

# Universality of one-dimensional Fermi systems, II. The Luttinger liquid structure.

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## Abstract

We complete the proof started in [1] of the universal Luttinger liquid relations for a general model of spinning fermions on a lattice, by making use of the Ward Identities due to asymptotically emerging symmetries. This is done by introducing an effective model verifying extra symmetries and by relating its critical exponents to those of the fermion lattice gas by suitable fine tuning of the parameters.

## 1 Main results

We consider a standard (generically non solvable) model of a gas of spinning fermions on a one dimensional lattice with a short range repulsive interaction, with Hamiltonian

$$H = -\frac{1}{2} \sum_{\substack{x \in \mathcal{C} \\ s = \pm}} (a_{x,s}^+ a_{x+1,s}^- + a_{x,s}^+ a_{x-1,s}^-) + \bar{\mu} \sum_{\substack{x \in \mathcal{C} \\ s = \pm}} a_{x,s}^+ a_{x,s}^- + \lambda \sum_{\substack{x,y \in \mathcal{C} \\ s,s' = \pm}} v_L(x-y) a_{x,s}^+ a_{x,s}^- a_{y,s'}^+ a_{y,s'}^- \quad (1.1)$$

where  $\mathcal{C} = \{1, 2, \dots, L\}$  is a one dimensional lattice of  $L$ ,  $a_{x,s}^\pm$  are fermion creation and annihilation operators at site  $x$  with spin  $s$ , and such that  $a_{1,s}^\pm = a_{L+1,s}^\pm$  (periodic boundary conditions),  $v_L(x)$  is a function on  $\mathbb{Z}$ , periodic of period  $L$ , such that  $v_L(x) = v(x)$  for  $-[L/2] \leq x \leq [(L-1)/2]$ ,  $v(x)$  being an even function on  $\mathbb{Z}$  satisfying the short range condition  $|v(x)| \leq C e^{-\kappa|x|}$ , and  $-\bar{\mu} \in (-1, +1)$  is the *chemical potential*. In the special case  $\lambda v(x-y) = U \delta_{x,y}$  the model is known as *Hubbard model*, which is exactly solvable by Bethe ansatz [2].

We define the operators  $a_{\mathbf{x},s}^\pm = e^{x_0 H} a_x^\pm e^{-H x_0}$ , with  $\mathbf{x} = (x, x_0)$ ,  $0 \leq x_0 < \beta$  for some  $\beta > 0$  ( $\beta^{-1}$  is the temperature) and the densities  $\rho_{\mathbf{x}}^\alpha$  with  $\alpha = C, S_i, SC_i, TC_i$  denoting respectively the *Charge density*, the *spin densities* and singlet and tripled *Cooper densities*

$$\begin{aligned} \rho_{\mathbf{x}}^C &= \sum_{s=\pm} a_{\mathbf{x},s}^+ a_{\mathbf{x},s}^- & \rho_{\mathbf{x}}^{S_i} &= \sum_{s,s'=\pm} a_{\mathbf{x},s}^+ \sigma_{s,s'}^{(i)} a_{\mathbf{x},s'}^- \\ \rho_{\mathbf{x}}^{SC} &= \frac{1}{2} \sum_{\substack{s=\pm \\ \varepsilon=\pm}} s a_{\mathbf{x},s}^\varepsilon a_{\mathbf{x},-s}^\varepsilon & \rho_{\mathbf{x}}^{TC_i} &= \frac{1}{2} \sum_{\substack{s,s'=\pm \\ \varepsilon=\pm}} a_{\mathbf{x},s}^\varepsilon \tilde{\sigma}_{s,s'}^{(i)} a_{\mathbf{x}+\mathbf{e},s'}^\varepsilon \quad , \quad \mathbf{e} = (1, 0) \end{aligned} \quad (1.2)$$

where  $\sigma^{(i)}$  are the Pauli matrices, while

$$\tilde{\sigma}^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \tilde{\sigma}^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \tilde{\sigma}^{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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Similarly we will introduce the *paramagnetic* and *diamagnetic* part of the current

$$\mathbf{J}_{\mathbf{x}} = \frac{1}{2i} \sum_{s=\pm} [a_{\mathbf{x}+\mathbf{e},s}^+ a_{\mathbf{x},s}^- - a_{\mathbf{x},s}^+ a_{\mathbf{x}+\mathbf{e},s}^-] \quad , \quad \tau_{\mathbf{x}} = -\frac{1}{2} \sum_{s=\pm} [a_{\mathbf{x},s}^+ a_{\mathbf{x}+\mathbf{e},s}^- + a_{\mathbf{x}+\mathbf{e},s}^+ a_{\mathbf{x},s}^-] \quad (1.3)$$

Defining  $\langle \cdot \rangle_{L,\beta} := \frac{\text{Tr}[e^{-\beta H} \cdot]}{\text{Tr}[e^{-\beta H}]}$ , the density and current *response functions* are defined by the following truncated correlations:

$$\begin{aligned} \Omega_{\alpha,\beta,L}(\mathbf{x} - \mathbf{y}) &:= \langle \mathbf{T} \rho_{\mathbf{x}}^{\alpha} \rho_{\mathbf{y}}^{\alpha} \rangle_{T;\beta,L} := \langle \mathbf{T} \rho_{\mathbf{x}}^{\alpha} \rho_{\mathbf{y}}^{\alpha} \rangle_{\beta,L} - \langle \rho_{\mathbf{x}}^{\alpha} \rangle_{\beta,L} \langle \rho_{\mathbf{y}}^{\alpha} \rangle_{\beta,L} \\ \Omega_{j,j,\beta,L}(\mathbf{x} - \mathbf{y}) &:= \langle \mathbf{T} J_{\mathbf{x}} J_{\mathbf{y}} \rangle_{T;\beta,L} := \langle \mathbf{T} J_{\mathbf{x}} J_{\mathbf{y}} \rangle_{\beta,L} - \langle J_{\mathbf{x}} \rangle_{\beta,L} \langle J_{\mathbf{y}} \rangle_{\beta,L} \end{aligned} \quad (1.4)$$

where, if  $O_{\mathbf{x}}$  is quadratic in the fermion operators,  $\mathbf{T}O_{\mathbf{x}}O_{\mathbf{y}} = O_{\mathbf{x}}O_{\mathbf{y}}$  if  $x_0 \geq y_0$  and  $O_{\mathbf{y}}O_{\mathbf{x}}$  if  $x_0 \leq y_0$ . If  $\mathbf{x} - \mathbf{y} = (\xi, \tau)$ , the response functions are defined in  $(-L, L) \times [-\beta, \beta]$  and are  $\beta$ -periodic in  $\tau$  and  $L$ -periodic in  $\xi$ . If  $F_{\beta,L}$  is any function of this type, we define its Fourier transform as

$$\widehat{F}_{\beta,L}(\mathbf{p}) = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x \in \mathcal{C}} e^{i\mathbf{p}\mathbf{x}} F_{\beta,L}(\mathbf{x}) \quad (1.5)$$

where  $\mathbf{p} = (p, p_0)$ , with  $p \in \frac{2\pi}{L}n$ ,  $-[L/2] \leq n \leq [(L-1)/2]$  and  $p_0 \in \frac{2\pi}{\beta}\mathbb{Z}$ .

In the following we will be interested in the zero temperature limit of some truncated correlation functions, in particular the two-point function  $\delta_{s,s'} S_2^{\beta,L}(\mathbf{x} - \mathbf{y}) := \langle \mathbf{T} \alpha_{\mathbf{x},s}^- \alpha_{\mathbf{y},s'}^+ \rangle_{\beta,L}$ , the density and current response functions and vertex functions (to be defined later), calculated in the thermodynamic limit. We shall denote these functions by the same symbols, without the  $\beta$  and  $L$  labels; for example, we shall write:  $\lim_{\beta \rightarrow \infty} \lim_{L \rightarrow \infty} \widehat{\Omega}_{L,\beta,\alpha}(\mathbf{p}) \equiv \widehat{\Omega}_{\alpha}(\mathbf{p})$ . Note that the thermodynamic limit  $L \rightarrow \infty$  is taken before the zero temperature limit  $\beta \rightarrow \infty$ ; this allows us to derive properties of the *thermal ground state*. To shorten the notation, in the following we shall use the definition  $\lim_{\beta,L \rightarrow \infty} \equiv \lim_{\beta \rightarrow \infty} \lim_{L \rightarrow \infty}$ .

Several important thermodynamic quantities can be derived from the knowledge of the response functions. In particular the *susceptibility*, which is given by

$$\kappa := \lim_{p \rightarrow 0} \lim_{p_0 \rightarrow 0} \widehat{\Omega}_C(\mathbf{p}) \quad (1.6)$$

and the *Drude weight*, related to the response of the system to an e.m. field, is defined as

$$D = \lim_{p_0 \rightarrow 0} \lim_{p \rightarrow 0} \widehat{D}(\mathbf{p}) \quad (1.7)$$

with

$$\widehat{D}_{\beta,L}(\mathbf{p}) = -\langle \tau_x \rangle - \int_{-\beta/2}^{\beta/2} dx_0 \sum_{x \in \Lambda} e^{i\mathbf{p}\mathbf{x}} \langle J_{\mathbf{x}} J_{\mathbf{0}} \rangle_{T;\beta,L} \quad (1.8)$$

where the first term is a constant independent of  $x$ . If one assumes analytic continuation in  $p_0$  around  $p_0 = 0$ , one can compute the conductivity in the linear response approximation by the Kubo formula, that is  $\sigma = \lim_{\omega \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{\widehat{D}(-i\omega + \delta, 0)}{-i\omega + \delta}$ . Therefore, a nonvanishing  $D$  indicates infinite conductivity.

The conservation law

$$\frac{\partial \rho_{\mathbf{x}}^C}{\partial x_0} = e^{Hx_0} [H, \rho_{\mathbf{x}}] e^{-Hx_0} = -i \partial_x^{(1)} J_{\mathbf{x}} \equiv -i [J_{x,x_0} - J_{x-1,x_0}] ,$$

where  $\partial_x^{(1)}$  denotes the lattice derivative, implies exact relations, called *Ward identities* (WI), among the Schwinger functions, the density correlations and the *vertex functions*, defined as

$G_{\rho,\beta,L}^{2,1}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \langle \mathbf{T} \rho_{\mathbf{x}}^{(C)} a_{\mathbf{y}}^- a_{\mathbf{z}}^+ \rangle_{T,\beta,L}$  and  $G_{j,\beta,L}^{2,1}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \langle \mathbf{T} J_{\mathbf{x}} a_{\mathbf{y}}^- a_{\mathbf{z}}^+ \rangle_{T,\beta,L}$ . Some Ward Identities, which will play an important role in the following, are

$$-ip_0 \widehat{G}_{\rho,\beta,L}^{2,1}(\mathbf{k}, \mathbf{k} + \mathbf{p}) - i(1 - e^{-ip}) \widehat{G}_{j,\beta,L}^{2,1}(\mathbf{k}, \mathbf{k} + \mathbf{p}) = \widehat{S}_2^{\beta,L}(\mathbf{k}) - S_2^{\beta,L}(\mathbf{k} + \mathbf{p}) \quad (1.9)$$

$$-ip_0 \widehat{\Omega}_{C,\beta,L}(\mathbf{p}) - i(1 - e^{-ip}) \widehat{\Omega}_{j,\rho,\beta,L}(\mathbf{p}) = 0 \quad (1.10)$$

$$-ip_0 \widehat{\Omega}_{\rho,j,\beta,L}(\mathbf{p}) - i(1 - e^{-ip}) \widehat{D}_{\beta,L}(\mathbf{p}) = 0 \quad (1.11)$$

where  $\Omega_{\rho,j,\beta,L}(\mathbf{x}, \mathbf{y}) = \langle \rho_{\mathbf{x}}^C J_{\mathbf{y}} \rangle_{T,\beta,L}$  and  $\Omega_{j,\rho,\beta,L}(\mathbf{x}, \mathbf{y}) = \langle J_{\mathbf{x}} \rho_{\mathbf{y}}^C \rangle_{T,\beta,L}$ .

In the previous paper [1] we have analyzed the model (1.1) in the repulsive non half-filled band case and we have proved, see Theorem 1.1 of [1], that the zero temperature response functions (1.4) in the thermodynamic limit decay at large distances with *critical exponents*, which are non trivial functions of the coupling and are denoted by  $X_C, X_{S_i}, X_{SC}, X_{SC}, X_{TC_i}$ , see (1.26) of [1]; finally  $\eta$  is the critical exponent of the interacting propagator, see (1.25) [1].

Such exponents, together with  $\kappa$  and  $D$ , depend on all microscopic details of the model, for instance the form of the two body interaction or the chemical potential. Nevertheless, according to the *Luttinger liquid conjecture* proposed by Haldane [3] (extending previous ideas by Kadanoff [4], and Luther and Peschel [5]) such quantities, through model dependent, are believed to satisfy a set of model independent relations. Such relations are true in a special solvable spinless models, the *Luttinger model*, whose solvability relies on the absence of the spin and on the linear dispersion relation of the fermions, see [6]. The content of the conjecture is that several of the relations valid in the Luttinger model are generically valid in a wide class of systems describing 1D fermions with *non linear* dispersion relation and in presence of *spin*. The following Theorem proves for the first time the validity of the universal Luttinger liquid relations in a wide class of models (including the 1D Hubbard model) of spinning fermions on a one dimensional lattice with a generic short range interaction satisfying a special positivity condition, in the non half filled band case.

**Theorem 1.1** *Given the Hamiltonian (1.1), if  $\bar{\mu} \neq 0$  and  $\widehat{v}(2 \arccos(\bar{\mu})) > 0$ , there exists  $\lambda_0 > 0$  such that, if  $0 \leq \lambda \leq \lambda_0$ , it is possible to find a continuous function  $p_F \equiv p_F(\bar{\mu}, \lambda) = \arccos(\bar{\mu}) + O(\lambda)$  verifying the conditions*

$$p_F \neq 0, \pi/2, \pi \quad , \quad \widehat{v}(2p_F) > 0 \quad (1.12)$$

so that there exist continuous functions

$$K \equiv K(\lambda) = 1 - c\lambda + O(\lambda^2), \quad \bar{K} \equiv \bar{K}(\lambda) = 1 - c\lambda + O(\lambda^2) \quad (1.13)$$

with  $c = 2[\widehat{v}(0) - \widehat{v}(2p_F)/2](\pi \sin p_F)^{-1}$ , such that

1. the critical exponents satisfy the extended scaling formulas

$$\begin{aligned} 4\eta &= K + K^{-1} - 2, & 2X_C &= 2X_{S_i} = K + 1, \\ 2X_{TC_i} &= 2X_{SC} = K^{-1} + 1, & 2\tilde{X}_{SC} &= K + K^{-1}; \end{aligned} \quad (1.14)$$

2. the small momentum asymptotic behavior of the response functions is

$$\begin{aligned} \widehat{\Omega}_C(\mathbf{p}) &= \frac{\bar{K}}{\pi v} \frac{v^2 p^2}{p_0^2 + v^2 p^2} + A(\mathbf{p}) \\ \widehat{D}(\mathbf{p}) &= \frac{v}{\pi} \bar{K} \frac{p_0^2}{p_0^2 + v^2 p^2} + B(\mathbf{p}) \end{aligned} \quad (1.15)$$

with  $A(\mathbf{p}), B(\mathbf{p})$  continuous and vanishing at  $\mathbf{p} = 0$ ,  $v = \sin p_F + O(\lambda)$ ; therefore the Drude weight  $D$  and the susceptibility  $\kappa$  are  $O(\lambda)$  close to their free values and verify the Luttinger liquid relation

$$v^2 = D/\kappa \quad (1.16)$$

The relations (1.14) were proposed in the spinless case by Kadanoff [7] and Luther and Peschel [5], and imply the exact determination of all the other exponents from the knowledge of a single one of them. The relation (1.16), proposed by Haldane [3], gives exact formulas relating the susceptibility and the Drude weight (connected to the *amplitudes* of the response functions) to the charge velocity  $v$ . The importance of such relations, in addition to the fact that they are among the very few cases in which the basic principle of universality in statistical mechanics can be rigorously established, rely in the fact that they allow for predictions which could be possibly experimentally verified in real anisotropic materials.

In order to prove such properties we introduce an effective model verifying some extra symmetries (which are only asymptotic in the lattice model) like the Lorentz symmetry and *chiral* local phase invariance implying many Ward Identities. In the model (1.1) such symmetries are not true, and there is a much smaller number of Ward Identities, given by (1.9), (1.10), (1.11), which are not sufficient by themselves to derive the universal relations. The critical exponents and the thermodynamic quantities of the effective model are related to the ones of the model (1.1), provided that a suitable fine tuning of the parameters is done; on the other hand the Ward Identities valid for the effective model imply relations from which at the end (1.14) and (1.16) follow. This method is a way to implement the concept of *emerging symmetries* in a rigorous mathematical setting. This fine tuning is possible thanks to the fact that both model are analyzed by Renormalization Group methods, and have a two dimensional manifold of fixed points. The strategy we followed can provide several extra information on the Hubbard model; an example is provided by the content of App. D, in which a detailed expression of the 2-point function of the Hubbard model is given, which looks in agreement with the so called *spin-charge separation*, that is the conjectured property that the spin and charge waves have different velocities.

## 2 Renormalization Group Analysis of the Effective Model

### 2.1 Definition of the effective model

The effective model which we introduce in order to prove Theorem 1.1 is not Hamiltonian and is defined directly in terms of *Grassmann variables*. Given  $L > 0$ , we consider the set  $\mathcal{D}'_L$  of space-time momenta  $\mathbf{k} = (k, k_0)$ , with  $k = \frac{2\pi}{L}(n + \frac{1}{2})$  and  $k_0 = \frac{2\pi}{L}(n_0 + \frac{1}{2})$ ; with each  $\mathbf{k} \in \mathcal{D}'_L$  we associate eight *Grassmann variables* (sometimes also called fields)  $\widehat{\psi}_{\mathbf{k},\omega,s}^+$ ,  $\widehat{\psi}_{\mathbf{k},\omega,s}^-$  with  $\omega = \pm$  a *quasi-particle* index and  $s = \pm$  a *spin* index. We also define (only *formally* for the moment, since the number of Grassmann variables is infinite) the *Grassmannian field* as

$$\psi_{\mathbf{x},\omega,s}^\pm = \frac{1}{L^2} \sum_{\mathbf{k} \in \mathcal{D}'_L} e^{\pm i\mathbf{k}\mathbf{x}} \widehat{\psi}_{\mathbf{k},\omega,s}^\pm \quad (2.1)$$

where  $\mathbf{x} = (x, x_0) \in \Lambda_L$ ,  $\Lambda_L$  being a two dimensional square torus of size  $L^2$ . To shorten notation we will also denote  $\int_{\Lambda_L} d\mathbf{x}$  by  $\int d\mathbf{x}$ ; moreover, in general we shall not stress the dependence on  $L$  of the various quantities we shall consider.

Given two integers  $l$  and  $N$  independent of  $L$ , such that  $l \ll 0 \ll N$ , the *Generating Function* of the effective model with *ultraviolet cutoff*  $\gamma^N$  and *infrared cutoff*  $\gamma^l$  is the following Grassmann integral:

$$e^{\mathcal{W}_{l,N}(J,\eta)} = \int P_Z^{[l,N]}(d\psi) \exp \left\{ -V(\sqrt{Z}\psi) + \sum_{\omega,s} \int d\mathbf{x} J_{\mathbf{x},\omega,s} \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,t}^- \right. \\ \left. + \sum_{\omega,s} \int d\mathbf{x} [\psi_{\mathbf{x},\omega,s}^+ \eta_{\mathbf{x},\omega,s}^- + \eta_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^-] \right\}, \quad (2.2)$$

where  $J, \eta$  are (respectively commuting and anticommuting) *external fields*,  $P_Z^{[l, N]}(d\psi)$  is the *Grassmann-valued Gaussian measure* with propagator  $\delta_{\omega, \omega'} \delta_{s, s'} g_{D, \omega}^{[l, N]}(\mathbf{x} - \mathbf{y})$ , where

$$g_{D, \omega}^{[l, N]}(\mathbf{x} - \mathbf{y}) = \frac{1}{Z} \frac{1}{L^2} \sum_{\mathbf{k} \in \mathcal{D}'_L} e^{-i\mathbf{k}(\mathbf{x} - \mathbf{y})} \frac{\chi_{l, N}^\varepsilon(|\tilde{\mathbf{k}}|)}{-ik_0 + \omega ck} \quad , \quad \tilde{\mathbf{k}} = (ck, k_0) \quad (2.3)$$

$Z, c > 0$  and  $\chi_{l, N}^\varepsilon(t)$  is a smooth cut-off function defined for  $t \geq 0$ , depending on a small positive parameter  $\varepsilon \geq 0$  with the following properties. If  $\varepsilon > 0$ , it is strictly positive for all  $t > 0$ , which reduce, as  $\varepsilon \rightarrow 0$ , to a compact support function  $\chi_{l, N}(t)$  equal to 1 for  $\gamma^l \leq t \leq \gamma^N$  and vanishing for  $t \leq \gamma^{l-1}$  or  $t \geq \gamma^{N+1}$ . The model is not really dependent of  $\varepsilon$ , since we plan to perform the limit  $\varepsilon \rightarrow 0$  in the expressions we get for the correlation functions and the kernels of the effective potential, at fixed values of  $L, N$  and  $l$ .

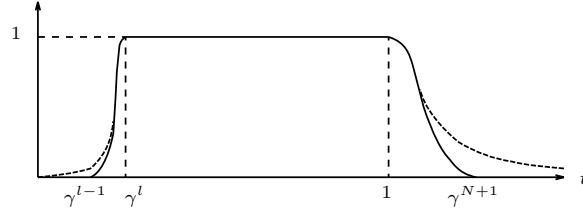


Figure 1: The cut-off functions  $\chi_{l, N}^\varepsilon(|\tilde{\mathbf{k}}|)$  (dashed line) and  $\chi_{l, N}(|\tilde{\mathbf{k}}|)$  (solid line)

The interaction is

$$V(\psi) = g_{1, \perp} V_{1, \perp}(\psi) + g_{\parallel} V_{\parallel}(\psi) + g_{\perp} V_{\perp}(\psi) + g_4 V_4(\psi) \quad (2.4)$$

with

$$\begin{aligned} V_{1, \perp}(\psi) &= \frac{1}{2} \sum_{\omega, s} \int d\mathbf{x} d\mathbf{y} h_{L, K}(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, -s}^- \psi_{\mathbf{y}, -\omega, s}^- \psi_{\mathbf{y}, -\omega, -s}^+ \\ V_{\parallel}(\psi) &= \frac{1}{2} \sum_{\omega, s} \int d\mathbf{x} d\mathbf{y} h_{L, K}(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, s}^- \psi_{\mathbf{y}, -\omega, s}^+ \psi_{\mathbf{y}, -\omega, s}^- \\ V_{\perp}(\psi) &= \frac{1}{2} \sum_{\omega, s} \int d\mathbf{x} d\mathbf{y} h_{L, K}(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, s}^- \psi_{\mathbf{y}, -\omega, -s}^+ \psi_{\mathbf{y}, -\omega, -s}^- \\ V_4(\psi) &= \frac{1}{2} \sum_{\omega, s} \int d\mathbf{x} d\mathbf{y} h_{L, K}(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, s}^- \psi_{\mathbf{y}, \omega, -s}^+ \psi_{\mathbf{y}, \omega, -s}^- \end{aligned} \quad (2.5)$$

where  $h_{L, K}(\mathbf{x})$  is defined in the following way. Let us fix an integer  $K$ <sup>1</sup> and a smooth function  $\hat{h}(\mathbf{p})$ , defined on  $\mathbb{R}^2$  and rotational invariant, such that  $|\hat{h}(\mathbf{p})| \leq C e^{-\mu|\mathbf{p}|}$  for some positive  $C$  and  $\mu$ , and  $\hat{h}(0) = 1$ ; moreover, let us call  $\mathcal{D}_L$  the set of space-time momenta  $\mathbf{k} = (k, k_0)$ , with  $k = \frac{2\pi}{L}n$  and  $k_0 = \frac{2\pi}{L}n_0$ . Then

$$h_{L, K}(\mathbf{x}) := \frac{1}{L^2} \sum_{\mathbf{p} \in \mathcal{D}_L} \hat{h}(\gamma^{-K} \mathbf{p}) e^{i\mathbf{p}\mathbf{x}} \quad (2.6)$$

<sup>1</sup>it is possible but not advisable to choose  $K = 0$ , because leaving a generic  $K$  will make clearer the ‘‘power counting’’ formulas: see for example Lemma 2.2

Note that  $h_{L,K}(\mathbf{x})$  is a smooth periodic function on  $\Lambda_L$ , which converges, as  $L \rightarrow \infty$ , to a smooth function  $h_K(\mathbf{x}) = \gamma^{2K} h(\gamma^K \mathbf{x})$ ,  $h(\mathbf{x})$  being the Fourier transform of  $\widehat{h}(\mathbf{p})$ ; this function is fast decreasing on scale  $\gamma^{-K}$ .

Even if the properties of (2.2) are largely independent of the exact form of  $\chi_{l,N}^\varepsilon(t)$  and  $\chi_{l,N}(t)$ , we find it convenient to choose them in the following way. We introduce a function  $\chi(t) \in C^\infty(\mathbb{R}^+)$  such that  $\chi(t) = 1$  if  $t \leq 1$  and  $\chi(t) = 0$  if  $|t| \geq \gamma$ ; then we define, for any integer  $j$ ,  $f_j(t) = \chi(\gamma^{-j}t) - \chi(\gamma^{-j+1}t)$  (hence the support of  $f_j(t)$  is contained in the interval  $[\gamma^{j-1}, \gamma^j]$ ) and we put

$$\chi_{l,N}^\varepsilon(|\tilde{\mathbf{k}}|) = \sum_{j=l}^N f_j^\varepsilon(|\tilde{\mathbf{k}}|) \quad (2.7)$$

where  $f_j^\varepsilon(t) = f_j(t)$ , if  $l+1 \leq j \leq N-1$ , while  $f_l^\varepsilon(t)$  and  $f_N^\varepsilon(t)$  are obtained by slightly modifying  $f_l(t)$  and  $f_N(t)$  in the following way, see Fig. 1. We put  $f_N^\varepsilon(t) = f_N(t) + \varepsilon \Delta_N(\gamma^{-N}t)$ , where  $\Delta_N(t)$  is a Schwartz function with support in  $[1, +\infty)$ , such that  $\Delta_N(t) > 0$ , if  $t > 1$ . Analogously,  $f_l^\varepsilon(t) = f_l(t) + \varepsilon \Delta_l(\gamma^{-l}t)$ , where  $\Delta_l(t)$  is a  $C^\infty$  function with support in  $[0, 1]$ , such that  $\Delta_l(t) > 0$ , if  $t \in (0, 1)$ .

In order to understand this definition, note that, if  $\varepsilon = 0$ , the model is well defined, since the family of Grassmann variables  $\widehat{\psi}_{\mathbf{k},\omega,s}^\pm$ , with  $\mathbf{k} \in \widetilde{\mathcal{D}}'_L = \{\mathbf{k} \in \mathcal{D}'_L : \chi_{l,N}(|\tilde{\mathbf{k}}|) > 0\}$ , is finite, so that we can restrict the sum in (2.3) and (2.1) to the set  $\widetilde{\mathcal{D}}'_L$  and we can write

$$P_Z^{[l,N]}(d\psi) = \frac{1}{\mathcal{N}} \exp \left\{ -\frac{Z}{L^2} \sum_{\substack{\omega,s \\ \mathbf{k} \in \widetilde{\mathcal{D}}'_L}} \chi_{l,N}^{-1}(|\tilde{\mathbf{k}}|) (-ik_0 + \omega ck) \widehat{\psi}_{\mathbf{k},\omega,s}^+ \widehat{\psi}_{\mathbf{k},\omega,s}^- \right\} \prod_{\substack{\omega,s \\ \mathbf{k} \in \widetilde{\mathcal{D}}'_L}} d\widehat{\psi}_{\mathbf{k},\omega,s}^+ d\widehat{\psi}_{\mathbf{k},\omega,s}^- \quad (2.8)$$

where  $\mathcal{N}$  is a suitable normalization constant. However, in the following we have to analyze the behavior of this measure under the local gauge transformation  $\psi_{\mathbf{x},\omega,s}^\pm \rightarrow e^{\pm i\alpha_{\mathbf{x},\omega,s}} \psi_{\mathbf{x},\omega,s}^\pm$  and this looks very difficult, since it is not possible to give a useful representation of the measure (2.8) in terms of the Grassmann field  $\psi_{\mathbf{x},\omega,s}^\pm$ , even if it is now well defined, if we restrict in (2.1) the sum to the set  $\widetilde{\mathcal{D}}'_L$ . In order to solve this problem, we put  $\varepsilon > 0$  and, at the same time, in order to keep the number of independent Grassmann variables finite, we introduce a *lattice cutoff*, by substituting the torus  $\Lambda_L$  with a lattice of spacing  $a$ , such that  $\gamma^{N+1} < \pi a^{-1}$  and  $La^{-1} = 2M$ ,  $M$  integer. We call  $\Lambda_L^a = \{\mathbf{x} = (ma, m_0a), m, m_0 \in [-M, M-1]\}$  the lattice and we substitute everywhere  $\int d\mathbf{x}$  with  $\sum_{\mathbf{x} \in \Lambda_L^a} a^2$  and the set  $\widetilde{\mathcal{D}}'_L$  with the set  $\widetilde{\mathcal{D}}'_{L,a} = \{\mathbf{k} \in \mathcal{D}'_L : |k|, |k_0| \leq \pi a^{-1} - \pi L^{-1}\}$ .

The outcome of this procedure is that the field  $\{\psi_{\mathbf{x},\omega,s}^\pm, \mathbf{x} \in \Lambda_L^a\}$  is related to the field  $\{\widehat{\psi}_{\mathbf{k},\omega,s}^\pm, \mathbf{k} \in \widetilde{\mathcal{D}}'_{L,a}\}$ , up to a constant, through the finite Fourier transform, so that

$$d\psi := \prod_{\omega,s,\mathbf{k} \in \widetilde{\mathcal{D}}'_{L,a}} d\widehat{\psi}_{\mathbf{k},\omega,s}^+ d\widehat{\psi}_{\mathbf{k},\omega,s}^- = \frac{1}{\mathcal{N}'} \prod_{\omega,s,\mathbf{x} \in \Lambda_L^a} d\psi_{\mathbf{x},\omega,s}^+ d\psi_{\mathbf{x},\omega,s}^- \quad (2.9)$$

where  $\mathcal{N}'$  is a normalization constant; (2.9) easily implies the invariance of the *Grassmann-valued Lebesgue measure*  $d\psi$  under the local gauge transformation (3.1). Of course, after writing the Ward identities following from the gauge invariance, we have to take the limit  $a \rightarrow 0$ , followed by the limit  $\varepsilon \rightarrow 0$ , while keeping fixed  $L$ ,  $N$  and  $l$ .

In agreement with these definitions, we shall define the Fourier transform of the external field  $\eta_{\mathbf{x},\omega,s}^\pm$  through the analogous of (2.1), while, for the external field  $J_{\mathbf{x},\omega,s}$ , we shall use the definition  $J_{\mathbf{x},\omega,s} = L^{-2} \sum_{\mathbf{p} \in \widetilde{\mathcal{D}}'_{L,a}} \widehat{J}_{\mathbf{p},\omega,s} e^{-i\mathbf{p}\mathbf{x}}$ , where  $\widetilde{\mathcal{D}}'_{L,a} := \{\mathbf{p} \in \mathcal{D}_L : p, p_0 \in [-\pi/a, \pi/a - 2\pi/L]\}$ . For the same reasons we modify the definition (2.6) of the function  $h_{L,K}(\mathbf{x})$ , by restricting the sum over  $\mathbf{p}$  to the set  $\widetilde{\mathcal{D}}'_{L,a}$ .

With this setup, as we shall prove, we can rigorously compute the correlations and the kernels of the effective potentials at fixed values of  $L$ ,  $N$ ,  $l$  and then perform the limit  $a \rightarrow 0$ , followed by the limit  $\varepsilon \rightarrow 0$ .

It is easy to see that the limit  $a \rightarrow 0$  is essentially trivial; in fact, in the scale decomposition that we shall describe below, the lattice spacing has a role only in two points:

- a) The bounds concerning the integration of the Grassmann variables with momenta larger than  $\gamma^N$ , which has to be performed in a single step using the propagator  $g_\omega^{(N)}(\mathbf{x})$  defined as the r.h.s. of (2.3) with  $f_N^\varepsilon$  in place of  $\chi_{L,N}^\varepsilon$ , can not be uniform in  $L$ . The reason is that we *do not* modify the free propagator, in order to make it periodic on the set  $\tilde{\mathcal{D}}'_{L,a}$ , see (2.8); it follows that boundary terms appear in the integration by part arguments which allow us to control the decreasing properties in  $\mathbf{x}$  of  $g_\omega^{(N)}(\mathbf{x})$ , see (2.15) below. However, thanks to the fast decreasing properties of the function  $\Delta_N(t)$  introduced in the definition of  $\tilde{f}_N^\varepsilon(|\tilde{\mathbf{k}}|)$ , it is easy to see that the boundary terms are negligible for  $a$  small enough, so that the dimensional bounds on  $g_\omega^{(N)}(\mathbf{x})$  are uniform in  $a \leq \bar{a}_{L,N}$  and  $\varepsilon \in [0, 1]$ , with  $\bar{a}_{L,N} \rightarrow 0$  for  $L, N \rightarrow \infty$ . This is not a source of trouble, since we have to perform the limit  $a \rightarrow 0$  at fixed values of  $N$  and  $L$ .
- b) The presence of the lattice introduces the so called *Umklapp terms*, when we write the kernels of the effective potentials and the correlations in terms of Fourier transforms, a procedure that is important in our analysis of the infrared scales. For example, our definitions imply that  $\sum_{\mathbf{x} \in \Lambda_L^q} J_{\mathbf{x},\omega,s} \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^- = L^{-2} \sum_{\mathbf{p} \in \tilde{\mathcal{D}}'_{L,a}} \hat{J}_{\mathbf{p},\omega,s} [\hat{\rho}_{\mathbf{p}} + \hat{\rho}_{\mathbf{p}+2\pi a^{-1}} + \hat{\rho}_{\mathbf{p}-2\pi a^{-1}}]$  with  $\hat{\rho}_{\mathbf{p}} = L^{-2} \sum_{\mathbf{k}^+, \mathbf{k}^- \in \tilde{\mathcal{D}}'_{L,a}: \mathbf{p} = \mathbf{k}^+ - \mathbf{k}^-} \hat{\psi}_{\mathbf{k}^+, \omega, s}^+ \hat{\psi}_{\mathbf{k}^-, \omega, s}^-$ . It is easy to see that the contribution of the terms proportional to  $\hat{\rho}_{\mathbf{p}+2\pi a^{-1}}$  and  $\hat{\rho}_{\mathbf{p}-2\pi a^{-1}}$ , as well of all similar Umklapp terms does not qualitatively modifies the structure of our multiscale expansion for  $a \leq \bar{a}_{L,N}$  and that all quantities of interest are well defined in the limit  $a \rightarrow 0$ , at fixed values of  $L$  and  $N$  and  $\varepsilon \leq 1$ .

Hence, in the following we shall write the results of our calculations directly in the limit  $a \rightarrow 0$ ; in particular we shall use the symbols  $\mathcal{D}'_L$  and  $\mathcal{D}_L$  in place of  $\tilde{\mathcal{D}}'_{L,a}$  and  $\tilde{\mathcal{D}}_{L,a}$ , respectively.

As concerns the functions  $\Delta_N(\gamma^{-N}|\tilde{\mathbf{k}}|)$  and  $\Delta_l(\gamma^{-l}|\tilde{\mathbf{k}}|)$ , their strict positivity has a role only when we discuss the Ward Identities following from the local gauge invariance. This calculation involves the expression in the exponent of (2.8), which is very singular for  $\varepsilon \rightarrow 0$ , but, as we shall see, the Ward Identities for the correlations and the kernels of the effective potentials have a simple well defined limit, which contains very important terms, which are at the origin of the anomalous critical exponents.

Let us now give a look at the interaction. The coupling  $g_{1,\perp}$  has a special role: if  $g_{1,\perp} = 0$  the model is invariant under the global phase transformation

$$\psi_{\mathbf{x},\omega,s}^\pm \rightarrow e^{\pm i\alpha_{\omega,s}} \psi_{\mathbf{x},\omega,s}^\pm \quad (2.10)$$

with the constant phase  $\alpha_{\omega,s}$  which can depend both on  $\omega$  and  $s$ . Otherwise, if  $g_{1,\perp} \neq 0$ , the spin invariance is broken; the invariance is under the transformation

$$\psi_{\mathbf{x},\omega,s}^\pm \rightarrow e^{\pm i\alpha_\omega} \psi_{\mathbf{x},\omega,s}^\pm \quad (2.11)$$

with the phase independent of  $s$ .

For finite values of  $N, l, L$  it is a consequence of the *Brydges-Battle-Federbush* formula and of the Gram-Hadamard inequality, see (2.43) and (2.47) of the companion paper [1], that the functional integral (2.2) is well defined and analytic in the couplings  $\vec{g} = (g_{1,\perp}, g_{\parallel}, g_{\perp}, g_4)$  in a disk in the complex plane. In order to get results valid in the limit of removed cut-off we need a multiscale analysis.

It will turn out that the limit  $N \rightarrow \infty$  at  $l$  or  $L$  finite (that is the solution of the *ultraviolet problem*) can be controlled by only assuming that  $\vec{g}$  is small enough. The analysis of the ultraviolet problem is somewhat similar to the one in §2.2, 2.3 of the companion paper [1] for the ultraviolet problem of the model (1.1); in particular a tree expansion and a multiscale analysis are necessary. However, in the case of the model (1.1) the lattice plays the role of an ultraviolet cut-off for spatial momenta and this makes the problem much simpler. In the case of the continuum model (2.2),

on the contrary, there is no such cut-off and the propagator (2.3) in the limit  $N \rightarrow \infty$  has a low  $O(\mathbf{k}^{-1})$  decay for large  $\mathbf{k}$ ; power counting arguments suggest that ultraviolet divergences, similar to those appearing in  $d = 1 + 1$  quantum Field Theory models, could appear. They are however avoided, as we will see, thanks to cancelations and the *non-locality* of the interaction (1.1), see lemma 2.2 below.

While the removal of the ultraviolet cut-off (at finite infrared cut-off) only requires that the couplings are small, the removal of the infrared cut-off is much more subtle and depends critically on the values of the couplings  $g_{\perp}, g_{1,\perp}, g_{\perp}, g_4$ ; this reflects the fact that different long distance decay properties of the correlations are expected to depend on the nature of the interaction. We will show that the infrared cut-off can be safely removed in the following two cases:

1. the case  $g_{1,\perp} = 0$
2. the case  $g_{\parallel} = g_{\perp} - g_{1,\perp}$  and  $g_{1,\perp} > 0$

## 2.2 The ultraviolet integration

For simplicity, we shall put in (2.2)  $\eta = 0$  and, for notational convenience we write (2.4) as

$$V(\psi) = \frac{1}{2} \sum_{\Theta, \Theta'} \int d\mathbf{x} d\mathbf{y} \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, t}^- h_{\Theta, \Theta'}^{L, K}(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{y}, \omega', s'}^+ \psi_{\mathbf{y}, \omega', t'}^- \quad (2.12)$$

with  $\Theta = (\omega, s, t)$ ,  $\Theta' = (\omega', s', t')$  and

$$h_{\Theta, \Theta'}^{L, K}(\mathbf{x} - \mathbf{y}) = \begin{cases} -g_{1,\perp} h_{L, K}(\mathbf{x} - \mathbf{y}) & \text{for } \omega' = -\omega \text{ and } s = t' = -s' = -t \\ g_{\parallel} h_{L, K}(\mathbf{x} - \mathbf{y}) & \text{for } \omega' = -\omega \text{ and } s = t = s' = t' \\ g_{\perp} h_{L, K}(\mathbf{x} - \mathbf{y}) & \text{for } \omega' = -\omega \text{ and } s = t = -s' = -t' \\ g_4 h_{L, K}(\mathbf{x} - \mathbf{y}) & \text{for } \omega' = \omega \text{ and } s = t = -s' = -t' \\ 0 & \text{otherwise} \end{cases}$$

To exploit certain identities, we have to put in (2.2) a more general source term; hence, we substitute the term proportional to the  $J$  field with  $\sum_{\Theta} \int d\mathbf{x} J_{\mathbf{x}, \Theta} \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, t}^-$ , where again  $\Theta = (\omega, s, t)$ , and we define:

$$\mathcal{V}(\psi, J) = \sum_{\Theta} \int d\mathbf{x} J_{\mathbf{x}, \Theta} \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, t}^- - V(\sqrt{Z}\psi) \quad (2.13)$$

The graphical representation of the interaction is given in Fig. 2.

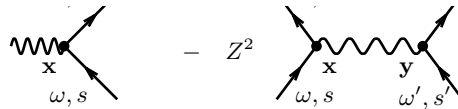


Figure 2: Graphical representation of (2.13)

By using (2.7), we can write

$$g_{D, \omega}^{[l, N]}(\mathbf{x}) = \sum_{j=l}^N g_{\omega}^{(j)}(\mathbf{x}) \quad (2.14)$$

where  $g_\omega^{(j)}(\mathbf{x})$  is defined as  $g_{D,\omega}^{[l,N]}(\mathbf{x})$  with  $\chi_{l,N}^\varepsilon(|\tilde{\mathbf{k}}|)$  replaced by  $f_j^\varepsilon(|\tilde{\mathbf{k}}|)$ . Therefore, for a positive constant  $c_0$ ,

$$\begin{aligned} \|g^{(k)}\|_{L_\infty} &:= \sup_{\mathbf{x},\omega} |g_\omega^{(k)}(\mathbf{x})| \leq c_0 \gamma^k \quad , \quad \|g^{(k)}\|_{L_1} := \max_\omega \int d\mathbf{x} |g_\omega^{(k)}(\mathbf{x})| \leq c_0 \gamma^{-k} \\ \|g^{(k)}\|_{\tilde{L}_1} &:= \max_\omega \int d\mathbf{x} \|\mathbf{x}\| |g_\omega^{(k)}(\mathbf{x})| \leq c_0 \gamma^{-2k} \end{aligned} \quad (2.15)$$

where  $\|\mathbf{x} - \mathbf{y}\|$  is the distance on the torus, and, from (2.6)

$$\begin{aligned} \|h_{L,K}\|_{L_\infty} &:= \sup_{\mathbf{x}} |h_{L,K}(\mathbf{x})| \leq c_0 \gamma^{2K} \quad , \quad \|\partial h_{L,K}\|_{L_1} := \sup_{j=0,1} \int d\mathbf{x} |\partial_j h_{L,K}(\mathbf{x})| \leq c_0 \gamma^K \\ \|h_{L,K}\|_{L_1} &:= \int d\mathbf{x} |h_{L,K}(\mathbf{x})| \leq c_0 \end{aligned} \quad (2.16)$$

The scales  $N, N-1, \dots, K$  are called *ultraviolet scales*, while the remaining ones are called *infrared scales*; if  $\psi^{(j)}$ ,  $j \geq K$  is the field with propagator  $g_\omega^{(j)}(\mathbf{x})$  and we call  $P_Z(d\psi^{(j)})$  the corresponding Grassmann measure, the integration of the fields  $\psi^{(N)}, \dots, \psi^{(K)}$  is done iteratively in the usual way, by using the decomposition  $P_Z(d\psi^{[l,N]}) \prod_{j=l}^K P_Z(d\psi^{(j)})$ , where we have modified the previous notation by writing  $P_Z(d\psi^{[l,N]})$  in place of  $P_Z^{[l,N]}(d\psi)$ . After integrating the fields  $\psi^{(N)}, \psi^{(N-1)}, \dots, \psi^{(h+1)}$ ,  $h \geq K$ , we get an expression of the form:

$$\int P_Z(d\psi^{[l,N]}) e^{\mathcal{V}(\psi^{[l,N]}, J)} = e^{-L^2 E_h + S_h(J)} \int P_Z(d\psi^{[l,h]}) e^{\mathcal{V}^{(h)}(\psi^{[l,h]}, J)} \quad (2.17)$$

where

$$\mathcal{V}^{(h)}(\psi, J) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{\underline{\Theta}, \underline{\Theta}'} \int d\underline{\mathbf{z}} d\underline{\mathbf{x}} d\underline{\mathbf{y}} W_{\underline{\Theta}, \underline{\Theta}'}^{(n; 2m)(h)}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}}) \left[ \prod_{j=1}^n J_{\mathbf{z}_j, \Theta_j} \right] \left[ \prod_{j=1}^m \psi_{\mathbf{x}_j, \omega'_j, s'_j}^+ \psi_{\mathbf{y}_j, \omega'_j, t'_j}^- \right] \quad (2.18)$$

and  $\underline{\Theta} = (\Theta_1, \dots, \Theta_n)$ ,  $\underline{\Theta}' = (\Theta'_1, \dots, \Theta'_m)$ ,  $\Theta'_j = (\omega'_j, s'_j, t'_j)$ .

**Remark** - The compact support properties of the single scale propagators  $g_\omega^{(j)}$  imply that  $\mathcal{V}^{(h)}(\psi^{[l,h]}, J)$  depends on  $N$  and  $L$ , but is independent of the IR cutoff  $l$ , if  $l < K$ .

The integration of the field  $\psi^{(j)}$  is done after resumming the marginal terms, which is done by rewriting the r.h.s. of (2.17) as

$$e^{-L^2 E_h + S_h(J)} \int P_Z(d\psi^{[l,h-1]}) \int P_Z(d\psi^{(h)}) e^{\mathcal{L}\mathcal{V}^{(h)}(\psi^{[l,h]}, J) + \mathcal{R}\mathcal{V}^{(h)}(\psi^{[l,h]}, J)} \quad (2.19)$$

where  $\mathcal{R} = 1 - \mathcal{L}$  and  $\mathcal{L}$  is a linear operation acting on the kernels of  $\mathcal{V}^{(h)}$  so that

$$\mathcal{L}W^{(n; 2m)(h)}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}}) := \begin{cases} W^{(n; 2m)(h)}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}}) & \text{if } n + m \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad (2.20)$$

Therefore, if we define, as in §2.3 of [1],  $e_h = E_h - E_{h+1}$  and  $s_h(J) = S_h(J) - S_{h+1}(J)$ , then  $-L^2 e_h + s_h(J) + \mathcal{V}^{(h)}(\psi, J)$  can be expressed as a tree expansion very similar to the one used in Lemma 2.2 of the companion paper [1] that here we briefly recall.

Let us consider the family of all trees which can be constructed by joining a point  $r$ , the *root*, with an ordered set of  $\bar{n} \geq 1$  points, the *endpoints* of the *unlabeled tree*, so that  $r$  is not a branching point.  $\bar{n}$  will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol  $<$  to denote the partial order. Two unlabeled trees are

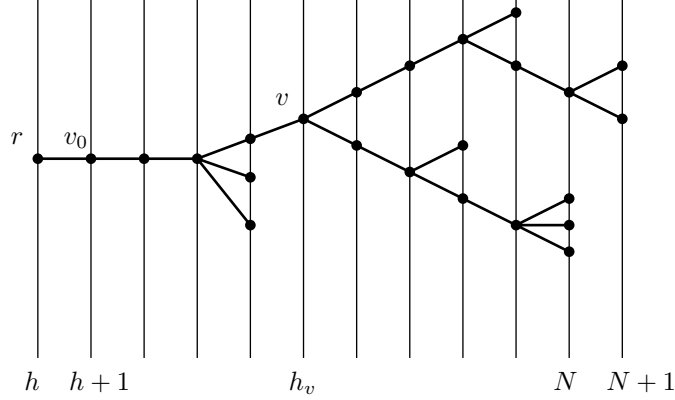


Figure 3: A renormalized tree appearing in the graphic representation of  $\mathcal{V}^{(k)}$

identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with  $\bar{n}$  end-points is bounded by  $4^{\bar{n}}$ . We shall also consider the set  $\mathcal{T}_{N,h,n_g,n_J}$  of the *labeled trees* with  $n_g + n_J$  endpoints (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

1) We associate a label  $J$  or  $g$  to each endpoint, so that there are  $n_g$  endpoints with label  $g$ , to be called *normal endpoints*, and  $n_J$  endpoints with label  $J$  to be called *special endpoints*.

2) We associate a label  $h \leq N$  with the root. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in  $[h, N + 1]$ , and we represent any tree  $\tau \in \mathcal{T}_{N,h,n_g,n_J}$  so that, if  $v$  is an endpoint or a non trivial vertex, it is contained in a vertical line with index  $h_v > h$ , to be called the *scale* of  $v$ , while the root  $r$  is on the line with index  $h$ . In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. The set of the *vertices* will be the union of the endpoints, of the trivial vertices and of the non trivial vertices; note that the root is not a vertex. Every vertex  $v$  of a tree will be associated to its scale label  $h_v$ , defined, as above, as the label of the vertical line whom  $v$  belongs to. Note that, if  $v_1$  and  $v_2$  are two vertices and  $v_1 < v_2$ , then  $h_{v_1} < h_{v_2}$ .

3) There is only one vertex immediately following the root, which will be denoted  $v_0$ ; its scale is  $h + 1$ . If  $v_0$  is an endpoint, the tree is called the *trivial tree*; this can happen only if  $n_g + n_J = 1$ .

4) Given a vertex  $v$  of  $\tau \in \mathcal{T}_{N,h,n_g,n_J}$  that is not an endpoint, we can consider the subtrees of  $\tau$  with root  $v$ , which correspond to the connected components of the restriction of  $\tau$  to the vertices  $w \geq v$ . If a subtree with root  $v$  contains only  $v$  and one endpoint on scale  $h_v + 1$ , it will be called a *trivial subtree*.

5) Given an end-point, the vertex  $v$  preceding it is surely a non trivial vertex, if  $n_g + n_J > 1$ .

Our expansion is build by associating a value to any tree  $\tau \in \mathcal{T}_{N,h,n,m}$  in the following way. First of all, given an endpoint  $v \in \tau$  with  $h_v = N + 1$ , we associate to it one of the terms contributing to the potential in (2.13), while, if  $h_v \leq N$ , we associate to it one of the terms appearing in

$$\sum_{\Theta, \omega'} \int d\mathbf{x} d\mathbf{x}' d\mathbf{y} W_{\Theta, (\omega', s, t)}^{(1;2)(h_v)}(\mathbf{z}; \mathbf{x}, \mathbf{y}) J_{\mathbf{z}, \Theta} \psi_{\mathbf{x}, \omega', s}^{+(\leq h_v)} \psi_{\mathbf{y}, \omega', t}^{-(\leq h_v)} + \sum_{\omega, s} \int d\mathbf{x} d\mathbf{y} W^{(0;2)(h_v)}(\mathbf{x}, \mathbf{y}) \psi_{\mathbf{x}, \omega, s}^{+(h_v)} \psi_{\mathbf{y}, \omega, s}^{-(h_v)}$$

$$+ \sum_{\underline{\omega}, \underline{s}} \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 W^{(0;4)(h_v)} \psi_{\mathbf{x}_1, \omega_1, s_1}^{+(h_v)} \psi_{\mathbf{x}_2, \omega_2, s_2}^{-(h_v)} \psi_{\mathbf{x}_3, \omega_3, s_3}^{+(h_v)} \psi_{\mathbf{x}_4, \omega_4, s_4}^{-(h_v)} \quad (2.21)$$

All these possible choices will be distinguished by a label  $\alpha$  in a set  $A_\tau$ , depending on  $\tau$ . Finally, the operator  $\mathcal{R}$  is associated with each non-trivial vertex  $v$ .

The previous definitions imply that the following iterative equations are satisfied:

$$\mathcal{V}^{(h)}(\psi^{(\leq h)}, J) + s_h(J) - L^2 e_h = \sum_{n=1}^{\infty} \sum_{\substack{\tau \in \mathcal{T}_{N, h, n_g, n_J} \\ \alpha \in A_\tau}} \mathcal{V}^{(h)}(\tau, \alpha, \psi^{(\leq h)}, J), \quad (2.22)$$

where, if  $v_0$  is the first vertex of  $\tau$  and  $\tau_1, \dots, \tau_s$ ,  $s \geq 1$ , are the subtrees with root in  $v_0$ ,

$$\begin{aligned} \mathcal{V}^{(h)}(\tau, \alpha, \psi^{(\leq h)}, J) = \\ \frac{1}{s!} \mathcal{E}_{h+1}^T [\bar{\mathcal{V}}^{(h+1)}(\tau_1, \alpha_1, \psi^{(\leq h+1)}, J); \dots; \bar{\mathcal{V}}^{(h+1)}(\tau_s, \alpha_s, \psi^{(\leq h+1)}, J)] \end{aligned} \quad (2.23)$$

where  $\bar{\mathcal{V}}^{(h+1)}(\tau_i, \alpha_i, \psi^{(\leq h+1)}, J)$  is equal to  $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i, \alpha_i, \psi^{(\leq h+1)}, J)$  if the subtree  $\tau_i$  contains more than one end-point (that is  $v$  is a non trivial vertex), otherwise it is given by one of the terms in (2.13), if  $h_v = N + 1$ , or one of the terms in (2.21), if  $h_v \leq N$ .

Note that, by the definition (2.20), the tree value can be different from 0 only if, for any non trivial vertex  $v \neq v_0$ ,  $|P_v|/2 + n_v^J > 2$ , where  $P_v$  is the number of  $\psi$  variables not yet contracted in the vertex  $v$  (see App. A for a precise definition) and  $n_v^J$  is the number of  $J$ -endpoints of the subtree with root  $v$ . Note also that  $|P_v|$  has to be positive for any  $v \neq v_0$  and that  $P_{v_0} = 0$  for the trees contributing to  $e_h$  and  $s_h(J)$ ; in the following, we shall denote by  $W^{(0;0)(h)}$  and  $W_{\underline{\Theta}}^{(n;0)(h)}(\underline{\mathbf{z}})$  the corresponding kernels, while  $\widetilde{W}^{(0;0)(h)} = \sum_{j=h}^N W^{(0;0)(j)}$  and  $\widetilde{W}_{\underline{\Theta}}^{(n;0)(h)}(\underline{\mathbf{z}}) = \sum_{j=h}^N W_{\underline{\Theta}}^{(n;0)(j)}(\underline{\mathbf{z}})$  will be the kernels of  $E_h$  and  $\mathcal{S}_h(J)$

By (2.22) and (2.23) it is straightforward to verify, see App. A, that the kernels  $W^{(n;2m)(h)}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}})$  in (2.18),  $h > K$  are represented by integrals of power series expansions in the running coupling functions  $W^{(0;2)(k)}$ ,  $W^{(1;2)(k)}$ ,  $W^{(0;4)(k)}$  with  $k > h$ . Consider now the  $L^1$  norm

$$\|W^{(n;2m)(k)}\| := \max_{\underline{\Theta}, \underline{\Theta}'} \frac{1}{L^2} \int d\mathbf{z} d\mathbf{x} d\mathbf{y} \left| W_{\underline{\Theta}, \underline{\Theta}'}^{(n;2m)(k)}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}}) \right| \quad (2.24)$$

Of course, since the kernels may contain delta functions, we extend, as usual, the definition of the  $L^1$  norm by treating the delta as positive functions. We also define

$$w_{\underline{\Theta}, \underline{\Theta}'}^{(1;2)(h)}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}}) := \delta_{\underline{\Theta}, \underline{\Theta}'} \delta(\underline{\mathbf{z}} - \underline{\mathbf{x}}) \delta(\underline{\mathbf{z}} - \underline{\mathbf{y}}) \quad (2.25)$$

$$w_{\underline{\Theta}, \underline{\Theta}'}^{(0;4)(h)}(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \underline{\mathbf{y}}_1, \underline{\mathbf{y}}_2) := Z^2 h_{\underline{\Theta}, \underline{\Theta}'}^{L, K}(\underline{\mathbf{x}}_1 - \underline{\mathbf{x}}_2) \delta(\underline{\mathbf{x}}_1 - \underline{\mathbf{y}}_1) \delta(\underline{\mathbf{x}}_2 - \underline{\mathbf{y}}_2) \quad (2.26)$$

that are equal to  $W_{\underline{\Theta}, \underline{\Theta}'}^{(1;2)(h)}$  and  $W_{\underline{\Theta}, \underline{\Theta}'}^{(0;4)(h)}$  when  $h = N$ .

**Theorem 2.1** *Let  $K \leq k \leq N$  and assume that*

$$\sup_{h > k} \left[ \gamma^{-h} \|W^{(0;2)(h)}\| + \|W^{(0;4)(h)}\| \right] \leq \varepsilon_0 \quad \sup_{h > k} \|W^{(1;2)(h)}\| \leq 2 \quad (2.27)$$

with  $\varepsilon_0$  independent of  $k, L, N$ ; there exists a constant  $\bar{\varepsilon}$  such that, if  $\varepsilon_0 \leq \bar{\varepsilon}$  then, for a suitable constant  $C$ ,

$$\|W^{(n;2m)(k)}\| \leq \varepsilon_0^{d_{n,m}} C^{n+m} \gamma^{k(2-n-m)} \quad (2.28)$$

where  $d_{n,m} = \max(m - 1, 0)$  if  $n > 0$ , and  $d_{n,m} = \max(m - 1, 1)$  if  $n = 0$ . Moreover the limits  $\lim_{N \rightarrow \infty} W_{\underline{\Theta}, \underline{\Theta}'}^{(n;2m)(k)}$  do exist and are reached uniformly in  $N$ .

The proof of this theorem is very similar to the proof of Lemma 2.2 of the companion paper [1], and is in appendix A. Note however a crucial difference; in the present case the scaling dimension, which can be read from the r.h.s. of (2.28), is  $n + m - 2$ ; this explains why, according to (2.20), we renormalize the kernels in  $\mathcal{V}^h$  such that  $n + m \leq 2$ , that is those which are the only ones with  $m > 0$  and scaling dimension  $\leq 0$ .

Note that Theorem 2.1 implies a bound also on the  $L^1$  norm of the kernels  $\widetilde{W}_{\underline{\Theta}}^{(n;0)(k)}(\underline{\mathbf{z}}) = \sum_{j=k}^N W_{\underline{\Theta}}^{(n;0)(j)}(\underline{\mathbf{z}})$ . This bound is finite uniformly in  $N$  for  $n \geq 3$ , while it is divergent for  $N \rightarrow \infty$ , if  $n \leq 2$ . The cases  $n = 0$  and  $n = 1$  do not give any trouble; in fact  $W^{(1,0)(h)} = 0$ , because of the oddness of the free propagator, while the divergence of the free energy is not a problem in a QFT model. As concerns  $\|\widetilde{W}^{(2;0)(k)}\|$ , the logarithmic divergence for  $N \rightarrow \infty$  of its bound is a problem in the following analysis. However, by using the symmetry  $g_{\omega}^{[k,N]}(x, x_0) = -i\omega g_{\omega}^{[k,N]}(-x_0, x)$ , one can see that  $\int d\mathbf{z}d\mathbf{z}' \widetilde{W}_{\underline{\Theta}, \underline{\Theta}'}^{(2;0)(k)}(\mathbf{z}, \mathbf{z}') = 0$ ; this identity (or better its validity for the term of order 0 in  $\bar{g}$ ) will be a crucial ingredient in the proof of Lemma 2.2 below, together with the non locality of the interaction, which allows us to improve some bounds by substituting the  $L_{\infty}$  norm of the propagator with its  $L_1$  norm. In this way we will be able to verify the assumptions (2.27) on the running coupling functions by improving their dimensional bound.

**Lemma 2.2** *There exist positive constants  $C_1, C_2, C_3$ , such that, if  $\bar{g} = \max\{|g_{1,\perp}|, |g_{\parallel}|, |g_{\perp}|, |g_4|\}$  is small enough and if  $K \leq h \leq N$ , then*

$$\|W^{(0;2)(h)}\| \leq C_1 \bar{g} \gamma^h \gamma^{-2(h-K)} \quad (2.29)$$

$$\|W^{(1;2)(h)} - w^{(1;2)(h)}\| \leq C_2 \bar{g} \gamma^{-(h-K)} \quad (2.30)$$

$$\|W^{(0;4)(h)} - w^{(0;4)(h)}\| \leq C_3 \bar{g} \varepsilon_0 \gamma^{-(h-K)} \quad (2.31)$$

By the above lemma we see that we can choose  $\bar{g}$  so small that Theorem 2.1 holds and the bound (2.28) is true. Note that the crux of the above bounds is the fact that  $C_1, C_2$  and  $C_3$  are independent of the scale  $h$ . In other words, we have standard ‘‘power counting’’ bounds even though we do not study beta functions of marginal or relevant terms: that is obtained by the cancelations that we will provide in the proof below. For  $K \leq h \leq N$ , the factor(s)  $\gamma^{-(h-K)}$  do not play any decisive role, and in applications will be bounded by 1. However, note that they are unbounded if, instead,  $h < K$ : this explains why this Lemma is stated only for  $K \leq h \leq N$ .

**Proof of Lemma 2.2** The proof is by induction. We assume that the bounds (2.29)-(2.31) hold for  $k + 1 \leq h \leq N$  (for  $h = N$  they are true with  $C_1 = C_2 = C_3 = 0$ ); then we can use Theorem 2.1 to have (2.28) on scale  $k$ , assuming that  $\bar{g} \leq \bar{\varepsilon}/\bar{C}$  where  $\bar{C}$  is the minimum among  $C_1, C_2, C_3$ ; finally, we prove that (2.29)-(2.31) holds for  $h = k$ . Note that the definition of the kernels (see remarks after (2.23)) imply that, if  $m > 0$  and  $n \geq 0$ ,

$$\begin{aligned} W_{\underline{\Theta}; \underline{\Theta}'}^{(n;2m)(h)}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}}) &:= \prod_{j=1}^n \frac{\partial}{\partial J_{\mathbf{z}_j, \Theta_j}} \times \prod_{j=1}^m \frac{\partial^2}{\partial \psi_{\mathbf{x}_j, \omega'_j, s'_j}^+ \partial \psi_{\mathbf{y}_j, \omega'_j, t'_j}^-} \mathcal{V}^{(h)}(\sqrt{Z} \psi^{[l,h]}, J) \Bigg|_{\psi=J=0} \\ \widetilde{W}_{\underline{\Theta}}^{(n+1;0)(h)}(\underline{\mathbf{z}}) &:= \prod_{j=1}^{n+1} \frac{\partial}{\partial J_{\mathbf{z}_j, \Theta_j}} S_h(J) \Bigg|_{J=0}, \quad \widetilde{W}^{(0,0)(h)} = E_h \end{aligned} \quad (2.32)$$

Let us now put  $\bar{V}^{(h)}(\psi, J) = \mathcal{V}^{(h)}(\psi, J) - L^2 E_h + S_h(J)$ . We will make repeated use of the following two identities. The first one, graphically represented in Fig. 4, says that, for any  $\Theta = (\omega, s, t_0)$ ,

$$\frac{\partial \mathcal{V}^{(k)}}{\partial \psi_{\mathbf{x}, \omega, s}^+}(\psi, J) = \sum_{t_0} J_{\mathbf{x}, \Theta} \psi_{\mathbf{x}, \omega, t_0}^- + \sum_{t_0} J_{\mathbf{x}, \Theta} \int d\mathbf{u} g_{\omega}^{[k+1, N]}(\mathbf{x} - \mathbf{u}) \frac{\partial \mathcal{V}^{(k)}}{\partial \psi_{\mathbf{u}, \omega, t_0}^+}(\psi, J)$$

$$\begin{aligned}
 & + \sum_{t_0, \Theta_1} \int d\mathbf{w} d\mathbf{u} h_{\Theta, \Theta_1}^{L, K}(\mathbf{x} - \mathbf{w}) g_{\omega}^{[k+1, N]}(\mathbf{x} - \mathbf{u}) \left[ \frac{\partial^2 \mathcal{V}^{(k)}}{\partial J_{\mathbf{w}, \Theta_1} \partial \psi_{\mathbf{u}, \omega, t_0}^+} + \frac{\partial \mathcal{V}^{(k)}}{\partial J_{\mathbf{w}, \Theta_1}} \frac{\partial \mathcal{V}^{(k)}}{\partial \psi_{\mathbf{u}, \omega, t_0}^+} \right] (\psi, J) \\
 & + \sum_{t_0, \Theta_1} \psi_{\mathbf{x}, \omega, t_0}^- \int d\mathbf{w} h_{\Theta, \Theta_1}^{L, K}(\mathbf{x} - \mathbf{w}) \frac{\partial \bar{V}^{(k)}}{\partial J_{\mathbf{w}, \Theta_1}} (\psi, J), \tag{2.33}
 \end{aligned}$$

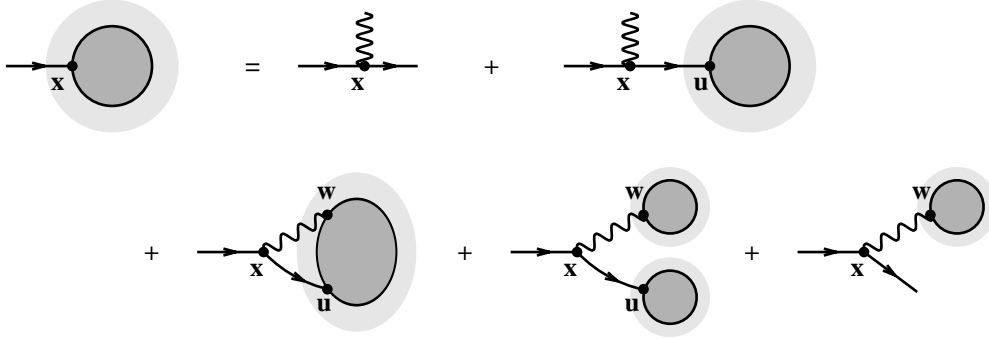


Figure 4: Graphical representation of (2.33). The wiggly line and the solid arrow between two points represent  $h_{\Theta, \Theta}^{L, K}$  and  $g_{\omega}^{[k+1, N]}$ , respectively; the external wiggly line and the external arrow represent  $J_{\mathbf{x}, \Theta}$  and  $\psi_{\mathbf{x}, \omega, s}^{[l, k]+}$ , respectively; the blobs represent the derivatives of  $\bar{V}^{(k)}$  with respect to their external fields, with a 'halo' that reminds the possibility of taking other derivatives w.r.t.  $J$  or  $\psi$  fields.

The second identity, graphically represented in Fig. 5, says that

$$\begin{aligned}
 \frac{\partial \mathcal{V}^{(k)}}{\partial J_{\mathbf{x}, \Theta}} (\psi, J) & = \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, t_0}^- + \int d\mathbf{u} g_{\omega}^{[k+1, N]}(\mathbf{x} - \mathbf{u}) \left[ \psi_{\mathbf{x}, \omega, s}^+ \frac{\partial \mathcal{V}^{(k)}}{\partial \psi_{\mathbf{u}, \omega, t_0}^+} - \frac{\partial \mathcal{V}^{(k)}}{\partial \psi_{\mathbf{u}, \omega, s}^-} \psi_{\mathbf{x}, \omega, t_0}^- \right] (\psi, J) \\
 & + \int d\mathbf{u} d\mathbf{u}' g_{\omega}^{[k+1, N]}(\mathbf{x} - \mathbf{u}) g_{\omega}^{[k+1, N]}(\mathbf{u}' - \mathbf{x}) \left[ \frac{\partial^2 \mathcal{V}^{(k)}}{\partial \psi_{\mathbf{u}, \omega, s}^+ \partial \psi_{\mathbf{u}', \omega, t_0}^-} + \frac{\partial \mathcal{V}^{(k)}}{\partial \psi_{\mathbf{u}, \omega, s}^+} \frac{\partial \mathcal{V}^{(k)}}{\partial \psi_{\mathbf{u}', \omega, t_0}^-} \right] (\psi, J). \tag{2.34}
 \end{aligned}$$

A proof of these identities is given in appendix B. Let us show that (2.28) can be improved to (2.29)-(2.31) case by case.

a) *Bound for  $W_{\Theta}^{(0;2)(k)}$ .* If we take in (2.33) one derivative w.r.t.  $\psi_{\mathbf{y}, \omega, t}^-$  and, after that, we put  $J = \psi = 0$ , we obtain an identity for  $W_{\Theta}^{(0;2)(k)}(\mathbf{x}, \mathbf{y})$ ,  $\Theta = (\omega, s, t)$ , which, since  $W_{\Theta}^{(1;0)}(\mathbf{0}) = 0$  by the oddness of the fermion propagator, reads

$$W_{\Theta}^{(0;2)(k)}(\mathbf{x} - \mathbf{y}) = \sqrt{Z} \sum_{t_0, \Theta'} \int d\mathbf{w} d\mathbf{w}' h_{(\omega, s, t_0), \Theta'}^{L, K}(\mathbf{x} - \mathbf{w}) g_{\omega}^{[k+1, N]}(\mathbf{x} - \mathbf{w}') W_{\Theta'; (\omega, t_0, t)}^{(1;2)(k)}(\mathbf{w}; \mathbf{w}', \mathbf{y}), \tag{2.35}$$

where  $\Theta = (\omega, s, t)$ . We now bound  $h_K$  by its  $L_{\infty}$ -norm, the fermion propagator by its  $L_1$ -norm and  $W^{(1;2)(k)}$  by (2.28),  $\|W^{(1;2)(k)}\| \leq C^2$ . We get

$$\|W_{\Theta}^{(0;2)(k)}\| \leq 16\bar{g} \|h_{L, K}\|_{L_{\infty}} \|W^{(1;2)(k)}\| \sum_{j \geq k+1} \|g^{(j)}\|_{L_1} \leq \bar{g} C_1 \gamma^k \gamma^{-2(k-K)}, \tag{2.36}$$

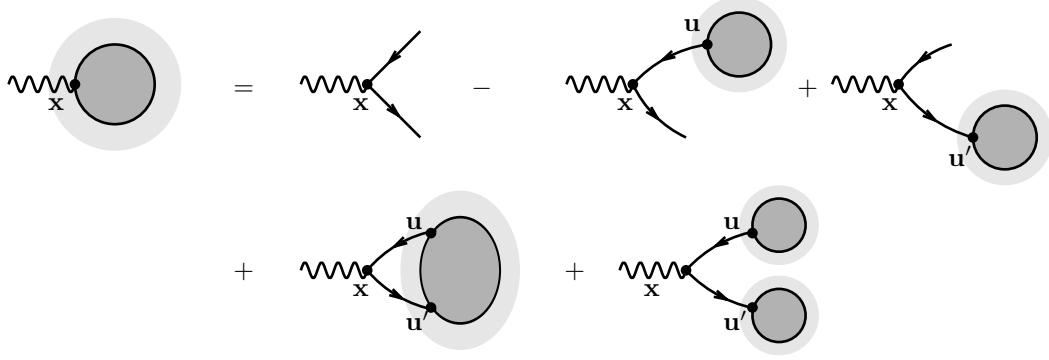


Figure 5: Graphical representation of (2.34). The notation is the same as in Fig. 4

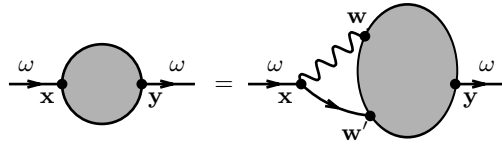


Figure 6: Graphical representation of (2.35); the gray blobs represent  $W_{\Theta}^{(0;2)(k)}$  and  $W_{\Theta';(\omega,t_o,t)}^{(1;2)(k)}$ ; the solid lines are fermion propagators  $g_{\omega}^{[k+1,N]}$  or external lines; the wiggly line is  $h_{\Theta,\Theta'}^{L,K}$ .

This proves (2.29).

b) *Bound for  $W^{(1;2)(k)}$ .* If we take in (2.33) two derivatives w.r.t.  $\psi_{y,\omega,t}^-$  and  $J_{z,\Theta'}$  and we put  $J = \psi = 0$ , we can decompose  $W_{\Theta';\Theta}^{(1;2)(k)} - w_{\Theta';\Theta}^{(1;2)(k)}$  into the four terms in Fig.7.

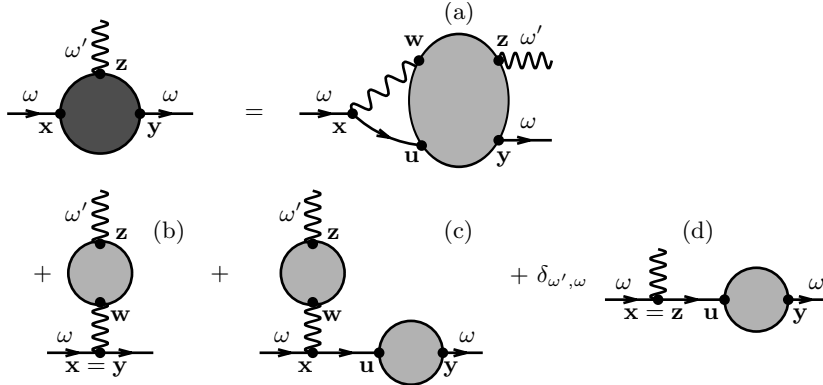


Figure 7: : Graphical representation of the decomposition of  $W_{\Theta';\Theta}^{(1;2)(k)}$ ; the darker bubble represents the difference  $W_{\Theta';\Theta}^{(1;2)(k)} - w_{\Theta';\Theta}^{(1;2)(k)}$ ; the other bubbles represent the kernels in (2.32); the internal vertices are integrated.

Consider first the graph (a). As in the previous case, take the  $L_1$ -norm of the fermion propagator and the  $L_{\infty}$ -norm of the function  $h_{\Theta,\Theta'}^{L,K}$ ; then use for  $W^{(2;2)(k)}$  the bound (2.28), that is

$\|W^{(2;2)(k)}\| \leq C^3 \varepsilon_0 \gamma^{-k}$ . We get:

$$\|W_{(a)}^{(1;2)(k)}\| \leq 16\bar{g}\|h_{L,K}\|_{L_\infty} \|W^{(2;2)(k)}\| \sum_{j \geq k+1} \|g^{(j)}\|_{L_1} \leq \frac{C_2}{4} \bar{g} \gamma^{-2(k-K)}. \quad (2.37)$$

Analogously, by using (2.36), we get for the graph (d) the bound

$$\|W_{(d)}^{(1;2)(k)}\| \leq 2\|W^{(0;2)(k)}\| \sum_{j \geq k+1} \|g^{(j)}\|_{L_1} \leq \frac{C_2}{4} \bar{g} \gamma^{-2(k-K)}. \quad (2.38)$$

Let us now consider the terms (b) and (c). In these cases the previous procedure, based on the non locality of the interaction, which allows us to improve the bound by substituting the  $L_\infty$  norm of the propagator with its  $L_1$  norm, can not be directly applied and we have to face the problem that the trivial bound is proportional to  $\|\widetilde{W}^{(2;0)(k)}\|$ , which is divergent for  $N \rightarrow \infty$  (see the remarks after Theorem 2.1). We shall bypass this difficulty, by exploiting the fact that  $\int dz dz' \widetilde{W}_{\Theta, \Theta'}^{(2;0)(k)}(\mathbf{z}, \mathbf{z}') = 0$  in the following way. Let us consider first the term (b) and note that

$$W_{(b)\Theta'; \Theta}^{(1;2)(k)}(\mathbf{z}; \mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) Z \sum_{\Theta''} \int d\mathbf{w} \widetilde{W}_{\Theta', \Theta''}^{(2;0)(k)}(\mathbf{z}, \mathbf{w}) h_{\Theta'', \Theta}^{L,K}(\mathbf{w} - \mathbf{x})$$

If we take one derivative w.r.t.  $J_{\mathbf{z}, \Theta'}$  in (2.34) and then we put  $\psi = J = 0$ , we get a representation of  $\widetilde{W}_{\Theta', \Theta''}^{(2;0)(k)}(\mathbf{z}, \mathbf{w})$ , that we insert in the previous expression; the result can be graphically represented as in the left side of the first line in Fig. 8. We further expand this expression by substituting the kernel  $W_{(b)\Theta'; \Theta''}^{(1;2)(k)}(\mathbf{z}; \mathbf{u}', \mathbf{u})$  with its expansion in Fig. 7; the result is represented in Fig.8.

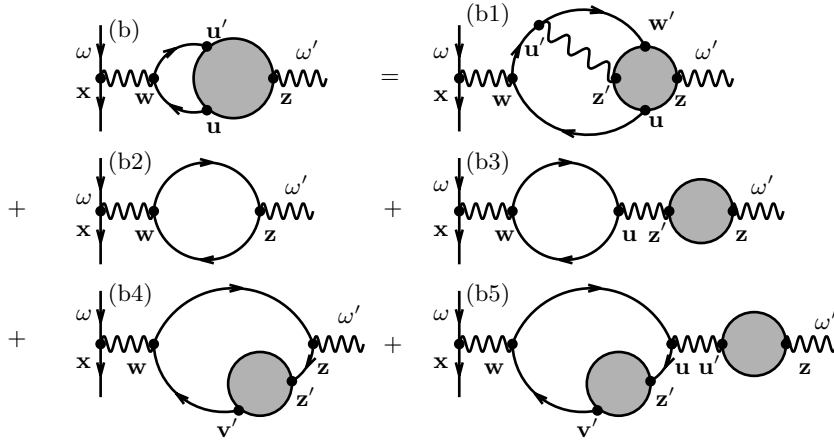


Figure 8: Graphical representation of the term (b) in the decomposition in Fig.7

Consider the term (b1). The new estimation procedure is now slightly more elaborated; decompose the three propagators  $g_{\omega''}^{[k+1, N]}(\mathbf{w} - \mathbf{u}')$ ,  $g_{\omega''}^{[k+1, N]}(\mathbf{u} - \mathbf{w})$  and  $g_{\omega''}^{[k+1, N]}(\mathbf{u}' - \mathbf{w}')$  into scales and take the  $L_\infty$ -norm of the lowest scale propagator, while the two others are used to control the integration over the inner space variables through their  $L_1$ -norm; then, for  $W^{(2;2)(k)}$  use again the bound (2.28), that is  $\|W^{(2;2)(k)}\| \leq C^3 \varepsilon_0 \gamma^{-k}$ . We get:

$$\begin{aligned} \|W_{(b1)}^{(1;2)(k)}\| &\leq 16\bar{g}^2 \|h_{L,K}\|_{L_1} \|h_{L,K}\|_{L_\infty} \|W^{(2;2)(k)}\|. \\ &\cdot 3! \sum_{i \geq j \geq i' \geq k+1} \|g^{(i)}\|_{L_1} \|g^{(j)}\|_{L_1} \|g^{(i')}\|_{L_\infty} \leq C_{2,1} \bar{g}^2 \gamma^{-2(k-K)} \end{aligned} \quad (2.39)$$

In a similar way, by using the bound (2.36), we see that  $\|W_{(b4)}^{(1;2)(k)}\| \leq C_{2,4}\bar{g}\gamma^{-2(k-K)}$

The bound of (b2) and (b3) looks as problematic as the bound of (b), but we can now use in a simple way the cancellations related to the propagator symmetry  $g_\omega^{[k,N]}(x, x_0) = -i\omega g_\omega^{[k,N]}(-x_0, x)$ , which implies that  $\int d\mathbf{z}d\mathbf{z}' \widetilde{W}_{\Theta, \Theta'}^{(2;0)(k)}(\mathbf{z}, \mathbf{z}') = 0$ , at any order in  $\bar{g}$ , in particular at order 0, that is

$$\int d\mathbf{u} \left[ g_{\omega'}^{[k+1,N]}(\mathbf{u}) \right]^2 = 0. \quad (2.40)$$

Let us consider first (b2); we get

$$W_{(b2)\Theta'; \Theta}^{(1;2)(k)}(\mathbf{z}; \mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \sum_{\Theta''} \int d\mathbf{w} h_{\Theta, \Theta''}^{L,K}(\mathbf{x} - \mathbf{w}) \left[ g_{\omega''}^{[k+1,N]}(\mathbf{w} - \mathbf{z}) \right]^2 \quad (2.41)$$

By using the identities (2.40) and

$$h_{\Theta, \Theta''}^{L,K}(\mathbf{x} - \mathbf{w}) = h_{\Theta, \Theta''}^{L,K}(\mathbf{x} - \mathbf{z}) + \sum_{j=0,1} (z_j - w_j) \int_0^1 ds (\partial_j h_{\Theta, \Theta''}^{L,K})(\mathbf{x} - \mathbf{z} + s(\mathbf{z} - \mathbf{w})) \quad (2.42)$$

we get:

$$\begin{aligned} W_{(b2)\Theta'; \Theta}^{(1;2)(k)}(\mathbf{z}; \mathbf{x}, \mathbf{y}) &= \delta(\mathbf{x} - \mathbf{y}) \cdot \\ &\cdot \sum_{\substack{\Theta'' \\ j=0,1}} \int_0^1 ds \int d\mathbf{w} (\partial_j h_{\Theta, \Theta''}^{L,K})(\mathbf{x} - \mathbf{z} + s(\mathbf{z} - \mathbf{w})) (z_j - w_j) \left[ g_{\omega'}^{[k+1,N]}(\mathbf{w} - \mathbf{z}) \right]^2 \end{aligned} \quad (2.43)$$

Hence,

$$\|W_{(b2)}^{(1;2)(k)}\| \leq 8\bar{g} \|\partial h_{L,K}\|_{L_1} \sum_{i \geq j \geq k+1} \|g^{(i)}\|_{\tilde{L}_1} \|g^{(j)}\|_{L_\infty} \leq C_{2,2}\bar{g}\gamma^{-(k-K)}. \quad (2.44)$$

Following the same strategy, we obtain a bound for (b3) which is obtained from the bound for (b2) by multiplying it by  $\|W_{(b)}^{(1;2)(k)}\|$ ; we get:

$$\|W_{(b3)}^{(1;2)(k)}\| \leq C_{2,2}\bar{g}\gamma^{-(k-K)} \|W_{(b)}^{(1;2)(k)}\| \quad (2.45)$$

Analogously, the bound for (b5) is obtained from the bound for (b4) by multiplying it by  $\|W_{(b)}^{(1;2)(k)}\|$ ; hence  $\|W_{(b5)}^{(1;2)(k)}\| \leq C_{2,4}\bar{g}\gamma^{-2(k-K)} \|W_{(b)}^{(1;2)(k)}\|$ .

Therefore, summing all the bounds for (b1) to (b5), we obtain

$$\|W_{(b)}^{(1;2)(k)}\| \leq \frac{C_2}{2}\bar{g}\gamma^{-(k-K)} + \bar{g}C_5 \|W_{(b)}^{(1;2)(k)}\|$$

with  $C_2 = 2(C_{2,1} + C_{2,2} + C_{2,4})$  and  $C_5 = C_{2,2} + C_{2,4}$ . Hence, if  $\bar{g}$  is small enough,

$$\|W_{(b)}^{(1;2)(k)}\| \leq C_2\bar{g}\gamma^{-(k-K)} \quad (2.46)$$

It is now easy to complete the proof of (2.30).

c) *Bound for  $W^{(0;4)(k)}$ .* If we take three derivatives w.r.t.  $\psi$  in (2.34) and then we put  $\psi = J = 0$ , we get the decomposition of  $W^{(0;4)(k)}$  depicted in Fig.9.

Let us consider first the graphs (a) and (d), which differ only for the interchange of  $\mathbf{y}$  with  $\mathbf{y}'$ ; by using (2.30), we get:

$$\|W_{(a)}^{(0;4)(k)}\| + \|W_{(d)}^{(0;4)(k)}\| \leq 16 \|h_{L,K}\|_{L_1} C_2\bar{g}\gamma^{-(h-K)} \leq C_{3,1}\bar{g}^2\gamma^{-(h-K)}$$

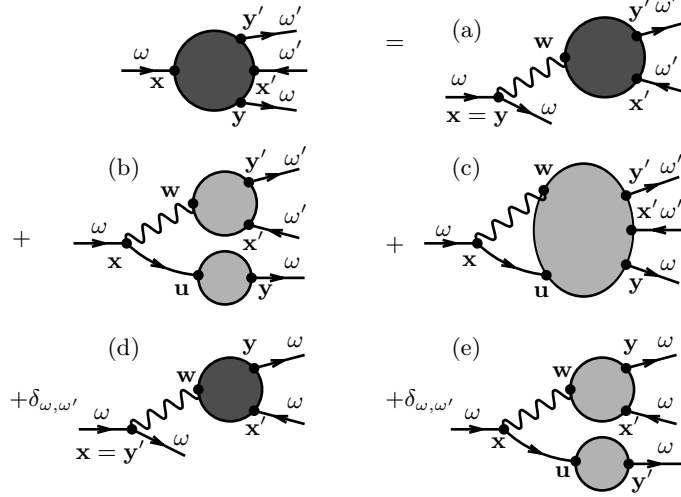


Figure 9: Decomposition of the class of graphs  $W_{\alpha, \alpha'}^{(0;4)(k)}$ . The darker bubbles represents  $W_{\alpha, \alpha'}^{(0;4)(k)} - w_{\alpha, \alpha'}^{(0;4)}$  or  $W_{\alpha; \alpha'}^{(1;2)(k)} - w_{\alpha; \alpha'}^{(1;2)}$ .

By using (2.30) and the bound in the right side of (2.27), we get

$$\|W_{(b)}^{(0;4)(k)}\| + \|W_{(e)}^{(0;4)(k)}\| \leq 16 \|h_{L,K}\|_{L_1} \|W^{(1;2)(k)}\| \|W^{(0;2)(k)}\| \sum_{j=k}^N \|g^{(j)}\|_{L_1} \leq C_{3,2} \bar{g}^2 \gamma^{-2(h-K)}$$

Finally, by using (2.27) to bound  $\|W^{(1;4)(k)}\|$ , we get

$$\|W_{(c)}^{(0;4)(k)}\| \leq 8 \|h_{L,K}\|_{L_\infty} \|W^{(1;4)(k)}\| \sum_{j \geq k} \|g^{(j)}\|_{L_1} \leq C_{3,3} \varepsilon_0 \bar{g} \gamma^{-2(k-K)}$$

Then, if  $\bar{g} \leq \varepsilon_0$ , by summing the three previous bounds, we get (2.31), with  $C_3 = C_{3,1} + C_{3,2} + C_{3,3}$ . The proof of Theorem 2.2 is complete. ■

### 2.3 The infrared integration

The parameter  $K$  has only the role of clarifying the dimensional content of the bounds in §2.2; hence, from now on, we shall put  $K = 0$ . For analogous reasons we shall also put  $Z = 1$ .

Note that Theorem 2.1 implies that the effective potential on scale 0 of the model (2.2) is equivalent, from the RG point of view, to the effective potential on scale 0 of the extended Hubbard model, which has been studied in detail in the companion paper [1]. The reason is in the following remarks:

- a) The free propagator  $g_{D,\omega}^{[l,0]}(\mathbf{x})$  has the same asymptotic behavior of the Hubbard free propagator, see eq. (2.100) in [1], at the infrared scales; the fact that, in [1], there is a different definition of the finite space region and of the infrared cutoff has negligible effects, as it is easy to see by looking at the description of the infrared integration in §2.4 of [1].
- b) The local parts of the effective potential in the two models have the same structure. As a consequence, the running coupling constants (r.c.c.) of the two models can be defined in a similar way, with two differences: in the model (2.2) there is one more quartic marginal term, but, thanks to the oddness of the free propagator, the r.c.c.  $\nu_h$  (that multiplying the relevant monomial  $\psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^-$ ) vanishes exactly, so that the singularity of the interacting propagator stays fixed at

$\mathbf{k} = (0, 0)$ . As concerns the renormalization constants (ren.c.), in the model (2.2) we put a restrict set of source terms, which are related to the charge and spin ren.c.'s of the Hubbard model; hence, we have to consider three ren.c.'s, that is  $Z_h$ , the *free measure ren.c.*,  $Z_h^{(1)}$  and  $Z_h^{(2)}$ , related to the source terms with  $s = t$  and  $s = -t$ , respectively.

Hence we can repeat word by word the analysis of §2.4 of [1], with the simplifications that the free propagator scales exactly and  $\nu_h = 0$ . In particular, we can define in a similar way the functionals  $\tilde{\mathcal{S}}_j(J, \eta)$ ,  $\widehat{\mathcal{V}}^{(j)}(\psi)$ ,  $\mathcal{V}^{(j)}(\psi)$ ,  $\widehat{\mathcal{B}}^{(j)}(\psi, J, \eta)$ ,  $\mathcal{B}^{(j)}(\psi, J, \eta)$  and the renormalized single scale free measure  $P_{Z_{j-1}, \tilde{f}_j^{-1}}(d\psi)$ , with propagator

$$\widehat{g}_\omega^{(j)}(\mathbf{k}) = \begin{cases} \frac{1}{Z_{j-1}} \frac{\tilde{f}_j(|\tilde{\mathbf{k}}|)}{D_\omega(\mathbf{k})} & l < j \leq 0 \\ \frac{1}{Z_{l-1}(\mathbf{k})} \frac{f_l(|\tilde{\mathbf{k}}|)}{D_\omega(\mathbf{k})} & j = l \end{cases} \quad (2.47)$$

where  $\tilde{f}_j(t)$  has the same support and smoothness properties of  $f_j(t)$  and  $\tilde{Z}_{l-1}(\mathbf{k}) = 1 + (Z_{l-1} - 1)f_l(|\tilde{\mathbf{k}}|)$ ; for details see also §3 of [8].

**Remark** - Even if  $\tilde{Z}_{l-1}(\mathbf{k})$  takes values between 1 and  $Z_{l-1}$ ,  $|\widehat{g}_\omega^{(j)}(\mathbf{k})|$  is bounded by  $c\gamma^{-j}Z_{j-1}^{-1}$  for all  $j \in [l, 0]$ . However, in the following, in particular in the prove of the bounds (3.9), (3.10) and (C.7), the strong dependence on  $\mathbf{k}$  of  $\tilde{Z}_{l-1}(\mathbf{k})$  will be relevant.

The r.c.c. are defined as the coefficients of the local part of  $\widehat{\mathcal{V}}^{(j)}(\psi)$ , which is given by

$$\mathcal{L}\widehat{\mathcal{V}}^{(j)}(\psi) = \delta_j F_\delta(\psi) + g_{1,\perp,j} F_{1,\perp}(\psi) + g_{\parallel,j} F_{\parallel}(\psi) + g_{\perp,j} F_{\perp}(\psi) + g_{4,j} F_4(\psi) \quad (2.48)$$

where  $F_\delta(\psi) = \sum_{\omega,s} \int d\mathbf{x} \psi_{\mathbf{x},\omega,s}^+ (i\omega \partial_x) \psi_{\mathbf{x},\omega,s}^-$ , while the other functions are defined as the potentials (2.5) with  $\delta(\mathbf{x} - \mathbf{y})$  in place of  $h_{L,K}(\mathbf{x} - \mathbf{y})$ . Moreover,

$$\mathcal{L}\widehat{\mathcal{B}}^{(j)}(\psi, J, \eta) = \sum_{\omega,s} \int d\mathbf{x} \left[ J_{\mathbf{x},\omega,s,s} \frac{Z_j^{(1)}}{Z_{j-1}} \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^- + J_{\mathbf{x},\omega,s,-s} \frac{Z_j^{(2)}}{Z_{j-1}} \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,-s}^- \right] \quad (2.49)$$

We introduce a label  $\alpha \in \{1, 2\}$  to distinguish, in the tree expansion, the endpoints of type  $J$  associated with the two terms of (2.49).

**Remark** - As we have remarked in §2.2 for the UV scales, the compact support properties of the single scale propagators  $\widehat{g}_\omega^{(j)}(\mathbf{k})$  imply that the effective potential on scale  $j$  depends on  $N$  and  $L$ , but is independent of the IR cutoff  $l$ , if  $j > l$ . The same remark is of course true for the r.c.c. and the ren.c.; however, since the r.c.c. are defined only up to scale  $l + 1$ , they are all independent of  $L$ .

Let us now call  $W_{\underline{\omega}, \underline{\alpha}, \underline{\varepsilon}, \underline{s}, m_\psi, m_J, m_\eta}^{(h)}(\underline{\mathbf{x}})$  the kernels of the various terms contributing to  $\mathcal{V}^{(j)}(\psi)$ ,  $\mathcal{B}^{(j)}(\psi, J, \eta)$  and  $\tilde{\mathcal{S}}_j(J, \eta)$ , where  $m_\psi$ ,  $m_J$ ,  $m_\eta$  are the numbers of external fields of type  $\psi$ ,  $J$ ,  $\eta$ , respectively;  $\underline{\omega}$ ,  $\underline{\varepsilon}$  and  $\underline{s}$  are the sets of  $\omega$ ,  $\varepsilon$  and  $s$  indices (possibly void) of the  $\psi$  and  $\eta$  fields, while  $\underline{\alpha}$  is the the set of  $\alpha$  indices of the  $J$  fields. Let us define

$$\varepsilon_h = \max_{h \leq j \leq 0} \max\{|\delta_j|, |g_{1,\perp,j}|, |g_{\parallel,j}|, |g_{\perp,j}|, |g_{4,j}|\} \quad (2.50)$$

Lemma 2.3 of [1] and the analysis of the ren.c. flow in §2.6 of the same paper imply the following Theorem.

**Theorem 2.3** *There is  $\bar{\varepsilon}$ ,  $c_1$ ,  $C$ , independent of  $N, l, L$ , such that, if  $\varepsilon_h \leq \bar{\varepsilon}$ , then*

$$\sup_{j>h} Z_j/Z_{j-1} \leq e^{c_1 \varepsilon_h^2} \quad , \quad \sup_{\substack{j>h \\ i=1,2}} Z_j^{(i)}/Z_{j-1} \leq e^{c_1 \varepsilon_h} \quad (2.51)$$

$$\int d\mathbf{x} |W_{\underline{\omega}, \underline{\alpha}, \underline{\varepsilon}, \underline{s}, m_\psi, m_J, m_\eta}^{(h)}(\mathbf{x})| \leq \beta LC^{m+m_S} \varepsilon_h^{k_{m, m_S}} \gamma^{-hD_{m_\psi, m_J, m_\eta}} \quad (2.52)$$

where  $2m = m_\psi + m_\eta$ ,  $m_S = m_J + m_\eta$ ,  $D_{m_\psi, m_J, m_\eta} = -2 + m + m_J(1 + c_1\varepsilon_0) + m_\eta(1 + \frac{1}{2}c_1\varepsilon_0^2)$ ,  $k_{m, m_S} = \max\{1, m-1\}$ , if  $m_S = 0$ , otherwise  $k_{m, m_S} = \max\{0, m-1\}$ .

This Theorem is sufficient to perform the limit  $N \rightarrow \infty$ , followed by the limit  $L \rightarrow \infty$ , at fixed  $l$ , but to prove that  $\bar{\varepsilon}$  is independent of  $l$  is not trivial. It can be proved only under one of the conditions: 1)  $g_{1,\perp} = 0$ ; 2)  $g_{\parallel} = g_{\perp} - g_{1,\perp}$  and  $g_{1,\perp} > 0$ . Moreover, we have to use the strong cancelations related to the local gauge invariance of the interaction (2.13) in the case  $g_{1,\perp} = 0$ . In the following sections, we shall explain how to use this symmetry in order to prove iteratively that  $\bar{\varepsilon}$  stays away from 0 for  $l \rightarrow -\infty$  and, at the same time, the important bound

$$\left| \frac{Z_j^{(1)}}{Z_j} - 1 \right| \leq C|\varepsilon_j^2| \quad (2.53)$$

The properties of the tree expansion allow us also to prove that all Schwinger functions and the correlation functions, at *fixed distinct values of the space variables*, are well defined in the limit  $L \rightarrow \infty$ , when all the cutoffs are removed (the condition that there are not coinciding space points is to avoid the singularities related with the UV divergence of the free propagator). The same is true for their Fourier transforms, if one puts the following condition on the corresponding momenta  $\{\mathbf{q}_i, i = 1, \dots, m\}$  of the  $m$  external fields of type  $J$  or  $\eta$  :

$$\sum_{i=1}^m \sigma_i \mathbf{q}_i \neq 0 \quad , \quad \forall \sigma_i = \pm 1 \quad (2.54)$$

If this condition is satisfied, we shall say that *the momenta are not exceptional*; it is imposed to avoid the singularities related to the possible divergence of the  $L^1$  norm for  $L, -l \rightarrow \infty$ , in agreement with the bound (2.52); these singularities are of course related to the divergence of the  $L^1$  norm of the free propagator. The proof of these properties depends only on the structure of the tree expansion and on the possibility of controlling the r.c.c.'s flow, while the details of the model are not important. Hence, we refer for their proof to paper [9], where they are precisely discussed, especially the first one, with a different localization procedure, which involves also the terms with one  $\eta$  field and no  $J$  field.

In §4.4 we shall also need a bound on some particular correlation functions in the limit  $L, N \rightarrow \infty$  at fixed  $l$ , when the external momenta are of order  $\gamma^l$ . A similar property, whose proof is based on a technique introduced in §3 of [8], has been used in [10], Theorem 1, in the case of a model with spin 0, local interaction and a fixed UV cutoff. To extend this theorem to the actual model is essentially trivial; hence we shall write the result without proof. Let us define  $\langle \cdot \rangle_T$  the truncated expectation of the model (2.2) and  $\rho_{\mathbf{x}, \omega, s} := \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, s}^-$ .

**Theorem 2.4** *In the hypotheses of Theorem 2.3, if  $|\tilde{\mathbf{k}}| = \gamma^l$ , then, in the limit  $L \rightarrow \infty$ , followed by the limit  $N \rightarrow \infty$ , at fixed  $l$ ,*

$$\begin{aligned} \langle \hat{\psi}_{\mathbf{k}, \omega, s}^- \hat{\psi}_{\mathbf{k}, \omega, s}^+ \rangle &= \frac{1}{Z_l D_\omega(\mathbf{k})} [1 + O(\varepsilon_l)] \\ \langle \hat{\rho}_{2\mathbf{k}, \omega, s}; \hat{\psi}_{\mathbf{k}, \omega, s}^- \hat{\psi}_{-\mathbf{k}, \omega, s}^+ \rangle_T &= -\frac{Z_l^{(1)}}{Z_l^2 D_\omega(\mathbf{k})^2} [1 + O(\varepsilon_l)] \\ \langle \hat{\psi}_{\mathbf{k}, \omega, s}^+; \hat{\psi}_{-\mathbf{k}, \omega, \mu s}^-; \hat{\psi}_{-\mathbf{k}, \omega', s'}^+; \hat{\psi}_{\mathbf{k}, \omega', \mu s'}^- \rangle_T &= -\frac{1}{Z_l^2 |\tilde{\mathbf{k}}|^4} \left\{ -g_{1,\perp, l} \delta_{\mu, -1} \delta_{\omega, -\omega'} \delta_{s, -s'} \right. \\ &\quad \left. + \delta_{\mu, 1} [g_{\parallel, l} \delta_{\omega, -\omega'} \delta_{s, s'} + g_{\perp, l} \delta_{\omega, -\omega'} \delta_{s, -s'} + g_{4, l} \delta_{\omega, \omega'} \delta_{s, -s'} + O(\varepsilon_l^2)] \right\} \end{aligned} \quad (2.55)$$

### 3 Ward Identities

#### 3.1 Ward Identities in the $g_{1,\perp} = 0$ case in presence of cut-offs

In this section we want to analyze the Ward Identities (WI) related to the local gauge invariance of the interaction (2.4) in the case  $g_{1,\perp} = 0$ . As explained in §2.1, this can only be done by putting  $\varepsilon > 0$  in (2.3) and by introducing a lattice spacing  $a$  in  $\Lambda_L$ . At the end, we have to perform the limit  $a \rightarrow 0$ , followed by the limit  $\varepsilon \rightarrow 0$ . In §2.1 we have also explained why the limit  $a \rightarrow 0$  is essentially trivial; hence, in the following, we shall write the WI directly in the continuum limit  $a = 0$ . As concerns the limit  $\varepsilon \rightarrow 0$ , it is trivial too, in the contributions coming from the effective potentials, that we shall then suppose evaluated at  $\varepsilon = 0$ , while it needs an accurate analysis for the terms coming from the free measure.

If  $g_{1,\perp} = 0$ , the model (2.2) is invariant under the global gauge transformation (2.10) and the interaction  $\mathcal{V}(\psi, J)$  (see (2.13), where  $\int d\mathbf{x}$  has to be understood as  $\sum_{\mathbf{x} \in \Lambda_L^a}$ ) is invariant even under the local gauge transformation

$$\psi_{\mathbf{x},\omega,s}^{\pm} \rightarrow e^{\pm i\alpha_{\mathbf{x},\omega,s}} \psi_{\mathbf{x},\omega,s}^{\pm}, \quad \mathbf{x} \in \Lambda_L^a \quad (3.1)$$

for any lattice periodic function  $\alpha_{\mathbf{x},\omega,s}$ . Hence, if we make in the r.h.s. of (2.2) this transformation, we get in the limit  $a \rightarrow 0$ , followed by the limit  $\varepsilon \rightarrow 0$  (see §2 of [8] for details in a similar problem), the following *functional Ward Identity (WI)*, where  $\mathbf{p} \in \mathcal{D}_L$  and  $\widehat{J}_{\mathbf{p},\omega,s}$  is defined so that  $J_{\mathbf{z},\omega,s} = L^{-2} \sum_{\mathbf{p} \in \mathcal{D}_L} e^{-i\mathbf{p}\mathbf{z}} \widehat{J}_{\mathbf{p},\omega,s}$ .

$$D_{\omega}(\mathbf{p}) \frac{\partial \mathcal{W}_{l,N}(J, \eta)}{\partial \widehat{J}_{\mathbf{p},\omega,s}} = B_{\mathbf{p},\omega,s}(J, \eta) + R_{\mathbf{p},\omega,s}(\mathbf{p}; J, \eta) \quad , \quad D_{\omega}(\mathbf{p}) = -ip_0 + c\omega p \quad (3.2)$$

where (since  $Z = 1$ )

$$B_{\mathbf{p},\omega,s}(J, \eta) := \frac{1}{L^2} \sum_{\mathbf{k} \in \mathcal{D}'_L} \left[ \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+ \frac{\partial \mathcal{W}_{l,N}(J, \eta)}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+} - \frac{\partial \mathcal{W}_{l,N}(J, \eta)}{\partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^-} \widehat{\eta}_{\mathbf{k},\omega,s}^- \right] \quad (3.3)$$

and

$$R_{\mathbf{p},\omega,s}(J, \eta) = e^{-\mathcal{W}_{l,N}(J, \eta)} \lim_{\varepsilon \rightarrow 0} \frac{1}{L^2} \sum_{\mathbf{q} \in \mathcal{D}'_L} C_{\omega}(\mathbf{q} + \mathbf{p}, \mathbf{q}) \int P_Z^{[l,N]}(d\psi) \widehat{\psi}_{\mathbf{q}+\mathbf{p},\omega,s}^+ \widehat{\psi}_{\mathbf{q},\omega,s}^- e^{\mathcal{V}(\psi, J, \eta)} \quad (3.4)$$

with

$$C_{\omega}(\mathbf{k}^+, \mathbf{k}^-) := \left[ \frac{1}{\chi_{l,N}^{\varepsilon}(|\mathbf{k}^-|)} - 1 \right] D_{\omega}(\mathbf{k}^-) - \left[ \frac{1}{\chi_{l,N}^{\varepsilon}(|\mathbf{k}^+|)} - 1 \right] D_{\omega}(\mathbf{k}^+) \quad (3.5)$$

Note that, here and in the following, the derivatives with respect to the momenta have to be understood as the ordinary derivatives times  $L^2$ , so that they formally become the usual functional derivatives when  $L \rightarrow \infty$ . Moreover, we are defining the derivatives w.r.t. the Grassman variables so that, if  $\{\chi_i^{\varepsilon}, i = 1, \dots, n\}$  is a set of distinct  $\psi_{\mathbf{k},\omega,s}^{\varepsilon}$  variables and  $n > 1$ , then

$$\frac{\partial}{\partial \chi_j^{\varepsilon}} \prod_{i=1}^n \chi_i^{\varepsilon} = \varepsilon_j (-1)^{j-1} \prod_{i \neq j} \chi_i^{\varepsilon}$$

The last term  $R_{\mathbf{p},\omega,s}(J, \eta)$  formally vanishes if we replace the cut-off function  $\chi_{l,N}^{\varepsilon}$  in (2.3) (necessary to make the functional integral (2.2) well defined) with 1. This term can be therefore considered a *correction* to the *formal* functional WI (the one in which  $R_{\omega,s}(\mathbf{p}; J, \eta)$  is neglected); the origin of this correction is in the fact that the momentum cut-offs break the (formal) *local*

gauge invariance. We will show that such corrections are not vanishing even in the limit of removed cut-offs, a phenomenon known as *quantum anomaly*.

A crucial role is played by the properties of the function  $C_\omega(\mathbf{k}^+, \mathbf{k}^-)$ . This function looks very singular; however, since it appears in the expansions always multiplied by two propagators, what it is really important are the properties of the function

$$\widehat{U}_{l,N,\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q}) := C_\omega(\mathbf{q} + \mathbf{p}, \mathbf{q}) \widehat{g}_\omega^{(i)}(\mathbf{q} + \mathbf{p}) \widehat{g}_\omega^{(j)}(\mathbf{q}) \quad (3.6)$$

with  $\widehat{g}_\omega^{(j)}(\mathbf{k})$  defined as in (2.47), for all  $j$ , if we put  $\widetilde{f}_j(|\widetilde{\mathbf{k}}|) = f_j(|\widetilde{\mathbf{k}}|)$  and  $Z_{j-1}^{-1} = 1$ , if  $j > 0$ . The main property, which we will extensively use in the following, is that, since  $\chi_{l,N}^\varepsilon(\mathbf{k}) = 1$  if  $\widehat{g}_\omega^{(j)}(\mathbf{k}) \neq 0$  and  $l < j < N$ , then

$$\widehat{U}_{l,N,\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q}) = 0 \quad l < i, j < N \quad (3.7)$$

In the following we shall use the function  $\widehat{U}_{l,N,\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q})$  only under the condition that  $|\widetilde{\mathbf{p}}| \leq 2\gamma^{N+1}$ ; hence we can multiply it by  $\widetilde{\chi}_N(\mathbf{p}) = \widetilde{\chi}_0(2^{-1}\gamma^{-N-1}|\widetilde{\mathbf{p}}|)$ , where  $\widetilde{\chi}_0(t)$  is a smooth positive function of support in  $[0, 2]$  and equal to 1 for  $t \leq 1$ . In App. C we show that it is possible to define two functions  $\widehat{S}_{l,N,\omega',\omega}^{(i,j)}(\mathbf{k}^+, \mathbf{k}^-)$ , such that (recall that  $Z_j = 1$  for  $j \geq 0$ )

$$\widetilde{\chi}_N(\mathbf{p}) \lim_{\varepsilon \rightarrow 0} \widehat{U}_{l,N,\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q}) = \frac{1}{Z_{i-1} Z_{j-1}} \sum_{\omega' = \pm \omega} D_{\omega'}(\mathbf{p}) \widehat{S}_{l,N,\omega',\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q}) \quad (3.8)$$

and, if we define the Fourier transform of  $\widehat{S}_{l,N,\omega',\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q})$  as

$$S_{l,N,\omega',\omega}^{(i,j)}(\mathbf{z}; \mathbf{x}, \mathbf{y}) = \frac{1}{L^4} \sum_{\mathbf{k}^+, \mathbf{k}^- \in \mathcal{D}'_L} e^{-i\mathbf{k}^+(\mathbf{x}-\mathbf{z})} e^{i\mathbf{k}^-(\mathbf{y}-\mathbf{z})} \widehat{S}_{l,N,\omega',\omega}^{(i,j)}(\mathbf{k}^+, \mathbf{k}^-)$$

then, if  $j \geq l$ , for any positive integers  $r$  (which in the following will be fixed large enough, e.g.  $r = 4$ ),

$$|S_{l,N,\omega',\omega}^{(N,j)}(\mathbf{z}; \mathbf{x}, \mathbf{y})| \leq (1 + \delta_{j,l} Z_{l-1}) b_{r,N}(\mathbf{x} - \mathbf{z}) b_{r,j}(\mathbf{y} - \mathbf{z}) \quad , \quad b_{r,k}(\mathbf{x}) := C_p \frac{\gamma^k}{1 + [\gamma^k \|\mathbf{y} - \mathbf{z}\|]^r} \quad (3.9)$$

where  $\|\mathbf{x} - \mathbf{y}\|$  is the distance on the torus  $\Lambda_L$ . The bound (3.9) has an essential role in the integration of the UV scales; in the integration of the IR scales, we shall need a bound on the Fourier transform of the functions (independent of  $N$ )

$$\widetilde{S}_{l,\omega',\omega}^{(j,l)}(\mathbf{q} + \mathbf{p}, \mathbf{q}) := \frac{1}{2} [1 - \widetilde{\chi}_l(\mathbf{p})] \lim_{\varepsilon \rightarrow 0} \frac{\widehat{U}_{l,N,\omega}^{(j,l)}(\mathbf{q} + \mathbf{p}, \mathbf{q})}{D_{\omega'}(\mathbf{p})} \quad , \quad j \geq l$$

where  $\widetilde{\chi}_l(\mathbf{p}) := \widetilde{\chi}_0[2^{-1}\gamma^{-l-1}|\widetilde{\mathbf{p}}|]$ , with  $\widetilde{\chi}_0(t)$  is the same function used in the definition of  $\widetilde{\chi}_N(\mathbf{p})$ . These functions allow to write an identity similar to (3.8) for  $\widehat{U}_{l,N,\omega}^{(j,l)}$ .

In App. C we also show that, for any positive integers  $r$ ,

$$\begin{aligned} |\widetilde{S}_{l,\omega',\omega}^{(j,l)}(\mathbf{z}; \mathbf{x}, \mathbf{y})| &\leq \frac{(1 - \delta_{j,l}) \widetilde{b}_{r,j}(\mathbf{x} - \mathbf{z}) \widetilde{u}_{r,l}(\mathbf{y} - \mathbf{z})}{Z_{j-1}} \\ \widetilde{b}_{r,j}(\mathbf{x}) &:= C_p \frac{1}{1 + [\gamma^k \|\mathbf{y} - \mathbf{z}\|]^r} \quad , \quad \widetilde{u}_{r,l}(\mathbf{x}) := C_p \frac{\gamma^{2l}}{1 + [\gamma^k \|\mathbf{y} - \mathbf{z}\|]^r} \end{aligned} \quad (3.10)$$

Finally, in App. C we show that, if we define (recall that we have put  $Z = 1$  and  $K = 0$ )

$$\tau_N^\pm := \sum_{i,j=K+1}^N \frac{1}{L^2} \sum_{\mathbf{q} \in \mathcal{D}'_L} \widehat{S}_{\pm\omega,\omega}^{(i,j)}(\mathbf{q}, \mathbf{q}) = \sum_{i,j=K+1}^N \int d\mathbf{u} S_{\pm\omega,\omega}^{(i,j)}(\mathbf{z}; \mathbf{u}, \mathbf{u}) \quad (3.11)$$

then

$$|\tau_N^+| \leq C \frac{\gamma^{-N}}{L} \quad , \quad |\tau_N^- - \tau| \leq C \frac{\gamma^{-N}}{L} \quad , \quad \tau = \frac{1}{4\pi c} \quad (3.12)$$

Let us now observe that, for any choice of the functions  $\nu_s^\omega(\mathbf{p})$ , we can write  $R_{\mathbf{p},\omega,s}(\mathbf{p}; J, \eta)$  in the following way:

$$\begin{aligned} R_{\mathbf{p},\omega,s}(\mathbf{p}; J, \eta) &= -\tau_N^- D_{-\omega}(\mathbf{p}) \widehat{J}_{-\mathbf{p},\omega,s} + \\ &+ D_{-\omega}(\mathbf{p}) \sum_{\omega',s'} \nu_{ss'}^{\omega\omega'}(\mathbf{p}) \frac{\partial \mathcal{W}_{l,N}(J, \eta)}{\partial \widehat{J}_{\mathbf{p},-\omega',s'}} + \lim_{\varepsilon \rightarrow 0} \frac{\partial \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p},\omega,s}}(0, J, \eta) \end{aligned} \quad (3.13)$$

where, if  $\mathcal{V}(\psi, J, \eta)$  is the expression inside the braces in the r.h.s. side of (2.2) and  $\alpha(\mathbf{z}, \omega, s)$  is a periodic function, whose Fourier transform is defined (analogously to  $J(\mathbf{z}, \omega, s)$ ) so that  $\alpha(\mathbf{z}, \omega, s) = L^{-2} \sum_{\mathbf{p} \in \mathcal{D}_L} e^{-i\mathbf{p}\mathbf{z}} \widehat{\alpha}(\mathbf{p}, \omega, s)$ , then

$$e^{\mathcal{H}_{l,N}(\alpha, J, \eta)} := e^{\frac{1}{L^2} \sum_{\mathbf{p} \in \mathcal{D}_L} \widehat{\alpha}_{\mathbf{p},\omega,s} \tau_N^- D_{-\omega}(\mathbf{p}) \widehat{J}_{-\mathbf{p},\omega,s}} \int P_Z^{[l,N]}(d\psi) e^{\mathcal{V}(\psi, J, \eta) + \mathcal{A}_0(\psi, \alpha) - \mathcal{A}_-(\psi, \alpha)} \quad (3.14)$$

and

$$\begin{aligned} \mathcal{A}_0(\alpha, \psi) &= \sum_{\omega,s} \frac{1}{L^2} \sum_{\mathbf{p} \in \mathcal{D}_L} \frac{1}{L^2} \sum_{\mathbf{q} \in \mathcal{D}'_L} C_\omega(\mathbf{q} + \mathbf{p}, \mathbf{q}) \widehat{\alpha}_{\mathbf{p},\omega,s} \widehat{\psi}_{\mathbf{q}+\mathbf{p},\omega,s}^+ \widehat{\psi}_{\mathbf{q},\omega,s}^- \\ \mathcal{A}_-(\alpha, \psi) &= \sum_{\substack{\omega,\omega' \\ s,s'}} \frac{1}{L^2} \sum_{\mathbf{p} \in \mathcal{D}_L} \frac{1}{L^2} \sum_{\mathbf{q} \in \mathcal{D}'_L} D_{-\omega}(\mathbf{p}) \nu_{ss'}^{\omega\omega'}(\mathbf{p}) \widehat{\alpha}_{\mathbf{p},\omega,s} \widehat{\psi}_{\mathbf{q}+\mathbf{p},-\omega',s'}^+ \widehat{\psi}_{\mathbf{q},-\omega',s'}^- \end{aligned}$$

Let us call *removed cutoffs limit* the limit  $L \rightarrow \infty$ , followed by the limit  $N \rightarrow \infty$  and, finally, from the limit  $l \rightarrow -\infty$ . In the following sections we shall prove that, in the removed cutoffs limit, the last term in the r.h.s. of (3.13) vanishes, if the functions  $\nu_s^\omega(\mathbf{p})$  are chosen as

$$\nu_s^\omega(\mathbf{p}) = \tau_N^- \widehat{h}_{L,K}(\mathbf{p}) [\delta_{s,1} (\delta_{s,-1} g_\perp + \delta_{s,1} g_\parallel) + \delta_{\omega,-1} \delta_{s,-1} g_4] \quad (3.15)$$

Note that the first two terms in the r.h.s. of (3.13) are, in this case, the corrections to the formal functional WI we are looking for. Hence, we shall rewrite the functional WI (3.2), by using (3.13), in the more convenient form

$$\begin{aligned} D_\mu(\mathbf{p}) \frac{\partial \mathcal{W}_{l,N}(J, \eta)}{\partial \widehat{J}_{\mathbf{p},\mu,s}} - D_{-\mu}(\mathbf{p}) \sum_{\sigma,r} \nu_{sr}^{\mu\sigma}(\mathbf{p}) \frac{\partial \mathcal{W}_{l,N}(J, \eta)}{\partial \widehat{J}_{\mathbf{p},-\sigma,r}} = \\ -\tau_N^- D_{-\omega}(\mathbf{p}) \widehat{J}_{-\mathbf{p},\omega,s} + B_{\mathbf{p},\mu,s}(J, \eta) + \lim_{\varepsilon \rightarrow 0} \frac{\partial \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p},\mu,s}}(0, J, \eta) \end{aligned} \quad (3.16)$$

Summing (3.16) over  $s$  we obtain the *charge Ward identity*:

$$\begin{aligned} [D_\mu(\mathbf{p}) - \nu_4(\mathbf{p}) D_{-\mu}(\mathbf{p})] \sum_s \frac{\partial \mathcal{W}_{l,N}(J, \eta)}{\partial \widehat{J}_{\mathbf{p},\mu,s}} - 2\nu_\rho(\mathbf{p}) D_{-\mu}(\mathbf{p}) \sum_s \frac{\partial \mathcal{W}_{l,N}(J, \eta)}{\partial \widehat{J}_{\mathbf{p},-\mu,s}} = \\ \sum_s \left[ -\tau_N^- D_{-\mu}(\mathbf{p}) \widehat{J}_{-\mathbf{p},\mu,s} + B_{\mathbf{p},\mu,s}(J, \eta) + \lim_{\varepsilon \rightarrow 0} \frac{\partial \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p},\mu,s}}(0, J, \eta) \right] \end{aligned} \quad (3.17)$$

with

$$\nu_4(\mathbf{p}) = \tau_N^- g_4 \widehat{h}_{L,K}(\mathbf{p}) \quad , \quad 2\nu_\rho(\mathbf{p}) = \tau_N^- (g_\parallel + g_\perp) \widehat{h}_{L,K}(\mathbf{p}) \quad (3.18)$$

Multiplying (3.16) by  $s$  and summing over  $s$  we obtain the *spin Ward identity*:

$$\begin{aligned} & \left[ D_\mu(\mathbf{p}) + \nu_4(\mathbf{p})D_{-\mu}(\mathbf{p}) \right] \sum_s s \frac{\partial \mathcal{W}_{l,N}(J, \eta)}{\partial \widehat{J}_{\mathbf{p}, \mu, s}} - 2\nu_\sigma(\mathbf{p})D_{-\mu}(\mathbf{p}) \sum_s s \frac{\partial \mathcal{W}_{l,N}(J, \eta)}{\partial \widehat{J}_{\mathbf{p}, -\mu, s}} \\ &= \sum_s s \left[ -\tau_N^- D_{-\mu}(\mathbf{p}) \widehat{J}_{-\mathbf{p}, \mu, s} + B_{\mathbf{p}, \mu, s}(J, \eta) + \lim_{\varepsilon \rightarrow 0} \frac{\partial \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p}, \mu, s}}(0, J, \eta) \right] \end{aligned} \quad (3.19)$$

with

$$2\nu_\sigma(\mathbf{p}) = \tau_N^-(g_\parallel - g_\perp) \widehat{h}_{L,K}(\mathbf{p}) \quad (3.20)$$

By doing suitable functional derivatives of the charge WI (3.17), we get many WI involving the correlation functions. For example, if we take two derivatives w.r.t.  $\widehat{\eta}_{\mathbf{p}+\mathbf{k}, \omega, s}^+$  and  $\widehat{\eta}_{\mathbf{k}, \omega, s}^-$  in both sides of (3.17), we sum over  $\mu$  and we put  $\eta = J = 0$ , we get the following *charge vertex Ward identity*:

$$\begin{aligned} -ip_0 \left[ 1 - 2\nu_\rho(\mathbf{p}) - \nu_4(\mathbf{p}) \right] G_{\rho; \omega, s}(\mathbf{p}; \mathbf{p} + \mathbf{k}) + cp \left[ 1 - 2\nu_\rho(\mathbf{p}) + \nu_4(\mathbf{p}) \right] G_{j; \omega, s}(\mathbf{p}; \mathbf{p} + \mathbf{k}) \\ = G_{2; \omega, s}(\mathbf{k}) - G_{2; \omega, s}(\mathbf{p} + \mathbf{k}) + R_{\omega, s}(\mathbf{p}; \mathbf{p} + \mathbf{k}) \end{aligned} \quad (3.21)$$

where

$$G_{\rho; \omega, s}(\mathbf{p}; \mathbf{p} + \mathbf{k}) = \sum_{\mu, t} \frac{\partial^3 \mathcal{W}}{\partial \widehat{J}_{\mathbf{p}, \mu, t} \partial \widehat{\eta}_{\mathbf{k}, \omega, s}^- \partial \widehat{\eta}_{\mathbf{p}+\mathbf{k}, \omega, s}^+}(0, 0) \quad (3.22)$$

$$G_{j; \omega, s}(\mathbf{p}; \mathbf{p} + \mathbf{k}) = \sum_{\mu, t} \mu \frac{\partial^3 \mathcal{W}}{\partial \widehat{J}_{\mathbf{p}, \mu, t} \partial \widehat{\eta}_{\mathbf{k}, \omega, s}^- \partial \widehat{\eta}_{\mathbf{p}+\mathbf{k}, \omega, s}^+}(0, 0) \quad (3.23)$$

$$G_{2; \omega, s} = \frac{\partial^2 \mathcal{W}_{l,N}}{\partial \widehat{\eta}_{\mathbf{k}, \omega, s}^- \partial \widehat{\eta}_{\mathbf{k}, \omega, s}^+}(0, 0) \quad , \quad R_{\omega, s}(\mathbf{p}; \mathbf{p} + \mathbf{k}) = \lim_{\varepsilon \rightarrow 0} \sum_{\mu, t} \frac{\partial^3 \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p}, \mu, t} \partial \widehat{\eta}_{\mathbf{k}, \omega, s}^- \partial \widehat{\eta}_{\mathbf{p}+\mathbf{k}, \omega, s}^+}(0, 0, 0) \quad (3.24)$$

Similarly, if we define  $\rho_{\mathbf{x}, \omega}^{(c)} := \sum_s \psi_{\mathbf{x}, \omega, s}^{[l, N]+} \psi_{\mathbf{x}, \omega, s}^{[l, N]-}$ , so that  $\langle \widehat{\rho}_{\mathbf{p}, \omega}^{(c)} \widehat{\rho}_{-\mathbf{p}, \omega'}^{(c)} \rangle_T = \sum_{s, s'} \frac{\partial^2 \mathcal{W}}{\partial \widehat{J}_{\mathbf{p}, \omega, s} \partial \widehat{J}_{-\mathbf{p}, \omega', s'}}(0, 0)$ , we get from (3.17):

$$\begin{aligned} & [D_\omega(\mathbf{p}) - \nu_4(\mathbf{p})D_{-\omega}(\mathbf{p})] \langle \widehat{\rho}_{\mathbf{p}, \omega}^{(c)} \widehat{\rho}_{-\mathbf{p}, \omega'}^{(c)} \rangle_T - 2\nu_\rho(\mathbf{p})D_{-\omega}(\mathbf{p}) \langle \widehat{\rho}_{\mathbf{p}, -\omega}^{(c)} \widehat{\rho}_{-\mathbf{p}, \omega'}^{(c)} \rangle_T \\ &= -\delta_{\omega, \omega'} \tau_N^- D_{-\omega}(\mathbf{p}) + R_{\omega, \omega'}^{(c)}(\mathbf{p}) \end{aligned} \quad (3.25)$$

with  $R_{\omega, \omega'}^{(c)}(\mathbf{p}) = \lim_{\varepsilon \rightarrow 0} \sum_{s, s'} \frac{\partial^2 \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p}, \omega, s} \partial \widehat{J}_{-\mathbf{p}, \omega', s'}}(0, 0, 0)$ .

Finally, from (3.17) and (3.19) and some algebra we get the following identity, which will be useful in the following:

$$\begin{aligned} \frac{\partial \mathcal{W}_{l,N}}{\partial \widehat{J}_{\mathbf{p}, \mu', s'}}(J, \eta) &= \sum_{\mu, s} \frac{M_{\mu', \mu}^\rho(\mathbf{p}) + s' s M_{\mu', \mu}^\sigma(\mathbf{p})}{2} \left[ -\tau_N^- D_{-\mu}(\mathbf{p}) \widehat{J}_{-\mathbf{p}, \mu, s} + \right. \\ &\quad \left. + B_{\mathbf{p}, \mu, s}(J, \eta) + \lim_{\varepsilon \rightarrow 0} \frac{\partial \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p}, \mu, s}}(0, J, \eta) \right] \quad , \quad \mathbf{p} \neq 0 \end{aligned} \quad (3.26)$$

where, if  $\gamma = \rho, \sigma$ , and setting  $\nu_{4, \rho} = -\nu_{4, \sigma} = \nu_4$ ,

$$\begin{aligned} M_{\mu', \mu}^\gamma(\mathbf{p}) &= \frac{\left[ D_{-\mu}(\mathbf{p}) - \nu_{4, \gamma}(\mathbf{p})D_\mu(\mathbf{p}) \right] \delta_{\mu', \mu} + \left[ 2\nu_\gamma(\mathbf{p})D_\mu(\mathbf{p}) \right] \delta_{\mu', -\mu}}{\left[ D_+(\mathbf{p}) - \nu_{4, \gamma}D_-(\mathbf{p}) \right] \left[ D_-(\mathbf{p}) - \nu_{4, \gamma}(\mathbf{p})D_+(\mathbf{p}) \right] - 4\nu_\gamma^2(\mathbf{p})D_+(\mathbf{p})D_-(\mathbf{p})} \\ &= \frac{u_{\gamma, +}(\mathbf{p})\delta_{\mu', \mu} + w_{\gamma, +}(\mathbf{p})\delta_{\mu', -\mu}}{-iv_{\gamma, +}(\mathbf{p})(p_0 + i\mu v_\gamma(\mathbf{p})cp_1)} + \frac{u_{\gamma, -}(\mathbf{p})\delta_{\mu', \mu} + w_{\gamma, -}(\mathbf{p})\delta_{\mu', -\mu}}{-iv_{\gamma, +}(\mathbf{p})(p_0 - i\mu v_\gamma(\mathbf{p})cp_1)} \end{aligned} \quad (3.27)$$

for

$$\begin{aligned} u_{\gamma,\mu}(\mathbf{p}) &= \frac{1}{2} \left[ \frac{1 - \nu_{4,\gamma}(\mathbf{p})}{v_{\gamma,+}(\mathbf{p})} + \mu \frac{1 + \nu_{4,\gamma}(\mathbf{p})}{v_{\gamma,-}(\mathbf{p})} \right] \\ w_{\gamma,\mu}(\mathbf{p}) &= \nu_{\gamma}(\mathbf{p}) \left[ \frac{1}{v_{\gamma,+}(\mathbf{p})} - \mu \frac{1}{v_{\gamma,-}(\mathbf{p})} \right] \end{aligned} \quad (3.28)$$

$$v_{\gamma,\mu}^2(\mathbf{p}) = \left(1 - \mu\nu_{4,\gamma}(\mathbf{p})\right)^2 - 4\nu_{\gamma}^2(\mathbf{p}) \quad , \quad v_{\gamma}(\mathbf{p}) = v_{\gamma,-}(\mathbf{p})/v_{\gamma,+}(\mathbf{p}) \quad (3.29)$$

### 3.2 Analysis of the correction term (3.14). Integration of the UV scales.

The functional  $\mathcal{H}_{l,N}(\alpha, J, \eta)$  defined in (3.14) has a form close to the functional integral  $\mathcal{W}_{l,N}(J, \eta)$  of (2.2); hence, the integration of the UV scales can be studied by a Renormalization Group analysis very similar to the one in §2. In particular, the external field  $\alpha$  in  $\mathcal{A}_0(\alpha, \psi)$  and  $\mathcal{A}_-(\alpha, \psi)$  plays the same role of the field  $J$  in (2.2); the main difference is due to the presence, in the definition of  $\mathcal{A}_0$ , of the singular function (3.5), whose peculiar properties will play a crucial role.

After the integration of the fields  $\psi^{(N)}, \dots, \psi^{(K+1)}$ , where  $K$  is the fixed integer defined in §2.1, we get

$$e^{\mathcal{H}_{l,N}(\alpha, J, \eta)} = \int P_Z^{[l,K]}(d\psi) e^{\mathcal{V}^{(K)}(\psi, J, \eta) + \mathcal{A}^{(K)}(\alpha, J, \eta, \psi)} \quad (3.30)$$

where  $\mathcal{V}^{(K)}(\psi, J, \eta)$  is defined as in (B.1) and  $\mathcal{A}^{(K)}(\alpha, J, \eta, \psi)$  satisfies the identity

$$e^{\mathcal{A}^{(K)}(\alpha, J, \eta, \psi)} = e^{\frac{1}{L^2} \sum_{\mathbf{p} \in \mathcal{D}_L} \widehat{\alpha}_{\mathbf{p}, \omega, s} \tau_N^- D_{-\omega}(\mathbf{p}) \widehat{J}_{-\mathbf{p}, \omega, s}} \int P_Z^{[K+1, N]}(d\zeta) e^{\mathcal{V}(\psi + \zeta, J, \eta) + \mathcal{A}_0(\psi + \zeta, \alpha) - \mathcal{A}_-(\psi + \zeta, \alpha)} \quad (3.31)$$

As in §2.2, we shall consider, for simplicity, only the contributions to  $\mathcal{A}^{(K)}(\alpha, J, \eta, \psi)$  with  $\eta = 0$ ; the result can be easily extended to the general case. Hence, since we have to evaluate only the terms linear in  $\alpha$ , we shall study in detail only the terms linear in  $\alpha$  of  $\mathcal{A}^{(K)}(\alpha, J, 0, \psi)$ , whose kernels are given, for  $\Omega$  a generic multi-index, such as  $(\underline{\omega}', \underline{\sigma}'; \underline{\omega}, \underline{\sigma})$ , by

$$H_{\omega'', s''; \Omega}^{(n+1; 2m)^{(K)}}(\mathbf{z}; \underline{\mathbf{w}}; \underline{\mathbf{x}}, \underline{\mathbf{y}}) := \prod_{i=1}^n \frac{\partial}{\partial J_{\mathbf{w}_i, \omega'_i, s'_i}} \times \prod_{i=1}^m \frac{\partial}{\partial \psi_{\mathbf{x}_i, \omega_i, s_i}^+} \frac{\partial}{\partial \psi_{\mathbf{y}_i, \omega_i, t_i}^-} \frac{\partial \mathcal{A}^{(K)}}{\partial \alpha_{\mathbf{z}, \omega'', s''}}(0, 0, 0, 0) . \quad (3.32)$$

By using the definitions of  $\mathcal{A}_0$  and  $\mathcal{A}_-$ , we can get an explicit formula for the Fourier transforms of the kernels  $H_{\omega'', s''; \Omega}^{(n+1; 2m)^{(h)}}$ . In fact, (3.31) implies that

$$\begin{aligned} \frac{\partial \mathcal{A}^{(K)}}{\partial \widehat{\alpha}_{\mathbf{p}, \omega, s}}(0, J, 0, \psi) &= \tau_N^- D_{-\omega}(\mathbf{p}) \widehat{J}_{-\mathbf{p}, \omega, s} + \\ + e^{-\mathcal{V}^{(K)}(\psi, J, 0)} &\left[ \frac{1}{L^2} \sum_{\mathbf{q} \in \mathcal{D}'_L} C_{\omega}(\mathbf{q} + \mathbf{p}, \mathbf{q}) \frac{\partial^2 e^{\mathcal{V}^{(K)}}}{\partial \widehat{\eta}_{\mathbf{q}, \omega, s}^+ \partial \widehat{\eta}_{\mathbf{p} + \mathbf{q}, \omega, s}^-}(\psi, J, 0) - \right. \\ &\left. - \sum_{\omega'', s''} D_{-\omega}(\mathbf{p}) \nu_{s s''}^{\omega \omega''}(\mathbf{p}) \frac{\partial e^{\mathcal{V}^{(K)}}}{\partial \widehat{J}_{\mathbf{p}, -\omega'', s''}}(\psi, J, 0) \right] \end{aligned} \quad (3.33)$$

In a similar way, see (B.3) in App. B, one can prove that:

$$\frac{\partial e^{\mathcal{V}^{(k)}}}{\partial \eta_{\mathbf{x}, \omega, s}^\varepsilon}(\psi, J, \eta) = \psi_{\mathbf{x}, \omega, s}^{-\varepsilon} e^{\mathcal{V}^{(k)}(\psi, J, \eta)} + \varepsilon \int d\mathbf{u} g_{\omega}^{[k+1, N]}(\mathbf{x} - \mathbf{u}) \frac{\partial e^{\mathcal{V}^{(k)}}}{\partial \psi_{\mathbf{u}, \omega, s}^\varepsilon}(\psi, J, \eta) \quad (3.34)$$

By using this identity, we can expand the r.h.s. of (3.33) and we get, by some simple algebra, that

$$\begin{aligned} \frac{\partial \mathcal{A}^{(K)}}{\partial \widehat{\alpha}_{\mathbf{p},\omega,s}}(0, J, 0, \psi) &= \tau_N^- D_{-\omega}(\mathbf{p}) \widehat{J}_{-\mathbf{p},\omega,s} + \frac{1}{L^2} \sum_{\mathbf{q} \in \mathcal{D}'_L} C_\omega(\mathbf{q} + \mathbf{p}, \mathbf{q}) \widehat{\psi}_{\mathbf{q}+\mathbf{p},\omega,s}^+ \widehat{\psi}_{\mathbf{q},\omega,s}^- \\ &+ W_{\mathcal{A},1}(J, \psi) + W_{\mathcal{A},2}(J, \psi) \end{aligned}$$

where  $W_{\mathcal{A},1}(J, \psi)$  and  $W_{\mathcal{A},2}(J, \psi)$  are graphically represented in Fig. 10 and Fig. 11, respectively; the filled triangle together with the wiggly line represents  $\partial \mathcal{A}_0 / \partial \widehat{\alpha}_{\mathbf{p},\omega,s}$ , while the wiggly line enclosed between two filled points represents  $\sum_{\omega'', s''} D_{-\omega}(\mathbf{p}) \nu_{ss''}^{\omega\omega''}(\mathbf{p})$ . Note that in the terms contributing to  $W_{\mathcal{A},1}(J, \psi)$  both  $\psi$  fields of the interaction  $\mathcal{A}_0$  or  $V(\psi)$  are contracted; in the case of  $W_{\mathcal{A},2}(J, \psi)$ , only one of the  $\psi$  fields of the interaction  $\mathcal{A}_0$  is contracted.

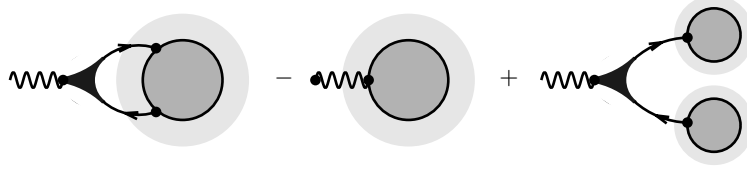


Figure 10: : Graphical representation of  $W_{\mathcal{A},1}(J, \psi)$ . The corresponding kernels  $H^{(n+1;2m)(K)}$  are obtained by substituting the halos with  $2m$  fermion lines and  $n$   $J$  lines. The full triangle represents the  $C_\omega$  operator; the wiggly line represents, in the first and the third graph, the  $\alpha$  external field; the wiggly line between two points represents the Fourier transform of  $D_{-\omega}(\mathbf{p}) \nu_{ss''}^{\omega\omega''}(\mathbf{p})$ .

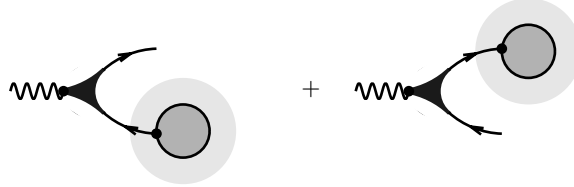


Figure 11: Graphical representation of  $W_{\mathcal{A},2}(J, \psi)$ . The corresponding kernels  $H^{(n+1;2m)(K)}$  are obtained by substituting the halos with  $2m - 1$  fermion lines and  $n$  wiggly lines.

For the analysis of the corresponding kernels, it is convenient to extract from the  $\mathcal{A}_0$  vertex the factors  $D_{\omega'}(\mathbf{p})$  of (3.8) and to separate the first two terms in Fig. 10 from the third one. Hence, we make the following decomposition of the kernels Fourier transforms (without the momentum conservation delta):

$$\begin{aligned} \widehat{H}_{\omega,s;\Omega}^{(n+1;2m)(K)}(\mathbf{p}; \underline{\mathbf{p}}'; \underline{\mathbf{k}}^+, \underline{\mathbf{k}}^-) &= \sum_{\sigma=\pm} D_{\sigma\omega}(\mathbf{p}) \widehat{K}_{\sigma;\omega,s;\Omega}^{(n+1;2m)(K)}(\mathbf{p}; \underline{\mathbf{p}}'; \underline{\mathbf{k}}^+, \underline{\mathbf{k}}^-) + \\ &+ \sum_{\sigma=\pm} D_{\sigma\omega}(\mathbf{p}) \widehat{H}_{\sigma;\omega,s;\Omega}^{(n+1;2m)(K)}(\mathbf{p}; \underline{\mathbf{p}}'; \underline{\mathbf{k}}^+, \underline{\mathbf{k}}^-) + \widehat{H}_{\#;\omega,s;\Omega}^{(n+1;2m)(K)}(\mathbf{p}; \underline{\mathbf{p}}'; \underline{\mathbf{k}}^+, \underline{\mathbf{k}}^-) \end{aligned} \quad (3.35)$$

where  $\widehat{K}_{\sigma;\omega,s;\Omega}^{(n+1;2m)(K)}$  is related to the sum of the first two graphs in Fig. 10,  $\widehat{H}_{\sigma;\omega,s;\Omega}^{(n+1;2m)(K)}$  is related to the third one and  $\widehat{H}_{\#;\omega,s;\Omega}^{(n+1;2m)(K)}$  is related to the sum of the graphs in Fig. 11. Our notation implies that  $\underline{\mathbf{p}}'$  are the momenta of the  $n$   $J$  fields, while  $\underline{\mathbf{k}}^\pm$  are the momenta of the  $m$  external  $\psi^\pm$  fields and  $\mathbf{p} = -\sum_{i=1}^n \mathbf{p}'_i + \sum_{j=1}^m (\mathbf{k}_j^+ - \mathbf{k}_j^-)$ . Note that  $\widehat{H}_{\sigma;\omega,s;\Omega}^{(n+1;0)(K)} = 0$ , for any  $n$ .

Let us consider first the kernels corresponding to the functional  $W_{\mathcal{A},1}(J, \psi)$ . The following lemma shows that, if  $\nu_s^\omega(\mathbf{p})$  is chosen as in (3.15) and  $m \geq 1$ , the kernels  $\widehat{K}_{\sigma;\omega,s;\Omega}^{(n;2m)(K)}$  and  $\widehat{H}_{\sigma;\omega,s;\Omega}^{(n;2m)(K)}$  are *vanishing* in the limit  $N \rightarrow \infty$  and that the same is true for  $\widehat{K}_{\sigma;\omega',s'}^{(2;0)(K)}(\mathbf{p})$ .

**Lemma 3.1** *If  $\nu_s^\omega(\mathbf{p})$  is chosen as in (3.15), there exist two constants  $C_0 > 0$  and  $\vartheta \in (0, 1)$ , such that, if  $\bar{g}$  is small enough,  $n \geq 1$  and  $m \geq 1$ , then, for  $\varepsilon \geq 0$ ,*

$$|\widehat{K}_{\sigma;\omega,s;\Omega}^{(n;2m)(K)}(\mathbf{p}; \underline{\mathbf{p}}'; \underline{\mathbf{k}}^+, \underline{\mathbf{k}}^-)| \leq C_0^{n+m} \bar{g}^{d_{1,n,m}} \gamma^{(2-m-n)K} \gamma^{-\vartheta(N-K)} \quad (3.36)$$

$$|\widehat{H}_{\sigma;\omega,s;\Omega}^{(n;2m)(K)}(\mathbf{p}; \underline{\mathbf{p}}'; \underline{\mathbf{k}}^+, \underline{\mathbf{k}}^-)| \leq C_0^{n+m} \bar{g}^{d_{2,n,m}} \gamma^{(2-m-n)K} \gamma^{-\vartheta(N-K)} \quad (3.37)$$

with  $d_{1,n,m} \geq 1$  and  $d_{2,n,m} \geq 2$ . Moreover,

$$|\widehat{K}_{\sigma;\omega,s;\omega',s'}^{(2;0)(K)}(\mathbf{p})| \leq C_0 \bar{g} \gamma^{-\vartheta(N-K)} \quad (3.38)$$

*Proof.* The bound (3.9) implies that  $|S_{\omega',\omega}^{(N,j)}(\mathbf{z}; \mathbf{x}, \mathbf{y})|$  has the same bound of  $|g_\omega^{(N)}(\mathbf{x} - \mathbf{z})| |g_\omega^{(j)}(\mathbf{y} - \mathbf{z})|$ . Therefore, if we bound  $|\widehat{K}_{\sigma;\omega,s;\Omega}^{(n;2m)(K)}|$  and  $|\widehat{H}_{\sigma;\omega,s;\Omega}^{(n;2m)(K)}|$  with the  $L_1$  norm of  $K_{\sigma;\omega,s;\Omega}^{(n;2m)(K)}$  and  $H_{\sigma;\omega,s;\Omega}^{(n;2m)(K)}$ , respectively, we can proceed as in the proof of Lemma 2.2.

An important role in the proof will have the property (3.7) of  $\widehat{U}_\omega^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q})$ , which implies that in the multiscale expansion of  $K_{\sigma;\omega,s;\Omega}^{(n;2m)(k)}$  and  $H_{\sigma;\omega,s;\Omega}^{(n;2m)(k)}$  at least one of the two  $\psi$ -fields in  $\mathcal{A}_0$  has to be integrated at scale  $N$ . This implies, in particular, that we can bound  $\widehat{H}_{\sigma;\omega,s;\Omega}^{(n;2m)(K)}$  by

$$C_2 \|b_N\|_{L_1} \sum_{j=K+1}^N \|b_j\|_{L_1} \sum_{\substack{n_1+n_2=n-1 \\ m_2+m_2=m+1}} \|W^{(n_1;2m_1)(K)}\| \|W^{(n_2;2m_2)(K)}\|;$$

Hence, if we use (2.28) with  $k = K$ , we easily get (3.37) with  $\vartheta = 1$ .

In order to prove the bounds (3.36) and (3.38), instead, we have to take advantage of partial cancelations. We can write

$$\begin{aligned} K_{\sigma;\omega,s;\Omega}^{(n;2m)(K)}(\mathbf{z}; \underline{\mathbf{x}}; \underline{\mathbf{y}}) &= \sum_{i,j=K}^N \int d\mathbf{u} d\mathbf{w} Z^2 S_{\sigma,\omega}^{(i,j)}(\mathbf{z}; \mathbf{u}, \mathbf{w}) W_{\Omega,\omega,s}^{(n-1;2m+2)(K)}(\underline{\mathbf{x}}; \underline{\mathbf{y}}, \mathbf{u}, \mathbf{w}) \\ &- \delta_{\sigma,-1} \sum_{\omega'',s''} \int d\mathbf{w} \nu_{ss''}^{\omega\omega''}(\mathbf{z} - \mathbf{w}) W_{-\omega'',s'',\Omega}^{(n;2m)(K)}(\mathbf{w}; \underline{\mathbf{x}}, \underline{\mathbf{y}}) + \delta_{\sigma,-1} \delta_{m,0} \delta_{n,2} \delta_{(\omega,s),\Omega} \frac{\tau_N^-}{Z^2} \end{aligned} \quad (3.39)$$

where  $\nu_{ss''}^{\omega\omega''}(\mathbf{z})$  is the Fourier transform of  $\nu_{ss''}^{\omega\omega''}(\mathbf{p})$ .

*a) Bound for  $K_{-\omega,s;\Omega}^{(n;2m)(K)}$  for  $n, m \geq 1$ .* If we take in (2.33)  $2m + 1$  derivatives with respect to the  $\psi$  field and  $n - 1$  derivatives with respect to the  $J$  field, calculated at  $J = \psi = 0$ , we get an expansion for  $W_{\Omega,\omega,s}^{(n;2m+2)(K)}$ , which has the structure of the r.h.s. of Fig. 4, without the first term (since  $m \geq 1$ ). If we now insert this expansion in the first term of the r.h.s. of (3.39), we get an expansion for  $K_{-\omega,s;\Omega}^{(n;2m)(K)}$ , which is graphically represented in Fig.12.

We shall bound the terms corresponding to the graphs in the r.h.s. of Fig. 12, by the same procedure used in §2; though this time we also want to find the exponential small factor  $\gamma^{-\vartheta(N-K)}$ . A crucial role plays a cancelation between the terms corresponding to the graphs (a) and (b), whose sum can be represented in the following way:

$$\sum_{\omega''} \int d\mathbf{u} \left[ \frac{Z^2}{\tau_N^-} \sum_{i,j=K+1}^N S_{-\omega,\omega}^{(i,j)}(\mathbf{z}; \mathbf{u}, \mathbf{u}) - \delta(\mathbf{z} - \mathbf{u}) \right] \int d\mathbf{w} \nu_{ss''}^{\omega\omega''}(\mathbf{u} - \mathbf{w}) W_{-\omega'',s'',\Omega}^{(n;2m)(K)}(\mathbf{w}; \mathbf{x}, \mathbf{y}) \quad (3.40)$$

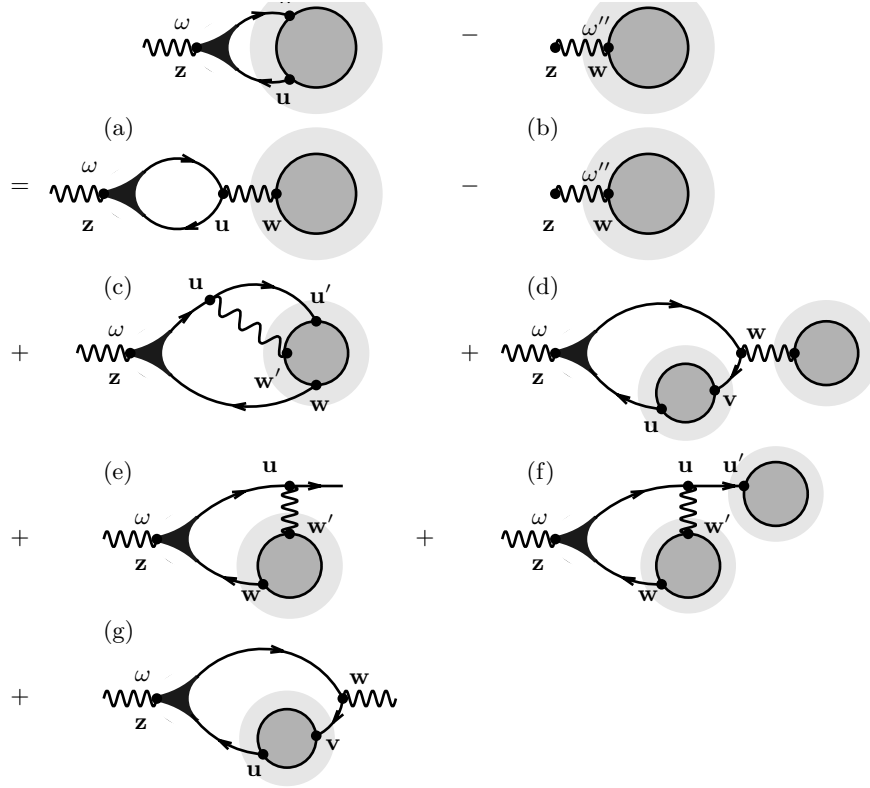


Figure 12: : Graphical representation of  $K_{-;\omega,s;\Omega}^{(n;2m)(k)}$ . The identity is valid if  $m \geq 1$ . The internal wiggly lines represent  $h_{\Theta,\Theta'}^{L,K}$ ; the others represent the  $\alpha$  or  $J$  fields or, if they are closed by two full points, the Fourier transform of  $\nu_{ss''}^{\omega\omega''}(\mathbf{p})$ ; the full triangle represents here the operator  $C_{\omega}(\mathbf{q} + \mathbf{p}, \mathbf{q})D_{-\omega}(\mathbf{p})$ . The graph (g) is present only if  $n \geq 2$  (one wiggly line being already attached to  $\mathbf{w}$ ).

Using the identity (2.42) and the definition (3.11) of  $\tau_N^-$ , we have

$$\begin{aligned}
 & \sum_{\omega''} \sum_{i,j=K+1}^N \int d\mathbf{u} d\mathbf{w} \frac{Z^2}{\tau_N} S_{-\omega,\omega}^{(i,j)}(\mathbf{z}; \mathbf{u}, \mathbf{u}) \nu_{ss''}^{\omega\omega''}(\mathbf{u} - \mathbf{w}) W_{-\omega'',s'';\Omega}^{(n;2m)(K)}(\mathbf{w}; \mathbf{x}, \mathbf{y}) \\
 &= \sum_{\omega''} \int d\mathbf{w} \nu_{ss''}^{\omega\omega''}(\mathbf{z} - \mathbf{w}) W_{-\omega'',s'';\Omega}^{(n;2m)(K)}(\mathbf{w}; \mathbf{x}, \mathbf{y}) + \sum_{p=0,1} \sum_{\omega''} \sum_{i,j=K+1}^N \int d\mathbf{u} S_{-\omega,\omega}^{(i,j)}(\mathbf{z}; \mathbf{u}, \mathbf{u}) \cdot \\
 & \cdot \int_0^1 dt \int d\mathbf{w} (u_p - z_p) (\partial_p \nu_{ss''}^{\omega\omega''})(\mathbf{z} - \mathbf{w} + t(\mathbf{u} - \mathbf{z})) W_{-\omega'',s'';\Omega}^{(n;2m)(K)}(\mathbf{w}; \mathbf{x}, \mathbf{y})
 \end{aligned} \tag{3.41}$$

The first term in the r.h.s. of (3.41), whose dimensional bound is divergent for  $N \rightarrow \infty$ , is indeed finite and is exactly canceled by the second term in (3.40), which then acts as a (finite) counterterm. Note that the interpolation (2.42) was used also to control the graphs (b2) and (b3) of Fig. 8; however, in that case the cancelation was due only to the symmetries of the propagator.

We are then left with the second term in the r.h.s. of (3.41), which has the estimate we wanted; in fact, by using that one between  $i$  and  $j$  is equal to  $N$ , the bound (2.28) on  $\|W^{(n;2m)(K)}\|$ , the

bound (3.9) on  $b_j(\mathbf{x})$  and the bound  $\|\partial h_{L,K}\|_{L_1} \leq c_0 \gamma^K$ , we can bound its norm by

$$C_3 \bar{g} \|W^{(n;2m)(K)}\| \|\partial h_K\|_{L_1} \|\mathbf{x}\|_{L_1} \|b_N\|_{L_1} \sum_{j=K+1}^N \|b_j\|_{L_\infty} \leq C_4 C^{n+m} (C_1 \bar{g})^m \gamma^{-(N-K)} \gamma^{(2-m-n)K} \quad (3.42)$$

Let us now consider the graph (c). Since either  $i$  or  $j$  of the kernel  $S_{-\omega,\omega}^{(i,j)}$  has to be equal to  $N$ , we obtain the bound

$$C_5 \bar{g} \|h_{L,K}\|_{L_\infty} \|W^{(n;2+2m)(K)}\| \sum_{i,j,k=K}^N \sup_{\mathbf{w}, \mathbf{u}'} \int d\mathbf{z} d\mathbf{u} b_i(\mathbf{z} - \mathbf{w}) b_j(\mathbf{z} - \mathbf{u}) |g_\omega^{(k)}(\mathbf{u} - \mathbf{u}')| \quad (3.43)$$

where  $*$  is the constraint that at least one between  $i$  and  $j$  has to be  $N$ . Since the  $L_\infty$  and  $L_1$  norm of  $b_j(\mathbf{x})$  and  $g_\omega^{(j)}(\mathbf{x})$  are dimensionally equivalent, the best bound for the integral in (3.43) (as in the analogous bound of the graph (b1) of Fig.8) is obtained by taking the  $L_\infty$  norm of the function with the smallest scale index and the  $L_1$  norm of the others; it is then easy to see that

$$E_{N,K} := \sum_{i,j,k=K}^N \sup_{\mathbf{w}, \mathbf{u}'} \int d\mathbf{z} d\mathbf{u} b_i(\mathbf{z} - \mathbf{w}) b_j(\mathbf{z} - \mathbf{u}) |g_\omega^{(k)}(\mathbf{u} - \mathbf{u}')| \leq C \gamma^{-N} (N - K) \quad (3.44)$$

By using also the bound (2.28) for  $\|W^{(n;2+2m)(K)}\|$  and the bound  $\|h_{L,K}\|_{L_\infty} \leq c_0 \gamma^{2K}$ , we see that (3.43) can be bounded by

$$C_6 C^{n+m+1} (C_1 \bar{g})^{m+1} \gamma^{-\vartheta(N-K)} \gamma^{(2-m-n)K} \quad (3.45)$$

for any  $0 < \vartheta < 1$  ( $C_6$  is divergent for  $\vartheta \rightarrow 1$ ).

For graph (d) a convenient estimate is given by

$$E_{N,K} \sum_{\substack{n_1+n_2=n, n_1 \geq 1 \\ m_1+m_2=m}} F_{n_1, 2m_1} \|W^{(n_2; 2+2m_2)(K)}\| \quad (3.46)$$

where  $F_{n_1, 2m_1}$  is equal to either  $\|h_K\|_{L_1} \|W^{(n_1; 2m_1)(K)}\|$ , if  $(n_1, 2m_1) \neq (2; 0)$ , or  $\|W_{(b), \Omega}^{(1; 2)(K)}\|$ , if  $(n_1, 2m_1) = (2; 0)$  (see Fig. 8 and bound (2.46)). Therefore, by using (2.28), (2.46) and (3.44), we can bound (3.46) by

$$C_7 C^{n+m+1} (C_1 \bar{g})^{\tilde{d}_{n,m}} \gamma^{-\vartheta(N-K)} \gamma^{(2-m-n)K} \quad , \quad \tilde{d}_{n,m} \geq 2 \quad (3.47)$$

For the graphs (e) and (f) the argument is similar; for (f), for example, the bound is

$$\|h_K\|_{L_\infty} \|b_N\|_{L_1} \sum_{j=K+1}^N \|b_j\|_{L_1} \|g^{[K+1, N]}\|_{L_1} \sum_{\substack{n_1+n_2=n-1 \\ m_1+m_2=m+1, m_i \geq 1}} \|W^{(n_1; 2m_1)(K)}\| \|W^{(n_2; 2m_2)(K)}\| \quad ,$$

which is less than  $C_8 C^{n+m+1} (C_1 \bar{g})^{\tilde{d}_{n,m}} \gamma^{-(N-K)} \gamma^{(2-m-n)K}$ , with  $\tilde{d}_{n,m} \geq 2$ . Finally, the graph (g) has to be considered only in the case  $n \geq 2$ ; it is easy to see that the wanted bound can be obtained with the same procedure used for the graph (d).

b) *Bound for  $K_{+, \Omega}^{(n; 2m)(K)}$  for  $n, m \geq 1$ .* The graph expansion of  $K_{+, \Omega}^{(n; 2m)(K)}$  is given again by Fig.12; the only difference is that the graph (b) is missing (because of  $\delta_{\sigma, -1}$  in (3.39)). Hence a bound can be obtained as before, with only one important difference: the contribution that in the previous analysis was compensated by (b) now is of order  $\gamma^{-N}/L$  by the first bound in (3.12). Hence, the result is the same.

c) *Bound for  $K_{\sigma;\omega,s;\Omega}^{(2;0)(K)}$* . In this case the last term in the r.h.s of (3.39) gives a contribution different from 0, for  $\sigma = -1$ , and the kernel  $W_{\Omega,\omega,s}^{(1;2)(K)}$  contains a term of order 0 is  $\bar{g}$ , corresponding to the kernel  $w_{\Theta,\Theta'}$ , defined in (2.25). Hence, if we expand  $W_{\Omega,\omega,s}^{(1;2)(K)}$ ,  $\Omega = (\omega', s')$ , as in Fig.7, we obtain the graphical representation of Fig.13. In particular, the graph (e) comes from the kernel  $w_{\Theta,\Theta'}$  that is included in the darker bubble of Fig.7; moreover, the graphs (d) and (f) are obtained by also using the identity in Fig.6 to extract a further wiggly line. It is evident that the terms

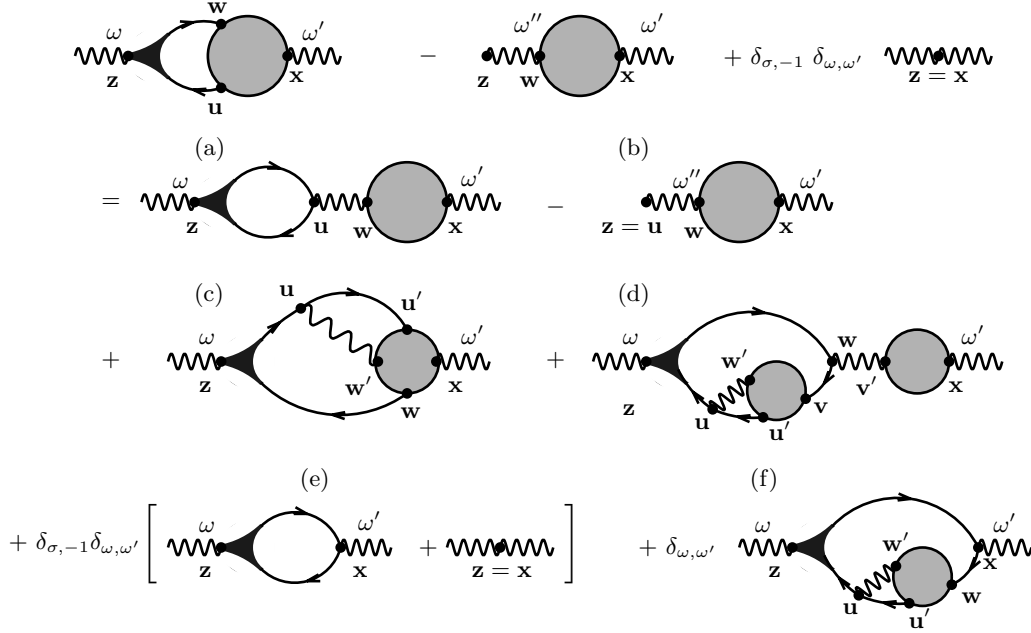


Figure 13: Graphical representation of  $K_{\sigma;\omega,s;\Omega}^{(2;0)(K)}$ . All graphical objects are defined as in the previous pictures.

corresponding to graphs (a), (b), (c), (d) and (f) can be bounded as those related to the graphs (a), (b), (c), (d) and (g) in Fig.(12), respectively; the only difference is in the number and nature of the possible external lines, but that does not change the topology of the graph, and therefore does not change the approach to the bound. The only term that is not bounded by the exponential small factor is that corresponding to graph (e), which is present only if  $\sigma = -1$ ; however it is easy to see that this terms is a constant equal to  $-\tau_N^-$ , hence it is exactly canceled by the constant represented by the graph with a single point between two wiggly lines. Therefore also (3.38) is proved. ■

Let us now analyze the kernels  $\widehat{H}_{\#;\omega,s;\Omega}^{(n+1;2m)(K)}(\mathbf{p}; \mathbf{p}'; \mathbf{k}^+, \mathbf{k}^-)$ , defined in (3.35); recall that they are the kernels of the functional  $W_{\mathcal{A},2}(J, \psi)$ , graphically represented in Fig. 11. Since the two graphs in this picture have the same structure, it is sufficient to analyze the contribution of the first one, which has the form

$$C_{\omega}(\mathbf{p} + \mathbf{q}, \mathbf{q}) \widehat{g}_{\omega}^{[K+1,N]}(\mathbf{q}) \psi^{+(\leq K)}(\mathbf{p} + \mathbf{q}) W_{\omega,s;\Omega}^{(n;2m)(K)}(\mathbf{p}'; (\mathbf{k}'^+, \mathbf{q}), \mathbf{k}^-) \quad (3.48)$$

where  $\mathbf{k}^+ = (\mathbf{k}'^+, \mathbf{p} + \mathbf{q})$ . By using (3.5) and the definitions of §2.1, it is easy to see that, if we define  $u_N(\mathbf{k}) = 1 - \chi_{[-\infty, N]}(|\mathbf{k}|)$  (see also App. C), the expression at left of  $W_{\omega,s;\Omega}^{(n;2m)(K)}$  in (3.48)

can be written as

$$u_N(\tilde{\mathbf{q}})\psi^{+(\leq K)}(\mathbf{p} + \mathbf{q}) - \widehat{g}_\omega^{[K+1, N]}(\mathbf{q}) \left[ \frac{1}{\chi_{l, N}^\varepsilon(|\tilde{\mathbf{p}} + \tilde{\mathbf{q}}|)} - 1 \right] D_\omega(\mathbf{p} + \mathbf{q})\psi^{+(\leq K)}(\mathbf{p} + \mathbf{q}) \quad (3.49)$$

In the following sections, when we shall analyze also the integration of the infrared scales, the following remarks will play an important role.

a) The first term in (3.49) gives no contribution, if  $\mathbf{p}$  is fixed and  $N$  is large enough (how large depending on  $\mathbf{p}$ ). In fact, the support properties of the field  $\psi$  implies that  $|\tilde{\mathbf{p}} + \tilde{\mathbf{q}}| \leq \gamma^{K+1}$ , while  $u_N(\tilde{\mathbf{q}}) = 0$  for  $|\tilde{\mathbf{q}}| \leq \gamma^N$ .

b) The second term can give a non zero contribution only if the field  $\psi^{+(\leq K)}(\mathbf{p} + \mathbf{q})$  is contracted on scale  $l$ , that is if  $|\tilde{\mathbf{p}} + \tilde{\mathbf{q}}| \leq \gamma^{l+1}$ . However, at finite volume,  $|\tilde{\mathbf{p}} + \tilde{\mathbf{q}}| \geq \sqrt{2(\pi/L)^2}$ , so that this condition can not be satisfied if  $\gamma^{l+1} \leq \sqrt{2(\pi/L)^2}$ . It follows that the second term gives no contribution, if  $L$  is fixed and  $l$  is small enough.

### 3.3 Analysis of the correction term (3.14). Integration of the IR scales and removed cutoffs limit.

The integration of the IR scales can be done as in §2.3, with  $K = 0$  for simplicity, by starting the localization procedure at scale 0. Since we need to analyze the correction term only for external momenta of order one, we shall impose the condition that

$$0 < |\tilde{\mathbf{p}}| \leq 1 \quad (3.50)$$

Moreover, we have to take into account that at scale 0 the effective potential can be expanded in terms of trees with one and only one  $\alpha$  endpoint, which can be divided into three classes:

a) Those graphically represented in Fig. 12, that is those containing either one  $\alpha$  endpoint with interaction  $\mathcal{A}_-(\alpha, \psi)$  or one  $\alpha$  endpoint with interaction  $\mathcal{A}_0(\alpha, \psi)$  (see (3.14), whose  $\psi$  fields are both contracted on the UV scales; their kernels, by Lemma 3.1, have the property that their Fourier transforms are bounded by a dimensional factor times a  $\gamma^{-\vartheta N}$  factor.

b) Those corresponding to the first term in (3.49), that is those with only one  $\psi$  field of the  $\alpha$  endpoint with  $\mathcal{A}_0$  interaction contracted on the UV scales and a smooth kernel proportional to  $u_N(\tilde{\mathbf{q}})$ ; these terms vanish for  $N \geq 2$ , under the condition (3.50), so that we can neglect them, since we want to send  $N$  to  $\infty$ .

c) Those with at most one  $\psi$  field of the  $\alpha$  endpoint with  $\mathcal{A}_0$  interaction contracted on the UV scales and singular kernel, that is  $L^{-2} \sum_{\mathbf{q} \in \mathcal{D}_L} C_\omega(\mathbf{q} + \mathbf{p}, \mathbf{q}) \widehat{\psi}_{\mathbf{q} + \mathbf{p}, \omega, s}^+ \widehat{\psi}_{\mathbf{q}, \omega, s}^-$  and those corresponding to the second term in (3.49).

Let us now suppose that the condition  $\varepsilon_h \leq \bar{\varepsilon}$  of Theorem 2.3 is satisfied, so that we can control the tree expansion by localizing, besides the terms involved in the tree expansion of  $W_{l, N}(J, \eta)$ , the only new marginal terms, that is those with one  $\alpha$ -vertex and two external  $\psi$  field. If we do not localize the trees having a subtree at scale 0 of class c) above (whose kernel is singular), the localization procedure gives rise to a local term of the form

$$\sum_{\omega, s} \int d\mathbf{x} \alpha_{\mathbf{x}, \omega, s, s} \frac{Z_j^{(3)}}{Z_{j-1}} \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, s}^-$$

with  $Z_j^{(3)}/Z_{j-1} \leq c_0 \gamma^{-\vartheta N}$ . As concerns the marginal terms whose trees have a subtree at scale 0 of class c), the identity (3.7) implies that one of the  $\psi$  fields of the  $\alpha$  vertex has to be contracted at scale  $l$ , while the other, by momentum conservation, can be contracted only on a finite set of scales around the scale  $j_{\mathbf{p}} = \max\{j : |\tilde{\mathbf{p}}| \leq \gamma^{j_{\mathbf{p}}}\}$ . It follows that this term, because of the compact

support properties of the single scale propagators, can be connected at a larger cluster with only two external  $\psi$  field only a finite number of times; hence, the fact that these terms can not be localized is irrelevant and the bound of the corresponding trees can be done as in the case where all the vertices dimensions are positive, see App. A. Moreover, since the scale  $j > l$  of the tree vertex where one of the  $\psi$  fields of the  $\alpha$  vertex is contracted is essentially fixed to the value  $j_{\mathbf{p}}$ , we can extract from the bound the "short memory" factor  $\gamma^{-\vartheta(j_{\mathbf{p}}-l)}$  ( $0 < \vartheta < 1$ ), which goes to 0 if  $l \rightarrow -\infty$  at fixed  $\mathbf{p}$ .

Let us now consider the kernels  $\widehat{\mathcal{H}}_{\omega,s;\Omega}^{(n+1;2m)}(\mathbf{p}; \underline{\mathbf{p}}'; \mathbf{k}^+, \mathbf{k}^-)$  of the functional  $\mathcal{H}_{l,N}(\alpha, J, \eta)$  defined in (3.14), in the limit  $L \rightarrow \infty$ ; we are using here a notation similar to that used for the kernels (3.35) (in terms of graphs, one has to substitute the  $2m$  external  $\psi$  fields with external propagators linked to  $\eta$  fields). Let us call  $\underline{\mathbf{q}}$  the set  $(\mathbf{p}; \underline{\mathbf{p}}'; \mathbf{k}^+, \mathbf{k}^-)$  of external momenta; the previous considerations imply that, if the momenta  $\underline{\mathbf{q}}$  are non exceptional, see (2.54), and we take the limit  $N \rightarrow \infty$ , the values of all the trees go to zero, except those containing a subtree at scale 0 of class c), which vanish in the limit  $l \rightarrow -\infty$ , as explained above. Hence, we have proved the following Theorem.

**Theorem 3.2** *If  $\bar{\varepsilon}$  is defined as in Theorem 2.3 and  $\varepsilon_h \leq \bar{\varepsilon}$ , uniformly in  $l$  and  $l \leq h \leq 0$ , the kernels  $\widehat{\mathcal{H}}_{\omega,s;\Omega}^{(n+1;2m)}(\mathbf{p}; \underline{\mathbf{p}}'; \mathbf{k}^+, \mathbf{k}^-)$  are well defined for any  $l$  and  $N$  and, if the momenta  $(\mathbf{p}; \underline{\mathbf{p}}'; \mathbf{k}^+, \mathbf{k}^-)$  are non exceptional,*

$$\lim_{l \rightarrow -\infty} \lim_{N \rightarrow \infty} \lim_{L \rightarrow \infty} \widehat{\mathcal{H}}_{\omega,s;\Omega}^{(n+1;2m)}(\mathbf{p}; \underline{\mathbf{p}}'; \mathbf{k}^+, \mathbf{k}^-) = 0 \quad (3.51)$$

so that, in the same limit, the WI (3.16), (3.17), (3.19) and (3.26) are satisfied without the correction term.

An application of this Theorem, which is important in this paper is the following. If we take (3.26) with  $\eta = 0$  and we perform a derivative w.r.t.  $\widehat{J}_{-\mathbf{p},\mu,s}$ , we obtain, in the removed cutoffs limit, a closed expression for the Fourier transform of the density operator correlation  $\rho_{\mathbf{k},\omega,s} = \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^-$ , that is:

$$\langle \widehat{\rho}_{\mathbf{p},\omega',s'} \widehat{\rho}_{-\mathbf{p},\omega,s} \rangle_T = -D_{-\omega}(\mathbf{p}) \frac{\widehat{h}(\mathbf{p}) M_{\omega',\omega}^\rho(\mathbf{p}) + s' s M_{\omega',\omega}^\sigma(\mathbf{p})}{4\pi Z^2 c} \quad , \quad \mathbf{p} \neq 0 \quad (3.52)$$

which implies that

$$\langle \rho_{\mathbf{x},\omega',s'} \rho_{\mathbf{y},\omega,s} \rangle_T = \frac{1}{2} \left[ G_{\omega',\omega}^\rho(\mathbf{x} - \mathbf{y}) + s' s G_{\omega',\omega}^\sigma(\mathbf{x} - \mathbf{y}) \right] \quad (3.53)$$

where

$$G_{\omega',\omega}^\gamma(\mathbf{x} - \mathbf{y}) = \frac{1}{4\pi Z^2 c} \int \frac{d\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \frac{p_0^2 + c^2 p^2}{D_\omega(\mathbf{p})} M_{\omega',\omega}^\gamma(\mathbf{p})$$

### 3.4 Ward Identities in the limit $L, N \rightarrow \infty$ at fixed $l$ .

If we take the limit  $L, N \rightarrow \infty$  at fixed  $l$  and we consider external momenta of order  $\gamma^l$ , (3.51) is of course not true, but we can say that the correlation values only depend on the trees (containing a subtree at scale 0 of class c), which can be easily estimated by trivial dimensional bounds. Note that this result would be true, even if there were a contribution from the other trees; what it is important is only that the effective potential on scale 0 is well defined in the limit  $N \rightarrow \infty$ . In §4.4 we shall use only the following bound. Let us define:

$$\widehat{\mathcal{H}}_{\omega,s}^{(1;2)}(\mathbf{p}; \mathbf{k}^+, \mathbf{k}^-) := \lim_{\varepsilon \rightarrow 0} \frac{\partial^3 \mathcal{H}_{l,N}}{\partial \widehat{\alpha}_{\mathbf{p},\mu,s} \partial \widehat{\eta}_{\mathbf{k}^+,\omega,s}^+ \partial \widehat{\eta}_{\mathbf{k}^-}^-} (0, 0, 0)$$

**Lemma 3.3** *If  $\mathbf{k}^+ = -\mathbf{k}^- = \mathbf{k}$  and  $\mathbf{p} = \mathbf{k}^+ - \mathbf{k}^- = 2\mathbf{k}$ , with  $|\tilde{\mathbf{k}}| = \gamma^l$ , then*

$$|D_\omega(2\mathbf{k})^{-1}\widehat{\mathcal{H}}_{\omega,s}^{(1;2)}(2\mathbf{k}; \mathbf{k}, -\mathbf{k})| \leq C\bar{\varepsilon}^2 \frac{\gamma^{-2l}}{Z_l} \quad (3.54)$$

*Proof.* Note that, if  $|\tilde{\mathbf{k}}^+| = |\tilde{\mathbf{k}}^-| = \gamma^l$ , the term of order 0 in  $\varepsilon$  is given by  $D_\omega(2\mathbf{k})^{-1}\widehat{U}_{l,N,\omega}^{(l,l)}(\mathbf{k}, -\mathbf{k})$ , which vanish, as follows from its explicit value (C.6), since  $u_l(\pm\mathbf{k}) = 0$ . The same is true for the terms of order 1 in  $\bar{\varepsilon}$ , which are obtained by calculating the truncated expectation on scale  $j = l, l+1$  of the product of  $T_\omega(\mathbf{k}^+, \mathbf{k}^-) = C_\omega(\mathbf{k}^+, \mathbf{k}^-)\psi_{\mathbf{k}^+, \omega, s}^+\psi_{\mathbf{k}^-, \omega, s}^-$  and the local effective interaction (2.48). In fact, the two terms proportional to  $\delta_j$  are obtained from the term of order 0 multiplying it by  $\delta_j k \widehat{g}_\omega^{(j)}(\mathbf{k})$ , while the others can be represented in terms of Feynmann graphs, which must contain a tadpole  $\widehat{g}_\omega^{(j)}(0) = 0$ . The factor  $\gamma^{-2l}$  simply reflects the dimension of  $D_\omega(2\mathbf{k})^{-1}\widehat{\mathcal{H}}^{(1;2)}$ , which is the same as the dimension of the correlation  $\langle \widehat{\rho}_{2\mathbf{k}, \omega, s} \widehat{\psi}_{\mathbf{k}, \omega, s}^- \widehat{\psi}_{-\mathbf{k}, \omega, s}^+ \rangle_T$ , see (2.55). On the contrary, the factor  $Z_l^{-1}$ , related to the field strength renormalization, is different from the corresponding factor of  $\langle \widehat{\rho}_{2\mathbf{k}, \omega, s} \widehat{\psi}_{\mathbf{k}, \omega, s}^- \widehat{\psi}_{-\mathbf{k}, \omega, s}^+ \rangle_T$ , that is  $Z_l^{(1)}Z_l^{-2}$ , for the following two reasons:

- a) as discussed before, the potential  $\mathcal{A}_0(\alpha, \psi)$  do not need any renormalization, hence we must put 1 in place of  $Z_l^{(1)}$ ;
- b) because of the bound (3.10), if  $j = l+1$ , or (C.7), if  $j = l$ , there is a  $Z_l^{-1}$  missing in the contractions of the two  $\psi$  fields of  $\mathcal{A}_0(\alpha, \psi)$ . ■

## 4 Closed equations and vanishing of the beta function

We want to prove two relevant properties of our model:

- a) If  $g_{1,\perp} = 0$ , the assumption  $\varepsilon_h \leq \bar{\varepsilon}$ , which allows us to remove the IR cutoff, by Theorem 2.3, is indeed satisfied. In order to prove such highly non trivial property, we first combine the *Schwinger-Dyson Equation* (SDE) with the WI, then we take the limit  $L \rightarrow \infty$ , followed by the limit  $N \rightarrow \infty$ , at fixed infrared cut-off  $\gamma^l$ . We get some relations among the Schwinger functions, which, if computed at momenta of order  $\gamma^l$ , imply that the condition  $\varepsilon_h \leq \bar{\varepsilon}$  is satisfied for  $h = l$ , if this is true for  $h = l+1$ , provide  $\bar{\varepsilon}$  is small enough.
- b) By using the previous results, we can take the limit  $l \rightarrow -\infty$  at fixed non exceptional momenta in the previous relations and we get *exact closed equations* among the correlations; some of them will be used in §5 to calculate the two point function and some response functions needed in the proof of Theorem 1.1.

### 4.1 Combination of Schwinger-Dyson equations and Ward Identities in the $g_{1,\perp} = 0$ case

If we put again  $Z = 1$ , for simplicity, and we define, as in §3.1,  $\tilde{\chi}_N(\mathbf{p}) = \tilde{\chi}_0(2^{-1}\gamma^{-N-1}|\tilde{\mathbf{p}}|)$ , where  $\tilde{\chi}_0(t)$  is a smooth positive function of support in  $[0, 2]$  and equal to 1 for  $t \leq 1$ , the Schwinger-Dyson equations (in the limit  $\varepsilon \rightarrow 0$ , see §2.1) are generated by the identity

$$\frac{\partial e^{\mathcal{W}_{l,N}(0,\eta)}}{\partial \widehat{\eta}_{\mathbf{k}, \omega, s}^+} = \frac{\chi_{l,N}(|\tilde{\mathbf{k}}|)}{D_\omega(\mathbf{k})} \left[ \widehat{\eta}_{\mathbf{k}, \omega, s}^- e^{\mathcal{W}_{l,N}(0,\eta)} - \sum_{\mu', s'} \frac{1}{L^2} \sum_{\mathbf{p}} \frac{\nu_{ss'}^{\omega\mu'}(\mathbf{p})}{\tau_N^-} \tilde{\chi}_N(\mathbf{p}) \frac{\partial^2 e^{\mathcal{W}_{l,N}(J,\eta)}}{\partial \widehat{J}_{\mathbf{p}, -\mu', s'} \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p}, \omega, s}^+} \Big|_{J=0} \right] \quad (4.1)$$

where  $\mathbf{k} \in \mathcal{D}'_L$  (hence  $\mathbf{k} \neq 0$ ),  $\mathbf{p} \in \mathcal{D}_L$  and  $\nu_s^\omega(\mathbf{p})$  is defined as in (3.15).

Note that  $\tilde{\chi}_N(\mathbf{p}) = 1$  in the finite set  $\{\mathbf{p} \in \mathcal{D}_L : |\mathbf{p}| \leq 2\gamma^{N+1}\}$ , so that it has been freely introduced to remind us that  $\mathbf{p} = \mathbf{k}^+ - \mathbf{k}^-$ , where  $\mathbf{k}^\pm$  are the momenta of the  $\widehat{\psi}^\pm$  variables linked to  $\widehat{\mathcal{J}}_{\mathbf{p}}$ , whose modula are both bounded by  $\gamma^{N+1}$ , since  $\varepsilon = 0$ . Hence, (4.1) is a straightforward consequence of the following property: given any  $F(\psi)$  which is a power series in the field, if  $\langle \cdot \rangle_0$  denotes the expectation w.r.t. the free measure,

$$\langle \widehat{\psi}_{\mathbf{k},\omega,s}^- F(\psi) \rangle_0 = \widehat{g}_{\mathbf{D},\omega}^{[l,N]}(\mathbf{k}) \langle \frac{\partial F(\psi)}{\partial \widehat{\psi}_{\mathbf{k},\omega,s}^+} \rangle_0. \quad (4.2)$$

so that

$$\frac{\partial e^{\mathcal{W}_{l,N}}}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+}(0, \eta) = \langle \widehat{\psi}_{\mathbf{k},\omega,s}^- e^{\mathcal{V}(\psi,0,\eta)} \rangle_0 = \widehat{g}_{\mathbf{D},\omega}^{[l,N]}(\mathbf{k}) \langle \frac{\partial}{\partial \widehat{\psi}_{\mathbf{k},\omega,s}^+} e^{\mathcal{V}(\psi,0,\eta)} \rangle_0 \quad (4.3)$$

We now combine the SDE (4.1) with the functional WI (3.26), in order to get relations among the correlations, in two different ways.

**a)** The first family of relations involves the non connected correlations; some of these relations will be used in §5 to calculate the two point function and some response functions in the removed cutoffs limit.

First of all, we decompose the sum over  $\mathbf{p}$  in two parts, by using the identity

$$\tilde{\chi}_N(\mathbf{p}) = \tilde{\chi}_l(\mathbf{p}) + \tilde{\chi}_{l,N}(\mathbf{p}) \quad , \quad \tilde{\chi}_{l,N}(\mathbf{p}) := \tilde{\chi}_N(\mathbf{p})[1 - \tilde{\chi}_l(\mathbf{p})] \quad (4.4)$$

where  $\tilde{\chi}_l(\mathbf{p}) := \tilde{\chi}_0[2^{-1}\gamma^{-l}|\tilde{\mathbf{p}}|]$ ; by this decomposition we divide in two parts the second term in the r.h.s. of (4.1). Then we multiply both sides of (3.26) by  $\exp(\mathcal{W}_{l,N})$ , we take one derivative w.r.t.  $\widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+$  and we insert the result in the part of the second term of (4.1) containing  $\tilde{\chi}_{l,N}(\mathbf{p})$ . By using the oddness of the function  $\widehat{F}_{\omega,s}^\mu(\mathbf{p})$  (defined below) and the fact that the sum over  $\mathbf{p}$  is a finite sum, we see that the term that we get, if we apply the derivative w.r.t.  $\widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+$  to the variable  $\widehat{\eta}_{\mathbf{k}+\mathbf{p},\mu,s}^+$  in the decomposition (3.3) of  $B_{\mathbf{p},\mu,s}(J, \eta)$ , vanishes. Hence, we get, if  $\mathbf{k} \neq 0$ :

$$D_\omega(\mathbf{k}) \frac{\partial e^{\mathcal{W}_{l,N}(0,\eta)}}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+} = \chi_{l,N}(|\mathbf{k}|) \left\{ \widehat{\eta}_{\mathbf{k},\omega,s}^- e^{\mathcal{W}_{l,N}(0,\eta)} - \frac{1}{\tau_N^-} \sum_{\mu,t} \frac{1}{L^4} \sum_{\mathbf{p},\mathbf{q}} \tilde{\chi}_{l,N}(\mathbf{p}) \widehat{F}_{-\omega,st}^{-\omega\mu}(\mathbf{p}) \cdot \right. \\ \left. \cdot \left[ \widehat{\eta}_{\mathbf{q}+\mathbf{p},\mu,t}^+ \frac{\partial^2 e^{\mathcal{W}_{l,N}(0,\eta)}}{\partial \widehat{\eta}_{\mathbf{q},\mu,t}^+ \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+} - \frac{\partial^2 e^{\mathcal{W}_{l,N}(0,\eta)}}{\partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+ \partial \widehat{\eta}_{\mathbf{q}+\mathbf{p},\mu,t}^-} \widehat{\eta}_{\mathbf{q},\mu,t}^- \right] + R_{\mathbf{k},\omega,s}(\eta) + \widetilde{R}_{\mathbf{k},\omega,s}(\eta) \right\} \quad (4.5)$$

where, if  $M_{\omega,\omega'}^\gamma$  is defined as in (3.27) and  $\mathcal{H}_{l,N}$  is defined as in (3.14),

$$\widehat{F}_{\omega,s}^\mu(\mathbf{p}) = \sum_{\omega',s'} \nu_{ss'}^{\omega\omega'}(\mathbf{p}) M_{\omega',\omega\mu}^{s'}(\mathbf{p}) \quad , \quad M_{\mu,\mu'}^s = \frac{1}{2}(M_{\mu,\mu'}^\rho + sM_{\mu,\mu'}^\sigma)$$

$$R_{\mathbf{k},\omega,s}(\eta) = \frac{1}{\tau_N^-} \sum_{\omega',s'} \frac{1}{L^2} \sum_{\mathbf{p}} \tilde{\chi}_{l,N}(\mathbf{p}) \widehat{F}_{-\omega,ss'}^{-\omega\omega'}(\mathbf{p}) \lim_{\varepsilon \rightarrow 0} \frac{\partial^2 e^{\mathcal{H}_{l,N}}}{\partial \widehat{\alpha}_{\mathbf{p},\omega',s'} \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+}(0,0,\eta) \quad (4.6)$$

$$\widetilde{R}_{\mathbf{k},\omega,s}(\eta) = -\frac{1}{\tau_N^-} \frac{1}{L^2} \sum_{\mu',s'} \sum_{\mathbf{p}} \tilde{\chi}_l(\mathbf{p}) \nu_{ss'}^{\omega\omega'}(\mathbf{p}) \left. \frac{\partial^2 e^{\mathcal{W}_{l,N}(J,\eta)}}{\partial \widehat{\mathcal{J}}_{\mathbf{p},-\mu',s'} \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+} \right|_{J=0} \quad (4.7)$$

We will show that

**Theorem 4.1** *The correlations generated by the functionals  $R_{\mathbf{k},\omega,s}(\eta)$  and  $\widetilde{R}_{\mathbf{k},\omega,s}(\eta)$  vanish in the removed cutoffs limit (that is the limit  $L \rightarrow \infty$ , followed by the limit  $N \rightarrow \infty$  and, finally, by the limit  $l \rightarrow -\infty$ ), if the external momenta are non exceptional (see (2.54)) and the condition  $\varepsilon_h \leq \bar{\varepsilon}$*

is satisfied for any  $h$ . Hence, under these conditions, we get from (4.5) and (3.12) a family of exact closed equations for the non connected correlations, generated by the functional identity

$$D_\omega(\mathbf{k}) \frac{\partial e^{\mathcal{W}_{l,N}(0,\eta)}}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+} = \widehat{\eta}_{\mathbf{k},\omega,s}^- e^{\mathcal{W}_{l,N}(0,\eta)} - 4\pi c \sum_{\mu,t} \int \frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^6} \widehat{F}_{-\omega,st}^{-\omega\mu}(\mathbf{p}) \cdot \left[ \widehat{\eta}_{\mathbf{q}+\mathbf{p},\mu,t}^+ \frac{\partial^2 e^{\mathcal{W}_{l,N}(0,\eta)}}{\partial \widehat{\eta}_{\mathbf{q},\mu,t}^+ \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+} - \frac{\partial^2 e^{\mathcal{W}_{l,N}(0,\eta)}}{\partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+ \partial \widehat{\eta}_{\mathbf{q}+\mathbf{p},\mu,t}^-} \widehat{\eta}_{\mathbf{q},\mu,t}^- \right] \quad (4.8)$$

b) The second family of relations, which is equivalent to the first one in the removed cutoffs limit, involves truncated correlations. We shall use one of them to prove the vanishing of the beta function.

First of all, we choose  $\mathbf{k}$  so that  $\chi_{l,N}(|\tilde{\mathbf{k}}|) = 1$  and we write (4.1) in the equivalent form

$$D_\omega(\mathbf{k}) \frac{\partial \mathcal{W}_{l,N}(0,\eta)}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+} = \widehat{\eta}_{\mathbf{k},\omega,s}^- - \sum_{\mu',s'} \frac{1}{L^2} \sum_{\mathbf{p}} \frac{\nu_{ss'}^{\omega\mu'}(\mathbf{p})}{\tau_N^-} \tilde{\chi}_N(\mathbf{p}) \frac{\partial \mathcal{W}_{l,N}(J,\eta)}{\partial \widehat{J}_{\mathbf{p},-\mu',s'}} \Big|_{J=0} \frac{\partial \mathcal{W}_{l,N}(J,\eta)}{\partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+} - \sum_{\mu',s'} \frac{1}{L^2} \sum_{\mathbf{p}} \frac{\nu_{ss'}^{\omega\mu'}(\mathbf{p})}{\tau_N^-} \tilde{\chi}_N(\mathbf{p}) \frac{\partial^2 \mathcal{W}_{l,N}(J,\eta)}{\partial \widehat{J}_{\mathbf{p},-\mu',s'} \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+} \Big|_{J=0} \quad (4.9)$$

Then we make in the last term of (4.9) the decomposition (4.4), we take on both sides of (3.26) one derivative w.r.t.  $\widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+$  and we insert the result in the term of (4.9) containing  $\tilde{\chi}_{l,N}(\mathbf{p})$ . We get

$$D_\omega(\mathbf{k}) \frac{\partial \mathcal{W}_{l,N}(0,\eta)}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+} = \widehat{\eta}_{\mathbf{k},\omega,s}^- - \sum_{\mu',s'} \frac{1}{L^2} \sum_{\mathbf{p}} \frac{\nu_{ss'}^{\omega\mu'}(\mathbf{p})}{\tau_N^-} \tilde{\chi}_N(\mathbf{p}) \frac{\partial \mathcal{W}_{l,N}(J,\eta)}{\partial \widehat{J}_{\mathbf{p},-\mu',s'}} \Big|_{J=0} \frac{\partial \mathcal{W}_{l,N}(J,\eta)}{\partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+} - \frac{1}{\tau_N^-} \sum_{\mu,t} \frac{1}{L^4} \sum_{\mathbf{p},\mathbf{q}} \tilde{\chi}_{l,N}(\mathbf{p}) \widehat{F}_{-\omega,st}^{-\omega\mu}(\mathbf{p}) \left[ \widehat{\eta}_{\mathbf{q}+\mathbf{p},\mu,t}^+ \frac{\partial^2 \mathcal{W}_{l,N}(0,\eta)}{\partial \widehat{\eta}_{\mathbf{q},\mu,t}^+ \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+} - \frac{\partial^2 \mathcal{W}_{l,N}(0,\eta)}{\partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+ \partial \widehat{\eta}_{\mathbf{q}+\mathbf{p},\mu,t}^-} \widehat{\eta}_{\mathbf{q},\mu,t}^- \right] + R'_{\mathbf{k},\omega,s}(\eta) + \widetilde{R}'_{\mathbf{k},\omega,s}(\eta) \quad (4.10)$$

where the correction terms  $R'_{\mathbf{k},\omega,s}(\eta)$  and  $\widetilde{R}'_{\mathbf{k},\omega,s}(\eta)$  are defined as (4.6) and (4.7), with  $\mathcal{H}_{l,N}$  and  $\mathcal{W}_{l,N}$  in place of the corresponding exponentials.

Since the truncated correlations can be written in terms of the untruncated ones, it is a priori true that, if the external momenta are non exceptional, the correlations generated by  $R'_{\mathbf{k},\omega,s}(\eta)$  and  $\widetilde{R}'_{\mathbf{k},\omega,s}(\eta)$  go to 0 in the removed cutoffs limit; hence, the relations (4.10) could be used to get directly exact closed equations even for the truncated correlations. However, in the following we are only interested to consider the limit  $L, N \rightarrow \infty$  at fixed  $l$ , for external momenta of order  $\gamma^l$ ; in such case the correction terms do not vanish. In particular, in §4.4 we shall use only the bound of the kernels we get if we take in (4.5) three derivatives w.r.t. the  $\eta$  variables at  $\eta = 0$ . Hence we define

$$\mathcal{R}'_{\omega,s,\omega',s'}(3)(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) := \frac{\partial^3}{\partial \eta_{\mathbf{k}_1,\omega,s}^- \partial \eta_{\mathbf{k}_2,\omega,s}^+ \partial \eta_{\mathbf{k}_3,\omega',s'}^-} R_{\mathbf{k}_4,\omega',s'}(0) \quad (4.11)$$

In a similar way we define  $\widetilde{\mathcal{R}}'_{\omega,s,\omega',s'}(3)(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ . We will show that

**Lemma 4.2** *Given a momentum  $\mathbf{k}$ , such that  $|\tilde{\mathbf{k}}| = \gamma^l$ ,*

$$|D_{\omega'}(\mathbf{k})|^{-1} \widetilde{\mathcal{R}}'_{\omega,s,\omega',s'}(3)(\mathbf{k}, -\mathbf{k}, \mathbf{k}, -\mathbf{k}) \leq C \bar{\varepsilon}^2 \gamma^{-4l} \frac{Z_l^{(1)}}{Z_l^3} \quad (4.12)$$

$$|D_{\omega'}(\mathbf{k})|^{-1} \mathcal{R}'_{\omega,s,\omega',s'}(3)(\mathbf{k}, -\mathbf{k}, \mathbf{k}, -\mathbf{k}) \leq C \bar{\varepsilon}^2 \gamma^{-4l} \frac{1}{Z_l^2} \quad (4.13)$$

## 4.2 Proof of Theorem 4.1

To prove the vanishing of the (untruncated) correlations generated by  $\tilde{R}_{\mathbf{k},\omega,s}(\eta)$  in the removed cutoffs limit is an easy task. In fact these correlations can be expressed as sums of products of truncated correlations independent of  $\mathbf{k}$  times the correlations generated by

$$-\frac{1}{\tau_N^-} \frac{1}{L^2} \sum_{\mu',s'} \sum_{\mathbf{p}} \tilde{\chi}_l(\mathbf{p}) \nu_{ss'}^{\omega\mu'}(\mathbf{p}) \left[ \frac{\partial^2 \mathcal{W}_{l,N}(J,\eta)}{\partial \hat{J}_{\mathbf{p},-\mu',s'} \partial \hat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+} \Big|_{J=0} + \frac{\partial \mathcal{W}_{l,N}(J,\eta)}{\partial \hat{J}_{\mathbf{p},-\mu',s'}} \Big|_{J=0} \frac{\partial \mathcal{W}_{l,N}(J,\eta)}{\partial \hat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+} \right]$$

All the correlations generated by the second term vanish for  $l \rightarrow -\infty$ , by momentum conservation; in fact  $\mathbf{p}$ , by the condition on the external momenta, has a fixed non zero value, so that  $\tilde{\chi}_l(\mathbf{p}) = 0$  for  $|l|$  large enough. On the other hand, under the hypotheses of the theorem, the kernels of the functional  $W_{l,N}(J,\eta)$  with one external  $J$  field (of momentum  $\mathbf{p}$ ) and  $2m$  external  $\eta$  fields have a bound divergent but integrable for  $\mathbf{p} \rightarrow 0$ . This claim, which is easy to understand on the base of rough dimensional arguments, can be made rigorous by using the (model independent) properties of the tree expansion used to prove Theorem 2.3. A detailed discussion in the case  $m = 1$  can be found in §2.4 of [9] for the Thirring model.

To get the same result for the *correction term* (4.6), we shall proceed as in §3.1, by introducing the following functional integral:

$$e^{\mathcal{T}_{\mu,t,l,N}(\beta,\eta)} = \int P_Z^{[l,N]}(d\psi) e^{\mathcal{V}(\psi,0,\eta) + \mathcal{B}_{\mu,t,0}(\psi,\beta) - \mathcal{B}_{\mu,t,-}(\psi,\beta)} \quad (4.14)$$

where, if  $\{\hat{\beta}_{\mathbf{k},\omega,s}, \mathbf{k} \in \mathcal{D}'_L\}$  are external Grassmann variables,

$$\mathcal{B}_{\mu,t,0}(\psi,\beta) = \sum_{\omega,s} \frac{1}{L^6} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \hat{G}_{-\omega,t}^{-\mu}(\mathbf{p}) C_{\mu\omega}(\mathbf{q} + \mathbf{p}, \mathbf{q}) \hat{\beta}_{\mathbf{k},\omega,s} \hat{\psi}_{\mathbf{k}+\mathbf{p},\omega,s}^- \hat{\psi}_{\mathbf{q}+\mathbf{p},\mu\omega,ts}^+ \hat{\psi}_{\mathbf{q},\mu\omega,ts}^- \quad (4.15)$$

$$\mathcal{B}_{\mu,t,-}(\psi,\beta) = \sum_{\substack{\omega,s \\ \omega',s'}} \sum_{\omega,s} \frac{1}{L^6} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \hat{G}_{-\omega,t}^{-\mu}(\mathbf{p}) D_{-\mu\omega}(\mathbf{p}) \nu_{tss'}^{\mu\omega\omega'}(\mathbf{p}) \hat{\beta}_{\mathbf{k},\omega,s} \hat{\psi}_{\mathbf{k}+\mathbf{p},\omega,s}^- \hat{\psi}_{\mathbf{q}+\mathbf{p},\omega',s'}^+ \hat{\psi}_{\mathbf{q},\omega',s'}^- \quad (4.16)$$

where

$$\hat{G}_{-\omega,t}^{-\mu}(\mathbf{p}) := \frac{\tilde{\chi}_{l,N}(\mathbf{p}) \hat{F}_{-\omega,t}^{-\mu}(\mathbf{p})}{\tau_N^-}$$

Then we have:

$$\frac{\partial e^{\mathcal{T}_{\mu,t,l,N}}}{\partial \hat{\beta}_{\mathbf{k},\omega,s}}(0,\eta) = \frac{1}{L^2} \sum_{\mathbf{p}} \hat{G}_{-\omega,t}^{-\mu}(\mathbf{p}) \frac{\partial^2 e^{\mathcal{H}_{l,N}}}{\partial \hat{\alpha}_{\mathbf{p},\mu\omega,ts} \partial \hat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+}(0,\eta)$$

The multiscale analysis of the r.h.s. of this identity, performed in the following subsections will allow us to complete the proof of the theorem.

### 4.2.1 Analysis of the correction term (4.6). Integration of the UV scales.

Let us put  $K = 0$ ; after the integration of the fields  $\psi^{(N)}, \dots, \psi^{(K+1)}$ , we get

$$e^{\mathcal{T}_{\mu,t,l,N}(\beta,\eta)} = \int P_Z^{[l,0]}(d\psi) e^{\mathcal{V}^{(0)}(\psi,0,\eta) + \mathcal{B}_{\mu,t}^{(0)}(\beta,\eta,\psi)} \quad (4.17)$$

and the functional  $\mathcal{B}_{\mu,t}^{(0)}(\beta,\eta,\psi)$  satisfies the identity:

$$e^{\mathcal{B}_{\mu,t}^{(0)}(\beta,\eta,\psi)} = \int P_Z^{[1,N]}(d\zeta) e^{\mathcal{V}(\psi+\zeta,0,\eta) + \mathcal{B}_{\mu,t,0}(\psi+\zeta,\beta) - \mathcal{B}_{\mu,t,-}(\psi+\zeta,\beta)} \quad (4.18)$$

As in §3.2 and §2.2, we shall consider, for simplicity, only the contributions to  $\mathcal{B}_{\mu,t}^{(0)}(\beta, \eta, \psi)$  with  $\eta = 0$ ; the result can be easily extended to the general case. Hence, since we have to evaluate only the terms linear in  $\beta$ , we shall study in detail only the terms linear in  $\beta$  of  $\mathcal{B}_{\mu,t}^{(0)}(\beta, 0, \psi)$ , whose kernels are given, for  $\Omega = (\underline{\omega}', \underline{\sigma}')$  a generic multi-index, by (one has to take an odd number of derivatives w.r.t. the  $\psi$  variables to get a non-zero result):

$$T_{\mu,t;\omega,s;\Omega}^{(2m+1)(0)}(\mathbf{v}; \mathbf{u}; \underline{\mathbf{x}}, \underline{\mathbf{y}}) := \prod_{i=1}^m \frac{\partial}{\partial \psi_{\mathbf{x}_i, \omega'_i, s'_i}^+} \frac{\partial}{\partial \psi_{\mathbf{y}_i, \omega'_i, s'_i}^-} \times \frac{\partial^2 \mathcal{B}_{\mu,t}^{(0)}}{\partial \beta_{\mathbf{u}, \omega, s} \partial \psi_{\mathbf{v}, \omega, s}^-} (0, 0, 0). \quad (4.19)$$

A more explicit formula for the kernels  $T_{\mu,t;\Omega}^{(2m+1)(0)}$  is obtained by using in (4.19) the identity:

$$\begin{aligned} \frac{\partial \mathcal{B}_{\mu,t}^{(0)}}{\partial \beta_{\mathbf{k}, \omega, s}}(0, 0, \psi) &= \frac{1}{L^4} \sum_{\mathbf{p}, \mathbf{q}} \widehat{G}_{-\omega, t}^{-\mu}(\mathbf{p}) C_{\mu\omega}(\mathbf{q} + \mathbf{p}, \mathbf{q}) e^{-\mathcal{V}^{(0)}(\psi, 0, 0)} \frac{\partial^3 e^{\mathcal{V}^{(0)}(\psi, 0, \eta)}}{\partial \widehat{\eta}_{\mathbf{q}, \mu\omega, ts}^+ \partial \widehat{\eta}_{\mathbf{q}+\mathbf{p}, \mu\omega, ts}^- \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p}, \omega, s}^+} \Big|_{\eta=0} \\ &+ \sum_{\omega', s'} \int \frac{d\mathbf{p}}{(2\pi)^2} \widehat{G}_{-\omega, t}^{-\mu}(\mathbf{p}) D_{-\mu\omega}(\mathbf{p}) \nu_{ts s'}^{\mu\omega\omega'}(\mathbf{p}) e^{-\mathcal{V}^{(0)}(\psi, 0, 0)} \frac{\partial^2 e^{\mathcal{V}^{(0)}(\psi, 0, \eta)}}{\partial \widehat{J}_{\mathbf{p}, \omega', s'} \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p}, \omega, s}^+} \Big|_{\eta=0} \end{aligned} \quad (4.20)$$

If we now use again (3.34) to expand the derivatives w.r.t. the  $\eta$  variables in the first line of (4.20), we get, by some simple algebra, that

$$\frac{\partial \mathcal{B}_{\mu,t}^{(0)}}{\partial \beta_{\mathbf{k}, \omega, s}}(0, 0, \psi) = W_{\mathcal{T}, 1}(\psi) + W_{\mathcal{T}, \#}(\psi) \quad (4.21)$$

where  $W_{\mathcal{T}, 1}(\psi)$ , which is graphically represented in Fig. 14, includes all terms such that both  $\widehat{\psi}$  variables of the interaction  $\mathcal{A}_0$  (that is the variables  $\widehat{\psi}_{\mathbf{q}+\mathbf{p}, \mu\omega, ts}^+$  and  $\widehat{\psi}_{\mathbf{q}, \mu\omega, ts}^-$  of (4.15)) are contracted, while  $W_{\mathcal{T}, \#}(\psi)$  includes the other terms. We shall call  $\widehat{T}_{1, \mu, t; \omega, s; \Omega}^{(2m+1)(0)}(\mathbf{k}; \mathbf{k}^-; \underline{\mathbf{q}}^+, \underline{\mathbf{q}}^-)$  and  $\widehat{T}_{\#, \mu, t; \omega, s; \Omega}^{(2m+1)(0)}(\mathbf{k}; \mathbf{k}^-; \underline{\mathbf{q}}^+, \underline{\mathbf{q}}^-)$  the corresponding kernels

**Lemma 4.3** *There exist two constants  $C_1 > 0$  and  $\vartheta \in (0, 1)$ , such that, if  $\bar{g}$  small enough, for  $\varepsilon = \pm 1$  and  $t = \pm 1$ ,*

$$|\widehat{T}_{1, \mu, t; \omega, s; \Omega}^{(2m+1)(0)}(\mathbf{k}; \mathbf{k}^-; \underline{\mathbf{q}}^+, \underline{\mathbf{q}}^-)| \leq C_0^m \bar{g}^{-2} \gamma^{-\vartheta N} \quad (4.22)$$

**Proof.** Let us put  $\widehat{H}_\sigma(\mathbf{p}) = \widehat{G}_{-\omega, t}^{-\mu}(\mathbf{p}) D_{-\sigma\mu\omega}(\mathbf{p})$  and let us call  $H_\sigma(\mathbf{x})$  its Fourier transform. It is easy to see that:

$$|\widehat{H}_\sigma(\mathbf{p})| \leq c_1 \bar{g} \quad , \quad \|H_\sigma\|_{L_\infty} \leq c_0 \bar{g} \quad (4.23)$$

However,  $H_\sigma(\mathbf{x})$  is not  $L_1$  uniformly in  $L$ ; the point is that the function  $\widehat{H}_\sigma(\mathbf{p})$ , in the limit  $L \rightarrow \infty$ , converges to a function which is bounded, but has a discontinuity in  $\mathbf{p} = 0$ . In any case, the second bound in (4.23) is sufficient to prove the bound (4.22) for the terms described, in Fig. 14, by graphs where the wiggling line belongs to a loop, since we can proceed in this case as in the proof of Lemma 3.1, by using the  $L_1$  norm of  $T_{1, \mu, t; \omega, s; \Omega}^{(2m+1)(0)}$  as an upper bound of  $|\widehat{T}_{1, \mu, t; \omega, s; \Omega}^{(2m+1)(0)}|$ .

Let us consider first the sum  $[\widehat{T}_{1, \mu, t; \omega, s; \Omega}^{(a)(2m+1)(0)} + \widehat{T}_{1, \mu, t; \omega, s; \Omega}^{(b)(2m+1)(0)}](\mathbf{k}; \mathbf{k}^-; \underline{\mathbf{q}}^+, \underline{\mathbf{q}}^-)$  of the kernels associated to the graphs (a) and (b) in Fig. 14; by using the identity (3.8) and the definition (3.35) of the kernels  $\widehat{K}_{\sigma; \omega, s; \Omega}^{(1; 2m)(0)}$  (see also (3.39)), we see that it can be written in the form:

$$\frac{1}{L^2} \sum_{\mathbf{p} \neq 0} g^{[1, N]}(\mathbf{k} + \mathbf{p}) \sum_{\sigma = \pm} \widehat{G}_{-\omega, t}^{-\mu}(\mathbf{p}) D_{-\sigma\mu\omega}(\mathbf{p}) \widehat{K}_{\sigma; \omega, s; \Omega}^{(1; 2m+2)(0)}(\mathbf{p}; 0; \underline{\widetilde{\mathbf{q}}}^+, \underline{\widetilde{\mathbf{q}}}^-) \quad (4.24)$$

if we put  $\underline{\mathbf{q}}^+ = (\underline{\widetilde{\mathbf{q}}}^+, \mathbf{k} + \mathbf{p})$  and  $\underline{\mathbf{q}}^- = (\underline{\widetilde{\mathbf{q}}}^-, \mathbf{k}^-)$ . By using the second bound in (4.23) and (3.36), we can bound the  $L_1$  norm of the Fourier transform of this expression by

$$c_0 \bar{g} \|g^{[1, N]}\|_{L_1} \sum_{\sigma} \|K_{\sigma}^{(1; 2m+2)(0)}\| \leq C_1^m \bar{g}^{-2} \gamma^{-\vartheta N}$$

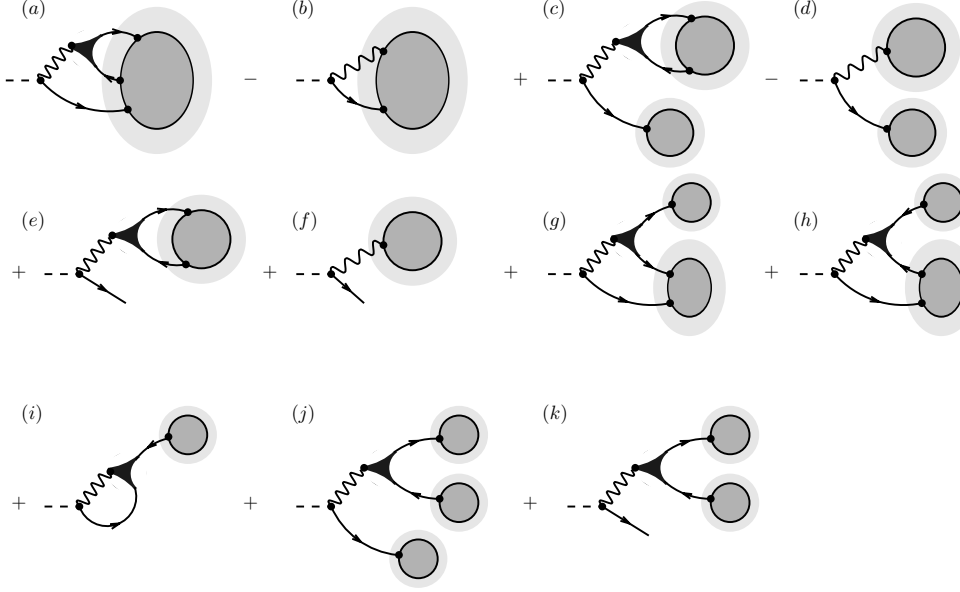


Figure 14: Graphical representation of  $W_{\tau,1}(\psi)$ . The dashed line represents the external Grassmann variable  $\beta$ ; the external full line represents a  $\psi$  variable. The full triangle represents the function  $C_{\mu\omega}(\mathbf{q} + \mathbf{p}, \mathbf{q})$ . The wiggling line represents the function  $\hat{F}_{-\omega,t}^{-\mu}(\mathbf{p})$ , if it is linked to a triangle vertex, otherwise  $\hat{F}_{-\omega,t}^{-\mu}(\mathbf{p})D_{-\mu\omega}(\mathbf{p})$ .

Let us now consider the sum of the kernels associated to the graphs (c) and (d) in Fig. 14. In this case we can not bound the l.h.s. of (4.22) by the  $L_1$  norm of the Fourier transform, but it is easy to see, by using the first bound in (4.23), (3.36) and (2.28), that we can bound it by

$$c_1 \bar{g} \|g^{[1,N]}\|_{L_1} \sum_{\substack{m_1+m_2=m \\ m_2 \geq 1}} \|W^{(0;2m_1+2)(0)}\| \|K^{(1;2m_2)(0)}\| \leq C_1^m \bar{g}^3 \gamma^{-\vartheta N}$$

Notice that the case  $m_2 = 0$  is excluded, because  $\hat{F}_{-\omega,t}^{-\mu}(0) = 0$ , but, in any case, for any fixed  $\mathbf{p} \neq 0$ ,  $K^{(1;0)(0)}(\mathbf{p})$  converges, for  $L \rightarrow \infty$ , to a function which vanishes for  $\mathbf{p} \rightarrow 0$ .

The bound of the sum of the kernels associated to the graphs (e) and (f) (that give a contribution only for  $m \geq 1$ ) is similar; we get

$$c_1 \bar{g} \|K^{(1;2m)(0)}\| \leq C_1^m \bar{g}^2 \gamma^{-\vartheta N}$$

The bound for the kernels associated to the remaining graphs can be done one at a time, since there are no cancellation among them. For (g), (h) and (i) we just use again that the second bound in (4.23), together with (2.28). Let us consider, for example, the graph (g); we get

$$\begin{aligned} c_0 \bar{g} \|g^{[1,N]}\|_{L_1} \|b_N\|_{L_1} \sum_{j=1}^N \|b_j\|_{L_1} \sum_{\substack{m_1+m_2=m \\ m_1 \geq 1}} \|W^{(0;2m_1+2)(0)}\| \|W^{(0;2m_2+2)(0)}\| \\ \leq C_1^m \bar{g}^3 \gamma^{-N} \end{aligned}$$

Finally, for the graphs (j) and (k) we have to use the first bound in (4.23), together to (3.36) and (2.28) and the constraint that one of the  $\psi$  field of  $\mathcal{A}_0$  has to be contracted on scale  $N$ ; we omit the details. ■

**4.2.2 Analysis of the correction term (4.6). Integration of the IR scales and completion of the proof of Theorem 4.1.**

The integration of the IR scales can be done as in §2.3, by starting the localization procedure at scale 0, where the effective potential can be expanded in terms of trees with one and only one  $\beta$ -vertex, which can be divided into three classes:

- a) Those graphically represented in Fig. 14, that is all those containing one  $\beta$  endpoint with interaction  $\mathcal{B}_{\mu,t,-}(\alpha, \psi)$  and those containing one  $\beta$  endpoint with interaction  $\mathcal{B}_{\mu,t,0}(\alpha, \psi)$ , whose  $\psi$  fields of the triangle vertex are both contracted on the UV scales; their kernels, by Lemma 4.3, have the property that their  $L^1$  norms are bounded by a dimensional factor times a  $\gamma^{-\vartheta N}$  factor.
- b) Those graphically represented in Fig. 15 (or with different orientation of the arrows in the lines linked to the triangle vertex), with the further condition the contracted line of the triangle vertex has scale  $N$  and gives a smooth contribution  $\tilde{\chi}_N(\mathbf{p})u_N(\tilde{\mathbf{q}})D_{\omega'}^{-1}$ , corresponding to the first term in (3.49).
- c) The remaining ones, that is the interaction  $\mathcal{B}_{\mu,t,0}(\alpha, \psi)$ , those graphically represented in Fig. 15, with the further condition the contracted line of the triangle vertex has scale  $j < N$  and gives a contribution corresponding to the second term in (3.49), and those with both lines of the triangle vertex not contracted.

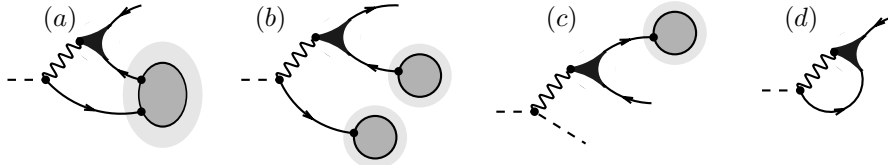


Figure 15: Graphical representation of the contributions to  $W_{\mathcal{T},\#}(\psi)$  with the property that one and only one of the  $\psi$  fields linked to the triangle vertex is contracted. One should add the graphs obtained by exchanging the arrows of the lines linked to the triangle vertex.

Let us now suppose that the condition  $\varepsilon_h \leq \bar{\varepsilon}$  of Theorem 2.3 is satisfied, so that we can control the tree expansion by localizing, besides the terms involved in the tree expansion of  $W_{l,N}(J, \eta)$ , the new marginal terms, that is those with one  $\beta$  external field and two or three external  $\psi$  fields. As we shall see, it is sufficient to localize only the terms with one external  $\psi$  field, by putting (in the limit  $L \rightarrow \infty$ )

$$\mathcal{L} \int d\mathbf{x}d\mathbf{y} \beta_{\mathbf{x},\omega,s} w(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{y},\omega,s} = \hat{w}(0) \int d\mathbf{x} \beta_{\mathbf{x},\omega,s} \psi_{\mathbf{x},\omega,s}$$

Since  $\hat{w}(0) = 0$ , by the oddness of the free propagator, this can be done without introducing any new renormalization constant and we are left with only marginal terms. The consequence is that, given any tree with root of scale  $l - 1$ , all the tree vertices have positive dimension, except those belonging to the path  $\mathcal{C}$  which connects the endpoint of type  $\beta$  with the root; these vertices may have dimension 0. However, if the tree has a subtree with root of scale 0 belonging to the class a) above, its value has a  $\gamma^{\vartheta N}$  factor in front of the dimensional bound. If we write  $\gamma^{\vartheta N} = \gamma^{\vartheta(N+l)/2} \gamma^{\vartheta(N-l)/2}$ , we can use the factor  $\gamma^{\vartheta(N-l)/2}$  to regularize the vertices on  $\mathcal{C}$ ; hence, if we consider the tree expansion of any particular correlation function, we can safely bound the sum over the trees of class a), and we are left with a factor  $\gamma^{\vartheta(N+l)/2}$  in front to it, which goes to 0, if  $N \rightarrow \infty$  at fixed  $l$ . A similar argument can be done for the trees of class b), by using the bound (3.9).

Let us now consider the trees of class c). Because of the identity (3.7), they are characterized by the fact that one of the two lines linked to the triangle vertex in Fig. 15 is contracted at scale  $l$ ,

while the other is contracted at a scale  $j \in [l, N-1]$  in a tree vertex that we call  $v^*$ , belonging to the path  $\mathcal{C}$  which connects the  $\beta$  endpoint  $v_\beta$ , of scale  $j_\beta \geq j+1$ , with the root. Note that  $j_\beta+1$  can be larger than  $j$  only if  $j > l$  and the two  $\psi$  fields linked to the triangle vertex are still not contracted in the effective potential of scale  $h \in [j+1, j_\beta]$ . However, these vertices may have dimension 0 only if the remaining  $\psi$  field of the  $\mathcal{B}_{\mu,t,0}$  interaction is contracted in a propagator which enters a cluster with two external fields; by the support properties of the free single scale propagators, this can happen at most two times; hence these vertices do not need a regularization. As concerns the vertices between  $v^*$  and the root, the regularization is ensured by the bound (3.10), which implies a “gain factor”  $\gamma^{-(j-l)}$  with respect to the dimensional bound; from this factor we can extract a factor  $\gamma^{-(j_\beta-l)}$ , without destroying the summability of the tree expansion of the correlation functions. On the other hand, if the external momenta are non exceptional, the momentum  $\mathbf{k}$  of the  $\beta$  field is different from 0 and this implies that  $j_\beta$  is bounded from below, so that even the contribution of the trees of class c) goes to 0 as  $l \rightarrow -\infty$ . This completes the proof of Theorem 4.1

### 4.3 Proof of Lemma 4.2

First of all, note that the connectivity structure of both  $\tilde{\mathcal{R}}'_{\omega,s,\omega',s'}{}^{(3)}$  and  $\mathcal{R}'_{\omega,s,\omega',s'}{}^{(3)}$  has to be the same as that of graph *a*) of Fig. 14 (with three  $\eta$  fields extracted from the halo and a  $J$ -vertex in place of the  $\alpha$ -vertex, for  $\tilde{\mathcal{R}}'_{\omega,s,\omega',s'}{}^{(3)}$ ), that is any Feymann graph has to be connected without using the  $\beta$  vertex. It immediately follows that the lowest order contributions are of the second order in  $\bar{\varepsilon}$ .

The bound for  $\tilde{\mathcal{R}}'_{\omega,s,\omega',s'}{}^{(3)}(-\mathbf{k}, \mathbf{k}, -\mathbf{k}, \mathbf{k})$  can be proved by the same arguments used in [11] for proving a similar result (see Lemma A1.2 of that paper) in the case of a model with spin 0, local interaction and a fixed UV cutoff. Hence, we shall skip the proof, but only note that the dimensional factors of the bound differ from those of the bound in (2.55) for the four point correlations, which have the same formal scaling dimension, only because

- a) the external propagator in the  $\beta$ -vertex is substituted with  $D_{\omega'}(\mathbf{k})$ , so that we “lose” a  $Z_l^{-1}$  factor;
- b) we have to take into account that the internal  $J$ -vertex is renormalized, so that we “gain” a factor  $Z_l^{(1)}/Z_l$ .

Let us finally consider  $\mathcal{R}'_{\omega,s,\omega',s'}{}^{(3)}(-\mathbf{k}, \mathbf{k}, -\mathbf{k}, \mathbf{k})$ . First of all note that, for a generic choice of the external momenta,  $\mathcal{R}'_{\omega,s,\omega',s'}{}^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$  is obtained from  $\tilde{\mathcal{R}}'_{\omega,s,\omega',s'}{}^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$  by subtracting the Feymann graphs which are only connected through the  $\beta$  vertex. However, these graphs give a vanishing contribution if  $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = (-\mathbf{k}, \mathbf{k}, -\mathbf{k}, \mathbf{k})$ , thanks to the cutoff function  $\tilde{\chi}_{l,N}(\mathbf{p})$  present in the definition (4.6); in fact, for these graphs  $\mathbf{p} = \mathbf{k}_4 - \mathbf{k}_3 = 2\mathbf{k}$  and  $\tilde{\chi}_{l,N}(\mathbf{p}) = 0$ , if  $|\mathbf{p}| \geq 2\gamma^l$ . It follows that the tree expansion of  $\mathcal{R}'_{\omega,s,\omega',s'}{}^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$  is that discussed in the proof of Theorem 4.1; in particular it only depends on the trees containing a subtree at scale 0 of class c), defined in §4.2.2. The contributions of these trees can be easily estimated by trivial dimensional bounds and we get (4.13); the “missing factor”  $Z_l^{(1)}/Z_l$  with respect to the bound (4.12) can be explained as in the proof of Lemma 3.3.

### 4.4 Vanishing of the Beta function

We want now to prove the property a), defined at the beginning of §4, of the model with  $g_{1,\perp} = 0$  (at  $N, L = +\infty$ ) and, at the same time, to analyze the dependence of the r.c.c. on a small perturbation of the “velocity”  $c$ , which appears in the free propagator (2.3). Hence, we put

$$c = c_0 + \delta \quad , \quad c_0 > 0 \quad , \quad |\delta| \leq c_0/2 \quad (4.25)$$

The introduction of the new parameter  $\delta$  is essential to derive from the property a) a property of the beta function, that we call *partial vanishing of the beta functions*, see (2.108), (2.159) and App. C of the companion paper [1]. In its turn, this property is essential to control the IR integration in the Hubbard model.

Lemma 2.2 and Theorem 2.1 imply that, if  $\bar{g} = \max\{|g_{1,\perp}|, |g_{\parallel}|, |g_{\perp}|, |g_4|\}$  is small enough and  $K = 0$ ,

$$e^{\mathcal{W}_{[l,N]}(0,0)} = e^{-L^2 E_1} \int P_Z(d\psi^{[l,1]}) e^{\mathcal{V}^{(1)}(\sqrt{Z}\psi^{[l,1]},0)} \quad (4.26)$$

where  $\mathcal{V}^{(1)}(\psi, 0)$  is given by (2.18) with  $W^{(n;2m)}(1)$  verifying (2.28). Starting from (4.26) we can move from the free integration to the effective interaction the term  $-\delta F_{\delta}(\sqrt{Z}\psi^{[l,1]})$ , with  $F_{\delta}(\psi)$  defined as in (2.48).

The new propagator differs from (2.3) because, in place of  $c$ , we have  $c_0 + \delta u_l(\mathbf{k})$ , with  $u_l(\mathbf{k}) = 1 - \chi^{[l,1]}(|(k_0, ck)|)$ . On the other hand, for  $\delta$  small enough,  $\chi^{[l,1]}(|(k_0, ck)|)$  and  $\chi^{[l,1]}(|(k_0, c_0 k)|)$  differ only for values of  $\mathbf{k}$  of size  $\gamma$  or  $\gamma^l$  and

$$u_l(\mathbf{k}) = 0 \quad , \quad \text{if } \chi^{[l+1,0]}(|(k_0, c_0 k)|) > 0 \quad (4.27)$$

Hence, we can write

$$\chi^{[l,1]}(|(k_0, ck)|) = \bar{\chi}^{(1)}(\mathbf{k}) + \chi^{[l+1,0]}(|(k_0, c_0 k)|) + \bar{\chi}^{(l)}(\mathbf{k}) \quad (4.28)$$

with  $\bar{\chi}^{(1)}(\mathbf{k})$  and  $\bar{\chi}^{(l)}(\mathbf{k})$  smooth functions, whose support is on values of  $\mathbf{k}$  of size  $\gamma$  or  $\gamma^l$ , respectively; moreover, if we define

$$\tilde{C}_l(\mathbf{k}) = \left[ \chi^{[l+1,0]}(|(k_0, c_0 k)|) + \bar{\chi}^{(l)}(\mathbf{k}) \right]^{-1} \quad (4.29)$$

then  $\tilde{C}_l(\mathbf{k}) = 1$ , if  $1 \geq |(k_0, c_0 k)| \geq \gamma^{l+1}$  and  $\tilde{C}_l(\mathbf{k})^{-1} \bar{\chi}^{(l)}(\mathbf{k}) \leq 1$ . It follows that the free measure  $P(d\psi^{[l,1]})$  can be written as  $P(d\bar{\psi}^{(1)})P(d\tilde{\psi}^{[l,0]})$ , where  $\bar{\psi}^{(1)}$  is a field whose covariances has the same scale properties of  $\psi^{(1)}$ , while

$$P(d\tilde{\psi}^{[l,0]}) = \mathcal{N}^{-1} \exp \left\{ -\frac{Z}{L^2} \sum_{\mathbf{k}, \omega, s} \tilde{C}_l(\mathbf{k}) [-ik_0 + \omega c_0 (1 + u_l(\mathbf{k})\delta)k] \tilde{\psi}_{\mathbf{k}, \omega, s}^{+[l,0]} \tilde{\psi}_{\mathbf{k}, \omega, s}^{-[l,0]} \right\} \quad (4.30)$$

The integration of the single scale field  $\bar{\psi}^{(1)}$  can be done without any problem. At this point, we start the multiscale integration of the field  $\tilde{\psi}^{[l,0]}$  as in §2.3 up to scale  $l+1$ , the only difference being that we have  $c_0$  in place of  $c$  in the single scale propagators (2.47), thanks to (4.27). This property is not true only in the last step,  $j = l$ , but this is not a problem, since we have to study the RG flow at fixed  $j$  and  $l \rightarrow -\infty$  and, moreover, the contribution of the IR scale fluctuations to the Schwinger functions at fixed space-time coordinates vanishes as  $l \rightarrow -\infty$ .

We prove the following lemma.

**Lemma 4.4** *Let  $\bar{\varepsilon}$  and  $\varepsilon_h$  be defined as in Theorem 2.3. Then there are constants  $\varepsilon_1$  and  $c_2$ , independent of  $l$ , such that, if  $\varepsilon_0 \leq \varepsilon_1$ , then  $\varepsilon_h \leq c_2 \varepsilon_0 \leq \bar{\varepsilon}$ , for any  $h \in [l+1, 0]$ .*

*Proof.* The proof is by contradiction. Assume that there exists a  $h \leq 0$  such that

$$\varepsilon_{h+1} \leq c_2 \varepsilon_0 < \varepsilon_h \leq 2c_2 \varepsilon_0 \leq \bar{\varepsilon} \quad (4.31)$$

We show that this is not possible, if  $\varepsilon_1, c_2$  are suitably chosen. Let us consider the model (2.2) with  $l = h$ , that is with infrared cutoff  $\gamma^h$ .

Let us consider the first equation in (2.55), which says that, in the limit  $L, N \rightarrow \infty$  at fixed  $l = h$ , if  $|\mathbf{k}| = \gamma^h$ ,  $\hat{S}_{\omega}(\mathbf{k}) = \langle \hat{\psi}_{\mathbf{k}, \omega, s}^{-} \hat{\psi}_{\mathbf{k}, \omega, s}^{+} \rangle = (Z_h D_{\omega}(\mathbf{k}))^{-1} [1 + O(\varepsilon_h)]$ . It is easy to see that,

the only term of order one in  $\varepsilon_h$  contributing to  $\widehat{S}_\omega(\mathbf{k})$  is given by the graph with one insertion of a  $\delta_l$  vertex in the renormalized free propagator. Hence, if we move from the interaction to the renormalized free measure the term  $-\delta_l F_\delta(\sqrt{Z_l}\psi^{(l)})$ , before the integration of the field  $\psi^{(l)}$ , we get

$$\widehat{S}_\omega(\mathbf{k}) \equiv \langle \widehat{\psi}_{\mathbf{k},\omega,s}^- \widehat{\psi}_{\mathbf{k},\omega,s}^+ \rangle_h = \frac{1}{Z_h D_{h,\omega}(\mathbf{k})} [1 + W_2^{(h)}(\mathbf{k})] \quad (4.32)$$

where  $\langle \cdot \rangle_l$  denotes the expectation with propagator (2.3) ( $c = c_0 + \delta$ ) and

$$D_{\omega,h}(\mathbf{k}) = -ik_0 + \omega k(c_0 + \delta_h) \quad , \quad |W_2^{(h)}(\mathbf{k})| \leq C\varepsilon_h^2 \quad (4.33)$$

In the same manner we get, by using the second equation in (2.55), that

$$\langle \widehat{\rho}_{2\mathbf{k},\omega,s} \widehat{\psi}_{\mathbf{k},\omega,s}^- \widehat{\psi}_{-\mathbf{k},\omega,s}^+ \rangle_h = -\frac{Z_h^{(1)}}{Z_h^2 D_{\omega,h}(\mathbf{k})^2} [1 + W_{2,1}^{(l)}(\mathbf{k})] \quad (4.34)$$

where  $|W_{2,1}^{(l)}(\mathbf{k})| \leq C\varepsilon_h^2$ . Moreover, by using the WI (3.16), we can write an equation relating (4.32) with (4.34). Let us put in (3.16)  $\mu = \omega$  and let us take two derivatives with respect to  $\widehat{\eta}_{\mathbf{k},\omega,s}^-$  (the first) and  $\widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+$  at  $J = \eta = 0$ .

$$D_\omega(\mathbf{p}) \langle \widehat{\rho}_{\mathbf{p},\omega,s} \widehat{\psi}_{\mathbf{k}+\mathbf{p},\omega,s}^- \widehat{\psi}_{\mathbf{k},\omega,s}^+ \rangle_h = \widehat{S}_\omega(\mathbf{k}) - \widehat{S}_\omega(\mathbf{k} + \mathbf{p}) + D_{-\omega}(\mathbf{p})R(\mathbf{p}, \mathbf{k}) + \widehat{\mathcal{H}}_{\omega,s}^{(1;2)}(\mathbf{p}; \mathbf{k} + \mathbf{p}, \mathbf{k}) \quad (4.35)$$

where  $D_{-\omega}(\mathbf{p})R(\mathbf{p}, \mathbf{k})$  is the contribution of the second term in the first line of (3.16) and  $\widehat{\mathcal{H}}_{\omega,s}^{(1;2)}(\mathbf{p}; \mathbf{k} + \mathbf{p}, \mathbf{k})$  is defined as in Lemma 3.3. Note that  $R_h(\mathbf{p}, \mathbf{k})$  is of the second order in  $\varepsilon_h$ , since it is a linear combination, with coefficients of order  $\varepsilon_h$  of the correlations  $\langle \widehat{\rho}_{\mathbf{p},\omega,s} \widehat{\psi}_{\mathbf{k}+\mathbf{p},\omega',s'}^- \widehat{\psi}_{\mathbf{k},\omega',s'}^+ \rangle_h$  with  $(\omega', s') \neq (\omega, s)$ , which vanish at order 0. It follows, by using again the second equation in (2.55), that, if  $\mathbf{k} = -\bar{\mathbf{k}}$ , with  $|\mathbf{k}| = \gamma^h$ , and  $\mathbf{p} = 2\bar{\mathbf{k}}$ , then

$$|R(2\bar{\mathbf{k}}, \bar{\mathbf{k}})| \leq C\gamma^{-2h} \frac{Z_h^{(1)}}{Z_h^2} \varepsilon_h^2 \quad (4.36)$$

Noreover, by Lemma 3.3,

$$|D_\omega(2\bar{\mathbf{k}}) \widehat{\mathcal{H}}_{\omega,s}^{(1;2)}(2\bar{\mathbf{k}}; \bar{\mathbf{k}}, -\bar{\mathbf{k}})| \leq C\gamma^{-2h} \frac{1}{Z_h} \varepsilon_h^2 \quad (4.37)$$

If we insert (4.32), (4.34), (4.36), (4.37) in (4.35), we easily get

$$\frac{Z_h^{(1)}}{Z_h} [-ik_0 + \omega k(c_0 + \delta)] [1 + O(\varepsilon_h^2)] = [-ik_0 + \omega k(c_0 + \delta_h)] [1 + O(\varepsilon_h^2)]$$

which implies that

$$\frac{Z_h^{(1)}}{Z_h} = 1 + O(\varepsilon_h^2) \quad , \quad \delta_h = \delta + O(\varepsilon_h^2) \quad (4.38)$$

Let us now consider the equation we get, if we take three derivatives w.r.t. the field  $\eta$  at  $\eta = 0$  on both sides of the WI (4.10), written in the limit  $N, L \rightarrow \infty$ . Let us define

$$\widehat{S}_4(\underline{\mathbf{k}}, \omega, s, \omega', s') := \langle \widehat{\psi}_{\mathbf{k}_1,\omega,s}^+ \widehat{\psi}_{\mathbf{k}_2,\omega,s}^- \widehat{\psi}_{\mathbf{k}_3,\omega',s'}^+ \widehat{\psi}_{\mathbf{k}_4,\omega',s'}^- \rangle_T$$

with  $\underline{\mathbf{k}} = (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ ,  $\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 = 0$ , and

$$\widehat{S}_{1,2}(\mathbf{p}, \mathbf{k}, \omega, s, \omega', s') = \langle \widehat{\rho}_{\mathbf{p},\omega,s} \widehat{\psi}_{\mathbf{k}-\mathbf{p},\omega',s'}^- \widehat{\psi}_{\mathbf{k},\omega',s'}^+ \rangle_T$$

Hence, if we consider (4.10) with  $\mathbf{k} = \mathbf{k}_4$  and  $(\omega', s')$  in place of  $(\omega, s)$  and we take the derivative w.r.t.  $\widehat{\eta}_{\mathbf{k}_3, \omega', s'}$  followed by the derivatives w.r.t.  $\widehat{\eta}_{\mathbf{k}_2, \omega, s}^+$  and  $\widehat{\eta}_{\mathbf{k}_1, \omega, s}$ , we get, by using (3.15), (3.12), (2.6) (with  $K = 0$ ) and the definition (4.11), if we put  $\Omega := (\omega, s, \omega', s')$ ,

$$\begin{aligned}
 D_\omega(\mathbf{k}_4)S_4(\underline{\mathbf{k}}, \Omega) &= - \sum_{\mu, t} \widehat{h}(\mathbf{k}_1 - \mathbf{k}_2) \left[ \delta_{\mu\omega', 1} \delta_{ts', -1} g_\perp + \right. \\
 &+ \delta_{\mu\omega', 1} \delta_{ts', 1} g_\parallel + \delta_{\mu\omega', -1} \delta_{ts', -1} g_4 \left. \right] S_{1,2}(\mathbf{k}_1 - \mathbf{k}_2, \mathbf{k}_2, -\mu, t, \omega, s) S_2(\mathbf{k}_3) - \\
 &- \sqrt{4\pi c} \int \frac{d\mathbf{p}}{(2\pi)^2} \left[ \widehat{F}_{-\omega', ss'}^{-\omega\omega'}(\mathbf{p}) S_4(\mathbf{k}_1, \mathbf{k}_2 - \mathbf{p}, \mathbf{k}_3, \mathbf{k}_4 + \mathbf{p}, \Omega) - \right. \\
 &- \widehat{F}_{-\omega', +1}^{-1}(\mathbf{p}) S_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 + \mathbf{p}, \mathbf{k}_4 + \mathbf{p}, \Omega) - \widehat{F}_{-\omega', ss'}^{-\omega\omega'}(\mathbf{p}) S_4(\mathbf{k}_1 + \mathbf{p}, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 + \mathbf{p}, \Omega) \left. \right] + \\
 &+ R'_\Omega{}^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) + \widetilde{R}'_\Omega{}^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)
 \end{aligned} \tag{4.39}$$

Let us now put  $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = (\mathbf{k}, -\mathbf{k}, \mathbf{k}, -\mathbf{k})$ , with  $|\widetilde{\mathbf{k}}| = \gamma^h$ . Under this condition, the terms in the third and fourth line of (4.39), can be bounded by  $C\varepsilon_h^2 \gamma^{-3h} Z_h^{-2}$ ; this can be proved as in the proof of the analogous bound (4.21) in the paper [11]. A similar bound is valid for  $R'_\Omega{}^{(3)}(\mathbf{k}, -\mathbf{k}, \mathbf{k}, -\mathbf{k})$  and  $\widetilde{R}'_\Omega{}^{(3)}(\mathbf{k}, -\mathbf{k}, \mathbf{k}, -\mathbf{k})$ , by Lemma 4.2. Hence, if we extract in both sides of (4.39) the terms of order one in  $\varepsilon_h$  and use Theorem 2.4 and the first identity in (4.38), we easily get the identity

$$[g_{\alpha, h} + O(\varepsilon_h^2)] \frac{\gamma^{-4h}}{Z_h^2} = g_{\alpha, 0} [1 + O(\varepsilon_h)] \frac{\gamma^{-4h}}{Z_h^2} + O(\varepsilon_h^2) \frac{\gamma^{-4h}}{Z_h^2}, \quad \alpha = \perp, \parallel, 4 \tag{4.40}$$

Note that, if we call  $v_j$  the set of r.c.c. of scale  $j$ , its value is independent of the IR cutoff scale up to  $j = h$ , see the remark after (2.49). Hence, (4.40) is valid also if we interpret  $\varepsilon_h$  as that calculated in the model with IR cutoff  $l < h$ . Hence, the second identity in (4.38) and (4.40) imply that, if the hypothesis (4.31) is satisfied, there exists a constant  $c_3$ , independent of  $c_2$  and  $\varepsilon_1$ , such that

$$\varepsilon_h - \varepsilon_0 \leq c_3 \varepsilon_h^2 \tag{4.41}$$

By the hypothesis (4.31),  $\varepsilon_h \leq 2c_2 \varepsilon_0$ ; hence, the bound (4.41) implies that  $\varepsilon_h \leq \varepsilon_0 (1 + 4c_3 c_2^2 \varepsilon_0)$ . On the other hand, the second member of this inequality can not be larger than  $c_2 \varepsilon_0$ , if, for instance,  $c_2 = 2$  and  $\varepsilon_0 \leq \varepsilon_1 := 1/(16c_3)$ . ■

Lemma 4.4 implies that the r.c.c.'s are well defined and uniformly bounded for any  $h$  and that, as a consequence, also the ren.c.'s, the effective potentials and the correlation functions are well defined. We can then write, for any  $h \leq 0$ , equations of the form

$$v_{h-1} = v_h + \underline{B}^{(h)}(v_h, \dots, v_0, v) \quad , \quad v := (g_\parallel, g_\perp, g_4, \delta) \tag{4.42}$$

$$\frac{Z_{h-1}^{(1)}}{Z_{h-1}} = \frac{Z_h^{(1)}}{Z_h} \left[ 1 + \widetilde{B}^{(h)}(v_h, \dots, v_0, v) \right] \tag{4.43}$$

Let us call  $\underline{b}^{(j)}(v')$  and  $\widetilde{b}^{(j)}(v')$  the functions which are obtained from  $\underline{B}^{(h)}$  and  $\widetilde{B}^{(h)}$ , by subtracting the contribution of the trees containing endpoints of scale greater than 0 and by putting everywhere a fixed value  $v'$ , with  $|v'| \leq \bar{\varepsilon}$  in place of  $v_j$ , for all  $j \leq 0$ , in the remaining trees contribution (which do not depend explicitly on the parameters  $v$  of the interaction). It is easy to see, by using the *short memory property* of the tree expansion and its analyticity in the r.c.c.'s, that  $\underline{b}^{(j)}(v')$  and  $\widetilde{b}^{(j)}(v')$  converge, as  $h \rightarrow -\infty$ , to analytic functions  $\underline{b}^{(-\infty)}(v')$  and  $\widetilde{b}^{(-\infty)}(v')$ , such that  $|\underline{b}^{(j)}(v') - \underline{b}^{(-\infty)}(v')|$  and  $|\widetilde{b}^{(j)}(v') - \widetilde{b}^{(-\infty)}(v')|$  are bounded by  $C|v'|^2 \gamma^{\vartheta j}$ , with  $\vartheta \in (0, 1)$ . On the other hand, it is not

too hard to prove that, as a consequence of the bound (4.41),  $\underline{b}^{(-\infty)}(\underline{v}') = \tilde{\underline{b}}^{(-\infty)}(\underline{v}') = 0$ ; see pag. 156 of [12]. Hence, we get the very important property:

$$\left| \underline{b}^{(j)}(\underline{v}') \right| \leq C |\underline{v}'|^2 \gamma^{\vartheta j} \quad , \quad \left| \tilde{\underline{b}}^{(j)}(g_{\parallel}, g_{\perp}, g_4, \delta) \right| \leq C |\underline{v}'|^2 \gamma^{\vartheta j} \quad (4.44)$$

that we call the *asymptotic vanishing of the beta function*. This property has been used to control the flow of the r.c.c.'s also in the companion paper [1], not only in the case  $g_{1,\perp} = 0$ , but also in the spin-symmetric and repulsive case, that is  $g_{\parallel} = g_{\perp} - g_{1,\perp}$  and  $g_{1,\perp} > 0$ .

As one can easily guess, the bound (4.44) allows us to prove that the r.c.c.'s, the correlation functions at fixed space coordinates and their Fourier transforms at non exceptional external momenta, converge as  $l \rightarrow -\infty$  to functions, which are analytic in  $\underline{v}$  around the origin. A detailed discussion of this point, which only depends on the structure of the tree expansion, can be found, in the case of the Thirring model, in [9].

## 5 Closed equations in the limit of removed cut-offs

In order to prove Theorem 1.1, we need to calculate the explicit expression of the two-points function and of the truncated correlations of some quadratic operators in the space coordinates. We shall do that by taking the inverse Fourier transform of the closed equations obtained in the previous sections for the Fourier transforms of the correlations.

There is in principle a problem in this procedure, because one could be afraid that the removed cutoffs limit does not commute with the Fourier transform because of the presence of delta functions in the calculations. However, we have shown in a paper on the Thirring model (see §A.3 of [13]) that this is not the case and the arguments given there are model independent.

### 5.1 The two-points function

If we perform in (4.8) one derivative w.r.t.  $\widehat{\eta}_{\omega,s}^-$  at  $\eta = 0$  and we put  $\langle \psi_{\mathbf{x},\omega,s}^- \psi_{\mathbf{y},\omega',s'}^+ \rangle = \delta_{\omega,\omega'} \delta_{s,s'} S_{\omega}(\mathbf{x}-\mathbf{y})$ , we get, in the space coordinates:

$$(\partial_{\omega} S_{\omega})(\mathbf{x}) - F_{-\omega,+}^-(\mathbf{x}) S_{\omega}(\mathbf{x}) = \frac{1}{Z} \delta(\mathbf{x}) \quad (5.1)$$

with  $\partial_{\omega} = \partial_{x_0} + i\omega c \partial_{x_1}$ ; note that we have added the dependence on  $Z$ , which was put equal to 1 in (4.8).

The solution of (5.1) is:

$$S_{\omega}(\mathbf{x}) = e^{\Delta_{+}^-(\mathbf{x}|0)} g_{\omega}(\mathbf{x}) \quad , \quad g_{\omega}(\mathbf{x}) = \frac{1}{2\pi Z} \frac{1}{cx_0 + i\omega x} \quad , \quad (5.2)$$

having defined  $\Delta_s^{\varepsilon}$  such that  $\partial_{\omega}^{\mathbf{x}} \Delta_s^{\varepsilon}(\mathbf{x}|\mathbf{z}) = F_{-\omega,s}^{\varepsilon}(\mathbf{x})$ :

$$\Delta_s^{\varepsilon}(\mathbf{x}|\mathbf{z}) = \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{e^{-i\mathbf{k}\mathbf{x}} - e^{-i\mathbf{k}\mathbf{z}}}{D_{\omega}(\mathbf{k})} \widehat{F}_{-\omega,s}^{\varepsilon}(-\mathbf{k}) = \Delta_{\rho}^{\varepsilon}(\mathbf{x}|\mathbf{z}) + s \Delta_{\sigma}^{\varepsilon}(\mathbf{x}|\mathbf{z}) \quad . \quad (5.3)$$

for

$$\begin{aligned} \widehat{\Delta}_{\rho}^{\varepsilon}(\mathbf{p}) &= g_{\rho} \widehat{h}(-\mathbf{p}) \frac{M_{-\omega,-\omega\varepsilon}^{\rho}(-\mathbf{p})}{D_{\omega}(\mathbf{p})} + \frac{g_4}{2} \widehat{h}(-\mathbf{p}) \frac{M_{\omega,-\omega\varepsilon}^{\rho}(-\mathbf{p})}{D_{\omega}(\mathbf{p})} \\ \widehat{\Delta}_{\sigma}^{\varepsilon}(\mathbf{p}) &= g_{\sigma} \widehat{h}(-\mathbf{p}) \frac{M_{-\omega,-\omega\varepsilon}^{\sigma}(-\mathbf{p})}{D_{\omega}(\mathbf{p})} - \frac{g_4}{2} \widehat{h}(-\mathbf{p}) \frac{M_{\omega,-\omega\varepsilon}^{\sigma}(-\mathbf{p})}{D_{\omega}(\mathbf{p})} \end{aligned}$$

In order to evaluate the asymptotic behavior of  $\Delta_s^\varepsilon(\mathbf{x}|0)$ , we need to study functions of the type

$$I_{\omega,\varepsilon}(\mathbf{x}) = \int \frac{d^2\mathbf{p}}{(2\pi)^2} a(\mathbf{p}) \frac{e^{-i\mathbf{p}\cdot\mathbf{x}} - 1}{(p_0 + i\omega cp_1)[v_+(\mathbf{p})p_0 - i\varepsilon\omega v_-(\mathbf{p})cp_1]} \quad (5.4)$$

where  $a(\mathbf{p})$  and  $v_s(\mathbf{p}) > 0$  are even smooth functions of fast decrease. It is easy to show that

$$I_{\omega,\varepsilon}(\mathbf{x}) = \frac{a(0)}{v_+(0)} \tilde{I}_{\omega,\varepsilon}(\mathbf{x}) + A + O(1/|\mathbf{x}|) \quad (5.5)$$

where  $A$  is a real constant and, if  $v = v_-(0)/v_+(0)$ ,

$$\tilde{I}_{\omega,\varepsilon}(\mathbf{x}) = \int_{-1}^{+1} \frac{dp_1}{(2\pi c)} \int_{-\infty}^{+\infty} \frac{dp_0}{(2\pi)} \frac{e^{-i(p_0 x_0 + p_1 x_1/c)} - 1}{(p_0 + i\omega p_1)(p_0 - i\varepsilon\omega v p_1)}$$

One can see that, if  $v > 0$ ,  $v \neq 1$  and  $\mathbf{x} \neq 0$ ,

$$\tilde{I}_{\omega,\varepsilon}(\mathbf{x}) = \frac{1}{2\pi c(1 + \varepsilon v)} [F(x_0, \omega x_1/c) + \varepsilon F(vx_0, -\varepsilon\omega x_1/c)]$$

where

$$F(x_0, x_1) = \int_0^1 \frac{dp_1}{p_1} \left[ e^{-p_1(|x_0| + i\text{sgn}(x_0)x_1)} - 1 \right] = \ln|z| + i\text{Arg}(\text{sgn}(x_0)z) + B + O(1/z)$$

where  $z = x_0 + ix_1$ ,  $B$  is a real constant and  $|\text{Arg}(z)| \leq \pi$ . Since

$$\text{Arg}(\text{sgn}(x_0)z) = \text{Arg}(z) - \vartheta(x_0)\text{sgn}(x_1)\pi$$

the function  $F(\mathbf{x})$  (considered only for  $|\mathbf{x}| > 1$ ) is discontinuous at  $x_0 = 0$ , while  $\tilde{I}_{\omega,\varepsilon}(\mathbf{x})$  is continuous. We can then write

$$\tilde{I}_{\omega,\varepsilon}(\mathbf{x}) = -\frac{1}{2\pi c(1 + \varepsilon v)} [\log(x_0 + i\omega x_1/c) + \varepsilon \log(vx_0 - i\varepsilon\omega x_1/c)] + C + O(1/|\mathbf{x}|) \quad (5.6)$$

where  $C$  is again a real constant. By using (3.27), (5.4), (5.5) and (5.6), one can easily check that

$$\begin{aligned} \Delta_\gamma^\varepsilon(\mathbf{x}|0) &= -\frac{H_{\gamma,\varepsilon}^\varepsilon}{4\pi c} \ln(v_\gamma^2 x_0^2 + (x_1/c)^2) - \frac{H_{\gamma,-}^\varepsilon + H_{\gamma,+}^\varepsilon}{4\pi c} \ln \frac{x_0 + i\omega x_1/c}{v_\gamma x_0 + i\omega x_1/c} \\ &\quad + C_\gamma^\varepsilon + O(1/|\mathbf{x}|) \end{aligned} \quad (5.7)$$

for

$$\begin{aligned} H_{\gamma,\varepsilon}^+ &= \frac{2g_\gamma u_{\gamma,\varepsilon} + g_{4,\gamma} w_{\gamma,\varepsilon}}{v_{\gamma,+} + \varepsilon v_{\gamma,-}} = \frac{\varepsilon g_\gamma}{v_{\gamma,+} + v_{\gamma,-}} \\ H_{\gamma,\varepsilon}^- &= \frac{2g_\gamma w_{\gamma,\varepsilon} + g_{4,\gamma} u_{\gamma,\varepsilon}}{v_{\gamma,+} - \varepsilon v_{\gamma,-}} = -\frac{4\pi\varepsilon}{2\nu_{\gamma,+} + \nu_{\gamma,-}} \left[ 1 - \frac{(v_{\gamma,-} - \varepsilon v_{\gamma,+})^2}{4} \right] \end{aligned}$$

where  $C_\gamma^\pm$  are real constants and  $v_\gamma = v_{\gamma,+}(0)/v_{\gamma,-}(0)$  (and  $g_{4,\rho} = g_4$  while  $g_{4,\sigma} = -g_4$ ).

By using (3.28) and (3.29),

$$\frac{H_{\gamma,+}^+}{4\pi c} = \frac{\nu_\gamma}{v_{\gamma,+} + v_{\gamma,-}} = \frac{\zeta_\gamma}{2} \quad \frac{H_{\gamma,-}^+ + H_{\gamma,+}^+}{4\pi c} = 0 \quad (5.8)$$

$$\frac{H_{\gamma,-}^-}{4\pi c} = \frac{1 - \frac{1}{4}(v_{\gamma,+} + v_{\gamma,-})^2}{2v_{\gamma,+} + v_{\gamma,-}} = \frac{\eta_\gamma}{2} \quad \frac{H_{\gamma,-}^- + H_{\gamma,+}^-}{4\pi c} = -\frac{1}{2} \quad (5.9)$$

Note that this expression is continuous in  $v_\gamma = 1$ , as one expects, and that, at least at small coupling,  $\eta_\gamma \geq 0$ .

By using (5.2) and (5.3), we finally get

$$S_\omega(\mathbf{x}) = \frac{1}{2\pi Z} \frac{(c^2 v_\rho^2 x_0^2 + x_1^2)^{-\eta_\rho/2} (c^2 v_\sigma^2 x_0^2 + x_1^2)^{-\eta_\sigma/2}}{(c v_\rho x_0 + i\omega x_1)^{1/2} (c v_\sigma x_0 + i\omega x_1)^{1/2}} e^{C+O(1/|\mathbf{x}|)} \quad (5.10)$$

where  $C$  is a real constant  $O(g)$  and  $z^{1/2} = |z|^{1/2} e^{i \text{Arg}(z)/2}$ . Note that the leading term is well defined and continuous at any  $\mathbf{x} \neq 0$ .

Note also that, if  $g_\perp = 0$ ,  $v_\rho = v_\sigma = 1$  and  $\eta_\rho = \eta_\sigma \equiv \eta/2$ , so that

$$S_\omega(\mathbf{x}) = \frac{1}{2\pi Z} \frac{(c^2 x_0^2 + x_1^2)^{-\eta/2}}{c x_0 + i\omega x_1} e^{C+O(1/|\mathbf{x}|)} \quad (5.11)$$

If we also put  $g_\perp = 0$  and  $g_\parallel = \lambda$ , we get for  $\eta$  the value found for the regularized Thirring model, that is  $\eta = 2\tau^2/(1 - \tau^2)$ , with  $\tau = \lambda/(4\pi c)$ ; see eq. (4.21) of [14].

## 5.2 The four point functions and the densities correlations

We want to calculate the truncated correlations  $\langle O_{\mathbf{x}}^{(t)} O_{\mathbf{y}}^{(t)} \rangle^T$  of the local quadratic operators  $O_{\mathbf{x}}^{(t)}$ ,  $t = (1, \alpha)$  or  $(2, \alpha)$ , defined as the analogous operators of the Hubbard model in §2.4 of the previous paper [1], that is

$$\begin{aligned} O_{\mathbf{x}}^{(1,C)} &= \sum_{\omega,s} \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^- , & O_{\mathbf{x}}^{(1,S_i)} &= \sum_{\omega,s,s'} \psi_{\mathbf{x},\omega,s}^+ \sigma_{s,s'}^{(i)} \psi_{\mathbf{x},\omega,s'}^- , & O_{\mathbf{x}}^{(1,SC)} &= \sum_{\varepsilon,\omega,s} s e^{2i\varepsilon\omega p_F x} \psi_{\mathbf{x},\omega,s}^\varepsilon \psi_{\mathbf{x},\omega,-s}^\varepsilon \\ O_{\mathbf{x}}^{(2,C)} &= \sum_{\omega,s} e^{2i\omega p_F x} \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},-\omega,s}^- , & O_{\mathbf{x}}^{(2,S_i)} &= \sum_{\omega,s,s'} e^{2i\omega p_F x} \psi_{\mathbf{x},\omega,s}^+ \sigma_{s,s'}^{(i)} \psi_{\mathbf{x},-\omega,s'}^- , \\ O_{\mathbf{x}}^{(2,SC)} &= \sum_{\varepsilon,\omega,s} s \psi_{\mathbf{x},\omega,s}^\varepsilon \psi_{\mathbf{x},-\omega,-s}^\varepsilon , & O_{\mathbf{x}}^{(2,TC_i)} &= \sum_{\varepsilon,\omega,s,s'} e^{-i\varepsilon\omega p_F x} \psi_{\mathbf{x},\omega,s}^\varepsilon \tilde{\sigma}_{s,s'}^{(i)} \psi_{\mathbf{x},-\omega,s'}^\varepsilon \end{aligned}$$

Note that  $p_F$  has no special meaning in the effective model, but it is left there since we want to compare the correlations in the two models, in the proof of Theorem 1.1.

Our UV regularization implies that  $\langle O_{\mathbf{x}}^{(t)} \rangle = 0$  for any  $t$ ; hence we can make the calculation very simply, by using the explicit expressions of the (untruncated) four points functions which follow from the closed equation (4.5) and then evaluating them so that the two coordinates corresponding to each  $O^{(t)}$  operator coincide, if this is meaningful. This works for all values of  $t$ , except  $(1, C)$  and  $(1, S_3)$ , where there is a singularity, related to the fact that the operators  $\rho_{\mathbf{x},\omega,s} = \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^-$  are not well defined in the limit  $N \rightarrow \infty$ , because of the singularity of the free propagator at  $\mathbf{x} = 0$ . However, in these cases we can use directly the WI (3.52) to calculate correctly, in the limit  $N \rightarrow \infty$ , the correlations of  $O_{\mathbf{x}}^{(1,C)}$  and  $O_{\mathbf{x}}^{(1,S_3)}$ , by using (3.53), (3.27) and the equations (5.4), (5.5), (5.6). We get, for  $|\mathbf{x}| > 1$ ,

$$\begin{aligned} G_{\omega,\omega}^\gamma(\mathbf{x}) &\simeq \frac{1 - v_\gamma^2}{8\pi^2 c^2 Z^2} \left[ \frac{u_{\gamma,+}}{v_{\gamma,+} - v_{\gamma,-}} \frac{1}{(v_\gamma x_0 + i\omega x_1/c)^2} - \frac{u_{\gamma,-}}{v_{\gamma,+} + v_{\gamma,-}} \frac{1}{(v_\gamma x_0 - i\omega x_1/c)^2} \right] \\ G_{-\omega,\omega}^\gamma(\mathbf{x}) &\simeq \frac{1 - v_\gamma^2}{8\pi^2 c^2 Z^2} \left[ \frac{w_{\gamma,+}}{v_{\gamma,+} - v_{\gamma,-}} \frac{1}{(v_\gamma x_0 + i\omega x_1/c)^2} - \frac{w_{\gamma,-}}{v_{\gamma,+} + v_{\gamma,-}} \frac{1}{(v_\gamma x_0 - i\omega x_1/c)^2} \right] \end{aligned}$$

the corrections being of order  $1/|\mathbf{x}|^3$ . This implies that, for  $|\mathbf{x}| > 1$ ,

$$\langle O_0^{(1,C)} O_{\mathbf{x}}^{(1,C)} \rangle^T = \frac{v_\rho^2(1 - \nu_4 + 2\nu_\rho) + (1 + \nu_4 - 2\nu_\rho)}{2\pi Z^2 c^2 v_{\rho,+} v_{\rho,-}} \frac{v_\rho^2 x_0^2 - x^2/c^2}{(v_\rho^2 x_0^2 + x^2/c^2)^2} + O(1/|\mathbf{x}|^3) \quad (5.12)$$

while  $\langle O_{\mathbf{0}}^{(1,S_3)} O_{\mathbf{x}}^{(1,S_3)} \rangle^T$  is obtained from this expression, by replacing  $\nu_4$  with  $-\nu_4$  and  $\nu_\rho$  with  $\nu_\sigma$  (hence also  $v_\rho$ ,  $v_{\rho,+}$  and  $v_{\rho,-}$  with  $v_\sigma$ ,  $v_{\sigma,+}$  and  $v_{\sigma,-}$ ).

In order to calculate the other correlations, we first note that the only four points functions different from zero are those defined by the equation

$$G_{s_1, s_2}^{\omega_1, \omega_2}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = \langle \psi_{\mathbf{x}, \omega_1, s_1}^- \psi_{\mathbf{y}, \omega_2, s_2}^- \psi_{\mathbf{u}, \omega_2, s_2}^+ \psi_{\mathbf{v}, \omega_1, s_1}^+ \rangle$$

By (4.5),  $G_{s_1, s_2}^{\omega_1, \omega_2}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v})$  is the solution of the equation:

$$\begin{aligned} (\partial_{\omega_1}^{\mathbf{x}} G_{s_1, s_2}^{\omega_1, \omega_2})(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) &= \delta(\mathbf{x} - \mathbf{v}) S_{\omega_2}(\mathbf{y} - \mathbf{u}) - \delta_{\omega_1, \omega_2} \delta_{s_1, s_2} \delta(\mathbf{x} - \mathbf{u}) S_{\omega_1}(\mathbf{y} - \mathbf{v}) + \\ &\left[ -F_{-\omega_1, s_1 s_2}^{-\omega_1, \omega_2}(\mathbf{x} - \mathbf{y}) + F_{-\omega_1, s_1 s_2}^{-\omega_1, \omega_2}(\mathbf{x} - \mathbf{u}) + F_{-\omega_1, +}^{-\omega_1, \omega_2}(\mathbf{x} - \mathbf{v}) \right] G_{s_1, s_2}^{\omega_1, \omega_2}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) \end{aligned} \quad (5.13)$$

For the two-points correlation of  $O_{\mathbf{x}}^{(2, \alpha)}$  we are interested in the case  $\omega_1 = -\omega_2 = \omega$ . For  $G_s^\omega(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = G_{s', s'}^{\omega, -\omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v})$  we find

$$G_s^\omega(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = e^{-\left[ \Delta_s^+(\mathbf{x} - \mathbf{y} | \mathbf{v} - \mathbf{y}) - \Delta_s^+(\mathbf{x} - \mathbf{u}, \mathbf{v} - \mathbf{u}) \right]} S_\omega(\mathbf{x} - \mathbf{v}) S_{-\omega}(\mathbf{y} - \mathbf{u}). \quad (5.14)$$

Therefore, for  $\alpha = C, S_3$  we set  $\mathbf{x} = \mathbf{u}$ ,  $\mathbf{y} = \mathbf{v}$  and  $s = +$ , while for  $\alpha = S_1, S_2$  we set  $s = -$ ; for  $TC_1, TC_3$  we set  $\mathbf{u} = \mathbf{v}$ ,  $\mathbf{x} = \mathbf{y}$  and  $s = +$ ; while for  $TC_2, SC$  we set  $s = -$ .

For the two-points correlation of  $O_{\mathbf{x}}^{(1, \alpha)}$ ,  $\alpha \neq C, S_3$ , we are interested in the case  $\omega_1 = \omega_2 = \omega$ . If  $\bar{G}_s^\omega(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = G_{s', s'}^{\omega, \omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v})$  we find

$$\begin{aligned} \bar{G}_s^\omega(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) &= e^{-\left[ \Delta_s^-(\mathbf{x} - \mathbf{y} | \mathbf{v} - \mathbf{y}) - \Delta_s^-(\mathbf{x} - \mathbf{u}, \mathbf{v} - \mathbf{u}) \right]} S_\omega(\mathbf{x} - \mathbf{v}) S_\omega(\mathbf{y} - \mathbf{u}) \\ &- \delta_{s, +} e^{-\left[ \Delta_+^-(\mathbf{x} - \mathbf{y} | \mathbf{u} - \mathbf{y}) - \Delta_+^-(\mathbf{x} - \mathbf{v}, \mathbf{u} - \mathbf{v}) \right]} S_\omega(\mathbf{x} - \mathbf{u}) S_\omega(\mathbf{y} - \mathbf{v}). \end{aligned} \quad (5.15)$$

For  $\alpha = SC$  we set  $\mathbf{x} = \mathbf{y}$ ,  $\mathbf{u} = \mathbf{v}$  and  $s = -$ ; for  $\alpha = S_1, S_2$  we set  $\mathbf{x} = \mathbf{u}$ ,  $\mathbf{y} = \mathbf{v}$  and  $s = -$ ; for  $\alpha = S_3, C$  we set  $\mathbf{x} = \mathbf{u}$ ,  $\mathbf{y} = \mathbf{v}$  and  $s = +$ .

Therefore, it is easy to see, by using (5.3) and (5.10), that, for  $|\mathbf{x}| > 1$ ,

$$\begin{aligned} \langle O_{\mathbf{0}}^{(2, \alpha)} O_{\mathbf{x}}^{(2, \alpha)} \rangle^T &= \frac{1}{\pi^2 Z^2 c^2} \frac{\cos(2p_F x)^{m_\alpha}}{(v_\rho^2 x_0^2 + x^2/c^2)^{x_\rho, t}} \frac{1}{(v_\sigma^2 x_0^2 + x^2/c^2)^{x_{\sigma, t}}} + O(1/|\mathbf{x}|^3), \quad \forall \alpha \\ \langle O_{\mathbf{0}}^{(1, SC)} O_{\mathbf{x}}^{(1, SC)} \rangle^T &= -\frac{1}{\pi^2 Z^2 c^2} \frac{\cos(2p_F x)}{(v_\rho^2 x_0^2 + x^2/c^2)^{2\eta_\rho}} \frac{v_\rho^2 x_0^2 - x^2/c^2}{(v_\rho^2 x_0^2 + x^2/c^2)^2} + O(1/|\mathbf{x}|^3) \\ \langle O_{\mathbf{0}}^{(1, \alpha)} O_{\mathbf{x}}^{(1, \alpha)} \rangle^T &= \frac{1}{\pi^2 Z^2 c^2} \frac{1}{(v_\sigma^2 x_0^2 + x^2/c^2)^{2\eta_\sigma}} \frac{v_\sigma^2 x_0^2 - x^2/c^2}{(v_\sigma^2 x_0^2 + x^2/c^2)^2} + O(1/|\mathbf{x}|^3), \quad \alpha = S_1, S_2 \end{aligned} \quad (5.16)$$

where  $m_\alpha = 1$ , if  $\alpha = C, S_i$ , while  $m_\alpha = 0$ , if  $\alpha = SC, TC_i$ , and

$$x_{\gamma, t} = \begin{cases} \eta_\gamma - \zeta_\gamma + 1/2 & t = (2, C), (2, S_3) \\ \eta_\gamma - s(\gamma)\zeta_\gamma + 1/2 & t = (2, S_1), (2, S_2) \\ \eta_\gamma + \zeta_\gamma + 1/2 & t = (2, TC_1), (2, TC_3) \\ \eta_\gamma + s(\gamma)\zeta_\gamma + 1/2 & t = (2, SC), (2, TC_2) \end{cases} \quad (5.17)$$

Let us now consider the special case  $g_\sigma = 0$  (i.e.  $\eta_\sigma = \zeta_\sigma = 0$ ), which we use as a effective model for the Hubbard model. In this case, the equations (5.16) imply that  $\langle O_{\mathbf{0}}^{(t)} O_{\mathbf{x}}^{(t)} \rangle$  decays, for  $|\mathbf{x}| \rightarrow \infty$ , as  $|\mathbf{x}|^{-2X_t}$ , with

$$2X_t = \begin{cases} 2 + 2\eta_\rho - 2\zeta_\rho & t = (2, C), (2, S_i) \\ 2 + 2\eta_\rho + 2\zeta_\rho & t = (2, SC), (2, TC_i) \\ 2 + 4\eta_\rho & t = (1, SC) \\ 2 & t = (1, C), (1, S_i) \end{cases} \quad (5.18)$$

Note that

$$\eta_\rho = -\frac{1}{2} + \frac{4 - v_{\rho+}^2 - v_{\rho-}^2}{4v_{\rho+}v_{\rho-}} \quad , \quad \xi_\rho = \frac{2\nu_\rho}{v_{\rho+}v_{\rho-}} \quad (5.19)$$

Let us now define  $K = 2X_{2,C} - 1$  and  $\tilde{K} = 2X_{2,SC} - 1$ . By using (3.29), we see that

$$K = \frac{(1 - 2\nu_\rho)^2 - \nu_4^2}{v_{\rho+}v_{\rho-}} = \sqrt{\frac{(1 - \nu_4) - 2\nu_\rho}{(1 - \nu_4) + 2\nu_\rho}} \sqrt{\frac{(1 + \nu_4) - 2\nu_\rho}{(1 + \nu_4) + 2\nu_\rho}} \quad (5.20)$$

$$\tilde{K} = \frac{(1 + 2\nu_\rho)^2 - \nu_4^2}{v_{\rho+}v_{\rho-}} = K^{-1}$$

$$4\eta_\rho = K + \tilde{K} - 2$$

These equations imply that all the critical indices  $X_t$  and the parameter  $\eta_\rho$  can be expressed in terms of the single parameter  $K$ , only depending on  $g_2/c$  and  $g_4/c$ . In the following section we will show that the coupling of the model (2.2) can be chosen so that its exponents coincide with the Hubbard ones; then, by some simple algebra, one can check the validity of the scaling relations (1.14).

## 6 Proof of Theorem 1.1

### 6.1 The extended scaling relations: proof of (1.14)

In this section we finally establish contact with the Renormalization Group analysis of the Hubbard model done in the companion paper [1], and we prove Theorem 1.1.

Let us call  $\tilde{v}_h = (\tilde{g}_{2,h}, \tilde{g}_{4,h}, \tilde{\delta}_h)$ ,  $h \leq 0$ , the running coupling constants in the effective model with ultraviolet cutoff  $\gamma^N$  and parameters

$$g_{1,\perp} = 0 \quad , \quad g_{\parallel} = g_{\perp} = \tilde{g}_{2,N} \quad , \quad g_4 = \tilde{g}_{4,N} \quad , \quad \delta = \tilde{\delta}_N \quad (6.1)$$

so that, in particular,  $c = c_0(1 + \tilde{\delta}_N)$ , and put  $\tilde{v}_N = (\tilde{g}_{2,N}, \tilde{g}_{4,N}, \tilde{\delta}_N)$ . We call  $v_h = (g_{2,h}, g_{4,h}, \delta_h)$ ,  $h \leq 0$ , the analogous constants in the Hubbard model, and  $\vec{v}_h = (v_h, g_{1,h}, \nu_h)$ . The analysis of the RG flow given above implies that, for  $h \leq 0$ ,

$$\tilde{v}_{h-1} = \tilde{v}_h + \beta^{(0,h)}(\tilde{v}_h, \dots, \tilde{v}_0) + \tilde{r}^{(h)}(\tilde{v}_h, \dots, \tilde{v}_0, \tilde{v}_N) \quad (6.2)$$

$$v_{h-1} = v_h + \beta^{(0,h)}(v_h, \dots, v_0) + r^{(h)}(\vec{v}_h, \dots, \vec{v}_0, \lambda) \quad (6.3)$$

where  $\beta^{(0,h)}(\tilde{v}_h, \dots, \tilde{v}_0)$  is the beta function of the effective model with parameters (6.1), modified so that the endpoints have scale  $\leq 0$ . Note that  $\beta^{(0,h)}(v_h, \dots, v_0)$  is the function  $\beta^{(h)}(v_h, \dots, v_0)$  modified so that, in its tree expansion, no trees containing endpoints of type  $g_1$  appear and the space integrals are done in terms of continuous variables, instead of lattice variables (the difference is given by exponentially vanishing terms). The crucial bound (4.44) and the short memory property imply that  $|\tilde{r}^{(h)}(\tilde{v}_h, \dots, \tilde{v}_N)| \leq C[\max_{k \geq h} |\tilde{v}_k|]^2 \gamma^{\theta h}$ , and a similar bound is verified from  $r^{(h)}(\vec{v}_h, \dots, \vec{v}_0, \lambda)$ .

**Lemma 6.1** *Given the Hubbard model with coupling  $\lambda$  such that  $g_{1,0} \in D_{\varepsilon,\delta}$ , it is possible to choose  $\tilde{v}_N$  as analytic function of  $\lambda$ , so that*

$$\tilde{g}_2 = 2\lambda \left[ \hat{v}(0) - \frac{1}{2}\hat{v}(2p_F) \right] + O(\lambda^{3/2}) \quad , \quad \tilde{g}_4 = 2\lambda\hat{v}(0) + O(\lambda^2) \quad , \quad \tilde{\delta} = O(\lambda) \quad (6.4)$$

and, if  $\tilde{v}_h$  are the r.c.c. of the effective model with parameters satisfying (6.1), while  $v_h$  are the r.c.c. of the Hubbard model, then,  $\forall h \leq 0$ ,

$$|v_h - \tilde{v}_h| \leq C \frac{|g_{1,0}|}{1 + (a/2)|g_{1,0}||h|} \quad (6.5)$$

Moreover, the r.c.c.  $\tilde{v}_h$  have a well definite limit as  $N \rightarrow +\infty$  and this limit still satisfies (6.5).

**Proof** - We have seen in the previous sections that the flows (6.2) and (6.3) have well defined limits  $\tilde{v}_{-\infty}$  and  $v_{-\infty}$ , as  $h \rightarrow -\infty$ , if the initial values are small enough and  $g_{1,0} \in D_{\varepsilon,\delta}$ . Moreover, the proof of this property for the flow (6.2) implies that  $\tilde{v}_{-\infty}$  is a smooth invertible function of  $\tilde{v}_N$ , such that  $\tilde{v}_{-\infty} = \tilde{v}_N + O(\tilde{v}_N^2)$ ; let us call  $\tilde{v}_N(\tilde{v}_{-\infty}) = \tilde{v}_{-\infty} + O(\tilde{v}_{-\infty}^2)$  its inverse. It is also clear that  $\tilde{v}_N(\tilde{v}_{-\infty})$  has a well defined limit as  $N \rightarrow \infty$ , that we shall call  $\tilde{v}(\lambda)$ , and that this is true also for the r.c.c.  $\tilde{v}_h$ ,  $h \leq 0$ .

The previous remarks imply that it is possible to choose  $\tilde{v}_N$ , satisfying (6.4), so that

$$\tilde{v}_{-\infty} - v_{-\infty} = 0 \quad (6.6)$$

In order to prove (6.5), we note that, because of the bound (4.44) and the short memory property, in the effective model with couplings satisfying (6.4),

$$|\tilde{v}_h - \tilde{v}_{-\infty}| \leq C \lambda^2 \gamma^{\theta h} \quad (6.7)$$

On the other hand

$$|v_h - v_{-\infty}| \leq C \sum_{j=-\infty}^h [|g_{1,j}|^2 + \lambda \gamma^{\frac{\theta}{2}j}] \leq C_1 \frac{|g_{1,0}|}{1 + (a/2)|g_{1,0}||h|} \quad (6.8)$$

These two bounds immediately imply (6.5). ▮

Let us now note that the critical indices of the effective model can be calculated in terms of  $\tilde{v}_{-\infty}$  by the same procedure used for the Hubbard model and that we get an equation *with the same function*  $\beta_t^{(0,j)}$ . Hence, the above lemma allows us to conclude that the critical indices in the Hubbard model and in the effective model coincide, provided that the value of  $\tilde{v} = \lim_{N \rightarrow \infty} \tilde{v}_N$  is chosen properly. It follows that all the indices are given by the equations (5.18), with

$$\begin{aligned} \nu_\rho &= \frac{\tilde{g}_2(\lambda)}{4\pi c} = \lambda \frac{\hat{v}(0) - \hat{v}(2p_F)/2}{2\pi \sin \bar{p}_F} + O(\lambda^{3/2}) \\ \nu_4 &= \frac{\tilde{g}_4(\lambda)}{4\pi c} = \lambda \frac{\hat{v}(0)}{2\pi \sin \bar{p}_F} + O(\lambda^2) \end{aligned} \quad (6.9)$$

where (6.4) has been used, together with  $c = \sin \bar{p}_F + O(\lambda)$ . Moreover, (6.9) and (5.20) imply that  $K = 1 - 2\lambda[\hat{v}(0) - \hat{v}(2p_F)/2]/(\pi \sin \bar{p}_F) + O(\lambda^{3/2})$ , in agreement with (1.13).

## 6.2 Susceptibility and Drude weight: proof of (1.15)

The effective model is not invariant under a gauge transformation with the phase depending both on  $\omega$  and  $s$ , if  $g_{1,\perp} > 0$ ; however, it is still invariant under a gauge transformation with the phase only depending on  $\omega$ . This is true, in particular, if the interaction is spin symmetric, that is if  $g_{\parallel} = g_{\perp} - g_{1,\perp}$ . Since also the Hubbard model is spin symmetric, it is natural to see if one can use this “restricted” gauge invariance to get some useful information on the asymptotic behavior of the Hubbard model, by comparing it with the effective model with  $g_{1,\perp} > 0$ .

Let us put  $g_{\perp} \equiv \bar{g}_2$ ,  $g_{1,\perp} \equiv \bar{g}_1$  and  $g_{\parallel} = \bar{g}_2 - \bar{g}_1$ . We want to show that we can choose the parameters of the effective model  $\bar{g}_1, \bar{g}_2, \bar{g}_4, \bar{\delta}$ , so that the running coupling constants are

asymptotically close to those of the Hubbard model. This result is stronger of the similar one contained in Lemma 6.1, since now all the running couplings are involved, and this implies also that the values of  $\bar{g}_2$ ,  $\bar{g}_4$  and  $\bar{\delta}$  are *different* with respect to the analogous constants defined in Lemma 6.1. The main consequence of these considerations is that we can use the restricted WI of this new effective model to get non trivial information on some Hubbard model correlation functions, not plagued by logarithmic corrections.

Let  $\vec{l}_h = (\bar{g}_{1,h}, \bar{g}_{2,h}, \bar{g}_{4,h}, \bar{\delta}_h)$ ,  $h \leq 0$ , be the running coupling constants appearing in the integration of the infrared part of the effective model. The smoothness properties of the integration procedure imply that, in the UV limit,  $\vec{l}_0$  is a smooth invertible function of the interaction parameters  $\vec{l} = (\bar{g}_1, \bar{g}_2, \bar{g}_4, \bar{\delta})$ , whose inverse we shall call  $\vec{l}(\vec{l}_0)$ ; hence we can fix the effective model by giving the value of  $\vec{l}_0$  and by putting  $\vec{l} = \vec{l}(\vec{l}_0)$ . In a similar way we call  $\vec{g}_h = (g_{1,h}, g_{2,h}, g_{4,h}, \delta_h)$ ,  $h \leq 0$ , the running couplings of the Hubbard model with coupling  $\lambda$ .

We now define  $\vec{x}_h = \vec{l}_h - \vec{g}_h$ ,  $h \leq 0$ . By using the decomposition for  $\vec{g}_h$  and the similar one for  $\vec{l}_h$ , we can write

$$\vec{x}_{h-1} = \vec{x}_h + [\bar{\beta}_h^{(1)}(\vec{g}_h, \dots, \vec{g}_0) - \bar{\beta}_h^{(1)}(\vec{l}_h, \dots, \vec{l}_0)] + \bar{\beta}_h^{(2)}(\vec{g}_h, \nu_h, \dots, \vec{g}_0, \nu_0, \lambda) + \bar{\beta}_h^{(3)}(\vec{l}_h, \dots, \vec{l}_0, \vec{l}) \quad (6.10)$$

In the usual way, one can see that

$$|\bar{\beta}_h^{(1)}(\vec{g}_h, \dots, \vec{g}_0) - \bar{\beta}_h^{(1)}(\vec{l}_h, \dots, \vec{l}_0)| \leq C \left[ \lambda + \sup_{k \geq h} |\vec{l}_k| \right] \sum_{k=h}^0 \gamma^{-\vartheta(k-h)} |\vec{x}_k| \quad (6.11)$$

and that  $|\bar{\beta}_h^{(2)}| \leq C|\lambda|\gamma^{\vartheta h}$ ,  $|\bar{\beta}_h^{(3)}| \leq C[\sup_{k \geq h} |\vec{l}_k|]^2$ . Note that the different power in the coupling of these two bounds is due to the terms linear in  $\lambda$  in the beta function for  $\delta_h$ , which are present in the Hubbard model, while similar terms are absent in the effective model.

We want to show that, given  $\lambda$  positive and small enough, it is possible to choose  $\vec{l}_0$ , hence  $\vec{x}_0$ , so that  $\vec{x}_{-\infty} = 0$ ; we shall do that by a simple fixed point argument. Note that  $\vec{x}_{-\infty} = 0$  if and only if

$$\vec{x}_h = - \sum_{h=-\infty}^{\bar{h}} \{ [\bar{\beta}_h^{(1)}(\vec{g}_h, \dots, \vec{g}_0) - \bar{\beta}_h^{(1)}(\vec{l}_h, \dots, \vec{l}_0)] + \bar{\beta}_h^{(2)} + \bar{\beta}_h^{(3)} \} \quad (6.12)$$

We consider the Banach space  $\mathcal{M}_\vartheta$ ,  $\vartheta < 1$ , of sequences  $\vec{x} = \{\vec{x}_h\}_{h \leq 0}$  with norm  $\|\vec{x}\| = \sup_{k \leq 0} |\vec{x}_k| \gamma^{-(\vartheta/2)k}$  and the operator  $\mathbf{T} : \mathcal{M}_\vartheta \rightarrow \mathcal{M}_\vartheta$ , defined as the r.h.s. of (6.12). Given  $\xi > 0$ , let  $\mathcal{B}_\xi = \{\vec{x} \in \mathcal{M}_\vartheta : \|\vec{x}\| \leq \xi\}$ ; if  $\lambda$  is small enough, say  $\lambda \leq \varepsilon_0$  and  $\xi\lambda \leq \varepsilon_0$ , and  $\vec{l}_h = \vec{g}_h + \vec{x}_h$ , the functions  $\bar{\beta}_h^{(1)}(\vec{l}_h, \dots, \vec{l}_0)$  and  $\bar{\beta}_h^{(3)}(\vec{l}_h, \dots, \vec{l}_0, \vec{l})$  are well defined and satisfy the bounds above, even if  $\vec{x}$  is not the flow of the effective model corresponding to  $\vec{l}_0$ . Hence, we have:

$$\gamma^{-(\vartheta/2)h} |\mathbf{T}(\vec{x})_h| \leq c_0 \lambda (\xi \lambda + 1) \sum_{k=-\infty}^{h-1} \gamma^{\frac{\vartheta}{2}k} \leq c_1 \lambda (1 + \xi \lambda) \quad (6.13)$$

so that  $\mathcal{B}_\xi$  is invariant if  $\xi = 2c_1$  and  $\lambda \leq \varepsilon_1 = \min\{\varepsilon_0, \varepsilon_0/(2c_1), 1/(2c_1)\}$ . Moreover

$$\begin{aligned} \mathbf{T}(\vec{x})_h - \mathbf{T}(\vec{x}')_h &= \sum_{h=-\infty}^{\bar{h}} \{ [\bar{\beta}_h^{(1)}(\{\vec{g}_k + \vec{x}_k\}_{k \geq h}) - \bar{\beta}_h^{(1)}(\{\vec{g}_k + \vec{x}'_k\}_{k \geq h})] \\ &\quad + [\bar{\beta}_h^{(3)}(\{\vec{g}_k + \vec{x}'_k\}_{k \geq h}) - \bar{\beta}_h^{(3)}(\{\vec{g}_k + \vec{x}_k\}_{k \geq h})] \} \end{aligned}$$

and  $|\mathbf{T}(\vec{x})_h - \mathbf{T}(\vec{x}')_h| \leq c_2 \lambda \|x - x'\|$ , thanks to the fact that all the terms in the r.h.s. of this equation are of the second order in the running couplings. It follows that, if  $c_2 \lambda < 1$ ,  $\mathbf{T}$  is a contraction in  $\mathcal{B}_\xi$ , so that (6.12) has a unique solution  $\vec{x}^{(0)}$  in this set; moreover, if we put  $\vec{l}_h = \vec{g}_h + \vec{x}_h^{(0)}$ ,  $\{\vec{l}_h\}_{h \leq 0}$  is the flow of the effective model corresponding to a value of  $\vec{l}$  such that

$$|\vec{g}_h - \vec{l}_h| \leq C|\lambda|\gamma^{\frac{\vartheta}{2}h} \quad (6.14)$$

Finally, this solution is such that  $\vec{l}$  is equal to  $\vec{g}_0$  at the first order; hence we get

$$\bar{g}_1 = 2\lambda\hat{v}(2p_F) + O(\lambda^2) \quad , \quad \bar{g}_2 = 2\lambda\hat{v}(0) + O(\lambda^2) \quad , \quad \bar{g}_4 = 2\lambda\hat{v}(0) + O(\lambda^2) \quad , \quad \bar{\delta} = O(\lambda) \quad (6.15)$$

Thanks to the bound (6.14), this choice of  $\vec{l}$ , allows us to say that there are constants  $Z = 1 + O(\lambda^2)$ ,  $Z_3 = 1 + O(\lambda)$  and  $\tilde{Z}_3 = v_F + O(\lambda)$  such that, if  $\kappa \leq 1$  and  $|\mathbf{p}| \leq \kappa$ ,

$$\begin{aligned} \widehat{\Omega}_C(\mathbf{p}) &= Z_3^2 \langle \widehat{\rho}_{\mathbf{p}} \widehat{\rho}_{-\mathbf{p}} \rangle^{(g)} + A_c + o(\mathbf{p}) \\ \widehat{D}(\mathbf{p}) &= -\tilde{Z}_3^2 \langle \widehat{j}_{\mathbf{p}} \widehat{j}_{-\mathbf{p}} \rangle^{(g)} + A_j + o(\mathbf{p}) \end{aligned} \quad (6.16)$$

where  $\langle \cdot \rangle^{(g)}$  denotes the expectation in the effective model satisfying (6.14),  $A_c$  and  $A_j$  are suitable  $O(1)$  constants and

$$\rho_{\mathbf{x}} = \sum_{\omega, s} \psi_{\mathbf{x}, \omega s}^+ \psi_{\mathbf{x}, \omega s} \quad , \quad j_{\mathbf{x}} = \sum_{\omega, s} \omega \psi_{\mathbf{x}, \omega s}^+ \psi_{\mathbf{x}, \omega s} \quad . \quad (6.17)$$

Moreover, if we put  $\mathbf{p}_F^\omega = (\omega p_F, 0)$  and we suppose that  $0 < \kappa \leq |\mathbf{p}|, |\mathbf{k}'|, |\mathbf{k}' - \mathbf{p}| \leq 2\kappa$ ,  $0 < \vartheta < 1$ , then

$$\begin{aligned} \widehat{G}_\rho^{2,1}(\mathbf{k}' + \mathbf{p}_F^\omega, \mathbf{k}' + \mathbf{p} + \mathbf{p}_F^\omega) &= Z_3 \langle \widehat{\rho}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)} [1 + O(\kappa^\vartheta)] \\ \widehat{G}_j^{2,1}(\mathbf{k}' + \mathbf{p}_F^\omega, \mathbf{k}' + \mathbf{p} + \mathbf{p}_F^\omega) &= \tilde{Z}_3 \langle \widehat{j}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)} [1 + O(\kappa^\vartheta)] \\ \widehat{S}_2(\mathbf{k}' + \mathbf{p}_F^\omega) &= \langle \psi_{\mathbf{k}', \omega, \sigma}^- \psi_{\mathbf{k}, \omega, \sigma}^+ \rangle^{(g)} [1 + O(\kappa^\vartheta)] \quad . \end{aligned} \quad (6.18)$$

where  $G_\rho^{2,1}(\mathbf{x})$  and  $G_j^{2,1}(\mathbf{x})$  are defined after (1.9), while the functions  $\langle \widehat{\rho}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)}$  and  $\langle \widehat{j}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)}$  coincide with the functions (3.22) and (3.23), respectively, with  $c = v_F(1 + \bar{\delta})$ . As already mentioned, if  $g_1 > 0$ , the effective model is still invariant under a spin-independent phase transformation; hence the WI (3.16) is satisfied, if we sum both sides over  $s$  and we substitute  $\nu_s^\mu(\mathbf{p})$  with

$$\bar{\nu}_s^\mu(\mathbf{p}) = \{ \delta_{\omega, 1} [\delta_{s, -1} \bar{g}_2 + \delta_{s, 1} (\bar{g}_2 - \bar{g}_1)] + \delta_{\omega, -1} \delta_{s, -1} \bar{g}_4 \} \frac{\widehat{h}(\mathbf{p})}{4\pi\bar{c}} \quad , \quad \bar{c} = v_F(1 + \bar{\delta}) \quad (6.19)$$

Therefore we get the a WI similar to (3.21), that is

$$\begin{aligned} &-ip_0 [1 - \bar{\nu}_4(\mathbf{p}) - 2\bar{\nu}_\rho(\mathbf{p})] \langle \widehat{\rho}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)} + \bar{c}p [1 + \bar{\nu}_4(\mathbf{p}) - 2\bar{\nu}_\rho(\mathbf{p})] \langle \widehat{j}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)} \\ &= \frac{1}{Z} \left[ \langle \psi_{\mathbf{k}, \omega, \sigma}^- \psi_{\mathbf{k}, \omega, \sigma}^- \rangle^{(g)} - \langle \psi_{\mathbf{k} + \mathbf{p}, \omega, \sigma}^- \psi_{\mathbf{k} + \mathbf{p}, \omega, \sigma}^- \rangle^{(g)} \right] \end{aligned} \quad (6.20)$$

where

$$\bar{\nu}_4(\mathbf{p}) = \bar{g}_4 \frac{\widehat{h}(\mathbf{p})}{4\pi\bar{c}} \quad , \quad \bar{\nu}_\rho(\mathbf{p}) = \frac{\bar{g}_2 - \bar{g}_1/2}{4\pi\bar{c}} \widehat{h}(\mathbf{p}) \quad (6.21)$$

By replacing (6.18) in (6.20), and comparing with (1.9) we get, if  $\bar{\nu}_4(0) \equiv \bar{\nu}_4$ ,  $\bar{\nu}_\rho(0) = \bar{\nu}_\rho$

$$\frac{Z_3}{Z} = (1 - \bar{\nu}_4 - 2\bar{\nu}_\rho) \quad , \quad \frac{\tilde{Z}_3}{Z} = \bar{c}(1 + \bar{\nu}_4 - 2\bar{\nu}_\rho) \quad (6.22)$$

Similarly we get:

$$\begin{aligned} D_\omega(\mathbf{p}) \langle \rho_{\mathbf{p}, \omega}^{(c)} \rho_{-\mathbf{p}, \omega'}^{(c)} \rangle^{(g)} - \bar{\nu}_4(\mathbf{p}) D_{-\omega}(\mathbf{p}) \langle \rho_{\mathbf{p}, \omega}^{(c)} \rho_{-\mathbf{p}, \omega'}^{(c)} \rangle^{(g)} \\ - 2\bar{\nu}_\rho(\mathbf{p}) D_{-\omega}(\mathbf{p}) \langle \rho_{\mathbf{p}, -\omega}^{(c)} \rho_{-\mathbf{p}, \omega'}^{(c)} \rangle^{(g)} = -\delta_{\omega, \omega'} \frac{D_{-\omega}(\mathbf{p})}{2\pi Z^2 \bar{c}} \end{aligned} \quad (6.23)$$

Hence, by some simple algebra, we get:

$$\langle \rho_{\mathbf{p},\omega}^{(c)} \rho_{-\mathbf{p},\omega}^{(c)} \rangle^{(g)} = \frac{1}{2\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2} \frac{D_{-\omega}(\mathbf{p}) [D_{-\omega}(\mathbf{p}) - \bar{\nu}_4 D_{\omega}(\mathbf{p})]}{p_0^2 + \bar{c}^2 \bar{v}_{\rho}^2 p^2} + O(\mathbf{p}) \quad (6.24)$$

$$\langle \rho_{\mathbf{p},\omega}^{(c)} \rho_{-\mathbf{p},-\omega}^{(c)} \rangle^{(g)} = \frac{1}{2\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2} \frac{2\bar{\nu}_{\rho} D_{\omega}(\mathbf{p}) D_{-\omega}(\mathbf{p})}{p_0^2 + \bar{c}^2 \bar{v}_{\rho}^2 p^2} + O(\mathbf{p}) \quad (6.25)$$

where

$$\bar{v}_{\rho} = \bar{v}_{\rho,-}/v_{\rho,+} \quad , \quad \bar{v}_{\rho,\mu}^2 = (1 - \mu \bar{\nu}_4)^2 - 4\bar{\nu}_{\rho}^2 \quad (6.26)$$

Therefore the charge and current density correlations are given by:

$$\begin{aligned} \langle \rho_{\mathbf{p}}^{(c)} \rho_{-\mathbf{p}}^{(c)} \rangle^{(g)} &= \frac{1}{\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2} \frac{-p_0^2(1 - \bar{\nu}_4 + 2\bar{\nu}_{\rho}) + \bar{c}^2 p^2(1 + \bar{\nu}_4 - 2\bar{\nu}_{\rho})}{p_0^2 + \bar{c}^2 \bar{v}_{\rho}^2 p^2} + O(\mathbf{p}) \\ \langle j_{\mathbf{p}}^{(c)} j_{-\mathbf{p}}^{(c)} \rangle^{(g)} &= \frac{1}{\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2} \frac{-p_0^2(1 - \bar{\nu}_4 - 2\bar{\nu}_{\rho}) + \bar{c}^2 p^2(1 + \bar{\nu}_4 + 2\bar{\nu}_{\rho})}{p_0^2 + \bar{c}^2 \bar{v}_{\rho}^2 p^2} + O(\mathbf{p}) \end{aligned} \quad (6.27)$$

Note the above equations are true also for  $g_1 < 0$ , provided that an infrared cut-off is present and for coupling small enough (vanishing removing the cut-off).

From the WI (1.9) we see that

$$\widehat{\Omega}_C(0, p_0) = 0 \quad , \quad \widehat{D}(p, 0) = 0 \quad (6.28)$$

and this fixes the values of the constants  $A_c$  and  $A_j$  in (6.16), so that

$$\begin{aligned} \widehat{\Omega}_C(\mathbf{p}) &= \frac{Z_3^2}{\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2} [(1 + \bar{\nu}_4 - 2\bar{\nu}_{\rho}) + \bar{v}_{\rho}^2(1 - \bar{\nu}_4 + 2\bar{\nu}_{\rho})] \frac{\bar{c}^2 p^2}{p_0^2 + \bar{v}_{\rho}^2 \bar{c}^2 p^2} + o(\mathbf{p}) \\ \widehat{D}(\mathbf{p}) &= \frac{\widetilde{Z}_3^2}{\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2 v_{\rho}^2} [(1 + \bar{\nu}_4 + 2\bar{\nu}_{\rho}) + \bar{v}_{\rho}^2(1 - \bar{\nu}_4 - 2\bar{\nu}_{\rho})] \frac{p_0^2}{p_0^2 + \bar{v}_{\rho}^2 \bar{c}^2 p^2} + o(\mathbf{p}) \end{aligned} \quad (6.29)$$

If we insert (6.22) in the previous equations, we get, for the susceptibility (1.6) and the Drude weight (1.7), the values

$$\begin{aligned} \kappa &= \frac{(1 - \bar{\nu}_4 - 2\bar{\nu}_{\rho})^2}{\pi \bar{c} \bar{v}_{\rho,+}^2 \bar{v}_{\rho}^2} [(1 + \bar{\nu}_4 - 2\bar{\nu}_{\rho}) + \bar{v}_{\rho}^2(1 - \bar{\nu}_4 + 2\bar{\nu}_{\rho})] = \frac{\bar{K}}{\pi \bar{c} \bar{v}_{\rho}} \\ D &= \frac{\bar{c}(1 + \bar{\nu}_4 - 2\bar{\nu}_{\rho})^2}{\pi \bar{v}_{\rho,+}^2 \bar{v}_{\rho}^2} [(1 + \bar{\nu}_4 + 2\bar{\nu}_{\rho}) + \bar{v}_{\rho}^2(1 - \bar{\nu}_4 - 2\bar{\nu}_{\rho})] = \frac{\bar{K} \bar{c} \bar{v}_{\rho}}{\pi} \end{aligned} \quad (6.30)$$

where

$$\bar{K} = \frac{(1 - 2\bar{\nu}_{\rho})^2 - \bar{\nu}_4^2}{\bar{v}_{\rho,+} \bar{v}_{\rho,-}} = \sqrt{\frac{(1 - \bar{\nu}_4) - 2\bar{\nu}_{\rho}}{(1 - \bar{\nu}_4) + 2\bar{\nu}_{\rho}}} \sqrt{\frac{(1 + \bar{\nu}_4) - 2\nu_{\rho}}{(1 + \bar{\nu}_4) + 2\bar{\nu}_{\rho}}} \quad (6.31)$$

so that

$$\frac{\kappa}{D} = \frac{1}{\bar{c}^2 \bar{v}_{\rho}^2} \quad (6.32)$$

and this completes the proof of Theorem 1.1.

**Remark 1** We are unable to see if  $\bar{c} \bar{v}_{\rho} = v_F(1 + \bar{\delta}) \bar{v}_{\rho}$  coincides with the velocity  $cv_{\rho} = v_F(1 + \bar{\delta})v_{\rho}$  appearing in the two-point function asymptotic behavior (5.10), with  $v_{\rho}$  given (see (3.29)) by

$$v_{\rho} = \frac{(1 + \nu_4)^2 - 4\nu_{\rho}^2}{(1 - \nu_4)^2 - 4\nu_{\rho}^2} \quad (6.33)$$

with  $\nu_\rho$  and  $\nu_4$  defined as in (6.9). In fact, it is easy to see that  $\tilde{\delta}$  is equal to  $\bar{\delta}$  at the first order and this is true also for  $\bar{v}_\rho$  and  $v_\rho$  by (6.15), (6.21) and (6.9); however, our arguments are not able to exclude that the two velocities are different. Moreover, (6.31) and (5.20) imply that  $\bar{K} = K$  at first order, but they also could be different. Note that the equality of  $\bar{K}$  and  $K$ , would imply that  $\kappa = K/v$ , with  $v = \bar{c}v_\rho$  being the charge velocity, a relation proposed in [3] which, together with (1.14) and (1.16), would allow for the exact determination of the exponents in terms of the susceptibility and the Drude weight.

**Remark 2** Note that the Ward Identities (6.27) are true also for  $g_{1,\perp} < 0$  and  $l = -\infty$ , provided that  $L$  is finite.

## A Proof of Theorem 2.1

By using (2.23) and the definition (2.20) of  $\mathcal{L}$ , we get in the following way a simple expression for  $\mathcal{V}^{(0)}(\tau, \alpha, \psi^{[l,k]}, J)$ , for any  $\tau \in \mathcal{T}_{N,k,n_g,n_J}$ .

We associate with any vertex  $v$  of the tree a subset  $P_v$  of  $I_v$ , the *external  $\psi$  fields* of  $v$ , and the set  $\mathbf{x}_v$  of all space-time points associated with one of the end-points following  $v$ ; moreover, we shall denote  $\mathbf{x}_v^J \subset \mathbf{x}_v$  the set of all space time points associated with the special endpoints following  $v$ . The subsets  $P_v$  must satisfy various constraints. First of all,  $|P_v| \geq 2$ , if  $v > v_0$ ; moreover, if  $v$  is not an endpoint and  $v_1, \dots, v_{s_v}$  are the  $s_v \geq 1$  vertices immediately following it, then  $P_v \subseteq \cup_i P_{v_i}$ ; if  $v$  is an endpoint,  $P_v = I_v$ . If  $v$  is not an endpoint, we shall denote by  $Q_{v_i}$  the intersection of  $P_v$  and  $P_{v_i}$ ; this definition implies that  $P_v = \cup_i Q_{v_i}$ . The union  $\mathcal{I}_v$  of the subsets  $P_{v_i} \setminus Q_{v_i}$  is, by definition, the set of the *internal fields* of  $v$ , and is not empty if  $s_v > 1$ . Given  $\tau \in \mathcal{T}_{N,k,n_g,n_J}$ , there are many possible choices of the subsets  $P_v$ ,  $v \in \tau$ , compatible with all the constraints. We shall denote  $\mathcal{P}_\tau$  the family of all these choices and  $\mathbf{P}$  the elements of  $\mathcal{P}_\tau$ . Then we can write:

$$\mathcal{V}^{(0)}(\tau, \alpha, \psi^{[l,k]}, J) = \sum_{\mathbf{P} \in \mathcal{P}_\tau} \mathcal{V}^{(0)}(\tau, \alpha, \mathbf{P}) \quad (\text{A.1})$$

$$\mathcal{V}^{(0)}(\tau, \alpha, \mathbf{P}) = \int d\mathbf{x}_{v_0} \tilde{\psi}^{[l,k]}(P_{v_0}) \tilde{J}(S_{v_0}) K_{\tau, \alpha, \mathbf{P}}^{(k+1)}(\mathbf{x}_{v_0}) \quad (\text{A.2})$$

where  $S_v$  denotes the set of special endpoints following  $v$ ,  $\tilde{\psi}^{[l,k]}(P_v) := \prod_{f \in P_v} \psi_{\mathbf{x}(f), s(f)}^{[l,k] \varepsilon(f)}$ ,  $\tilde{J}(S_v) = \prod_{v \in S_v} J_{\mathbf{x}_v, \Theta_v}$  and  $K_{\tau, \alpha, \mathbf{P}}^{(k+1)}(\mathbf{x}_{v_0}, \mathbf{y}_{v_0})$  is defined inductively (recall that  $h_{v_0} = k+1$ ) by the equation, valid for any  $v \in \tau$  which is not an endpoint,

$$K_{\tau, \alpha, \mathbf{P}}^{(h_v)}(\mathbf{x}_v) = \frac{1}{s_v!} \prod_{i=1}^{s_v} [K_{v_i}^{(h_v+1)}(\mathbf{x}_{v_i})] \mathcal{E}_{h_v}^T[\tilde{\psi}^{(h_v)}(P_{v_1} \setminus Q_{v_1}), \dots, \tilde{\psi}^{(h_v)}(P_{v_{s_v}} \setminus Q_{v_{s_v}})] \quad (\text{A.3})$$

If  $v$  is an endpoint,  $K_v^{(h_v)}(\mathbf{x}_v)$  is equal to one of the kernels in  $\mathcal{L}\mathcal{V}^{(h_v)}$ , otherwise  $K_v^{(h_v)} = \mathcal{R}K_{\tau_i, \mathbf{P}_i}^{(h_v)}$ , where  $\tau_1, \dots, \tau_{s_v}$  are the subtrees of  $\tau$  with root  $v$ ,  $\mathbf{P}_i = \{P_v, v \in \tau_i\}$ . Hence, if we use the *Brydges-Battle-Federbush* identity (see [15]) to expand the truncated expectation in (A.3), the kernel in (A.2) can be rewritten as

$$K_{\tau, \alpha, \mathbf{P}}^{(k+1)}(\mathbf{x}_{v_0}) = \sum_{T \in \mathbf{T}} \left[ \prod_{i=1}^n K_{v_i^*}^{h_i}(\mathbf{x}_{v_i^*}) \right] \left\{ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \cdot \det G^{h_v, T_v}(\mathbf{t}_v) \left[ \prod_{l \in T_v} g_{\omega_l}^{(h_v)}(\mathbf{x}_l - \mathbf{y}_l) \right] \right\} \quad (\text{A.4})$$

where ‘‘e.p.’’ is an abbreviation of ‘‘endpoint’’,  $v_i^*$  are the end-points,  $K_{v_i^*}^{h_i}(\mathbf{x}_{v_i^*})$  is  $W^{(0,4), h_i}$ ,  $W^{(0,2), h_i}$  or  $W^{(1,2), h_i}$ ,  $\mathbf{T}$  is the set of the tree graphs on  $\mathbf{x}_{v_0}$ , obtained by putting together an

anchored tree graph  $T_v$  for each non trivial vertex  $v$  and adding the lines connecting the space-time points belonging to the sets  $\mathbf{x}_{v_i^*}$ ; moreover,  $dP_{T_v}(\mathbf{t}_v)$  is a probability measure with support on the set of  $\mathbf{t}_v = \{\mathbf{t}_{v,i,i'} \in [0, 1], \mathbf{1} \leq \mathbf{i}, \mathbf{i}' \leq \mathbf{s}_v\}$  such that  $t_{v,i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$  for some family of vectors  $\mathbf{u}_i \in \mathbb{R}^s$  of unit norm; finally, if  $L_v = |\mathcal{I}_v|/2 - s_v + 1$ ,  $G^{h_v, T_v}(\mathbf{t}_v)$  is a  $L_v \times L_v$  matrix with elements of the form

$$G_{v,ij,i'j'}^{h_v, T_v} = t_{v,i,i'} g_{\omega_{ij,i'j'}}^{(h_v)}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) \quad (\text{A.5})$$

with  $g_{\omega}^{(h)}(\mathbf{x})$  admitting a Gram representation:  $g_{\omega}^{(h)}(\mathbf{x} - \mathbf{y}) = \frac{1}{Z} \int d\mathbf{z} A_h^*(\mathbf{x} - \mathbf{z}) \cdot B_h(\mathbf{y} - \mathbf{z})$ , with

$$\begin{aligned} A_h(\mathbf{x}) &= \frac{1}{L^2} \sum_{\mathbf{k}} \sqrt{f_h(\tilde{\mathbf{k}})} \frac{e^{-i\mathbf{k}\mathbf{x}}}{k_0^2 + c^2 k^2} \\ B_h(\mathbf{x}) &= \frac{1}{L^2} \sum_{\mathbf{k}} \sqrt{f_h(\tilde{\mathbf{k}})} e^{-i\mathbf{k}\mathbf{x}} (ik_0 + \omega ck) \end{aligned} \quad (\text{A.6})$$

and

$$\|A_h\|^2 = \int d\mathbf{z} |A_h(\mathbf{z})|^2 \leq C\gamma^{-2h}, \quad \|B_h\|^2 \leq C\gamma^{4h}, \quad (\text{A.7})$$

for a suitable constant  $C$ . Therefore the Gram–Hadamard inequality, combined with the dimensional bound on  $g_{\omega}^{(h)}(\mathbf{x})$ , implies that

$$|\det G^{h_v, T_v}(\mathbf{t}_v)| \leq C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v| - 2(s_v - 1)} \cdot \gamma^{h_v(\sum_{i=1}^{s_v} |P_{v_i}| - |P_v| - 2(s_v - 1))}. \quad (\text{A.8})$$

By the decay properties of  $g_{\omega}^{(h)}(\mathbf{x})$ , it also follows that

$$\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int \prod_{l \in T_v} d(\mathbf{x}_l - \mathbf{y}_l) \|g_{\omega_l}^{(h_v)}(\mathbf{x}_l - \mathbf{y}_l)\| \leq C^{n+m} \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{-h_v(s_v - 1)}. \quad (\text{A.9})$$

Therefore, proceeding as in the proof of Lemma 2.2 in the companion paper [1], we get the bound

$$\|W^{(n; 2m)(h)}\| \leq \sum_{n \geq d_{n,m}} C^{n+m} \varepsilon_0^n \sum_{\substack{\tau \in \mathcal{T}_{N,h,n,m} \\ \alpha \in A_{\tau}}} \sum_{\substack{\mathbf{P} \in \mathcal{P}_{\tau} \\ |P_{v_0}| = 2m}} \gamma^{-D_{v_0} h} \left[ \prod_{\substack{v \text{ not e.p.} \\ v > v_0}} \gamma^{-D_v} \right], \quad (\text{A.10})$$

where

$$D_v = -2 + \frac{|P_v|}{2} + n_v^J \quad (\text{A.11})$$

where  $n_v^J$  is the number of  $J$ -endpoints of the subtree with root  $v$ . The definition of the  $\mathcal{R}$  operation (which is applied to all vertices, except the endpoints and  $v_0$ ) implies that the tree value vanish, if there is even one vertex, except the endpoints and  $v_0$ , with  $D_v \leq 0$ . Hence, we can suppose that in (A.10),  $D_v > 0$  in all vertices of the product and the bound (2.28) follows from (A.10) by the combinatorial arguments used in the proof of Lemma 2.2 of companion paper [1].

As concerns the proof of the uniform convergence as  $N \rightarrow \infty$ , which only depends on the structure of the tree expansion and on the corresponding bounds, we again refer to Lemma 2.2 of [1]. ■

## B Proof of (2.33) and (2.34)

Let us denote by  $\mathcal{V}(\psi, J, \eta)$  the expression inside the braces in the r.h.s. side of (2.2) and let us define  $\mathcal{V}^{(0)}(\psi, J, \eta)$  so that

$$e^{\mathcal{V}^{(0)}(\psi, J, \eta)} = \int P_{1,N}(d\zeta) e^{\mathcal{V}(\psi + \zeta, J, \eta)} \quad (\text{B.1})$$

where  $P_{1,N}(d\zeta) \equiv P(d\psi^{[1,N]})$ . Note that  $\mathcal{V}^{(N)}(\psi, J, \eta) = \mathcal{V}(\psi, J, \eta)$  and that the functional  $\bar{V}^{(0)}(\psi, J)$  used in (2.33) and (2.34) is equal to  $\mathcal{V}^{(0)}(\psi, J, 0)$ . The properties of the Grassmannian integral imply that, for  $\varepsilon = \pm$ ,

$$\int P_{k+1,N}(d\zeta) \zeta_{\mathbf{x},\omega,s}^{-\varepsilon} F(\zeta) = \varepsilon \int d\mathbf{u} g_{\omega}^{[1,N]}(\mathbf{x} - \mathbf{u}) \int P_{k+1,N}(d\zeta) \frac{\partial F(\zeta)}{\partial \zeta_{\mathbf{u},\omega,s}^{\varepsilon}} \quad (\text{B.2})$$

and therefore

$$\begin{aligned} \frac{\partial e^{\mathcal{V}^{(0)}}}{\partial \eta_{\mathbf{x},\omega,s}^{\varepsilon}}(\psi, J, \eta) &= \int P_{k+1,N}(d\zeta) (\psi_{\mathbf{x},\omega,s}^{-\varepsilon} + \zeta_{\mathbf{x},\omega,s}^{-\varepsilon}) e^{\mathcal{V}(\psi+\zeta, J, \eta)} \\ &= \psi_{\mathbf{x},\omega,s}^{-\varepsilon} e^{\mathcal{V}^{(0)}(\psi, J, \eta)} + \varepsilon \int d\mathbf{u} g_{\omega}^{[1,N]}(\mathbf{x} - \mathbf{u}) \frac{\partial e^{\mathcal{V}^{(0)}}}{\partial \psi_{\mathbf{u},\omega,s}^{\varepsilon}}(\psi, J, \eta) \end{aligned} \quad (\text{B.3})$$

which is (3.34). Besides, using the explicit expression of  $\mathcal{V}(\psi, J, \eta)$ , obtained by adding to (2.13) the terms linear in  $\eta$ , by explicit computation of the derivatives on both sides of (B.1) we find the following two identities; if  $\Theta = (\omega, s, t_0)$ ,

$$\begin{aligned} \frac{\partial e^{\mathcal{V}^{(0)}}}{\partial \psi_{\mathbf{x},\omega,s}^{\pm}}(\psi, J, 0) &= \sum_{t_0} J_{\mathbf{x},\Theta} \frac{\partial e^{\mathcal{V}^{(0)}}}{\partial \eta_{\mathbf{x},\omega,t_0}^{\pm}}(\psi, J, 0) \\ &+ \sum_{t_0, \Theta'} \int d\mathbf{w} h_{\Theta, \Theta'}^{L,K}(\mathbf{x} - \mathbf{w}) \frac{\partial^2 e^{\mathcal{V}^{(0)}}}{\partial J_{\mathbf{w}, \Theta'} \partial \eta_{\mathbf{x},\omega,t_0}^{\pm}}(\psi, J, 0) \end{aligned} \quad (\text{B.4})$$

$$\frac{\partial e^{\mathcal{V}^{(0)}}}{\partial J_{\mathbf{x},\Theta}}(\psi, J, 0) = - \frac{\partial^2 e^{\mathcal{V}^{(0)}}}{\partial \eta_{\mathbf{x},\omega,t_0}^{\pm} \partial \eta_{\mathbf{x},\omega,s}^{\mp}}(\psi, J, 0) \quad (\text{B.5})$$

Finally, plug (B.3) for  $\varepsilon = +$  into (B.4) and get (2.33). Also from (B.3) and (B.5) we obtain:

$$\begin{aligned} \frac{\partial e^{\mathcal{V}^{(0)}}}{\partial J_{\mathbf{x},\Theta}}(\psi, J, \eta) &= -\psi_{\mathbf{x},\omega,t_0}^{-} \frac{\partial e^{\mathcal{V}^{(0)}}}{\partial \eta_{\mathbf{x},\omega,s}^{-}}(\psi, J, \eta) \\ &- \int d\mathbf{u} g_{\omega}^{[1,N]}(\mathbf{x} - \mathbf{u}) \frac{\partial^2 e^{\mathcal{V}^{(0)}}}{\partial \psi_{\mathbf{u},\omega,t_0}^{+} \partial \eta_{\mathbf{x},\omega,s}^{-}}(\psi, J, \eta) \end{aligned} \quad (\text{B.6})$$

which gives (2.34) by applying (B.3) again and using that  $g_{\omega}^{[1,N]}(0) = 0$ .

## C Some properties of the functions $\widehat{U}_{l,N,\omega}^{(i,j)}(\mathbf{k}^+, \mathbf{k}^-)$

By using (3.6) and (3.7), we easily see that, if  $i, j > l$ ,

$$\lim_{\varepsilon \rightarrow 0} \widehat{U}_{l,N,\omega}^{(i,j)}(\mathbf{k}^+, \mathbf{k}^-) = \frac{1}{Z_{i-1} Z_{j-1}} \frac{\delta_{j,N} u_N(\tilde{\mathbf{k}}^-) \tilde{f}_i(|\tilde{\mathbf{k}}^+|) D_{\omega}(\mathbf{k}^-) - \delta_{i,N} u_N(\tilde{\mathbf{k}}^+) \tilde{f}_j(|\tilde{\mathbf{k}}^-|) D_{\omega}(\mathbf{k}^+)}{D_{\omega}(\mathbf{k}^+) D_{\omega}(\mathbf{k}^-)} \quad (\text{C.1})$$

where  $U_N(\mathbf{k})$  is a smooth function such that  $u_N(\mathbf{k}) = 0$  for  $|\mathbf{k}| \leq \gamma^N$  and  $u_N(\mathbf{k}) = 1 - f_N(|\mathbf{k}|)$  for  $|\mathbf{k}| \geq \gamma^N$ ; note that  $u_N(\mathbf{k})$  is a smooth function and that  $u_N(\mathbf{k}) = 1 - \chi_{[-\infty, N]}(|\mathbf{k}|)$ . If we put  $\mathbf{p} = \mathbf{k}^+ - \mathbf{k}^-$  and we use that  $D_{\omega}(\mathbf{p}) = D_{\omega}(\mathbf{k}^+) - D_{\omega}(\mathbf{k}^-)$ , we see that  $\widehat{U}_{l,N,\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q})$  satisfies the identity (3.8), if we put

$$\widehat{S}_{l,N,\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q}) = -\tilde{\chi}_N(\mathbf{p}) \frac{\delta_{i,N} u_N(\tilde{\mathbf{q}} + \tilde{\mathbf{p}}) \tilde{f}_j(|\tilde{\mathbf{q}}|)}{D_{\omega}(\mathbf{q}) D_{\omega}(\mathbf{q} + \mathbf{p})} \quad (\text{C.2})$$

$$\widehat{\mathcal{S}}_{l,N,-\omega,\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q}) = \widetilde{\chi}_N(\mathbf{p}) \frac{\delta_{j,N} u_N(\widetilde{\mathbf{q}}) \widetilde{f}_i(|\widetilde{\mathbf{q}} + \widetilde{\mathbf{p}}|) - \delta_{i,N} u_N(\widetilde{\mathbf{q}} + \widetilde{\mathbf{p}}) \widetilde{f}_j(|\widetilde{\mathbf{q}}|)}{D_{-\omega}(\mathbf{p}) D_{\omega}(\mathbf{q} + \mathbf{p})} \quad (\text{C.3})$$

The proof of the bound (3.9) for  $i = N$  and  $j > l$  easily follows from the smoothness and support properties of the functions  $f_i(|\widetilde{\mathbf{k}}|)$ ,  $u_N(\mathbf{k})$  and  $\widetilde{\chi}_N(\mathbf{p})$ . Note that this definition of the functions  $\widehat{\mathcal{S}}_{l,N,\omega',\omega}^{(i,j)}$  is useful only in the case  $j > N - 2$ , in order to exploit the fact that  $\widehat{U}_{l,N,\omega}^{(i,j)}(\mathbf{k}, \mathbf{k}) = 0$ ; see §4.2 of [8] for more details in the case  $N = 0$ . If  $i = N$  and  $l < j \leq N - 2$ , we could write the simpler decomposition

$$\widetilde{\chi}_N(\mathbf{p}) \lim_{\varepsilon \rightarrow 0} \widehat{U}_{l,N,\omega}^{(N,j)}(\mathbf{k}^+, \mathbf{k}^-) = -\frac{1}{2} \sum_{\omega' = \pm\omega} D_{\omega'}(\mathbf{p}) \frac{1}{Z_{j-1}} \widetilde{\chi}_N(\mathbf{p}) \frac{\widetilde{f}_j(|\widetilde{\mathbf{k}}^-|) u_N(\widetilde{\mathbf{k}}^+)}{D_{\omega}(\mathbf{k}^-) D_{\omega'}(\mathbf{p})} \quad (\text{C.4})$$

which also satisfies the bound (3.9).

If  $i = N$  and  $j = l$ , one has to add to the r.h.s. of (C.4) the term

$$\frac{\widetilde{\chi}_N(\mathbf{p})}{\widetilde{Z}_{l-1}(\mathbf{k}^-)} \frac{f_N(|\widetilde{\mathbf{k}}^+|) u_l(\widetilde{\mathbf{k}}^-)}{D_{\omega}(\mathbf{k}^+)} = \frac{1}{2} \sum_{\omega' = \pm\omega} D_{\omega'}(\mathbf{p}) \frac{1}{\widetilde{Z}_{l-1}(\mathbf{k}^-)} \widetilde{\chi}_N(\mathbf{p}) \frac{f_N(|\widetilde{\mathbf{k}}^+|) u_l(\widetilde{\mathbf{k}}^-)}{D_{\omega}(\mathbf{k}^+) D_{\omega'}(\mathbf{p})}$$

with  $\widetilde{Z}_{l-1}(\mathbf{k})$  defined as in (2.47); then we get the bound (3.9) even for  $j = l$ , since  $\widetilde{Z}_{l-1}(\mathbf{k}) \geq 1$ .

In order to prove the bound (3.10), note that, if  $N > j > l$ ,

$$[1 - \widetilde{\chi}_l(\mathbf{p})] \lim_{\varepsilon \rightarrow 0} \widehat{U}_{l,N,\omega}^{(j,l)}(\mathbf{k}^+, \mathbf{k}^-) = \frac{1}{2} \sum_{\omega' = \pm\omega} D_{\omega'}(\mathbf{p}) \frac{[1 - \widetilde{\chi}_l(\mathbf{p})]}{Z_{j-1} \widetilde{Z}_{l-1}(\mathbf{k}^-)} \frac{\widetilde{f}_j(|\widetilde{\mathbf{k}}^+|) u_l(\widetilde{\mathbf{k}}^-)}{D_{\omega}(\mathbf{k}^+) D_{\omega'}(\mathbf{p})} \quad (\text{C.5})$$

where  $u_l(\mathbf{k})$  is a smooth function such that  $u_l(\mathbf{k}) = 0$  for  $|\mathbf{k}| \geq \gamma^l$  and  $u_l(\mathbf{k}) = 1 - f_l(|\mathbf{k}|)$  for  $|\mathbf{k}| \leq \gamma^l$ ; the identity is valid also for  $j = l$ , with  $\widetilde{Z}_{l-1}(\mathbf{k}^-)$  in place of  $Z_{j-1}$ . Then the bound (3.10) follows from the remarks that  $\widetilde{Z}_{l-1}(\mathbf{k}) \geq 1$  and that  $[1 - \widetilde{\chi}_l(\mathbf{p})] \widetilde{f}_j(|\widetilde{\mathbf{k}}^+|) u_l(\widetilde{\mathbf{k}}^-) \neq 0$  only if  $j > l$  and, in such case  $|D_{\omega}(\mathbf{p})^{-1}| \leq c\gamma^{-j}$ .

We are also interested in the bound of

$$\frac{\widehat{U}_{l,N,\omega}^{(l,l)}(\mathbf{k}^+, \mathbf{k}^-)}{D_{\omega'}(\mathbf{p})} = \frac{1}{\widetilde{Z}_{l-1}(\mathbf{k}^+) \widetilde{Z}_{l-1}(\mathbf{k}^-)} \left[ \frac{\widetilde{f}_l(|\widetilde{\mathbf{k}}^+|) u_l(\widetilde{\mathbf{k}}^-)}{D_{\omega}(\mathbf{k}^+) D_{\omega'}(\mathbf{p})} - \frac{\widetilde{f}_l(|\widetilde{\mathbf{k}}^-|) u_l(\widetilde{\mathbf{k}}^+)}{D_{\omega}(\mathbf{k}^-) D_{\omega'}(\mathbf{p})} \right] \quad (\text{C.6})$$

By using the remark after (2.47), we see that its Fourier transform  $\overline{S}_{\omega,\omega'}^{l,l}(\mathbf{z}; \mathbf{x}, \mathbf{y})$  satisfies, for any positive integer  $r$ , the bound

$$|\overline{S}_{\omega,\omega'}^{l,l}(\mathbf{z}; \mathbf{x}, \mathbf{y})| \leq \frac{1}{Z_{l-1}} b_{r,l}(\mathbf{x} - \mathbf{z}) b_{r,l}(\mathbf{x} - \mathbf{z}) \quad (\text{C.7})$$

with  $b_{r,l}(\mathbf{x})$  defined as in (3.9).

Finally, we want to prove the bounds (3.12). To begin with, note that the r.h.s. of (C.2) and (C.3) do not depend on  $l$ , if  $i, j \geq K + 1$ , and that  $\sum_{i=K+1}^N f_i(|\widetilde{\mathbf{q}}|) = \chi_{[-\infty, N]}(|\widetilde{\mathbf{q}}|)$ , if  $|\widetilde{\mathbf{p}}| \leq 1$  and  $N$  is large enough. Hence, if  $|\widetilde{\mathbf{p}}| \leq 1$ ,

$$\sum_{i,j=K+1}^N \frac{1}{L^2} \sum_{\mathbf{q} \in \mathcal{D}'_L} \widehat{\mathcal{S}}_{l,N,\omega,\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q}) = -\frac{1}{L^2} \sum_{\mathbf{q} \in \mathcal{D}'_L} \frac{u_N(\widetilde{\mathbf{q}} + \widetilde{\mathbf{p}}) \chi_{[-\infty, N]}(|\widetilde{\mathbf{q}}|)}{D_{\omega}(\mathbf{q}) D_{\omega}(\mathbf{q} + \mathbf{p})} \quad (\text{C.8})$$

$$\sum_{i,j=K+1}^N \frac{1}{L^2} \sum_{\mathbf{q} \in \mathcal{D}'_L} \widehat{\mathcal{S}}_{l,N,-\omega,\omega}^{(i,j)}(\mathbf{q} + \mathbf{p}, \mathbf{q}) = \frac{1}{L^2} \sum_{\mathbf{q} \in \mathcal{D}'_L} \frac{\chi_{[-\infty, N]}(|\widetilde{\mathbf{q}} + \widetilde{\mathbf{p}}|) - \chi_{[-\infty, N]}(|\widetilde{\mathbf{q}}|)}{D_{-\omega}(\mathbf{p}) D_{\omega}(\mathbf{q} + \mathbf{p})} \quad (\text{C.9})$$

where we also used the fact that  $u_N(\mathbf{k}^-)\chi_{[-\infty,N]}(\mathbf{k}^+) - u_N(\mathbf{k}^+)\chi_{[-\infty,N]}(\mathbf{k}^-) = \chi_{[-\infty,N]}(\mathbf{k}^+) - \chi_{[-\infty,N]}(\mathbf{k}^-)$ . Since  $|\tilde{\mathbf{q}}|$  is of order  $\gamma^N$ , it is easy to see that we can substitute  $L^{-2} \sum_{\mathbf{q} \in \mathcal{D}'_L}$  with  $(2\pi)^{-2} \int d\mathbf{q}$  in the r.h.s. of (C.8), for any small  $\mathbf{p}$ , and in the r.h.s. of (C.9), for  $\mathbf{p} \neq 0$  and small, by making an error of order  $\gamma^{-N}/L$ . On the other hand, by using the fact that under the change of variables  $\tilde{\mathbf{q}} = (cq, q_0) \rightarrow (q_0, -cq)$ , which leaves invariant  $|\tilde{\mathbf{q}}|$ ,  $D_\omega(\mathbf{q}) \rightarrow i\omega D_\omega(\mathbf{q})$ , we see that

$$\int \frac{d\mathbf{q}}{(2\pi)^2} \frac{u_N(\tilde{\mathbf{q}})\chi_{[-\infty,N]}(|\tilde{\mathbf{q}}|)}{D_\omega(\mathbf{q})^2} = 0$$

This proves (3.12) for  $\tau_N^+$ . As concerns  $\tau_N^-$ , an easy calculation shows that

$$\begin{aligned} \lim_{\mathbf{p} \rightarrow 0} \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{\chi_{[-\infty,N]}(|\tilde{\mathbf{q}} + \tilde{\mathbf{p}}|) - \chi_{[-\infty,N]}(|\tilde{\mathbf{q}}|)}{D_{-\omega}(\mathbf{p})D_\omega(\mathbf{q} + \mathbf{p})} = \\ - \frac{1}{2c} \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{\chi'(|\mathbf{q}|)}{|\mathbf{q}|} = - \frac{1}{4\pi c} \int_0^\infty dt \chi'(t) = - \frac{1}{4\pi c} [\chi(\infty) - \chi(0)] = \frac{1}{4\pi c} \end{aligned}$$

which proves (3.12) for  $\tau_N^-$ .

## D Approximate Spin-Charge Separation

**Theorem D.1** *Under the same condition of the previous Theorem, the Fourier transform of the 2-point Schwinger function is given by*

$$\widehat{S}_2(\mathbf{k} + \mathbf{p}_F^\omega) = Z(\mathbf{k})\widehat{S}_{M,\omega}(\mathbf{k})[1 + R(\mathbf{k})] \quad , \quad \mathbf{p}_F^\omega = (\omega p_F, 0) \quad (\text{D.1})$$

where

$$|R(\mathbf{k})| \leq C \frac{\lambda^2}{1 + a|\lambda \log |\mathbf{k}||} \quad , \quad a \geq 0 \quad , \quad (\text{D.2})$$

$$Z(\mathbf{k}) = L(|\mathbf{k}|^{-1})^{\zeta_z} [1 + R'(\mathbf{k})] \quad , \quad |R'(\mathbf{k})| \leq C|\lambda| \quad (\text{D.3})$$

$L(t)$ ,  $t \geq 1$ , is the function defined in Theorem 1.1 of the companion paper [1] and  $\widehat{S}_{M,\omega}(\mathbf{k})$  is a function whose Fourier transform is of the form

$$S_{M,\omega}(\mathbf{x}) = \frac{1}{2\pi v_F} \frac{[v_\rho^2 x_0^2 + (x_1/v_F)^2]^{-\eta_\rho/2}}{(v_\rho x_0 + i\omega x_1/v_F)^{1/2} (v_\sigma x_0 + i\omega x_1/v_F)^{1/2}} e^{C+O(1/|\mathbf{x}|)} \quad (\text{D.4})$$

with  $v_{\rho,\sigma} = 1 + O(\lambda)$ ,  $\eta_\rho = O(\lambda^2)$ ,  $v_\rho - v_\sigma = c_v \lambda + O(\lambda^2)$ , with  $c_v \neq 0$ .

Similar expressions are true also for the density correlations (the explicit formulae are in §5 and are not reported here for brevity). The above theorem says that the two point function can be written, up to a logarithmic correction, as the 2-point function of the Mattis model [16], a model which shows an *anomalous dimension* and the phenomenon of *spin-charge separation*. This last property means that there is, in the Mattis model, an exact decoupling of the hamiltonian as sum of two independent hamiltonians describing the spin and charge degrees of freedom. A manifestation of spin charge separation is that the 2-point function is factorized in the product of two functions, similar to Schwinger functions of particles with different velocities. In this sense, the above theorem says that the spin-charge separation occurs approximately also in the Hubbard model, but is valid only at large distances and up to logarithmic corrections.

If  $\mathbf{k} \neq 0$ , the Fourier transform  $\widehat{S}_2(\mathbf{k} + \omega \mathbf{p}_F)$  of the two-point Schwinger function  $S_2(\mathbf{x})$  in the Hubbard model can be written as a tree expansion, in a way similar to eq. (2.64) of [9], whom we shall refer to for the notation:

$$\widehat{S}_2(\mathbf{k} + \mathbf{p}_F^\omega) = \sum_{n=0}^{\infty} \sum_{j_0=-\infty}^0 \sum_{\tau \in \mathcal{T}_{j_0, n, 2, 0}} \sum_{\substack{\mathbf{P} \in \mathcal{P} \\ |P_{v_0}|=2}} \widehat{G}_{\tau,\omega}^2(\mathbf{k}) \quad (\text{D.5})$$

where  $\mathbf{p}_F^\omega = (\omega p_F, 0)$ ,  $\omega = \pm$ . Here  $\widehat{G}_{\tau,\omega}^2(\mathbf{k})$  represents the contribution of a single tree  $\tau$  with  $n$  endpoints and root of scale  $j_0$ ; if  $|\mathbf{k}| \in [\gamma^{h_{\mathbf{k}}}, \gamma^{h_{\mathbf{k}}+1})$ , it obeys the bound:

$$|\widehat{G}_{\tau,\omega}^2(\mathbf{k})| \leq C \gamma^{-(h_{\mathbf{k}}-j_0)} \frac{\gamma^{-h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \sum_{n=0}^{\infty} \sum_{j_0=-\infty}^0 \sum_{\tau \in \mathcal{T}_{j_0,n,\mathbf{k}}} \sum_{\substack{\mathbf{P} \in \mathcal{P} \\ |P_{v_0}|=2}} (C|\lambda|)^n \prod_{v \text{ not e.p.}} \gamma^{-d_v} \frac{Z_{h_v}}{Z_{h_v-1}}, \quad (\text{D.6})$$

where  $\mathcal{T}_{j_0,n,\mathbf{k}}$  denotes the family of trees whose special vertices (those associated with the external lines) have scale  $h_{\mathbf{k}}$  or  $h_{\mathbf{k}+1}$ . Moreover,  $d_v > 0$ , except for the vertices belonging to the path connecting the root with  $v^*$ , the higher vertex (of scale  $h^*$ ) preceding both the two special endpoints, where  $d_v$  can be equal to 0. These vertices can be regularized by using a factor  $\gamma^{-(h^*-j_0)}$ , extracted from the factor  $\gamma^{-(h_{\mathbf{k}}-j_0)}$ , so that we can safely perform the sum over all the trees with a fixed value of  $h^*$  and we get

$$|\widehat{S}_2(\mathbf{k} + \mathbf{p}_F^\omega)| \leq C \frac{\gamma^{-h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \sum_{h^*=-\infty}^{h_{\mathbf{k}}} \gamma^{-(h_{\mathbf{k}}-h^*)} \leq C \frac{\gamma^{-h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \quad (\text{D.7})$$

A similar bounds can be obtained for the effective model with  $g_{1\perp} = 0$  and couplings chosen as in Lemma 6.1. We shall call  $\widehat{S}_\omega^M(\mathbf{k})$  and  $\widetilde{Z}_h$  the two-point function Fourier transform and the renormalization constants, respectively, in this model.

Let us put

$$\widehat{S}_2(\mathbf{k} + \mathbf{p}_F^\omega) = \frac{1}{Z_{h_{\mathbf{k}}}} \overline{G}_\omega^2(\mathbf{k}) \quad , \quad \widehat{S}_\omega^M(\mathbf{k}) = \frac{1}{\widetilde{Z}_{h_{\mathbf{k}}}} \overline{G}_\omega^{2,M}(\mathbf{k}) \quad (\text{D.8})$$

We can write

$$\widehat{S}_2(\mathbf{k} + \mathbf{p}_F^\omega) = \frac{\widetilde{Z}_{h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \frac{\overline{G}_\omega^2(\mathbf{k})}{\widetilde{Z}_{h_{\mathbf{k}}}} = \frac{\widetilde{Z}_{h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \widehat{S}_\omega^M(\mathbf{k}) + \frac{1}{Z_{h_{\mathbf{k}}}} [\overline{G}_\omega^2(\mathbf{k}) - \overline{G}_\omega^{2,M}(\mathbf{k})] \quad (\text{D.9})$$

Note now that  $\overline{G}_\omega^2(\mathbf{k})$  differs from  $\overline{G}_\omega^{2,M}(\mathbf{k})$  for three reasons:

- 1) the propagators are different, which produces a difference exponentially small thanks to the bounds (2.102) and (2.103) of [1] and the short memory property;
- 2) the r.c.c.  $v_h$  and  $\widetilde{v}_h$  are different, which produces a difference of order  $\widetilde{g}_{1,h_{\mathbf{k}}}$ , thanks to (6.5) and the short memory property;
- 3) in the tree expansion of  $\overline{G}_\omega^2(\mathbf{k})$  and of the ratios  $Z_j/Z_{j-1}$  there are trees with endpoints of type  $g_1$ , not present in the tree expansion of  $\overline{G}_\omega^{2,M}(\mathbf{k})$  and  $\widetilde{Z}_j/\widetilde{Z}_{j-1}$ ; this fact produces again a difference of order  $\widetilde{g}_{1,h_{\mathbf{k}}}$ .

These remarks, together with the fact that there is no tree with only one endpoint in the tree expansion, implies that

$$\left| \frac{1}{Z_{h_{\mathbf{k}}}} [\overline{G}_\omega^2(\mathbf{k}) - \overline{G}_\omega^{2,M}(\mathbf{k})] \right| \leq C |\lambda \widetilde{g}_{1,h_{\mathbf{k}}}| \frac{\gamma^{-h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \quad (\text{D.10})$$

For similar reason, we have

$$\frac{\widetilde{Z}_{h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} = \prod_{j=h_{\mathbf{k}}}^0 \frac{\widetilde{Z}_{h_j}}{Z_{h_{j-1}}} \frac{\widetilde{Z}_{h_0}}{Z_{h_0}} = [1 + O(\lambda^2)] e^{O(\lambda) \sum_{j=h_{\mathbf{k}}}^0 \widetilde{g}_{1,j}} = [1 + O(\lambda)] L(|\mathbf{k}|^{-1})^{O(\lambda)} \quad (\text{D.11})$$

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