear space with complicated topology. To define the integration theory, we will begin by constructing a sub-algebra of the Abelian C^* -algebra of bounded functions on $\mathcal{A}/\mathcal{G}$ and the desired measures will arise from positive linear functionals on this C^* -algebra. Given any closed loop α on the 3-manifold, we can define the Wilson-loop functional T_α on $\mathcal{A}/\mathcal{G}$:

$$(4.2) \quad T_\alpha(A) := \frac{1}{2} \text{Tr} \, \mathcal{P} \exp G \oint_\alpha A \cdot dl,$$

where the trace is taken in the fundamental representation of $SU(2)$ and the Newton's constant G appears because, in general relativity, it is GA_a^i that has the dimensions of a connection; in gauge theories, of course, this factor would be absent⁵. For technical reasons, we will have to restrict ourselves to piecewise analytic loops α . (This is why we needed Σ to be analytic. Note that the loops need not be smooth; they can have kinks and intersections but only at a finite number of points.) It turns out that, due to $SU(2)$ trace identities, product of any two Wilson-loop functionals can be expressed as a sum of other Wilson-loop functionals. Therefore, the vector space generated by finite complex-linear combinations of these functions has the structure of a $\star$ -algebra. The functionals T_α are all bounded (between -1 and 1). Hence, the sup-norm (over $\mathcal{A}/\mathcal{G}$) is well-defined and we can take the completion to obtain a C^* -algebra. We will call it the *holonomy C^* -algebra* and denote it by $\mathcal{HA}$. Elements of $\mathcal{HA}$ are to be thought of as the configuration variables of the theory.

Since $\mathcal{HA}$ is an Abelian C^* -algebra with identity, we can apply the Gel'fand theory and conclude that $\mathcal{HA}$ is isomorphic with the C^* -algebra of all continuous functions

⁴Results reported in this section for which explicit references are not provided are all taken from Ashtekar and Isham (1992) and Ashtekar and Lewandowski (1993a, b). The last two papers and those by Baez (1993a, b) contain significant generalizations which include allowing more general gauge groups, allowing the manifold Σ to be of arbitrary dimension and allowing the connections to live in non-trivial bundles.

⁵The letter T stands for trace. We will later define other T -variables which have the information about the "electric field" E_a^i as well. Although defined on $\mathcal{A}$, being gauge invariant, functions T_α project down to $\mathcal{A}/\mathcal{G}$ unambiguously.

on a compact, Hausdorff space $\text{sp}(\mathcal{HA})$, the spectrum of the given C^* -algebra $\mathcal{HA}$. Furthermore, since elements of $\mathcal{HA}$ suffice to separate points of $\mathcal{A}/\mathcal{G}$, it follows that $\mathcal{A}/\mathcal{G}$ is *densely embedded* in $\text{sp}(\mathcal{HA})$. To emphasize this point, from now on, we will denote the spectrum by $\overline{\mathcal{A}/\mathcal{G}}$ and regard it as a completion of $\mathcal{A}/\mathcal{G}$ (in the Gel'fand topology). Integration theory will be defined on $\overline{\mathcal{A}/\mathcal{G}}$. This is in accordance with the common occurrence in quantum field theory: while the classical configuration (or phase) space may contain only smooth fields (typically taken to belong to be the Schwartz space), the domain space of quantum states is a completion of this space in an appropriate topology (the space of distributions).

A key difficulty with the use of the Gel'fand theory is that one generally has relatively little control on the structure of the spectrum. In the present case, however, we are more fortunate: a simple and complete characterization of the spectrum is available. To present it, I first need to introduce a key definition. Fix a base point x_o in the 3-manifold Σ and regard two (piecewise analytic) closed loops α and α' to be equivalent if the holonomy of any connection in $\mathcal{A}$, evaluated at x_o , around α is the same as that around α' . We will call each equivalence class (a holonomically equivalent loop or) a *hoop* and denote the hoop to which a loop α belongs by $\tilde{\alpha}$. For example, α and α' define the same hoop if they differ by a reparametrization or by a line segment which is immediately re-traced⁶. The set of hoops has, naturally, the structure of a group. We will call it the *hoop group* and denote it by $\mathcal{HG}$. In terms of this group, we can now present a simple characterization of the Gel'fand spectrum $\overline{\mathcal{A}/\mathcal{G}}$:

Every homomorphism $\hat{H}$ from the hoop group $\mathcal{HG}$ to the gauge group $SU(2)$ defines an element $\bar{A}$ of the spectrum $\overline{\mathcal{A}/\mathcal{G}}$ and every $\bar{A}$ in the spectrum defines a homomorphism $\hat{H}$ such that $\bar{A}(\tilde{\alpha}) = \frac{1}{2} \text{Tr } \hat{H}(\tilde{\alpha})$. This is a 1-1 correspondence modulo the trivial ambiguity that homomorphisms $\hat{H}$ and $g^{-1} \cdot \hat{H} \cdot g$ define the same element $\bar{A}$ of the spectrum.

Clearly, every regular connection A in $\mathcal{A}$ defines the desired homomorphism simply through the holonomy operation: $\hat{H}(\tilde{\alpha}) := \mathcal{P} \exp \int_{\alpha} A \cdot dl$, where α is any loop in the hoop $\tilde{\alpha}$. However, there are many homomorphisms which do not arise from smooth connections. This leads to “generalized connections”—i.e., elements in $\overline{\mathcal{A}/\mathcal{G}} - \mathcal{A}/\mathcal{G}$. In particular, there exist $\bar{A}$ in $\overline{\mathcal{A}/\mathcal{G}}$ which have support at a single point and are thus “distributional”. Note that this characterization of the spectrum $\overline{\mathcal{A}/\mathcal{G}}$ is *completely algebraic*; there is no continuity assumption on the homomorphisms. This property makes the characterization very useful in practice.

From the general representation theory of C^* -algebras, it follows that positive linear functionals on $\mathcal{HA}$ are in 1-1 correspondence with regular measures on (the compact Hausdorff space) $\overline{\mathcal{A}/\mathcal{G}}$. It turns out that the positive linear functions, in turn, are determined completely by certain “generating functionals” $\Gamma(\alpha)$ on the space $\mathcal{L}_{x_o}$ of loops based at x_o :

There is a 1-1 correspondence between positive linear functionals on $\mathcal{HA}$ (and hence regular measures on $\overline{\mathcal{A}/\mathcal{G}}$) and functional $\Gamma(\alpha)$ on $\mathcal{L}_{x_o}$ satisfying:

⁶For piecewise analytic loops and $SU(n)$ connections, these two are the most general operations; two loops define the same hoop if and only if they are related by a combination of reparametrizations and retracings.

- i) $\sum_i a_i T_{a_i} = 0 \Rightarrow \sum_i a_i \Gamma(\alpha_i) = 0$; and,
 ii) $\sum_{i,j} \bar{a}_i a_j (\Gamma(\alpha_i \circ \alpha_j) + \Gamma(\alpha_i \circ \alpha_j^{-1})) \geq 0$.
 for all loops α_i and complex numbers a_i .

The first condition implies that the functional Γ is well-defined on hoops. Hence we could have taken it to be a functional on $\mathcal{HG}$ from the beginning. Thus, we see that there is a nice “non-linear duality” between the spectrum $\overline{\mathcal{A}/\mathcal{G}}$ and the hoop group $\mathcal{HG}$: Elements of $\overline{\mathcal{A}/\mathcal{G}}$ are homomorphisms from $\mathcal{HG}$ to $SU(2)$ and regular measures on $\overline{\mathcal{A}/\mathcal{G}}$ correspond to certain functionals on $\mathcal{HG}$. Finally, if one is interested in measures on $\overline{\mathcal{A}/\mathcal{G}}$ which are invariant under the (induced) action of diffeomorphisms on Σ , one is led to seek functionals $\Gamma(\alpha)$ which depend not on the individual loops α but rather on the (generalized) knot class to which α belongs. (The qualification “generalized” refers to the fact that here we are considered loops which can have kinks, overlaps and self-intersections. Until recently, knot theorists considered only smoothly embedded loops.) Thus, there is an interesting—and potentially powerful—interplay between knot theory and representations of the holonomy algebra $\mathcal{HA}$ in which the diffeomorphism group of Σ is unitarily implemented.

Finally, we can make the integration theory more explicit. Consider a subgroup S_n of the hoop group $\mathcal{HG}$ which is generated by n (independent) hoops. We can introduce the following equivalence relation on $\overline{\mathcal{A}/\mathcal{G}}$: $\bar{A} \equiv \bar{A}'$ if and only if their action on all elements of S_n coincides, i.e., if and only if $\bar{A}(\tilde{\alpha}) = g^{-1} \cdot \bar{A}'(\tilde{\alpha}) \cdot g$ for all $\tilde{\alpha} \in S_n$ and some (hoop independent) $g \in SU(2)$. It turns out that the quotient space is isomorphic to $[SU(2)]^n / Ad$. Therefore we can introduce a notion of cylindrical functions on $\overline{\mathcal{A}/\mathcal{G}}$: A function f on $\overline{\mathcal{A}/\mathcal{G}}$ will be said to be *cylindrical* if it is the pull-back to $\overline{\mathcal{A}/\mathcal{G}}$ of a smooth function $\tilde{f}$ on $[SU(2)]^n / Ad$ for some sub-group S_n of the hoop group. Finally, we can define integrals of these functions f on $\overline{\mathcal{A}/\mathcal{G}}$ through their integrals on $[SU(2)]^n / Ad$, provided of course we equip $[SU(2)]^n / Ad$ with suitable measures $d\mu_n$ for each n . We can then define a positive linear functional Γ' on the space of cylindrical functions f via:

$$(4.3) \quad \Gamma'(f) := \int_{[SU(2)]^n / Ad} \tilde{f} d\mu_n.$$

For the functional to be well-defined, the family of measures $d\mu_n$ on $[SU(2)]^n / Ad$ must satisfy certain consistency conditions. It turns out that these requirements can be met and, furthermore, the resulting functionals $\Gamma'(f)$ define regular measures on $\overline{\mathcal{A}/\mathcal{G}}$. A particularly natural choice (and, not surprisingly, the first to be discovered) is to let $d\mu_n$ be simply induced on $[SU(2)]^n / Ad$ by the Haar-measure on $SU(2)$. We then have the following results:

- i) The consistency conditions are satisfied; the left side of (4.3) is well-defined for all cylindrical functions f on $\overline{\mathcal{A}/\mathcal{G}}$;
 ii) The generalized holonomies $T_{\tilde{\alpha}}$ are cylindrical functionals on $\overline{\mathcal{A}/\mathcal{G}}$ and $\Gamma(\alpha) := \Gamma'(T_{\tilde{\alpha}})$ defined via (4.3) serves as a generating functional for a faithful, cyclic representation of the homomony C^* -algebra $\mathcal{HA}$ which ensures that $d\mu$ is a regular, strictly positive measure on $\overline{\mathcal{A}/\mathcal{G}}$;
 iii) The measure $d\mu$ is invariant under the induced action of the diffeomorphism group on Σ .

(The knot invariant defined by $d\mu$ is a genuinely generalized one; roughly, it counts the number of self-overlaps in any given loop.)

This measure is in some ways analogous to the Gaussian measure on linear vector spaces. Both can be obtained by a “cylindrical construction.” The Gaussian measure uses the natural metric on $\mathbb{R}^n$ while the above measure uses the natural (induced) Haar measure on $[SU(2)]^n/Ad$. They are both regular and strictly positive. This leads us to ask if other properties of the Gaussian measure are shared. For instance, we know that the Gaussian measure is concentrated on distributions; although the smooth fields are dense in the space of distributions in an appropriate topology, they are contained in a set whose total measure is zero. Is the situation similar here? The answer turns out to be affirmative. The classical configuration space $\mathcal{A}/\mathcal{G}$ with which we began is dense in the domain space $\overline{\mathcal{A}/\mathcal{G}}$ of quantum states in the Gel’fand topology. However, $\mathcal{A}/\mathcal{G}$ is contained in a set whose total measure is zero. The measure is again concentrated on “generalized” connections in $\overline{\mathcal{A}/\mathcal{G}}$ (Marolf & Mourão (1993)). In a certain sense, just as the Gaussian measures on linear spaces originate in the harmonic oscillator, the new measure on $\overline{\mathcal{A}/\mathcal{G}}$ originates in a (generalized) rotor (whose configuration space is the $SU(2)$ group-manifold). However, the measure is, so to say, “genuinely” tailored to the underlying non-linearity. It is *not* obtained by “perturbing” the Gaussian measure.

With the measure $d\mu$ at hand, we can consider the Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu)$ and introduce operators on it. This is *not* the Hilbert space of physical states of quantum gravity since we have not imposed constraints. It is a fiducial, kinematical space which enables us to regularize various operators (in particular, the quantum constraint operators). The configuration operators are associated with the generalized Wilson loop functionals: $\hat{T}_{\tilde{\alpha}} \circ \Psi(\tilde{A}) = \tilde{A}(\tilde{\alpha})\Psi(\tilde{A})$. One can show that there are bounded, self-adjoint operators on the Hilbert space. There are also “momentum operators”—associated with closed, 2-dimensional ribbons or strips in the 3-manifold Σ —which are gauge invariant and linear in the electric field. One can show that these are also self-adjoint (but unbounded). Finally, since $d\mu$ is invariant under the induced action of the diffeomorphism group of Σ , this group acts unitarily.

The next task is to represent quantum constraints as well-defined operators on the Hilbert space and then solve them, i.e., find their kernels. The first step has been completed for the three vector constraints of general relativity. As for the kernel, typically, the constraint operators are self-adjoint on the kinematical Hilbert space and zero is in the continuous part of their spectrum. The physical states—elements of the kernel—are thus *not* normalizable; they do not belong to the Hilbert space. Rather, they belong to the (appropriately constructed) rigged Hilbert spaces (Hajiček, 1993). Consider for example, the simple case of a free relativistic particle, where the classical constraint is $P^\alpha P_\alpha + \mu^2 = 0$. In this case, the kinematical Hilbert space can be taken to be $L^2(\mathbb{R}^4)$. This space is needed to translate the classical constraint function to a well-defined operator (whose kernel can then be found). The operator, of course, is $\eta^{\alpha\beta} \partial_\alpha \partial_\beta - \mu^2$. No (non-zero) element in its kernel is normalizable in $L^2(\mathbb{R}^4)$. These elements belong to the rigged Hilbert space; in the momentum space, they are distributions with support on the mass shell. One wishes to carry out a similar construction in the gravitational case. For this, one needs to introduce the appropriate rigged Hilbert spaces. This is an open problem where input from the mathematical physics community would be most useful. To summarize, the present status is that the

operators representing the vector constraints of general relativity are well-defined and self-adjoint on $L^2(\mathcal{A}/\mathcal{G}, d\mu)$. As we will see in the next sub-section, it is intuitively clear what their kernel is. What is needed is a precise, rigorous result. For the Hamiltonian constraint, we are yet to show that the operator is well-defined and all considerations are heuristic at this stage.

Thus, the integration theory based on the measure $d\mu$ is being used as the mathematical basis in the quantization of general relativity in the connection-dynamics approach. However, these techniques may be used also in other theories of connections which are diffeomorphism invariant and perhaps even in Yang-Mills theory which is not diffeomorphism invariant. We saw that the full domain space of quantum theory, $\overline{\mathcal{A}/\mathcal{G}}$, can be thought of as the space of homomorphisms from the full hoop group $\mathcal{H}\mathcal{G}$ to the gauge group $SU(2)$. Given a finitely generated sub-group S_n of the hoop group, we can consider the space of homomorphisms from it to $SU(2)$. This provides the space $[SU(2)]^n/Ad$ which is precisely the domain space of quantum states of a lattice gauge theory where the lattice is not rectangular but tailored to the given subgroup S_n of the gauge group. Thus, what we have is a set of "floating lattices," each associated with a finitely generated subgroup of the hoop group. The space $\overline{\mathcal{A}/\mathcal{G}}$ can be rigorously recovered as a projective limit of the configuration spaces of lattice theories (Marolf & Mourão 1993). This construction is potentially quite powerful; it may enable one to take continuum limits of operators of lattice theories in a completely new fashion. The limit is obtained not by taking the lattice separation to zero but by enlarging lattices to probe the continuum connections better and better, i.e., by considering larger and larger subgroups of the hoop group.

4.3 Loop representation of quantum gravity

In non-perturbative quantum gravity, to date, most progress has been made in the so-called "loop representation" in which *quantum states arise as functionals of closed loops on the 3-manifold*. This representation can also be used in, e.g., Yang-Mills theories (Loll 1992) and has led to concrete results in the lattice formulations (Brügmann 1991). However, the representation is particularly well-suited in the gravitational case since it seems to be best suited for solving the quantum constraints. In particular, the solutions to the vector (or diffeomorphism) constraints, referred to in the last sub-section, are explicit in this representation. In this sub-section, I will outline various results that have been obtained so far in this framework. From the perspective of mathematical physics, some of the results I will report here are still heuristic. However, there does not seem to be any difficulty of principle in making them rigorous. This may well be a fertile area for young researchers in the field.

The loop representation was introduced as a heuristic device by Rovelli and Smolin (1990) in the context of quantum gravity (and somewhat earlier, but in a somewhat different fashion, by Gambini and Trias (1986) in the context of Yang-Mills theory.) These ideas can be now made rigorous using the framework outlined in the last sub-section. Consider, as before, the Hilbert space $L^2(\mathcal{A}/\mathcal{G}, d\mu)$ the elements of which are functionals $\Psi(\bar{A})$ of generalized connections $\bar{A}$. Since the generalized Wilson loop functionals

$T_{\tilde{\alpha}}(\bar{A}) := \text{Tr } \bar{A}(\tilde{\alpha})$ belong to this space, we can define the following transform:

$$(4.4) \quad \psi(\tilde{\alpha}) := \int_{\mathcal{A}/\mathcal{G}} \overline{T_{\tilde{\alpha}}(\bar{A})} \Psi(\bar{A}) \, d\mu,$$

to pass from functionals $\Psi(\bar{A})$ of generalized connections to functionals $\psi(\tilde{\alpha})$ of hoops⁷. This has some similarities with the Fourier transform

$$(4.5) \quad \psi(k) := \int_{-\infty}^{\infty} e^{ik \cdot x} \Psi(x) \, dx,$$

that enables one to pass from the position to the momentum representation. The role of the integral kernel, $\exp ik \cdot x$ is now played by the generalized Wilson loop functional. The Rovelli-Smolín transform is faithful. However, a nice characterization of the space of functionals $\psi(\tilde{\alpha})$ obtained through this transform is not available. In particular, integration theory on the hoop group $\mathcal{HG}$ has not yet been developed and therefore, at present, there is no analog of the Plancharel theorem which makes the Fourier transform so powerful. This is one of the open problems looking longingly to mathematical physicists for help.

The transform as it is written above is however well-defined. Therefore one can take operators from the connection side and write them on the hoop side. Typically, these involve simple geometric operations on the loop arguments—composing, breaking, and rerouting loops. For example, the generalized Wilson-loop operators which act on $L^2(\mathcal{A}/\mathcal{G}, d\mu)$ simply by multiplication, $\hat{T}_{\tilde{\beta}} \circ \Psi(\bar{A}) = \bar{A}(\tilde{\beta}) \Psi(\bar{A})$, can be transported to the loop states and their action is given by:

$$(4.6) \quad (\hat{T}_{\tilde{\beta}} \circ \psi)(\tilde{\alpha}) = \frac{1}{2} \left(\psi(\tilde{\alpha} \cdot \tilde{\beta}) + \psi(\tilde{\alpha} \cdot \tilde{\beta}^{-1}) \right),$$

where $\tilde{\alpha} \cdot \tilde{\beta}$ is the hoop obtained by composing $\tilde{\alpha}$ and $\tilde{\beta}$ in the hoop group. Similarly, one can transport other operators. Of particular interest are the vector or the diffeomorphism constraint operators. There is one such operator $\hat{C}(V)$ associated with every vector field V^a on Σ and their action is given simply by:

$$(4.7) \quad \hat{C}(V) \circ \psi(\alpha) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\psi(\alpha_{\epsilon}^V) - \psi(\alpha) \right),$$

where α_{ϵ}^V is the loop obtained by displacing α along the integral curves of V^a an affine parameter distance ϵ . Thus, the regularized operator corresponding to the diffeomorphism constraint does what one intuitively expects it to do: it drags the loop in the argument of the wave function along the diffeomorphism. Therefore, it is intuitively clear that the kernel of this constraint consists of functionals of loops which remain unchanged if the loop is replaced by a diffeomorphic one. That is, the elements of the kernel are functions of generalized knot classes on the 3-manifold. It is remarkable that in the loop representation, one can write down the *general* solution to the three of the four constraints of general relativity in a simple, geometrical way and the solutions relate quantum gravity with knot theory. As I remarked earlier, one would like

⁷Since the left side is a function of hoops, we should, strictly, use the terms hoop representation and the hoop transform. However, in various calculations, it is often convenient to lift these functionals from the hoop group $\mathcal{HG}$ to the space of loops $\mathcal{L}_{x_0}$. Therefore, as in most of the literature on the subject, we will not make a careful distinction between loops and hoops in what follows.

to make these results rigorous by specifying the regularity conditions on the permitted knot invariants using rigged Hilbert spaces.

The remaining (Hamiltonian) constraints are more difficult. They *have been* transcribed in the loop representation and various apparently distinct methods of doing so have led to equivalent results (Brügmann & Pullin (1993)). Thus, there is a general feeling that one is on the "right track." However, these results are not rigorous. A promising new direction is being pursued by Gambini and his collaborators where the hoops group is replaced by a group of "smoothed out" hoops (Di Bortolo et al., 1993). This group has the structure of a Lie group and offers a new approach to the problem of regularization of various operators including the Hamiltonian constraints. This approach has already led to some interesting, new solutions to the Hamiltonian constraints which are related to some well-known knot invariants (Brügmann et al., 1992a, b). This general area of research related to the Hamiltonian constraint is one of the centers of current activities although one seems quite far from finding the generic solution to these constraints even at a heuristic level.

5 Weaving a Classical Geometry with Quantum Threads

I will now discuss two striking results that have emerged—already at a kinematical level, prior to the imposition of quantum constraints—from the loop representation (Ashtekar et al. 1992). The first is that certain operators representing *geometrical observables* can be regulated in a way that respects the diffeomorphism invariance of the underlying theory. What is more, these regulated operators are finite *without any renormalization*. Using these operators, one can ask if there exist loop states which approximate smooth geometry at large scales. One normally takes for granted that the answer to such questions would be obviously "yes." However, in genuinely non-perturbative treatments, this is by no means clear a priori; one may be working in a sector of a theory which does not admit the correct or unambiguous classical limit. For example, the sector may correspond to a confined phase which has no classical analog or the limit may yield a wrong number even for the macroscopic dimensions of space-time! The second main result of this subsection is that not only is the answer to the question raised above in the affirmative but, furthermore, *these states exhibit a discrete structure of a definite type at the Planck scale*. (For further details, see, e.g., Rovelli & Smolin (1990) and Ashtekar (1992), Smolin (1993).)

Let us begin with the issue of regularization. As noted in section 4, in the present framework, the spatial metric is constructed from products of "electric fields" E_i^a . It is a "composite" field given by⁸ $q^{ab}(x) = E^{ai}(x)E_i^b(x)$. In the quantum theory, therefore, this operator must be regulated. The obvious possibility is point splitting. One might set $q^{ab}(x) = \lim_{y \rightarrow x} E^{ai}(x)E_i^b(y)$. However, the procedure violates gauge invariance since

⁸More precisely, the situation is as follows. Since there is no background metric, the "momenta" E_i^a are actually vector densities of weight one, whence the composite field q^{ab} is of density weight two; it is the determinant of the covariant metric multiplied by the contravariant metric. In what follows, these density weights are important for the details of the arguments. However, for brevity, I will not dwell on this point any further.

the internal indices at two *different* points have been contracted. There is, however, a suitable modification that will ensure gauge invariance. Consider the field $T^{aa'}[\alpha](y', y)$, labelled by a closed loop α and points y and y' thereon, defined in the classical theory by:

$$(5.1) \quad T^{aa'}[\alpha](y, y') := \frac{1}{2} \text{Tr} \left[(\mathcal{P} \exp G \int_{y'}^y A_b dl^b) E^a(y') (\mathcal{P} \exp G \int_y^{y'} A_c dl^c) E^{a'}(y) \right].$$

In the limit α shrinks to zero, $T^{aa'}[\alpha](y, y')$ tends to $-4q^{aa'}$.

Now, in quantum theory, one can define the action of the operator $\hat{T}^{aa'}[\alpha](y, y')$ directly on the loop states $\psi(\beta)$. The explicit form will not be needed here. We only note that using the bra-ket notation, $\psi(\beta) = \langle \beta | \psi \rangle$ the action can be specified easily. Indeed, $\langle \beta | \circ \hat{T}^{aa'}[\alpha](y, y')$ is rather simple: if a loop β does not intersect α at y or y' , the operator simply annihilates the bra $\langle \beta |$ while if an intersection does occur, it breaks and re-routes the loop β , each routing being assigned a specific weight. One may therefore try to define a quantum operator $\hat{q}^{aa'}$ as a limit of $\hat{T}^{aa'}[\alpha]$ as α shrinks to zero. The resulting operator does exist after suitable regularization *and* renormalization. However (because of the density weights involved) the operator necessarily carries a memory of the background metric used in regularization. Thus, the idea of defining the metric operator again fails. In fact one can give general qualitative arguments to say that there are no *local*, operators which carry the metric information *and* which are independent of background fields (used in the regularization). Thus, in quantum theory, the absence of background fields introduces new difficulties. That such difficulties would arise was recognized quite early by Chris Isham and John Klauder.

There do exist, however, *non-local* operators which can be regulated in a way that respects diffeomorphism invariance.

As the first example, consider the function $Q(\omega)$ —representing the smeared 3-metric—on the classical phase space, defined by

$$(5.2) \quad Q(\omega) := \int d^3x \ (q^{ab} \omega_a \omega_b)^{\frac{1}{2}},$$

where ω_a is any smooth 1-form of compact support⁹. It is important to emphasize that, in spite of the notation, $Q(\omega)$ is *not* obtained by smearing a *distribution* with a test field; because of the square-root, $Q(\omega)$ is *not* linear in ω . We can, nonetheless define the corresponding quantum operator as follows. Let us choose on Σ test fields $f_\epsilon(x, y)$ (which are densities of weight one in x and) which satisfy:

$$(5.3) \quad \lim_{\epsilon \rightarrow 0} \int_{\Sigma} d^3x f_\epsilon(x, y) g(x) = g(y)$$

for all smooth functions of compact support $g(x)$. If Σ is topologically $\mathbb{R}^3$, for example, we can construct these test fields as follows:

$$(5.4) \quad f_\epsilon(x, y) = \frac{\sqrt{h(x)}}{\pi^{\frac{3}{2}} \epsilon^3} \exp -\frac{|\vec{x} - \vec{y}|^2}{2\epsilon^2},$$

⁹Note that the integral is well-defined without the need of a background volume element because q^{ab} is a density of weight two.

where $\vec{x}$ are the cartesian coordinates labeling the point x and $h(x)$ is a "background" scalar density of weight 2. Next, let us define

$$(5.5) \quad q_\epsilon^{aa'}(x) = -\frac{1}{4} \int_\Sigma d^3y \int_\Sigma d^3y' f_\epsilon(x, y) f_\epsilon(x, y') T^{aa'}(y, y').$$

As ϵ tends to zero, the right side tends to q^{ab} because the test fields force both the points y and y' to approach x , and hence the loop passing through y, y' , used in the definition of $T^{aa'}(y, y')$, to zero. It is now tempting to try to define a local metric operator $\hat{q}^{aa'}$ corresponding to $q^{aa'}$ by replacing $T^{aa'}(y, y')$ in (5.5) by its quantum analog and then taking the limit. One finds that the limit does exist provided we first renormalize $\hat{q}_\epsilon^{aa'}$ by an appropriate power of ϵ . However, as before, the answer depends on the background structure (such as the density $h(x)$) used to construct the test fields $f_\epsilon(x, y)$. If, however, one tries to construct the quantum analog of the *non-local* classical variable $Q(\omega)$, this problem disappears. To see this, let us first express $Q(\omega)$ using (5.5) as:

$$(5.6) \quad Q(\omega) = \lim_{\epsilon \rightarrow 0} \int_\Sigma d^3x (q_\epsilon^{aa'} \omega_a \omega_{a'})^{\frac{1}{2}}.$$

The required quantum operator $\hat{Q}(\omega)$ on the loop states can now be obtained by replacing $T^{aa'}(y, y')$ by the operator $\hat{T}^{aa'}(y, y')$. A careful calculation shows that: i) the resulting operator is well-defined on loop states; ii) no renormalization is necessary, i.e., the limit is automatically *finite*; and, iii) the final answer carries no imprint of the background structure (such as the density $h(x)$ or, more generally, the specific choice of the test fields $f_\epsilon(x, y)$) used in regularization. To write out its explicit expression, let me restrict myself to smooth loops α without any self-intersection. Then, the action is given simply by:

$$(5.7) \quad \langle \alpha | \circ \hat{Q}(\omega) = l_P^2 \oint_\alpha ds |\dot{\alpha}^a \omega_a| \cdot \langle \alpha |,$$

where $l_P = \sqrt{G\hbar}$ is the Planck length, s , a parameter along the loop and $\dot{\alpha}^a$ the tangent vector to the loop. In this calculation, the operation of taking the square-root is straightforward because the relevant operators are diagonal in the loop representation. This is analogous to the fact that, in the position representation of non-relativistic quantum mechanics, we can set $\langle x | \circ (\hat{X}^2)^{1/2} = \langle x | \cdot |x|$ without recourse to the detailed spectral theory. The G in l_P of (5.7) comes from the fact that GA_a^i has the usual dimensions of a connection while $\hbar$ comes from the fact that $\hat{E}_i^a$ is $\hbar$ times a functional derivative. The final result is that, on non-intersecting loops, the operator acts simply by multiplication: the loop representation is well-suited to find states in which the 3-geometry—rather than its time evolution—is sharp.

The second class of operators corresponds to the area of 2-surfaces. Note first that, given a smooth 2-surface S in Σ , its area $\mathcal{A}_S$ is a function on the classical phase space. We first express it using the classical loop variables. Let us divide the surface S into a large number N of area elements $S_I, I = 1, 2, \dots, N$, and set $\mathcal{A}_I^{\text{appr}}$ to be

$$(5.8) \quad \mathcal{A}_I^{\text{appr}} = -\frac{1}{4} \left[\int_{S_I} d^2S^{bc}(x) \eta_{abc} \int_{S_I} d^2S^{b'c'}(x') \eta_{a'b'c'} T^{aa'}(x, x') \right]^{\frac{1}{2}},$$

where η_{abc} is the (metric independent) Levi-Civita density of weight -1 . It is easy to show that $\mathcal{A}_I^{\text{appr}}$ approximates the area function (on the phase space) defined by the

surface elements S_I , the approximation becoming better as S_I —and hence loops with points x and x' used in the definition of $T^{aa'}$ —shrink. Therefore, the total area $\mathcal{A}_S$ associated with S is given by

$$(5.9) \quad \mathcal{A}_S = \lim_{N \rightarrow \infty} \sum_{I=1}^N \mathcal{A}_I^{\text{appr}}.$$

To obtain the quantum operator $\hat{\mathcal{A}}_S$, we simply replace $T^{aa'}$ in (5.8) by the quantum loop operator $\hat{T}^{aa'}$. This somewhat indirect procedure is necessary because, as indicated above, there is no well-defined operator-valued distribution that represents the metric or its area element *at a point*. Again, the operator $\hat{\mathcal{A}}_S$ turns out to be finite. Its action, evaluated on a nonintersecting loop α (for simplicity), is given by:

$$(5.10) \quad \langle \alpha | \circ \hat{\mathcal{A}}_S = \frac{l_P^2}{2} I(S, \alpha) \cdot \langle \alpha |,$$

where $I(S, \alpha)$ is simply the *unoriented* intersection number between the 2-surface S and the loop α . (One obtains the *unoriented* intersection number here and the absolute sign in the integrand of (5.7) because of the square-root operation involved in the definition of these operators.) Thus, in essence, “a loop α contributes half a Planck unit of area to any surface it intersects.”

The fact that the area operator also acts simply by multiplication on non-intersecting loops lends further support to the idea that the loop representation is well-suited to “diagonalize” operators describing the 3-geometry. Indeed, we can immediately construct a large set of simultaneous eigenbras of the smeared metric and the area operators. There is one, $\langle \alpha |$, associated to every nonintersecting loop α . Note that the corresponding eigenvalues of area are *quantized* in integral multiples of $l_P^2/2$. There are also eigenstates associated with intersecting loops which, however, I will not go into to since the discussion quickly becomes rather involved technically.

With these operators on hand, we can now turn to the construction of quantum loop states that approximate the classical metric h_{ab} on Σ on a scale large compared to the Planck length. The basic idea is to weave the classical metric out of quantum loops by spacing them so that, on an average, precisely one line crosses *any* surface element whose area, *as measured by the given h_{ab}* is one Planck unit. Such loop states will be called *weaves*. Note that these states are not uniquely picked out since our requirement is rather weak. Indeed, given a weave approximating a given classical metric, one can obtain others, approximating the same classical metric.

Let us begin with a concrete example of such a state which will approximate a *flat* metric h_{ab} . To construct this state, we proceed as follows. Using this metric, let us introduce a random distribution of points on $\Sigma = \mathbb{R}^3$ with density n (so that in any given volume V there are $nV(1 + \mathcal{O}(1/\sqrt{nV}))$ points). Center a circle of radius $a = (1/n)^{1/3}$ at each of these points, with a random orientation. We assume that $a \ll L$, so that there is a large number of (non-intersecting but, generically, *linked*) loops in a macroscopic volume L^3 . Denote the collection of these circles by Δ_a . As noted in section 4, due to trace identities, products of Wilson loop functionals $T_{\tilde{\alpha}}$ can be expressed as linear combinations of Wilson loop functionals. As a consequence, it turns out that the bras defined by multi-loops are equivalent to linear combinations of

single loop bras. Therefore, for each choice of the parameter a , there is a well-defined bra $\langle \Delta_a |$. This is our candidate weave state.

Let us consider the observable $\hat{Q}[\omega]$. To see if $\langle \Delta_a |$ reproduces the geometry determined by the classical metric h_{ab} on a scale $L \gg l_p$, let us introduce a 1-form ω_a which is *slowly varying on the scale L* and compare the value $Q[\omega](h)$ of the classical $Q[\omega]$ evaluated at the metric h_{ab} , with the action of the quantum operator $\hat{Q}[\omega]$ on $\langle \Delta_a |$. A detailed calculation yields:

$$(5.11) \quad \langle \Delta_a | \circ \hat{Q}[\omega] = \left[\frac{\pi}{2} \left(\frac{l_p}{a} \right)^2 Q[\omega](h) + \mathcal{O}(a/L) \right] \cdot \langle \Delta_a |.$$

Thus, $\langle \Delta_a |$ is an eigenstate of $\hat{Q}[\omega]$ and the corresponding eigenvalue is closely related to $Q[\omega](h)$. However, even to the leading order, the two are unequal *unless* the parameter a —the average distance between the centers of loops—equals $\sqrt{\pi/2} l_p$. More precisely, (5.11) can be interpreted as follows. Let us write the leading coefficient on the right side of this equation as $(1/4)(2\pi a/l_p)(n/l_p^3)$. Since this has to be unity for the weave to reproduce the classical value (to leading order), we see that Δ_a should contain, on an average, one fourth Planck length of curve per Planck volume, where lengths and volumes are measured using h_{ab} .

The situation is the same for the area operators $\hat{A}_S$. Let S be a 2-surface whose extrinsic curvature varies slowly on a scale $L \gg l_p$. One can evaluate the action of the area operator on $\langle \Delta_a |$ and compare the eigenvalue obtained with the value of the area assigned to S by the given flat metric h_{ab} . Again, the eigenvalue can be re-expressed as a sum of two terms, the leading term which has the desired form, except for an overall coefficient which depends on the mean separation a of loops constituting Δ_a , and a correction term which is of the order of $\mathcal{O}(a/L)$. We require that the coefficient be so adjusted that the leading term agrees with the classical result. This occurs, again, precisely when $a = \sqrt{\pi/2} l_p$. It is interesting to note that the details of the calculations which enable one to express the eigenvalues in terms of the mean separation are rather different for the two observables. In spite of this, the final constraint on the mean separation is *precisely* the same.

Let us explore the meaning and implications of these results.

1. It is generally accepted that, to obtain classical behavior from quantum theory, one needs two things: i) an appropriate coarse graining, and, ii) special states. In our procedure, the slowly varying test fields ω_a and surfaces S with slowly varying extrinsic curvature enable us to perform the appropriate coarse graining while weaves—with the precisely tuned mean separation a —are the special states. There is, however, something rather startling: The restriction on the mean separation a —i.e., on the *short distance* behavior of the multi-loop Δ_a —came from the requirement that $\langle \Delta_a |$ should approximate the classical metric h_{ab} on *large scales L* !
2. In the limit $a \rightarrow \infty$, the eigenvalues of the two operators on $\langle \Delta_a |$ go to zero. This is not too surprising. Roughly, in a state represented by any loop α , one expects the quantum geometry to be excited just at the points of the loops. If the loops are *very* far away from each other as measured by the fiducial h_{ab} , there would

be macroscopic regions devoid of excitations where the quantum geometry would seem to correspond to a zero metric.

3. The result of the opposite limit, however, is surprising. One might have naively expected that the best approximation to the classical metric would occur in the continuum limit in which the separation a between loops goes to zero. However, the explicit calculation outlined above shows that this is not the case: as a tends to zero, the leading terms in the eigenvalues of $\hat{Q}[\omega]$ and $\mathcal{A}_S$ actually diverge! (One's first impulse from lattice gauge theories may be to say that the limit is divergent simply because we are not rescaling, i.e., renormalizing the operator appropriately. Note, however, that, in contrast to the calculations one performs in lattice theories, here, we *already* have a well defined operator in the continuum. We are only probing the properties of its eigenvectors and eigenvalues, whence there is nothing to renormalize.) It is, however, easy to see the reason underlying this behavior. Intuitively, the factors of the Planck length in (5.7) and (5.10) force each loop in the weave to contribute a Planck unit to the eigenvalue of the two geometrical observables. In the limit $a \rightarrow 0$, the number of loops in any fixed volume (relative to the fiducial h_{ab}) grows unboundedly and the eigenvalue diverges.
4. It is important to note the structure of the argument. In non-perturbative quantum gravity, there is no background space-time. Hence, terms such as "slowly varying" or "microscopic" or "macroscopic" have, a priori, no physical meaning. One must do some extra work, introduce some extra structure to make them meaningful. The required structure should come from the very questions one wants to ask. Here, the questions had to do with approximating a classical geometry. Therefore, we could *begin* with classical metric h_{ab} . We used it repeatedly in the construction: to introduce the length scale L , to speak of "slowly varying" fields ω_a and surfaces S , and, to construct the weave itself. The final result is then a consistency argument: If we construct the weave according to the given prescription, then we find that it approximates h_{ab} on macroscopic scales L provided we choose the mean separation a to be $\sqrt{\pi/2}l_p$, where all lengths are measured relative to the same h_{ab} .
5. Note that there is a considerable non-uniqueness in the construction. As we noted already, a given 3-geometry can lead to distinct weave states; our construction only serves to make the existence of such states explicit. For example, there is no reason to fix the radius r of the individual loops to be a . For the calculation to work, we only need to ensure that the loops are large enough so that they are generically linked and small enough so that the values of the slowly varying fields on each loop can be regarded as constants plus error terms which we can afford to keep in the final expression. Thus, it is easy to obtain a 2-parameter family of weave states, parametrized by r and a . The condition that the leading order terms reproduce the classical values determined by h_{ab} then gives a relation between r , a and l_p which again implies discreteness. Clearly, one can further enlarge this freedom considerably: There are a lot of eigenbras of the smeared-metric and the area operators whose eigenvalues approximate the classical values determined by

h_{ab} up to terms of the order $\mathcal{O}(l_p/L)$ since this approximation ignores Planck scale quantum fluctuations.

6. Finally, I would like to emphasize that, at a conceptual level, the important point is that the eigenvalues of $\hat{Q}[\omega]$ and $\mathcal{A}[S]$ can be *discrete*.

Let me conclude the discussion on weaves with two remarks. First, it is not difficult to extend the above construction to obtain weave states for curved metrics g_{ab} which are slowly varying with respect to a flat metric h_{ab} . Given such a metric, one can find a slowly varying tensor field t_a^b , such that the metric g_{ab} can be expressed as $t_a^c t_b^d h_{cd}$. Then, given a weave of the type $\langle \Delta |$ considered above approximating h_{ab} , we can “deform” each circle in the multi-loop Δ using t_a^b to obtain a new weave $\langle \Delta |_t$ which approximates g_{ab} in the same sense as $\langle \Delta |$ approximates h_{ab} . (See, also, Zegwaard (1992) for the weave corresponding to the Schwarzschild black-hole.) The second remark is that since the weaves are eigenbras of the operators that capture the 3-geometry, they do not approximate 4-geometries. To obtain a state that can approximate Minkowski space-time, for example, one has to consider a loop state that resembles a “coherent state” peaked at the weave Δ_a . In that state, neither the 3-geometry nor the time-derivative thereof would be sharp; but they would have minimum spreads allowed by the uncertainty principle. This issue has been examined in detail by Iwasaki and Rovelli (1993).

Since these results are both unexpected and interesting, it is important to probe their origin. We see no analogous results in familiar theories. For example, the eigenvalues of the fluxes of electric or magnetic fields are not quantized in QED nor do the linearized analogs of our geometric operators admit discrete eigenvalues in spin-2 gravity. Why then did we find qualitatively different results? The technical answer is simply that the familiar results refer to the *Fock representation* for photons and gravitons while we are using a completely different representation here. Thus, the results *are* tied to our specific choice of the representation. Why do we not use Fock or Fock-like states? It is not because we insist on working with loops rather than space-time fields such as connections. Indeed, one *can* translate the Fock representation of gravitons and photons to the loop picture. (See, e.g., Ashtekar et al. (1991) and Ashtekar & Rovelli, (1992).) And then, as in the Fock space, the discrete structures of the type we found in this section simply disappear. However, to construct these loop representations, one must use a flat background metric and essentially every step in the construction violates diffeomorphism invariance. Indeed, there is simply no way to construct “familiar, Fock-like” representations without spoiling the diffeomorphism invariance. Thus, the results we found are, in a sense, a direct consequence of our desire to carry out a genuinely non-perturbative quantization without introducing any background structure. However, we do not have a uniqueness theorem singling out the measure $d\mu$ which was used to define the loop transform and hence to construct the loop representation used here. There do exist other diffeomorphism invariant measures which will lead to other loop representations. The measure we have used is the most natural and the simplest among the known ones. Whether the results presented here depend sensitively on the choice of the measure is not known. Therefore, it would be highly desirable to have a uniqueness theorem which tells us that, on physical grounds, we should restrict ourselves to a specific (class of) measure(s).

My overall viewpoint is that one should simultaneously proceed along two lines: i) one should take these results as an indication that we are on the right track and push heuristic calculations within this general framework; and, ii) one should try to put the available heuristic results on rigorous mathematical footing to avoid the danger of “wandering off” in unsound directions.

6 Conclusion

In this article I have reported on some recent developments in non-perturbative quantum general relativity in 4 dimensions. Due to the space limitation, I could not go in to details, and, to avoid the discussion of certain technical points in the Hamiltonian framework of general relativity, I have omitted a couple of important issues. However, I hope I have given the general flavor of the type of problems we face and the strategies that have been devised. Many of the problems may indeed be unfamiliar in the world of rigorous quantum field theory since they are peculiar to general relativity. There is no background space-time, no obvious notion of microcausality, no reference to the asymptotic states of scattering theory. And yet, significant progress could be made. In 3 space-time dimensions, in particular, a complete, rigorous, mathematical treatment is available.

In 4 space-time dimensions, the program is far from being complete. But as I have tried to argue in the last two sections, a number of interesting results have emerged. The loop representation, in particular, furnishes brand new tools to tackle problems arising from the diffeomorphism invariance. Furthermore, the general program has matured at least to the extent that there is a well-defined mathematical framework in the background. Indeed, a number of open problems are precisely of the type that mathematical physicists can now make major contributions to the field. Finally, the results obtained so far may themselves have interesting implications for quantum field theory in general, quite outside the realm of quantum gravity.

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SOME OLD QUESTIONS IN QUANTUM FIELD THEORY AND A FEW NEW ANSWERS

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1 Introduction

When Streater and I were writing our book PCT, Spin and Statistics, and All That [1] in 1962-3, we felt that we were summarizing an intellectually coherent subject. The notion of a quantized field as an operator-valued distribution with an associated transformation law under a physical representation of the Poincaré group seemed to capture the essence of what had been learned up to that time about what a quantized field in relativistic field theory ought to be. Suitably elaborated, the theory seemed to deserve the name Jost gave it in his book [2]; the General Theory of Quantized Fields.

The situation had been quite different a decade earlier when Gårding and I first thought of using L. Schwartz's theory of distributions in the mathematical definition of the notion of quantized fields. Then the phrase "to find the solution of the equations of quantum electrodynamics" was assigned widely varying meanings, or, more commonly, accepted as meaningful without meaningful definitions of the terms appearing in it. The appearance of Schwartz's treatise on distributions and their applications was providential for our purposes [3]. There had been detailed mathematical studies of special classes of generalized functions before (Hadamard, Sobolev, et al.), but until Schwartz's theory of distributions appeared, there was no systematic calculus for widely applicable classes of generalized functions. On the other hand, in physics the advisability, indeed the necessity, of coping with what were then called singular functions was a commonplace after Dirac's introduction of the Dirac δ function. For example, in the context of the free electromagnetic field, the quantization scheme of Jordan and Pauli [4] had led to the commutation relations

$$[F_{k\lambda}(x), F_{\mu\nu}(y)] = [g_{k\mu}\partial_\lambda\partial_\nu - g_{k\nu}\partial_\lambda\partial_\mu - g_{\lambda\mu}\partial_k\partial_\nu + g_{\lambda\nu}\partial_k\partial_\mu]\frac{1}{i}D(x-y)$$

where

$$D(x) = \frac{1}{2\pi} \operatorname{sgn} x^0 \delta(x^2).$$

Now the fields appearing in this example actually make sense as operators in Hilbert space, when smeared with the characteristic functions of bounded regions in space-time:

$$F_{k\lambda}(x) \rightarrow \int \chi_V(x) F_{k\lambda}(x) d^4x = F_{k\lambda}(\chi_V)$$

where

$$\chi_V(x) = \begin{cases} 1 & \text{if } x \in V \\ 0 & \text{if } x \notin V. \end{cases}$$

There seemed to be no necessity to assume that a test function, f , had to have any smoothness properties to guarantee that

$$F_{k\lambda}(f) = \int d^4x f(x) F_{k\lambda}(x)$$

is an operator in Hilbert space. The discovery that this is not true in general is an interesting historical episode which I have related in detail elsewhere [5]. In capsule form, the story is this. The famous three-man paper by Born, Heisenberg and Jordan [6] was notable not only for its formulation of quantum mechanics as matrix mechanics but also for the construction of the first theory of a quantized field, the quantized vibrating string. Jordan was primarily responsible for this section of the paper.

He showed among other things, that the energy fluctuation in any finite length of the string is identical with that derived by Einstein in 1909 [7], by an astute application of statistical mechanics and the old quantum theory. This was generally regarded as a very satisfactory resolution of what had been a disputed issue. Unfortunately, five years later, Heisenberg discovered that the calculation was incorrect; the fluctuation is actually infinite [8]. In an attempt to understand this infinity, Heisenberg made an analogous calculation for the energy fluctuation in a finite region in the case of a Schrödinger particle moving freely on a line; it was also infinite. Heisenberg concluded that, whatever the cause of the infinity, it was not connected with the infinite zero-point energy of the vacuum because there is no such zero-point energy for a free Schrödinger particle. To get to the origin of the problem, he replaced the energy density integrated over a finite volume by the energy density smeared with a function, f , equal to one in the region and dropping smoothly to zero outside. He showed that the fluctuation in this quantity is finite but contains a term proportional to $(\partial f / \partial x)$ which blows up as f approaches the characteristic function of the region. Here was clear-cut evidence that quantized fields have to be smeared with functions having some degree of smoothness before one can expect finite fluctuations. Needless to say, Heisenberg was not in a position to exclaim: "Aha, fields in general have to be operator-valued Schwartz distributions", even if he had been so inclined. But Gårding and I did just that.

In our Preface, Streater and I said "... The Main Problem of quantum field theory turned out to be to kill it or cure it: either to show that the idealizations involved in the fundamental notions of the theory (relativistic invariance, quantum mechanics, local fields, etc.) are incompatible in some physical sense, or to recast the theory in such a form that it provides a practical language for the description of elementary particle dynamics." How close have we come to solving the Main Problem? Well, in the revised edition of our book in 1978, Streater and I took the position that a decade of constructive quantum field theory had effectively solved a restricted version of the Main Problem. What we said was: "... Thus, in our opinion, constructive quantum field theory has provided a very satisfactory solution of the Main Problem, but, so far, only for superrenormalizable theories in space-times of dimension ≤ 3 ." Since we made a conscientious attempt to describe what had happened, with reasonably full references, in our Appendix pp. 179-181, I will not repeat that here. What I would like to emphasize is that attempts to extend the methods that had been successful in the treatment of the $P(\phi)_2$, Y_2 , and ϕ_3^4 models to renormalizable theories and to theories in four-dimensional space-time were initially unsuccessful. What followed was a period of digestion in which the objective was to simplify the overwhelmingly complicated proofs

of Glimm, Jaffe, and Spencer using new ideas from heuristic field theory, especially the renormalization group.

I should also emphasize that the long and grueling process of incorporation of new techniques of proof, assimilation of hard-won results and simplification of their proofs seems to be indispensable in a subject as difficult and complicated as quantum field theory. In any case, by the early 1980's, the fruits of application of rigorous renormalization group methods began to appear. For example, the solution of the Gross-Neveu model in two-dimensional space-time, GN_2 , was constructed by Kupiainen and Gawedzki [9] and independently by Feldman, Magnen, Sénéor and Rivasseau [10]. It is renormalizable (not super-renormalizable). Constructive gauge field theory, which was a particularly attractive target if one started in the manifestly gauge-invariant lattice formulation, made impressive progress. Indeed, the cutoff-free limit of Higgs₃ was shown to exist by C. King [11], and Balaban took the first step towards a treatment of non-abelian YM_4 , a proof of ultraviolet stability [12].

And old things really did get simpler. The Brydges, Fröhlich, Sokal proof of the existence of a solution of the Euclidean ϕ_3^4 model for small values of the coupling constant [13] can be given in three hours of lecture. Even perturbative renormalization theory of QED_4 and ϕ_4^4 has been drastically simplified. Using expansions introduced by Gallavotti in his rigorous renormalization group discussion of ϕ_3^4 , [14], Feldman, Hurd, Rosen and Wright [15] gave a mathematically rigorous and conceptually clear treatment of the perturbative renormalization of QED_4 . But even that is not the end of the story. The idea has been floating around, since the work of Polchinski [16] and Brydges and Kennedy [7], that renormalization group theory might be simpler if one worked with the differential equations rather than the difference equations satisfied by finite transformations. G. Keller, C. Kopper and M. Salmhofer have worked this out in detail for QED_4 and ϕ_4^4 and it's true; things are very much simpler [18].

As new non-perturbative results resulting from improved techniques (and a lot of hard work), I will mention only two examples. The heroic thesis of Paulo de Veiga (Ecole Polytechnique [19] 1990) has given us the first example of the solution of a non-trivial non-renormalizable field theory, GN_3 . Magnen, Rivasseau and Sénéor have announced the construction of a nonabelian gauge field theory (with $SU(2)$ as gauge group) free of ultra violet cutoffs but in a finite box [20].

With all these achievements in mind, it is natural to ask the question: Is there some obstacle of principle in the passage to non-trivial theories in four-dimensional space-time? In particular, is there an obstruction to the construction of the solutions of the theories that dominate particle physics today: QCD, electroweak theory and their combination in the standard model? We do not yet know but most of the partial results so far obtained are consistent with the idea that such theories exist, but are hellishly complicated to construct using present techniques. I say most of the evidence because there is the set of no-go theorems of Aizenman [21] and Fröhlich [22]. Under some assumptions, they show that when the cutoffs are removed in ϕ_4^4 , the result is a physically trivial theory. I will have something to say about these results, which surely are among the most interesting of the field theory of the 1980's, in the following section.

My personal opinion about the current situation in quantum field theory is that we will need more analysis of concrete examples before we will be able to develop a reliable intuition about the general structure of field theories, especially those in three

and four dimensional space-time. Much more work will likely be needed before rigorous quantum field theory is a practical working tool of particle physics.

In the following section, I compile a list of questions which seem to me natural given this point of view.

2 Some Questions

Structure Theory and Two-Dimensional Conformal Field Theories

When Borchers introduced his algebra in 1958 [23], one of the motivations was the idea that the so-called reconstruction theorem of quantum field theory can be expressed as the analogue for a general topological $*$ algebra of the GNS construction for a C^* algebra. More explicitly, if the $*$ algebra is $\mathcal{A}$ and ρ is a positive linear form on $\mathcal{A}$ then there exists a cyclic representation π_ρ of $\mathcal{A}$ in a Hilbert space $\mathcal{H}_\rho$, with a cyclic vector ψ_ρ , such that

$$\rho(A) = (\psi_\rho, \pi_\rho(A)\psi_\rho).$$

(That π_ρ is cyclic with cyclic vector ψ_ρ means that the set of vectors $\pi_\rho(\mathcal{A})\psi_\rho$ is dense in $\mathcal{H}_\rho$.) It was natural to hope that a systematic application of the ideas of representation theory to the Borchers algebra $\mathcal{A}$ would yield some kind of classification of quantum field theories.

This aspiration of the general theory of quantized fields remained essentially unachieved despite some nice results of Borchers such as the constancy of the S -matrix in equivalence classes of relatively local fields. However, in a more specific context, that of scale-invariant field theories in two-dimensional space-time there has been great progress in the last decade, which can be interpreted as a classification of conformal invariant field theories in two-dimensional space-time. Since other speakers will have a great deal to say about those results, I will limit myself here to remarks necessary to set the scene for my questions.

It was already recognized in the 1970's that, in two-dimensional space-time, conformally invariant theories give rise to a special algebra, because the energy-momentum tensor is a sum of contributions depending on one light cone variable: $x_+ = x^0 + x^1$, or the other: $x_- = x^0 - x^1$, but not both. I should mention in this connection at least Ferrara, Grillo and Gatto [24] (who seem to have been the first to call it the Virasoro algebra, by analogy with the algebra previously constructed in dual models), Schroer [25], and Mack and Lusscher [26]. The idea behind the argument is that Lorentz boosts, which act on $x_\pm$ by

$$x_\pm \rightarrow e^{\pm\lambda} x_\pm$$

and scaling transformations which act as

$$x_\pm \rightarrow \rho x_\pm$$

can be combined to yield the transformation

$$x_+ \rightarrow e^{2\lambda} x_+, x_- \rightarrow x_-$$

and

$$x_+ \rightarrow x_+, x_- \rightarrow e^{-2\lambda} x_-$$

so that the dependence of suitable linear combinations of the $T_{\mu\nu}$ on $x_{\pm}$ can be studied separately. The result is that

$$T_{00} = T_{11}$$

and that

$$T_{\pm} = T_{00} \pm T_{01}$$

depend only on $x_{\pm}$. With this simplified dependence of $T_{\pm}(x)$ on x the commutator $[T_+(x_+), T_-(y_-)]$ can vanish for spacelike $x - y$ only if it vanishes for all x and y . The two commutators $[T_+(x_+), T_+(y_+)]$ and $[T_-(x_-), T_-(y_-)]$ do not vanish, but can be evaluated for all values of x and y . The result is

$$[T_+(x_+), T_+(y_+)] = \frac{c_+}{6\pi} i^3 \delta'''((x-y)_+) + 4i\delta'((x-y)_+) - 2i\delta((x-y)_+) \partial_y T_+(y_+)$$

and similarly for $T_-(x_-)$. Here $c_+ \geq 0$ is a constant called the central charge.

With these results in mind I ask my first question.

Q1: What becomes of the Virasoro algebra when a mass is introduced?

A good example on which to begin to study this question is the Thirring model with mass m . For $m = 0$, it illustrates the above general discussion. What happens when $m \neq 0$? Does the free wave equation

$$\square T_{\mu\nu} = 0,$$

valid by virtue of the above argument, become a local partial differential equation for T ?

The decompositions of the energy-momentum tensor of a two dimensional conformal invariant theory:

$$(2.1) \quad \begin{aligned} T_{00}(x) &= \frac{1}{2} [T_+(x_+) + T_-(x_-)] \\ T_{01}(x) &= \frac{1}{2} [T_+(x_+) - T_-(x_-)] \end{aligned}$$

are reminiscent of but not quite the same as solutions of the characteristic initial value problem for the wave equation. See Figure 1.

If the second partial $F_{vu}(u, v) = 0$, then

$$\begin{aligned} 0 &= \int_0^u da \int_0^v db \frac{\partial^2}{\partial a \partial b} F(a, b) = \int_0^u da \left(\frac{\partial F(a, v)}{\partial a} - \frac{\partial F(a, 0)}{\partial a} \right) \\ &= F(u, v) - F(0, v) - F(u, 0) + F(0, 0) \end{aligned}$$

so instead of (2.1) one would have

$$T^{00}(x_+, x_-) = T^{00}(x_+, 0) + T^{00}(0, x_-) - T^{00}(0, 0)$$

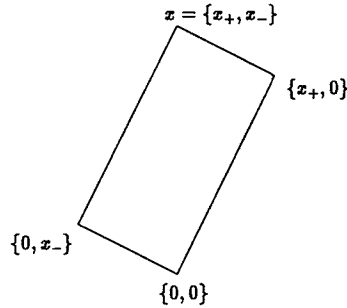


Figure 1: The characteristic initial value problem for the wave equation in two-dimensional space-time

Now, for the application to quantum field theory, the term $T^{00}(0,0)$ is abhorrent, but it is there to cancel abhorrent parts of $T^{00}(x_+, 0)$ and $T^{00}(0, x_-)$ as becomes clear if the right hand side is rewritten

$$\frac{1}{2}[T_+(x_+) + T_-(0)] + \frac{1}{2}[T_+(0) + T_-(x_-)] - \frac{1}{2}[T_+(0) + T_-(0)] = \frac{1}{2}[T_+(x_+) + T_-(x_-)]$$

This interpretation of (2.1) in terms of the characteristic initial value problem suggests my second question:

Q2: What is the analogue of the Virasoro algebra for conformal field theories in space-times of dimensions three and four? In particular, is there an analogue of the decomposition of the energy momentum tensor in which $T^{\mu\nu}$ is written as a weighted integral over contributions from the generating lines of a light cone? See Figure 2:

It probably won't come as a surprise to you that the proposed Q2 is really very old, although I only realized that when I was writing up this talk. The idea expressed in Figure 2 was already described 20 years ago in [28] (see especially equation (2)), but, so far as I know, nobody has actually made it work for the energy momentum tensor.

As far as the first part of Q2 is concerned, Gerhard Mack has already provided his answer in his Cargèse lectures of 1987 [27]. He starts from the tensor algebra defined by the test functions $f_{\mu\nu}$ for the energy momentum tensor in space-time dimension $d \geq 2$ and divides out by the equivalence relations expressing locality,

$$[T_{\mu\nu}(x), T_{k\lambda}(y)] \equiv 0 \quad \text{for } (x - y)^2 < 0$$

symmetry,

$$T_{\mu\nu}(x) - T_{\nu\mu}(x) \equiv 0$$

conservation law,

$$\partial_\mu T^{\mu\nu}(x) \equiv 0$$

and the transformation law of $T_{\mu\nu}$ under infinitesimal conformal transformations. Representations of the resulting algebra define energy momentum tensors of conformally

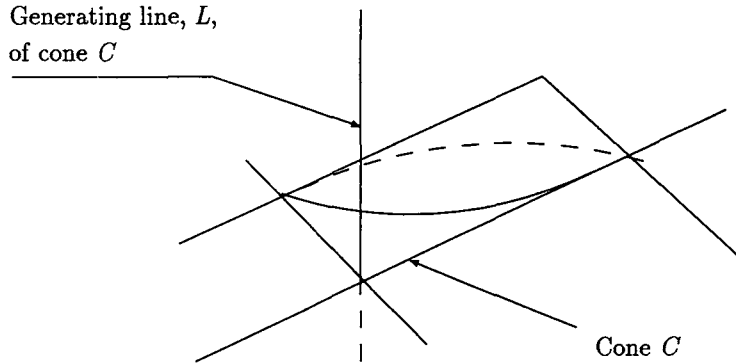


Figure 2: The cone C is generated by a $d - 2$ parameter family of lines L for d -dimensional space-time. Perhaps the energy-momentum tensor decomposes in an analogous way

invariant quantum field theories. For $d = 2$ the θ -Borchers algebra reduces to the enveloping algebra of $\text{Virasoro} \otimes \text{Virasoro}$. So why shouldn't Mack's algebra be exactly what is needed for $d \geq 3$? Perhaps it is but there is one thing missing: some analogue of the decomposition (2.1). That is one reason why people are led to the ideas expressed in Figure 2.

The situation again illustrates the point that we lack exactly soluble models for $d \geq 3$. Such models could serve as a guide in attempts to answer Q2.

When a conformally invariant theory has a set of conserved currents, there is an analogue of the above discussion of the Virasoro algebra. Since Ivan Todorov will give a critical review of the present state of that theory, I will say no more here.

How to Approach Quantum Gravity, Gently

In a recent paper, Buchholz and Summers [29] have shown that one can tell from the algebraic structure of the quasi-local algebra of a Haag-Kastler theory that it is the quasi-local algebra, $\mathcal{A}$, of an essentially uniquely determined Minkowski space. In other words, the Minkowski geometry is reflected in and determined by the algebraic structure of $\mathcal{A}$. This suggests the following question (it's really Buchholz's and Summers' question).

Q3: Can the geometry of other (curved) space-times also be characterized in terms of the algebraic structure of their quasi-local algebras?

My only contribution to this matter is the following remark. If the answer to Q3 is positive, the algebraic structure involved could be regarded as the old quantum theory of a theory of gravitation. The parallel may be displayed as in Figure 3.

What the quantization rules of the old quantum theory do is to single out a special orbit, the quantized orbit from among all possible classical orbits. What the putative algebraic characterization would do is to single out a classical space-time from among

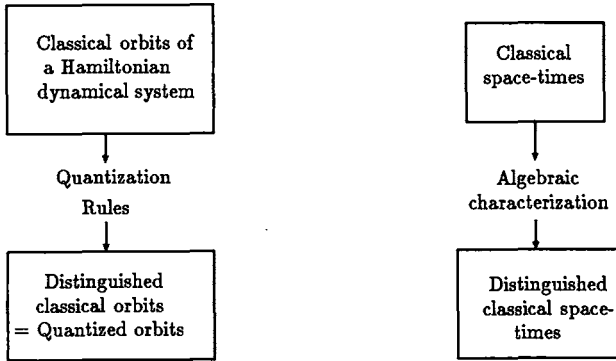


Figure 3: An algebraic characterization of space-time is an old quantum theory of gravitation

all possible classical space-times. This might be regarded as a preliminary to a true quantum theory of gravitation which would work with non-commutative differential geometry throughout.

Phase Diagrams and No-Go Theorems

The no-go theorems of Aizenman and Fröhlich, mentioned above [21] [22], are surely among the most remarkable results of the constructive or, as Alan Sokal would have us say, destructive quantum field theory of the 1980's. The theorems state, under rather general assumptions, that, if one starts from the (surely non-trivial) ferromagnetic lattice approximation to ϕ^4 , the limit: *lattice spacing* $\rightarrow 0$ yields a physically trivial theory. I have been puzzling over the significance of these results and others of a similar character for a long time, [30] [31], without coming to any very well defined conclusion. At the risk of boring those of you who have heard me discuss these problems several times before, I would like to take them up again.

Let me begin with the ϕ^4_2 theory. It is, of course, based on the Lagrangian density

$$\mathcal{L} = \frac{1}{2} [: \partial_\mu \phi \partial^\mu \phi : - m_0^2 : \phi^2 :] - \frac{\lambda}{4!} : \phi^4 :$$

and so one expects solutions to depend on the parameter m_0^2 , and for $m_0^2 > 0$, on the dimensionless parameter λ/m_0^2 . It was a triumph of the constructive field theory of the 1970's that these solutions were constructed with full mathematical rigor. Furthermore, it was shown that there is a value $(\lambda/m_0^2)_{\text{crit}}$ such that for $m_0^2 > 0$ and $0 \leq \lambda/m_0^2 < (\lambda/m_0^2)_{\text{crit}}$ the solution is unique, while for $\lambda/m_0^2 > (\lambda/m_0^2)_{\text{crit}}$ there are two solutions ϕ^I and ϕ^{II} with corresponding vacuum states ψ_0^I and ψ_0^{II} such that

$$(\psi_0^I, \phi^I(x) \psi_0^I) = -(\psi_0^{II}, \phi^{II}(x) \psi_0^{II}) \neq 0$$

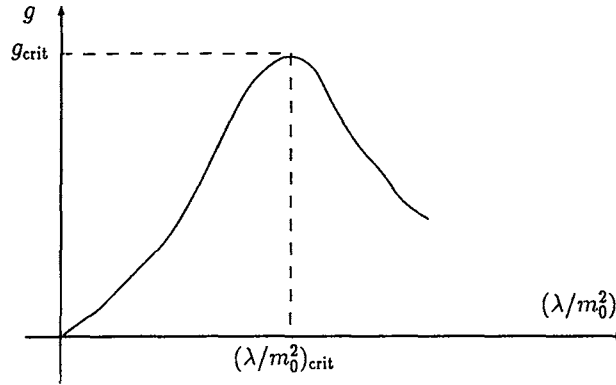


Figure 4: The scale invariant dimensionless coupling constant, g , is a function of the bare dimensionless coupling constant λ/m_0^2

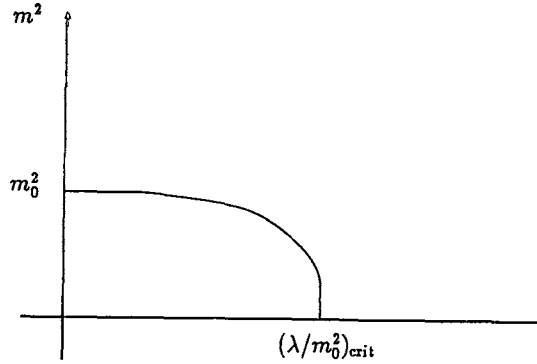


Figure 5: Square of the mass gap

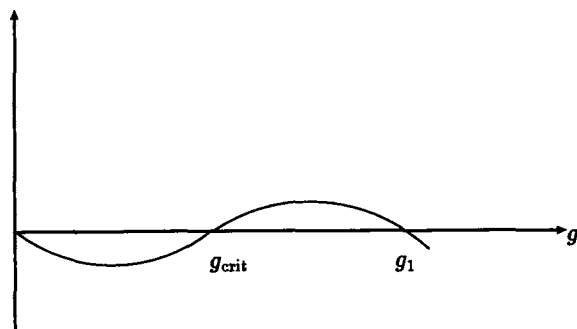
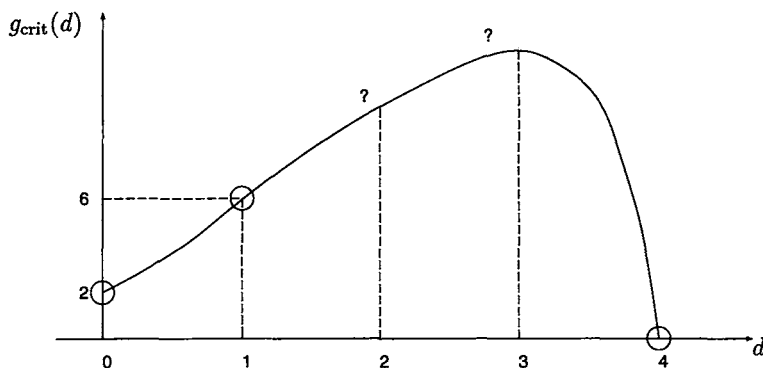
Here $(1234)_T$ is the truncated vacuum expectation value of the product of four fields, and $(12)_T$ the analogous quantity for two fields, the bar denotes integration over all but one of the variables. The coupling constant g is independent of m_0^2 and so depends on (λ/m_0^2) alone. Glimm and Jaffe proved that g is bounded as a function of (λ/m_0^2) [32]. They conjectured that g assumes its maximum only for $(\lambda/m_0^2)_{\text{crit}}$; this they called critical point dominance. Qualitatively, the graph of g is expected to look like Figure (4). Notice that a single value of the coupling constant g corresponds to three distinct theories if $g < g_{\text{crit}} = g(\lambda/m_0^2)_{\text{crit}}$ or none if $g > g_{\text{crit}}$. Meanwhile, the mass gap, m , can be written as

$$m^2 = m_0^2 f(\lambda/m_0^2)$$

and is expected to look like Figure (5).

It has been proved that $\lim f(\lambda/m_0^2) = 0$ as λ/m_0^2 approaches $(\lambda/m_0^2)_{\text{crit}}$ from below but what happens for $\lambda/m_0^2 > (\lambda/m_0^2)_{\text{crit}}$ is not known except asymptotically for large (λ/m_0^2) [33].

Now this behavior is quite different from that discussed in old papers on the renormalization group where one plotted the famous function β , as in Figure (6).

Figure 6: The g -dependence of the β -functionFigure 7: Bounds on the coupling constant, g , proved and conjectured

According to the corresponding conventional wisdom, for g satisfying $g_{\text{crit}} < g < g_1$ there should be a theory whose ultra-violet behavior is determined by that of the theory at $g = g_1$ while its infrared behavior is determined by that of the theory at $g = g_{\text{crit}}$. The Glimm-Jaffe bound for g shows that there is *no* solution for sufficiently large g and their conjecture amounts to the statement: no solution for $g > g_{\text{crit}}$. Thus, the Glimm-Jaffe bound provides the first of the no-go theorems. Later on, Newman was able to verify the analogue of this behavior for $d = 0, 1$ and to show that for $d = 0$, $g_{\text{crit}} = 2$ and for $d = 1$, $g_{\text{crit}} = 6$. So we have the graph of Figure (7):

The dashed curve is purely conjectural at the moment, but it suggests that it might be reasonable to try to interpolate in dimension d , and to regard each point $\{m, g, d\}$ as a label for a solution in the single phase region.

What were the physical assumptions that permitted Glimm and Jaffe to prove the boundedness of g ? The crucial assumption appears to be

$$\int_0^\infty \frac{d\rho(a^2)}{a} < \infty$$

where the spectral representation of the two-point distribution is

$$(\psi_0, \phi(x)\phi(y)\psi_0) = \int_0^\infty d\rho(a^2) \frac{1}{i} \Delta_d^{(+)}(a^2, x-y)$$

with

$$\frac{1}{i} \Delta_d^{(+)}(a^2, x) = \frac{1}{2(2\pi)^{d-1}} \int d\Omega_{a^2}(p) \exp(-ip - x)$$

and

$$d\Omega_{a^2}(p) = \frac{d^{d-1}p}{\sqrt{a^2 + \vec{p}^2}}$$

the invariant measure on the positive energy hyperboloid of mass $a\{p; p^0 = \sqrt{a^2 + \vec{p}^2}\}$. This leads to the general question

Q4: Does there exist a solution of ϕ_d^4 theory for $g > g_{\text{crit}}(d)$? In particular, could such a solution evade the coupling constant bound by being a non-tempered field?

There is also a more specific question:

Q5: What are the numerical values of $g_{\text{crit}}(2)$ and $g_{\text{crit}}(3)$? Does constructive quantum field theory provide existence theorems for solutions of the ϕ_2^4 and ϕ_3^4 models for all g satisfying $0 \leq g \leq g_{\text{crit}}(d)$?

There appears to be no known reason why there should not exist a non-tempered solution for $g > g_{\text{crit}}$. On the other hand, having a phase boundary at $g = g_{\text{crit}}$ with no phase beyond it is not excluded by any known principle.

It is instructive to examine exactly soluble models for the light they throw on this question. For the Thirring model, K. Johnson gave an expression for the two-point distribution which contains a factor

$$\sin \left[\pi \left(\frac{\lambda}{2\pi} \right) \left(1 - \left(\frac{\lambda}{2\pi} \right)^2 \right)^{-1} \right],$$

where λ is the Fermi interaction coupling constant [35]. This factor is non-negative for $0 \leq \left(\frac{\lambda}{2\pi} \right)^2 \leq \frac{1}{2}$ but negative for $\frac{1}{2} < \left(\frac{\lambda}{2\pi} \right)^2 < \frac{2}{3}$. As a consequence, the Thirring model has a no-go theorem for the latter range of the coupling constant. The solutions exist there but not in

a Hilbert space with a positive definite scalar product.

3 Dual Potentials of Conserved Currents

I have been working on three exactly soluble model quantum field theories recently, one in each of the space-time dimensions 2, 3, and 4 [36] [37] [38]. They have the unusual feature that what worked for the two-dimensional case thirty years ago appears to be generalizable to higher dimensions.

The fields in question are the dual potentials of conserved currents K for $d = 2$, K_λ for $d = 3$, and $K_{\lambda\mu}$ for $d = 4$ defined as follows. If the current is j^μ , $\mu = 0, 1, \dots, d-1$ in space-time dimension d , then its dual is defined by

$$(3.1) \quad J_{\mu_1 \dots \mu_{d-1}} = \varepsilon_{\mu_1 \dots \mu_d} j^{\mu_d}$$

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$$(3.1) \quad J_{\mu_1 \dots \mu_{d-1}} = \varepsilon_{\mu_1 \dots \mu_d} j^{\mu_d}$$

where $\varepsilon_{\mu_1 \dots \mu_d}$ is the totally antisymmetric symbol in d dimensions and

$$\varepsilon^{01 \dots (d-1)} = 1.$$

Because (3.1) can be inverted to give

$$j^{\mu_1} = \frac{1}{(d-1)!} \varepsilon^{\mu_1 \dots \mu_d} J_{\mu_2 \dots \mu_d}$$

the conservation law

$$\partial_\mu j^\mu = 0$$

can be written as

$$\partial_{\mu_1} \varepsilon^{\mu_1 \dots \mu_d} J_{\mu_2 \dots \mu_d} = 0$$

which says, if J is regarded as a $(d-1)$ -form, that

$$dJ = 0.$$

The dual potential K is then a $(d-2)$ -form satisfying

$$J = dK.$$

Explicitly,

$$\begin{aligned} d = 2 & \quad J_\mu = \partial_\mu K, & j^\mu &= \varepsilon^{\mu\nu} \partial_\nu K \\ d = 3 & \quad J_{\lambda\mu} = \partial_\lambda K_\mu - \partial_\mu K_\lambda, & j^\lambda &= \varepsilon^{\lambda\mu\nu} \partial_\mu K_\nu \\ d = 4 & \quad J_{k\lambda\mu} = \partial_k K_{\lambda\mu} + \partial_\lambda K_{\mu k} + \partial_\mu K_{k\lambda}, & j^k &= \frac{1}{2} \varepsilon^{k\lambda\mu\nu} \partial_\lambda K_{\mu\nu} \end{aligned}$$

The existence of K , K_λ , and $K_{\lambda\mu}$ can be established by direct construction for the special case of the currents arising from a free massive Dirac spinor field, ψ :

$$j^\mu(x) =: \psi^\dagger \gamma^\mu \psi : (x)$$

where

$$(3.2) \quad (-i\gamma^{k\mu} \partial_\mu + m)\psi(x) = 0$$

and

$$(3.3) \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$g^{00} = +1 = -g^{jj}, \quad j = 1, \dots, d-1 \quad g^{\mu\nu} = 0, \quad \mu \neq \nu.$$

I will not describe the details of that construction but instead will show how the Dirac equation can be used to derive equations of motion for K , K_λ , and $K_{\lambda\mu}$.

$d=2$

Multiply (3.2) from the left by $\gamma^5 = \gamma^0 \gamma^1$ and the adjoint equation for $\psi^+ = \bar{\psi} \gamma^0$

$$(3.4) \quad i(\partial_\mu \psi^+) \gamma^\mu + m \psi^+ = 0$$

from the right by γ^5 . Contract with ψ^+ and ψ respectively and add the two equations. The result is after the subtraction of a vacuum expectation value

$$(3.5) \quad \partial_\mu (: \psi^+ \gamma^5 \gamma^\mu \psi :) = -2m i : \psi^+ \gamma^5 \psi :$$

But

$$J_\mu = \varepsilon_{\mu\nu} : \psi^+ \gamma^\nu \psi := \psi^+ \gamma^5 \gamma^\mu \psi := \partial_\mu K$$

so (3.5) may be written

$$(3.6) \quad \square K = -2m i : \psi^+ \gamma^5 \psi :$$

Coleman recognized that the right hand side can be written as a local function of K :

$$(3.7) \quad \square K = -\frac{m^2}{\sqrt{4\pi}} \sin \sqrt{4\pi} K$$

which is the Sine-Gordon equation

$$\square K = -\frac{m^2}{\beta} \sin \beta K$$

written for the special value $\beta^2 = 4\pi$.

$d=3$

The procedure is analogous to that used for $d=2$. Contract (3.2) from the left with $\psi^+ \gamma^\lambda$ and (3.4) from the right with $\gamma^\lambda \psi$. Use

$$\gamma^\lambda \gamma^\mu = g^{\lambda\mu} - i\sigma^{\lambda\mu}$$

where

$$\sigma^{\lambda\mu} = \frac{i}{2} (\gamma^\lambda \gamma^\mu - \gamma^\mu \gamma^\lambda)$$

to get

$$\begin{aligned} -i\psi^+\partial^\lambda\psi - \psi^+\sigma^{\lambda\mu}\partial_\mu\psi + m\psi^+\gamma^\lambda\psi &= 0 \\ i(\partial^\lambda\psi^+)\psi - \psi^+\sigma^{\lambda\mu}\partial_\mu\psi + m\psi^+\gamma^\lambda\psi &= 0. \end{aligned}$$

Add to get

$$(3.8) \quad -i[\psi^+(\partial^\lambda\psi) - (\partial^\lambda\psi^+)\psi] - \partial_\mu(\psi^+\sigma^{\lambda\mu}\psi) + 2m\psi^+\gamma^\lambda\psi = 0.$$

This is Gordon's decomposition of the Dirac current into the sum of a convection current

$$\frac{i}{2m}[\psi^+(\partial^\lambda\psi) - (\partial^\lambda\psi^+)\psi]$$

and a spin current

$$\frac{1}{2m}\partial_\mu(\psi^+\sigma^{\lambda\mu}\psi).$$

We write the Dirac current in terms of K_λ , and restore the double dots

$$:\psi^+\gamma^\lambda\psi := \varepsilon^{\lambda\mu\nu}\partial_\mu K_\nu.$$

Now to deal with the spin current we use the identity

$$\varepsilon^{\lambda\mu\nu}\gamma_\nu = -\tilde{\gamma}^5\sigma^{\lambda\mu}$$

where

$$\tilde{\gamma}^5 = -i\gamma^0\gamma^1\gamma^2$$

$\tilde{\gamma}^5$ is +1 for the irreducible representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of (3.3), and -1 for the irreducible representation given by the negatives of these. Thus

$$J^{\lambda\mu} = \partial^\lambda K^\mu - \partial^\mu K^\lambda = \varepsilon^{\lambda\mu\nu} : \psi^+\gamma_\nu\psi := -\tilde{\gamma}^5 : \psi^+\sigma^{\lambda\mu}\psi :$$

and (3.8) can be written as

$$(3.9) \quad \tilde{\gamma}^5[\partial^\lambda(\partial_\mu K^\mu) - \square K^\lambda] + 2m\varepsilon^{\lambda\mu\nu}\partial_\mu K_\nu = i : [\psi^+\partial^\lambda\psi - \partial^\lambda\psi^+\psi] :$$

This is the analogue for $d = 3$ of equation (3.6) for $d = 2$. The remaining problem (unsolved as of this writing) is to write the convection current on the right hand side as a function of K_λ .

d = 4

Contract the Dirac equation from the left with $\psi^+\gamma^5\gamma^\kappa\gamma^\lambda$ and its adjoint from the right with $\gamma^5\gamma^\kappa\gamma^\lambda\psi$:

$$\begin{aligned} -i\psi^+\gamma^5\gamma^\kappa\gamma^\lambda\gamma^\mu\partial_\mu\psi + m\psi^+\gamma^5\gamma^\kappa\gamma^\lambda\psi &= 0, \\ i(\partial_\mu\psi^+)\gamma^\mu\gamma^5\gamma^\kappa\gamma^\lambda\psi + m\psi^+\gamma^5\gamma^\kappa\gamma^\lambda\psi &= 0. \end{aligned}$$

Insert the identity

$$(3.10) \quad \gamma^\kappa \gamma^\lambda \gamma^\mu = g^{\lambda\mu} \gamma^\kappa - g^{\kappa\mu} \gamma^\lambda + g^{\kappa\lambda} \gamma^\mu + \varepsilon^{\kappa\lambda\mu\nu} \gamma^5 \gamma_\nu$$

and add

$$\begin{aligned} & -i\partial_\mu [g^{\kappa\lambda} \psi^+ \gamma^5 \gamma^\mu \psi + \psi^+ \varepsilon^{\kappa\lambda\mu\nu} \gamma_\nu \psi] - i\psi^+ [\gamma^5 \gamma^\kappa \partial^\lambda \psi - \gamma^5 \gamma^\lambda \partial^\kappa \psi] \\ & + i[\partial^\lambda \psi^+] \gamma^5 \gamma^\kappa - (\partial^\kappa \psi^+) \gamma^5 \gamma^\lambda \psi + 2m\psi^+ \gamma^5 (g^{\kappa\lambda} - i\sigma^{\kappa\lambda}) \psi = 0 \end{aligned}$$

The terms proportional to $g^{\kappa\lambda}$ cancel. When the dual current $J^{\kappa\lambda\mu}$ is written in terms of $K^{\kappa\lambda}$:

$$\partial_\mu (\varepsilon^{\kappa\lambda\mu\nu} \psi^+ \gamma_\nu \psi) = \square K^{\kappa\lambda} + \partial^\kappa \partial_\mu K^{\lambda\mu} - \partial^\lambda \partial_\mu K^{\kappa\mu}$$

so that

$$\begin{aligned} \square K^{\kappa\lambda} + \partial^\kappa \partial_\mu K^{\lambda\mu} - \partial^\lambda \partial_\mu K^{\kappa\mu} = & -2m : \psi^+ \sigma^{\kappa\lambda} \psi : + : (\partial^\lambda \psi^+) \gamma^5 \gamma^\kappa \psi : \\ & - : (\partial^\kappa \psi^+) \gamma^5 \gamma^\lambda \psi : - : \psi^+ \gamma^5 \gamma^\kappa \gamma^\lambda \psi : + : \psi^+ \gamma^5 \gamma^\lambda \gamma^\kappa \psi : \end{aligned}$$

This is the equation of motion for $K^{\kappa\lambda}$; the problem of rewriting the right hand side as a function of $K^{\kappa\lambda}$ is unsolved as of the moment.

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PERTURBATION THEORY*

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The seed ye sow, another reaps[‡]

1 Mathematical Physics In General

It is difficult to find a subject to discuss as afterthought about a conference on *Mathematical Physics towards the XXIst century*. I will describe some personal viewpoints and discuss a recent work (with some variations with respect to the previous versions) to stress some of the points made.

If I were to say what is the legacy of Mathematical Physics to the next century I would point to a subject which has been left out of the talks by most of the speakers: The understanding and formalization of perturbation techniques for problems quite different from classical mechanics (and for classical mechanics as well).

It is of course still true that the so called “non-perturbative techniques” are more impressive and glamorous; they also look deeper. And there are other aspects of Mathematical Physics that have received a large impetus in this century.

Let me begin by stressing the similarities between the perturbative and non-perturbative techniques in Mathematical Physics. What I see as common to the two is what I consider to be a very basic aspect of Mathematical Physics which, although a going back to the XIXth century, has only become clear during the XXth century: the attempt, sometimes successful, to put on a really rational (*i.e.*, mathematically rigorous) basis the fundamental problems of Physics.

It is not an overstatement to claim that many important results came out of attempts to set in a clear form problems which had been treated too empirically. For instance the Dirac formulation of quantum mechanics is comparable to the Lagrange and Hamilton formalization of classical mechanics. The latter made available to everybody the possibility of computing in an unambiguous way the solution to problems of mechanics which, until then, had been accessible only to few people, who managed to dominate the rather non-systematic state of the matter at the time. After Dirac’s work “everybody” could work out Atomic Physics computations and predictions.

* *Keywords*: Classical mechanics, Quantum field theory, Solid state physics, Statistical mechanics

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[‡]P. B. Shelley, *Song to the Men of England*

The attempt to axiomatize quantum field theory and renormalization theory (by Wightman and the Zürich school, [SW]) led first (as a byproduct) to the clear formulation of the major problems of Statistical Mechanics (Ruelle, Fisher and the Moscow school, [R]), and generated a great number of new results and a deep understanding of the phenomena of phase transitions. This opened the way to the renormalization group theory of scale invariance and universality (Widom, Kadanoff, Fisher, Wilson, [Ma], [P]) which, eventually, brought about a good understanding of renormalization theory and quantum field theory themselves.

Turbulence is a phenomenon which was typically only empirically described and scarcely "analysed objectively". Attempting its precise definition produced surprisingly new results and points of view (Lorenz, Ruelle, [R2], [ER]): and nowadays everybody knows about "chaos", which was essentially unknown only twenty years ago.

General relativity is even more exceptional as it is an example (the only one I know) of a theory born as a mathematically well formulated theory, [W].

All the above examples, which are not exhaustive and only reflect my personal interests, are remarkable: But I see many colleagues (and have seen many more of them in the past) saying that all of the above were not so important. The real problems would have been, or had been, indeed solved without the need for refined mathematics or precise formulations. This attitude is quite erroneous in my view at least as a general statement, as the examples of the theory of chaos or of general relativity show, in a clear way, to those who wish to see.

It is nevertheless true that most progress was achieved before a precise mathematical formulation: for instance the Schrödinger equation came before Dirac's work. Schwinger, Feynman and Dyson understood renormalization theory before the work of the mathematical physicists (Hepp, [H], Glimm, Jaffe, Nelson, Guerra, Spencer, [Si], [E]). In the same way the work of Newton, D'Alembert, Laplace, Gauss, Euler¹ preceded analytical mechanics.

The reason I see the mathematical formulations as important, even in fields in which "progress has already been achieved", is that the rigorization attempts had a "democratic nature": They made, and make, accessible to the vast majority ideas and methods that were (obviously) perfectly understood by only a few scientists. And, if science has to advance, it needs democracy", *i.e.*, many scientists who control what they are doing.

Mathematical Physics in this century had mostly the role of making accessible results that were very deep and difficult to understand because the scientists who developed them found no time (or no need) to explain them in a form that could be widely understood. The mathematical language is really universal, as Galileo explicitly noted, and is very well suited for the transmission of knowledge.

2 Fundamental, Exact, Non-perturbative and Constructive Methods

One can distinguish roughly four aspects of Mathematical Physics. I will call the first "fundamental": it consists of the analysis of the general structures of various problems,

¹I close the list only for lack of space!

and of their proper mathematical formulation. As examples we have the formalisms of statistical mechanics (general notion of equilibrium states, thermodynamic limit, phase transitions, ground states, correlations), field theory (general notion of relativistic quantum field, interaction) or fluid mechanics (general notion of regular and chaotic motions). In this conference the above aspects have been stressed and discussed in considerable detail.

I consider the “fundamental” aspects as essential, in the same sense as Hamiltonian mechanics is essential for classical mechanics. It is true that one could just ignore the formalisms: but at least in the case of Hamiltonian mechanics hardly anybody would dare to say that it is useless².

But others could still insist that the formalisms are useless (it is not uncommon to meet people who refuse to try to understand the general theory of chaos which has emerged from the work of Ruelle on the grounds that it is a formalism while “science deals with concrete problems”). The reason I am not too impressed by such statements is that Mathematical Physics has been able to go far beyond the formalisms in all the examples I quoted. The formalism has been used to provide a framework into which to formulate precisely concrete problems to be solved, *and their solutions*. I would, in fact, also agree in considering of little interest a formalism which did not attack (and solve, at least partially) some physical problems.

A way to make use of a formalism is by discussing in its framework the properties of concrete models. As some models can be studied “exactly”, this brings up a second aspect of Mathematical Physics.

This century will leave the legacy of many exactly soluble models which play, and will continue to play, the role of the basic mechanical models like harmonic oscillators, central forces and two body problems, rigid body, vibrating string, heat equation, linear waves: they clarify the meaning of the formalism and provide examples of physical significance as well as terms of firm comparison for checks of approximate theories and methods. I just mention the work of Onsager on the Ising model, or the work of Lieb, Yang and Baxter on the ground state of the Heisenberg model, [B]. Such models, and many others (Schwarzschild and other metrics in general relativity, [W], Thirring model in field theory, Luttinger model in solid state Physics, one-dimensional dynamics of hard rods, [ML], Korteweg-de Vries equation, Toda lattice *etc.*) are not only non-trivial and illustrative of rather complicated behaviour but have also provided important new ideas in probability theory, PDE's and other fields.

Another way of making use of the formalism is via the so called “non-perturbative methods”. They consist of studies of concrete models for which no exact solution can be provided. One tries to obtain information about some general properties by using various special features of the models. An example of this is the theory of the thermodynamic limit or the theory of statistical ensembles for systems with short-range forces in statistical mechanics (Fisher, Ruelle, [R]). More specific (non-exhaustive) examples are the existence of thermodynamics for systems with electromagnetic forces (Dyson, Lenard, Lieb, Thirring, Fefferman, [L], [Fe]), the theory of coexistence in the Ising model (Russo, Aizenman, Higuchi, [Ai], [Hi]), the theory of percolation (Russo, Kesten, [Gr]), the theory of the critical point in ferromagnets (Aizenman, [Ai]), structure of “large atoms” (Lieb, [L2]), existence of phase transitions in some quantum statistical

²Although the book of Laplace could easily provide arguments in support of such a thesis!

mechanics models (Dyson, Lieb, Simon, [DLS]), and the theory of “incommensurate crystals” (Aubry, Mather, [Au], [M]).

Very often the above general non-perturbative methods are based on the use of convexity inequalities or of other classes of inequalities (GKS, FKG, *etc.*). Therefore very often (though not always) the results are “non constructive”, *i.e.*, they are intrinsically unable to provide estimates for the speed at which some limits are reached, or for the value of various constants whose existence is proved. They describe general, non-trivial properties which are very useful to know, particularly for checking approximate theoretical results.

The fourth aspect consists of the fully constructive results for specific models, based on approximations by convergent series expansion to quantities that, by general arguments, can be shown to exist. Such theories, usually called “perturbative theories”, are often considered “too technical” and the word “perturbative” has very often an ironic and negative connotation. But they are certainly among the greatest achievements of the subject and essentially they provide the only cases in which the “problem” is completely solved. The latter aspect should not be underestimated: In fact there have been examples of problems “solved” by general theorems of “fundamental nature” but whose solution has been subject to criticism. One of the dangers of Mathematical Physics is that, in order to solve a problem, one has to formulate it in a precise mathematical way; but this can lead to subtle changes of the problem itself or to too strict a formulation of it. Thus the solution obtained by one might not be regarded as such by others. This has certainly damaged the image of research in Mathematical Physics in many respects (for instance, at the end of the sixties, claims about supposed “theorems” on nonexistence of crystals only brought discredit to the field).

A very illustrious example of a “theorem” which, although mathematically correct and of great importance, missed the point completely is the generic non-integrability of perturbations of integrable Hamiltonian systems and in particular of the three-body problem, [G2]. This theorem of Poincaré has been poorly interpreted and referred to only by quotations for about half a century, until it was shown that a slight modification of the statement gave rise to a non-trivial and very interesting class of results (the KAM theory, see [G2]) which stated the truth of almost opposite properties.

Another important example is the “triviality theorem” of scalar field theories in space-time dimension 4, ([Fr], [Ai]): this theorem is valid under some very strong assumptions, so far, and skeptics may interpret it as saying that the assumptions are too strong or inappropriate. It is nevertheless very often quoted as a theorem established in full generality, see [GR].

Unfortunately the tendency of quoting the “theorems” of Mathematical Physics out of context and as absolute truths, proving or disproving some fundamental result, is still quite widespread. This happens mostly because non-specialists try to use Mathematical Physics non-critically, refusing to recall that theorems need assumptions and the assumptions have a physical meaning which is not always entirely conveyed by the words which are used to state them. Therefore a technical analysis of the assumptions is always necessary and useful. The above attitude is unfortunately found even inside Mathematical Physics itself: Specialists in one area tend to quote noncritically results concerning other areas. I am afraid that this will not be cured by us and it will be a problem for the next century as well.

In some sense only “positive” results should really matter. But “no go” results are also in some sense positive results. And sometimes positive results might be of little physical interest, even if they come in a wholly constructive form, with all desirable error or remainder estimates as they always do in perturbation-theoretic approaches. Nevertheless perturbation theory is usually non-trivial, and I will illustrate some of its achievements which I find remarkable not only for their intrinsic interest, but for the surprising unity of methods of solution of different problems that they make evident.

Another legacy that we shall certainly leave to the next century is our unwillingness to consider seriously the work done in other fields even inside the domain of Mathematical Physics. Hence we shall certainly not be able to find a way to avoid that results found in some area will reappear, derived under different words but in essentially identical manner, or regarded as open problems in other areas (or even in the same area): I am not thinking here of plagiarism (which is of little interest except from a moral point of view, totally ridiculous in our present world except on individual grounds), but just of honest ignorance. I am always surprised when I meet such cases, and am afraid that I too have exhibited this behaviour. But this might in fact be a “good thing”: It is more likely that the results will not be forgotten if they are duplicated often enough; the main works of Greek science were lost because there were not enough copies of them.

3 A Review of Perturbation Theory Methods

Perturbation theory arises when one studies a problem dependent on a parameter ε , and for $\varepsilon = 0$ the problem is exactly soluble (at least in some aspects). Until recently a perturbation treatment meant finding a series in ε convergent or asymptotic for small ε , whose sum was equal or asymptotic to a quantity of predeclared interest.

The simplest examples of perturbation theory dealt with the problems of celestial mechanics and in the last century not much attention was devoted to the actual convergence of the series, the main problem being whether they could be defined as formal power series (*i.e.*, their coefficients were well defined). In fact Poincaré proved that some of the most-used series in celestial mechanics could not possibly be convergent or even defined to all orders and that they were, at best, asymptotic (see [G2]).

The convergence problem never seemed to bother the theoretical physicists; perhaps because good enough practical results seemed to come out of the series, if regarded as asymptotic, or perhaps because the series met in the theory of atoms were actually convergent (see [K], [RS], [T]). The question of convergence of the series started playing a role when it was discovered that the series expansions for quantum field theory were not even well defined, at least not from a naive viewpoint. Renormalization theory provided a way to define them properly, but it also generated the puzzle of how proper the definition really was; whether it was a fundamental definition or just an arbitrary prescription. This was a particularly pertinent question as it was in any event clear that in most cases the series could not be convergent but at best could only be asymptotic.

The first series to be studied was precisely that arising in the theory of perturbations of integrable Hamiltonian systems. This led to the theorem of Kolmogorov on quasi-periodic Hamiltonian motions. In the same years the virial series for statistical-mechanical systems with short range forces was proved to converge for small enough density (Morrey, [Mo], Groeneveld, Penrose, Ruelle, [R]; independently within a decade).

In my opinion the latter was the keystone for the future developments. It was, I remember, a great surprise for many that such series describing the behaviour of systems with infinitely many degrees of freedom could be "proved" convergent. The technique soon found applications in other fields, for instance the theory of phase transitions and the study of pure phases and phase coexistence (started by Dobrushin, Minlos, Sinai, [S]).

At the end of the sixties the situation was ripe for attempting the proof of convergence, or asymptoticity, of the formal series generated by renormalization theory in quantum field theory, say for the computation of the Schwinger functions. The work on the foundations had allowed the quantities to be studied to be defined precisely. Giving formally their values, the series had been mathematically proven to be well defined, at least in simple cases (Hepp, [H]), a fact that the discoverers of renormalization theory had certainly perceived clearly enough to consider a formal proof unnecessary. The asymptoticity proof came from the work by Glimm, Jaffe, Spencer, [E], and it was achieved by showing the convergence of another series in terms of *functions* of the expansion parameter ε , which were singularly dependent on ε , but in a controlled way. The difficult part was to set up the appropriate convergent expansion, whose convergence turned out to be, essentially, a consequence of the same ideas leading to the convergence of the virial series.

I will never understand why such a result, which among other things solved the question of whether a relativistic quantum theory with non-trivial S -matrix was possible, i.e., showed the compatibility of non-trivial quantum mechanics and special relativity (at least for some models and in "low" space time dimension, (i.e., 3)), has been very often ignored and sometimes considered irrelevant: an important later paper provides us with a rather typical example, see [PW].

The above work was the last major one before the new ideas generated by the renormalization group started playing a role, [Po], [G3]; but a typical tool of the renormalization group is already present: The proof of asymptoticity (or convergence) of a perturbation series by checking convergence of series in other quantities with controlled (albeit singular) analyticity properties in terms of the expansion parameter ε .

This method can be called "proof of asymptoticity by resummations" and is a classic mathematical method. What is not classic and is non-trivial is its combination with the theory of convergence of the virial series, known nowadays as the *cluster expansion*, [E], [G2].

The cluster expansion led also to the solution of some old problems, like the proof of existence of the Debye screening in classical systems interacting at large distance by an electrostatic Coulomb potential (Brydges, [Br]).

Thus it is interesting to see that the resummation techniques could all be considered part of the same basic idea rooted in the *renormalization group* approach to the critical point in statistical mechanics. The approach is particularly interesting as it shows that the basic idea of *asymptotic freedom*, which is one of the really new ingredients in renormalization theory and which had escaped the founding fathers, is the one which allows us to build the rigorous proofs of asymptoticity of the perturbation expansion of various field theories.

In fact the renormalization group can be viewed as a method for resumming formal series: This was realized through various works on the rigorous mathematical meaning

of the renormalization group “calculations” (see the review papers [CE], [Po], [G3], [GK1]) in which the “old” results were rederived and a host of new ones obtained (among them see [FS] for the Anderson localization problem³, [GK2] for the theory of the Gross-Neveu model, [BGPS] for the theory of the ground state of spin 0 one-dimensional Fermi gases, just to mention a few). Other applications are promising to come, see [A], [Sh], [Po2], [W2]. All these problems have one feature in common, namely they have some scaling property playing a key role. However a fully unified treatment is not yet available, although it seems that we possess all the necessary knowledge.

4 An Example of Modern Perturbation Theory: Twistless Invariant Tori in Hamiltonian Mechanics

The resummation is not always possible, or not always known. In particular the asymptoticity of the (well defined) perturbation expansions cannot be treated rigorously in important cases such as 4-dimensional field theories like quantum electrodynamics or the standard model of elementary particles.

Scaling properties, more or less hidden, are the reason why problems of perturbation theory appear to be similar in statistical mechanics and field theory and other apparently unrelated problems like the Feigenbaum cascades in the theory of maps of the interval (see [F], [CE2], [ER]), or the KAM theory, or the invariant tori breakdown theory of Kadanoff, McKay, Aubry, Mather, ([Mc], [AL], [M]). The same is true of the theory of regularity of the solutions of the Navier-Stokes equations in 3 dimensions; see the review [G4].

I will describe here the connection between the Kolmogorov theorem on the existence of quasi-periodic motions and the resummations of perturbation expansions used in quantum field theory. The basic idea goes back to Eliasson: the idea has a particularly simple application as the series studied turns out to be a convergent series because of some remarkable cancellations. Although a part of the problem is therefore trivial (the series being convergent rather than asymptotic), what is left is nontrivial enough to be interesting and to show the relation with the renormalization group methods in quantum field theory.

The connection might have been suspected already from the relation between the KAM theory and the renormalization group methods, pointed out in various independent works ([G2], [Mc], [ED]). I find it nevertheless quite surprising that the KAM problem can in fact be formulated as a field theory problem, in which interesting scaling properties appear, and treated as such.

The explanation of the field theory model given below requires, to be understood, some familiarity with the theory of the dipole gas and of the Sine-Gordon model. But the analysis following it is an independent and self-contained proof of the KAM theorem, not requiring any knowledge of field theory or statistical mechanics. The field theory model linked to the KAM theorem was pointed out to me by G. Parisi, who showed me

³A remarkable new non-perturbative method, for this theory has been presented at this conference, see [AM].

what I should have done in my attempts to interpret field theoretically my version of Eliasson's ideas.

To establish a connection with field theory consider first a vector field $\vec{F}_{\vec{\psi}}$, with zero average, defined on the torus $\mathbb{T}^l$ with free propagator given by the operator $(\vec{\omega}_0 \cdot \vec{\partial})^2$, instead of the usual Laplacian, with $\vec{\omega}_0 \in \mathbb{R}^l$ being a vector satisfying a strong diophantine condition, see below.

Consider the function of the field corresponding to the "action potential" $V(\vec{F}) = \varepsilon \int f(\vec{\psi} + \vec{F}_{\vec{\psi}}) d\vec{\psi}$, where the f is an even trigonometric polynomial. The partition function Z is then formally expressed in terms of the Gaussian process $P(d\vec{F})$ generated by the differential operator $(\vec{\omega}_0 \cdot \vec{\partial})^2$ on the functions $\vec{F}_{\vec{\psi}}$ with zero average: it is the functional integral of the functional $\exp V(\vec{F})$. The Schwinger function $\vec{h}(\vec{\psi}) = \langle \vec{F} \rangle_{\vec{\psi}}$ is the formal average of $\vec{F}_{\vec{\psi}}$ with respect to the measure $Z^{-1} P(d\vec{F}) e^{V(\vec{F})}$.

I shall show below that the evaluation of $\vec{h}(\vec{\psi})$ via perturbation theory, if performed *neglecting all Feynman diagrams which are not tree diagrams*, gives exactly the perturbation series solution to the problem of finding an invariant KAM torus for the model considered below (that I call the Thirring model, or rotators model, see (2) below).

It was pointed out to me by G. Parisi that the approximation to $\vec{h}(\vec{\psi})$ obtained by evaluating $\vec{h}(\vec{\psi})$ via a perturbation expansion in ε , in which only the Feynman diagrams with tree structure (*i.e.*, no loops) are retained, yields the solution to the equation: $(\vec{\omega}_0 \cdot \vec{\partial})^2 \vec{h} = -\varepsilon \vec{f}'(\vec{\psi} + \vec{h}(\vec{\psi}))$, where $\vec{f}'$ denotes the gradient of the function f . I shall show below that the latter equation is the equation that has to be solved to determine the KAM tori for the model introduced in the following eqn. (1). Hence a field theoretic interpretation of the KAM theorem will easily follow. A further connection with field theory was pointed out to me by A. Berretti who remarked that the tree approximation is exact in mean field theory: Therefore the model provided by Parisi, solved in the mean field, approximation gives rise to a non-trivial equation whose solution is equivalent to the Thirring model.

The above formal field theoretic interpretation is interesting, but it has the drawback of being "an approximation" and one can ask whether one can find a field theory whose one-point Schwinger function is exactly, at least formally, the function $\vec{h}(\vec{\psi})$ describing the invariant tori of the Thirring model.

This can be done as follows (a procedure probably known in field theory): let $\vec{F}_{\vec{\psi}}^{\pm}$ be two complex vector fields, with zero average on the torus $\mathbb{T}^l$, and suppose that their free propagator is $\langle F_{i,\vec{\psi}}^+ F_{j,\vec{\psi}'}^+ \rangle = 0 = \langle F_{i,\vec{\psi}}^- F_{j,\vec{\psi}'}^- \rangle$ while $\langle F_{i,\vec{\psi}}^+ F_{j,\vec{\psi}'}^- \rangle = \delta_{ij} S(\vec{\psi} - \vec{\psi}')$ where S has Fourier transform equal to $(\vec{\omega}_0 \cdot \vec{\nu})^{-2}$. The fields $\vec{F}_{\vec{\psi}}^{\pm}$ can be realized by considering two independent real vector fields $\vec{\Phi}^q$, $q = 1, 2$, with free propagator $\delta_{ij} S(\vec{\psi} - \vec{\psi}')$. One simply sets: $\vec{F}^{\pm} = \vec{\Phi}^1 \pm i\vec{\Phi}^2$.

Then consider the Schwinger function:

$$(1) \quad \vec{h}(\vec{\psi}) = \frac{\int P(d\vec{F}) \vec{F}_{\vec{\psi}}^+ e^{\varepsilon \int \vec{F}_{\vec{\psi}}^-, \vec{\partial} f(\vec{\psi}' + \vec{F}_{\vec{\psi}'}^+) d\vec{\psi}'}}{\int P(d\vec{F}) e^{\varepsilon \int \vec{F}_{\vec{\psi}}^-, \vec{\partial} f(\vec{\psi}' + \vec{F}_{\vec{\psi}'}^+) d\vec{\psi}'}}$$

and it is easy to check that the linearity in $\vec{F}^-$ of the potential implies that only the tree

diagrams of the perturbation expansion of (1) do not vanish. And the tree diagrams *have the same value* as the ones of the field theory of the single field $\vec{F}$ previously considered. Hence $\vec{h}(\vec{\psi})$ is a (formal) sum of the tree diagrams for (1).

One should not think that the functional integrals in (1) are easy to define rigorously. In fact the fields $\vec{F}^\pm$ are complex valued and therefore the argument of the exponentials is unbounded, so that summability is by no means obvious. The KAM theorem discussed below can be interpreted as a theory of the above functional integral and as a proof of the convergence of the perturbation expansion for it.

To avoid references to field theory I begin by formulating from scratch the problem of the version of the KAM theorem that I will discuss. It will be a particularization of the Eliasson method, [El], for KAM tori to a special model: the Thirring model. This is a system of rotators interacting via a potential. It is described by the Hamiltonian:

$$(2) \quad \frac{1}{2} J^{-1} \vec{A} \cdot \vec{A} + \varepsilon f(\vec{\alpha})$$

where J is the (diagonal) matrix of the moments of inertia, $\vec{A} = (A_1, \dots, A_l) \in \mathbb{R}^l$ are their angular momenta and $\vec{\alpha} = (\alpha_1, \dots, \alpha_l) \in \mathbb{T}^l$ are the angles describing their positions. The matrix J will be assumed nonsingular, but we only suppose that $\min_{j=1, \dots, l} J_j = J_0 > 0$, and *no assumption* is made on the size of the *twist rate* $T = \min J_j^{-1}$; the results will be uniform in T (hence the name “twistless” that can be given to the above tori). We suppose f to be an even trigonometric polynomial of degree N :

$$(3) \quad f(\vec{\alpha}) = \sum_{0 < |\vec{\nu}| \leq N} f_{\vec{\nu}} \cos \vec{\nu} \cdot \vec{\alpha}, \quad f_{\vec{\nu}} = f_{-\vec{\nu}}$$

We shall consider a “rotation vector” $\vec{\omega}_0 = (\omega_1, \dots, \omega_l) \in \mathbb{R}^l$ having a *strong diophantine property* with diophantine constants C_0, τ, γ, c ; this means that:

$$(4) \quad \begin{aligned} C_0 |\vec{\omega}_0 \cdot \vec{\nu}| &\geq |\vec{\nu}|^{-\tau}, \quad \vec{0} \neq \vec{\nu} \in \mathbb{Z}^l & 1) \\ \min_{0 \leq p \leq n} |C_0 |\vec{\omega}_0 \cdot \vec{\nu}| - \gamma^p| &> \gamma^{n+1} \text{ if } n \leq 0, \quad 0 < |\vec{\nu}| \leq (\gamma^{n+c})^{-\tau-1} & 2) \end{aligned}$$

and it is easy to see that the *strongly diophantine vectors* have full measure in $\mathbb{R}^l$ if $\gamma > 1$ and c are fixed and if τ is fixed $\tau > l - 1$; we take $\gamma = 2, c = 3$ for simplicity. Note that 2) is empty if $n > -3$ or $p < n + 3$. We shall set $\vec{A}_0 = J\vec{\omega}_0$. A special example can be the model $f_0(\vec{\alpha}) = J_0 \vec{\omega}_0^2 (\cos \alpha_1 + \cos(\alpha_1 + \alpha_2))$.

We look for an ε -analytic family of motions starting at $\alpha = \vec{0}$ and having the form:

$$(5) \quad \vec{A} = \vec{A}_0 + \vec{H}(\vec{\omega}_0 t; \varepsilon), \quad \vec{\alpha} = \vec{\omega}_0 t + \vec{h}(\vec{\omega}_0 t; \varepsilon)$$

with $\vec{H}(\vec{\psi}; \varepsilon), \vec{h}(\vec{\psi}; \varepsilon)$ analytic in $\vec{\psi} \in \mathbb{T}^l$ and in ε close to 0. We shall prove that such functions exist and are analytic for $|\Im \psi_j| < \xi$ for $|\varepsilon| < \varepsilon_0$, with:

$$(6) \quad \varepsilon_0^{-1} = b J_0^{-1} C_0^2 f_0 N^{2+l} e^{cN} e^{\xi N}$$

where b, c are l -dependent positive constants, $f_0 = \max_{\vec{\nu}} |f_{\vec{\nu}}|$. This means that the set $\vec{A} = \vec{A}_0 + \vec{H}(\vec{\psi}; \varepsilon), \vec{\alpha} = \vec{\psi} + \vec{h}(\vec{\psi}; \varepsilon)$ described as $\vec{\psi}$ varies in $\mathbb{T}^l$ is, for ε small enough, an

invariant torus for (2), which is run quasi-periodically with angular velocity vector $\vec{\omega}_0$. It is a family of invariant tori coinciding, for $\varepsilon = 0$, with the torus $\vec{A} = \vec{A}_0$, $\vec{\alpha} = \vec{\psi} \in \mathbb{T}^l$. One recognizes a version of the KAM theorem. The proof that follows simplifies the one reported in [G1].

Supposing $J_0 \equiv J_1 < J_2$ the uniformity in J_2 (i.e., what we call the *twistless* property) implies that the same ε_0 can be used as an estimate of the radius of convergence in ε of the power series describing the KAM tori with rotation vector $\vec{\omega}_0 = (\omega_1, \omega_2)$ in the system $(2J_1^{-1}A_1^2 + \omega_2 A_2 + \varepsilon f_0(\alpha_1, \alpha_2))$, which is one of the most-studied Hamiltonian systems. The estimate can be improved. Note that a careful analysis of the proof of the KAM theorem also reveals the uniformity in the twist rate in the case (2).

Calling $\vec{H}^{(k)}(\vec{\psi})$, $\vec{h}^{(k)}(\vec{\psi})$ the k -th order coefficients of the Taylor expansion of $\vec{H}$, $\vec{h}$ in powers of ε and writing the equation of motion as $\dot{\vec{\alpha}} = J^{-1}\vec{A}$ and $\vec{A} = -\varepsilon \partial_{\vec{\alpha}} f(\vec{\alpha})$ we get immediately the equation (mentioned above and having a field theory interpretation) $(\vec{\omega}_0 \cdot \vec{\partial})^2 \vec{h}(\vec{\psi}) = -\varepsilon \partial_{\vec{\alpha}} f(\vec{\psi} + \vec{h}(\vec{\psi}))$, or the (equivalent) recursion relations for $\vec{H}^{(k)}$, $\vec{h}^{(k)}$. They are $\vec{\omega}_0 \cdot \vec{\partial} \vec{h}_j^{(k)} = J_j^{-1} H_j^{(k)}$ and, for $k > 1$:

$$(7) \quad \vec{\omega}_0 \cdot \vec{\partial} H_j^{(k)} = - \sum_{\substack{m_1, \dots, m_l \\ |\vec{m}| > 0}} \frac{1}{\prod_{s=1}^l m_s!} \partial_{\alpha_j} \partial_{\alpha_1}^{m_1} \dots \partial_{\alpha_l}^{m_l} f(\vec{\omega}_0 t) \cdot \sum^* \prod_{s=1}^l \prod_{j=1}^{m_s} h_s^{(k_j)}(\vec{\omega}_0 t)$$

where the $\sum^*$ denotes summation over the integers $k_j \geq 1$ with $\sum_{s=1}^l \sum_{j=1}^{m_s} k_j^s = k - 1$.

The trigonometric polynomial $\vec{h}^{(k)}(\vec{\psi})$ will be completely determined (if possible at all) by requiring it to have $\vec{0}$ average over $\vec{\psi}$, (note that $\vec{H}^{(k)}$ has to have zero average over $\vec{\psi}$). For $k = 1$ one easily finds:

$$(8) \quad \vec{h}^{(1)}(\vec{\psi}) = - \sum_{\vec{\nu} \neq \vec{0}} \frac{i J^{-1} \vec{\nu}}{(i \vec{\omega}_0 \cdot \vec{\nu})^2} f_{\vec{\nu}} e^{i \vec{\nu} \cdot \vec{\psi}}$$

Suppose that $\vec{h}^{(k)}(\vec{\psi})$ is a trigonometric polynomial of degree $\leq kN$, odd in t , for $1 \leq k < k_0$. Then we see immediately that the r.h.s. of (7) is odd in t . This means that the r.h.s. of (7) has zero average in t , hence in $\vec{\psi}$, and the second of (7) can be solved for $k = k_0$. It yields an even function $\vec{H}^{(k_0)}(\vec{\psi})$ which is defined up to a constant which, however, must be taken such that $\vec{H}^{(k_0)}(\vec{\psi})$ has zero average, to make $\vec{\omega} \cdot \vec{\partial} \vec{h}_j^{(k)} = J_j^{-1} H_j^{(k)}$ soluble. Hence the equation for $\vec{h}^{(k)}$ can be solved (because the r.h.s. has zero average) and its solution is a trigonometric polynomial in $\vec{\psi}$, odd if $\vec{h}^{(k)}$ is determined by imposing that its average over $\vec{\psi}$ vanishes.

Hence the (8) provide an algorithm to evaluate a formal power series solution to our problem. It has been remarked, [E], see also [G1], that (7) yields a *diagrammatic expansion* of $\vec{h}^{(k)}$. We simply "iterate" it until only $\vec{h}^{(1)}$, given by (8), appears.

Let ϑ be a tree diagram. It will consist of a family of "lines" (i.e., segments) numbered from 1 to k taken from a "box" containing k lines with a label, *number label*, going from 1 to k (i.e., distinguishable), arranged to form a (rooted) tree diagram as in Fig. 1.

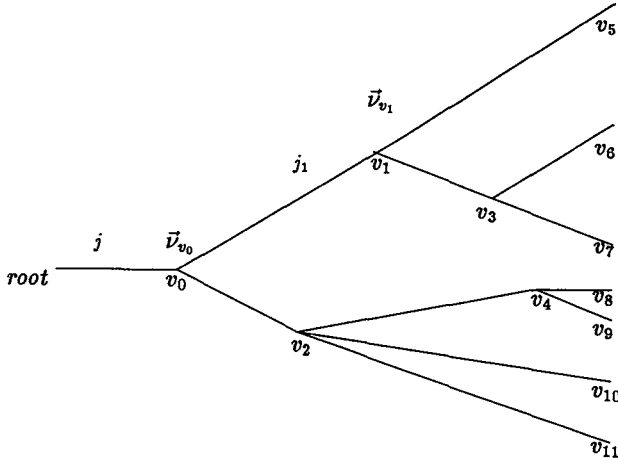


Figure 1: A tree diagram ϑ with $m_{v_0} = 2, m_{v_1} = 2, m_{v_2} = 3, m_{v_3} = 2, m_{v_4} = 2$ and $m = 12, \prod m_v! = 2^4 \cdot 6$, and some labels. The line numbers, distinguishing the lines, are not shown.

To each vertex v we attach a “mode label” $\vec{\nu}_v \in \mathbb{Z}^l, |\vec{\nu}_v| \leq N$ and to each branch leading to v we attach a “branch label” $j_v = 1, \dots, l$. The order of the diagram will be $k = \text{number of vertices} = \text{number of branches}$ (the tree root will not be regarded as a vertex).

We imagine that all lines in the diagram have the same length (even though they have been drawn with arbitrary lengths in Fig. 1). A group acts on the set of diagrams generated by the permutations of the subdiagrams having the same vertex as root. Two diagrams that can be superposed by the action of a transformation of the group will be regarded as identical (recall however that the lines of the diagram are numbered, *i.e.*, are regarded as distinct, and the superposition has to be such that all the labels of the diagram match: the branch label, the mode label *and* the line label). Tree diagrams are regarded as partially ordered sets of vertices (or lines) with a minimal element given by the root (or the root line). As usual the order relation will be denoted $\leq$ and, in general, not all pairs of vertices will be comparable. We shall imagine that each branch carries also an arrow pointing to the root (“gravity” direction, opposite to the order).

We define the “momentum” entering v as $\vec{\nu}(v) = \sum_{w \geq v} \vec{\nu}_w$. If from a vertex v there emerge m_1 lines carrying the label $j = 1, m_2$ lines carrying $j = 2, \dots$, it follows that (7) can be rewritten as

$$(9) \quad \tilde{h}_{\vec{\nu}_j}^{(k)} = \frac{1}{k!} \sum^* \prod_{v \in \vartheta} \frac{(-iJ^{-1}\vec{\nu}_v)_{j_v} f_{\vec{\nu}_v} \prod_{s=1}^l (i\vec{\nu}_v)_s^{m_s}}{(i\vec{\omega}_0 \cdot \vec{\nu}(v))^2}$$

with the sum running over the diagrams ϑ of order k and with $\vec{\nu}(v_0) = \vec{\nu}$; and the combinatorics can be checked from (7), by taking into account that we regard the lines of the diagram as all different (to fix the factorials). Basically the introduction of the

line labels increases the number of trees, but at the same time they acquire all the same combinatorial weight $k!^{-1}$, instead of the more complicated $\prod m_v!^{-1}$ that one would expect from (7).

The * on Σ^* recalls that the diagram ϑ can and will be supposed such that $\vec{\nu}(v) \neq \vec{0}$ for all $v \in \vartheta$ (by the parity properties noted above). Observe that (9) is implied by the corresponding (6.23) of [G1]: one can check that the two formulae coincide (by summing over what in [G1] are called the “fruit values”). The theory in [G1] is in fact a little more general, although it is really applied to the same Thirring model.

There are other diagrams, however, which we would like to eliminate. They are the diagrams with nodes v', v , with $v' < v$, not necessarily nearest neighbours, such that $\vec{\nu}(v) = \vec{\nu}(v')$. Such diagrams are, according to Eliasson's terminology, *in resonance*.

The first remark is that if the sum in (9) was restricted to a sum over diagrams such that:

I $\vec{\nu}(v) \neq \vec{0}$ if v is any vertex.

II $\vec{\nu}(v) \neq \vec{\nu}(v')$ for all pairs of comparable vertices v', v , (not necessarily next to each other in the diagram order, however), with $v \geq v_0$.

then the result would be easily reduced to known convergence theorems, like the basic Brjuno's lemma, (see [G1]).

However *there are* resonant diagrams. The key remark is that they cancel *almost exactly*. The reason is very simple: Imagine detaching from a diagram ϑ the subdiagram ϑ_2 with first node v . Then consider it attached it to all the *remaining* vertices $w \geq v', w \in \vartheta/\vartheta_2$. We obtain a family of diagrams whose contributions to $h^{(k)}$ differ because:

- 1) Some of the branches above v' have changed total momentum by the amount $\vec{\nu}(v)$: this means that some of the denominators $(\vec{\omega} \cdot \vec{\nu}(w))^{-2}$ have become $(\vec{\omega} \cdot \vec{\nu}(w) \pm \varepsilon)^{-2}$ if $\varepsilon \equiv \vec{\omega}_0 \cdot \vec{\nu}(v)$; and:
- 2) Because one of the vertex factors changes by taking successively the values ν_{wj} , j being the line label of the line leading to v , and $w \in \vartheta/\vartheta_2$ is the vertex to which this line is reattached.

Hence if $\vec{\omega} \cdot \vec{\nu} = \varepsilon = 0$ we would build in this *resummation* a quantity proportional to $\sum \vec{\nu}_w = \vec{\nu}(v') - \vec{\nu}(v)$, which is zero, because $\vec{\nu}(v) = \vec{\nu}(v')$ means that the sum of the $\vec{\nu}_w$'s vanishes. Since $\vec{\omega} \cdot \vec{\nu} = \varepsilon \neq 0$ we can expect to see a sum of order ε^2 , if we sum as well on a overall change of sign of the $\vec{\nu}_w$ values (which sum up to $\vec{0}$).

But this can be true only if $\varepsilon \ll \vec{\omega} \cdot \vec{\nu}'$, for any line momentum $\vec{\nu}'$ of a line in ϑ/ϑ_2 . If the latter property is not true this means that $\vec{\omega} \cdot \vec{\nu}'$ is small and that there are many vertices in ϑ/ϑ_2 of order of the amount needed to create a momentum with small divisors of order ε .

Examining carefully the proof of Brjuno's lemma one sees that such an extreme case would also be essentially treatable. Therefore the problem is to show that the two regimes just envisaged (and their “combinations”) indeed exhaust all possibilities.

Such problems are very common in renormalization theory and are called “overlapping divergences”. Their systematic analysis is made through the renormalization

group methods. We argue here that Elliasson's method can be interpreted in the same way.

The diagrams introduced above will play the role of *Feynman diagrams*; and they will be plagued by overlapping divergences. They will therefore be collected into another family of graphs, that we shall call *trees*, on which the bounds are easy. The $(\vec{\omega} \cdot \vec{\nu})^{-2}$ are the *propagators*, in our analogy.

We fix a *scaling* parameter γ , which we take $\gamma = 2$ for consistency with (4), and we also define $\vec{\omega} \equiv C_0 \vec{\omega}_0$: it is a dimensionless frequency. Then we say that a propagator $(\vec{\omega} \cdot \vec{\nu})^{-2}$ is *on scale* n if $2^{n-1} < |\vec{\omega} \cdot \vec{\nu}| \leq 2^n$, for $n \leq 0$, and we set $n = 1$ if $1 < |\vec{\omega} \cdot \vec{\nu}|$.

Proceeding as in quantum field theory, see [G3], given a diagram ϑ we can attach a *scale label* to each line $v'v$ in (9) (with v' being the vertex preceding v): it is equal to n if n is the scale of the line propagator. Note that the labels thus attached to a diagram are uniquely determined by the diagram: they will have only the function of helping to visualize the orders of magnitude of the various diagram lines.

Looking at such labels we identify the connected clusters T of vertices that are linked by a continuous path of lines with the same scale label n_T or a higher one. We shall say that *the cluster T has scale n_T* .

Among the clusters we consider the ones with the property that there is only one diagram line entering them and only one exiting and both carry the same momentum. Here we use the fact that the diagram lines carry an arrow pointing to the root; this gives a meaning to the words "incoming" and "outgoing".

If V is one such cluster we denote λ_V the incoming line. The line scale $n = n_{\lambda_V}$ is smaller than the smallest scale $n' = n_V$ of the lines inside V . We call w_1 the vertex into which the line λ_V ends, inside V . We say that such a V is a *resonance* if the number of lines contained in V is $\leq E 2^{-n\epsilon}$, where $n = n_{\lambda_V}$, and E, ϵ are defined by: $E \equiv 2^{-3\epsilon} N^{-1}$, $\epsilon = \tau^{-1}$. We call n_{λ_V} the *resonance scale*, and λ_V a *resonant line*.

Let us consider a diagram ϑ and its clusters. We wish to estimate the number N_n of lines with scale $n \leq 0$ in it, assuming $N_n > 0$.

Denoting by T a cluster of scale n , let m_T be the number of resonances of scale n contained in T (i.e., with incoming lines of scale n); we have the following inequality, valid for any diagram ϑ :

$$(10) \quad N_n \leq \frac{4k}{E 2^{-\epsilon n}} + \sum_{T, n_T=n} (-1 + m_T)$$

with $E = N^{-1} 2^{-3\epsilon}$, $\epsilon = \tau^{-1}$. This "combinatorial" inequality is a version of Brjuno's lemma; a proof can be found in [G1].

Consider a diagram ϑ^1 ; we define the family $\mathcal{F}(\vartheta^\infty)$ generated by ϑ^1 as follows. Given a resonance V of ϑ^1 we detach the part of ϑ^1 above λ_V and attach it successively to the points $w \in \tilde{V}$, where $\tilde{V}$ is the set of vertices of V (including the endpoint w_1 of λ_V contained in V) outside the resonances contained in V . Note that all the lines λ in $\tilde{V}$ (i.e., contained in V and with at least one point in $\tilde{V}$) have a scale $n_\lambda \geq n_V$.

For each resonance V of ϑ^1 we shall call M_V the number of vertices in $\tilde{V}$. To the set of diagrams just defined we add the diagrams obtained by reversing simultaneously the signs of the vertex modes $\vec{\nu}_w$, for $w \in \tilde{V}$; the sign change is performed independently for the various resonant clusters. This defines a family of $\prod 2M_V$ diagrams that we call $\mathcal{F}(\vartheta_\infty)$. The number $\prod 2M_V$ will be bounded by $\exp \sum 2M_V \leq e^{2k}$.

It is important to note that the definition of resonance is such that the above operation (of shift of the vertex to which the line entering V is attached) does not change too much the scales of the diagram lines inside the resonances: the reason is simply that inside a resonance of scale n the number of lines is not very large, being $\leq \bar{N}_n \equiv E 2^{-n\epsilon}$.

Let λ be a line, in a cluster T , contained inside the resonances $V = V_1 \subset V_2 \subset \dots$ of scales $n = n_1 > n_2 > \dots$; then the shifting of the lines λ_{V_i} can cause at most a change in the size of the propagator of λ by at most $2^{n_1} + 2^{n_2} + \dots < 2^{n+1}$.

Since the number of lines inside V is smaller than $\bar{N}_n$ the quantity $\vec{\omega} \cdot \vec{\nu}_\lambda$ of λ has the form $\vec{\omega} \cdot \vec{\nu}_\lambda^0 + \sigma_\lambda \vec{\omega} \cdot \vec{\nu}_{\lambda_V}$ if $\vec{\nu}_\lambda^0$ is the momentum of the line λ "inside the resonance V ", i.e., it is the sum of all the vertex modes of the vertices preceding λ in the sense of the line arrows, but contained in V ; and $\sigma_\lambda = 0, \pm 1$.

Therefore not only $|\vec{\omega} \cdot \vec{\nu}_\lambda^0| \geq 2^{n+3}$ (because $\vec{\nu}_\lambda^0$ is a sum of $\leq \bar{N}_n$ vertex modes, so that $|\vec{\nu}_\lambda^0| \leq N \bar{N}_n$) but $\vec{\omega} \cdot \vec{\nu}_\lambda^0$ is "in the middle" of the dyadic interval containing it and by (4) does not get out of it if we add a quantity bounded by 2^{n+1} (like $\sigma_\lambda \vec{\omega} \cdot \vec{\nu}_{\lambda_V}$). Hence no line changes scale as ϑ varies in $\mathcal{F}(\vartheta^\infty)$, if $\vec{\omega}_0$ satisfies (4).

This implies, by the strong diophantine hypothesis on $\vec{\omega}_0$, (4), that the resonant clusters of the diagrams in $\mathcal{F}(\vartheta^\infty)$ all contain the same sets of lines, and the same lines go in or out of each resonance (although they are attached to generally distinct vertices inside the resonances: the identity of the lines is here defined by the number label that each of them carries in ϑ^1). Furthermore the resonance scales and the scales of the resonant clusters, and of all the lines, do not change.

Let ϑ^2 be a diagram not in $\mathcal{F}(\vartheta^\infty)$ and construct $\mathcal{F}(\vartheta^\epsilon)$, etc.. We define a collection $\{\mathcal{F}(\vartheta^l)\}_{l=\infty, \epsilon, \dots}$ of pairwise disjoint families of diagrams. We shall sum all the contributions to $\tilde{h}^{(k)}$ coming from the individual members of each family. This is *Eliasson's resummation*.

We call ϵ_V the quantity $\vec{\omega} \cdot \vec{\nu}_{\lambda_V}$ associated with the resonance V . If λ is a line in $\tilde{V}$, see above, the quantity $\vec{\omega} \cdot \vec{\nu}_\lambda$ may be written as $\vec{\omega} \cdot \vec{\nu}_\lambda^0 + \sigma_\lambda \epsilon_V$, with $\sigma_\lambda = 0, \pm 1$. Since $|\vec{\omega} \cdot \vec{\nu}_\lambda| > 2^{n_V-1}$ we see that the product of the propagators is holomorphic⁴ in ϵ_V for $|\epsilon_V| < 2^{n_V-3}$. While ϵ_V varies in such a complex disk the quantity $|\vec{\omega} \cdot \vec{\nu}_\lambda|$ does not become smaller than $2^{n_V-1} - 2 \cdot 2^{n_V-3} \geq 2^{n_V-2}$. Note the main point here; the quantity 2^{n_V-3} will usually be $\gg 2^{n_{\lambda_V}}$, which is the value ϵ_V can actually reach in every diagram in $\mathcal{F}(\vartheta^\infty)$; this can be exploited for applying the maximum principle, as done below.

It follows that, calling n_λ the scale of the line λ in ϑ^1 , each of the $\prod 2M_V \leq e^{2k}$ products of propagators of the members of the family $\mathcal{F}(\vartheta^\infty)$ can be bounded above by $\prod_\lambda 2^{-2(n_\lambda-2)} = 2^{4k} \prod_\lambda 2^{-2n_\lambda}$, if regarded as a function of the quantities $\epsilon_V = \vec{\omega} \cdot \vec{\nu}_{\lambda_V}$, for $|\epsilon_V| \leq 2^{n_V}$, associated with the resonant clusters V . This even holds if the ϵ_V are regarded as independent complex parameters.

By construction it is clear that the sum of the $\prod 2M_V \leq e^{2k}$ terms, giving the contribution to $\tilde{h}^{(k)}$ from the trees in $\mathcal{F}(\vartheta^\infty)$, vanishes to second order in the parameters ϵ_V (by the approximate cancellation discussed above). Hence, by the maximum principle, and recalling that each of the scalar products in (9) can be bounded by N^2 , we can

⁴In fact $|\vec{\omega} \cdot \vec{\nu}_\lambda^0| \geq 2^{n+3}$ because V is a resonance; therefore $|\vec{\omega} \cdot \vec{\nu}_\lambda| \geq 2^{n+3} - 2^{n+1} \geq 2^{n+2}$ so that $n_V \geq n+3$. On the other hand note that $|\vec{\omega} \cdot \vec{\nu}_\lambda^0| > 2^{n_V-1} - 2^n$ so that $|\vec{\omega} \cdot \vec{\nu}_\lambda^0 + \sigma_\lambda \epsilon_V| \geq 2^{n_V-1} - 2^n - 2^{n_V-3} \geq 2^{n_V-1} - 2^{n_V-3} \geq 2^{n_V-2}$, for $|\epsilon_V| < 2^{n_V-3}$.

bound the contribution from the family $\mathcal{F}(\vartheta^\infty)$ by:

$$(11) \quad \left[\frac{1}{k!} \left(\frac{f_0 C_0^2 N^2}{J_0} \right)^k 2^{4k} e^{2k} \prod_{n \leq 0} 2^{-2n N_n} \right] \left[\prod_{n \leq 0} \prod_{T, n_T = n} \prod_{i=1}^{m_T} 2^{2(n-n_i+3)} \right]$$

where:

1) N_n is the number of propagators of scale n in ϑ^1 ($n = 1$ does not appear as $|\vec{\omega} \cdot \vec{\nu}| \geq 1$ in such cases).

2) The first square bracket is the bound on the product of individual elements in the family $\mathcal{F}(\vartheta^\infty)$ times the bound e^{2k} on their number.

3) The second term is the part coming from the maximum principle, applied to bound the resummations, and is explained as follows:

i) The dependence on the variables $\varepsilon_{V_i} \equiv \varepsilon_i$ relative to resonances $V_i \subset T$ with scale $n_{\lambda_{V_i}} = n$ is holomorphic for $|\varepsilon_i| < 2^{n_i-3}$ if $n_i \equiv n_{V_i}$, provided $n_i > n + 3$ (see above).

ii) The resummation says that the dependence on the ε_i 's has a second order zero in each. Hence the maximum principle tells us that we can improve the bound given by the first factor in (11) by the product of factors $(|\varepsilon_i| 2^{-n_i+3})^2$ if $n_i > n + 3$. If $n_i \leq n + 3$ we cannot gain anything; but since the contribution to the bound from such terms in (11) is > 1 we can leave them in it to simplify the notation (of course this means that the gain factor can be important only when $\ll 1$).

Hence substituting (10) into (11) we see that the m_T is cancelled by the first factor in $2^{2n} 2^{-2n_i}$, while the remaining 2^{-2n_i} are compensated by the -1 before the $+m_T$ in (10), taken from the factors with $T = V_i$ (note that there are always enough -1 's).

Therefore the product (11) is bounded by:

$$(12) \quad \frac{1}{k!} (C_0^2 J_0^{-1} f_0 N^2)^k e^{2k} 2^{4k} 2^{6k} \prod_{n=-\infty}^0 2^{-4nk E^{-1} 2^{en}} \leq \frac{1}{k!} B_0^k$$

with B_0 suitably chosen.

To sum over the trees we note that, once ϑ is fixed the collection of clusters is fixed. Therefore we only have to multiply (12) by the number of diagram shapes for ϑ , ($\leq 2^{2k} k!$), by the number of ways of attaching mode labels, ($\leq (3N)^{ik}$), so that we can bound $|h_{\vec{\nu}_j}^{(k)}|$ by (6).

Remark:

It is interesting to remark that we know, from the proof of [Th] of the KAM theorem for the model (2) (for instance, and most directly, see [Th]), that there is a canonical transformation with generating function having the form $\Phi(\vec{A}', \vec{\alpha}) = (\vec{A}' - \vec{A}_0) \cdot \vec{\alpha} + (\vec{A}' - \vec{A}_0) \cdot \vec{g}(\vec{\alpha}) + \gamma(\vec{\alpha})$ with suitable analytic functions $\vec{g}(\vec{\alpha})$ and $\gamma(\vec{\alpha})$, changing coordinates from $(\vec{A}, \vec{\alpha})$ to $(\vec{A}', \vec{\psi})$, and giving the invariant torus, (5), the form:

$$(13) \quad \vec{A} = \vec{A}_0 + \vec{\partial}_{\vec{\alpha}} \gamma(\vec{\alpha}), \quad \vec{\psi} = \vec{\alpha} + \vec{g}(\vec{\alpha})$$

Comparing with (5), and with the equation $(\vec{\omega}_0 \cdot \vec{\partial}_{\vec{\psi}})^2 \vec{h}(\vec{\psi}) = -\varepsilon \vec{\partial}_{\vec{\alpha}} f(\vec{\psi} + \vec{h}(\vec{\psi}))$ (meaning that (5) is an invariant torus and generating (7)) we see that $\vec{g}(\vec{\alpha}) = -\vec{h}(\vec{\psi})$ and $\vec{\partial}_{\vec{\alpha}} \gamma(\vec{\alpha}) = \vec{H}(\vec{\psi}) = (\vec{\omega}_0 \cdot \vec{\partial}_{\vec{\psi}}) J \vec{h}(\vec{\psi})$ if $\vec{\alpha} = \vec{\psi} + \vec{h}(\vec{\psi})$.

This implies that $\vec{\partial}_{\vec{\alpha}} \gamma(\vec{\alpha}) = -\varepsilon (\vec{\omega}_0 \cdot \vec{\partial}_{\vec{\psi}})^{-1} \vec{\partial}_{\vec{\alpha}} f(\vec{\alpha})$, if $\vec{\alpha} = \vec{\psi} + \vec{h}(\vec{\psi})$, and it would be nice to see which identities among tree values are equivalent to the last relation (i.e., to the identity of the second order cross derivatives of $\gamma(\vec{\alpha})$).

5 Some Problems that Will Be Left Over

The following are a few of the problems that have interested me and many of us which I believe (having no reason to believe the contrary) will be left over to the next century:

1) Develop a perturbation theory that applies to the simplest dynamical problems in statistical mechanics, such as a perturbation expansion in the density of the transport coefficients. Only “recently” it has been realized that the classical expansions were simply not well-defined (*i.e.*, had infinite coefficients), if the computation was attempted and resummation schemes proposed, [CD].

2) Develop a theory of the (incompressible) Navier-Stokes equation, like some constructive theorem for its solution in $d = 3$ dimensions, see [G4]. This seems to be the same as understanding a way to extend the theory of the $d = 2$ case, based on vorticity conservation. No mathematical theory of the $d = 3$ Navier-Stokes equation incorporates the fact that, for zero viscosity, there should be vorticity conservation in the form of Thomson’s theorem. The best known results [CKN] from the point of view of existence and regularity are based only on energy conservation, or, more mathematically, on the equation obtained by taking the scalar product of the velocity field with the difference of the two sides of the equation, and setting this equal to zero (with the pressure being given by the well known formula); *i.e.*, on just too little information, see also [G4].

3) Is it possible to construct (*i.e.*, prove existence of) a relativistic scalar field theory in dimension 4 which has an asymptotic series for the Schwinger functions given by the renormalized series for the $\lambda\varphi^4$ model? see [GR].

4) Can one understand Bose condensation? Definition, existence and basic properties, see [ADG].

5) Can a Fermi liquid be normal at $T = 0$, in 2 or more dimensions? See [A]. In fact the definition, existence, and basic properties of the ground state of Fermi systems are poorly understood, particularly if $d \geq 2$.

6) Can one show the existence of a liquid-solid phase transition in the statistical mechanics of one-component systems interacting via short range translation and rotation invariant interactions?

7) Can developed, stationary turbulence be understood more quantitatively in terms of the observed scaling properties and in conformity with the phenomenological theory of Kolmogorov?

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FROM SCHRÖDINGER SPECTRA TO ORTHOGONAL POLYNOMIALS, VIA A FUNCTIONAL EQUATION

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Abstract

The main difference between certain spectral problems for linear Schrödinger operators, e.g. the almost Mathieu equation, and three-term recurrence relations for orthogonal polynomials is that in the former the index ranges across $\mathbb{Z}$ and in the latter only across $\mathbb{Z}^+$. We present a technique that, by a mixture of Dirichlet and Taylor expansions, translates the almost Mathieu equation and its generalizations to three term recurrence relations. This opens up the possibility of exploiting the full power of the theory of orthogonal polynomials in the analysis of Schrödinger spectra.

The aforementioned three-term recurrence relations share the property that their coefficients are almost periodic. We generalize a method of proof, due originally to Jeff Geronimo and Walter van Assche, to investigate the essential support of the Borel measure of associated orthogonal polynomials, thereby deriving information on the underlying absolutely continuous spectra of Schrödinger operators.

1 The Almost Mathieu Equation and Orthogonal Polynomials

The point of departure of our analysis is the *almost Mathieu equation* (also known as the *Harper equation*). We seek $\lambda \in \mathbb{R}$ and $\{a_n\}_{n \in \mathbb{Z}} \in \ell_1[\mathbb{Z}]$ that satisfy

$$(1.1) \quad a_{n-1} - 2\kappa \cos(\alpha n + \beta) a_n + a_{n+1} = \lambda a_n, \quad n \in \mathbb{Z}.$$

α, β and $\kappa \neq 0$ being given real constants.

The almost Mathieu equation features in a number of applications [9] and has been already extensively studied [1, 3, 14, 15]. The purpose of our analysis is not to reveal new features of the spectrum of (1.1) *per se*, since the latter is quite comprehensively known. Instead, we intend to demonstrate that the almost Mathieu equation exhibits an intriguing connection with orthogonal polynomials, a connection that lends itself to far reaching generalizations.

Let $\omega = e^{i\alpha}$, $b_1 = \kappa e^{i\beta}$ and $b_2 = \kappa e^{-i\beta} = \bar{b}_1$. We rewrite (1.1) as

$$(1.2) \quad (b_1 \omega^{2n} + b_2) a_n = \omega^n (a_{n-1} - \lambda a_n + a_{n+1}), \quad n \in \mathbb{Z}$$

and consider the *Dirichlet expansion*

$$y(z) = \sum_{n=-\infty}^{\infty} a_n \exp \left\{ b_1^{\frac{1}{2}} \omega^n z \right\}, \quad z \in \mathbb{C}.$$

The choice of a specific branch of the square root of b_1 is arbitrary. Note that, since $|\omega| = 1$, it is easy to demonstrate that $\{a_n\}_{n \in \mathbb{Z}} \in \ell_1[\mathbb{Z}]$ implies convergence of the series for all $z \in \mathbb{C}$ [7, 10].

We multiply (1.2) by $\exp \left\{ b_1^{\frac{1}{2}} \omega^n z \right\}$ and sum for $n \in \mathbb{Z}$. Since

$$y'(z) = b_1^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} a_n \omega^n e^{b_1^{\frac{1}{2}} \omega^n z}, \quad y''(z) = b_1 \sum_{n=-\infty}^{\infty} a_n \omega^{2n} e^{b_1^{\frac{1}{2}} \omega^n z},$$

we readily deduce that y obeys the functional differential equation

$$(1.3) \quad y''(z) + b_2 y(z) = b_1^{-\frac{1}{2}} \left\{ \omega^{-1} y'(\omega^{-1} z) - \lambda y'(z) + \omega y'(\omega z) \right\}.$$

The solution of (1.3) is determined uniquely by the values of $y(0)$ and $y'(0)$.

Dirichlet expansions have been employed by Gregori Defel and Stanislav Molchanov [4] to investigate the simplified spectral problem

$$a_{n-1} - 2\kappa q^n a_n + a_{n+1} = \lambda a_n, \quad n \in \mathbb{Z},$$

where $\kappa \in \mathbb{R} \setminus \{0\}$ and $q > 0$, and they have demonstrated that it obeys another functional differential equation. Both the Defel–Molchanov equation and (1.3) are a generalization of the *pantograph equation*, that has been extensively analysed in [13] and [10]. However, it is important to emphasize that (1.3) has an important feature that sets it apart from other functional equations of the pantograph type, namely that, unless $\omega \in \mathbb{R}$, its evolution makes sense only for complex z and proceeds along circles of constant $|z|$, emanating from the origin.

Inasmuch as the equation (1.3) can be analysed directly, our next step entails expanding it in Taylor series. Thus, letting

$$y(z) = \sum_{m=0}^{\infty} \frac{y_m}{m!} z^m,$$

substitution in (1.3) readily yields

$$(1.4) \quad y_{m+1} = \sqrt{b_1}^{-\frac{1}{2}} (2 \cos \alpha m - \lambda) y_m - b_2 y_{m-1}, \quad m = 1, 2, \dots$$

It is beneficial to treat y_m as a function of the spectral parameter λ and to define

$$\tilde{y}_m(t) := b_2^{-m/2} y_m(-(b_1 b_2)^{1/2} t), \quad m = 0, 1, \dots$$

Brief manipulation affirms that (1.4) is equivalent to

$$(1.5) \quad \tilde{y}_{m+1}(t) = \left(t + \frac{2}{\kappa} \cos \alpha m \right) \tilde{y}_m(t) - \tilde{y}_{m-1}(t), \quad m = 0, 1, \dots$$

To specify the solution of (1.5) in a unique fashion we need to choose $\tilde{y}_0$ and $\tilde{y}_1$, which, of course, corresponds to equipping (1.3) with requisite initial conditions. We note in passing that $t = -\kappa\lambda$, hence real initial values in (1.3) correspond to $\tilde{y}_0, \tilde{y}_1 \in \mathbb{R}$.

Each solution of (1.5) is a linear combination of two linearly independent solutions. Setting $\sigma = 2/\kappa$, we let

$$(1.6) \quad \begin{aligned} r_{-1}(t) &\equiv 0, \\ r_0(t) &\equiv 1, \\ r_{m+1}(t) &= (t + \sigma \cos \alpha m) r_m(t) - r_{m-1}(t), \quad m = 0, 1, \dots, \end{aligned}$$

and

$$(1.7) \quad \begin{aligned} s_{-1}(t) &\equiv 0 \\ s_0(t) &\equiv 1 \\ s_{m+1}(t) &= (t + \sigma \cos \alpha(m+1)) s_m(t) - s_{m-1}(t), \quad m = 0, 1, \dots \end{aligned}$$

It is trivial to verify that $\{r_m\}_{m \in \mathbb{Z}^+}$ and $\{s_{m+1}\}_{m \in \mathbb{Z}^+}$ are linearly independent, hence they span all solutions of (1.5). Moreover, we observe that each $r_m(t)$ and $s_m(t)$ is an m th degree monic polynomial in t . This is a crucial observation, by virtue of the Favard theorem [2]: Given any three-term recurrence relation of the form

$$\begin{aligned} p_{-1}(t) &\equiv 0, \\ p_0(t) &\equiv 1, \\ p_{m+1}(t) &= (t + c_m) p_m(t) - d_m p_{m-1}(t), \quad m = 0, 1, \dots, \end{aligned}$$

the monic polynomial sequence $\{p_m\}_{m \in \mathbb{Z}^+}$ is orthogonal with respect to some Borel measure $d\varphi$, i.e.

$$\int_{\mathbb{R}} p_m(t) p_n(t) d\varphi(t) = 0, \quad m \neq n.$$

The essential support Ξ of $d\varphi$ is of central importance to the matter in hand, since it is an easy consequence of, for example, the n th root asymptotics of orthogonal polynomials [16] that, as long as Ξ is bounded, the sum $\sum_{m=0}^{\infty} p_m(t) z^m / m!$ converges for $z \in \mathbb{C}$, whereas convergence fails for $z \neq 0$ for $t \notin \Xi$. Thus, travelling all the way back from orthogonal polynomials to functional equations, and hence to the almost Mathieu equation, we deduce that, subject to the linear transformation $\lambda = -t/\kappa$, essential supports of the Borel measures corresponding to (1.6) and (1.7) result in the essential spectrum of (1.1).

We note that, in principle, the Favard theorem falls short of producing a unique measure and it is entirely possible that there exist many Borel measures that produce an identical set of monic orthogonal polynomials. This, however, is ruled out by the determinacy of the underlying Hamburger moment problem and the latter can be affirmed for both (1.6) and (1.7) by the Carleman criterion [2].

The remainder of this paper is devoted to the determination of Ξ . In Section 2 we demonstrate, by extending a technique due to Jeff Geronimo and Walter van Assche, that the essential support – and, indeed, the underlying Borel measure – can be specified explicitly when α/π is rational. We prove in essence that $d\varphi$ is a linear combination of Chebyshev measures of the second kind, supported on a set of disjoint intervals.

Various results on irrational α/π are reported in Section 3. Inasmuch as the general form of $d\varphi$ is currently a matter for conjecture, we derive a number of results that, beside being of interest on their own merit, are fully consistent with known results about the almost Mathieu operator, in particular with the theorem that the absolutely continuous spectrum of (1.1) is (for irrational α/π) a Cantor set.

The equation (1.1) has been extensively studied in the past and, inasmuch as there are few outstanding conjectures, our knowledge of the spectrum of the almost Mathieu operator is quite comprehensive. Although the approach of this paper introduces a new perspective, there is no claim that Sections 2–3 add to the current state of knowledge of Schrödinger spectra. This state of affairs is remedied in Section 4, where the framework of our discussion undergoes a far reaching generalization. Firstly, we demonstrate that general periodic potentials with a finite number of Fourier harmonics lend themselves to similar analysis, except that, instead of orthogonal polynomials, the outcome is a generalized eigenvalue problem for a certain matrix pencil. Secondly, we prove that a multivariate extension of the almost Mathieu equation can be ‘transformed’ by our techniques to a problem in (univariate) orthogonal polynomials.

A possible application of our analysis, and in particular of Section 4, is numerical computation of the essential spectrum of (1.1) and of its generalizations. We do not pursue this further in the present paper.

The observation that (1.1) is ‘almost’ a three-term recurrence relation – only the index range is wrong – hence that the almost Mathieu equation might be connected with orthogonal polynomials, is not new. Most prominently, it had been made in [12], where it motivated a very interesting generalization of Chebyshev polynomials. The most important innovation in the present paper is in a technique that reduces the index range from $\mathbb{Z}$ to $\mathbb{Z}^+$ by passing from Dirichlet to Taylor expansions and which can be extended to cater for a substantially more general problem.

2 Orthogonal Polynomials with Periodic Recurrence Coefficients

The focus of our attention in this section is the three-term recurrence

$$(2.1) \quad \begin{aligned} p_{-1}(t) &\equiv 0, \\ p_0(t) &\equiv 1, \\ p_{m+1}(t) &= (t - \alpha_m)p_m(t) - p_{m-1}(t), \quad m = 0, 1, \dots, \end{aligned}$$

where the sequence $\{\alpha_m\}_{m \in \mathbb{Z}^+}$ is K -periodic,

$$(2.2) \quad \alpha_{m+K} = \alpha_m, \quad m = 0, 1, \dots$$

Note that both (1.6) and (1.7) assume this form when α/π is rational. Our objective is to determine the Borel measure that renders $\{p_m\}_{m \in \mathbb{Z}^+}$ into an orthogonal polynomial system (OPS).

In [6] the authors consider the following problem. Let $\{Q_m\}_{m \in \mathbb{Z}^+}$ be an OPS whose measure has an essential support $\Xi_0 \subseteq [-1, 1]$ and let T be a given N th degree polynomial. Setting $\Xi = T^{-1}(\Xi_0)$ (the latter set is, generically, a union of $\leq N$ disjoint

intervals), they derive a new OPS, whose Borel measure is supported in Ξ , explicitly in terms of $\{Q_m\}_{m \in \mathbb{Z}^+}$. This construction is intimately related to the discussion of this section, except that we need, in a manner of speech, to travel in the opposite direction. As it turns out, the Borel measure associated with (2.1) inhabits a sets of disjoint intervals and we identify it by choosing an appropriate polynomial transformation T .

Let

$$q_n(t) := p_{(n+1)K-1}(t), \quad n = 0, 1, \dots$$

Note that $q_{-1} \equiv 0$ and, moreover, (2.1) and (2.2) imply

$$(2.3) \quad p_{nK}(t) = (t - \alpha_0)q_{n-1} - p_{nK-2}(t), \quad n = 1, 2, \dots$$

We seek polynomials α_ℓ, β_ℓ , $\ell = 0, 1, \dots, K-1$, such that

$$p_{nK+\ell}(t) = a_\ell(t)q_{n-1}(t) - b_\ell(t)p_{nK-2}(t), \quad \ell = 0, 1, \dots, K-1, \quad n = 1, 2, \dots$$

Because of (2.3) and the definition of q_n , we have

$$(2.4) \quad \begin{aligned} a_{-1}(t) &\equiv 1, & b_{-1}(t) &\equiv 0, \\ a_0(t) &= t - \alpha_0, & b_0(t) &\equiv 1. \end{aligned}$$

We next substitute in the recurrence relation (2.1) and, by virtue of (2.2), obtain

$$\begin{aligned} p_{nK+\ell+1}(t) &= (t - \alpha_{\ell+1})p_{nK+\ell}(t) - p_{nK+\ell-1}(t) \\ &= (t - \alpha_{\ell+1})\{a_\ell(t)q_{n-1}(t) - b_\ell(t)p_{nK-2}(t)\} \\ &\quad - \{a_{\ell-1}(t)q_{n-1}(t) - b_{\ell-1}(t)p_{nK-2}(t)\}. \end{aligned}$$

Thus, comparing coefficients, we derive the recurrences

$$(2.5) \quad a_{\ell+1}(t) = (t - \alpha_{\ell+1})a_\ell(t) - a_{\ell-1}(t),$$

$$(2.6) \quad b_{\ell+1}(t) = (t - \alpha_{\ell+1})b_\ell(t) - b_{\ell-1}(t), \quad \ell = 0, 1, \dots, K-2,$$

which, in tandem with (2.4), determine $\{a_\ell, b_\ell\}_{\ell=0}^{K-1}$.

Let $\ell = K-1$, then

$$p_{nK-2}(t) = \frac{a_{K-1}(t)q_{n-1}(t) - q_n(t)}{b_{K-1}(t)}$$

and, shifting the index,

$$p_{(n+1)K-2}(t) = \frac{a_{K-1}(t)q_n(t) - q_{n+1}(t)}{b_{K-1}(t)}$$

Substituting both expressions into

$$p_{(n+1)K-2}(t) = a_{K-2}(t)q_{n-1}(t) - b_{K-2}(t)p_{nK-2}(t)$$

yields the recurrence relation

$$(2.7) \quad q_{n+1}(t) = (a_{K-1}(t) - b_{K-2}(t))q_n(t) - \Delta_{n-2}(t)q_{n-1}(t),$$

where

$$\Delta_\ell(t) = \det \begin{bmatrix} a_\ell(t) & a_{\ell+1}(t) \\ b_\ell(t) & b_{\ell+1}(t) \end{bmatrix}, \quad \ell = 0, 1, \dots, K-1.$$

We multiply (2.6) by $a_\ell(t)$, (2.5) by $b_\ell(t)$ and subtract from each other. This readily affirms by induction that

$$\Delta_\ell(t) = \Delta_{\ell-1}(t) = \dots = 1$$

and (2.7) simplifies into

$$(2.8) \quad q_{n+1}(t) = (a_{K-1}(t) - b_{K-2}(t))q_n(t) - q_{n-1}(t), \quad n = 0, 1, \dots$$

Note that (2.8) is consistent with $n = 0$, since $q_{-1} \equiv 0$. To further simplify the recurrence, we observe that $q_0(t) = a_{K-1}(t) - b_{K-2}(t)$, hence, letting

$$\tilde{q}_n(x) := \frac{q_n(t)}{q_0(t)}, \quad n = -1, 0, \dots,$$

where $x = q_0(t)$, we obtain the three-term recurrence

$$\begin{aligned} \tilde{q}_{-1}(x) &\equiv 0, \\ \tilde{q}_0(x) &\equiv 1, \\ \tilde{q}_{n+1}(x) &= x\tilde{q}_n(x) - \tilde{q}_{n-1}(x), \quad n = 0, 1, \dots \end{aligned}$$

Thus, each $\tilde{q}_n$ is an n th degree monic polynomial and, by virtue of the Favard theorem, $\{\tilde{q}_n\}_{n \in \mathbb{Z}^+}$ is an OPS. It can be easily identified as a shifted and scaled *Chebyshev polynomial of the second kind*,

$$\tilde{q}_n(x) = 2^n U_n\left(\frac{1}{2}x\right), \quad n = 0, 1, \dots$$

We thus deduce that

$$(2.9) \quad q_n(t) = 2^n q_0(t) U_n\left(\frac{1}{2}q_0(t)\right), \quad n = 0, 1, \dots$$

Before we identify the underlying Borel measure, let us 'fill in' the remaining values of p_m . By definition, $p_{nK-1} = q_{n-1}$, $p_{n(K+1)-1} = q_n$, hence the recurrence (2.1) gives

$$\begin{aligned} (x - \alpha_1)p_{nK}(t) - p_{nK+1}(t) &= q_{n-1}(t), \\ -p_{nK+\ell-1}(t) + (t - \alpha_{\ell+1})p_{nK+\ell}(t) - p_{nK+\ell+1}(t) &= 0, \quad \ell = 1, 2, \dots, K-3, \\ -p_{(n+1)K-3}(t) + (t - \alpha_{K-1})p_{(n+1)K-2}(t) &= q_n(t). \end{aligned}$$

This is a linear system of equations, which we write as

$$(2.10) \quad A_{K-1} \mathbf{p}_n = \mathbf{q}_n,$$

where

$$A_m = \begin{bmatrix} t - \alpha_1 & -1 & & & & \\ -1 & t - \alpha_2 & -1 & & & \\ & -1 & t - \alpha_3 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & t - \alpha_{m-1} & -1 \\ & & & & -1 & t - \alpha_m \end{bmatrix}, \quad m = 1, 2, \dots, K-1,$$

$$\mathbf{p}_n = \begin{bmatrix} p_{nK}(t) \\ p_{nK+1}(t) \\ \vdots \\ p_{(n+1)K-3}(t) \\ p_{(n+1)K-2}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{q}_n = \begin{bmatrix} q_{n-1}(t) \\ 0 \\ \vdots \\ 0 \\ q_n(t) \end{bmatrix}.$$

We expand the determinant of A_m in its bottom row and rightmost column. This results in a three-term recurrence relation and comparison with (2.4) and (2.6) affirms that $\det A_m = b_m(t)$. Hence, solving (2.10) with Cramer's rule, we deduce that there exist $(K-2)$ -degree polynomials $\tilde{a}_\ell$ and $\tilde{b}_\ell$, $\ell = 0, 1, \dots, K-2$, such that

$$(2.11) \quad p_{nK+\ell}(t) = \frac{\tilde{a}_\ell(t)q_{n-1}(t) + \tilde{b}_\ell(t)q_n(t)}{b_{K-1}(t)}, \quad \ell = 0, 1, \dots, K-2.$$

Bearing in mind the definition of a_ℓ and b_ℓ ,

$$p_{nK+\ell}(t) = a_\ell(t)q_{n-1}(t) - b_\ell(t)p_{nK-2}(t),$$

we obtain from (2.11) the identity

$$\begin{aligned} \frac{\tilde{a}_\ell(t)q_{n-1}(t) + \tilde{b}_\ell(t)q_n(t)}{b_{K-1}(t)} &= a_\ell(t)q_{n-1}(t) \\ &\quad - \frac{b_\ell(t)(\tilde{a}_{K-2}(t)q_{n-2}(t) + \tilde{b}_{K-2}(t)q_{n-1}(t))}{b_{K-1}(t)}. \end{aligned}$$

We next substitute

$$q_{n-2}(t) = (a_{K-1}(t) - b_{K-2}(t))q_{n-1}(t) - q_n(t)$$

(pace (2.8)) and rearrange terms, whereby

$$\begin{aligned} &\{\tilde{b}_\ell(t) - b_\ell(t)\tilde{a}_{K-2}(t)\}q_n(t) \\ &= \{-\tilde{a}_\ell + a_\ell(t)b_{K-1}(t) - (a_{K-1}(t)b_\ell(t) - b_\ell(t)b_{K-2}(t))\tilde{a}_{K-2}(t) \\ &\quad - b_\ell(t)\tilde{b}_{K-2}(t)\}q_{n-1}(t). \end{aligned}$$

However, consecutive orthogonal polynomials q_{n-1} and q_n cannot share zeros [2], therefore both sides of the last equality identically vanish and we derive the explicit expressions

$$(2.12) \quad \begin{aligned} \tilde{a}_\ell(t) &= a_\ell(t)b_{K-1}(t) - (a_{K-1}(t)b_\ell(t) - b_\ell(t)b_{K-2}(t))\tilde{a}_{K-2}(t) \\ &\quad - b_\ell(t)\tilde{b}_{K-2}(t), \end{aligned}$$

$$(2.13) \quad \tilde{b}_\ell(t) = b_\ell(t)\tilde{a}_{K-2}(t).$$

Letting $\ell = K-2$ in (2.13) gives $\tilde{b}_{K-2}(t) = b_{K-2}(t)\tilde{a}_{K-2}(t)$ and we substitute this into (2.12). The outcome is

$$(2.14) \quad \tilde{a}_\ell(t) = a_\ell(t)b_{K-1}(t) - a_{K-1}(t)b_\ell(t)\tilde{a}_{K-2}(t).$$

In particular, $\ell = K - 2$ and the definition of Δ_m result in

$$(1 + a_{K-1}(t)b_{K-2}(t))\tilde{a}_{K-2}(t) = a_{K-2}(t)b_{K-1}(t) = a_{K-1}(t)b_{K-2}(t) \\ + \Delta_{K-2}(t) = 1 + a_{K-1}(t)b_{K-2}(t).$$

Since $a_{K-1}b_{K-2} \neq -1$, we conclude that $\tilde{a}_{K-2} \equiv 1$ and substitution in (2.13) and (2.14) yields the explicit formulae

$$\tilde{a}_\ell = \det \begin{bmatrix} a_\ell(t) & a_{K-1}(t) \\ b_\ell(t) & b_{K-1}(t) \end{bmatrix}, \quad \tilde{b}_\ell(t) = b_\ell(t), \quad \ell = 0, 1, \dots, K-2.$$

Theorem 1 The OPS $\{p_m\}_{m \in \mathbb{Z}^+}$ has an explicit representation in the form

$$(2.15) \quad p_{nK+\ell}(t) = \frac{1}{b_{K-1}(t)} \left\{ \det \begin{bmatrix} a_\ell(t) & a_{K-1}(t) \\ b_\ell(t) & b_{K-1}(t) \end{bmatrix} q_{n-1}(t) + b_\ell(t)q_n(t) \right\},$$

where $n = 0, 1, \dots$, $\ell = 0, 1, \dots, K-1$ and the OPS $\{q_n\}_{n \in \mathbb{Z}^+}$ satisfies the three-term recurrence (2.8). $\square$

Note that letting $\ell = K-1$ or $\ell = -1$ in (2.15), in tandem with (2.4), results in $p_{(n+1)K-1} = q_n$ and $p_{nK-1} = q_{n-1}$ respectively, as required.

Let $t \in \Xi$. Then, by the discussion preceding the representation (2.9), we know that $\frac{1}{2}q_0(t) \in [-1, 1]$, and there exists $\theta \in [-\pi, \pi]$ such that $\frac{1}{2}q_0(t) = \cos \theta$. Since, by the definition of Chebyshev polynomials of the second kind,

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta},$$

(2.9) implies that

$$q_n(t) = 2^n q_0(t) \frac{\sin(n+1)\theta}{\sin \theta}.$$

Substitution into (2.15) results in

$$p_{nK+\ell}(t) = \frac{2^n q_0(t)}{b_{K-1}(t) \sin \theta} \left\{ \det \begin{bmatrix} a_\ell(t) & a_{K-1}(t) \\ b_\ell(t) & b_{K-1}(t) \end{bmatrix} \sin n\theta + b_\ell(t) \sin(n+1)\theta \right\}.$$

We next proceed to determine the essential support Ξ of the Borel measure $d\varphi$ which corresponds to the OPS $\{p_n\}_{n \in \mathbb{Z}^+}$. As we have already mentioned, this is very similar to the construction of Geronimo and van Assche in [6], with $\{q_n\}_{n \in \mathbb{Z}^+}$, $\frac{1}{2}q_0$ and $[-1, 1]$ playing the role of $\{Q_n\}_{n \in \mathbb{Z}^+}$, T and Ξ_0 respectively. This similarity notwithstanding, there are some important differences – not least that our argument advances in an opposite direction to that of Geronimo and van Assche – and we present here a complete derivation of Ξ . We commence by observing that, by virtue of Theorem 1, everything depends on the support of $\tilde{q}_n(x(t)) = 2^n U_n(\frac{1}{2}q_0(t))$, $n = 0, 1, \dots$. Thus, we seek a Borel measure $d\psi$ such that

$$I_{n,m} = \int_{-\infty}^{\infty} \tilde{q}_n(x(t)) \tilde{q}_m(x(t)) d\psi(t) = 0, \quad n, m = 0, 1, \dots, \quad n \neq m.$$

It follows from the definition of $\tilde{q}_n$ that

$$I_{n,m} = 2^{n+k} \int_{-\infty}^{\infty} U_n \left(\frac{1}{2} q_0(t) \right) U_m \left(\frac{1}{2} q_0(t) \right) d\psi(t).$$

Similarly to [6], we seek the inverse function to $x = \frac{1}{2} q_0(t)$. Let $\xi_1 < \xi_2 < \dots < \xi_s$ be all the minima and maxima of q_0 in $\mathbb{R}$ (of course, $s \leq K-2$) and define $\xi_0 = -\infty$, $\xi_{s+1} = \infty$. In each interval $[\xi_j, \xi_{j+1}]$, $j = 0, 1, \dots, s$, the function $\frac{1}{2} q_0(t)$ is monotone, hence it possesses there a well-defined inverse. We denote it by $X_j(x)$, hence $\frac{1}{2} q_0(X_j(x)) = x$. Changing the integration variable, we have

$$\begin{aligned} I_{n,m} &= 2^{n+m} \sum_{j=0}^s \int_{\xi_j}^{\xi_{j+1}} U_n \left(\frac{1}{2} q_0(t) \right) U_m \left(\frac{1}{2} q_0(t) \right) d\psi(t) \\ (2.16) \quad &= 2^{n+m} \sum_{j=0}^s \int_{\frac{1}{2} q_0(\xi_j)}^{\frac{1}{2} q_0(\xi_{j+1})} U_n(x) U_m(x) d\psi(X_j(x)). \end{aligned}$$

We recall that $\{U_n\}_{n \in \mathbb{Z}^+}$ is an OPS with respect to the Borel measure $(1-x^2)^{\frac{1}{2}} dx$, supported by $x \in [-1, 1]$. Thus, for every $j = 0, 1, \dots, s$ we distinguish among the following cases:

Case 1: $q_0(\xi_j) \leq -2$ and $2 \leq q_0(\xi_{j+1})$.

We stipulate that $d\psi(X_j(x))$ vanishes for all

$$x \in \left[\frac{1}{2} q_0(\xi_j), \frac{1}{2} q_0(\xi_{j+1}) \right] \setminus [-1, 1].$$

Since X_j increases monotonically in $[\xi_j, \xi_{j+1}]$, the contribution of this interval to (2.16) is

$$(2.17) \quad 2^{n+m} \int_{-1}^1 U_n(x) U_m(x) d\psi(|X_j(x)|).$$

Case 2: $q_0(\xi_{j+1}) \leq -2$ and $2 \leq q_0(\xi_j)$.

Likewise, we require that the support of $d\psi(X_j)$ is restricted to $[-1, 1]$. q_0 decreases monotonically within $[\xi_j, \xi_{j+1}]$ and straightforward manipulation affirms that (2.17) represents the contribution of this interval to (2.16).

Case 3: $\min\{q_0(\xi_j), q_0(\xi_{j+1})\} > -2$ or $\max\{q_0(\xi_j), q_0(\xi_{j+1})\} < 2$.

In that case we cannot fit $[-1, 1]$ into $[\xi_j, \xi_{j+1}]$, hence we stipulate that $d\psi(X_j)$ is not supported in $[\xi_j, \xi_{j+1}]$.

Let $\nu_1 < \nu_2 < \dots < \nu_r$ be all the indices in $\{0, 1, \dots, s\}$ such that either Case 1 or Case 2 holds. We require that $r \geq 1$. Then (2.16) reduces to

$$I_{n,m} = \int_{-1}^1 U_n(x) U_m(x) \sum_{\ell=1}^r d\psi(|X_{\nu_\ell}(x)|).$$

Since the Hamburger moment problem for the Chebyshev measure of the second kind is determinate, it follows that necessarily

$$\sum_{\ell=1}^r d\psi(|X_{\nu_\ell}(x)|) = (1-x^2)^{\frac{1}{2}} dx, \quad x \in [-1, 1].$$

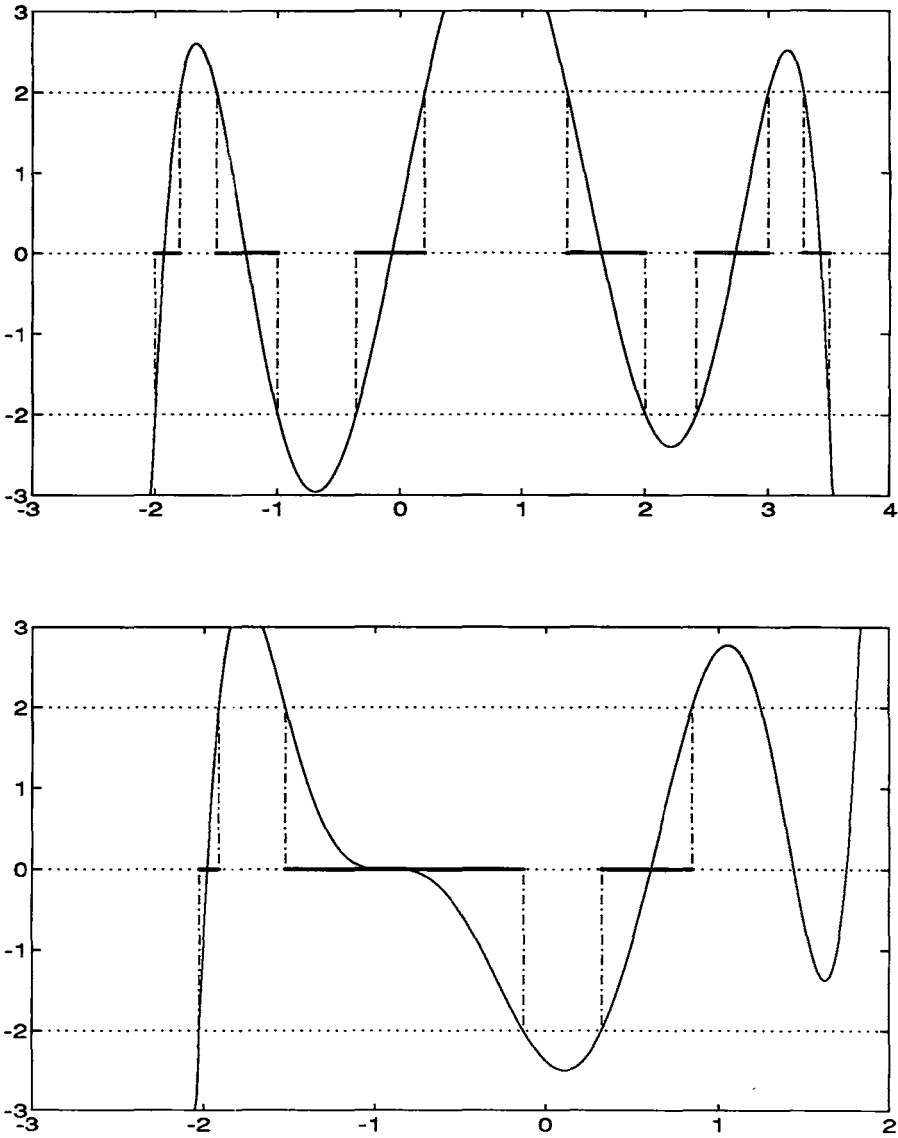


Figure 1: The sets Ξ for two different polynomials q_0 .

Theorem 2 The orthogonality measure corresponding to the OPS $\{p_m\}_{m \in \mathbb{Z}^+}$ is supported by

$$\Xi = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_r,$$

where for each $\ell = 1, 2, \dots, r$ $\mathcal{I}_\ell \subseteq [\xi_{\nu_\ell}, \xi_{\nu_{\ell+1}}]$ is the unique interval such that $|q_0(t)| = 2$ at its endpoints.

Proof. Follows at once from our construction. $\square$

Figure 1 displays two examples of the present construction, for different cases of q_0 . In each case Ξ is the union of the ‘thick’ intervals.

Harking back to (1.6) and (1.7), we let

$$\alpha_m = -\sigma \cos \alpha m \quad \text{and} \quad \alpha_m = -\sigma \cos(m+1)\alpha, \quad m = 0, 1, \dots,$$

respectively, where $\alpha = 2\pi L/K$. Thus, $\{\alpha_m\}_{m \in \mathbb{Z}^+}$ is indeed K -periodic. Unsurprisingly, the outcome of our analysis are the familiar spectral bounds [1, 3]. The merits of our approach are, however, not just in providing an alternative proof of known results but also in extending the framework to the multivariate case in Section 4.

We mention in passing that the analysis of this section can be easily extended to recurrences of the form

$$\begin{aligned} p_{-1}(t) &\equiv 0, \\ p_0(t) &\equiv 1, \\ p_{m+1}(t) &= (t - \alpha_m)p_m(t) - \beta_m p_{m-1}(t), \quad m = 0, 1, \dots, \end{aligned}$$

where both $\{\alpha_m\}_{m \in \mathbb{Z}^+}$ and $\{\beta_m\}_{m \in \mathbb{Z}^+}$ are K -periodic. This, however, is of little relevance to the theme of this paper.

3 Orthogonal Polynomials with Almost Periodic Recurrence Coefficients

The three-term recurrence relations (1.6) and (1.7) assume, α/π being irrational, almost-periodic recurrence coefficients and this state of affairs is even more important in a multivariate generalization of (1.1) in Section 4. Unfortunately, no general theory exists to cater for orthogonal polynomials with almost periodic recurrence coefficients. The theme of the present section is a preliminary and – in the nature of things – incomplete investigation of the case when an irrational α/π is approximated by rationals. In other words, we commence with the K -periodic recurrence (2.1), except that we will allow the period K to become unbounded.

The motivation for our analysis is an observation which is interesting on its own merit. Denote by σ_K the value of $q_{K-1}(0)$ for $\alpha_\ell = -\cos \frac{2\pi\ell}{K}$, $\ell = 0, 1, \dots, K-1$, i.e.

$$\sigma_K = \det \begin{bmatrix} \cos \frac{2\pi}{K} & 1 & & & \\ 1 & \cos \frac{4\pi}{K} & & & \\ & & \ddots & \ddots & \\ & & & 1 & \cos \frac{2(K-2)\pi}{K} \\ & & & & 1 & \cos \frac{2(K-1)\pi}{K} \end{bmatrix}, \quad K = 1, 2, \dots$$

Computation indicates that

$$(3.1) \quad \sigma_{4L} \equiv 0, \quad \lim_{L \rightarrow \infty} \sigma_{4L+2} = 0, \quad \lim_{L \rightarrow \infty} \sigma_{4L+1} = - \lim_{L \rightarrow \infty} \sigma_{4L+3} = \frac{2\sqrt{3}}{3}.$$

It is easy to prove that $\sigma_{4L} = 0$ for all $L \geq 1$. Thus, let $K = 4L$ and set $\ell = L$ in

$$(3.2) \quad \nu_\ell = \cos \frac{2\pi\ell}{K} \nu_{\ell-1} - \nu_{\ell-2}, \quad \ell = 1, 2, \dots, 4L-1$$

(note that (3.2), in tandem with $\nu_{-1} = 0$, $\nu_0 = 1$, yields $\sigma_K = \nu_{K-1}$). This yields $\nu_L + \nu_{L-2} = 0$ and we claim that, in general,

$$(3.3) \quad \nu_{L+k} + (-1)^k \nu_{L-k} = 0, \quad k = -1, 0, \dots, L-1.$$

We have already proved (3.3) for $k = 0$ and it is trivially true for $k = -1$. We continue by induction and assume that (3.3) is true for $k = -1, 0, \dots, s-1$. Letting $\ell = L \pm s$ in (3.2), we have

$$\begin{aligned} \nu_{L+s} &= -\sin \frac{\pi s}{2L} \nu_{L+s-1} - \nu_{L+s-2}, \\ \nu_{L-s} &= +\sin \frac{\pi s}{2L} \nu_{L-s-1} - \nu_{L-s-2}. \end{aligned}$$

We multiply the second equation by $(-1)^s$ and add to the first, thus

$$\begin{aligned} (\nu_{L+s} + (-1)^s \nu_{L-s}) &= -\sin \frac{\pi s}{2L} (\nu_{L+s-1} + (-1)^{s-1} \nu_{L-s+1}) \\ &\quad - (\nu_{L+s-2} + (-1)^{s-2} \nu_{L-s+2}) \end{aligned}$$

and (3.3) follows at once.

Hence, letting $k = L-1$ in (3.3) and recalling that $\nu_{-1} = 0$, we obtain $\nu_{2L-1} = 0$. Moreover, similarly to (3.3), we can prove that

$$\nu_{3L+k} + (-1)^k \nu_{3L-k} = 0, \quad k = -1, 0, \dots, L-1$$

and $k = L-1$ gives

$$\sigma_{4L} = \nu_{4L-1} = (-1)^L \nu_{2L-1} = 0.$$

This completes the proof of the lemma. $\square$

Other observations in (3.1) are also true and in the sequel we prove them in a generalized setting.

Given $\sigma \in (-1, 1)$, we define

$$(3.4) \quad \begin{aligned} A_{-1}(t) &\equiv 0, & A_0(t) &\equiv 1, \\ A_n(t) &= t\xi_n A_{n-1}(t) - A_{n-2}(t), & n &= 1, 2, \dots, \end{aligned}$$

where $\{\xi_n\}_{n=1}^\infty$ is a given real sequence. To emphasize the dependence on parameters, we write, as and when necessary, $A_n(\cdot) = A_n(\cdot; \xi_1, \xi_2, \dots, \xi_n)$.

An alternative representation of A_n is

$$A_n(t) = \det \begin{bmatrix} t\xi_1 & 1 & & & \\ 1 & t\xi_2 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & & 1 & t\xi_n \end{bmatrix}, \quad n = 1, 2, \dots,$$

hence A_n is an n th degree polynomial. We observe that

$$A_n(0) = \begin{cases} (-1)^s & : n = 2s, \\ 0 & : n = 2s + 1. \end{cases}$$

Moreover, differentiating with respect to t , we obtain

$$A'_n(t) = \sum_{k=1}^n \det \begin{bmatrix} t\xi_1 & 1 & & & & & \\ & 1 & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & & 1 & t\xi_{k-1} & 1 & \\ & & & & 0 & \xi_k & 0 \\ & & & & 1 & t\xi_{k+1} & 1 \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & \ddots & \ddots & 1 \\ & & & & & & & 1 & t\xi_n \end{bmatrix},$$

therefore, expanding in the k th row, we derive the identity

$$(3.5) \quad A'_n(t; \xi_1, \dots, \xi_n) = \sum_{k=1}^n \xi_k A_{k-1}(t; \xi_1, \dots, \xi_{k-1}) A_{n-k}(t; \xi_{k+1}, \dots, \xi_n).$$

Proposition 3 $A_n^{(r)}(0) = 0$ whenever $n + r$ is odd, hence the polynomial A_n has the same parity as n .

Proof. By induction on r . The assertion is true for $r = 0$. Moreover, repeatedly differentiating (3.5) with the Leibnitz rule and letting $t = 0$, we obtain

$$\begin{aligned} & A_{2n}^{(2r+1)}(0; \xi_1, \dots, \xi_{2n}) \\ &= \sum_{\ell=0}^{2r} \binom{2r}{\ell} \sum_{k=1}^{2n} \xi_k A_{k-1}^{(\ell)}(0; \xi_1, \dots, \xi_{k-1}) A_{2n-k}^{(2r-\ell)}(0; \xi_{k+1}, \dots, \xi_{2n}). \end{aligned}$$

But

$$(k-1) + \ell \text{ is even} \iff (2n-k) + (2r-\ell) \text{ is odd},$$

therefore for all $\ell = 0, 1, \dots, 2r$ and $k = 1, 2, \dots, 2n$ the induction hypothesis affirms that at least one of the terms in the product vanishes. Similar argument demonstrates that $A_{2n+1}^{(2r)}(0) = 0$. $\square$

We therefore let for all $n = 0, 1, \dots, s = 1, 2, \dots$,

$$\begin{aligned} A_{2n}(t; \xi_s, \dots, \xi_{2n+s-1}) &= \sum_{r=0}^n B_{2n,s}^{(2r)} t^{2r} \\ A_{2n+1}(t; \xi_s, \dots, \xi_{2n+s}) &= \sum_{r=0}^n B_{2n+1,s}^{(2r+1)} t^{2r+1}. \end{aligned}$$

Substitution into (3.4) (where we replace ξ_n by ξ_{n+s-1}) results in the recurrences

$$(3.6) \quad B_{2n,s}^{(2r)} = \xi_{s+2n-1} B_{2n-1,s}^{(2r-1)} - B_{2n-2,s}^{(2r)},$$

$$(3.7) \quad B_{2n+1,s}^{(2r+1)} = \xi_{s+2n} B_{2n,s}^{(2r)} - B_{2n-1,s}^{(2r+1)}.$$

Given $q \in \mathbb{C}$, we recall that the q -factorial symbol is defined as

$$(z; q)_n = \prod_{k=0}^{n-1} (1 - q^k z), \quad z \in \mathbb{C}, \quad n \in \mathbb{Z} \cup \{\infty\},$$

whereas the q -binomial reads

$$\begin{bmatrix} n \\ m \end{bmatrix} := \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}, \quad 0 \leq m \leq n.$$

Lemma 4 Let $\xi_s = q^{\frac{1}{2}s} + q^{-\frac{1}{2}s}$, $s = 1, 2, \dots$, where $q \in \mathbb{C}$ is given. Then, for every $n = 0, 1, \dots$, $s = 1, 2, \dots$ and $r = 0, 1, \dots, n$,

$$(3.8) \quad B_{2n,s}^{(2r)} = (-1)^{n+r} q^{-r(2n+s-r-\frac{1}{2})} \begin{bmatrix} n+r \\ 2r \end{bmatrix} (-q^{n+s-r}; q)_{2r},$$

$$(3.9) \quad B_{2n+1,s}^{(2r+1)} = (-1)^{n+r} q^{-(r+\frac{1}{2})(2n+s-r)} \begin{bmatrix} n+r+1 \\ 2r+1 \end{bmatrix} (-q^{n+s-r}; q)_{2r+1}.$$

Proof. By induction on n , using (3.8) and (3.9). Obviously, the assertion of the lemma is true for $n = 0$. Otherwise, for even values,

$$\begin{aligned} & \left(q^{n+\frac{s-1}{2}} + q^{-n-\frac{s-1}{2}} \right) B_{2n-1}^{(2r-1)} - B_{2n-2,s}^{(2r)} \\ &= (-1)^{n+r} q^{-(r-\frac{1}{2})(2n+s-r-1)} \left(q^{n+\frac{s-1}{2}} + q^{-n-\frac{s-1}{2}} \right) \begin{bmatrix} n+r-1 \\ 2r-1 \end{bmatrix} (-q^{n+s-r}; q)_{2r-1} \\ & \quad - (-1)^{n+r-1} q^{-r(2n+s-r-\frac{s}{2})} \begin{bmatrix} n+r-1 \\ 2r \end{bmatrix} (-q^{n+s-r-1}; q)_{2r} \\ &= (-1)^{n+r} q^{-r(2n+s-r-\frac{1}{2})} \frac{(q; q)_{n+r-1}}{(q; q)_{2r} (q; q)_{n-r}} (-q^{n+s-r}; q)_{2r-1} \\ & \quad \times \left\{ (1 + q^{2n+s-1})(1 - q^{2r}) + q^{2r}(1 - q^{n-r})(1 + q^{n+s-r-1}) \right\} \\ &= (-1)^{n+r} q^{-r(2n+s-r-\frac{1}{2})} \begin{bmatrix} n+r \\ 2r \end{bmatrix} (-q^{n+s-r}; q)_{2r}. \end{aligned}$$

This accomplishes a single inductive step for (3.8). We prove (3.9) in an identical manner, by considering odd values of n . $\square$

Recall that our interest in the polynomials A_n has been sparked by the observation (3.1). Thus, we require to recover cosine terms, and to this end we choose q of unit modulus.

Proposition 5 Suppose that $q^{n+1} = 1$ and $q^m \neq 1$ for $m = 1, 2, \dots, n$. Then $B_{2n+1,1}^{(2r+1)} = 0$ for all $0 \leq r \leq \frac{n}{2}$.

Proof. Let $2r \leq n$. We have from (3.9) that

$$B_{2n+1,1}^{(2r+1)} = (-1)^{n+r} q^{(r+\frac{1}{2})(r+1)} \frac{(q; q)_{n+r+1}}{(q; q)_{2r+1} (q; q)_{n-r}} (-q^{-r}; q)_{2r+1}.$$

However,

$$\frac{(q; q)_{n+r+1}}{(q; q)_{n-r}} = \prod_{\ell=-r}^r (1 - q^\ell) = 0,$$

whereas, because of our restriction on r ,

$$(q; q)_{2r+1} = \prod_{\ell=1}^{2r+1} (1 - q^\ell) \neq 0,$$

since q is a root of unity of *minimal* degree $n+1$. The proposition follows. $\square$

Corollary Let $q = \exp \frac{2\pi im}{n+1}$, where m and n are relatively prime. Then it is true that

$$(3.10) \quad \lim_{n \rightarrow \infty} A_{2n+1}(t) = 0$$

for every $t \in (-1, 1)$.

Proof. Straightforward, since $A_{2n+1}(t) = \mathcal{O}(t^{\frac{1}{2}n})$. $\square$

Proposition 6 Suppose that $\omega = q^{\frac{1}{2}}$ is a root of unity of minimal degree $2n+1$. Then, for all $r = 0, 1, \dots, n$ it is true that

$$(3.11) \quad (-1)^n B_{2n,1}^{(2r)} = \prod_{\ell=1}^r \frac{\sin(2\ell-1)\phi}{\sin 2\ell\phi},$$

where $\phi = \arg \omega$.

Proof. Since $q^{n+\frac{1}{2}} = 1$, it follows from (3.8) that

$$(-1)^n B_{2n,1}^{(2r)} = (-1)^r q^{r(r+\frac{1}{2})} \frac{(q; q)_{n+r}}{(q; q)_{2r}(q; q)_{n-r}} (-q^{-r+\frac{1}{2}}; q)_{2r}.$$

But

$$\frac{(q; q)_{n+r}(q^{-r+\frac{1}{2}}; q)_{2r}}{(q; q)_{n-r}} = \prod_{\ell=-r}^{r-1} (1 - q^{2\ell+1}).$$

Moreover, $(q; q)_{2r} \neq 0$ for $r = 0, 1, \dots, n$, since $2n+1$ is the least nontrivial degree of the root of unity q , and we deduce that

$$(-1)^n B_{2n,1}^{(2r)} = (-1)^r q^{r(r+\frac{1}{2})} \frac{\prod_{\ell=-r}^{r-1} (1 - q^{2\ell+1})}{\prod_{\ell=1}^{2r} (1 - q^\ell)}.$$

But

$$\prod_{\ell=1}^{2r} (1 - q^\ell) = \prod_{\ell=1}^{2r} \omega^\ell (\omega^\ell - \omega^{-\ell}) = q^{r(r+\frac{1}{2})} \prod_{\ell=1}^{2r} (\omega^\ell - \omega^{-\ell})$$

and, likewise,

$$\prod_{\ell=-r}^{r-1} (1 - q^{2\ell+1}) = \prod_{\ell=-r}^{r-1} \omega^{2\ell+1} (\omega^{2\ell+1} - \omega^{-2\ell-1}) = (-1)^r \prod_{\ell=0}^{r-1} (\omega^{2\ell+1} - \omega^{-2\ell-1})^2.$$

Consequently,

$$(-1)^n B_{2n,1}^{(2r)} = \frac{\prod_{\ell=0}^{r-1} (\omega^{2\ell+1} - \omega^{-2\ell-1})^2}{\prod_{\ell=1}^{2r} (\omega^\ell - \omega^{-\ell})}.$$

This is precisely the identity (3.11). $\square$

Next, we consider progression to a limit as $n \rightarrow \infty$ – (3.1) is a special case. Thus, suppose that we have a sequence $\Phi = \{\phi_n\}_{n \in \mathcal{I}}$, where $\phi_n = 2\pi m_n/(n+1)$, $m_n \in \mathbb{Z}^+$, $\mathcal{I} \subseteq \mathbb{Z}^+$ is a set of infinite cardinality and

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{I}}} \phi_n = \phi \in [0, 2\pi).$$

Set

$$C(t, \Phi) = \lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{I}}} (-1)^{[n/2]} A_n(t; \xi_1^{(n)}, \xi_2^{(n)}, \dots, \xi_n^{(n)}),$$

where $\xi_\ell^{(n)} = 2 \cos \ell \phi_n$, $\ell = 1, 2, \dots, n$, $n = 1, 2, \dots$. Thus, $\xi_\ell^{(n)} = q_n^\ell + q_n^{-\ell}$, where $q_n = \exp \frac{4\pi i}{n+1}$. Consequently, according to Proposition 5, if $\mathcal{I}$ consists of only odd indices, necessarily $C(t, \Phi) \equiv 0$. This proves, incidentally, that $\sigma_{2L} \rightarrow 0$ in (3.1).

Lemma 7 Suppose that $\mathcal{I} \subseteq 2\mathbb{Z}^+$ and that $\phi = 0$. Then, provided that $m_n = o(n^{\frac{2}{3}})$, it is true that $C(t, \Phi) = (1 - t^2)^{-\frac{1}{2}}$.

Proof. Since

$$\frac{\sin(2\ell - 1)\phi_n}{\sin 2\ell\phi_n} = \frac{2\ell - 1}{2\ell} + \mathcal{O}(\phi_n^3),$$

we deduce from (3.11) that

$$(-1)^n B_{2n,1}^{(2r)} = r^{-r} \binom{2r}{r} + \mathcal{O}(n\phi_n^3).$$

According to the assumption, $\mathcal{O}(n\phi_n^3) = o(1)$ and the lemma follows from

$$\sum_{r=0}^{\infty} \binom{2r}{r} \frac{t^{2r}}{4^r} = \frac{1}{\sqrt{1-t^2}}.$$

$\square$

Letting $t = \frac{1}{2}$ affirms the remaining part of (3.1).

Similarly to the last proposition, it is possible to derive an explicit expression for $C(t, \Phi)$, provided that ϕ/π is rational and that $n(\phi - \phi_n)^3 = o(1)$ as $n \rightarrow \infty$. The derivation is long and it will be published elsewhere. It suffices to mention here the remarkable sensitivity of $C(t, \Phi)$ to both the choice of $\mathcal{I}$ and to the specific nature of ϕ .

What happens when ϕ/π is irrational? This is, as things stand, an open problem. It is possible to show that, formally,

$$C(t, \Phi) = \sum_{\ell=0}^{\infty} t^{2\ell} \prod_{k=1}^{\ell} \frac{\sin(2k-1)\phi}{2k\phi} = \sum_{\ell=0}^{\infty} t^{2\ell} q^{\frac{1}{4}\ell} \frac{(q^{\frac{1}{2}}; q)_{\ell}}{(q; q)_{\ell}},$$

where $q = e^{4i\phi}$. The latter series can be summed up by means of the Gauß–Heine theorem [5] for $|q| < 1$ and, after simple manipulation, for $|q| > 1$. Unfortunately,

because of a breakdown in Hölder-continuity across $|q| = 1$, it is impossible to deduce its value on the unit circle from the values within and without by means, for example, of the Sokhotsky formula [8].

Clearly, there is much to be done to understand better the behaviour of the p_n 's when the period K becomes infinite. In this section we have established few results with regard to the values at the origin. They should be regarded as a preliminary foray into an interesting problem in orthogonal polynomial theory *cum* linear algebra and we hope to return to this theme in the future.

4 Generalizations

There are two natural ways of generalizing an almost Mathieu equation (1.1), by either specifying a more general periodic potential or replacing the index by a multi-index. Remarkably, the basic framework of this paper – replacing a doubly-infinite recurrence by a functional equation which, in turn, is replaced by a singly-infinite recurrence – survives both generalizations! In the present section we describe briefly this state of affairs.

Firstly, suppose that the cosine term in (1.1) is replaced by a more general harmonic term and we consider the spectral problem

$$(4.1) \quad a_{n-1} - 2 \left\{ \sum_{\ell=1}^m \kappa_\ell \cos(n\ell\theta + \psi_\ell) \right\} a_n + a_{n+1} = \lambda a_n, \quad n \in \mathbb{Z}.$$

We assume that $\kappa_1, \kappa_2, \dots, \kappa_m \in \mathbb{R}$ and, without loss of generality, that $\kappa_m \neq 0$. Letting $q = e^{i\theta}$, we set

$$\kappa_\ell^* = e^{i\psi_\ell} \kappa_\ell, \quad \kappa_{-\ell}^* = e^{-i\psi_\ell} \kappa_\ell, \quad \ell = 1, 2, \dots, m,$$

and $\kappa_0^* = 0$. Therefore (4.1) assumes the form

$$(4.2) \quad q^{mn} a_{n-1} - \left\{ \sum_{\ell=0}^{2m} \kappa_{\ell-m}^* q^{n\ell} \right\} a_n + q^{mn} a_{n+1} = \lambda q^{mn} a_n, \quad n \in \mathbb{Z}.$$

Let $c \in \mathbb{C} \setminus \{0\}$ and consider the Dirichlet series $y(t) = \sum_{n=-\infty}^{\infty} a_n \exp\{cq^n t\}$. Since, formally,

$$y^{(\ell)}(t) = c^\ell \sum_{n=-\infty}^{\infty} q^{n\ell} a_n e^{cq^n t}, \quad \ell = 0, 1, \dots,$$

we obtain from (4.2) the functional differential equation

$$(4.3) \quad \sum_{\ell=0}^{2m} \kappa_{\ell-m}^* c^{m-\ell} y^{(\ell)}(t) = q^m y^{(m)}(qt) - \lambda y^{(m)}(t) + q^{-m} y^{(m)}(q^{-1}t).$$

The derivation is identical to that of (1.3) and is left to the reader.

In line with Section 1, we next expand the solution of (4.3) in Taylor series, $y(t) = \sum_{n=0}^{\infty} p_n t^n / n!$. This readily yields

$$\sum_{\ell=0}^{2m} \kappa_{\ell-m}^* c^{m-\ell} p_{n+\ell} = (q^{n+m} + q^{-n-m} - \lambda) p_{n+m}, \quad n = 0, 1, \dots,$$

hence, replacing $n + m$ by n ,

$$\sum_{\ell=-m}^m \kappa_\ell^* c^{-\ell} p_{n+\ell} = (2 \cos n\theta - \lambda) p_n, \quad \ell = m, m+1, \dots$$

Finally, we choose $c = \exp\{i\psi_m/m\}$, hence $\kappa_m^* c^{-m} = \kappa_m^* = \kappa_m \in \mathbb{R} \setminus \{0\}$. We thus define $\alpha_\ell = c^{-\ell} \kappa_\ell^* / \kappa_m$, $|\ell| \leq m$ and replace λ by $-\lambda \kappa_m$. This results in the recurrence

$$(4.4) \quad \sum_{\ell=-m}^m \alpha_\ell p_{n+\ell} = (\lambda - \beta_n) p_n, \quad n = m, m+1, \dots,$$

where

$$\beta_n = -\frac{\cos n\theta}{\kappa_m}, \quad n = m, m+1, \dots$$

Note that, inasmuch as the α_ℓ s may be complex, we have $\alpha_{-\ell} = \bar{\alpha}_\ell$, $\ell = 1, 2, \dots, m$, $\alpha_0 = 0$.

The recurrence (4.4) is spanned by $2m$ linearly independent solutions. However, unless $m = 1$, it is no longer true that, for appropriate choice of $p_0, p_1, \dots, p_{2m-1}$, each p_n is a polynomial of degree $n + k$ for some k , independent of n . Indeed, it is easy to verify that the degree of p_n increases roughly as $[n/m]$. Hence, orthogonality is lost. Fortunately, an important feature of orthogonal polynomials, namely that their zeros are eigenvalues of a truncated Jacobi matrix [2], can be generalized to the present framework. It is possible to show that the zeros of p_n are generalized eigenvalues of a specific pencil of 'truncated' matrices and this provides a handle on their location. We expect to address ourselves to this issue in a future publication.

Another generalization of (1.1) allows the index n to be replaced by a multi-index $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$. Thus, let $\mathbf{e}_\ell \in \mathbb{Z}^d$ be the ℓ th unit vector, $\ell = 1, 2, \dots, d$, and consider the spectral problem

$$(4.5) \quad \sum_{\ell=1}^d (a_{\mathbf{n}+\mathbf{e}_\ell} + a_{\mathbf{n}-\mathbf{e}_\ell}) - 2\kappa \cos \left(\sum_{\ell=1}^d \alpha_\ell n_\ell + \beta \right) a_{\mathbf{n}} = \lambda a_{\mathbf{n}}, \quad \mathbf{n} \in \mathbb{Z}^d.$$

In line with Section 1, we let

$$b_1 = be^{i\beta}, \quad b_2 = e^{-i\beta}, \quad q_\ell = e^{i\alpha_\ell}, \quad \ell = 1, 2, \dots, d,$$

whereupon (4.5) becomes

$$(4.6) \quad \sum_{\ell=1}^d \mathbf{q}^{\mathbf{n}} (a_{\mathbf{n}+\mathbf{e}_\ell} + a_{\mathbf{n}-\mathbf{e}_\ell}) - (b_1 \mathbf{q}^{2\mathbf{n}} + b_2) a_{\mathbf{n}} = \lambda \mathbf{q}^{\mathbf{n}} a_{\mathbf{n}}, \quad \mathbf{n} \in \mathbb{Z}^d.$$

The last formula employs standard multi-index notation, e.g.

$$\mathbf{q}^{\mathbf{n}} = q_1^{n_1} q_2^{n_2} \cdots q_d^{n_d}.$$

We let formally

$$y(t) = \sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \exp \left\{ b_1^{\frac{1}{2}} \mathbf{q}^{\mathbf{n}} t \right\}$$

and note that

$$\begin{aligned}y'(t) &= b_1^{\frac{1}{2}} \sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \mathbf{q}^{\mathbf{n}} \exp \left\{ b_1^{\frac{1}{2}} \mathbf{q}^{\mathbf{n}} t \right\}, \\y''(t) &= b_1^{\frac{1}{2}} \sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \mathbf{q}^{2\mathbf{n}} \exp \left\{ b_1^{\frac{1}{2}} \mathbf{q}^{\mathbf{n}} t \right\}.\end{aligned}$$

Therefore, multiplying (4.6) by $\exp \left\{ b_1^{\frac{1}{2}} \mathbf{q}^{\mathbf{n}} t \right\}$ and summing up for $\mathbf{n} \in \mathbb{Z}^d$ yields, after brief manipulation, the complex functional differential equation

$$(4.7) \quad y''(t) + b_2 y(t) = b_1^{-\frac{1}{2}} \left\{ \sum_{\ell=1}^d \left(q_{\ell}^{-1} y'(q_{\ell}^{-1} t) + q_{\ell} y'(q_{\ell} t) \right) - \lambda y'(t) \right\}.$$

Equation (4.7) is of independent interest, being a special case of the equation

$$y''(t) + c_1 y'(t) + c_2 y(t) = \int_0^{2\pi} y(e^{i\theta} t) d\mu(\theta),$$

where $d\mu$ is a complex-valued Borel measure. This, in turn, is similar to the functional integro-differential equations of the form

$$y''(t) + c_1 y'(t) + c_2 y(t) = \int_0^1 y(qt) d\eta(q),$$

say, where $d\eta$ is, again, a complex-valued Borel measure. Equations of this kind have been considered by the present author, jointly with Yunkang Liu [11], with an emphasis on their dynamics and asymptotic behaviour. In the present paper, however, we are interested in the spectral problem for (4.7), and to this end we again expand y in Taylor series, $y(t) = \sum_{m=0}^{\infty} y_m t^m / m!$. It is easy to affirm by substitution into (4.7) the three-term recurrence relation

$$(4.8) \quad y_{m+1} = b_1^{-\frac{1}{2}} \left(2 \sum_{\ell=1}^d \cos \alpha_{\ell} m - \lambda \right) y_m - b_2 y_{m-1}, \quad m = 1, 2, \dots$$

Note a most remarkable phenomenon – although (4.5) is d -dimensional, the index in (4.8) lives in $\mathbb{Z}^+$! In other words, the dimensionality of the resultant three-term recurrence is independent of d – it is, instead, expressed as the number of harmonics in the recurrence coefficient. Moreover, inasmuch as (4.8) is more complicated for $d \geq 2$ than its one-dimensional counterpart (1.4), both recurrences display similar qualitative characteristics. In particular, we can use the theory of Section 2 to cater for the case of $\alpha_1/\pi, \alpha_2/\pi, \dots, \alpha_d/\pi$ being all rational.

In line with the analysis of Section 1, we let $\tilde{y}_m(t) = b_2^{\frac{1}{2}m} y_m(-(b_1 b_2)^{\frac{1}{2}} t)$, $m \in \mathbb{Z}^+$, whereupon (4.8) becomes

$$(4.9) \quad \tilde{y}_{m+1}(t) = \left(t + \frac{2}{\kappa} \sum_{\ell=1}^d \cos \alpha_{\ell} m \right) \tilde{y}_m(t) - \tilde{y}_{m-1}(t), \quad m = 1, 2, \dots$$

To recover all solutions of (4.9) we need to consider a linearly independent two-dimensional set of solutions. Letting $r_{-1} = 0$, $r_0 = 1$ and $s_0 = 0$, $s_1 = 1$, we recover,

similarly to (1.6–7), two sequences $\{r_m\}_{m \in \mathbb{Z}^+}$ and $\{s_m\}_{m \in \mathbb{Z}^+}$ that span all solutions of (4.9) and such that $\deg r_m = m$, $\deg s_m = m - 1$. In other words, by the Favard theorem both $\{r_m\}_{m \in \mathbb{Z}^+}$ and $\{s_{m+1}\}_{m \in \mathbb{Z}^+}$ are OPS and, in line with our analysis of the one-dimensional almost Mathieu equation (1.1), we are in position to exploit the theory of orthogonal polynomials.

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MODULAR THEORY AND SYMMETRY IN QFT

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Abstract

Application of the Tomita-Takesaki modular theory to the Haag-Kastler net approach in QFT yields external (space-time) symmetries as well as internal ones (internal gauge para-groups) and their dual counterparts (the superselection para-group). We make an attempt to develop a (speculative) picture of “quantum symmetry” which links space-time symmetries in an inexorable way with internal symmetries. We obtain several theorems in the process, and in particular derive the Kac-Wakimoto formula which links Jones inclusion indices with the asymptotics of expectation values in physical temperature states. This formula is a special case of a new asymptotic Gibbs-state representation of mapping class group matrices (in a Haag-Kastler net indexed by intervals on the circle!) as well as braid group matrices.

1 Introduction

In line with the title of this conference I will give a rather broad introduction touching on several open physical problems of low-dimensional QFT. The main part will present results which are related to (external and internal) symmetry. Finally, again in the vein of the conference title, I will express my opinions on the future of mathematical physics.

The topics to be discussed in this paper belong to the so-called algebraic QFT. This terminology refers to the mathematical methods used in the approach. However, the methods are not the most characteristic feature of this theory. It could incorporate any mathematical method which is consistent with relativistic quantum physics. A more intrinsic characterization would be to say that it is founded on mathematically formulated physical principles [1].

There is a better known approach, based on functional integrals, which grew out of a formal extension of quantization procedures which were successful in perturbation theory (founded on the presumed existence of a classical-quantum parallelism, i.e., a kind of inverse of the Bohr correspondence principle).

Algebraic QFT is, in some physical sense, closely linked with the older “dispersion theoretical approach” the modest success of which culminated in the theoretical derivation of Kramers-Kronig dispersion relation for the scattering amplitudes of relativistic particles from the two main principles of relativistic local quantum physics [1]: Einstein

causality and spectral properties (Dirac stability by filling negative energy states.) as well as their subsequent experimental verification in the case of $p-p$ forward scattering.

Algebraic QFT uses the same physical principles, but employs a vastly more subtle mathematical framework (von Neumann Algebras, Jones Inclusion Theory etc.) than just Cauchy's theorem in dispersion relations. A rough look at these old principles reveals that they can be divided into two groups as mentioned above: Causality properties expressed in terms of commutativity properties of local algebras, and stability properties expressed in terms of states on observable algebras (to be a little bit more precise, in terms of spectral or modular (KMS) properties of representation of algebras canonically (GNS) affiliated with these states). There are also related properties which join algebras and states as, e.g., the notion of "statistical independence" of quantum physics in space-like separated regions leading, among other things, to limitation on multiplicities of high energy states and yielding the existence of temperature states from the knowledge of the vacuum states [2].

Since in no other area of theoretical physics (e.g., string theory, 2- d quantum gravity) does anyone explain in detail the principles on which his approach is founded, neither will I. In my case, however, these principles do not represent an "industrial secret"; they have been exposed nicely in the literature [3]. The main idea is to avoid making assumptions on charge-carrying fields but rather to define them and derive their properties from observables. (In that respect algebraic QFT is different even from QFT in the spirit of Wightman).

Algebraic QFT has been very successful at that. It is, e.g., quite easy to define a composition of charged localized states on the observable algebra $\mathcal{A}$ (which is really a net of algebras):

$$(1) \quad \omega_1, \omega_2 \longrightarrow \omega_1 \times \omega_2(A) \neq \omega_2 \times \omega_1(A), \quad A \in \mathcal{A}$$

which commutes (the equality sign holds) if the localization is space-like, i.e., $\text{loc } \omega_1 \times \text{loc } \omega_2$.

The statistics operator ε shows up if one tries to lift states to state vectors. For spacelike separations one encounters the so called statistics operator ε as an obstruction [4]:

$$(2) \quad \psi_{\omega_2 \times \omega_1} = \varepsilon \psi_{\omega_1 \times \omega_2}$$

One immediately realizes the analogy with the Wigner phenomenon of ray representations of symmetry groups which arise if one lifts the algebraic automorphisms to the space of vector states, or of Berry's phase (the latter being of a more phenomenological nature) which is a well-known measurable obstruction in atomic physics. Statistics is (with the exception of the abelian case, the so called anyons) not described by a phase, but by a unitary operator.

With a bit of additional work one finds that the operator ε generates the braid group (a special case of which is the permutation group) in low dimensional QFT [5] and the permutation group otherwise [6]. Furthermore, there are Markov traces leading (after rescaling) to knot invariants, and by considering " ∞ " knots (i.e., suitable limits resembling infinite-order perturbation theory for vacuum polarization in QFT), one obtains invariants of 3-manifolds [7].

There are however two physically important areas which have remained rather difficult territory even for algebraic QFT. Unfortunately these are just the areas in which the new low-dimensional theoretical discoveries could be confronted with experimental reality (e.g., of condensed matter physics).

One is the problem of constructing and classifying the simplest fields which obey the new statistics in the “freest” way allowed by braid group statistics i.e., the analogues of free bosons and fermions (where the statistics refers to the permutation group) which may be called “free anyons” (abelian), or, more generally “free plektons” (nonabelian).

In conformal field theory, according to our present understanding, only free objects (which describe, not particles, but rather “infraparticles”) exist.

The second important physical problem is symmetry. In the old theories of fermions and bosons one had a sharp distinction between external (space-time) and internal (isospin, flavour) symmetries. Any attempt to “marry” them in a nontrivial way was doomed to failure [8]. As opposed to this, in the new theory there are obstructions against a complete divorce, e.g., in conformal field theory some global conformal transformations ($2\pi n$ rotations on $\hat{S}^1$) are inexorably linked with internal charge transformations [7].

According to our present understanding of 3- d theories with braid group statistics, it is perfectly conceivable that the limiting invariants of 3-manifolds already mentioned show up in some new perturbation theories of plektons in infinite order [7]. In other words, curved 3-manifolds could appear in ordinary (infinite-order) plekton scattering (even though the “living space” of the 3- d theory remains Minkowskian). In this case the 3-manifolds are expected to have something to do with a “space” formed by external charge-type variables together with (subgroups of) the Möbius or the 3- d Poincaré group: *Transl.* $\propto O(2, 1)$.

Topological Field Theory [10] in itself offers no concept of a physical “living space” of fields, i.e., the localization necessary for a physical interpretation is absent and therefore the physical interpretation of its topological content is difficult and ambiguous. This, however, in no way limits its mathematical usefulness.

These problems can be traced back to the global nature of the Feynman-Kac functional integral representation which (from the point of view of general QFT) constitutes an attempt to simultaneously guess a state and an operator algebra (e.g., its vacuum-representation) on the basis of perturbative experience. Causality and locality properties can only be checked at the end after a complete GNS reconstruction to the real-time analytically-continued expectation values has been carried out. The formal locality properties of the classical action are necessary (for quantum locality), but by no means sufficient. This is exemplified by the use of non-regular states on nets of Weyl algebras¹ which convert the “would be” translationally covariant III_1 factor into II_1 hyperfinite tracial factors as used by V. Jones (the model thus becoming “topological”): All continuous space-time covariances are “knocked out” by the singular (local gauge invariant) states [11], i.e., are not unitarily implementable in a continuous fashion².

¹The formal canonical quantization of Chern-Simons actions yields Weyl-like algebras on 1-forms as their operator-algebraic content.

²This singular state description is presently only known for the abelian Chern-Simons action [11]. For illustrating the above warning, this is sufficient. In quantum mechanics such singular states destroy the continuous translations and violate the uniqueness theorem of the Schrödinger representation.

There is, as yet, no study of singular states on suitably defined nonabelian Weyl algebras over 1-forms in 3d which could explain: (a) the tantalizing numerical coincidence [62] of the semiclassical approximations in the continuous Witten approach with the exact Turaev-Viro combinatorial theory³, and (b) yield also the correct $\{\textit{algebra}, \textit{state}\}$ pair behind Topological Field Theory. However I am not aware of any alternative idea which captures the intuitive picture of averaging over “gauge copies” in operator language (i.e., without resorting to semiclassical or quantization ideas as the BRST formalism).

Singular states have first been applied in QFT by Buchholz and Fredenhagen [see 11], and by Haag [1, page 180].

The geometrical approach based on functional integrals and formal (Verma module) representation theory is in no way limited by positivity requirements since it only explores free-field-like (integrable) systems. In physical “real life” the von Neumann algebra methods of algebraic QFT are, however, indispensable. By contrast, in algebraic QFT the locality principles are taken care of at every step of the argument, viz. the title and the content of a recent book [1].

In this paper we will use only regular states on nets and then topological data will appear only in suitably defined intertwiner subalgebras (centralizer algebras) which are of the II_1 Jones type and contain mapping class group matrices. The numerical invariants of 3-manifolds are, from our viewpoint, computable by suitable limits in the spirit of vacuum polarization effects, which amounts to the same as “ $\rho_{\text{reg.}}$ ” in [7].

If one adds spinor matter to 3-d pure Chern-Simons actions, one expects a remaining genuine III_1 net as the “gauge invariant” physical part of the model [12]. In fact one expects the main role of the formal local gauge principle in low-dimensional QFT’s to be just this: A method of finding interesting observable (plektonic) subalgebras inside the well-known algebras of Bosons and Fermions (but unfortunately not for the construction of the very non-classical and yet unknown field algebra). In the 4-d gauge theories, there is of course the additional problem of understanding the true quantum nature of “Magnetic” fields and “Stokes theorem” (on the same conceptual level as the notion of charges in the theory of superselection sectors) and the deeper meaning of “quark confinement”.

The present picture of 4-d QFT from the algebraic viewpoint is less rich than that of low-dimensional QFT. The reason for this is that the standard spectral gap assumption for the energy-momentum spectrum leads to the Buchholz-Fredenhagen semi-infinite space-like cone localization (the cones can be made arbitrarily thin) [63], which in turn yields permutation group statistics and compact group symmetry. The situation would change dramatically if one finds a physical argument in favor of infinite energy objects which allow no better localization than that around a core being a space-like hypersurface. The quark idea (outside the short-distance regime) suggests that it may be advantageous to think about quantum states (i.e., no gauge artifacts) with infinite energy. Quarks as Wigner-states with a finite mass and spin $\frac{1}{2}$ are too naive and not very well digestible by algebraic QFT. With the exception of this short interlude the paper will be focussed on low-dimensional QFT only.

While on the topic of physical interpretation, we find it worthwhile to mention the

³Which is identical with that of algebraic QFT if one specializes to models with 2-channel (Hecke-Jones) and 3-channel (Bierman-Wenzl) statistics.

difficulties which algebraic QFT meets with the notion of chiral conformal QFT “on Riemann surfaces”. In the special case of genus $g = 1$, the algebraic approach (as well as the Wightman approach based on the use of point-like covariant fields) yields two nets: one “living” (in the sense of localizability) on the Minkowskian cycle and the other (being also non-commutative and lacking a Feynman-Kac representation) on the Euclidean cycle. Between these two cycles there exists (as in standard QFT’s) the analogue of the Bargmann-Hall-Wightman region [13] of analyticity of correlation functions,⁴ which, at least a priori, has nothing to do with the localizability implicitly contained in the phrase “living on...”. For this reason, it is the more surprising that we find the presence of mapping class matrices belonging to all geni , even if we do not yet know the physical interpretation of this finding. How do nets on S^1 know about mapping class groups for all geni ? Does this observation mean that plektonic physics is better prepared (than fermionic or bosonic) to accept ideas about quantum gravity? Is it related to the absence of degrees of freedom of gauge- and metric-fields in a quantization approach to low-dimensional QFT’s?

Concerning the first problem of anyonic or plektonic free fields, one has an extremely useful physical picture [14], but as yet without sufficiently general explicit constructions. The picture emerges from a combination of the old Coleman-Mandula [8] ideas with results from the bootstrap programme of introducing 2- d integrable models via factorizing S -matrices and form factors of the would-be fields [15].

In S -matrix language the important observation (for our purposes) may be phrased as follows. If, in a 4- d local QFT one takes the “complete cluster limit” on the S -matrix, i.e., the limit in which all the centers of individual particle wave packets become infinitely space-like separated, then all scattering effects (first the inelastic ones and then also the particle conserving processes) die out and only $S = 1$ remains in the limit.

A trivial S -matrix, however, is believed to belong to the local equivalence class (Borchers class) of a free field, although a mathematical proof exists only under special additional conditions [17]. The complete clustering idea of S is used here to produce the “freest” permutation group statistics QFT, which turns out to be identical with a free field theory in the usual technical sense (i.e., the higher n -point functions are products of two-point functions, with the appropriate Wick-combinatorics).

Let us now jump from 4 down to 2 space-time dimensions. In that case the same extreme cluster limit of the S -matrix is also expected to lead to the asymptotic vanishing of all inelastic processes and part of the elastic ones. Only the elastic two particle scattering $S_{\text{limit}}^{(2)}(\theta)$ (θ =rapidity) should survive⁵ and then, for consistency reasons, all the higher elastic processes have to go through 2-particle scattering.

$S_{\text{limit}}^{(2)}(\theta)$ allows no further asymptotic space-time simplification since a separation of an interaction T -part from a “free” part in the decomposition $S = 1 + iT$ is physically meaningless (both parts contain the same momentum space δ -functions).

The Yang-Baxter equation emerges as a consistency condition from this picture. Therefore this picture is “quantum integrability” par excellence without recourse to a

⁴ $g = 1$ belongs to the L_∞ -Gibbs state correlation functions. A conceptually acceptable description of $g \geq 2$ in terms of algebras and states is not yet available, although there are some formal observations relating conformal correlations to Fuchsian groups [64].

⁵ The higher elastic processes have a smoother threshold behaviour leading to a faster decrease in the spatial separation of wave packets.

classical parallelism, conserved currents etc. The nontrivial part of this picture is of course the statement that the limiting S -matrix belongs to a full-fledged localizable QFT in its own right (partially obtained via the so-called form-factor programme [18]).

So quantum integrability is conceptually much easier than classical integrability which, in the spirit of Bohr's correspondence principle, should be derivable in an appropriate semiclassical approximation (not an easy matter!).

The particles to which the limiting S -matrix and its affiliated localizable QFT belong are plektons in special cases (i.e., the internal structure of S may be more complicated than that describable by symmetry groups); generically, they are "kinks" (which are not DHR-localizable but rather belong to half-space localized solitons) to which statistics notions are not applicable (statistical "schizos", whose operator commutation relations can be changed by neutral operators, i.e., without changing charges [19]). In the calculational scheme of Smirnov [18] this last observation is turned into a virtue by selecting for each interpolating field a bosonic representative for which the form-factor computations seem to be easier.

Now, finally, let us apply the picture to 3- d QFT. In that case we know that the generically admissible statistics are braid group statistics. On the other hand the complete cluster limit does not allow for an energy dependent limiting S -matrix as was the case in $d = 4$. This leaves only the possibility of a piecewise constant S -matrix (in momentum space) with discontinuous jumps if asymptotic directions (directions of momenta) go through degenerate situations where at least two direction become parallel. For such an S -matrix, the properly-defined cross-section would vanish. The values in the different nondegenerate components are given in terms of constant braid- R -matrices. This picture gets additional support by looking at the structure of scattering states [20].

Insufficient knowledge of low energy analytic momentum space properties in the presence of braid group statistics has, up to now, prevented a proof *à la* Coleman-Mandula which is based on the usual (permutation group statistics) analytic properties of dispersion theory.

An explicit construction of anyonic or plektonic free fields (i.e., their particle form-factors) belonging to that limiting piecewise constant S -matrix is presently even more remote. Without doubt such objects, if they exist, should be considered as the 3- d version of quantum integrability (i.e., if those piecewise constant S -matrices admit of localized interpolating fields).

Note that such a picture leads to a new class division of theories according to their long distance properties: Each interacting theory, complicated as it may be, has precisely one integrable companion, namely that belonging to S_{limit} . Up to now one only knows a short-distance universality principle formulated in terms of "conformal companions" of quantum field theoretical models.

The second problem, namely (internal) symmetry, is closely related to the previous one. In order to see this, I remind the reader of a well-understood (but unfortunately not so well-known) fact that in addition to the standard description of QFT with permutation group statistics in terms of local boson or fermion fields with tensorial compact transformation group, there exists a physically equivalent formalism (regarding local observables and the scattering matrix of charged particles) based on nonlocal para-statistics fields (the exchange-algebra fields for the permutation group) which have

no group transformation properties and satisfy R -matrix commutation relations with $R^2 = 1$ [43].

If one wants to prove, e.g., that the free field equation (considered as a defining 2-sided ideal) or $S = 1$ leads to the analytically correct correlation functions of a free field theory, it would not be advisable to use para-fields. But with local tensorial fields, the result follows easily from the field equation and the well-known analytic properties of correlation functions which follow from causality and spectral properties of tensorial fields⁶.

The analytic structure of $d = 2, 3$ exchange algebra fields is much more complicated than that of para-fermions or bosons.

So, if one could find a symmetry concept which would replace the tensorial formalism in the presence of braid group statistics, then it could be of great practical value. Nonrelativistic plektons and their (not understood) "quantum" symmetries could allow a new symmetry breaking which may be responsible for the fractional Hall effect and high- T_c superconductivity. Braid group statistics, even at its present state of poor physical understanding, has already produced an extremely rich collection of numbers: rational statistical phases (related to electromagnetic properties?) and algebraic integers in the form of statistical dimensions (related to amplification factors in cross-section or thermodynamical properties?) which are generalized Casimir factors counting degrees of freedom⁷. But one lacks a good understanding of their localized field carriers, which would be necessary to work out experimental consequences.

Present attempts using quantum group ideas do not seem to go into this direction. They often contain artificial integer numbers (the dimensionality of the representation q -deformed group), treat the notion of conjugates (antiparticles) in an extremely unsymmetric way, and generally lead to non-unique systems of fields not relatable by Klein Transformations [22].

There are rather convincing arguments that the quantum symmetry aspect should be discussed together with space-time symmetries, including the TCP symmetry and also with modular Tomita-Takesaki theory, and not as an isolated kinematical (combinatorial) property being apart from the rest of QFT.

In the following I will present some recent results on symmetries obtained by modular theory (in most parts within the framework of algebraic 2- d conformal QFT) and forget the presently inaccessible problem of free plektons. But it should be clear that my motivation (for presenting these partial results on symmetry) is entirely physical.

The paper is organized as follows.

I start in Section 2 with a net indexed by intervals on a line and build up a conformal net on the circle. The modular Tomita-Takesaki structure, which is heavily used in this construction, also leads to a Euclidean net whose construction is only sketched very briefly.

Section 3 recalls the derivation of the universal observable algebra with its nontrivial

⁶By a trivial extension of the Jost-Schroer theorem [13].

⁷Statistics of (quasi-)particles and the ensuing change of composition laws of spins and charges is the only difference between 4- d and low-dimensional QFT. This is expected to lead to significant differences of plektonic phases from the Fermi-liquid phases in condensed matter physics. The problem of classifying these phases is similar (but in its analytic aspects more difficult) to the classification of 2- d conformal QFT's.

center. It contains an “invariant symmetry algebra” which lends itself to a generalization of the Kac-Wakimoto [23] formula.

In Section 4 we recall the proof of the TCP theorem in the framework of the exchange algebra. In the case of the $\mathbb{Z}_{2N}$ -anyon algebra we show the coalescence of this theorem with the standard Tomita-Takesaki modular theory on half-circles and argue that this state of affairs prevails for all models with vanishing “defect projector” which measures the deviation of statistical dimensions from integers.

In Section 5 we study “self-duality”, observed globally as the symmetry of Verlinde’s matrix S , on the local net level. Here another tool of modular theory is used: Longo’s “canonical endomorphism”. Again we employ the $\mathbb{Z}_{2N}$ anyonic model as an illustration of perfect self-duality.

Finally in the last Section we use an old, but still mysterious observation of Vaughn Jones in order to make some speculative remarks on the possible nature of the non-invariant aspect (i.e., beyond the symmetry algebra of Section 3) of a non-commutative link between external and internal symmetry. This section belongs to the realm of fantasy and only serves the purpose to open the eyes for radical new possibilities of field algebras.

2 Modular Theory on Light Rays

Just as standard QFT starts from canonical or path-space-quantization of classical field Lagrangians, algebraic QFT starts from nets of observable, i.e., maps of space-time regions into von Neumann algebras⁸

$$(3) \quad \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$$

with certain physically motivated properties.

Particular models of QFT (apart from trivial illustrations) are expected to appear through classification of nets (similar to conformal QFT which was not obtained by quantization, but rather by classifying representations of certain algebras).

Already in the early stage of the net (Haag-Kastler) theory it was (more or less) clear that the physics is not to be looked for in the single algebras (which in many cases have been shown to be type III₁ hyperfinite von Neumann factors), but rather in their interrelations, in particular in their mutual inclusions.

This change of paradigm also leads to a de-emphasis of local covariant fields which were crucial in the Lagrangian approach. They are “just like local coordinates for the nets” (Haag [1]) and if they play a more significant role, this is still to be discovered.

In the following we explain some results which can be obtained by restricting to the simplest case in which the indexing of the net is not done by space-time regions in Minkowski-space, but by intervals on a line (a right or left light ray of a 2- d conformal QFT). We assume a translation-invariant vacuum state ω_0 and construct, via the GNS construction, the vacuum representation $\pi_0(\mathcal{A})$ and the affiliated von Neumann algebra

⁸As emphasized by Rudolf Haag on various occasions, the net theory fits Leibniz’s thinking about “monads”, whereas the standard picture about geometrical spaces with a material content belongs to Newton’s world-view.

$M = \pi_0(\mathcal{A})''$ as well as the unitary positive energy translation $U(a)$ implementing the assumed translation automorphism and leaving the GNS vacuum vector Ω invariant.

We consider the half-line algebra $M_+ = \pi_0(\mathcal{A}(0, \infty))''$ on which the right translation $U(a), a > 0$ acts like a one-sided compression. Under these circumstances Borchers proved the following theorem [24]:

Theorem (Borchers): *The modular reflection J and the modular operator Δ^{it} of the pair (M_+, Ω) have the following commutation relation with $U(a)$:*

$$(4) \quad \begin{aligned} JU(a)J &= U(-a) \\ \Delta^{it}U(a)\Delta^{-it} &= U(e^{-2\pi t}a), \quad t, a \in \mathbb{R} \end{aligned}$$

It was already known that the abstract Tomita-Takesaki modular theory (which affiliates a reflection J and a one-parameter group Δ^{it} with a von Neumann algebra and a cyclic and separating state vector) has a deep physical and geometric significance in ordinary Wightman QFT if one looks at algebras belonging to specific regions ("wedges"): the Tomita reflection J becomes (up to a π -rotation) the TCP operator related to the appearance of antiparticles and Δ^{it} a special L -subgroup leaving that wedge region invariant. The operator Δ^{it} in Borchers' theorem is clearly the dilation.

From the translations and J one can build up a new net (as Borchers did) by starting with M_+ which is, by construction, covariant under the Möbius subgroup leaving ∞ invariant. If the new net has the self-reproducing property, i.e., is equal to the original net, the original net is admissible for subsequent considerations. If, however, the original net is perverse and fails to have this property, we work with the new net (which is canonically affiliated with the old one and has the self-reproducing property by construction) and forget the old one.

For the full Möbius group including the change of ∞ one needs the following "quarter-circle assumption" (we use S^1 instead of $R \cup \{\infty\}$):

Assumption: *Essential duality holds for the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$: i.e.,*

$$(5) \quad \begin{aligned} \mathcal{L}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) &= \mathcal{L}\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)'\right)' \\ \text{with } \mathcal{L}(I) &:= M(I)', \end{aligned}$$

and the modular conjugation $\tilde{J}$ of $\mathcal{L}$ permutes the quarter-circles, which one obtains by intersection the old half-circle with the new one, so that we obtain

$$(6) \quad \tilde{J}M_+\tilde{J} = M_+.$$

Then one finds [25]:

Theorem: *The group generated by $U(a), \Delta^{it}$ and $J \cdot \tilde{J}$ is the full Möbius group and the net generated from the four-quadrant algebra is Möbius-covariant.*

Again we discard the input net if it is not identical with this final fully covariant net.

This theorem has recently been derived from a much wider and natural mathematical scheme than in [25] and called “half-sided modular inclusion” by H. W.-Wiesbrock [26].

Note that essential duality is weaker than Haag duality. It also holds for nets generated from the energy-momentum tensor for $c_{\text{Virasoro}} > 1$ where, according to Buchholz and Schulz-Mirbach [27], Haag duality breaks down for the net on $\mathbb{R}$ generated by the energy-momentum tensor.

The above theorem suggests that the B.-S. M. mechanism of violating Haag duality and full conformal invariance by “drilling a hole” into S^1 (i.e., going from S^1 to $\mathbb{R}$) is the only one, by which translation covariant nets indexed by intervals on the line $\mathbb{R}$ can fail to be chiral conformal field theories. But then there always exists a canonically affiliated Möbius covariant S^1 -net. For such a covariant net one can show directly that the stronger Haag duality holds for each interval [28, 29].

Replacing the assumed geometric translation by the product of two abstract Tomita-reflections one could even remove the last vestiges of geometry from the quantum-physical assumptions [26].

Since the T.-T. modular theory was shown to build up Möbius-invariance from pure algebraic inclusion data (indexed by intervals on S^1), it is tempting to ask whether diffeomorphisms beyond the Möbius transformation can also be generated from algebraic net inclusions and suitably chosen states.

Let us look at the simplest such transformation (which together with the Möbius transformation generates the diffeomorphism group):

$$(7) \quad \text{Dil}(t)^{(2)}z := \sqrt{\text{Dil}(t)}z^2.$$

Here we use the complex coordinate z for the circle, $\text{Dil}(t)$ is the usual dilation, i.e., the modular group of the pair $(\mathcal{A}(S_+), \Omega)$ and the resulting $\text{Dil}^{(2)}$ denotes the “dilation” in the “second Virasoro sheet”, i.e., a geometric transformation built from infinitesimal generators $L_{\pm 2}, L_0$ which “dilates” with 4 fixed points: $z = 1, i, -1, -i$ instead of the ± 1 fixed points of Dil .

This suggest the consideration of a doubly localized algebra

$$(8) \quad \mathcal{A}(S_1) \vee \mathcal{A}(S_3)$$

where S_1 and S_3 are the first and third quadrants of the circle (counterclockwise).

The corresponding state vector candidate for a modular pair cannot be the vacuum state-vector Ω . Here the split property [30] gives us the clue. In conformal field theory this property follows from a control of the asymptotic spectral density of the rigid rotation operator L_0 [28], and holds in all known conformal models [31]. It has the consequence that algebras like the previous one are spatially isomorphic to tensor products.

$$(9) \quad \mathcal{A}(S_1) \vee \mathcal{A}(S_3) \sim \mathcal{A}(S_1) \otimes \mathcal{A}(S_3) \sim \mathcal{A}_I(S_+) \otimes \mathcal{A}_{II}(S_+).$$

where $\mathcal{A}_{I,II}$ are two copies of the half-circle algebra.

It is well-known that tensor-product algebras together with tensor product state vectors have also a tensor product modular operator. But our state vector is prevented from having this form by the presence of a unitary equivalence transformation. We define the state on the doubly localized algebra by the vacuum product state on the

single components. The use of the “natural cone” of the doubly localized algebra allows for a unique representation of that state in terms of a state vector ϕ . In order to understand the modular theory for this situation we investigate the KMS property of

$$(10) \quad (\phi, A_I A_{II} B_I B_{II} \phi) := (\Omega, \hat{A}_I \hat{B}_I \Omega)(\Omega, \hat{A}_{II} \hat{B}_{II} \Omega)$$

with $A_I, B_I \in \mathcal{A}(S_1)$, $A_{II}, B_{II} \in \mathcal{A}(S_3)$, $\hat{A}_I, \hat{B}_I \in \mathcal{A}_I(S_+)$, $\hat{A}_{II}, \hat{B}_{II} \in \mathcal{A}_{II}(S_+)$.

The modular operator Δ appearing in the KMS relation on the left hand side:

$$(11) \quad (\phi, A_I A_{II} B_I B_{II} \phi) = (\phi, \Delta B_I B_{II} \Delta^{-1} A_I A_{II} \phi)$$

can easily be expressed in terms of the U -transformed modular operator $\hat{\Delta} \otimes \hat{\Delta}$ belonging to the right hand side (where U is the unitary equivalence transformation).

$$(12) \quad \phi = U(\Omega \otimes \Omega).$$

The problem of finding the correct state vector has therefore been reduced to whether one can construct a unitary operator with a geometric interpretation. Imagine that we start with the two copies on S_+ , i.e., $\hat{\mathcal{A}}_{I,II}(S_+)$ and the vacuum state vector Ω . These algebras are certainly isomorphic to $\mathcal{A}(S_1)$ or $\mathcal{A}(S_3)$. Then the covering transformation $z \rightarrow z^2$ maps S_1 and S_3 onto S_+ . As an automorphism of $\mathcal{A}(S)$ this is meaningless; it is not a diffeomorphism. However, restricted to the doubly localized algebras it makes sense. So there could exist a “partial automorphism” (i.e., not extendable to $\mathcal{A}(S^1)$ by the geometric formula) of the doubly localized von Neumann algebra which can be implemented in the vacuum representation by a unitary operator U_{cov} transforming from the tensor-product space to the original Hilbert space, such that (12) with $U^{-1} = U_{\text{cov}}$ is the state vector yielding $\Delta^it = \text{Dil}^{(2)}(t)$ as a modular object. Conditions for the uniqueness of the implementing unitary are known by the “natural cone” construction [1, Thm. 2.2.4]

The only piece which is lacking from the proof is the existence of the partial automorphism entailing that of U_{cov} with the geometric action on fields. Unfortunately we have not been able to find a structural analytic argument similar to [27]. However, in models which have a “first” quantization like the Weyl algebra one has a control of the partial automorphism. We will treat this problem in a future publication.

Note that this method which is based on the use of the KMS property deals only with the modular group and does not determine the Tomita reflection and its possible geometric manifestation in the form of Haag duality.

The rigorous clarification of the general situation would be important because it would allow for a modular (i.e., pure quantum physical) understanding of the Virasoro charge, the origin of diffeomorphisms and the intrinsic meaning of energy-momentum tensors in the framework of the algebraic net theory.

The way in which the Tomita-Takesaki modular theory converts formal net indexing by intervals into geometry and physics is really surprising. There is more to come later.

For later use we also explain schematically the construction of Euclidean chiral theories. The starting point is the observation that the existence of a positive natural cone (Araki, Connes [32]) in the Tomita-Takasaki modular theory entails the existence

of another scalar product between state vectors of the form⁹

$$(13) \quad \psi_A = \Delta^{1/4} A \Omega$$

$$(14) \quad \langle A, B \rangle := (\psi_A, \psi_B) = (JA^* J \Omega, B \Omega)$$

In our field-theoretic application $A \in M_+$, Δ^{it} is the dilation and Ω is (with respect to M_+) the cyclic and separating vacuum state vector. This yields another state-space $\mathcal{H}_{\text{Eucl}}$ with a different involutive and positive star-operation on which the old translation in positive direction becomes a contraction operator on $\mathcal{H}_{\text{Eucl}}$. This in turn allows for a unitary analytic continuation which should be thought of as a translation on the imaginary axis. Using this Euclidean translation, one can build up another net, whose localization intervals are to be placed on the imaginary axis and which is very non-local relative to the original real-time net. The only common elements are the non-local analytic (with respect to the modular operator Δ^{it}) elements.

We call this Euclidean theory the Cartesian Euclidean theory. A richer and more interesting Euclidean theory is the radial Euclidean theory. Its construction also starts from the upper half circle, but instead of the one-sided translation one uses a two-sided compression defined in terms of the modular dilation $\tilde{\Delta}^{it}$ of the right-hand half circle. In this case the analytic continuation of the two-sided contraction yields the Euclidean rigid rotation, i.e., a rotation whose localization properties should be pictured in terms of a radial periodized coordinate. It is remarkable that the Tomita reflection of the Euclidean theory is identical with the original star-operation which now acts like a charge conjugation, i.e., as a bounded operator on $\mathcal{H}_{\text{Eucl}}$ (and in turn the Euclidean-star becomes related to the real-time Tomita reflection).

3 Kac-Wakimoto Formula and the Invariant Symmetry Algebra

For a net indexed by intervals on a circle (rather than on a line), the vacuum representation previously constructed does not admit sectors, i.e., one cannot find interesting endomorphism [7]. On the line $\mathbb{R}$ one could find such an endomorphism on the quasilocal C^* -algebra $\mathcal{A}_{\text{quasi}}$ (the DHR algebra) which is directed towards infinity [6], but then one would wreck conformal invariance.

It turns out that the correct observable algebra on S^1 which does not have these flaws can be obtained by a universal construction: It is the free C^* -algebra generated from the vacuum-represented net amalgamated over all local relations [34]. Using the information one has from superselection theory, one finds that this universal algebra $\mathcal{A}_{\text{univ}}$ has global self-intertwiners which lead to a nontrivial center generated by invariant (abelian) charges [7].

Like the old (DHR) quasilocal algebra $\mathcal{A}_{\text{quasi}}$, it is uniquely determined from the net data in the vacuum representation. The universal observable algebra $\mathcal{A}_{\text{univ}}$ also has a

⁹The inner product is the same as that which appears in the Connes bimodule theory if one restricts the bimodule to the diagonal action.

representation theory in terms of localized endomorphisms analogous the DHR theory; only the meanings of “localized” and “transportable” are somewhat more restricted [7].

The derivation of the braid group structure is also similar to the standard theory [7]; there is, however, one additional item as already mentioned; the appearance of a global self-intertwiner V_ρ which “flips” the statistics operator $\varepsilon(\rho, \sigma)$ to $\varepsilon^*(\sigma, \rho)$.

Together with the change of localization this fact accounts for the different properties of the exchange algebra, which, as in the standard theory, is constructed in a canonical fashion from the $\mathcal{A}_{\text{vac}}$ data together with the endomorphisms and constitutes a kind of “nonlocal” extension of $\mathcal{A}_{\text{univ}}$.

Theorem (FRSII [7]): *In theories with a finite number of irreducible endomorphisms (finite depth in the Jones theory, rational theory in conformal QFT) the exchange algebra is localizable on an N -fold covering (related to the depth) of S^1 and its R -matrices represent the braid group of the cylinder (generated by applying endomorphisms to the generators $\varepsilon(\rho, \rho)$ and V_ρ).*

We have adapted QFT in this formulation to chirally conformal field theory; a generalization to the 3- d case is straightforward [7].

In the framework of the DHR theory the exchange algebra (see also the beginning of the next section) is the reduced “field bundle” [5] (a bimodule over $\mathcal{A}$ with a C^* algebra structure), whereas in the Wightman framework it was first abstracted from nontrivial conformally invariant models [35].

In the following we review the description of the intertwiner algebra (often referred to as the centralizer algebra) with all superselection sector being present. We first pick one endomorphism ρ_i for each sector (the “rational” situation being assumed always) where all ρ_i ’s are localized in one proper interval on S^1 . The theory then assures the existence of localized isometric intertwiners V_i such that

$$(15) \quad \hat{\rho} := \sum_i V_i \rho_i V_i^* \\ \text{with} \quad \rho(\mathcal{A}_{\text{univ}}) \subset \mathcal{A}_{\text{univ}} \text{ and } V_i \in \mathcal{A}_{\text{univ}}.$$

It is known that such a master endomorphism $\hat{\rho}$ can be chosen as self-conjugate: $\hat{\rho} = \bar{\hat{\rho}}$ [36].

Let $\hat{\phi}$ be the unique standard left inverse of $\hat{\rho}$, i.e., a positive map of $\mathcal{A}_{\text{univ}}$ with

$$\begin{aligned} \hat{\phi} \cdot \hat{\rho} &= id \\ \hat{\rho} \cdot \hat{\phi} &= \epsilon_{\text{extr}}, \\ \epsilon_{\text{extr}} &= \text{extremal conditional expectation of } \mathcal{A}_{\text{univ}} \rightarrow \hat{\rho}(\mathcal{A}_{\text{univ}}) \end{aligned}$$

This unique left inverse can be represented in terms of a conjugate $\bar{\rho}$ in the well-known form (using the self-conjugacy of $\hat{\rho}$):

$$(16) \quad \hat{\phi}(\cdot) = R^* \hat{\rho}(\cdot) R$$

with R a suitable intertwiner $R \in (\hat{\rho}^2, id)$. The iteration of this left inverse evaluated in the vacuum representation π_0 gives a tracial state on the centralizer algebra of $\hat{\rho}$:

$$(17) \quad \text{tr} := \lim_{m \rightarrow \infty} \pi_0 \cdot \hat{\phi}^m \text{ is a trace on } M = \cup_m (\rho^m, \rho^m).$$

Since M is a hyperfinite II_1 factor, this trace is unique. Its restriction to the colored ribbon-braid-groupoid algebra $RB_\infty \subset M$ yields a ribbon-Markov trace [3].

We now define a “master Gibbs state” as:

$$(18) \quad \omega_\beta(A) = \frac{1}{Z} \text{Tr } \pi_0 \hat{\rho}(A e^{-\beta L_0}) = \frac{1}{Z} \sum_i \text{Tr } \pi_0 \rho_i(A) e^{-\beta L_0^{\rho_i}}$$

with $Z = \sum_i \text{Tr } \pi_0 e^{-\beta L_0^{\rho_i}}$. Here $\rho_i(e^{-\beta L_0})$ is a shorthand notation for $e^{-\beta L_0^{\rho_i}}$, where $L_0^{\rho_i}$ implements the (infinitesimal) rigid rotation inside the endomorphism $\rho(\cdot)$.

The Gibbs state becomes formally a tracial state in the limit of infinite temperature. This limit does not exist on $\mathcal{A}_{\text{univ}}$. To be more precise, the sequence of states for increasing temperatures possesses subsequences with limit points (since the state space on a C^* -algebra is compact). But the limit states are generally (and in particular in our case) nonnormal on the net. A good example is the infinite temperature state of the circular Weyl algebra. The representation of the space-time transformation is lost in this limit. It is easy to see that it retains its good properties if one restricts to the subalgebra M of intertwiners. In fact on each finite subalgebra M_m the limit exists as a normal state since there are no nonnormal states on finite dimensional algebras.

Hence the limit must be equal to the tracial state φ on M (uniqueness on a factor!)

$$(19) \quad \lim_{\beta \rightarrow 0} \omega_\beta|_M = \varphi.$$

In other words the tracial state of algebraic QFT may be replaced by a more physical infinite-temperature Gibbs state. In particular, we obtain for the central projection

$$(20) \quad \lim \frac{\omega_\beta(P_\sigma)}{\omega_\beta(P_\kappa)} = \frac{\varphi(P_\sigma)}{\varphi(P_\kappa)} = \frac{d_\sigma}{d_\kappa}$$

where the last equation results from

$$(21) \quad \pi_0 \phi(P_\sigma) = \sum_i R^* W_i \pi_0 \rho_i(P_\sigma) W_i^* R$$

$$(22) \quad = R^* W_\sigma W_\sigma^* R = \frac{d_\sigma}{\sum d_{\rho_i}}.$$

This asymptotic formula is the Kac-Wakimoto formula.

We conjecture that M is the maximal algebra on which the infinite temperature state is regular. This should be seen as an analogue of the statement that singular states, which implement the idea of gauge invariance on 3- d suitably defined Weyl-like algebras, become regular on the gauge invariant intertwiner subalgebra, a phenomenon which is expected to be related to “topological QFT”. In physical terminology: Large temperature limits select the skeleton of intertwiners, whereas the analogous gauge invariant states produce gauge invariant subalgebras (and in special cases of Chern-Simons Weyl algebras, these are expected to consist of intertwiners only, see the Introduction).

It is well-known [65] that in $\mathcal{A}_{\text{univ}}$ one may construct nontrivial global self-intertwiners $V_{\rho_i} \in (\hat{\rho}, \hat{\rho})$. With the help of the R -intertwiners between the vacuum and $\bar{\rho}_i, \rho_i$ one then obtains a system of invariant charge operators C_{ρ_i} in the center of $\mathcal{A}_{\text{univ}}$. Assuming a non-degenerate situation [7], these charges span the center and their numerical values in the various irreducible representations $\pi_0 \rho_j$ are the entries of the Verlinde matrix S_{ij} . The process by which they have been obtained may be described in physical terms as a $\bar{\rho}_i - \rho_i$ charge pair creation from the vacuum, a subsequent charge transport of ρ_i with the help of V_{ρ_i} once around the circle and finally a recombination leading to the initial vacuum situation. This process may be generalized. Instead of applying the charge transporter V to one charge of a pair, one may apply V to an endomorphism which arises in a more complicated split of the vacuum into many charges, with subsequent recombination. This leads to multi-indexed invariant charge operators $C_{\rho_i}(\{\sigma\})$ where the σ 's label the other charges which are present while the charge transport around the circle V_{ρ_i} is being performed. These new charge operators are proportional to the old C_{ρ_i} :

$$(23) \quad C_{\rho_i}(\{\sigma\}) = S(\rho_i, \{\sigma\}) C_{\rho_i}.$$

The ‘‘polarization coefficients’’ S turn out to be the building blocks of the higher genus mapping class group matrices. This identification follows by comparing the graphical representation (in terms of the algebraic path basis for intertwiners) with the graphical representation for the combinatorial approach to invariants of 3-manifolds. For details the reader is referred to [65].

It is advantageous to define a subalgebra $\mathcal{G}_{\text{inv}} \subset M$ as the algebra¹⁰ generated by self-intertwiners and endomorphisms applied to self-intertwiners. This subalgebra is obtained from the coloured cylinder braid groupoid algebra by retaining only the elements which go around the cylinder. The iterated left inverse $\hat{\phi}$ leads to a tracial state on M whose restriction to $\mathcal{G}_{\text{inv}}$ yields not only the numerical entries of Verlinde’s matrix S , but also the above polarization coefficients (24) and hence the entries of all higher mapping class group elements, the analogues of Verlinde matrices S (the mapping class group also contains diagonal generators corresponding to the T -matrix). They are all algebraic matrix elements of global self-intertwiners in the algebraic basis given in terms of fixed reference endomorphisms [65]. According to our remarks on the asymptotic behaviour of L_0 -temperature states, all the data contained in the representations $\pi(M)$ and $\pi(\mathcal{G}_{\text{inv}})$ obtained via the tracial state can also be obtained as an infinite temperature limit in the form of generalized Kac-Wakimoto formulas.

A brief historical reminder is in order.

This matrix (together with a diagonal matrix T), first abstracted from characters of chiral conformal models by Cardy, Capelli, Itzykson and Zuber, then related to fusion rules by Verlinde [38] and generalized to a wider geometrical picture of chiral conformal field theory by Moore and Seiberg [39], was finally shown to relate fusion rules and monodromy properties (statistics character) independent of conformal QFT by Rehren [40]. By the present work we may now add the remark that the universal algebra on the circle (as well as the 3- d universal algebra) also contains mapping class group matrices for all geni in the form of polarization coefficients of self-intertwiners (24).

¹⁰Strictly speaking, only the GNS representation of $\mathcal{G}_{\text{inv}}$ with class invariant states (Gibbs or tracial states only depend on the sector $[\rho]$) deserve this name.

This observation on the presence of mapping class matrices for all geni in a chiral conformal field theory on a circle is very startling. It should be taken as a hint of a new yet unknown quantum symmetry of which we presently only see the invariant part.

4 The TCP Theorem of the Exchange Algebra and its Modular Interpretation for Anyons

We first recall our findings [7] on the TCP operator of the exchange algebra. The latter is generated by operators F which act on the Hilbert space consisting of N copies of the vacuum space:

$$(24) \quad \mathcal{H} = \bigoplus_{\alpha} (\alpha, \mathcal{H}_0)$$

according to:

$$(25) \quad F(e, A)(\alpha, \mathcal{H}_0) = (\beta, \pi_0(T_e^* \rho_{\alpha}(A))\psi)$$

with $A \in \mathcal{A}_{\text{univ}}$, $T_e \in \mathcal{A}_{\text{univ}} : \rho_{\beta} \rightarrow \rho_{\alpha} \cdot \rho$ and $\rho_{\alpha}, \rho_{\beta}, \rho$ taken from an arbitrary but fixed finite set of reference endomorphisms.

If e runs through the finite set of all “edges” e of the form $(\rho_{\alpha}, \rho, \rho_{\beta})$ and A through $\mathcal{A}_{\text{univ}}$, then the F generate a C^* algebra (exchange algebra or reduced field bundle) which, since it is an operator algebra in a concrete H -space, also allows for the von Neumann closure.

In contrast to Boson-(CCR) or Fermion-(CAR) algebras, the charge conjugation in this algebra differs significantly from the $*$ -operation.

The $*$ -operation has the form:

$$(26) \quad F(e, A)^* = \frac{d_{\rho}}{\chi_{\rho}} \sum_{e^*} \eta_{ee^*} F(e^*, \bar{\rho}(A^*) R_{\rho})$$

where e^* is of the adjoint form $(\beta, \bar{\rho}, \alpha)$ and the coefficients (d_{ρ} statistical dimension, χ_{ρ} a phase and η a structure constant matrix) are as in [7]. On the other hand the antilinear charge conjugation of the exchange algebra reads as:

$$(27) \quad \overline{F(e, A)} := \sqrt{\frac{d_{\alpha}}{d_{\beta}} \frac{d_{\rho}}{\chi_{\rho}}} \sum_{\bar{e}} \zeta_{e\bar{e}} F(\bar{e}, \bar{\rho}(A^*) R_{\rho})$$

with $\bar{e}$ of the form $(\bar{\alpha}, \bar{\rho}, \bar{\beta})$ (and ζ another structure matrix). It satisfies

$$(28) \quad \overline{\bar{F}} = \frac{\chi_{\beta}}{\chi_{\alpha}} \cdot F$$

where the phase can be adjusted to be ± 1 (+1 in the non-self-conjugate case).

By analogy with the standard case, one may define an operator:

$$(29) \quad S_{\pm} : F(e, A)\Omega \rightarrow \overline{F(e, A)}\Omega, \quad F \in \mathcal{F}_{\text{red}}(R_{\pm})$$

which together with its adjoint turns out to be densely defined, hence closable, and therefore permits a polar decomposition [9]. Simple considerations using “coordinates” of point-like fields [7] reveal that this polar decomposition is geometric:

$$(30) \quad S_{\pm} = \kappa^{\pm \frac{1}{2}} \theta V(\pm i\pi)$$

where κ is a statistics phase, θ the TCT operator and $V(\pm i\pi)$ the analytic continuation of the dilation on $\mathcal{H}$. This geometric result parallels the previously obtained result based on the use of the Tomita-Takesaki modular theory for the observable algebra [27].

However, in the case of our exchange algebra these formulae cannot be backed up by Tomita-Takesaki modular theory, since this theory requires the (positive-definite) $*$ -operation (and cannot be done with the involutive antilinear charge-conjugation).

In our paper [7] we speculated about the possible existence of a “twisted” Tomita-Takesaki theory relating a twisted involution with a twisted commutator. But perhaps a more conservative idea is to follow the mechanism by which the “would-be modular theory” in the case of trivial monodromy $\varepsilon^2 = 1$, (i.e., the permutation group exchange algebra) can be converted into a true modular theory.

In that case we know, according to Doplicher and Roberts [42], that there exists a genuine field algebra of Bosons and Fermions from which the exchange algebra can be obtained by removing multiplicities [43], i.e., making the Hilbert-space smaller.

The price one pays for converting the Bosons and Fermions with a non-abelian symmetry group into para-Bosons and para-Fermions (on which no nonabelian compact group can act since the H -space lacks the necessary multiplicities) is a certain amount of non-locality, now expressed in terms of R -matrix commutation relation ($R^2 = 1$) instead of local Boson or Fermion (anti)commutation relations for space-like distances [43].

In fact, one could contemplate that the insistence on a geometric (local) version of the standard Tomita-Takesaki modular theory would determine the smallest enlargement of the H -space of the permutation statistics exchange algebra and in this way could yield a direct operator-algebraic proof of the D. R. theorem (which in its present form contains a significant amount of C^* -category theory).

In any case, the previous considerations strongly suggest that the search for a new “quantum symmetry” behind braid group statistics should be pursued in conjunction with the modular Tomita-Takesaki theory and the TCP properties of QFT.

It is interesting to note that the special abelian case of Z_{2N} anyons can still be covered by the standard Tomita-Takesaki theory. As in the case of Fermions [3], one has to introduce a suitably “twisted” or “quasi”-commutant (and no “twisted star” which would lead away from the T.-T. theory).

The starting point is the field algebra $\mathcal{F}_N$ associated with the $\mathcal{A}_N^{\text{univ}}$ -observable algebra. The latter is the Z_{2N} maximal extension of the Weyl algebra on S^1 as constructed in [44].

This algebra has $2N$ central charges C_l and its faithful representation requires a Hilbert space:

$$(31) \quad \mathcal{H} = \bigoplus_{n=0}^{2N-1} \mathcal{H}_n, \quad C\mathcal{H}_n = e^{\frac{i\pi n}{N}} \mathcal{H}_n$$

$$(32) \quad C = e^{\frac{2\pi i Q}{\sqrt{2}N}}, \quad C_l = C^l.$$

The field algebra $\mathcal{F}_N$ is generated by sector-intertwining operators which “live” on a finite covering of S^1 and obey the cylinder-braid-group commutation relations of Section 3, which in this anyonic case reads as:

$$(33) \quad \phi_{\rho_1} \phi_{\rho_2} = e^{\frac{i\pi n_1 n_2}{2N}} e_{\text{mon}}^M \phi_{\rho_2} \phi_{\rho_1}$$

with e_{mon} = monodromy phase-factor, n_i = charge-values, and

$$(34) \quad \begin{aligned} \text{loc } \phi_{\rho_i} &\subset J_i, \quad \text{proj } J_1 \cap \text{proj } J_2 = \emptyset \\ J_1 + 2\pi M &< J_2 < J_1 + 2\pi(M+1). \end{aligned}$$

Here *loc* refers to the localization region.

We restrict our discussion to the first sheet and choose

$$(35) \quad \begin{aligned} \phi_{\rho_1} &\in \mathcal{F}_1 = \mathcal{F}(S_+), \\ \phi_{\rho_2} &\in \mathcal{F}_2 = \mathcal{F}(S_-). \end{aligned}$$

By analogy with Fermions, we construct a twist operator Z as a suitable square root of the 2π -rotation:

$$(36) \quad \begin{aligned} e^{-2\pi i L_0} &= e^{-i\pi Q^2} \\ Z &:= e^{-i\pi \frac{Q^2}{2}} = \sum_{n=0}^{2N-1} e^{-i\pi \frac{n^2}{4N}} P_n \\ &\text{with } P_n = \text{central projector onto } \mathcal{H}_n. \end{aligned}$$

The twist transforms ϕ_{ρ_2} according to

$$(37) \quad Z \phi_{\rho_2} Z^{-i} = \phi_{\rho_2} \sum_n e^{\frac{-i\pi(n+n_2)^2}{4N}} P_n \sum e^{\frac{i\pi n^2}{4N}} P_n$$

$$(38) \quad = e^{\frac{-i\pi n^2}{4N}} \phi_{\rho_2} V_{n_2}$$

$$(39) \quad \text{with } V_{n_2} := \sum_n e^{-\frac{i\pi n_2 n}{2N}} P_n.$$

As a consequence of

$$(40) \quad V_{n_2} \phi_{\rho_1} = e^{-i \frac{n_1 n_2}{2N} \pi} \phi_{\rho_1} V_{n_2}$$

this V_{n_2} commutation phase precisely compensates the statistics phase, and we obtain for the $\mathcal{F}_N$ -generators

$$(41) \quad \phi_{\rho_1} Z \phi_{\rho_2} Z^{-1} = Z \phi_{\rho_2} Z^{-1} \phi_{\rho_1}.$$

This result does not change if the localizations of the ϕ_{ρ_i} , $i = 1, 2$ (each in one sheet) are in general relative position.

From the commutation relations of the generators we abstract the so-called “twisted locality”, i.e., the statement:

$$(42) \quad \begin{aligned} \mathcal{F}(S_+) &\subset \mathcal{F}(S_-)^q \\ S_{\pm} &: \text{half-circles} \end{aligned}$$

$$\text{with } \mathcal{F}(S_-)^q := (Z\mathcal{F}(S_-)Z^{-1})'.$$

In addition we have the involution property

$$(43) \quad \mathcal{F}(S_-)^{qg} = \mathcal{F}(S_-)$$

since $Z^2 = 2\pi$ rotation.

The derivation of the twisted modularity from the geometrically twisted form of causality for $\mathcal{F}(S_-)$ is a well-known procedure [3] and identifies the dilation as the modular group.

The KMS property of the dilation manifests itself in terms of point-like covariant fields (and only in terms of these) as a quasiperiodicity in the analytic continuation of their correlation functions, thus paralleling the well-known $\text{Dil}(2\pi i)$ periodicity properties of local generators in $\mathcal{A}_N^{\text{univ}}$. The quasiperiodicity (as opposed to periodicity) accounts for the braid-group commutation relations (as opposed to bosonic commutation relations).

The modular reflection J contains, as for fermions, the twist Z in addition to the TCP operator. The well-known modular theory of the observable algebra $\mathcal{A}_N(S_+)$ in the vacuum state, and the existence of a state-preserving conditional expectation $\mathcal{F} \rightarrow \mathcal{A}$ follows then by the "Takesaki devissage theorem" [31].

In $\mathcal{F}_N$ the charge conjugation and the star-operation coalesce. This is the main difference between this special model and the general exchange algebra with noninteger statistical dimensions, where a geometric Tomita-Takesaki modular theory which is consistent with the geometric TCP theorem is still lacking.

It is, however, my conviction that all models with integer statistical dimension or (using a concept recently developed by Rehren [45]) with vanishing "defect projector" can be treated with the above method.

There are arguments [46] that Hopf algebras exhaust the integer statistical dimension sector category of properly infinite von Neumann algebras. It has been known for some time that Hopf C^* -algebras in the form of Drinfeld "doubles" [47] also fulfil braid-group requirements (perhaps even the complete list of the braid-group-category axioms?). So, such doubles may have field theoretic models with a similar twisted geometric modular theory as above.

In this connection it is interesting to point out that a framework for a systematic study of C^* -Hopf-algebra categories with the additional braid-statistics category requirements has been formulated recently [48].

5 Self-duality and QFT-Inclusions

For our purposes the most remarkable property of the matrix S (i.e., the numerical values of the various invariant charges C_{ρ_i} in the irreducible representations $\pi_0 \cdot \rho_j(\cdot)$) is its symmetry.

In the case of finite nonabelian groups, the invariant charges are obtained by averaging the group representatives on the physical Hilbert space over the various conjugacy classes, i.e., the index on C_ρ refers to a conjugacy class. Whereas the number of irreducible representations equals the number of conjugacy classes, thus leading to a square matrix S which is the same as the character table, the two indices remain conceptually

different (e.g., the fusion of irreducible sectors is not the same as the fusion of conjugacy classes).

So, already in a very early state of development of low dimensional QFT, it became clear that the new symmetry must be more symmetric than, e.g., finite non-abelian groups. It should be analogous to abelian symmetry groups even in cases where the statistical dimensions d_ρ are bigger than one.

There are some interesting lessons which one can learn from the abelian Z_{2N} model of Section 3 if one analyses its duality structure from the modular point of view, as proposed by Fredenhagen, Longo, Rehren and Roberts [45]. From a field algebra $\mathcal{F}(\mathcal{O})$ with a known geometric T.-T.-modular theory and a smaller observable algebra $\mathcal{A}(\mathcal{O})$ which is assumed to be irreducibly contained in $\mathcal{F}(\mathcal{O})$ with finite Jones index, one may construct the beginning of a Jones tower (tunnel):

$$(44) \quad \mathcal{F}(\mathcal{O}) \overset{\mu}{\supset} \mathcal{A}(\mathcal{O}) \overset{\nu}{\supset} \gamma(\mathcal{F}(\mathcal{O}) \supset \gamma(\mathcal{A}(\mathcal{O})) \dots$$

Here μ and ν are (unique) conditional expectations which exist according to the finite index assumption, and γ is Longo's canonical endomorphism which is only unique as a sector [49] and whose construction requires modular theory (it may be represented as the product of two Tomita reflections belonging to $\mathcal{F}(\mathcal{O})$ and $\mathcal{A}(\mathcal{O})$). The inclusion:

$$(45) \quad \mathcal{A}(\mathcal{O}) \supset \gamma(\mathcal{A}(\mathcal{O})) \equiv \rho(\mathcal{A}(\mathcal{O}))$$

is that on which the superselection theory is based (to see this, one has to decompose ρ into its irreducible pieces). It leads to the superselection "paragroup" ¹¹ (using the language of Ocneanu who first analysed such inclusions in the case of hyperfinite II_1 algebras from the point of view of an analogue of group theory), whereas the paragroup going with the inclusion

$$(46) \quad \mathcal{F}(\mathcal{O}) \supset \gamma(\mathcal{F}(\mathcal{O}))$$

deserves the name "symmetry" paragroup [45]. Self-duality means that the two paragroups behind (46) and (47) are isomorphic.

Guido and Longo have shown that γ allows one to transfer the "Mackey reduction-induction machine" to the sector theory of von Neumann algebras (by using Connes bimodules), thus generalizing an earlier observation of M. Rieffel.

Instead of giving a more detailed explanation of the various concepts used in these inclusions, let us specialize to the abelian Z_n -model. In that case $\mathcal{F}(\mathcal{O})$ and $\mathcal{A}(\mathcal{O})$ are subalgebras of well-known global algebras $\mathcal{F}_{\text{univ}}$ and $\mathcal{A}_{\text{univ}}$. In addition to localized anyonic or bosonic operators, they also contain global operators Γ (in case of $\mathcal{F}_{\text{univ}}$) and Q (in the case of $\mathcal{A}_{\text{univ}}$) [43].

Q is the "global Z_{2N} charge-measurer" and Γ the global "charge-creator":

$$(47) \quad \begin{aligned} Q\Gamma Q^* &= e^{\frac{2\pi i}{N}} \Gamma \\ [Q, e^{iyL_0}] &= 0, \quad e^{iL_0 y} \Gamma e^{-iyL_0} = e^{\frac{2\pi i}{N}} \Gamma. \end{aligned}$$

¹¹Ocneanu would also introduce endomorphisms for the single-step inclusion, i.e., $\gamma = \sigma \bar{\sigma}$, $\bar{\sigma}$ conjugate to σ . For a single algebra (but not coherently for the whole net) this is possible whenever $\mathcal{F}(0) \sim \mathcal{A}(0)$. But an extension of σ to the net is generally not possible.

The last relation is the transformation property of Γ under rigid rotations.

The conditional expectations on the von Neumann algebras μ and ν may be obtained directly from those of the global C^* algebras:

$$(48) \quad \begin{aligned} \mu(b) &= \frac{1}{|G|} \sum_n Q^n b Q^{-n} \quad b \in \mathcal{F}_{\text{univ}} \\ \nu(a) &= \frac{1}{|G|} \sum \Gamma^n a \Gamma^{-n} \quad a \in \mathcal{A}_{\text{univ}}. \end{aligned}$$

From this one obtains two global fixed-point algebras

$$(49) \quad \mathcal{F}_{\text{univ}} \xrightarrow{\mu} \mathcal{A}_{\text{univ}}, \quad \mathcal{A}_{\text{univ}} \xrightarrow{\nu} \gamma(\mathcal{F}_{\text{univ}})''.$$

The image under ν can be easily described as the extended Weyl algebra with the Q charge removed (it is still bigger than the ordinary Weyl algebra on the circle!). However, its interpretation in terms of γ fails (the reason for the parenthesis), since γ can only be constructed via the T.-T.-modular theory in case of properly infinite (e.g., type III) algebras whereas the global algebras are type I.

To be more precise: Whereas it is possible to compute the Jones projector of the global (type I) inclusion $\mathcal{A}_{\text{univ}} \supset \nu(\mathcal{A}_{\text{univ}})$ as an operator in $\mathcal{F}_{\text{univ}}$ as

$$(50) \quad E_0 = \sum_n \frac{1}{|G|} \Gamma^n, \quad E_i = Q^i E_0 Q^{-i} \quad , \quad |G| = 2N$$

with

$$(51) \quad \mu(E_0) = \frac{1}{|G|} 1,$$

one cannot split E_0 into an isometry

$$(52) \quad E_0 \neq VV^*$$

which is the "would-be" generator of the Popa-Pimsner basis in QFT [50]

$$(53) \quad \mu(bV^*)V = V^*\mu(Vb) = \lambda^{-1}b$$

with $\lambda = |G|$ in our case.

In order to obtain a projector E_0 as part of a local (type III₁) algebra $\mathcal{F}(\mathcal{O})$ with the split into Popa-Pimsner isometries V , one must replace the Γ 's by anyonic fields localized in say (without loss of generality) a half-circle $S_{\pm}$.

But then the isometry V (whose existence is secured by the properties of type III von Neumann algebras) does not act in a completely local way (i.e., does not transform all subalgebras of $\mathcal{A}(A_{\pm})$ into themselves as μ does).

Once V is known, one can define a corresponding (reducible, with index $|G|$) canonical endomorphism γ and a condition expectation ν [50]:

$$(54) \quad \gamma(b) := \lambda \mu(VbV^*) \quad b \in \mathcal{F}$$

$$(55) \quad \nu(a) := \lambda \mu(E_0 a E_0), \quad \lambda = |G|, \quad E_0 = VV^*.$$

It is easy to see that the invariance requirement $\nu(a) = a$ again imposes a linear condition on the generating charge distribution of the extended Weyl algebras $\mathcal{A}(S_{\pm})$, but this condition is less homogeneous as compared with the global algebra, i.e., cannot be simply expressed as the absence of certain zero modes Q .

So the self-duality picture, while valid (by the choice of the model) for the group theory of the charge-measures Q and charge-creators Γ system, cannot be pushed to the local level: μ remains completely local whereas ν is only partially local.

In contradistinction to the known ergodic behaviour of canonical endomorphisms constructed from double reflections of the geometric inclusions of the Section 2, the inclusions of this Section lead to a Jones tunnel with a nontrivial limiting III_1 algebra which in the case of the present $\mathbb{Z}_{2N}$ model turns out to be a subalgebra of the circular Weyl algebra. This remarkable distinction between “deep inclusions” (leading to external symmetries) and “shallow inclusions” (yielding inner symmetries) clearly asks for a more profound understanding.

It is a curious fact that the Euclidean theory, briefly sketched in Section 2, allows for linearly rising charge distributions which vanish outside a half circle (similar to “blips” [51]): such discontinuous charge distributions become finite operators with the Euclidean star-operation.

We take this as an indication that Euclidean quantum field theory (i.e., the operator algebra formulation of the Euclidean analytic continuation of correlation functions) may play an important role in the quest for a perfect (local) self-duality.

We also think of the analytic modular relation for temperature correlation functions found in conformal models (generalizing the modular relation between characters from which Verlinde [38] found the relation between S and the fusion matrices) as an analytic consequence of an algebraic local self-duality obtained by using class-invariant temperature states. The validity of this conjecture would then entail that those mysterious modular identities from Jacobi to Ramanujan, which presently are (in most cases) derived by Poisson-resummation techniques, have their ultimate conceptual understanding via T.-T. modular theory in terms of the two physical principles mentioned in the introduction: Einstein causality and Dirac stability¹².

The reader will find further speculative remarks resulting from our formulas (55) and (56) in a recent paper by Rehren. In particular Rehren shows that nonconfined nonabelian group-symmetries (never observed as exact symmetries in nature!) would destroy the pretty picture of selfduality.

6 Speculative Remarks on “Quantum Symmetry”

The observations about $\mathcal{G}^{\text{inv}}$ in Section 3 and the subsequent discussions of self-duality invite speculation about a full symmetry algebra $\mathcal{G}$.

This problem has been previously discussed in the work of Mack and Schomerus [52] as well as Szlachànyi [53] and Vecsernyès [54]. For the conformed Ising model, the nontrivial $d = \frac{1}{16}$ sector appears in their global approach with (assumed) multiplicity 2

¹²Then the findings of Verlinde and others would return from their present algebra-geometric setting (not understandable by physicists) to their quantum-physical roots (understandable by some mathematicians).

in their field algebra, which then has an underlying weak quasi-Hopf algebra structure. Apart from the actual physical problem of whether such quasi-Hopf algebras, which lack strict associativity, can be used in practical calculation (symmetry breaking, Hartree-Fock approximations etc.), these proposals deviate significantly, in other aspects, from our picture obtained on the basis of algebraic QFT emphasizing "localization" at every step. Actually a detailed classification of global "rational" symmetry operations and an understanding of their possible local shortcomings may be quite helpful.

To get an idea about $\mathcal{G}$, we contemplate that the overlap of the spatial abelian 2π -rotations (only nontrivial as a covering transformation) with the invariant charges [7] is only the invariant projection of a bigger non-commutation overlap between space-time and internal symmetries. The completely unexpected appearance of mapping class matrices of all geni in the $\mathcal{A}_{\text{univ}}(S^1)$ observable algebra lends credibility to such a conjecture.

In that case there should exist a non-commutative subgroup of the Möbius group (and not just of the covering) which contains the TCP reflection and whose relative size of conjugacy classes for the remaining generators should be measured by the Jones numbers (thinking about minimal models).

Surprisingly, Vaughn Jones found such a purely group theoretical quantization in which those numbers appear, based on the following observation:

- (i) Any subgroup generated by the reflection $\tilde{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and a parabolic subgroup $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ leads to quantization $\lambda = 2 \cos \frac{\pi}{q}$ $q \geq 3$ for $\lambda < 2$ if the group is supposed to act in a discrete fashion in the sense of hyperbolic geometry on $\mathbb{R} \cup \{\infty\}$
- (ii) Such groups Γ are free group on two cyclic generators $\tilde{J}$ and

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \cdot \tilde{J} = \begin{pmatrix} -\lambda & 1 \\ -1 & 0 \end{pmatrix} = K, \text{ with } K^q = 1$$

with K a rotation in $PSL(2, \mathbb{R})$ and Γ the non-amenable free product $\Gamma = \tilde{J} * K$.

The group (von Neumann) algebras of such Γ 's are non-hyperfinite factors of type II_1 [55]. This means that the standard representation theory via block-decomposition of the regular representation does not work. So, if Γ appears as a part of the unknown $\mathcal{G}$, it should come together with some yet unknown representation theory.

From the viewpoint of physical principles there is nothing against non-hyperfinite field algebras. In fact, the proof of hyperfiniteness only applies to local observable algebras $\mathcal{A}(\mathcal{O})$ [56]. The fact that the field algebras of the DR construction are also hyperfinite III_1 algebras is just a fringe benefit of the permutation group statistics which in turn results from the assumption of 4- d finite energy particle states with mass gaps [58] and has no other physical principle behind it. Statistical mechanics also leads to infinite hyperfinite algebras in the thermodynamic limit (this is true by construction). The construction of physically viable non-hyperfinite field algebras does not seem to be an easy matter.

In this context, one should mention an idea of Voiculescu [57]. In his approach to noncommutative probability theory based on the concept of “freeness”, the Bose field (at a fixed time) belongs to the standard commutative measure-theoretical situation (type I), whereas the CAR fermionic algebra, which admits a tracial state (not describable in measure-theoretic terms), is a hyperfinite type II_1 factor. Beyond this he expects an ever-increasing wealth of more and more non-hyperfinite algebras with more complicated Fock spaces (the plektons of algebraic QFT?).

In the spirit of Voiculescu one expects that localizability and statistical independence for space-like separation in the exchange-algebra description ought to be replaced by space-like “freeness” for the unknown field algebras, similar to the transition from statistical independence to “freeness” in his noncommutative probability theory. I believe that the startling self-duality and finiteness of the new symmetry (outside abelian groups) can only be properly understood with the help of very big (non-hyperfinite) von Neumann algebras: The simpler the properties, the bigger the objects carrying them. For the physical use of such ideas one must of course ensure that the finite-region observable algebras (expected to be the gauge invariant subalgebras of a generalized unknown gauge principle) remain hyperfinite.

Last but not least, the wealth of semiclassical observations made about the new invariants of 3-manifold on the basis of Witten’s topological QFT [10], Atiyah’s axiomatic approach to this problem [59] and the Reshtkhin-Turaev-Viro combinatorial approach [60], as well as some speculative ideas of Ocneanu [61], all point into the same direction: An inexorable new link between space-time and inner symmetries outwitting all the old No-Go theorems [8] based on analytic and algebraic properties of 4- d QFT’s!

The title of this conference invites me to express my thoughts (or rather speculations) on 21st century mathematical physics. I think that the problems of quantum symmetry and the related physics of “plektons” (with potentially interesting applications to condensed matter quasi-particles) will be solved in this decade and not remain as an open problem to next century. I expect that their solution will pave the way towards a profound understanding of 4-dimensional physics, in particular the problems related to quark confinement and a more profound understanding of gauge theories in the nonperturbative algebraic setting. This and further progress in the understanding of quantum gravitation will be the main 21st century topics (at least in my guess about the future). The development in physics will be matched by new mathematics, notably in the area of (nets of) von Neumann algebras, which will make the Jones inclusion theory appear as the first step into a new mathematical landscape. The corresponding physical theories will be based more on Leibnitz’ concepts of “monads” (von Neumann factors) and relations (inclusions, extensions) and less on Newton’s concepts of spatial geometry with a material content.

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ON THE COLLAPSE POSTULATE OF QUANTUM MECHANICS

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Abstract

A stochastic model for the measurement process of a quantum mechanical observable is discussed and its consequences, particularly for an observable with purely absolutely continuous spectrum, studied.

1 Introduction

The collapse postulate of von Neumann-Lüders [1, 2] gives the quantum state of a quantum mechanical system after an observable is measured on it. This prescription works well for observables with pure point spectra (bounded or unbounded). However, the attempts to extend it to those with continuous spectra are beset with many difficulties. Various authors have tried to rectify this situation with varying degrees of success (see, for example, Davies [3]). Srinivas [4, 5] studied the joint probabilities of successive observations of non-compatible observables and concluded that one may have to admit finitely additive probabilities. He also gave a prescription for a "collapse map" in terms of an invariant mean on the one-parameter automorphism group on $\mathcal{B}(\mathfrak{H})$ (the $*$ -algebra of bounded linear operators on a separable Hilbert space $\mathfrak{H}$), generated by the self-adjoint operator describing the observable in question.

In this paper, a stochastic model for coupling the system with the measuring apparatus is proposed, which leads to a completely positive (CP) semigroup evolution of the algebra of observables of the system. If the measured observable has a pure point spectrum, this semigroup, in the large time limit, gives the "collapse map" of von Neumann-Lüders. Finally, this idea is extended to the case of observables with continuous spectra and this gives the collapse map as a kind of 'transformer' of the type discussed by Birman and Solomyak [6, 7]. There have been many other attempts in this direction; in particular, those by Gisin [8] and Belavkin [9] may be mentioned.

2 A Model for Measurement of an Observable with Pure Point Spectrum

Let $\mathfrak{H}$ be a separable Hilbert space of states of a quantum system and let B be a selfadjoint operator (associated with the observable which is being measured) on $\mathfrak{H}$

with a pure point spectrum. In this case there exists a unique countable family of orthogonal projections $\{P_i\}_{i=1}^\infty$ on $\mathfrak{H}$ such that

$$(2.1) \quad B = \sum_{i=1}^{\infty} \lambda_i P_i, \quad I = \sum_{i=1}^{\infty} P_i,$$

where $\{\lambda_i\}$ are the eigenvalues of B and in the strongly convergent sums in (2.1), the eigenvalues are counted as many times as their multiplicities. Thus each P_i is one-dimensional; let the associated normalized eigenvector be φ_i .

The measuring apparatus is modelled as a Brownian bath or equivalently a bosonic Fock space. The states of the apparatus are given by vectors in $\mathcal{H} \equiv \Gamma(L^2(\mathbb{R}_+, \mathfrak{H}))$, which is generated by the total set of coherent (or exponential) vectors of the form

$$(2.2) \quad e(f) \equiv 1 \oplus f \oplus \frac{f^{\otimes 2}}{\sqrt{2!}} \oplus \cdots \oplus \frac{f^{\otimes n}}{\sqrt{n!}} \oplus \cdots,$$

with $f \in L^2(\mathbb{R}_+, \mathfrak{H}) \equiv \mathfrak{H}_1$. The Hilbert space for the system and apparatus is given by $\tilde{\mathcal{H}} = \mathfrak{H} \otimes \mathcal{H}$, and the annihilation and creation operators in $\tilde{\mathcal{H}}$ are defined as

$$(2.3) \quad \begin{aligned} a(f)u \otimes e(g) &= (f, g)u \otimes e(g) \\ a^\dagger(f)u \otimes e(g) &= u \otimes \frac{d}{d\varepsilon} e(g + \varepsilon f)|_{\varepsilon=0} \end{aligned}$$

where $u \in \mathfrak{H}$ and $f, g \in \mathfrak{H}_1$. The basic quantum stochastic processes (martingales) of the theory are, for $i = 1, 2, \dots$

$$(2.4) \quad \begin{aligned} A_i(t) &= a(\chi_{[0,t]} \otimes \varphi_i) \\ A_i^\dagger(t) &= a^\dagger(\chi_{[0,t]} \otimes \varphi_i), \end{aligned}$$

and the quantum Ito formula is:

$$(2.5) \quad \begin{aligned} dA_i(t)dA_j^\dagger(t) &= \delta_{ij} dt, \\ dA_i^\dagger(t)dA_j(t) &= dt dA_i(t) = dt dA_i^\dagger(t) = 0. \end{aligned}$$

For the basic theory of quantum stochastic calculus and solutions of quantum stochastic differential equations, the reader is referred to [10, 11].

We next state a proposition the proof of which is a special case of Theorem 27.8 of ref. [10].

Proposition 2.1: Let $L_i \in \mathcal{B}(\mathfrak{H})$ for each $i = 1, 2, \dots$, such that $\sum_i \|L_i u\|^2 < \infty \forall u \in \mathfrak{H}$. Then the quantum stochastic initial value problem

$$(2.6) \quad dU = \sum_i (L_i^* dA_i - L_i dA_i^\dagger - \frac{1}{2} L_i^* L_i dt) U, U(0) = I$$

has a unique unitary operator-valued process in $\tilde{\mathcal{H}}$ as its solution.

If we now define the *expectation semigroup* T_t as:

$$(2.7) \quad (u, T_t(X)v) = (u \otimes e(0), U(t)^* X U(t) v \otimes e(0))$$

for $u, v \in \mathfrak{H}, X \in \mathcal{B}(\mathfrak{H})$, then T_t is a semigroup of completely positive maps with (Lindblad) generator

$$(2.8) \quad \mathcal{L}(X) = \sum_i (L_i^* X L_i - \frac{1}{2} X L_i^* L_i - \frac{1}{2} L_i^* L_i X),$$

the series converging strongly.

Next we make a crucial assumption of the model.

A1: The evolution of the system and the apparatus is given by an unitary process in $\tilde{\mathcal{H}}$ satisfying (2.6) with $L_i = P_i$, the eigenprojections of B .

Under this assumption A1, the evolution equation is

$$(2.9) \quad dU = \left(\sum_i P_i dA_i - \sum_i P_i dA_i^\dagger - \frac{1}{2} dt \right) U,$$

and the generator $\mathcal{L}$ of the expectation semigroup T_t can easily be computed to be

$$(2.10) \quad \mathcal{L}(X) = \sum_i P_i X P_i - X.$$

We expect that the large-time asymptotics of T_t , or rather its predual τ_t on $\mathcal{B}_1(\mathfrak{H})$ (the trace-class operators on $\mathfrak{H}$) will describe the collapse map $\mathcal{E}$ of a state after an observation of B is made on it. The next theorem summarizes the results.

Theorem 2.2: Let B, P_i be as above and let $U(t)$ be given by (2.9). Then

(i) for $X \in \mathcal{B}(\mathfrak{H})$

$$\mathcal{E}(X) \equiv n - \lim_{t \rightarrow \infty} T_t(X) = \sum_i P_i X P_i,$$

(ii) for $\rho \in \mathcal{B}_1(\mathfrak{H})$,

$$\mathcal{E}(\rho) \equiv B_1 - \lim_{t \rightarrow \infty} \tau_t(\rho) = \sum_i P_i \rho P_i.$$

(iii) If one observes only the values of the observable B in a Borel set $\Delta \subseteq \sigma(B)$, then we replace (2.9) by

$$(2.11) \quad dU^\Delta = \left(\sum_{\lambda_i \in \Delta} P_i dA_i - \sum_{\lambda_i \in \Delta} P_i dA_i^\dagger - \frac{1}{2} P(\Delta) dt \right) U^\Delta$$

and obtain the corresponding expectation semigroup T_t^Δ ; in that case

$$\begin{aligned} \mathcal{E}^\Delta(X) &= n - \lim_{t \rightarrow \infty} T_t^\Delta(X) = \sum_{\lambda_i \in \Delta} P_i X P_i \\ &= P(\Delta) \mathcal{E}(X) P(\Delta). \end{aligned}$$

Proof: Note that $\mathcal{L}$ given by (2.10) is a bounded map on $\mathcal{B}(\mathfrak{H})$ and $\mathcal{L}^2 = -\mathcal{L}$. Thus for $X \in \mathcal{B}(\mathfrak{H})$, $T_t(X) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}^n(X) = X + \mathcal{L}(X) - e^{-t} \mathcal{L}(X)$ from which (i) follows. Part

(ii) is an easy consequence of (i) by duality. For part (iii), we need only to observe that

$$P(\Delta)U(t) = P(\Delta)U(t)P(\Delta) = U^\Delta(t).$$

■

The result of Theorem 2.2 (ii) is precisely the von Neumann-Lüders postulate for the collapse of a state after measurement of an observable in the state given by a density matrix ρ .

3 Measurement of an Observable with Arbitrary Spectrum

Let B be a bounded self-adjoint operators in $\mathfrak{H}$ representing a quantum observable with $P(\cdot)$ as its spectral measure:

$$(3.1) \quad B = \int_{\Lambda} \lambda P(d\lambda), \quad I = \int_{\Lambda} P(d\lambda).$$

As in Section 2, we work in $\tilde{\mathcal{H}} \equiv \mathfrak{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathfrak{H}))$ and adapt (2.9) to the general case. For this, we choose a family $\{\varphi(\lambda)\}_{\lambda \in \Lambda}$ of vectors in $\mathfrak{H}$ such that the map $\lambda \rightarrow \varphi(\lambda)$ is strongly continuous and, as in (2.4), set $A_\lambda(t) = a(\chi_{[0,t]} \otimes \varphi(\lambda))$ and consider the stochastic initial value problem

$$(3.2) \quad \begin{aligned} dU(t) &= \left(\int P(d\lambda) dA_\lambda(t) - \int P(d\lambda) dA_\lambda^\dagger(t) - \frac{1}{2} \int \|\varphi(\lambda)\|^2 P(d\lambda) dt \right) U(t), \\ U(0) &= I. \end{aligned}$$

We embed Λ in a compact interval $[a, b]$ and consider a partition $\pi \equiv \{a = \lambda_0 < \lambda_1 < \dots < \lambda_n < \lambda_{n+1} = b\}$, with $\Delta_i = (\lambda_i, \lambda_{i+1}]$. Then the Riemann-sum approximation to $U(t)$ in (3.2) is given by

$$(3.3) \quad \begin{aligned} U_\pi(t) &= I + \sum_{i=1}^n \int_0^t \{P(\Delta_i) dA_{\mu_i}(s) - P(\Delta_i) dA_{\mu_i}^\dagger(s) \\ &\quad - \frac{1}{2} P(\Delta_i) \|\varphi(\mu_i)\|^2 ds\} U_\pi(s). \end{aligned}$$

The next theorem describes the solution of (3.2).

Theorem 3.1:

- (i) For each partition π , there exists a unique unitary process $U_\pi(t)$ satisfying (3.3).
- (ii) For each $u \in \mathfrak{H}, f \in \mathfrak{H}_1$ and $t \geq 0$, $U_\pi(t)u \otimes e(f)$ converges strongly in $\tilde{\mathcal{H}}$ as $|\pi| = \max_i (\text{length } \Delta_i) \rightarrow 0$.
- (iii) The limit is (ii) defines a unitary process $U(t)$ in $\tilde{\mathcal{H}}$ which solves the stochastic initial value problem (3.2).

Proof: Part (i) follows from Proposition 2.1 and for (ii) we need only to note that if π' is another partition finer than π , then

$$(U_{\pi'}(t)u \otimes e(f), U_\pi(t)u \otimes e(f)) \rightarrow \|u \otimes e(f)\|^2$$

as $|\pi| \rightarrow 0$. Since U_π^* also satisfies a stochastic differential equation very similar to (3.3), $U_\pi^*u \otimes e(f)$ converges strongly as $|\pi| \rightarrow 0$, proving (iii). $\blacksquare$

Using the Ito formula (2.5), it is easy to compute the expectation semigroup $T_{\pi,t}$ and its generator $\mathcal{L}_\pi$, associated with the unitary evolution $U_\pi(t)$; we get for $X \in \mathcal{B}(\mathfrak{H})$

$$(3.4) \quad \mathcal{L}_\pi(X) = \sum_{i,j:\pi} [\psi(\lambda_i, \lambda_j) - \frac{1}{2}\psi(\lambda_i, \lambda_i) - \frac{1}{2}\psi(\lambda_j, \lambda_j)] P(\Delta_i) X P(\Delta_j),$$

where $\psi(\lambda, \mu) = (\phi(\lambda), \phi(\mu))$. Also,

$$(3.5) \quad \begin{aligned} T_{\pi,t}(X) &= \sum_{i,j:\pi} \exp\left[\left(\psi(\lambda_i, \lambda_j) - \frac{1}{2}\psi(\lambda_i, \lambda_i) - \frac{1}{2}\psi(\lambda_j, \lambda_j)\right)P(\Delta_i)XP(\Delta_j)\right] \\ &= \sum_{i,j:\pi} \exp\left[-\frac{t}{2}\|\varphi(\lambda_i) - \varphi(\lambda_j)\|^2 + it \operatorname{Im}(\varphi(\lambda_i), \varphi(\lambda_j))\right] P(\Delta_i)XP(\Delta_j). \end{aligned}$$

In order to understand the limits of the expressions in (3.4) and (3.5) as $|\pi| \rightarrow 0$, we need to assume more on the map $\lambda \rightarrow \varphi(\lambda) \in \mathfrak{H}$ and to use the theory of iterated double spectral integrals as developed in [6, 7], leading to the next theorem.

Theorem 3.2: Let $\varphi(\lambda) \in \mathfrak{H}$ be such that $\|\varphi(\lambda) - \varphi(\mu)\| \leq \text{const.}|\lambda - \mu|^\alpha$ with $\alpha > \frac{1}{2}$. Then for $X \in \mathcal{B}(\mathfrak{H})$, $T_{\pi,t}(X)$ and $\mathcal{L}_\pi(X)$ converge in norm to $T_t(X)$, the expectation semigroup of $U(t)$, and $\mathcal{L}(X)$ respectively, where

$$(3.6) \quad T_t(X) = \int_{\Lambda} \int_{\Lambda} \exp\left[-\frac{t}{2}\|\varphi(\lambda) - \varphi(\mu)\|^2 + it \operatorname{Im}(\varphi(\lambda), \varphi(\mu))\right] P(d\lambda)XP(d\mu),$$

and

$$(3.7) \quad \mathcal{L}(X) = \int_{\Lambda} \int_{\Lambda} \Lambda[(\varphi(\lambda), \varphi(\mu)) - \frac{1}{2}\|\varphi(\lambda)\|^2 - \frac{1}{2}\|\varphi(\mu)\|^2] P(d\lambda)XP(d\mu)$$

The integrals in (3.6) and (3.7) exist as double spectral integrals in the operator norm topology.

Next we need to investigate the large time limit of T_t or its pre-dual τ_t . Here we have only some partial results which form the next theorem.

Theorem 3.3: Assume furthermore that the observable B has purely absolutely continuous spectrum and let

$$(3.8) \quad \mathcal{E}(X) = \int \int \chi\{\lambda, \mu \in \Lambda | \varphi(\lambda) = \varphi(\mu)\} P(d\lambda)XP(d\mu)$$

for $X \in \mathcal{B}(\mathfrak{H})$, where $\chi\{\cdot\}$ is the characteristic function. Then, as $t \rightarrow \infty$,

(i) $T_t(X)$ converges to $\mathcal{E}(X)$ in $\mathcal{B}_2$ -norm for each $X \in \mathcal{B}_2(\mathfrak{H})$, the set of Hilbert-Schmidt operators in $\mathcal{H}$;

(ii) $\tau_t(\rho)$ converges to $\mathcal{E}(\rho)$ strongly for each $\rho \in \mathcal{B}_i(\mathfrak{H})$;

(iii) $\operatorname{tr} \rho T_t(X) \rightarrow \operatorname{Tr} \rho \mathcal{E}(X)$ for each $\rho \in \mathcal{B}_1(\mathfrak{H})$ and for X a closed operator in $\mathfrak{H}$ such that $|X|^{\frac{1}{2}}$ is locally smooth relative to B (i.e., $\| |X|^{\frac{1}{2}} P(\Delta) |X|^{\frac{1}{2}} \| \leq \text{const.}$ Lebesgue measure of Δ) (see [12]).

Proof: It suffices to show that in each of the three cases we have a complex measure involved and then apply the dominated convergence theorem. This also means that

$\mathcal{E}(X)$ is well defined in each case. In (i), we note that $X \rightarrow P(\cdot)XP(\cdot)$ defines a spectral measure on $[a, b] \times [a, b]$ in the Hilbert space $\mathcal{B}_2(\mathfrak{H})$. In the expression for $\|T_i(X)u - \mathcal{E}(X)u\|^2$ in (ii), one has

$$\begin{aligned} & \sum_{ijk} \|(P(\Delta_i)XP(\Delta_j)u, XP(\Delta_k)u)\| \\ & \leq \sum_i \| |X|^{\frac{1}{2}} P(\Delta_i) |X|^{\frac{1}{2}} \|_1 (\sum_j \| |X|^{\frac{1}{2}} P(\Delta_j) u \|^2) \\ & \leq \|X\|_1^2 \|u\|^2. \end{aligned}$$

A very similar computation is valid for case (iii). ■

Remarks:

(i) In case (iii) of Theorem 3.3, one can have for example (a) B = Hamiltonian of a free non-relativistic particle and X = projection operator for the position observable in a fixed bounded set in $\mathbb{R}^3$ or (b) B = momentum operator while X is the same as in (a).

(ii) It is interesting to note a few obvious properties of the expectation map $\mathcal{E}$ as defined in (3.8). If we define $\mathcal{E}^\Delta(X)$ for $\Delta \subset \Lambda$ as $\mathcal{E}^\Delta(X) = \int \int \chi\{\lambda, \mu \in \Delta\} \varphi(\lambda) = \varphi(\mu)\} P(d\lambda)XP(d\mu)$, then

(a) $\mathcal{E}^\Delta(X) = P(\Delta)\mathcal{E}(X)P(\Delta)$ and $\mathcal{E}^{\Delta_1 \cup \Delta_2}(X) \neq \mathcal{E}^{\Delta_1}(X) + \mathcal{E}^{\Delta_2}(X)$ if Δ_1 and Δ_2 are disjoint,

(b) $\mathcal{E}$ is an idempotent, i.e., $\mathcal{E}^2(X) = \mathcal{E}(X)$

(c) $\mathcal{E}$ is positive linear,

(d) $\text{Tr } \mathcal{E}(\rho) = \text{Tr } \rho \quad \forall \rho \in \mathcal{B}_1(\mathfrak{H})$.

4 Applications

If we write $\varphi(\lambda) = \sum m_i(\lambda)\varphi_i$, where $\{\varphi_i\}$ is an orthonormal basis in $\mathfrak{H}$ chosen so as to represent an exhaustive catalogue of various possibilities of measurement of B by the apparatus, then $\varphi(\lambda) = \varphi(\mu)$ iff $m_i(\lambda) = m_i(\mu)$ for each i . For simplicity, if we choose only one m_i (and write just m for it) to be non-zero (which means that our apparatus has only one window of measurement) and if we furthermore choose the function $m \in C_0^\infty(\mathbb{R})$ with $0 \leq m(\lambda) \leq 1$ and $m(\lambda) = 1$ for $|\lambda| \leq \frac{\delta}{2}$, then it is clear that $m(\lambda) = m(\mu)$ if and only if $\lambda, \mu \in [-\frac{\delta}{2}, \frac{\delta}{2}]$ except for sets of Lebesgue measure zero. In such a case, the collapsed state is given by

$$(4.1) \quad \mathcal{E}(\rho) = P(\Delta)\rho P(\Delta),$$

where $\Delta = [-\frac{\delta}{2}, \frac{\delta}{2}]$. In the more general situation we shall have

$$(4.2) \quad \mathcal{E}(\rho) = \sum_i P(\Delta_i)\rho P(\Delta_i),$$

where Δ_i is the 'window' of observation associated with the basis vector φ_i . Thus the collapse map $\mathcal{E}$ associated with the measurement of an observable with continuous spectrum depends on the precision involved in the measurement process, quite

unlike that for an observable with pure point spectrum. This is essentially because of the assumption implicit in the model that eigenvalues can be measured with absolute precision.

Note: A plain $\text{\LaTeX}$ source file for this article may be obtained by e-mail from the author: `kbs@isid.ernet.in`

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LANGUAGE AND NATURE IN TWENTIETH CENTURY PHYSICS: AN OUTLINE

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I My “qualifications”

- A. I am not a physicist, a mathematician, or a mathematical physicist.
- B. I am not even a “philosopher of physics.”
 - 1. These are people who analyze the concepts of contemporary physics, occasionally critically.
 - 2. Or, they give a philosophical interpretation of modern physical theory.
 - 3. Physicists are often impatient with this sort of “philosopher.”
 - 4. They don’t like non-physicists telling them that a successful theory is logically flawed, etc.
- C. My lecture is not really about modern physics, or about the Universe, nor am I concerned to tell anybody what the meaning of modern physics is. I am trying to learn what the success of physical inquiry tells us about the human race, what used to be called “Man.”
 - 1. Unfortunately, to get clear about this, I have had to struggle with some of the details of actual physical theories to a greater degree than I would have wished, or wished upon my friends.
 - 2. For example, there are people in this room who think that those who don’t understand something called “symplectic geometry” don’t have a clue as to what’s going on the Universe.
- D. My views are not even those of mainstream philosophy, so don’t use this lecture as a proof that “philosophy” is of no use.

II This Lecture

- A. This lecture is about an old philosophical problem, which has tended to be forgotten in the analytical turn of contemporary philosophy: What is the place of Mind in the Universe? And, in particular, what does the success of modern scientific discovery tell us about our place in the Universe?
 - 1. I am not assuming that the theories of modern physics are true.
 - 2. The only thing I am assuming about modern physics is that it has been successful in predicting the results of certain experiments to an astonishing accuracy.
 - a. A successful theory need not be true.
 - b. Quantum electrodynamics is staggeringly successful, but is widely believed to be inconsistent; hence, *a fortiori*, false.

III The Crisis

- A. At the end of the 19th Century, there was confusion about the progress of science.
- B. Physics was attempting to describe the unobservable even in principle.
- C. Some physicists (Mach, Duhem are the ones known to philosophers) rejected the idea of the atom for this reason.
- D. Others simply worried that the human faculties are not adequate to task.
- E. Charles Peirce put the matter as follows (Peirce 1958, 7.506):
 - 1. Physics progresses by a series of good (recall, I don't use the word "true") guesses which can be checked later by experiment and observation.
 - 2. Since guessing at random is futile, physics can progress only if *homo sapiens* has an innate ability to make good guesses.
 - 3. The guessing facility is a product of Natural Selection.
 - 4. Natural Selection is a process controlled by laws on the scale of the human body itself.
 - 5. These laws are compatible with indefinitely many atomic laws.
 - 6. Hence there is no natural faculty of guessing the truth about the atom.
- F. Of course, we know that this obstacle was overcome. How did scientists do it? How did they guess the laws of the atom and the subatomic laws?
- G. My answer is: by analogy to laws that "worked" (again, I don't necessarily mean "true"): either classical laws; or, once some of the laws were in place, to already known atomic and subatomic laws.

IV Analogies

- A. Any two things bear some analogy and some disanalogy.
- B. Every analogy presupposes a standard of relevant similarity, a classification, a taxonomy.
- C. Note that the theories of science also presuppose taxonomies—according to species in biology, the periodic table of elements in chemistry, the various schemes to classify elementary particles.
- D. Consider the hypothesis that, beginning 1840, and continuing every 20 years, the President of the United States then elected dies when in office. This hypothesis is rejected by rational people, not because it has no confirmation (it worked 7 times), but because it is not even a candidate. The categories it works with are improper.
- E. What is “improper” is itself a function of higher level, or background hypotheses, accepted by the scientific community.
- F. The major reason for disqualifying the hypothesis (IV.D) from consideration, in my opinion, is that it is anthropocentric:
 - 1. The concept “president” has to do with human institutions and goals.
 - 2. What year it is, or whether it is divisible by 20 is the result of human conventions.
- G. Similarly for astrology. The real reason why astrology is rejected is because the concepts are anthropocentric.
 - 1. Why should there be pictures of earthly things in the sky?
 - 2. These very pictures would disappear in a different reference frame (say, in a different galaxy).
 - 3. This is the real reason why astrology is rejected, not because of empirical refutation.
- H. In sum, anthropocentric categories are rejected in science, as not expressing valid analogies.

V Anthropocentric Classifications (not “anthropic”)

- A. Anthropocentric classifications were once considered reasonable in physics—but Copernicus, Darwin, Freud shook this idea.
- B. Of course it is true that physics is a human invention, since it is a product of human discourse.

- C. Nevertheless, human language is capable of expressing concepts that are not anthropocentric.
1. Newtonian mechanics would perhaps not exist if not for the human race; but Newtonian mechanics refers to things that would exist without problems without us, such as the gravitational attraction of the planets by the sun.
 2. On the other hand, Aristotelian mechanics, which accords a special function to the place where human beings happen to reside, is considered anthropocentric.
- D. One of the most anthropocentric types of classification is that based on language. There are two kinds of classification based on language:
1. Semantical Classification
 - a. This embodies the idea that the concepts expressed by a given language (Greek, Hebrew) mirror the "natural kinds" of reality.
 - b. This is a standard medieval idea.
 2. Syntactical Classification
 - a. The syntactical categories of language mirror reality.
 - b. This is magical thinking, considered today even more primitive than the other form.
 - c. An Example
 - 1) A palindrome is a sentence spelled the same forward and backward ("Madam, I'm Adam").
 - 2) Sentences like this played a role in ancient magic.
 - 3) To think, for example, that palindromes are more likely to be true is certainly magical thinking.
 3. John Locke "secularized" language.
 - a. He argued that the categories of thought, as expressed in language, are mere "nominal essences."
 - b. Language is basically a conventional tool to express ideas.
 - c. So even the semantic classification of a natural language need not come anywhere near the "real" classification of things.
 - d. One need not work hard to think of what he would say about magic.

VI Mathematical Language

- A. Mathematicians have their own taxonomy, their own classification, internal to mathematics.
- B. The mathematical taxonomy is grounded on the semantics of mathematical language and also on its syntax.

1. The semantic categories are the "models" of the various mathematical postulates: groups, fields, and the other structures.
 2. The syntactic categories are the rules by which we form mathematical expressions, combine them, manipulate them—without regard to their meaning. (Most high school students regard mathematics in this way.)
 - a. Consider Euler's manipulation of divergent series, or today's use of the Feynman path integral for which nobody has a model as a legitimate mathematical structure.
 3. It seems that the term "algebra" has these two components. Ideas like "left identity," "free group," etc., seem to belong to the syntax of expressions. What is called "laws of commutation" refer both to mathematical operations and syntactical transformation of expressions. "Substitution" expresses functional composition or a syntactical operation of replacement.
- C. I would now like to maintain that the classification of the various mathematical concepts: semantic (i.e., structures) and syntactic are internal to mathematics, serve the needs of mathematicians primarily, and, insofar, are anthropocentric. Mac Lane has a nice survey of the genesis of (certain) mathematical concepts as an outcome of mathematical activity itself.

1. Semantical Classification

- a. I would argue, with Wigner, von Neumann, Hardy, and many others, that mathematical concept formation serves an aesthetic function.
 - 1) Many of the concepts of mathematics have an intrinsic beauty.
 - 2) What is more important, though, is that the concepts of mathematics must lend themselves to beautiful proofs.
- b. But I would also argue that aesthetic properties are based on human preferences.
- c. I would add that many mathematical ideas are/were introduced for calculational convenience:
 - 1) Complex wave functions were originally introduced because it is easier to work with the exponential function than with trigonometric functions.
 - 2) The (scalar) potential function is more convenient to work with than the (vector) field, which can always be retrieved later by differentiation.
 - 3) If our brains had been larger, we might have had no need for these concepts.
- d. Hence, in either case (aesthetic or convenience) the choice of concepts as "mathematical" is anthropocentric.
- e. Note that the mathematician chooses not only concepts, but families of concepts. Thus, for example, $U(1)$, $SU(2)$, $SU(3)$, are all in the same mathematical category. The choice of grouping the structures together is anthropocentric, either because the criterion is aesthetic or convenience, or because of notational motivation, as we now discuss.

2. Syntactic Classification

- a. The syntax of every human symbol system is largely anthropocentric.
- b. In the case of mathematics, we have various rules of symbol combination, etc.
- c. These rules often precede the interpretation of the rules.
- d. Analogy, convenience, calculation are the motivation.
 - 1) An example is the imaginary numbers. The symbols for these were introduced before the interpretation was given hundreds of years later. Cardano used them to calculate even the real roots of a cubic equation over the reals.
- e. Even when a symbol system has a standard interpretation, its syntax may have a "life of its own."
 - 1) Matrices of imaginary (complex) numbers—as symbols—may have geometrical symmetries even though the matrices themselves do not denote geometrical symmetries.
 - 2) Maxwell's equations—considered as symbols—have a symmetrical look which can be appreciated by the layman.

3. The difference between mathematics and games

- a. Games (e.g., chess) also induce anthropocentric classifications, but it does not follow that mathematical theories are games.
 - 1) Games like chess serve other human functions than does mathematics.
 - 2) Thus concept formation in the field of games follows different criteria than in the field of mathematics.
 - 3) Games of competition, for example, can be seen as substitutes for war (of course I am not trying to be comprehensive).
 - 4) In the modern age, however, the invention of concepts in mathematics is dominated by aesthetic criteria, as the great mathematicians testify.
- b. Nor is every game a mathematical theory.
 - 1) Of course it is true that every game can be represented by an axiomatic theory of a structure.
 - 2) It would be a theorem of (the first-order theory of) chess, for example, that checkmate can be forced in this or that situation.
 - 3) I believe, however, that theorems of that type would not be regarded as mathematics and would not be published in mathematical journals.
 - a) I'm not speaking here, of course, about the (second-order) theory of games in general, which is a mathematical theory.
 - 4) As before (cf. 4), aesthetic criteria would prevail.

VII Pythagoreanism and Formalism

- A. Often enough, a mathematical classification has a known physical correlate.
 - 1. I am assuming, of course, that there is such a thing as a "physical" concept.
 - 2. These have to do with mechanisms, cause and effect, etc.
- B. Some examples:
 - 1. Linearity
 - a. Linearity is associated with the principle of superposition.
 - b. This is the physical principle that causes operate independently of one another.
 - 2. Covariance
 - a. This requirement can be formulated as a mathematical principle in differential geometry.
 - b. But, according to Einstein, it codifies the physical principle that space is not a cause.
- C. It happens that a mathematical classification at first has no known physical correlate, but one is discovered later.
 - 1. An example of this is the relation between the Lie Algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$, where the relation is given by the theory of quarks.
 - 2. On the other hand, that both the spin of the electron and isospin of the nucleon are governed by $\mathfrak{su}(2)$ has no physical underpinning. This could either be an "accident" or an unexplained fact.
- D. Definition: a Pythagorean classification relative to time t is one which can, at time t , be expressed only in mathematical language, i.e., has no known physical basis.
- E. A formalist classification (also relative to time) is one based on the syntax of the mathematical formalism.
 - 1. Thus a formalist classification is a special case of Pythagorean.
- F. In light of VI.C, both Pythagorean and formalist classifications are anthropocentric.

VIII Physical Discovery—My Claim

- A. Physical discovery in our century was dominated by Pythagorean and even formalist classifications.

1. For formalist classifications, cf. Thom in Tymoczko 1986: "Any mathematician endowed with a modicum of intellectual honesty will recognize that in each of his proofs he is capable of giving a meaning to the symbols he uses. Because of this, his work differs from that of the theoretical physicist, who very frequently does not hesitate to put his trust magically in the virtues of blind formalism in the hope (often deceived) that the light at the end of the tunnel will dispell the intervening darkness."
- B. Thus, anthropocentric considerations were at the heart of contemporary physics.
- C. The content of the theories themselves was not, presumably, anthropocentric.
- D. Also, "naturalist" scientists and philosophers reject anthropomorphism in all forms.
- E. Hence, there is a conflict between the content of the theories and the mode of their discovery; to put it a more provocative way, there is a connection between the human mind and the Universe, not apparently explicable by Natural Selection, that makes physical discovery possible.
- F. I cannot do more here than give a couple of examples in support of this thesis.

IX Two Examples

A. Dirac's equation for the electron

1. Schrödinger and the Klein-Gordon equation

- a. Schrödinger noticed that there is a formal relation of substitution between the classical energy equation and his own equation: you substitute differential operations for magnitudes.
- b. "Doing the same thing" in the Einstein equation gives a relativistic equation, the one known as the Klein-Gordon equation, though Schrödinger published it first.
- c. We have already noted that the concept "doing the same thing" presupposes a theory of classification, perhaps covert.
- d. As far as Schrödinger was concerned, the Einstein energy equation and the Hamiltonian energy equation were two different mathematical structures, and he knew no physical justification for his procedure.
 - 1) Actually, mathematical considerations show that the number of quantum equations obeying relativity is small.
 - 2) But Schrödinger, of course, was unaware of this.
 - 3) It very often happens, in the history of both mathematics and physics, that what seemed to be a free invention was in fact forced.
 - 4) This fact, rather than explaining the mystery, is a special case of it.
- e. Thus, Schrödinger's procedure was either "Pythagorean" or "formalist" in the sense of VII.D and VII.E. (I won't decide which here.)

- 1) Schrödinger himself calls this "derivation" purely formal in (Schrödinger 1978).
 - 2) But we would have to explore what he means by this.
2. Dirac: from the KGE to the electron equation
- a. Dirac wanted a linear equation. His reasons themselves do not seem too physical: he wants a single unit vector (not two vectors) to give the state, just as in the Schrödinger equation.
 - 1) This makes quantum mechanics explicable only in terms of Hilbert spaces.
 - 2) The concept of a Hilbert space does not correspond to a physical intuition the way linearity corresponds to superposition.
 - b. But in any case, he factors the Einstein equation formally using non-standard "numbers" which obey queer rules of "arithmetic."
 - c. A representation of these rules, though, he found using 4×4 matrices.
 - d. Thus, Dirac "factored" the Klein-Gordon equation using matrices as coefficients.
 - e. But this meant that the variable had to be a 4×1 matrix, so that Dirac's equation gives four kinds of solutions, for four kinds of electrons.
 - f. Whatever link to physical intuition remained in this was snapped.
 - g. Later the positron was discovered as predicted by this equation. Thus the four solutions are electron with spin up or down, positron with spin up or down.

B. Yang-Mills

1. I follow the original treatment in Yang and Mills 1954 as well as the retrospective Mills 1989.
2. Weyl discovered that there is a connection between three concepts: invariance under local change of phase, invariance under change of gauge, Maxwell's equations (Cf. Weyl 1950). This connection can be used to generate the quantum theory of the electromagnetic field by a pure symmetry argument:
 - a. Consider a free quantum particle. Multiplying its state by a phase factor leaves the physics unaltered.
 - b. Mathematically this is called global $U(1)$ symmetry, and it is trivial.
 - c. The next step is to assume that $U(1)$ symmetry is local—i.e., we can multiply by a different phase factor at every point in space.
 - d. But changing the phase factors will introduce interference effects which amount to changing other physical properties like momentum. Where does the momentum come from?
 - e. Thus, invariance under this local symmetry requires a compensating potential field, whose physical action is invariant under change of gauge (which can be interpreted as a change of phase).
 - f. If we require also Lorentz invariance from the potential field we get Maxwell's equations for these potentials.

- g. To quantize the electromagnetic field (photons), we use a standard "recipe" for getting "commutation relations" or rather a continuum of commutation relations.
3. Yang and Mills decided to generalize this procedure for deriving quantized field equation to a "related" group, namely $SU(2)$ [the group of 2×2 complex matrices with determinant $+1$]. This was a known global symmetry of the nuclear field.
- a. Starting with a free nuclear particle, they stipulated that the $SU(2)$ symmetry should be local.
 - b. As before, this requires an external potential, gauge invariant (here the gauge is two-dimensional).
 - c. The potential obeys a classical differential equation, but (unlike Maxwell's equation) it is non-linear, with the major physical differences this implies.
 - d. The non-linearity is related to a major difference between $U(1)$ and $SU(2)$: $SU(2)$ is a nonabelian group. $U(1)$, meaning simply multiplying by a complex number $\exp(ix)$, is of course abelian.
 - e. The last step is, of course, "quantizing" the non-linear field equation.
 - f. Yang and Mills did not actually arrive at the correct theory of the nuclear field by this procedure.
 - g. Nevertheless, physicists were eventually persuaded that the procedure they used is correct.
 - 1) For example, they should have used the group $SU(3)$ in their argument.
 - h. Namely, the way to construct a physical theory is to begin with a global symmetry group, making it local, find "classical field equations" for the local symmetry, and then quantize them.
 - i. Hence in what follows, I shall disregard the failure of the actual gauge group $SU(2)$, and, for simplicity, write as if Yang and Mills had actually succeeded in writing down the quantized field theory for the atomic nucleus.
4. Here are the remarkable aspects of the story:
- a. The relation between Maxwell and YM equations is unintelligible without the above story which begins with the Hilbert space formalism of quantum mechanics.
 - 1) These are two different kinds of equations (one is linear, one not), related only in that the linear is a limiting form of the nonlinear.
 - 2) To put the matter another way, Maxwell (without the benefit of the quantum-mechanical formalism) would not have seen the analogy.
 - 3) Had Maxwell—or Yang—used the concept of a fiber bundle he would not have needed quantum mechanics, but Yang insists adamantly (in several places in the comments to his collected papers (Yang 1983)) that this notion played no role in the discovery.

- 4) He even claims that he was incapable of learning this concept even years after the paper was published.
 - 5) It was only in Yang 1977 that the physical meaning of the concept of a fiber bundle became clear to him as a physicist.
- b. The idea that “global symmetries should be local” was a Pythagorean analogy between one symmetry and another.
- 1) I rely here on Mills himself, who argued (Mills 1989) that there was no physical reasoning behind it, but was suggested formally by one or two examples.
 - 2) It is possible, though, that Yang had a different point of view, since Yang and Mills 1954 contends that given “the local field concept” one should be allowed to decide, at each point in spacetime, what counts as a proton and what as a neutron. (I.e., that the $SU(2)$ symmetry should be local.)
 - 3) This argument would be analogous to Einstein’s argument that at each point in spacetime one can decide whether we are in a gravitational field or not.
 - 4) Note, though, that if the quantum field concept itself is only a mathematical abstraction—not corresponding to the intuitive classical notion of a field—the argument above (IX.B.4.b.2) would be just as Pythagorean as Mills (IX.B.4.b).
 - 5) And, at least with respect to the electron (fermionic) fields, the analogy to the physical (causal) concept of a field seems largely mathematical.
- c. The YM “classical” equation is purely hypothetical, since nobody can detect the nuclear field as a “classical” type field. (Quark theory explains why.) It exists only to be quantized away. Mills himself argues that we ought to be able today to get rid of the intermediate classical field.
- d. The commutation rules amount to infinitely many operator equations. The experts in the field tell me that even today it is not clear whether these equations are consistent. To me this suggests a formalist analogy.
- e. Indeed, the entire idea of quantization, which is an assumption of a recipe to go from “false” classical equations to “true” quantum equations seems to be formalist. We do the “same thing” to every classical equation regardless.

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