Quasi periodic Hamiltonian Motions, Scale Invariance, Harmonic Oscillators

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Abstract

The work of Kolmogorov, Arnold and Moser appeared just before the renormalization group approach to statistical mechanics was proposed by [1]: it can be classified as a multiscale approach which also appeared in works on the convergence of Fourier's series, [2, 3], or construction of Euclidean quantum fields, [4], or the scaling analysis of the short scale behaviour of Navier-Stokes fluids, [5], to name a few which originated a great variety of further problems. In this review the KAM theorem proof will be presented as a classical renormalization problem with the harmonic oscillator as a "trivial" fixed point.

1 Introduction

The KAM theorem can be regarded as a multiscale analysis of the stability of the harmonic oscillator viewed as a fixed point of a transformation which enlarges a region of phase space focused around a nonresonant quasi periodic motion. The problem considers a Hamiltonian

$$H_0(\mathbf{A},\boldsymbol{\alpha}) = \frac{1}{2}(\mathbf{A}, J_0\mathbf{A}) + \boldsymbol{\omega}_0\cdot\mathbf{A} + f_0(\mathbf{A},\boldsymbol{\alpha}) \equiv h_0 + f_0 \tag{1.1}$$

real analytic for $(\mathbf{A},\boldsymbol{\alpha}) \in (\mathcal{D}_\varrho \times \mathcal{T}^\ell)$ with: $\mathcal{D}_\varrho = \{\mathbf{A} \in R^\ell, |A_j| < \varrho\}$, $\mathcal{T}^\ell$ the ℓ-dimensional torus $[0,2\pi]^\ell$ identified with unit circle $\{\mathbf{z}|\, z_j = e^{i\alpha_j}, j = 1,\dots,\ell\}$, $\boldsymbol{\omega}_0 \in R^\ell$ and J_0 could be a $\ell\times\ell$ *non degenerate* symmetric matrix ($\det J_0 \neq 0$) but here it will be just the identity matrix time a constant, to simplify notations.

The Hamiltonian is supposed holomorphic in the complex region $\mathcal{C}_{\varrho_0,\kappa_0}$ with size of the perturbation f_0 measured by ε_0:

$$\begin{aligned}
&\mathcal{C}_{\varrho_0,\kappa_0} \stackrel{def}{=} \{(\mathbf{A},\mathbf{z})||A_j| \le \varrho_0,\ e^{-\kappa_0} \le |e^{i\alpha_j}| \le e^{\kappa_0}, j = 1,\dots,\ell\} \subset \mathcal{C}^{2\ell}\\
&\varepsilon_0 = ||\boldsymbol{\partial}_{\mathbf{A}} f_0||_{\varrho_0,\kappa_0} + \frac{1}{\varrho_0}||\boldsymbol{\partial}_{\boldsymbol{\alpha}} f_0||_{\varrho_0,\kappa_0}, \qquad \text{with :}\\
&||f||_{\varrho_0,\kappa_0} \stackrel{def}{=} \max_{\mathcal{C}_{\varrho_0,\kappa_0}} |f(\mathbf{A},\mathbf{z})|, \qquad \forall f \text{ holomorphic in } \mathcal{C}_{\varrho_0,\kappa_0}
\end{aligned} \tag{1.2}$$

with $\varrho_0 > 0, \kappa_0 > 0$, $z_j \equiv e^{i\alpha_j}$; generally $\mathcal{C}_{\varrho,\kappa}(\overline{\mathbf{A}})$ will denote a polydisk centered at $\overline{\mathbf{A}}$, *i.e.* defined as in Eq.(1.2) with $|A_j - \overline{A}_j| \leq \varrho$ replacing $|A_j| \leq \varrho$ and $e^{-\kappa} \leq |z_j| \leq e^{\kappa}$; polydisks centered at the "origin" will be simply denoted $\mathcal{C}_{\varrho,\kappa}$ and called "centered polydisks".

It is supposed, no loss of generality, that the $\boldsymbol{\alpha}$-average $\overline{f}_0(\mathbf{A})$ of $f_0(\mathbf{A}, \boldsymbol{\alpha})$ vanishes at $\mathbf{A} = \mathbf{0}$.

Set $|\mathbf{A}| = \max |A_j|, |\mathbf{z}| = \max |z_j|, \ \forall \mathbf{A}, \mathbf{z} \in \mathcal{C}^\ell$.

The idea is to focus attention on the center of $\mathcal{C}_{\varrho_0,\kappa_0}$ where, if $\varepsilon_0 = 0$, a motion ("free motion") takes place which is quasi periodic "with spectrum" $\boldsymbol{\omega}_0$. This is done by changing variables in a small polydisk $C_{\widetilde{\varrho},\widetilde{\kappa}}(\mathbf{a}) \subset C_{\varrho_0,\kappa_0}$, eccentric if $\mathbf{a} \neq \mathbf{0}$, that is then recentered and enlarged back to the original size so that it contains $\mathcal{C}_{\varrho_0,\kappa_0'}$ with $\kappa_0' > \frac{1}{2}\kappa_0$.

The motions developing in the initial polydisk can be studied as "through a microscope": in the good cases (*i.e.* under suitable assumption on the initial parameters $J_0, \boldsymbol{\omega}_0$ and f_0) the Hamiltonian will turn out to be substantially closer to that of a harmonic oscillator (described by its "normal" Hamiltonian $\boldsymbol{\omega}_0 \cdot \mathbf{A}$ in the variables $\mathbf{A}, \boldsymbol{\alpha}$).

Iterating the process the Hamiltonian changes but, *remaining analytic in the same polydisk* $C_{\varrho_0,\frac{1}{2}\kappa_0}$, converges to that of a harmonic oscillator: the interpretation will be that, looking very carefully in the vicinity of the torus $\mathcal{T}_{\boldsymbol{\omega}_0} = \{\mathbf{A} = \mathbf{0}, \boldsymbol{\alpha} \in [0, 2\pi]^\ell\}$, also the perturbed Hamiltonian exhibits a harmonic motion with spectrum $\boldsymbol{\omega}_0$: the result, proved below, is the KAM theorem.

This is not only reminiscent of the methods called "renormalization group", RG, in quantum field theory but in this review it will be shown to be just a realization of them, correcting an error in [6] and adapting the correction to more recent views on the RG: see concluding remarks where the mentioned error is recalled.

2 A formal coordinate change

The Hamiltonian Eq.(1.1), considered as a holomorphic function on a domain $\mathcal{C}_{\varrho_0,\kappa_0}$ (Eq.(1.2)), will be denoted $H_0 = h_0 + f_0$. The label 0 is attached since the beginning because $H_n, f_n, \varrho_n, \kappa_n$ will arise later with $n = 1, 2, \ldots$.

The frequency spectrum $\boldsymbol{\omega}_0$ will be supposed "Diophantine", *i.e.* for some $C_0 > 0$ it is, for all $\mathbf{0} \neq \boldsymbol{\nu} \in \mathcal{Z}^\ell$ where $\mathcal{Z}^\ell$ is the lattice of the integers:

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}|^{-1} < C_0 |\boldsymbol{\nu}|^\ell, \quad \forall \boldsymbol{\nu} \neq \mathbf{0} \tag{2.1}$$

and the latter inequality will be repeatedly used to define canonical transformations with generating functions of the form $\Phi(\mathbf{A}', \boldsymbol{\alpha}) + \mathbf{A}' \cdot \boldsymbol{\alpha}$:

$$\mathbf{A} = \mathbf{A}' + \mathbf{a} + \boldsymbol{\partial}_{\boldsymbol{\alpha}} \Phi(\mathbf{A}', \boldsymbol{\alpha}), \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha} + \boldsymbol{\partial}_{\mathbf{A}'} \Phi(\mathbf{A}', \boldsymbol{\alpha}) \tag{2.2}$$

with the function Φ chosen so that in the new coordinates $(\mathbf{A}', \boldsymbol{\alpha}')$ the perturbation is *weaker*, at the price that the new coordinates will cover a (much) smaller domain, inside the $\mathcal{D}_\varrho \times \mathcal{T}^\ell$.

To simplify the notations the functions of $\boldsymbol{\alpha}$ will always be implicitly regarded as functions of $z_j = e^{i\alpha_j}$ whenever referring to their holomorphy properties, and without further comments their arguments will be written as $\mathbf{z}$ or $\boldsymbol{\alpha}$, as convenient.

At first the natural choice for Φ, *temporarily forgetting* the determination of the domain of definition of the transformation would be

$$\begin{aligned} \Phi(\mathbf{A}', \boldsymbol{\alpha}) &= -\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{Z}^\ell} \frac{f_{0,\boldsymbol{\nu}}(\mathbf{A}'+\mathbf{a})}{i(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu} + ((\mathbf{A}'+\mathbf{a})\cdot J_0\boldsymbol{\nu})} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}} \\ \mathbf{a} &= -J_0^{-1}\boldsymbol{\partial}\overline{f}_0(\mathbf{a}) \end{aligned} \tag{2.3}$$

where $f_{0,\boldsymbol{\nu}}(\mathbf{A})$ is Fourier's transform of $f_0(\mathbf{A},\boldsymbol{\alpha})$, and $\overline{f}_0(\mathbf{A}')$ denotes the average of $f_0(\mathbf{A}',\boldsymbol{\alpha})$ over $\boldsymbol{\alpha}$. The symbol $\boldsymbol{\partial}$ with no further labels means gradient with respect to the argument of the function to which it is applied.

Then inserting Eq.(2.2) into H_0 the Hamiltonian is transformed into

$$\begin{aligned} (0):\quad & H'(\mathbf{A}',\boldsymbol{\alpha}') = \frac{1}{2}(\mathbf{A}'\cdot J_0\mathbf{A}') + \boldsymbol{\omega}_0\cdot\mathbf{A}' \\ (1):\quad & +(\boldsymbol{\omega}_0 + J_0(\mathbf{A}'+\mathbf{a}))\cdot\partial_{\boldsymbol{\alpha}}\Phi + f_0(\mathbf{A}'+\mathbf{a},\boldsymbol{\alpha}) - \overline{f}_{\mathbf{0}}(\mathbf{A}'+\mathbf{a}) \\ (2):\quad & +\overline{f}_0(\mathbf{A}'+\mathbf{a}) - \overline{f}_0(\mathbf{a}) - \boldsymbol{\partial}\overline{f}_0(\mathbf{a})\cdot\mathbf{A}' \\ (3):\quad & +f_0(\mathbf{A}'+\mathbf{a}+\boldsymbol{\partial}\Phi,\boldsymbol{\alpha}) - f_0(\mathbf{A}'+\mathbf{a},\boldsymbol{\alpha}) + \frac{1}{2}\boldsymbol{\partial}\Phi\cdot J_0\boldsymbol{\partial}\Phi \end{aligned} \tag{2.4}$$

where the second of Eq.(2.3) has been used and a few terms have been added or subtracted (including free addition or substraction of constants) so that:

- (0) The unperturbed Hamiltonian,
- (1) This term vanishes if Φ is defined via Eq.(2.3);
- (2) The term is of $O(\varepsilon_0(\mathbf{A}')^2)$, hence it is a higher order term if $|\mathbf{A}'|$ is small enough.
- (3) The contributions are *formally* of higher order in the size ε_0 of f_0.

In a domain in which the transformation Eq.(2.2) could be defined, the motions would be described by a simpler Hamiltonian which is still an integrable Hamiltonian plus a perturbation of higher order in ε_0.

However to make sense of the transformation in Eq.(2.2) it is not only necessary to restrict the variables $(\mathbf{A}',\boldsymbol{\alpha})$ to a smaller domain, but it has to be possible to solve the implicit functions problem in Eq.(2.2),(2.3) (namely to express $(\mathbf{A},\boldsymbol{\alpha})$ in terms of $(\mathbf{A}',\boldsymbol{\alpha}')$ and viceversa, and finding $\mathbf{a}$), but also the denominator in Eq.(2.3) will have to be modified to avoid dividing by 0: which will happen, for generic f_0 and for some $\boldsymbol{\nu}$, on a dense set of $\mathbf{A}' \in \mathcal{D}_{\varrho_0}$, if J_0 is not singular (as it is being supposed). Therefore the map in Eq.(2.2) will now be modified and defined properly after recalling the notion of dimensional estimate.

3 Dimensional estimates

The very nature of the stability of quasi periodic motions is that it is a multiscale problem: like many other problems in analysis, from the almost everywhere convergence of Fourier series of $L_2([0,2\pi])$-functions ([3]), to the study of the possible singularities of the Navier-Stokes problem ([5]), to the convergence of the functional integrals arising in quantum field theory ([7]), to name a few. The *renormalization group* method, [8, 9], unifies the approaches developed to study such problems.

The main feature of the renormalization group applications is their being based on what will be called here "*dimensional estimates*".

Dimensional estimates deal with elementary bounds on holomorphic functions. Let $g(z)$ be any holomorphic function in a closed domain $C \subset \mathcal{C}$ (domain $\Rightarrow$ closure of an open set in the complex plain $\mathcal{C}$). The function g can be bounded, toghether with its Taylor coefficients, in terms of $||g||_C = \max_{z\in C} |g(z)|$, inside the region C_δ consisting of the points in C at distance $\geq \delta$ from the boundary of C:

$$|\partial_z^n g(z)| \leq n! \, ||g||_C \, \delta^{-n}, \qquad \forall z \in \mathcal{C}_\delta, \; n \geq 0 \tag{3.1}$$

A consequence is that if g is holomorphic in a disk $C_\varrho = \{z \,|\, |z| \leq \varrho\}$ or in an poli-annulus $\Gamma_\kappa = \{\mathbf{z} | e^{-\kappa} \leq |z_j| \leq e^\kappa, \, j = 1, \dots, \ell\}$ then the following elementary bounds on the derivatives of g or, respectively, the Fourier coefficients $g_{\boldsymbol{\nu}}$ of the function $g(e^{i\alpha})$ hold

$$\begin{aligned} &||\partial_z^n g||_{C_{\varrho'}} \leq n! \, ||g||_{C_\varrho} \, (\varrho - \varrho')^{-n}, \qquad \forall n \geq 0 \\ &|g_{\boldsymbol{\nu}}| \leq ||g||_{\Gamma_\kappa} \, e^{-\kappa|\boldsymbol{\nu}|}, \qquad \forall \boldsymbol{\nu} \in \mathcal{Z}, \; |\boldsymbol{\nu}| = \sum_{i=1}^{\ell} |\nu_i| \end{aligned} \tag{3.2}$$

Holomorphic functions g of ℓ or 2ℓ arguments will be considered, in the following, in domains

$$\begin{aligned} &\mathcal{C}_\varrho = \{\mathbf{A} | |A_j| \leq \varrho, j = 1, .., \ell\}, \quad \Gamma_\kappa = \{\mathbf{z} | e^{-\kappa} \leq |z_j| \leq e^\kappa, j = 1, .., \ell\} \\ &\mathcal{C}_{\varrho,\kappa} = \mathcal{C}_\varrho \times \Gamma_\kappa \end{aligned} \tag{3.3}$$

and their maxima will be denoted by appending labels ϱ or κ or ϱ, κ, as appropriate, to the symbol $||g||$.

Hence if $||g||_{\varrho,\kappa} = \varepsilon$ the bounds

$$\begin{aligned} &||g_{\boldsymbol{\nu}}||_\varrho \leq \varepsilon \, e^{-\kappa|\boldsymbol{\nu}|}, \qquad\qquad \forall \boldsymbol{\nu} \in \mathcal{Z}^\ell, \mathbf{A} \in \mathcal{C}_{\varrho'} \\ &||\boldsymbol{\partial}_{\mathbf{A}}^n g_{\boldsymbol{\nu}}||_{\varrho',\kappa} \leq n! \, \varepsilon \, e^{-\kappa|\boldsymbol{\nu}|} (\varrho - \varrho')^{-n}, \quad \forall \boldsymbol{\nu} \in \mathcal{Z}^\ell, \mathbf{A} \in \mathcal{C}_{\varrho'} \end{aligned} \tag{3.4}$$

hold and will be called *dimensional bounds*.

Summarizing: the dimensional bounds say that the n-th derivatives of a function holomorphic in a domain C are bounded, at a point z at distance δ

from the boundary of C, by the maximum of the function in C divided by the n-th power of the distance of z to the boundary ∂C of C times $n!$ ("Cauchy's theorem").

4 A canonical map

The renormalization group generates a map $\mathcal{R}$ whose iterations can be interpreted as successive magnifications zooming on ever smaller regions of phase space in which motions develop closer and closer to the searched quasi periodic motion of spectrum $\boldsymbol{\omega}_0$.

At step $n = 0, 1, \dots$ the motions will be described by a Hamiltonian $H_n + f_n$ which will be the sum of three terms

$$\frac{1}{2}\mathbf{A} \cdot J_n \mathbf{A} + \boldsymbol{\omega}_0 \cdot \mathbf{A} + f_n(\mathbf{A}, \boldsymbol{\alpha}), \tag{4.1}$$

see Eq.(1.1). In the renormalization group nomenclature and *under the conditions Eq.(2.1) and* $\det J_0 \neq 0$ the first and third terms would be called "*irrelevant*" and the intermediate (*i.e.* the normal form for the ℓ-dimensional harmonic oscillators Hamiltonian) would be called a "*marginal trivial fixed point*": the reason behind the latter names will become clear.

Introducing the parameters $\varepsilon_n, J_n, \varrho_n, C_n, \kappa_n$, characterizing H_n in the same sense in which $\varepsilon_0, J_0, \varrho_0, C_0, \kappa_0$ characterize H_0, it is convenient, for the purpose of a rapid evalutation of several estimates, to keep in mind that the following "dimensionless" quantities,

$$\eta_n = \varepsilon_n C_n, \qquad e^{\kappa_n} \tag{4.2}$$

will naturally occurr in the dimensional estimates: the latter will, therefore, be expressed as products of selected dimensionless quantities times a suitable factor chosen among the dimensional parameters $\varepsilon_n, \varrho_n, C_n, J_n$.

All bounds will be carefully written so that they will involve only dimensionless constants and, when needed, a factor to fix the dimensions. Furthermore the construction of the sequence H_n will be so designed that

$$C_n \equiv C_0, \varrho_n \ll \varrho_0, \kappa_n = \kappa_{n-1} - 4\delta_n > \frac{1}{2}\kappa_0 \tag{4.3}$$

with δ_n defined so that $\kappa_0 \geq \kappa_n \geq \frac{1}{2}\kappa_0$; to fix the ideas δ_n will be fixed as $\delta_n = (n+10)^{-2}\kappa_0$. f_n will tend to 0, with $\varepsilon_n \ll \varepsilon_0$, while $J_n = J_0$.

It will not be restrictive to suppose, *initially*:

$$C_0 \varrho_0 J_0 < 1, \qquad 2^{-1} < e^{\frac{\kappa_0}{2}} < e^{\kappa_n} < e^{\kappa_0} < 2 \tag{4.4}$$

because the theorem will apply for ε_0 small enough and ϱ_0, κ_0 can be *initially* restricted as needed. Furthermore it is important to keep in mind that the bounds that follow are naive dimensional bounds derived *without any optimization attempt*, yet they will suffice for a complete proof.

To define properly a tranformation inspired by Eq.(2.2) and to eliminate the mentioned possible divisions by 0, *while still keeping H' in Eq.(2.4) formally close to H_0* as in Sec.2, the first task is to determine the shift $\mathbf{a}$, Eq.(2.3).

The implicit equation Eq.(2.3) for $\mathbf{a}$, $\mathbf{a} = -J_0^{-1}\boldsymbol{\partial}\overline{f}_0(\mathbf{a}) \stackrel{def}{=} \mathbf{n}(\mathbf{a})$, can be solved under a smallness condition on ε_0 obtaining $\mathbf{a}$ close to $\mathbf{a}_0 = -J_0^{-1}\boldsymbol{\partial}\overline{f}_0(\mathbf{0})$.

This follows from an application of a general implicit function theorem yielding the existence of a constant χ such that the smallness condition $|\mathbf{n}|_{\varrho_0}|\varrho_0^{-1} < \chi$ implies existence of a solution. Since $|\mathbf{n}|_{\varrho_0}|\varrho_0^{-1}$ is dimensionally bounded by $\varepsilon_0 J_0^{-1}\varrho_0^{-1} \stackrel{def}{=} \theta_0$ the condition for the solubility of the equation is:

$$\theta_0 = \varepsilon_0\varrho_0^{-1}J_0^{-1} < \chi \qquad \Rightarrow \qquad |\mathbf{a}| < \theta_0\varrho_0 < \chi\varrho_0 < \frac{1}{16}\varrho_0 \tag{4.5}$$

see, for instance, proposition 19 in [10, Sec.5.11]. The further bound $\theta_0 < \frac{1}{16}$ is an extra property useful for the coming analysis: it holds if $\chi < \frac{1}{16}$ which in general is implied by the estimate of χ presented in the just quoted reference.

The function $f_0(\mathbf{A}'+\mathbf{a}, \boldsymbol{\alpha})$ will then be defined and analytic in $C_{\frac{3}{4}\varrho_0,\kappa_0}$ (from $\frac{3}{4}+\frac{1}{16} < 1$). Then proceed to build Φ, but replace Eq.(2.3) with its second order expansion in J_0:

$$\begin{aligned}\Phi_0(\mathbf{A}', \boldsymbol{\alpha}) = -\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{Z}^\ell} &\frac{f_{0,\boldsymbol{\nu}}(\mathbf{A}'+\mathbf{a})e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}}}{i\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}} \\ &\cdot\big(1 - \frac{J_0(\mathbf{A}'+\mathbf{a})\cdot\boldsymbol{\nu}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}} + (\frac{J_0(\mathbf{A}'+\mathbf{a})\cdot\boldsymbol{\nu}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}})^2\big)\end{aligned} \tag{4.6}$$

The function Φ_0 is well defined in the polydisk $\mathcal{C}_{\frac{3}{4}\varrho_0,\kappa_0-\delta_0}$ as seen via the following general dimensional bounds (given in Eq.(3.4) on functions bounded by ε_0 and holomorphic in a domain $\mathcal{C}_{\varrho_0,\kappa_0}$).

Taking into account the Diophantine inequality Eq.(2.1), for $0 \le \delta_0 < \kappa_0$, the definitions Eq.(4.2),(4.3),(4.4) and the dimensional inequality Eq.(3.4), with the restrictions Eq.(4.4), leads to:

$$\begin{aligned}||\Phi_0||_{\frac{3}{4}\varrho_0,\kappa_0-\delta_0} \le &\varepsilon_0\varrho_0\sum_{\boldsymbol{\nu}\neq\mathbf{0}}\frac{e^{-\delta_0|\boldsymbol{\nu}|}}{|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|}(1+\frac{|J_0|\varrho_0|\boldsymbol{\nu}|}{|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|} \\ &+(\frac{|J_0|\varrho_0|\boldsymbol{\nu}|}{|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|})^2) \le \gamma_1\eta_0\varrho_0\,\delta_0^{-c_1} \\ |\Phi_{0,\boldsymbol{\nu}}(\mathbf{A}')| <&\gamma_1\eta_0\,\varrho_0\,\delta_0^{-c_1}\,e^{-\kappa_0|\boldsymbol{\nu}|}, \qquad \forall|\mathbf{A}'| < \varrho_0\end{aligned} \tag{4.7}$$

with γ_1 a dimensionless constant, $\eta_0 = \varepsilon_0 C_0$ and $J_0 C_0 \varrho_0 < 1$ have been used and $c_1 = 5\ell + 2$

Hence the functions in the *r.h.s* of Eq.(2.2) admit the dimensional bounds:

$$\begin{aligned}&||\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi_0||_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0} \le \gamma_2\eta_0\,\varrho_0\delta_0^{-c_2},\ ||\boldsymbol{\partial}_{\mathbf{A}'}\Phi_0||_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0} \le \gamma_3\eta_0\,\delta_0^{-c_3} \\ &||\boldsymbol{\partial}^2_{\boldsymbol{\alpha}\mathbf{A}'}\Phi_0||_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0} \le \gamma_4\eta_0\,\delta_0^{-c_4},\ ||\boldsymbol{\partial}^2_{\mathbf{A}'\mathbf{A}'}\Phi_0||_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0} \le \gamma_5\eta_0\varrho_0^{-1}\delta_0^{-c_5}\end{aligned} \tag{4.8}$$

where the derivatives with respect to α_j should be interpreted as $iz_j\partial_{z_j}$ for $z_j = e^{i\alpha_j}$ in the domain $(\mathbf{A}', \boldsymbol{\alpha}) \in C_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0}$, and a possible choice for the constants γ_i can be found in the appendix while c_i can be taken $c_2 = 5\ell+3, c_3 = 5\ell+2, c_4 = 5\ell+4, c_5 = 5\ell+2$. The radius is reduced to $\frac{2}{3}\varrho_0$ to allow simple dimensional bounds using $\frac{3}{4} - \frac{1}{16} > \frac{2}{3}$ (taking into account the second inequality in Eq.(4.5)).

To define the canonical transformation $(\mathbf{A}', \boldsymbol{\alpha}') \to (\mathbf{A}, \boldsymbol{\alpha})$ the implicit functions in Eq.(2.2) have to be solved. This can be done quite easily if one is willing to define the map only for $(\mathbf{A}', \boldsymbol{\alpha}')$ contained in a small enough domain.

The condition of solubility for $(\mathbf{A}', \boldsymbol{\alpha}') \in \mathcal{C}_{\varrho',\kappa'}$ for $\varrho' = \frac{1}{2}\varrho_0, \kappa' = \kappa_0 - 3\delta_0$ is prescribed (simple implicit function theorem for analytic functions, see for instance propositions 20,21 in Sec.5.11 and Appendix N in [10]) on dimensional grounds simply by

$$||\boldsymbol{\partial}^2_{\mathbf{A}'\boldsymbol{\alpha}}\Phi_0||_{\frac{2}{3}\varrho_0,\kappa_0-2\delta_0} < \gamma_6\eta_0\,\delta_0^{-c_6} < 1 \tag{4.9}$$

where the first inequality is just the bound Eq.(4.8) on the *l.h.s.* with γ_4 modified into a larger γ_6 and c_6 a constant (*e.g.* $5\ell+4$).

The solution can be obtained, again reducing the radius from $\frac{2}{3}\varrho_0$ to $\frac{1}{2}\varrho_0$ for ease of dimensional bounds in what follows, by first fixing $\mathbf{A}' \in C_{\frac{1}{2}\varrho_0}$ so that the second inequality in Eq.(4.9) simply *implies injectivity* of the map $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + \partial_{\mathbf{A}'}\Phi_0(\mathbf{A}', \boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in C_{\kappa_0-2\delta_0}$, for all $\mathbf{A}'$ fixed in $C_{\frac{1}{2}\varrho_0}$; it implies also $\boldsymbol{\alpha} \in C_{\kappa_0-2\delta_0}$ for $\boldsymbol{\alpha}' \in C_{\kappa_0-3\delta_0}$ if γ_6 is large enough. Therefore, given $\mathbf{A}' \in C_{\frac{1}{2}\varrho_0}$ and using the injectivity, $\boldsymbol{\alpha}$ can be computed from $\boldsymbol{\alpha}'$ in the form

$$\begin{aligned}
&\boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}'), \qquad \boldsymbol{\alpha}' \in C_{\kappa_0-3\delta_0}, \forall \mathbf{A}' \in C_{\frac{1}{2}\varrho_0} \\
&\boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}') \equiv -\partial_{\mathbf{A}'}\Phi_0(\mathbf{A}', \boldsymbol{\alpha}) \\
&||\boldsymbol{\Delta}||_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0} < \gamma_7\eta_0\delta_0^{-c_7} < \delta_0,
\end{aligned} \tag{4.10}$$

where the second line in Eq.(4.10) is an identity which implies, via Eqs.(4.8),(4.9), the inequalities in the third line, where γ_7, c_7 are suitable positve constants.

The second inequality in Eq.(4.9) also insures the injectivity of $\mathbf{A} = \mathbf{A}' + \partial_{\boldsymbol{\alpha}}\Phi_0(\mathbf{A}', \boldsymbol{\alpha})$ for $\mathbf{A}'$ in $C_{\frac{1}{2}\varrho_0}$, for all $\boldsymbol{\alpha}$ fixed in $C_{\kappa_0-3\delta_0}$. Hence having defined $\boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}')$ the angles $\boldsymbol{\alpha}$ can be expressed in terms of $\boldsymbol{\alpha}', \mathbf{A}'$; and it is possible to express, for each $\boldsymbol{\alpha}' \in C_{\kappa_0-3\delta_0}$, $\mathbf{A}$ in terms of $\mathbf{A}'$, $\forall \mathbf{A}' \in C_{\frac{1}{2}\varrho_0}$: simply by substituting $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}')$ and finding:

$$\begin{aligned}
&\mathbf{A} = \mathbf{A}' + \mathbf{a} + \boldsymbol{\Xi}(\mathbf{A}', \boldsymbol{\alpha}') \\
&\boldsymbol{\Xi}(\mathbf{A}', \boldsymbol{\alpha}') \equiv \partial_{\boldsymbol{\alpha}}\Phi_0\Big(\mathbf{A}', \boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}')\Big),
\end{aligned} \tag{4.11}$$

and for $(\mathbf{A}', \boldsymbol{\alpha}') \in C_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0}$ the $(\mathbf{A}, \boldsymbol{\alpha})$ will vary inside the original domain.

Then again Eqs.(4.8),(4.9), if $\gamma_2'\eta_0\delta_0^{-c_2} < \frac{1}{16}$, for γ_2' suitably larger then γ_2 yield

$$|\mathbf{A}| < \frac{1}{2}\varrho_0 + ||\boldsymbol{\Xi}||_{\frac{1}{2}\varrho_0,\kappa_0-3\delta_0} \le \frac{1}{2}\varrho_0 + \gamma_3\eta_0\varrho_0\delta_0^{-c_3} < \frac{3}{4}\varrho_0 \tag{4.12}$$

Collecting all conditions to define $\mathbf{a}, \boldsymbol{\Delta}, \boldsymbol{\Xi}$ a canonical map

$$\begin{aligned} \mathbf{A} =& \mathbf{A}' + \mathbf{a} + \boldsymbol{\Xi}(\mathbf{A}', \boldsymbol{\alpha}'), \qquad \boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}') \\ & ||\boldsymbol{\Xi}||_{\frac{1}{2}\varrho_0, \kappa_0 - 3\delta_0} < \gamma_8 \eta_0 \, \varrho_0 \delta_0^{-c_8} \\ & ||\boldsymbol{\Delta}||_{\frac{1}{2}\varrho_0, \kappa_0 - 3\delta_0} < \gamma_8 \eta_0 \, \delta_0^{-c_8} \end{aligned} \tag{4.13}$$

will be defined, for suitably chosen γ_8, c_8, changing $(\mathbf{A}', \boldsymbol{\alpha}') \in C_{\frac{1}{2}\varrho_0, \kappa_0 - 3\delta_0}$ into $C_{\frac{3}{4}\varrho_0, \kappa_0 - \delta_0}$.

The new function $f_0'(\mathbf{A}', \boldsymbol{\alpha}') = f_0(\mathbf{A}' + \mathbf{a} + \boldsymbol{\partial}\Phi, \boldsymbol{\alpha}) - \overline{f}_0(\mathbf{a})$ will be expressed by the three terms in Eq.(2.4) as discussed in the next section in terms of η_0, δ_0; the conditions imposed in the construction can be all implied by the conditions

$$\begin{aligned} & C_0 \varrho_0 J_0 < 1, \quad e^{\kappa_0} < 2 \qquad \text{initial restrictions} \\ & \varepsilon_0 J_0^{-1} \varrho_0^{-1} < \chi, \qquad \text{to define} \qquad \mathbf{a} = -J_0^{-1} \boldsymbol{\partial} \overline{f}_0(\mathbf{a}) \\ & \gamma_9 \eta_0 \delta_0^{-c_9} < 1, \qquad \text{to define } \boldsymbol{\Delta}, \boldsymbol{\Xi} \end{aligned} \tag{4.14}$$

for γ_9, c_9 large enough and χ small enough, see Eq.(4.5).

The domain of variability in the initial variables $(\mathbf{A}, \boldsymbol{\alpha})$, where the canonical map is defined, will now contain (at least) a small domain of shape close to a polydisk (*eccentric* because of the translation by $\mathbf{a}$) inside the initial domain C_{ϱ_0, κ_0} of the Hamiltonian H_0. The small eccentric polydisk is the image of a *centered* polydisk $\mathcal{C}_{\frac{1}{2}\varrho_0, \kappa_0 - 3\delta_0}$ in the new variables $(\mathbf{A}', \boldsymbol{\alpha}')$.

5 Renormalization

The Hamiltonian $H_0 + f_0$ in the new coordinates $\mathbf{A}', \boldsymbol{\alpha}'$ becomes:

$$H'(\mathbf{A}', \boldsymbol{\alpha}') = \frac{1}{2} \mathbf{A}' \cdot J_0 \mathbf{A}' + \boldsymbol{\omega}_0 \cdot \mathbf{A}' + f', \qquad (\mathbf{A}', \boldsymbol{\alpha}') \in \mathcal{C}_{\frac{1}{2}\varrho_0, \kappa_0 - 3\delta_0} \tag{5.1}$$

in the domain $(\mathbf{A}', \boldsymbol{\alpha}') \in C_{\frac{1}{2}\varrho_0, \kappa_0 - 3\delta_0}$. The function f' is defined, in the mixed variables $(\mathbf{A}', \boldsymbol{\alpha})$, by Eq.(2.4).

- The contribution 1) in Eq.(2.4), *does not vanish: but it carries the key cancellation* showing that the sum of terms individually formally $O(\varepsilon_0)$ is in fact of higher order in ε_0 as can be seen via the Fourier's transform of $f_0 - \overline{f}_0 = \sum_{\mathbf{0} \neq \boldsymbol{\nu}} f_{0,\boldsymbol{\nu}} e^{i\boldsymbol{\alpha} \cdot \boldsymbol{\nu}}$ which, after a few simplifications, is:

$$\begin{aligned} & (\boldsymbol{\omega}_0 + J_0(\mathbf{A}' + \mathbf{a}) \cdot \boldsymbol{\partial}_{\boldsymbol{\alpha}} \Phi_0 + f_0(\mathbf{A}' + \mathbf{a}, \boldsymbol{\alpha}) - \overline{f}_0(\mathbf{A}' + \mathbf{a}) \\ & = \sum_{\mathbf{0} \neq \boldsymbol{\nu}} f_{0,\boldsymbol{\nu}}(\mathbf{A}' + \mathbf{a}) \frac{(J_0(\mathbf{A}' + \mathbf{a}) \cdot \boldsymbol{\nu})^3}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu})^3} e^{i\boldsymbol{\alpha} \cdot \boldsymbol{\nu}} \end{aligned} \tag{5.2}$$

- using $J_0 \mathbf{a} = -\boldsymbol{\partial} f_0(\mathbf{a})$, admits, if $|\mathbf{A}'| < \widetilde{\varrho}$, the dimensional bound, *in the sense of Eq.(1.2)*, (using $(J_0 C_0 \varrho_0) < 1$, and Eq.(3.4) together with the

bound $|f_{0,\boldsymbol{\nu}}| < \varepsilon_0 \varrho_0 e^{-\kappa_0|\boldsymbol{\nu}|}$ from $\frac{1}{\varrho_0}|\boldsymbol{\partial}_{\boldsymbol{\alpha}} f_0| \le \varepsilon_0$):

$$\begin{aligned} &\gamma_{10}\varepsilon_0 \frac{\varrho_0}{\widetilde{\varrho}}((J_0\widetilde{\varrho}C_0)^3 + (C_0\varepsilon_0\varrho_0^{-1})^3)\delta_0^{-c_{10}} \\ &= \gamma_{10}\varepsilon_0 \frac{\varrho_0}{\widetilde{\varrho}}(\frac{\widetilde{\varrho}^3}{\varrho_0^3} + \eta_0^3)\delta_0^{-c_{10}} < \gamma_{11}\varepsilon_0\eta_0^{2\sigma}\delta_0^{-c_{11}} \end{aligned} \tag{5.3}$$

in the polydisk $\mathcal{C}_{\widetilde{\varrho},\kappa_0-3\delta_0}$, if $\widetilde{\varrho} = \eta_0^\sigma$ and $0 < \sigma \le \frac{1}{2}$.

- The contribution 2) in Eq.(2.4), is bounded, *still in the sense of Eq.(1.2)*, in a disk of radius $\widetilde{\varrho} = \eta_0^\sigma \varrho_0$ by

$$\gamma_{13}\varepsilon_0\eta_0 \tag{5.4}$$

making use of its $\boldsymbol{\alpha}$-independence (which permits to estimate the second derivative of $\overline{f}_0(\mathbf{A})$ in a disk of radius $\frac{1}{2}\varrho_0$) also yielding a contribution to the higher order terms if $\widetilde{\varrho} = \varrho_0\eta_0^\sigma$ is suitably small.

- The terms in the contribution 3) are also bounded, *still in the sense of Eq.(1.2)*, by:

$$\gamma_{14}\varepsilon_0\eta_0\delta_0^{-c_{14}} \tag{5.5}$$

in the polydisk $\mathcal{C}_{\widetilde{\varrho},\kappa_0-3\delta_0}$, using $C_0\varrho_0 J_0 < 1$.

Adding the bounds Eq.(5.3),(5.4),(5.5) it is found, for $\sigma = \frac{1}{2}$:

$$\varepsilon_1 = (|\boldsymbol{\partial}_{\mathbf{A}'} f'|_{\varrho_0\eta_0^{\frac{1}{2}},\kappa_0-4\delta_0} + \frac{1}{\varrho_0\eta_0^{\frac{1}{2}}}|\boldsymbol{\partial}_{\boldsymbol{\alpha}'} f'|_{\varrho_0\eta_0^{\frac{1}{2}},\kappa_0-4\delta_0} < \gamma\varepsilon_0\eta_0\delta_0^{-c}) \tag{5.6}$$

for $\gamma, c > 0$ suitably fixed, if $C_0\varrho_0 J_0 < 1$, (seealso Eq.(4.5)).

The result is that in the coordinates $\mathbf{A}_1, \boldsymbol{\alpha}_1$ the motion is Hamiltonian with Hamiltonian H_1, recalling the definitions of the dimensionless quantities in Eq.(4.2):

$$\begin{aligned} &H_1 = \mathbf{A}_1 \cdot J_0\mathbf{A}_1 + \boldsymbol{\omega}_0 \cdot \mathbf{A}_1 + f_1(\mathbf{A}_1, \boldsymbol{\alpha}_1) \\ &\varrho_1 = \varrho_0\eta_0^{\frac{1}{2}}, \quad \kappa_1 = \kappa_0 - \overline{\delta}_0, \quad J_1 = J_0, \quad C_1 = C_0 \\ &\eta_1 = \gamma\eta_0^2\delta_0^{-c}, \qquad \theta_1 = \gamma\theta_0\eta_0^{\frac{1}{2}}\delta_0^{-c}, \qquad J_1C_1\varrho_1 < J_0C_0\varrho_0 < 1 \end{aligned} \tag{5.7}$$

where γ, c are constants and $\overline{\delta}_n = 4\delta_n = \kappa_0(n+10)^{-2}$.

The above transformation of coordinates, which will be denoted $\mathcal{K}_0$, is well defined and holomorphic in the domain $C_{\frac{1}{2}\varrho_0,\kappa_0-4\delta_0}$ *provided* ε_0 *is small enough* so that the conditions imposed during the construction, namely Eq.(4.14), and the ones following it, are satisfied and *remain satisfied under iteration* allowing to define the sequence of maps $\mathcal{K}_n$, $n \ge 0$: because if η_0 (*i.e.* ε_0) is small

enough the map in Eq.(5.7) generates a sequence with $C_n\varepsilon_n = C_0\varepsilon_n = \eta_n$ tending to 0, fixed arbitrarlily $\mu \in (0, \frac{1}{2})$ and a corresponding suitable constant $\overline{\gamma}$, superexponentially with

$$\eta_n \sim (\overline{\gamma}\eta_0)^{(1+\mu)^n}, \quad \overline{\gamma} > 0,\ 0 < \mu < \frac{1}{2} \tag{5.8}$$

and θ_n also tend to 0 at similar rates (*e.g.* $\theta_n \sim (\overline{\gamma}'\eta_0)^{(1+\mu)^n c'}, c' < 1$), as can be checked by induction from Eq.(5.7) with suitable $\overline{\gamma}', c'$. This implies that for all $n \geq 0$ the transformations $\mathcal{K}_n$ can be defined if ε_0 (*i.e.* its dimensionless version η_0) is small enough.

Furthermore $\mathcal{K}_n$ is seen from Eq.(4.13) to be close to the identity within $\gamma_8\eta_n\delta_n^{-c_8}$. Hence the iteration of the renormalization procedure defines a sequence of transformations $\mathcal{K}_n$ under the only initial condition in Eq.(4.14) with γ_9, c_9, χ^{-1} large enough.

In the polydisk $\mathcal{C}_{\varrho_n,\kappa_n}$ the motions starting with $\mathbf{A}_n = \mathbf{0}$ and (say)$\boldsymbol{\alpha} = \mathbf{0}$ become closer and closer to the motion of a harmonic oscillator with frequency spectrum $\boldsymbol{\omega}_0$ and in the limit $n \to \infty$ all motions in the "polydisk" (degenerated to a torus $\mathbf{0} \times \mathcal{T}^\ell$) are harmonic with spectrum $\boldsymbol{\omega}_0$ This is checked simply by remaking that the motion of the initial data is, if observed in an *arbitrarily fixed time* t, is superxponentially close to the harmonic motion $\mathbf{A} = \mathbf{0}, \boldsymbol{\alpha}(t) = \boldsymbol{\alpha} + \boldsymbol{\omega}_0 t$. The torus on which the motion is quasi periodic is the limit of the tori with equations $\mathbf{A} = \mathbf{a}_n + \boldsymbol{\Xi}_n(\mathbf{a}_n, \boldsymbol{\alpha}'), \boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\Delta}_n(\mathbf{a}_n, \boldsymbol{\alpha}')$ which is the torus which at the n-th iteration of the renormalization has coordinates $\mathbf{a}_n, \boldsymbol{\alpha}'$). The successive corrections to $\mathbf{a}_n$ and to the functions $\boldsymbol{\Xi}_n, \boldsymbol{\Delta}_n$ tend to 0 superxponentially and the limits

$$\mathbf{a}_\infty,\ \boldsymbol{\Xi}_\infty(\boldsymbol{\alpha}'),\ \boldsymbol{\Delta}_\infty(\boldsymbol{\alpha}'), \qquad \boldsymbol{\alpha}' \in T^\ell \tag{5.9}$$

define an invariant torus on which motion is $\boldsymbol{\alpha}' \underset{\sim}{\to} \boldsymbol{\alpha}' + \boldsymbol{\omega} t$.

It is also possible to define a sequence of maps $\widetilde{\mathcal{K}}_n$ defined in the *fixed domain* $\mathcal{C}_{\frac{1}{2}\varrho_0,\frac{1}{2}\kappa_0}$ by rescaling the polydisks by a factor $h_{n-1}^{\frac{1}{2}} = \varrho_{n-1}/\varrho_n$, $n \geq 1$ so that they are all turned into $\mathcal{C}_{\frac{1}{2}\varrho_0,\frac{1}{2}\kappa_0}$: the rescaling transformation will change $\mathbf{A}_n$ into $\mathbf{A}'_n = \eta_n^{-\frac{1}{2}}\mathbf{A}_n$ and the Hamiltonian into

$$\widetilde{H}_n = \boldsymbol{\omega}_0 \cdot \mathbf{A}'_n + \eta_n^{\frac{1}{2}}\frac{1}{2}(\mathbf{A}'_n \cdot J_0\mathbf{A}'_n) + \eta_n^{-\frac{1}{2}} f_n(\eta_n^{\frac{1}{2}}\mathbf{A}'_n, \boldsymbol{\alpha}'_n) \xrightarrow[n\to\infty]{} \boldsymbol{\omega}_0 \cdot \mathbf{A}_\infty \tag{5.10}$$

This shows that the perturbation f and the twist j_0 are, after renormalization, a "irrelevant operators", while the harmonic oscillator is a "fixed point": in some sense the transformation has the harmonic oscillator as an *attractive fixed point. This completes a proof of the KAM theorem,* [11, 13, 14, 15].

Remarks: (1) a simpler analysis (and an instructive warm-up exercise) can be carried also if $J_0 = 0$ provided the perturbation depends only on the angles $\boldsymbol{\alpha}$. The independence of f_0 from $\mathbf{A}$ has the consequence, in the proof development, that all terms appearing to involve J_0^{-1} actually do not arise at all (and the

system is integrable).
(2) Since the condition $\det J_0 \neq 0$ is called "anisochrony condition" or "twist condition" the invariant tori that may exist when f_0 depends only on $\boldsymbol{\alpha}$ and $\det J_0 \neq 0$ are called "twistless", [16]. Their construction, if f_0 is an even function of $\boldsymbol{\alpha}$ (*i.e.* $f_0(\boldsymbol{\alpha}) = \sum_n f_n \cos(\boldsymbol{\nu}\cdot\boldsymbol{\alpha})$), can proceed via a simple graphical algorithm leading to a new "direct" proof of the KAM theorem, [17, 18, 19], that can even be extended to the general case, [20].
(3) The estimates in the above analysis are far from optimal and optimization is desirable.

6 Harmonic oscillator

- (1) If $\boldsymbol{\partial}\overline{f}_0(\mathbf{0}) = \mathbf{0}$ then $\mathbf{a} = \mathbf{0}$ is a solution to $\mathbf{a} = -J_0^{-1}\boldsymbol{\partial}\overline{f}_0(\mathbf{a})$ and no conditions on $\varepsilon_0, J_0, \varrho_0$ are required to obtain it; of course after the first canonical transformation the $\mathbf{a}_1 = -J_0^{-1}\boldsymbol{\partial}\overline{f}_1(\mathbf{a}_1)$ will have to be solved: therefore conditions on the size of $\varepsilon_n J_n^{-1}\varrho_n^{-1}$ can be avoided only if $\boldsymbol{\partial}\overline{f}_n(\mathbf{a}_n) = \mathbf{0}$ remains true for all $n \geq 0$.

  Are there cases in which the latter property holds for all n? if so the consequnce is that the invariant torus, after rescaling, is not eccentric but it is located at $\mathbf{a} = 0$ and the time average of $\mathbf{A}$ on the torus is $\mathbf{0}$.

  This seems to be the case of the *twistless Hamiltonians*: $H_0 = \frac{1}{2}(\mathbf{A}\cdot J_0\mathbf{A}) + \boldsymbol{\omega}_0 \cdot \mathbf{A} + f_0(\boldsymbol{\alpha})$ with f_0 even in $\boldsymbol{\alpha}$ and depending only on $\boldsymbol{\alpha}$, [16]). For such systems it is known that the average action on the invariant torus is $\mathbf{0}$ and furthermore it is true that the angles average of the pertubation after m perturbation theory steps is identically 0 if evaluated at $\mathbf{A} = \mathbf{0}$. A formal proof is desirable.

- (2) In general the Hamiltonian H_0 has, for ε_0 small, an invariant torus $\mathbf{A} = \mathbf{a}_\infty + \boldsymbol{\Xi}_\infty(\boldsymbol{\alpha}'), \boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\Delta}_\infty(\boldsymbol{\alpha}'), \boldsymbol{\alpha}' \in T^\ell$ on which the motion is $\boldsymbol{\alpha}' \to \boldsymbol{\alpha}' + \boldsymbol{\omega}t$. Then the change of variables

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\Delta}_\infty(\boldsymbol{\alpha}'), \mathbf{A} = (1 + \boldsymbol{\partial}_{\boldsymbol{\alpha}'}\boldsymbol{\Delta}_\infty(\boldsymbol{\alpha}'))(\mathbf{A}' + \mathbf{a}_\infty) \tag{6.1}$$

  describes for $\mathbf{A}' = \mathbf{0}$ the $\boldsymbol{\Xi}_\infty(\boldsymbol{\alpha}')$, hence the invariant torus; and is a map $(\mathbf{A}', \boldsymbol{\alpha}') \to (\mathbf{A}, \boldsymbol{\alpha})$ holomorphic and canonical (of course the tori with constant $\mathbf{A}' \neq \mathbf{0}$ in general are not invariant).

  The tranformation in Eq.(6.1) tranforms the initial Hamiltonian into a new one $H_{ren} = \frac{1}{2}(\mathbf{A}' \cdot J_0\mathbf{A}') + \boldsymbol{\omega}_0 \cdot \mathbf{A}' + f_{0,ren}(\mathbf{A}', \boldsymbol{\alpha}')$, holomorphic in a polydisk $C_{\varrho', \frac{1}{4}\kappa_0}$ with $\varrho' \geq \frac{1}{8}\varrho_0$ (determined by the condition that $|\mathbf{a}_\infty| + \varrho' > \frac{1}{4}\varrho_0$ so that f_0 is defined in the new variables: which the bounds in Sec.5 imply if ε_0 is small enough).

  Applying the theory of Sec.5 to H_{ren} it follows that $\boldsymbol{\partial}\overline{f}_{0,ren}(\mathbf{0}) = 0$: for consistency with perturbation theory and the property $\boldsymbol{\alpha}'$ evolving as $\boldsymbol{\alpha}' + \boldsymbol{\omega}t$ for $\mathbf{A}' = \mathbf{0}$. The property $\boldsymbol{\partial}\overline{f}_{n,ren}(\mathbf{0}) = 0, \ \forall n \geq 0$ remains true, again

for consistency, at leading order: therefore the KAM tori constructed in Sec.5 can be imagined described by a sequence of Hamoltionians $\widetilde{H}_{ren,n}$ which remain defined in a fixed polydisk, see Sec.6, $\mathcal{C}_{\frac{1}{8}\varrho_0,\frac{1}{4}\kappa_0}$ inside which they converge to $\boldsymbol{\omega}\cdot\mathbf{A}$. This is a different, and *a posteriori*, way of conceiving the convergence to the harmonic motions found by the KAM theorem.

A question arises: suppose that a system has an invariant torus of equations $\boldsymbol{\alpha}=\boldsymbol{\alpha}'+\boldsymbol{\Delta}(\boldsymbol{\alpha}'),\mathbf{A}=\mathbf{a}+\boldsymbol{\Xi}(\boldsymbol{\alpha}'),\boldsymbol{\alpha}'\in T^\ell$, on which the motion is $\alpha'\to\boldsymbol{\alpha}'+\boldsymbol{\omega}_0 t$ with $\boldsymbol{\omega}_0$ non resonant and $\boldsymbol{\Xi}$ has $\mathbf{0}$ average over $\boldsymbol{\alpha}'$. Change variables via the generating function $\mathbf{A}'\cdot\boldsymbol{\alpha}+(1+\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Gamma(\boldsymbol{\alpha}))^{-1}(\mathbf{A}'+\mathbf{a})$ where $\boldsymbol{\alpha}'=\boldsymbol{\alpha}+\Gamma(\boldsymbol{\alpha})$ is the inverse function to $\boldsymbol{\alpha}'+\boldsymbol{\Delta}(\boldsymbol{\alpha}')$. Is the torus a KAM torus? *i.e.* can it be costructed as a fixed point of a sequence of canonical maps?

7 Comments

The analysis in Sec.5 is a reformulation of the original proof by Kolmogorov, [11], reproduced in full detail in [15] and used to build a rigorous computation algorithm in [21]. The feature of the approach, common also to Moser's work, [14], is to use canonical maps with fixed small denominators: this avoids dealing with $\mathbf{A}$ dependent divisors appearing in [13, p.105], reproduced in [10].

The renormalization group interpretation has been proposed in in [22] with prefixed divisors and [23, 6] still dealing with $\mathbf{A}$-dependent divisors: the approach developed in Sections 4,5 is inspired by the latter development but avoids $\mathbf{A}$-dependent divisors, hence it is close to [11, 14, 15, 21, **?**, 24] and several other approaches. In [6] the definition of ε_n, see Eq.(1.2), is replaced by $\varepsilon_0=\max_{\mathcal{C}}|f_0|$: this choice would be possible here too, with obvious notational changes.

The relation between the KAM theorem and the renormalization group has been used in various forms for its proof, in several papers, for instance [23, 22, 25, 6, 26, 27, 28, 29].

The difference between the approach of Kolmogorov and Moser, with respect to Arnold's, [13], is that in the second the small divisors are $\mathbf{A}$-dependent and are controlled by an increasing sequence of cut-offs on $\boldsymbol{\nu}$, at each order of the perturbation expansion.

The analysis of the singularity at $\varepsilon_0=0$, in the case of resonant quasi periodic motions (*i.e.* motions which dwell on lower dimensional tori), can also be pursued via multiscale methods conveniently interpreted as methods of performing the resummations of the perturbative series, which unlike the KAM case, are divergent power series, [30, 31, 32].

A Appendix

Remark that the distance of the boundary of the polyannulus Γ_{κ_0} to that of $\Gamma_{\kappa_0-\delta_0}$ is bounded, if $\frac{1}{2} \le e^{\pm\kappa_0} \le 2$, below by $\frac{1}{2}\delta_0$ and above by $2\delta_0$.

The injectivity follows by writing $z_j = e^{i\alpha_j}$ and, integrating along the shortest path enclosed in $C_{\kappa_0-\delta_0}$ connecting $\mathbf{z}_1$ and $\mathbf{z}_2$:

$$\begin{aligned} |\mathbf{z}'_{1,j} - \mathbf{z}'_{2,j}| &= \int_{\mathbf{z}_1}^{\mathbf{z}_2} \sum_j dz_j \, \partial_{z_j} \Big(z_j \exp(i\partial_{A'_j} \Phi_0(\mathbf{A}', \boldsymbol{\alpha})) \Big) \\ &\ge |\mathbf{z}_1 - \mathbf{z}_2| \Big(1 - \ell e^{\gamma_3 \eta_0 \delta_0^{-c_3}} \gamma_4 \eta_0 \delta_0^{-3\ell-3} - \frac{\pi\ell}{2} \gamma_4 \eta_0 \delta_0^{-c_4} \Big) \\ &\ge \frac{1}{2} |\mathbf{z}_1 - \mathbf{z}_2| \end{aligned} \tag{A.1}$$

deduced after taking into account the inequalities Eq.(4.8),(4.9).

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