Quasi periodic Hamiltonian Motions, Scale Invariance, Harmonic Oscillators

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Abstract

The work of Kolmogorov, Arnold and Moser appeared just before the renormalization group approach to statistical mechanics was proposed by [1]: it can be classified as a multiscale approach which also appeared in works on the convergence of Fourier's series, [2, 3], or construction of Euclidean quantum fields, [4], or the scaling analysis of the short scale behaviour of Navier-Stokes fluids, [5], to name a few which originated a great variety of further problems. In this review the KAM theorem proof will be presented as a classical renormalization problem with the harmonic oscillator as a "trivial" fixed point.

1 Introduction

The KAM theorem can be regarded as a multiscale analysis of the stability of the harmonic oscillator viewed as a fixed point of a transformation which enlarges a region of phase space focused around a nonresonant quasi periodic motion. The problem considers a Hamiltonian

$$H_0(\mathbf{A},\boldsymbol{\alpha}) = \frac{1}{2}(\mathbf{A}\cdot J_0\mathbf{A}) + \boldsymbol{\omega}_0\cdot\mathbf{A} + f_0(\mathbf{A},\boldsymbol{\alpha}) \equiv h_0 + f_0 \tag{1.1}$$

real analytic for $(\mathbf{A},\boldsymbol{\alpha}) \in (\mathcal{D}_\varrho \times \mathcal{T}^\ell)$ with: $\mathcal{D}_\varrho = \{\mathbf{A} \in R^\ell, |A_j| < \varrho\}$, $\mathcal{T}^\ell$ the ℓ-dimensional torus $[0,2\pi]^\ell$ identified with unit circle $\{\mathbf{z}|\, z_j = e^{i\alpha_j}, j = 1,\ldots,\ell\}$, $\boldsymbol{\omega}_0 \in R^\ell$ and J_0 is a $\ell\times\ell$ positive matrix with eigenvalues $J_0^+ \ge J_0 \ge J_0^- > 0$.

The Hamiltonian is supposed holomorphic in the complex region $\mathcal{C}_{\varrho_0,\kappa_0}$ with size of the perturbation f_0 measured by ε_0:

$$\begin{aligned}
&\mathcal{C}_{\varrho_0,\kappa_0} \stackrel{def}{=} \{(\mathbf{A},\mathbf{z})| |A_j| \le \varrho_0,\ e^{-\kappa_0} \le |e^{i\alpha_j}| \le e^{\kappa_0}, j = 1,\ldots,\ell\} \subset \mathcal{C}^{2\ell}\\
&\varepsilon_0 = ||\boldsymbol{\partial}_{\mathbf{A}} f_0||_{\varrho_0,\kappa_0} + \frac{1}{\varrho_0}||\boldsymbol{\partial}_{\boldsymbol{\alpha}} f_0||_{\varrho_0,\kappa_0}, \qquad \text{with :}\\
&||f||_{\varrho_0,\kappa_0} \stackrel{def}{=} \max_{\mathcal{C}_{\varrho_0,\kappa_0}} |f(\mathbf{A},\mathbf{z})|, \qquad \forall f \text{ holomorphic in } \mathcal{C}_{\varrho_0,\kappa_0}
\end{aligned} \tag{1.2}$$

with $\varrho_0 > 0, \kappa_0 > 0$, $z_j \equiv e^{i\alpha_j}$; generally $\mathcal{C}_{\varrho,\kappa}(\overline{\mathbf{A}})$ will denote a polydisk centered at $\overline{\mathbf{A}}$, *i.e.* defined as in Eq.(1.2) with $|A_j - \overline{A}_j| \le \varrho$ replacing $|A_j| \le \varrho$ and $e^{-\kappa} \le |z_j| \le e^{\kappa}$; polydisks centered at the "origin" will be simply denoted $\mathcal{C}_{\varrho,\kappa}$ and called "centered polydisks".

It is supposed, no loss of generality, that the $\boldsymbol{\alpha}$-average $\overline{f}_0(\mathbf{A})$ of $f_0(\mathbf{A},\boldsymbol{\alpha})$ vanishes at $\mathbf{A}=\mathbf{0}$.

Set $|\mathbf{A}| = \max|A_j|, |\mathbf{z}| = \max|z_j|,\ \forall \mathbf{A}, \mathbf{z} \in \mathcal{C}^\ell$.

The idea is to focus attention on the center of $\mathcal{C}_{\varrho_0,\kappa_0}$ where, if $\varepsilon_0 = 0$, a motion ("free motion") takes place which is quasi periodic "with spectrum" $\boldsymbol{\omega}_0$. This is done by changing variables in a small polydisk $C_{\widetilde{\varrho},\widetilde{\kappa}}(\mathbf{a}) \subset C_{\varrho_0,\kappa_0}$, eccentric if $\mathbf{a} \neq \mathbf{0}$, that is then recentered and enlarged back to the original size so that it contains $\mathcal{C}_{\varrho_0,\kappa'_0}$ with $\kappa'_0 > \frac{1}{2}\kappa_0$.

The motions developing in the initial polydisk can be studied as "through a microscope": in the good cases (*i.e.* under suitable assumption on the initial parameters $J_0, \boldsymbol{\omega}_0$ and f_0) the Hamiltonian will turn out to be substantially closer to that of a harmonic oscillator (described by its "normal" Hamiltonian $\boldsymbol{\omega}_0 \cdot \mathbf{A}$ in the variables $\mathbf{A}, \boldsymbol{\alpha}$).

Iterating the process the Hamiltonian changes but, *remaining analytic in the same polydisk* $C_{\varrho_0,\frac{1}{2}\kappa_0}$, converges to that of a harmonic oscillator: the interpretation will be that, looking very carefully in the vicinity of the torus $\mathcal{T}_{\boldsymbol{\omega}_0} = \{\mathbf{A}=\mathbf{0}, \boldsymbol{\alpha} \in [0,2\pi]^\ell\}$, also the perturbed Hamiltonian exhibits a harmonic motion with spectrum $\boldsymbol{\omega}_0$: the result, proved below, is the KAM theorem.

This is not only reminiscent of the methods called "renormalization group", RG, in quantum field theory but in this review it will be shown to be just a realization of them, [6] to more recent views on the RG.

2 A formal coordinate change

The Hamiltonian Eq.(1.1), considered as a holomorphic function on a domain $\mathcal{C}_{\varrho_0,\kappa_0}$ (Eq.(1.2)), will be denoted $H_0 = h_0 + f_0$. The label 0 is attached since the beginning because $H_n, f_n, \varrho_n, \kappa_n$ will arise later with $n = 1, 2, \ldots$.

The frequency spectrum $\boldsymbol{\omega}_0$ will be supposed "Diophantine", *i.e.* for some $C_0 > 0$ it is, denoting $\mathcal{Z}^\ell$ is the lattice of the integers, for all $\mathbf{0} \neq \boldsymbol{\nu} \in \mathcal{Z}^\ell$:

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}|^{-1} < C_0\,|\boldsymbol{\nu}|^\ell, \qquad |\boldsymbol{\nu}| \equiv \sum_{i=1}^{\ell} |\nu_i| > 0 \tag{2.1}$$

and the latter inequality will be repeatedly used to define canonical transformations with generating functions of the form $\Phi(\mathbf{A}',\boldsymbol{\alpha}) + (\mathbf{A}'+\mathbf{a})\cdot\boldsymbol{\alpha}$:

$$\mathbf{A} = \mathbf{A}' + \mathbf{a} + \boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi(\mathbf{A}',\boldsymbol{\alpha}), \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha} + \boldsymbol{\partial}_{\mathbf{A}'}\Phi(\mathbf{A}',\boldsymbol{\alpha}) \tag{2.2}$$

with the function Φ chosen so that in the new coordinates $(\mathbf{A}',\boldsymbol{\alpha}')$ the perturbation is *weaker*, at the price that the new coordinates will cover a (much) smaller domain, inside the $\mathcal{D}_\varrho \times \mathcal{T}^\ell$.

To simplify the notations the functions of $\boldsymbol{\alpha}$ will always be implicitly regarded as functions of $z_j = e^{i\alpha_j}$ whenever referring to their holomorphy properties and, without further comments, their arguments will be written as $\mathbf{z}$ or $\boldsymbol{\alpha}$, as convenient.

At first the natural choice for Φ, *temporarily* forgetting the determination of the domain of definition of the transformation, would be

$$\begin{aligned}\Phi(\mathbf{A}',\boldsymbol{\alpha}) = & -\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{Z}^\ell} \frac{f_{0,\boldsymbol{\nu}}(\mathbf{A}'+\mathbf{a})}{i(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}+((\mathbf{A}'+\mathbf{a})\cdot J_0\boldsymbol{\nu})} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}}\\ \mathbf{a} = & -J_0^{-1}\boldsymbol{\partial}_{\mathbf{a}}\overline{f}_0(\mathbf{a})\end{aligned} \tag{2.3}$$

where $f_{0,\boldsymbol{\nu}}(\mathbf{A})$ is Fourier's transform of $f_0(\mathbf{A},\boldsymbol{\alpha})$, and $\overline{f}_0(\mathbf{A}')$ denotes the average of $f_0(\mathbf{A}',\boldsymbol{\alpha})$ over $\boldsymbol{\alpha}$.

Then inserting Eq.(2.2) into H_0 the Hamiltonian is transformed, setting $J_1 = J_0 + \boldsymbol{\partial}^2_{\mathbf{a}} f_0(\mathbf{a})$, into

$$\begin{aligned}
(0): &\quad H'(\mathbf{A}',\boldsymbol{\alpha}') = \frac{1}{2}(\mathbf{A}'\cdot J_1\mathbf{A}') + \boldsymbol{\omega}_0\cdot\mathbf{A}'\\
(1): &\quad + (\boldsymbol{\omega}_0 + J_0(\mathbf{A}'+\mathbf{a}))\cdot\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi + f_0(\mathbf{A}'+\mathbf{a},\boldsymbol{\alpha}) - \overline{f}_{\mathbf{0}}(\mathbf{A}'+\mathbf{a})\\
(2): &\quad + \overline{f}_0(\mathbf{A}'+\mathbf{a}) - \overline{f}_0(\mathbf{a}) - \boldsymbol{\partial}_{\mathbf{a}}\overline{f}_0(\mathbf{a})\cdot\mathbf{A}' - \frac{1}{2}\boldsymbol{\partial}^2_{\mathbf{a}} f_0(\mathbf{a})\mathbf{A}'\mathbf{A}'\\
(3): &\quad + (f_0(\mathbf{A}'+\mathbf{a}+\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi,\boldsymbol{\alpha}) - f_0(\mathbf{A}'+\mathbf{a},\boldsymbol{\alpha})) + \frac{1}{2}\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi\cdot J_0\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi
\end{aligned} \tag{2.4}$$

where the second of Eq.(2.3) has been used and a few terms have been added or subtracted (including free addition or subtraction of constants) so that:

- (0) The integrable part of the Hamiltonian,
- (1) This term vanishes if Φ is defined via Eq.(2.3);
- (2) The term is of $O(\varepsilon_0(\mathbf{A}')^3)$, hence it is a higher order term if $|\mathbf{A}'|$ is small enough.
- (3) The two terms are *formally* of higher order in the size ε_0 of f_0.

In a domain in which the transformation Eq.(2.2) could be defined, the motions would be described by a still integrable and quadratic Hamiltonian plus a perturbation of higher order in ε_0.

However to make sense of the transformation in Eq.(2.2) it is not only necessary to restrict the variables $(\mathbf{A}',\boldsymbol{\alpha})$ to a smaller domain, but it has to be possible to solve the implicit functions problem in Eq.(2.2),(2.3) (namely to express $(\mathbf{A},\boldsymbol{\alpha})$ in terms of $(\mathbf{A}',\boldsymbol{\alpha}')$ and viceversa, and finding $\mathbf{a}$), but also the denominator in Eq.(2.3) will have to be modified to avoid dividing by 0: which will happen, for generic f_0 and for some $\boldsymbol{\nu}$, on a dense set of $\mathbf{A}'\in\mathcal{D}_{\varrho_0}$, if J_0 is not singular (as it is being supposed). Therefore the map in Eq.(2.2) will now be modified and defined properly after recalling the notion of dimensional estimate.

3 Dimensional estimates

The very nature of the stability of quasi periodic motions is that it is a multiscale problem: like many other problems in analysis, from the almost everywhere convergence of Fourier series of $L_2([0,2\pi])$-functions ([3]), to the study of the possible singularities of the Navier-Stokes problem ([5]), to the convergence of the functional integrals arising in quantum field theory ([7]), to name a few. The *renormalization group* method, [8, 9], unifies the approaches developed to study such problems.

The main feature of the renormalization group applications is their being based on what will be called here "*dimensional estimates*".

Dimensional estimates deal with elementary bounds on holomorphic functions. Any holomorphic function in a closed domain $C\subset\mathcal{C}$ (domain $\Rightarrow$ closure of an open connected set in the complex plane $\mathcal{C}$) can be bounded, toghether with its Taylor coefficients, in terms of $||g||_C=\max_{z\in C}|g(z)|$, inside the region C_δ consisting of the points in C at distance $\ge\delta$ from the boundary of C.

Here holomorphic functions g of ℓ or 2ℓ arguments will, in the following, be considered in domains

$$\begin{aligned}&\mathcal{C}_\varrho=\{\mathbf{A}||A_j|\le\varrho, j=1,..,\ell\},\quad \Gamma_\kappa=\{\mathbf{z}|e^{-\kappa}\le|z_j|\le e^\kappa, j=1,..,\ell\}\\&\mathcal{C}_{\varrho,\kappa}=\mathcal{C}_\varrho\times\Gamma_\kappa\end{aligned}\tag{3.1}$$

and their maxima will be denoted by appending labels ϱ or κ or ϱ,κ, as appropriate, to the symbol $||g||$. The functions of $\mathbf{z}$ will be naturally considered a periodic in the annulus by setting $z_j=e^{i\alpha_j}, |\mathrm{Im}\,\alpha_j|\le\kappa$.

For such a function g the following bound for the derivatives of g or, respectively, its Laurents coefficients $g_{\boldsymbol{\nu}}, \boldsymbol{\nu}\in Z^\ell$ (or Fouriers coefficients regarding it as a function of $z_j=e^{i\alpha_j}$), hold.

If $||g||_{\varrho,\kappa}=\varepsilon$ and $|\boldsymbol{\nu}|=\sum_{j=1}^\ell|\nu_j|$, $\boldsymbol{\nu}\in Z^\ell$:

$$\begin{aligned}&||g_{\boldsymbol{\nu}}||_\varrho\le\,\varepsilon\,e^{-\kappa|\boldsymbol{\nu}|}, &&\forall\boldsymbol{\nu}\in\mathcal{Z}^\ell,\mathbf{A}\in\mathcal{C}_{\varrho'}\\&||\boldsymbol{\partial}^n_{\mathbf{A}}g_{\boldsymbol{\nu}}||_{\varrho',\kappa}\le\,n!\,\varepsilon\,e^{-\kappa|\boldsymbol{\nu}|}(\varrho-\varrho')^{-n}, &&\forall\boldsymbol{\nu}\in\mathcal{Z}^\ell,\mathbf{A}\in\mathcal{C}_{\varrho'}\end{aligned}\tag{3.2}$$

hold and will be called *dimensional bounds.*

Summarizing: the dimensional bounds say that the n-th derivatives of a function holomorphic in a domain C are bounded, at a point z at distance δ from the boundary of C, by the maximum of the function in C divided by the n-th power of the distance of z to the boundary ∂C of C times $n!$ ("Cauchy's theorem").

In the following essentially all bounds will be "dimensional": and each new bound presented may contain some new constants labeled c_i,γ_i; such constants will only depend on the number of degrees of freedom ℓ and, for simplicity, will be chosen so that $c_i\le c_{i+1},\gamma_i\le\gamma_{i+1}$.

4 A canonical map

The "renormalization group" is a map $\mathcal{R}$ whose iterations can be interpreted as successive magnifications, zooming on ever smaller regions of phase space in which motions develop closer and closer to the searched quasi periodic motion of spectrum $\boldsymbol{\omega}_0$.

At step $n = 0, 1, \ldots$ the motions will be described by a Hamiltonian $H_n + f_n$ which will be the sum of three terms

$$\frac{1}{2}\mathbf{A}\cdot J_n\mathbf{A} + \boldsymbol{\omega}_0\cdot\mathbf{A} + f_n(\mathbf{A},\boldsymbol{\alpha}), \tag{4.1}$$

see Eq.(1.1). In the renormalization group nomenclature and *under the conditions Eq.(2.1) and* $J_0 > 0$ the first and third terms would be called "*irrelevant*" and the intermediate (*i.e.* the normal form for the ℓ-dimensional harmonic oscillators Hamiltonian) would be called a "*marginal trivial fixed point*": the reason behind the latter names will be be mentioned below.

Introducing the parameters $\varepsilon_n, J_n^\pm, \varrho_n, C_n, \kappa_n$, characterizing H_n in the same sense in which $\varepsilon_0, J_0^\pm, \varrho_0, C_0, \kappa_0$ characterize H_0 in Eq.(1.1), it is convenient, for the purpose of a rapid evalutation of several estimates, to keep in mind that the following "dimensionless" quantities,

$$\eta_n = \varepsilon_n C_n, \qquad \theta_n = \varepsilon_n (J_n^- \varrho_n)^{-1}, \qquad e^{\kappa_n} \tag{4.2}$$

will naturally occurr in the dimensional estimates: the latter will, therefore, be expressed as products of selected dimensionless quantities times a suitable factor chosen among the dimensional parameters $\varepsilon_n, \varrho_n, C_n, J_n^\pm$.

All bounds will be carefully written so that they will involve only dimensionless constants and, when needed, a factor to fix the dimensions. Furthermore the construction of the sequence H_n will be so designed that

$$C_n \equiv C_0, \varrho_n \ll \varrho_0, \kappa_n = \kappa_{n-1} - 4\delta_n > \frac{1}{2}\kappa_0 \tag{4.3}$$

with δ_n defined so that $\kappa_0 \geq \kappa_n \geq \frac{1}{2}\kappa_0$; to fix the ideas δ_n will be fixed as $\delta_n = (n+10)^{-2}\kappa_0$, the size ε_n of f_n will tend to 0 provided ε_0 is small enough, while the matrix J_n will remain $\simeq J_0$.

It will not be restrictive to suppose, *initially*:

$$C_0\varrho_0 J_0^+ < 1, \qquad 2^{-1} < e^{\frac{\kappa_0}{2}} \leq e^{\kappa_n} \leq e^{\kappa_0} < 2 \tag{4.4}$$

because the theorem will apply for ε_0 small enough and ϱ_0, κ_0 can be *initially* restricted as needed. Furthermore it is important to keep in mind that the bounds that follow are naive dimensional bounds derived *without any optimization attempt*, yet they will suffice for a complete proof.

To define properly a tranformation inspired by Eq.(2.2) and to eliminate the mentioned possible divisions by 0, *while still keeping H' in Eq.(2.4) formally close to* H_0 as in Sec.2, the first task is to determine the shift $\mathbf{a}$, Eq.(2.3).

The implicit equation Eq.(2.3) for $\mathbf{a}$, $\mathbf{a} = -J_0^{-1}\boldsymbol{\partial}_{\mathbf{a}}\overline{f}_0(\mathbf{a}) \stackrel{def}{=} \mathbf{a}_0 + \mathbf{n}(\mathbf{a})$, with $\mathbf{a}_0 = -J_0^{-1}\boldsymbol{\partial}_{\mathbf{a}}\overline{f}_0(\mathbf{0})$ can be solved under a smallness condition on ε_0 obtaining $\mathbf{a}$ close to $\mathbf{a}_0$.

This follows from an application of a general implicit function theorem yielding the existence of a constant χ such that the smallness condition $|\mathbf{n}|_{\varrho_0}\varrho_0^{-1} < \chi$ implies existence of a solution (as it bounds how close to 1 is the Jacobian of the equation). Then $|\mathbf{n}|_{\varrho_0}\varrho_0^{-1}$ is dimensionally bounded by $\varepsilon_0(J_0^-\varrho_0)^{-1} \stackrel{def}{=} \theta_0$, a condition for the solubility of the equation is:

$$\theta_0 = \varepsilon_0/(J_0^-\varrho_0) < \chi \qquad \Rightarrow \qquad |\mathbf{a}| < \theta_0\varrho_0 < \chi\varrho_0 < \frac{1}{16}\varrho_0 \tag{4.5}$$

The choice $\chi = \frac{1}{16}$ (implied in general by the estimate of χ reproduced for completeness in Appendix A below) is useful for the coming analysis (with no attention to an optimal χ-value).

The function $f_0(\mathbf{A}'+\mathbf{a},\boldsymbol{\alpha})$ will then be defined and analytic in $C_{\frac{3}{4}\varrho_0,\kappa_0}$ (from $\frac{3}{4}+\frac{1}{16} < 1$). Then proceed to build Φ, but replace Eq.(2.3) with its second order expansion in J_0:

$$\begin{aligned}\Phi_0(\mathbf{A}',\boldsymbol{\alpha}) = &-\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{Z}^\ell}\frac{f_{0,\boldsymbol{\nu}}(\mathbf{A}'+\mathbf{a})e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}}}{i\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\\ &\cdot\Big(1-\frac{J_0(\mathbf{A}'+\mathbf{a})\cdot\boldsymbol{\nu}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}+(\frac{J_0(\mathbf{A}'+\mathbf{a})\cdot\boldsymbol{\nu}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}})^2\Big)\end{aligned} \tag{4.6}$$

The function Φ_0 is well defined in the polydisk $\mathcal{C}_{\frac{3}{4}\varrho_0,\kappa_0-\delta_0}$ as seen via the following general dimensional bounds (given in Eq.(3.2) on functions bounded by ε_0 and holomorphic in a domain $\mathcal{C}_{\varrho_0,\kappa_0}$).

Given any $0 \le \delta_0 < \kappa_0$, taking into account the Diophantine inequality Eq.(2.1), the definitions Eq.(4.2),(4.3),(4.4), the dimensional inequality Eq.(3.2), with the restrictions Eq.(4.4), and $\eta_0 = \varepsilon_0 C_0$ and $J_0^+ C_0\varrho_0 < 1$, leads to:

$$\begin{aligned}||\Phi_0||_{\varrho_0,\kappa_0-\delta_0} \le&\varepsilon_0\varrho_0\sum_{\boldsymbol{\nu}\neq\mathbf{0}}\frac{e^{-\delta_0|\boldsymbol{\nu}|}}{|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|}(1+\frac{|J_0\boldsymbol{\nu}|\varrho_0}{|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|}\\ &+(\frac{|J_0\boldsymbol{\nu}|\varrho_0}{|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|})^2) \le \gamma_1\eta_0\varrho_0\,\delta_0^{-c_1}\\ |\Phi_{0,\boldsymbol{\nu}}(\mathbf{A}')| \le&\gamma_1\eta_0\,\varrho_0\,\delta_0^{-c_1}\,e^{-(\kappa_0-\delta_0)\boldsymbol{\nu}|}, \qquad \forall|\mathbf{A}'| < \varrho_0\end{aligned} \tag{4.7}$$

with γ_1, c_1 are dimensionless constants. The latter depend only on the number of degrees of freedom ℓ, *e.g.* $\gamma_1 = 3\,\Omega_\ell\,(4\ell)!$, Ω_ℓ =volume of the ℓ dimensional ball, and $c_1 = 4\ell+1$: so, if $\ell = 2$, the $c_1 = 9, \gamma_1 = 6\pi 8!$ are possible.

Hence the functions in the *r.h.s* of Eq.(2.2) admit the dimensional bounds:

$$\begin{aligned}&||\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi_0||_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0} \le \gamma_2\eta_0\,\varrho_0\delta_0^{-c_2},\ ||\boldsymbol{\partial}_{\mathbf{A}'}\Phi_0||_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0} \le \gamma_3\eta_0\,\delta_0^{-c_3}\\ &||\boldsymbol{\partial}^2_{\boldsymbol{\alpha}\mathbf{A}'}\Phi_0||_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0} \le \gamma_4\eta_0\,\delta_0^{-c_4},\ ||\boldsymbol{\partial}^2_{\mathbf{A}'\mathbf{A}'}\Phi_0||_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0} \le \gamma_5\eta_0\varrho_0^{-1}\delta_0^{-c_5}\\ &c_2 = c_1+1, \gamma_2 = (4\ell+1)\gamma_1, \quad c_3 = c_1, \gamma_3 = 3\gamma_1,\\ &c_4 = c_1, \gamma_4 = 3\gamma_1, \qquad c_5 = c_1, \gamma_5 = 9\gamma_1\end{aligned} \tag{4.8}$$

where the derivatives with respect to α_j should be interpreted as $iz_j\partial_{z_j}$ for $z_j = e^{i\alpha_j}$ in the domain $(\mathbf{A}',\boldsymbol{\alpha}) \in C_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0}$, and the constants γ_i, c_i can be fixed to depend only on ℓ. The radius is reduced to $\frac{2}{3}\varrho_0$ to allow simple dimensional bounds using $\frac{3}{4}-\frac{1}{16} > \frac{2}{3}$ (taking into account the second inequality in Eq.(4.5)).

To define the canonical transformation $(\mathbf{A}',\boldsymbol{\alpha}') \to (\mathbf{A},\boldsymbol{\alpha})$ the implicit functions in Eq.(2.2) have to be solved. This can be done quite easily if one is willing to define the map only for $(\mathbf{A}',\boldsymbol{\alpha}')$ contained in a small enough domain.

The condition to express $(\mathbf{A},\boldsymbol{\alpha})$ in terms of $(\mathbf{A}',\boldsymbol{\alpha}') \in \mathcal{C}_{\varrho',\kappa'}$ with $\varrho' = \frac{1}{2}\varrho_0, \kappa' = \kappa_0 - 3\delta_0$ is prescribed via an implicit function theorem for analytic functions, see for instance propositions 20,21 in Sec.5.11 and Appendix N in [10], or [11, Appendix3].

The theorem is proved following the lines of the analogous result "for disks" leading to Eq.(4.5) (discussed in Appendix A below) adapting it to polydisks and the condition is obtained on dimensional grounds as the bound (on the Jacobian of the implicit equations Eq.(2.2))

$$||\boldsymbol{\partial}^2_{\mathbf{A}'\boldsymbol{\alpha}}\Phi_0||_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0} < \gamma_6\eta_0\,\delta_0^{-c_6} < 1 \tag{4.9}$$

where the first inequality is just the bound Eq.(4.8) on the *l.h.s.* with γ_4 modified into $\gamma_6 = 32\ell^2$ and $c_6 = c_4$ (see Appendix **??**).

This can be obtained, again reducing the radius from $\frac{2}{3}\varrho_0$ to $\frac{1}{2}\varrho_0$ for ease of dimensional bounds, by first fixing $\mathbf{A}' \in C_{\frac{1}{2}\varrho_0}$ so that the second inequality in Eq.(4.9) simply *implies injectivity* of the map $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + \partial_{\mathbf{A}'}\Phi_0(\mathbf{A}',\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in C_{\kappa_0-2\delta_0}$, for all $\mathbf{A}'$ fixed in $C_{\frac{1}{2}\varrho_0}$; it implies also $\boldsymbol{\alpha} \in C_{\kappa_0-2\delta_0}$ for $\boldsymbol{\alpha}' \in C_{\kappa_0-3\delta_0}$ if γ_6 is large enough, see appendix B. Therefore, given $\mathbf{A}' \in C_{\frac{1}{2}\varrho_0}$ and using the injectivity, $\boldsymbol{\alpha}$ can be computed from $\boldsymbol{\alpha}'$ in the form

$$\begin{aligned}
&\boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}',\boldsymbol{\alpha}'), \qquad \boldsymbol{\alpha}' \in C_{\kappa_0-3\delta_0}, \forall \mathbf{A}' \in C_{\frac{1}{2}\varrho_0}\\
&\boldsymbol{\Delta}(\mathbf{A}',\boldsymbol{\alpha}') \equiv -\partial_{\mathbf{A}'}\Phi_0(\mathbf{A}',\boldsymbol{\alpha})\\
&||\boldsymbol{\Delta}||_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0} < \gamma_3\eta_0\delta_0^{-c_3} < \delta_0,
\end{aligned} \tag{4.10}$$

where the second line in Eq.(4.10) is an identity which implies, via Eqs.(4.8),(4.9), the inequalities in the third line.

The second inequality in Eq.(4.9) also insures the injectivity of $\mathbf{A} = \mathbf{A}' + \partial_{\boldsymbol{\alpha}}\Phi_0(\mathbf{A}',\boldsymbol{\alpha})$ for $\mathbf{A}'$ in $C_{\frac{1}{2}\varrho_0}$, for all $\boldsymbol{\alpha}$ fixed in $C_{\kappa_0-2\delta_0}$, therefore for all $\boldsymbol{\alpha}'$ in $C_{\kappa_0-3\delta_0}$.

Hence $\boldsymbol{\Delta}(\mathbf{A}',\boldsymbol{\alpha}')$ is defined in $C_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0}$ and the angles $\boldsymbol{\alpha}$ can be expressed in terms of $\boldsymbol{\alpha}',\mathbf{A}'$; and it is possible to express, for each $\boldsymbol{\alpha}' \in C_{\kappa_0-3\delta_0}$, $\mathbf{A}$ in terms of $\mathbf{A}'$, $\forall \mathbf{A}' \in C_{\frac{1}{2}\varrho_0}$: simply by substituting $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}',\boldsymbol{\alpha}')$ to find:

$$\begin{aligned}
&\mathbf{A} = \mathbf{A}' + \mathbf{a} + \boldsymbol{\Xi}(\mathbf{A}',\boldsymbol{\alpha}')\\
&\boldsymbol{\Xi}(\mathbf{A}',\boldsymbol{\alpha}') \equiv \partial_{\boldsymbol{\alpha}}\Phi_0\Big(\mathbf{A}',\boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}',\boldsymbol{\alpha}')\Big),
\end{aligned} \tag{4.11}$$

For $(\mathbf{A}',\boldsymbol{\alpha}') \in C_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0}$ the $(\mathbf{A},\boldsymbol{\alpha})$ will vary inside the original domain.

Then again Eqs.(4.8),(4.9), if $\gamma_7\eta_0\delta_0^{-c_7} < 1$ for $\gamma_7 = 4\gamma_2$, $c_7 = c_2$, yield

$$|\mathbf{A}| < \frac{1}{2}\varrho_0 + ||\Xi||_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0} \le \frac{1}{2}\varrho_0 + \gamma_2\eta_0\varrho_0\delta_0^{-c_2} < \frac{3}{4}\varrho_0 \tag{4.12}$$

Collecting definitions of $\mathbf{a}, \boldsymbol{\Delta}, \Xi$ a canonical map

$$\begin{aligned}\mathbf{A} =& \mathbf{A}' + \mathbf{a} + \Xi(\mathbf{A}', \boldsymbol{\alpha}'), \qquad \boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}')\\ &||\Xi||_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0} < \gamma_8\eta_0\, \varrho_0\delta_0^{-c_8}\\ &||\boldsymbol{\Delta}||_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0} < \gamma_8\eta_0\, \delta_0^{-c_8}\end{aligned} \tag{4.13}$$

for suitable g_8, c_8 will be defined, under the condition Eq.(4.9), and will tranform $(\mathbf{A}', \boldsymbol{\alpha}') \in C_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0}$ into $(\mathbf{A}, \boldsymbol{\alpha}) \in C_{\frac{3}{4}\varrho_0,\kappa_0-\delta_0}$: for instance $\gamma_8 = \gamma_2, c_8 = c_2$.

The perturbation function $f_0 - \overline{f}_0(\mathbf{a})$ is tranformed, in the new coordinates, $f_0'(\mathbf{A}', \boldsymbol{\alpha}') = f_0(\mathbf{A}' + \mathbf{a} + \Xi(\mathbf{A}', \boldsymbol{\alpha}'), \boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}'))$ and the new Hamiltonian can expressed by replacing $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}')$ in the three terms in Eq.(2.4). This is discussed in the next section in terms of η_0, δ_0; the conditions imposed, so far, on the construction can be all implied by the conditions

$$\begin{aligned}&C_0\varrho_0 J_0^+ < 1, \quad e^{\kappa_0} < 2 \qquad \text{initial restrictions}\\ &\varepsilon_0(J_0^-\varrho_0)^{-1} < \chi, \qquad \text{to define} \qquad \mathbf{a} = -J_0^{-1}\boldsymbol{\partial}_{\mathbf{a}}\overline{f}_0(\mathbf{a})\\ &\gamma_9\eta_0\delta_0^{-c_9} < 1, \qquad \text{to define } \boldsymbol{\Delta}, \Xi\end{aligned} \tag{4.14}$$

for $\gamma_9 = \gamma_6, c_9 = c_6$ large enoughand χ small enough, see Eq.(4.5).

The domain of variability in the initial variables $(\mathbf{A}, \boldsymbol{\alpha})$, where the canonical map is defined, will now contain (at least) a small domain of shape close to a polydisk (*eccentric* because of the translation by $\mathbf{a}$) inside the initial domain C_{ϱ_0,κ_0} of the Hamiltonian H_0. The small eccentric polydisk is the image of a *centered* polydisk $\mathcal{C}_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0}$ in the new variables $(\mathbf{A}', \boldsymbol{\alpha}')$.

5 Renormalization

The Hamiltonian $H_0 + f_0$ in the new coordinates $\mathbf{A}', \boldsymbol{\alpha}'$ becomes:

$$H'(\mathbf{A}', \boldsymbol{\alpha}') = \frac{1}{2}\mathbf{A}' \cdot J_1\mathbf{A}' + \boldsymbol{\omega}_0 \cdot \mathbf{A}' + f', \qquad (\mathbf{A}', \boldsymbol{\alpha}') \in \mathcal{C}_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0} \tag{5.1}$$

in the domain $(\mathbf{A}', \boldsymbol{\alpha}') \in C_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0}$. The function f' is defined, in the mixed variables $(\mathbf{A}', \boldsymbol{\alpha})$, by Eq.(2.4).

- The contribution 1) in Eq.(2.4), *does not vanish: but it carries the key cancellation* showing that the sum of terms individually formally $O(\varepsilon_0)$ is in fact of higher order in ε_0 as can be seen via the Fourier's transform of $f_0 - \overline{f}_0 = \sum_{\mathbf{0}\ne\boldsymbol{\nu}} f_{0,\boldsymbol{\nu}} e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}}$ which, after a few simplifications, is:

$$\begin{aligned}F &\stackrel{def}{=} (\boldsymbol{\omega}_0 + J_0(\mathbf{A}' + \mathbf{a}) \cdot \boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi_0 + f_0(\mathbf{A}' + \mathbf{a}, \boldsymbol{\alpha}) - \overline{f}_0(\mathbf{A}' + \mathbf{a}))\\ &= -\sum_{\mathbf{0}\ne\boldsymbol{\nu}} f_{0,\boldsymbol{\nu}}(\mathbf{A}' + \mathbf{a})\frac{(J_0(\mathbf{A}' + \mathbf{a}) \cdot \boldsymbol{\nu})^3}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu})^3} e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}}\end{aligned} \tag{5.2}$$

- if $|\mathbf{A}'| < \widetilde{\varrho}$, F admits a dimensional bound *in the sense of Eq.(1.2)*, *i.e.* on $\|\boldsymbol{\partial}_{\mathbf{A}'}F\|_{\widetilde{\varrho},\kappa_0-3\delta_0} + \frac{1}{\widetilde{\varrho}}\|\boldsymbol{\partial}_{\boldsymbol{\alpha}'}F\|_{\widetilde{\varrho},\kappa_0-3\delta_0}$. Using $J_0\mathbf{a} = -\boldsymbol{\partial} f_0(\mathbf{a})$, $J_0^+C_0\varrho_0 < 1$, and Eq.(3.2) together with the bound $|f_{0,\boldsymbol{\nu}}| < \varepsilon_0\varrho_0 e^{-\kappa_0|\boldsymbol{\nu}|}$ (derived from $\frac{1}{\varrho_0}|\boldsymbol{\partial}_{\boldsymbol{\alpha}} f_0| \le \varepsilon_0$) a dimensional bound on F follows as:

$$\begin{aligned}&\le \gamma_{10}\varepsilon_0\,\frac{\varrho_0}{\widetilde{\varrho}}((J_0^+\widetilde{\varrho}C_0)^3 + (C_0\varepsilon_0)^3)\delta_0^{-c_{10}}\\ &\quad= \gamma_{10}\varepsilon_0\,\frac{\varrho_0}{\widetilde{\varrho}}(\frac{\widetilde{\varrho}^3}{\varrho_0^3} + \eta_0^3)\delta_0^{-c_{10}} < \gamma_{11}\varepsilon_0\eta_0^{2(1-\lambda)}\delta_0^{-c_{11}}\end{aligned} \tag{5.3}$$

  in the polydisk $\mathcal{C}_{\widetilde{\varrho},\kappa_0-3\delta_0}$. The $\widetilde{\varrho}$ will be determined as $\widetilde{\varrho} = \eta_0^{1-\lambda}\varrho_0$ with $0 < \lambda < 1$ so that $\widetilde{\varrho} < \frac{1}{2}\varrho_0$ provided η_0 is small enough. The constants could be chosen $c_{10} = c_{11} = 7\ell$ and $\gamma_{10} = 4\Omega_\ell(7\ell-1)!$, $\gamma_{11} = 2\gamma_{10}$.

- The contribution 2) in Eq.(2.4), is bounded, *still in the sense of Eq.(1.2)*, in a disk of radius $\widetilde{\varrho} = \eta_0^{1-\lambda}\varrho_0$, if, as above, η_0 is small enough, by

$$\gamma_{12}\varepsilon_0\eta_0^{2(1-\lambda)} \tag{5.4}$$

  making use of its $\boldsymbol{\alpha}$-independence, which permits to estimate dimensionally the second derivative of $\overline{f}_0(\mathbf{A}+\mathbf{a})$ in a disk of radius $\frac{1}{2}\varrho_0$ (rather than of radius $\widetilde{\varrho}$): thus it also yields a contribution to the higher order terms and $\gamma_{12} = 8, c_{12} = 0$.

- The terms in the contribution 3) are also dimensionally bounded, *still in the sense of Eq.(1.2)*, by:

$$\gamma_{13}\varepsilon_0\eta_0^{\lambda}\delta_0^{-c_{13}} \tag{5.5}$$

  in the polydisk $\mathcal{C}_{\widetilde{\varrho},\kappa_0-3\delta_0}$, using $C_0\varrho_0J_0^+ < 1$: $g_{13} = 12\gamma_2^2, c_{13} = 2c_2$.

Adding the bounds Eq.(5.3),(5.4),(5.5) it is, for $\lambda = \frac{2}{3}$ (*i.e.* $2(1-\lambda) = \lambda$):

$$\varepsilon_1 = (|\boldsymbol{\partial}_{\mathbf{A}'}f'|_{\widetilde{\varrho},\kappa_0-4\delta_0} + \widetilde{\varrho}^{-1}|\boldsymbol{\partial}_{\boldsymbol{\alpha}'}f'|_{\widetilde{\varrho},\kappa_0-4\delta_0} < \gamma\varepsilon_0\eta_0^{2(1-\lambda)}\delta_0^{-c}) \tag{5.6}$$

for $\gamma, c > 0$ suitably fixed, if $C_0\varrho_0J_0^+ < 1, \eta_0(J_0^-\varrho_0C_0)^{-1} < \chi$, (see also Eq.(4.2), 4.5): a simple choice is $\gamma = \max_{j\in[1,13]}\gamma_j, c = \max_{j\in[1,13]}c_j$.

A further dimensional estimate on (the eigenvalues of) the matrix $\boldsymbol{\partial}_a^2\overline{f}_0(\mathbf{a})$ in Eq.(2.4) is $\overline{c}\varepsilon_0\varrho_0^{-1}$, *e.g.* $\overline{c} = 2\ell^2$, hence $J_0^-(1-\overline{c}\theta_0) < J_1 < J_0^+(1+\overline{c}\theta_0)$.

The result is that in the coordinates $(\mathbf{A}',\boldsymbol{\alpha}') \stackrel{def}{=} (\mathbf{A}_1,\boldsymbol{\alpha}_1)$ the motion is Hamiltonian with Hamiltonian H_1; and recalling the definitions of the dimensionless quantities in Eq.(4.2),(4.5):

$$\begin{aligned}&H_1 = \frac{1}{2}\mathbf{A}_1\cdot J_1\mathbf{A}_1 + \boldsymbol{\omega}_0\cdot\mathbf{A}_1 + f_1(\mathbf{A}_1,\boldsymbol{\alpha}_1)\\ &\varrho_1 = \varrho_0\eta_0^{1-\lambda},\quad \kappa_1 = \kappa_0 - \overline{\delta}_0,\quad C_1 = C_0\\ &\eta_1 = \gamma\eta_0^{3-2\lambda}\delta_0^{-c},\qquad \theta_1 = \gamma\theta_0\eta_0^{1-\lambda}\delta_0^{-c},\\ &J_0^-(1-\overline{c}\theta_0) < J_1 < J_0^+(1+\overline{c}\theta_0)\end{aligned} \tag{5.7}$$

where γ, c are constants, $\overline{\delta}_n = 4\delta_n = \kappa_0(n+10)^{-2}$ and $\lambda = \frac{2}{3}$; the last inequality is intended as bounds on the eigenvalues of J_1.

The above transformation of coordinates $(\mathbf{A}, \boldsymbol{\alpha}) \to (\mathbf{A}_1, \boldsymbol{\alpha}_1)$, which will be denoted $\mathcal{K}_0$, is well defined and holomorphic in the domain $C_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0}$ whose $\mathcal{K}_0$-image contains the small polydisk $C_{\widetilde{\varrho},\kappa_0-4\delta_0}$ *provided* ε_0 *is small enough* so that the conditions imposed during the construction, namely Eq.(4.14) and $C_0\varrho_0 J_0^+ < 1, \eta_0(J_0^-\varrho_0 C_0)^{-1} < \chi$ the ones requiring η_0 to be small so that $\widetilde{\varrho}$ can lead to obtain Eq.5.3-5.6, are satisfied and *remain satisfied under iteration* leading to define the sequence of maps $\mathcal{K}_n$, $n \ge 0$.

This is possible because, if η_0 (*i.e.* ε_0) is small enough, the map in Eq.(5.7) generates a sequence with $C_n\varepsilon_n = C_0\varepsilon_n = \eta_n$ tending to 0, fixed arbitrarlily $\mu \in (0, \frac{2}{3})$ and a corresponding suitable constant $\overline{\gamma}$, superexponentially with

$$\eta_n \sim (\overline{\gamma}\eta_0)^{(1+\mu)^n}, \quad \overline{\gamma} > 0,\ 0 < \mu < \frac{2}{3}, \qquad J_n < 2J_0^+,\ J_n > \frac{1}{2}J_0^- \tag{5.8}$$

and θ_n also tend to 0 at similar rates (*e.g.* $\theta_n \sim (\overline{\gamma}'\eta_0)^{(1+\mu)^n c'}$), as can be checked by induction from Eq.(5.7) with suitable $c' < 1, \overline{\gamma}'$. This implies that for all $n \ge 0$ the transformations $\mathcal{K}_n$ can be defined if ε_0 (*i.e.* its dimensionless version η_0) is small enough.

Furthermore $\mathcal{K}_n$ is seen from Eq.(4.13) to be close to the identity within $\gamma_8\eta_n\delta_n^{-c_8}$. Hence the iteration of the renormalization procedure defines a sequence of transformations $\mathcal{K}_n$ under the only initial condition in Eq.(4.14) with γ_9, c_9, χ^{-1} large enough.

In the polydisk $\mathcal{C}_{\varrho_n,\kappa_n}$ the motions starting with $\mathbf{A}_n = \mathbf{0}$ and (say)$\boldsymbol{\alpha} = \mathbf{0}$ become closer and closer to the motion of a harmonic oscillator with frequency spectrum $\boldsymbol{\omega}_0$ and in the limit $n \to \infty$ all motions in the "polydisk" (degenerated to a torus $\mathbf{0} \times \mathcal{T}^\ell$) are harmonic with spectrum $\boldsymbol{\omega}_0$. This is checked simply by remarking that the motion of the initial data is, if observed in an *arbitrarily fixed time* t, is superxponentially close to the harmonic motion $\mathbf{A} = \mathbf{0}, \boldsymbol{\alpha}(t) = \boldsymbol{\alpha} + \boldsymbol{\omega}_0 t$. The torus on which the motion is quasi periodic is the limit of the tori with equations $\mathbf{A} = \mathbf{a}_n + \boldsymbol{\Xi}_n(\mathbf{a}_n, \boldsymbol{\alpha}'), \boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\Delta}_n(\mathbf{a}_n, \boldsymbol{\alpha}')$ which is the torus which at the n-th iteration of the renormalization has coordinates $\mathbf{a}_n, \boldsymbol{\alpha}'$). The successive corrections to $\mathbf{a}_n$ and to the functions $\boldsymbol{\Xi}_n, \boldsymbol{\Delta}_n$ tend to 0 superexponentially and their limits

$$\mathbf{a}_\infty,\ \boldsymbol{\Xi}_\infty(\boldsymbol{\alpha}'),\ \boldsymbol{\Delta}_\infty(\boldsymbol{\alpha}'), \qquad \boldsymbol{\alpha}' \in T^\ell \tag{5.9}$$

define an invariant torus on which motion is $\boldsymbol{\alpha}' \to \boldsymbol{\alpha}' + \boldsymbol{\omega}_0 t$.

It is also possible to define a sequence of maps $\widetilde{\mathcal{K}}_n$ defined in the *fixed domain* $\mathcal{C}_{\frac{1}{2}\varrho_0,\frac{1}{2}\kappa_0}$ by rescaling the polydisks by a factor $\eta_{n-1}^{\frac{1}{3}} = \varrho_{n-1}/\varrho_n$, $n \ge 1$ so that they are all turned into $\mathcal{C}_{\frac{1}{2}\varrho_0,\frac{1}{2}\kappa_0}$: the rescaling transformation will change $\mathbf{A}_n$ into $\mathbf{A}'_n = \eta_n^{-\frac{1}{3}}\mathbf{A}_n$ and the Hamiltonian into

$$\widetilde{H}_n = \boldsymbol{\omega}_0 \cdot \mathbf{A}'_n + \eta_n^{\frac{1}{3}}\frac{1}{2}(\mathbf{A}'_n \cdot J_n\mathbf{A}'_n) + \eta_n^{-\frac{1}{3}} f_n(\eta_n^{\frac{1}{3}}\mathbf{A}'_n, \boldsymbol{\alpha}'_n) \xrightarrow[n\to\infty]{} \boldsymbol{\omega}_0 \cdot \mathbf{A}_\infty \tag{5.10}$$

and in the rescaled variables the sizes of the anharmonic terms tend to 0 super-exponentially, taking into account the recursion defined in Eq.(5.7) (and that the size of f_n is η_n).

This shows that the perturbation f_0 and the twist J_0 are, after renormalization, "irrelevant operators" (in Eq.(5.10) both tend to 0 as $n \to \infty$), while the harmonic oscillator is a "fixed point": in some sense the transformation has the harmonic oscillator as an *attractive fixed point.* This completes a proof of the KAM theorem, *interpreted in the Renormalization Group frame* [12, 13, 14, 15]: it can be classified as a "super-renormalizable" problem, as it requires only a second order perturbation analysis, Eq.(4.6), around the trivial fixed point.

Remarks: (1) a simpler analysis (and an instructive warm-up exercise) can be carried also if $J_0 = 0$ provided the perturbation depends only on the angles $\boldsymbol{\alpha}$. The independence of f_0 from $\mathbf{A}$ has the consequence, in the proof development, that all terms appearing to involve J_0^{-1} actually do not arise at all (*but* the system is elementarily integrable).
(2) The condition $\det J_0 \neq 0$ is called "anisochrony condition" or "twist condition": the size $\neq 0$ of $\det J_0$ plays an important role in the above analysis. However invariant diophantine tori, may in certain cases, exist just for ε_0 smaller than a quantity independent on the size of $\det J_0$; such tori are called "twistless", because they can be shown to exist without invoking the twist condition. This happens in cases in which f_0 depends on $\boldsymbol{\alpha}$ only: and the tori can be constructed via a simple graphical algorithm, [16]. The graphical algorithm led, in the twistless cases, also to a new "direct" proof of the KAM theorem, [17, 18, 19], that was later extended to the general case, [20].
(3) The estimates in the above analysis are far from optimal and optimization is desirable.

6 Comments

The analysis in Sec.5 is a reformulation of the original proof by Kolmogorov, [12], reproduced in full detail in [15] and used to build a rigorous computation algorithm in [21]. The feature of the approach, common also to Moser's work, [14], is to use canonical maps with fixed small denominators: this avoids dealing with $\mathbf{A}$ dependent divisors appearing in [13, p.105], reproduced in [10].

The renormalization group interpretation has been proposed in in [22] with prefixed divisors and [23, 6] still dealing with $\mathbf{A}$-dependent divisors: the approach developed in Sections 4,5 is inspired by the latter development but avoids $\mathbf{A}$-dependent divisors, hence it is close to [12, 14, 15, 21, 24, 25] and several other approaches. The definition of ε_n, see Eq.(1.2), can be replaced by $\varepsilon_0 = \max_{\mathcal{C}} |f_0|$: this choice would be possible, jsut with obvious notational changes.

The relation between the KAM theorem and the renormalization group has been used in various forms for its proof, in several papers, for instance [23, 22, 26, 6, 27, 28, 24, 29, 30].

The difference between the approach of Kolmogorov and Moser, with respect to Arnold's, [13], is that in the second the small divisors are **A**-dependent and are controlled by an increasing sequence of cut-offs on $\boldsymbol{\nu}$, at each order of the perturbation expansion.

The analysis of the singularity at $\varepsilon_0 = 0$, in the case of resonant quasi periodic motions (*i.e.* motions which dwell on lower dimensional tori), can also be pursued via multiscale methods conveniently interpreted as methods of performing the resummations of the perturbative series, which unlike the KAM case, are divergent power series, [31, 32, 33].

A Implicit functions in (4.5) (and (4.10),(4.11))

This appendix is presented for completeness (see also proposition 19 in [10, Sec.5.11], and [11, Appendix3]). The equation $\mathbf{a} = \mathbf{a}_0 + \mathbf{n}(\mathbf{a})$, with $\mathbf{a}_0 = -J_0^{-1}\boldsymbol{\partial}\overline{f}(\mathbf{0})$ and $\mathbf{n}(\mathbf{a}) = -J_0^{-1}\boldsymbol{\partial}_{\mathbf{a}}\overline{f}(\mathbf{a}) + J_0^{-1}\boldsymbol{\partial}_{\mathbf{a}}\overline{f}(\mathbf{0})$ is written as equation for $\mathbf{b} = \mathbf{a} - \mathbf{a}_0$ with $\mathbf{a}_0 = -J_0^{-1}\boldsymbol{\partial}_{\mathbf{a}}\overline{f}(\mathbf{0})$:

$$\mathbf{b} = \mathbf{n}(\mathbf{a}_0 + \mathbf{b}), \quad i.e. \quad \mathbf{b} = \mathbf{c} + \widetilde{\mathbf{n}}(\mathbf{b}) \tag{A.1}$$

with $\widetilde{\mathbf{n}} = \mathbf{n}(\mathbf{a}_0 + \mathbf{b}) - \mathbf{n}(\mathbf{a}_0)$ defined in $C_{\frac12\varrho_0}$ if $|\mathbf{a}_0| < \mu\,\varrho_0$, with $\mu < \frac12$, and $\mathbf{c} = \mathbf{n}(\mathbf{a}_0)$.dimensional estimates hold:

$$\begin{aligned}
&|\mathbf{a}_0| \le \varepsilon_0 J_0^{-1} \le \mu\varrho_0, && \text{if } \theta_0 < \mu \le \frac14\\
&|\mathbf{c}| \ \le \varepsilon_0 J_0^{-1} 2\varrho_0^{-1}|\mathbf{a}_0| \le 2\theta_0\varepsilon_0 J_0^{-1} \equiv 2\theta_0^2\varrho_0 &&\\
&|\widetilde{\mathbf{n}}| \ \le \varepsilon_0 J_0^{-1} 2\varrho_0^{-1}\,|\mathbf{b}| = 2\theta_0|\mathbf{b}|, && \text{if } |\mathbf{b}| < \frac14\varrho_0
\end{aligned} \tag{A.2}$$

Consider $\mathbf{b}$ moving on the circle $|\mathbf{b}| = \lambda\varrho_0$, $\lambda = \frac18$; then:

$$|\mathbf{b} - \widetilde{\mathbf{n}}(\mathbf{b}) - \mathbf{c}| \begin{cases} \ge \lambda\varrho_0(1 - 2\theta_0 - 2\lambda^{-1}\theta_0^2)\\ \le \lambda\varrho_0(1 + 2\theta_0 + 2\lambda^{-1}\theta_0^2)\end{cases} \tag{A.3}$$

If the problem is in dimension 1 (*i.e.* a, b, c, n are scalars) this means that the image of the circle delimiting $C_{\lambda\varrho_0}$ is contained in the disk with radius $\lambda\varrho_0(1 + 2\theta_0 + 2\lambda^{-1}\theta_0^2)$ and contains the disk with radius $\lambda\varrho_0(1 - 2\theta_0 - 2\lambda^{-1}\theta_0^2)$ hence the equation has a solution $\mathbf{b}_0$ contained in the larger disk *if its radius is* $< \varrho_0$ *and if the smaller radius is* > 0 (hence the latter contains the origin).

Choosing $\mu = \frac14, \lambda = \frac18$ and assuming $\theta_0 < \frac{1}{16}$ all the conditions are fulfilled; so the equation has a solution $\mathbf{a} = \mathbf{a}_0 + \mathbf{b}_0 = -J_0^{-1}\boldsymbol{\partial}_{\mathbf{a}}\overline{f}(\mathbf{a})$ in C_{ϱ_0} (hence $|\mathbf{a}| < \theta_0\varrho_0$) and the statement in Eq.(4.5) is proved if $\ell = 1$. It also follows that Eq.(A.1), as an implicit equation for $\mathbf{b}$ in terms of $\mathbf{c}$, gives $\mathbf{b}$ analytic in terms of $\mathbf{c}$.

The multidimensional case can be proved in the same way: it might be expected that the bound χ_ℓ depends on ℓ but a careful examination of the above argument not only works if $\ell > 1$ (replacing the disk with a product of ℓ disks) but also shows that the constant χ_ℓ can be chosen ℓ-independent, [10, 11].

Likewise the implicit equations discussed in Eq.(4.10),(4.11) can be solved along the same lines, replacing disks with polydisks (see details in propositions 20,21 in [10] or appendix 3 in [11].

B Injectivity in (4.10)

Remark that the distance of the boundary of the polyannulus Γ_{κ_0} to that of $\Gamma_{\kappa_0-\delta_0}$ is bounded, if $\frac{1}{2} \le e^{\pm\kappa_0} \le 2$, below by $\frac{1}{2}\delta_0$ and above by $2\delta_0$.

The injectivity follows by integrating, along the shortest path enclosed in $C_{\kappa_0-\delta_0}$ connecting $\mathbf{z}_1$ and $\mathbf{z}_2$, and defining $\|z_2-z_1\| = \sum_{j=1}^{\ell} |z_{k,2}-z_{k,1}| \le 2\ell e^{\kappa_0}$, from:

$$\boldsymbol{\alpha}'_{1,k} - \boldsymbol{\alpha}'_{2,k} = \boldsymbol{\alpha}_{1,k} - \boldsymbol{\alpha}_{2,k} + \int_{\boldsymbol{\alpha}_1}^{\boldsymbol{\alpha}_2} \sum_{j=1}^{\ell} d\alpha_j \, \partial_{\alpha_j} \partial_{A'_k} \Phi_0(\mathbf{A}', \boldsymbol{\alpha})) \tag{B.1}$$

Taking into account the inequalities Eq.(4.8),(4.9)the constants c_6, γ_6 can be determined by rewriting Eq.(B.1) as

$$\frac{z'_{k,2}}{z'_{k,1}} = \frac{z_{k,2}}{z_{k,1}} e^{i\partial_{A'_k}\Phi_2 - i\partial_{A'_k}\Phi_1} \tag{B.2}$$

and, by Eq.(4.8), $|\frac{z'_{k,2}-z'_{k,1}}{z'_{k,1}}|$ is bounded below by

$$|\frac{z_{k,2}-z_{k,1}}{z_{k,1}}| e^{-2\gamma_3\eta_0\delta_0^{-c_3}} - |e^{i\int_{z_1}^{z_2}\sum_{j=1}^{\ell}\frac{d\zeta_j}{\zeta_j}\partial_{\boldsymbol{\alpha}_j, A'_k}\Phi(\boldsymbol{\alpha},\mathbf{A}')} - 1| \tag{B.3}$$

The $\frac{1}{2} \le |z_k| \le 2$ and the integration path length $\le (e^{\kappa_0} + \pi e^{-\kappa_0})\ell \le (2+\frac{\pi}{2})\ell \le 4\ell$ imply:

$$\|z'_2 - z'_1\| \ge \frac{1}{4}\|z_2 - z_1\| e^{-2g_3\eta_0\delta_0^{-c_3}} - \ell(e^{4\ell\|z_2-z_1\|\gamma_4\eta_0\delta_0^{-c_4}} - 1) \tag{B.4}$$

Hence, using $c_3 = c_4, \gamma_3 = \gamma_4, \|z_2 - z_1\| \le 2\ell e^{\kappa_0} \le 4\ell$, setting $\xi = \gamma_4\delta_0^{-c_4}$

$$\|z'_2 - z'_1\| \ge (\frac{1}{4}e^{-2\xi} - 4\ell^2\xi e^{(4\ell)^2\xi}\|z_2 - z_1\|) \ge \frac{1}{25}\|z_2 - z_1\| \tag{B.5}$$

if $\xi < (32\ell^2)^{-1}$ (so that $\frac{1}{4}e^{-2\xi} - \frac{\xi}{8}e^{\frac{1}{2}} \ge \frac{1}{25}$). This shows that γ_6, c_6 can be taken $\gamma_6 = 32\ell^2\gamma_4, c_6 = c_4$.

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References

[1] K. Wilson. Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture. *Physical Review B*, 4:3174–3183, 1971.

[2] L. Carleson. On convergence and growth of partial sums of fourier series. *Acta Mathematica*, 116:135–157, 1966.

[3] C. Fefferman. Pointwise Convergence of Fourier Series. *Annals of Mathematics*, 98:551–571, 197.

[4] E. Nelson. A quartic interaction in two dimensions. *In Mathematical Theory of elementary particles, ed. R. Goodman, I. Segal*, pages 69–73, 1966.

[5] L. Caffarelli and L. Nirenberg and L. Kohn. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Communications on Pure and Applied Mathematics*, 35:771–831, 1982.

[6] G. Gallavotti. Quasi integrable mechanical systems. *Phenomènes Critiques, Systèmes aleatories, Théories de jauge, Proceedings, Les Houches, XLIII (1984), North Holland, Amsterdam*, II:539–624, 1986.

[7] G. Gallavotti. Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods. *Reviews of Modern Physics*, 57:471–562, 1985.

[8] K. Wilson and J. Kogut. The renormalization group and the ε-expansion. *The renormalization group and the ε-expansion, Physics Reports*, 12:75–199, 1973.

[9] G. Benfatto and G. Gallavotti. *Renormalization Group*. Princeton U. Press, Princeton, 1995.

[10] G. Gallavotti. *The Elements of Mechanics (I edition);*. Springer Verlag, New York, 1983 [I edition].

[11] G. Gallavotti. Perturbation theory for classical Hamiltonian systems. *in Scaling and self similarity in Physics, Ed. J. Fröhlich, Birkhäuser, Boston*, pages 359–426, 1985.

[12] A.N. Kolmogorov. On the preservation of conditionally periodic motions. *In Lecture Notes in Physics, Stochastic behavior in classical and quantum Hamiltonians, ed. G. Casati, J. Ford, Vol. 93, 1979*, 93, 1979.

[13] V. Arnold. Small denominators and problems of stability of motion in classical and celestial mechanics. *Russian Mathematical Surveys*, 18:85–191, 1963.

[14] J. Moser. On invariant curves of an area preserving mapping of the annulus. *Nachrichten Akademie Wissenshaften Göttingen*, 11:1–20, 1962.

[15] G. Benettin, L. Galgani, A. Giorgilli, and J. Strelcyn. A proof of Kolmogorov's theorem on invariant tori using canonical transformations defined by the Lie method. *Nuovo Cimento B*, 79:201–223, 1984.

[16] G. Gallavotti. Twistless KAM tori. *Communications in Mathematical Physics*, 164:145–156, 1994.

[17] L.H. Eliasson. Absolutely convergent series expansions for quasi periodic motions. *MPEJ (Mathematical Physics Electronic Journal)*, 2, n.4:1–33, 1986-96.

[18] G. Gallavotti and G. Gentile. Majorant series convergence for twistless kam tori. *Ergodic Theory and Dynamical Systems*, 15:857–869, 1995.

[19] G. Gallavotti, F. Bonetto, and G. Gentile. *Aspects of the ergodic, qualitative and statistical theory of motion*. Springer Verlag, Berlin, 2004.

[20] G. Gentile and V. Mastropietro. Construction of periodic solutions of the nonlinear wave equation under strong irrationality conditions by the Lindstedt series method. *Journal de Mathématiques Pures et Appliqués*, 83:1019–1065, 2004.

[21] A. Giorgilli and U. Locatelli. Kolmogorov theorem and classical perturbation theory. *NATO ASI series, Hamiltonian systems with three or more degrees of freedom*, 533:72–89, 1999.

[22] R. MacKay. A renormalization approach to invariant circles in area-preserving maps. *Physica D*, 7:283–300, 1983.

[23] G. Gallavotti. A criterion of integrability for perturbed nonresonant harmonic oscillators. Wick Ordering of the perturbations in classical mechanics and invariance of the frequency spectrum. *Communications in Mathematical Physics*, 87:365–382, 1982.

[24] C. Chandre, H. Jauslin, and G. Benfatto. An Approximate KAM-Renormalization-Group Scheme for Hamiltonian Systemsa. *Journal of statistical physics*, 94:241–251, 1999.

[25] J. Hubbard and Y. Ilyashenko. A proof of Kolmogorov's theorem. *Discrete and continuous dynamical systems*, 10:367–385, 2004.

[26] G. Gallavotti. Invariant tori: a field theoretic point of view on Eliasson's work. *Advances in Dynamical Systems and Quantum Physics,* Ed. R. Figari, World Scientific, 164:117–132, 1995.

[27] J. Bricmont, K. Gawedzki, and A. Kupiainen. Kam theorem and quantum field theory. *Communications in Mathematical Physics*, 201:699–727, 1999.

[28] C. Chandre, M. Govin, and H. R. Jauslin. Kolmogorov-Arnold-Moser renormalization-group approach to the breakup of invariant tori in Hamiltonian systems. *Physical Review E*, 57:1536–1543, 1998.

[29] H. Koch. A renormalization group fixed point associated with the breakup of golden invariant tori. *Discrete and continuous dynamical systems*, 101:881909, 2004.

[30] G. Gentile. Quasi-periodic motions in dynamical systems. Review of a renormalisation group approach. *Journal of Mathematical Physics*, 51:015207 (+34), 2010.

[31] G. Gallavotti and G. Gentile. Hyperbolic low-dimensional invariant tori and summations of divergent series. *Communications in Mathematical Physics*, 227:421–460, 2002.

[32] G. Gallavotti and G. Gentile. Degenerate elliptic resonances. *Communications in Mathematical Physics*, 257:319–362, 2005.

[33] O. Costin, G. Gallavotti, G. Giuliani, and G. Gentile. Borel summability and Lindstedt series. *Communications in Mathematical Physics*, 269:175–193, 2006.