# Quasi-periodic motions in strongly dissipative forced systems

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**Abstract**

We consider a class of ordinary differential equations describing one-dimensional systems with a quasi-periodic forcing term and in the presence of large damping. We discuss the conditions to be assumed on the mechanical force and the forcing term for the existence of quasi-periodic solutions which have the same frequency vector as the forcing.

## 1 Introduction

In this paper we study the same problem considered in [7, 8], that is the existence of quasi-periodic motions in strongly dissipative forced systems, with the aim of removing as far as possible the non-degeneracy condition on the mechanical force and the forcing.

We consider one-dimensional systems with a quasi-periodic forcing term in the presence of strong damping, described by ordinary differential equations of the form

$$\varepsilon\ddot{x} + \dot{x} + \varepsilon g(x) = \varepsilon f(\boldsymbol{\omega} t), \tag{1.1}$$

where $g(x)$ is the *mechanical force*, $f(\boldsymbol{\omega} t)$ is the *forcing term*, $\boldsymbol{\omega} \in \mathbb{R}^d$ is the *frequency vector* of the forcing, and $\gamma = 1/\varepsilon > 0$ is the *damping coefficient*. Systems of the form (1.1) naturally arise in classical mechanics and electronic engineering; we refer to [2, 7] for physical motivations. A classical question in the case of forced systems asks for *response solutions*, that is solutions which are quasi-periodic with the same frequency vector as the forcing. Note that in (1.1) the forcing is not assumed to be small, as usually done [16] (see also [3] for a review of recent developments): it is the inverse of the damping coefficient which plays the role of the perturbation parameter.

Both functions $g$ and $f$ will be assumed to be analytic in their arguments, with $f$ quasi-periodic, i.e.

$$f(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} f_{\boldsymbol{\nu}}, \qquad \boldsymbol{\psi} \in \mathbb{T}^d, \tag{1.2}$$

with average $\langle f \rangle = f_{\mathbf{0}}$, and $\cdot$ denoting the scalar product in $\mathbb{R}^d$. By the analyticity assumption on $f$, one has $|f_{\boldsymbol{\nu}}| \le \Phi \mathrm{e}^{-\xi|\boldsymbol{\nu}|}$ for suitable positive constants $\Phi$ and $\xi$.

A Diophantine condition is assumed on $\boldsymbol{\omega}$. Define the *Bryuno function* [4]

$$\mathfrak{B}(\boldsymbol{\omega}) = \sum_{n=0}^{\infty} \frac{1}{2^n} \log \frac{1}{\alpha_n(\boldsymbol{\omega})}, \qquad \alpha_n(\boldsymbol{\omega}) = \inf\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : \boldsymbol{\nu} \in \mathbb{Z}^d \text{ such that } 0 < |\boldsymbol{\nu}| \le 2^n\}. \tag{1.3}$$

**Assumption 1** *The frequency vector $\boldsymbol{\omega}$ satisfies the Bryuno condition $\mathfrak{B}(\boldsymbol{\omega}) < \infty$.*

Note that if $\boldsymbol{\omega} \in \mathbb{R}^d$ satisfies the standard Diophantine condition $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq \gamma_0 |\boldsymbol{\nu}|^{-\tau}$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d_*$, where $|\boldsymbol{\nu}| := |\boldsymbol{\nu}|_1 \equiv |\nu_1| + \ldots + |\nu_d|$ and $\mathbb{Z}^d_* := \mathbb{Z}^d \setminus \{\mathbf{0}\}$, and for some positive constants $\gamma_0$ and $\tau$, then it also satisfies (1.3). Recently, the Bryuno condition has received a lot of attention in the theory of small divisor problems; see for instance [13, 10, 11, 12, 15] and papers cited therein.

The following assumption will be made on the functions $g$ and $f$.

**Assumption 2** *There exists $c_0 \in \mathbb{R}$ such that $x = c_0$ is a zero of odd order $\mathfrak{n}$ of the equation*

$$g(x) - f_{\mathbf{0}} = 0, \tag{1.4}$$

*that is $\mathrm{d}^{\mathfrak{n}} g / \mathrm{d}x^{\mathfrak{n}}(c_0) \neq 0$ and, if $\mathfrak{n} > 1$, $\mathrm{d}^k g / \mathrm{d}x^k (c_0) = 0$ for $k = 1, \ldots, \mathfrak{n} - 1$.*

Of course, for given force $g(x)$, one can read Assumption 2 as a condition on the forcing term.

In [7, 8] we considered Assumption 2 with $\mathfrak{n} = 1$, and, in that case, we proved that for $\varepsilon > 0$ small enough there exists a quasi-periodic solution with frequency vector $\boldsymbol{\omega}$, reducing to $c_0$ as $\varepsilon$ tends to 0, and that such a solution is analytic in a circle tangent at the origin to the vertical axis.

In this paper we show that the same result of existence extends under the weaker Assumption 2. We also show that in the case of even $\mathfrak{n}$ a quasi-periodic solution oscillating around $x = c_0$ fails to exist. More formal statements are given in Section 2.

The paper is organised as follows. In Section 2 we split the problem into two equations, which, using standard terminology, will be called the *range equation* and the *bifurcation equation*. The first one involves small denominator problems, and will be solved iteratively in Section 3 by using techniques of multiscale analysis [9, 10, 11]; from a technical point of view this is the core of the paper. The second one is an implicit function equation, and will be discussed in Section 4. In Section 5 we show that in the case of zeroes of even order for the equation (1.4), a quasi-periodic solution of the form $x(t) = c_0 + O(\varepsilon)$ does not exist. Finally in Section 6 we draw some conclusions and remarks. The paper is fully self-contained, and no acquaintance with previous works is required.

## 2 Setting the problem

We are interested in the existence of a quasi-periodic solution with frequency vector $\boldsymbol{\omega}$, hence we expand $x$ as

$$x(t) = c + X(\boldsymbol{\omega} t), \qquad X(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i} \boldsymbol{\nu} \cdot \boldsymbol{\psi}} X_{\nu}, \tag{2.1}$$

where $c = x_{\mathbf{0}}$ is the average of $x$ (hence $X$ is a zero-average function). Thus, we can rewrite (1.1) in Fourier space as

$$\begin{cases} (\mathrm{i}\boldsymbol{\omega} \cdot \boldsymbol{\nu})\,(1 + \mathrm{i}\varepsilon \boldsymbol{\omega} \cdot \boldsymbol{\nu})\, X_{\boldsymbol{\nu}} + \varepsilon [g \circ (c + X)]_{\boldsymbol{\nu}} = \varepsilon f_{\boldsymbol{\nu}}, & \boldsymbol{\nu} \neq \mathbf{0}, \\ [g \circ (c + X)]_{\mathbf{0}} = f_{\mathbf{0}}, & \boldsymbol{\nu} = \mathbf{0}, \end{cases} \tag{2.2}$$

where $[F]_{\boldsymbol{\nu}}$ denotes the $\boldsymbol{\nu}$-th Fourier coefficient of the function $F$.

We shall adopt the following strategy. We shall look for a solution $c + X$ of the range equation, i.e. the first equation in (2.2), with $c$ arbitrary, and thereafter we shall fix $c$ in such a way that the bifurcation equation, i.e. the second equation in (2.2), be satisfied. This suggests us to consider, besides the equations (2.2), also the equation

$$(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\,(1+\mathrm{i}\varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\nu})\,X_{\boldsymbol{\nu}} + \varepsilon[g\circ(c+X)]_{\boldsymbol{\nu}} = \varepsilon f_{\boldsymbol{\nu}}, \qquad \boldsymbol{\nu}\neq\mathbf{0}, \tag{2.3}$$

and we shall look for a solution of the form (2.1) to (2.3). In Section 3 we shall prove that, for any $c\in\mathbb{R}$ close enough to $c_0$ and all $\varepsilon$ small enough, there exist such a solution $x = c + X(\boldsymbol{\omega}t;\varepsilon,c)$. Then in Section 4 we shall study the bifurcation equation

$$[g(c+X(\cdot;\varepsilon,c)]_{\mathbf{0}} = f_{\mathbf{0}}, \tag{2.4}$$

and we shall see that for $\varepsilon$ small enough there exists a solution $c$ to (2.4), tending to $c_0$ as $\varepsilon$ tends to 0. More precisely we shall prove the following result.

**Theorem 2.1** *Under the Assumptions 1 and 2 for the ordinary differential equation (1.1), for all $\varepsilon$ small enough there exist a continuous function $c(\varepsilon)$ and a response solution $x(t) = c(\varepsilon) + X(\boldsymbol{\omega}t;\varepsilon,c(\varepsilon))$ to (1.1), with $c(0) = c_0$ and the function $X(\boldsymbol{\psi};\eta,c)$ which is $C^\infty$ in $\eta$ and $c$, vanishing at $\eta = 0$, and $2\pi$-periodic, analytic and zero-average in $\boldsymbol{\psi}$.*

If $c_0$ is not a zero of the equation (1.4), obviously there is no quasi-periodic solution to (1.1) reducing to $c_0$ as $\varepsilon$ tends to 0. We shall show that the same non-existence result holds if $c_0$ is a zero of even order of (1.4). Therefore, the following result strengthens Theorem 2.1.

**Theorem 2.2** *Under Assumption 1 for the ordinary differential equation (1.1), there exists a quasi-periodic solution of the form as in Theorem 2.1 if and only if Assumption 2 is satisfied. In particular if $c_0$ is a zero of even order of (1.4) such a solution does not exists.*

# 3 The small denominator equation and multiscale analysis

A graph is a connected set of points and lines. A *tree* $\theta$ is a graph with no cycle, such that all the lines are oriented toward a unique point (*root*) which has only one incident line (root line). All the points in a tree except the root are called *nodes*. The orientation of the lines in a tree induces a partial ordering relation ($\preceq$) between the nodes. Given two nodes $v$ and $w$, we shall write $w \prec v$ every time $v$ is along the path (of lines) which connects $w$ to the root.

We call $E(\theta)$ the set of *end nodes* in $\theta$, that is the nodes which have no entering line, and $V(\theta)$ the set of *internal nodes* in $\theta$, that is the set of nodes which have at least one entering line. Set $N(\theta) = E(\theta)\amalg V(\theta)$. With each end node $v$ we associate a *mode* label $\boldsymbol{\nu}_v \in \mathbb{Z}^d_*$. For all $v\in N(\theta)$ denote with $s_v$ the number of lines entering the node $v$.

We denote with $L(\theta)$ the set of lines in $\theta$. Since a line $\ell$ is uniquely identified with the node $v$ which it leaves, we may write $\ell = \ell_v$. With each line $\ell$ we associate a *momentum* label $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d_*$ and a *scale* label $n_\ell \in \mathbb{Z}_+$.

The modes of the end nodes and and the momenta of the lines are related as follows: if $\ell = \ell_v$ one has

$$\boldsymbol{\nu}_\ell = \sum_{w\in E(\theta):w\preceq v} \boldsymbol{\nu}_w. \tag{3.1}$$

If $v$ is an internal node then (3.1) gives $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}_{\ell_1} + \ldots + \boldsymbol{\nu}_{\ell_{s_v}}$, where $\ell_1, \ldots, \ell_{s_v}$ are the lines entering $v$.

We call *equivalent* two trees which can be transformed into each other by continuously deforming the lines in such a way that they do not cross each other. Let $\mathcal{T}_{k,\boldsymbol{\nu}}$ be the set of inequivalent trees of order $k$ and total momentum $\boldsymbol{\nu}$, that is the set of inequivalent trees $\theta$ such that $|N(\theta)| = |V(\theta)| + |E(\theta)| = k$ and the momentum of the root line is $\boldsymbol{\nu}$.

A cluster $T$ on scale $n$ is a maximal set of nodes and lines connecting them such that all the lines have scales $n' \le n$ and there is at least one line with scale $n$. The lines entering the cluster $T$ and the possible line coming out from it (unique if existing at all) are called the external lines of the cluster $T$. Given a cluster $T$ on scale $n$, we shall denote by $n_T = n$ the scale of the cluster. We call $V(T)$, $E(T)$, and $L(T)$ the set of internal nodes, of end nodes, and of lines of $T$, respectively; note that the external lines of $T$ do not belong to $L(T)$.

We call self-energy cluster any cluster $T$ such that $T$ has only one entering line $\ell_T^2$ and one exiting line $\ell_T^1$, and one has $\sum_{v\in E(T)} \boldsymbol{\nu}_v = \mathbf{0}$ (and hence $\boldsymbol{\nu}_{\ell_T^1} = \boldsymbol{\nu}_{\ell_T^2}$). Call $\mathcal{P}_T$ the path of lines $\ell \in L(T)$ connecting $\ell_T^2$ to $\ell_T^1$, and set $x_T = \boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_T^1} = \boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_T^2}$. Let $\mathfrak{T}_{k,\boldsymbol{\nu}}$ be the set of *renormalised trees* in $\mathcal{T}_{k,\boldsymbol{\nu}}$, i.e. of trees in $\mathcal{T}_{k,\boldsymbol{\nu}}$ which do not contain any self-energy clusters.

If we write

$$g(x) = \sum_{s=0}^{\infty} g_s(c)(x-c)^s, \qquad g_s(c) := \frac{1}{s!}\frac{\mathrm{d}^s}{\mathrm{d}x^s} g(c), \tag{3.2}$$

then we can choose $r > 0$ such that $|g_s(c)| \le \Gamma^s$ for all $c \in B_r(c_0)$, with the constant $\Gamma$ independent of $c$.

Let $\psi$ be a non-decreasing $C^\infty$ function defined in $\mathbb{R}_+$, such that

$$\psi(u) = \begin{cases} 1, & \text{for } u \ge 1, \\ 0, & \text{for } u \le 1/2, \end{cases} \tag{3.3}$$

and set $\chi(u) := 1 - \psi(u)$. For all $n \in \mathbb{Z}_+$ define $\chi_n(u) := \chi(u/4\alpha_n(\boldsymbol{\omega}))$ and $\psi_n(u) := \psi(u/4\alpha_n(\boldsymbol{\omega}))$, and set

$$\Xi_n(x) = \chi_0(|x|)\ldots\chi_{n-1}(|x|)\chi_n(|x|), \qquad \Psi_n(x) = \chi_0(|x|)\ldots\chi_{n-1}(|x|)\psi_n(|x|). \tag{3.4}$$

We associate with each node $v$ a *node factor*

$$F_v = \begin{cases} -\dfrac{1}{s_v!} g_{s_v}(c), & v \in V(\theta), \\ f_{\boldsymbol{\nu}_v}, & v \in E(\theta), \end{cases} \tag{3.5}$$

and we associate with each line $\ell$ a *propagator*

$$G_\ell = G^{[n_\ell]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell; \varepsilon, c), \tag{3.6}$$

where the functions $G^{[n]}(x;\varepsilon,c)$ are recursively defined for $n \ge 0$ as

$$G^{[n]}(x;\varepsilon,c) = \frac{\Psi_n(x)}{\mathrm{i}x(1+\mathrm{i}\varepsilon x) - \mathcal{M}^{[n-1]}(x;\varepsilon,c)}, \tag{3.7a}$$

$$\mathcal{M}^{[n]}(x;\varepsilon,c) = \mathcal{M}^{[n-1]}(x;\varepsilon,c) + \Xi_n(x) M^{[n]}(x;\varepsilon,c), \quad M^{[n]}(x;\varepsilon,c) = \sum_{T\in\mathfrak{R}_n} \mathrm{Val}(T,x;\varepsilon,c), \tag{3.7b}$$

where $\mathcal{M}^{[-1]}(x;\varepsilon,c)=\varepsilon g_1(c)$, $\mathfrak{R}_n$ is the set of renormalised self-energy clusters, i.e. of self-energy clusters which do not contain any further self-energy clusters, on scale $n$, and

$$\mathrm{Val}(T,x;\varepsilon,c)=\Big(\prod_{\ell\in L(T)} G_\ell\Big)\Big(\prod_{v\in N(T)} F_v\Big) \tag{3.8}$$

is called the value of the self-energy cluster $T$. Note that $\mathcal{M}^{[-1]}(x;\varepsilon,c)=0$ for $\mathfrak{n}>1$ in Assumption 2.

Set

$$X_{\boldsymbol{\nu}}^{[k]}=\sum_{\theta\in\mathfrak{T}_{k,\nu}}\mathrm{Val}(\theta;\varepsilon,c),\qquad \mathrm{Val}(\theta;\varepsilon,c)=\Big(\prod_{\ell\in L(\theta)} G_\ell\Big)\Big(\prod_{v\in N(\theta)} F_v\Big), \tag{3.9}$$

where $\mathrm{Val}(\theta;\varepsilon,c)$ is called the value of the tree $\theta$, and define the *renormalised series*

$$\overline{X}(\boldsymbol{\psi};\varepsilon,c)=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\overline{X}_{\boldsymbol{\nu}},\qquad \overline{X}_{\boldsymbol{\nu}}=\sum_{k=1}^{\infty}\varepsilon^k X_{\boldsymbol{\nu}}^{[k]}. \tag{3.10}$$

Set also

$$M(\theta)=\sum_{v\in E(\theta)}|\boldsymbol{\nu}_v|,\qquad M(T)=\sum_{v\in E(T)}|\boldsymbol{\nu}_v|, \tag{3.11}$$

and call $\mathfrak{N}_n(\theta)$ the number of lines $\ell\in L(\theta)$ such that $n_\ell\geq n$, and $\mathfrak{N}_n(T)$ the number of lines $\ell\in L(T)$ such that $n_\ell\geq n$.

Finally define

$$n(\boldsymbol{\nu})=\inf\left\{n\in\mathbb{Z}_+ : |\boldsymbol{\nu}|\leq 2^n\right\}. \tag{3.12}$$

Note that $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|\geq\alpha_{n(\boldsymbol{\nu})}(\boldsymbol{\omega})$, and $\alpha_{n'}(\boldsymbol{\omega})<\alpha_n(\boldsymbol{\omega})$ implies $n'>n$.

**Lemma 3.1** *For any renormalised tree $\theta$, one has $\mathfrak{N}_n(\theta)\leq 2^{-(n-2)}M(\theta)$.*

*Proof.* We prove that $\mathfrak{N}_n(\theta)\leq\max\{0,2^{-(n-2)}M(\theta)-1\}$ by induction on the number of nodes of $\theta$. If $N(\theta)=1$ and $\mathfrak{N}_n(\theta)=1$, then $\theta$ has only one line $\ell$ and $n_\ell\geq n$. Thus, $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|\leq\alpha_{n-1}(\boldsymbol{\omega})/4$, so that $n(\boldsymbol{\nu}_\ell)\geq n$, and hence $|\boldsymbol{\nu}_\ell|>2^{n-1}$, which implies $2^{-(n-2)}M(\theta)=2^{-(n-2)}|\boldsymbol{\nu}_\ell|\geq 2$.

If $N(\theta)>1$, let $\ell_0$ be the root line of $\theta$ and set $\boldsymbol{\nu}=\boldsymbol{\nu}_{\ell_0}$. If $n_{\ell_0}<n$ the assertion follows from the inductive hypothesis. If $n_{\ell_0}\geq n$, call $\ell_1,\ldots,\ell_m$ the lines with scale $\geq n$ which are closest to $\ell_0$. The case $m=0$ is trivial. If $m\geq 2$ the bound follows once more from the inductive hypothesis. Finally, if $m=1$, then $\ell_1$ is the entering line of a cluster $T$ and $\boldsymbol{\nu}'\neq\boldsymbol{\nu}$, where $\boldsymbol{\nu}'=\boldsymbol{\nu}_{\ell_1}$. Then $|\boldsymbol{\omega}\cdot(\boldsymbol{\nu}-\boldsymbol{\nu}')|\leq\alpha_{n-1}(\boldsymbol{\omega})/2$, so that $n(\boldsymbol{\nu}-\boldsymbol{\nu}')\geq n-1$, and hence $M(T)\geq|\boldsymbol{\nu}-\boldsymbol{\nu}'|>2^{n-2}$. Therefore, if $\theta_1$ is the tree with root line $\ell_1$, one has $M(\theta)=M(T)+M(\theta_1)$ and hence

$$\mathfrak{N}_n(\theta)=1+\mathfrak{N}_n(\theta_1)\leq 2^{-(n-2)}M(\theta_1)\leq 2^{-(n-2)}M(\theta)-2^{-(n-2)}M(T)\leq 2^{-(n-2)}M(\theta)-1.$$

Therefore the assertion follows also in this case. ∎

**Lemma 3.2** *Assume there exists a constant $C_0$ such that $|G^{[n]}(x;\varepsilon,c)|\leq C_0/\alpha_n(\boldsymbol{\omega})$ for all $n\in\mathbb{Z}_+$. Then there exists $\varepsilon_0>0$ such that, for all $c\in B_r(c_0)$ and all $|\varepsilon|<\varepsilon_0$, the series $c+\overline{X}(\boldsymbol{\omega}t;\varepsilon,c)$ converges.*

*Proof.* Set $D_0 = \max\{\Gamma, \Phi\}$. By assumption for all $\theta \in \mathfrak{T}_{\boldsymbol{\nu},k}$ one has

$$
\begin{aligned}
|\mathrm{Val}(\theta;\varepsilon,c)| &\le C_0^k D_0^k \mathrm{e}^{-\xi M(\theta)} \Big( \prod_{\ell \in L(\theta)} \alpha_{n_\ell}^{-1}(\boldsymbol{\omega}) \Big) \le C_0^k D_0^k \mathrm{e}^{-\xi M(\theta)} \alpha_{n_0}^{-k}(\boldsymbol{\omega}) \prod_{n=n_0+1}^{\infty} \mathrm{e}^{\mathfrak{N}_n(\theta) \log 1/\alpha_n(\boldsymbol{\omega})} \\
&\le C_0^k D_0^k \mathrm{e}^{-\xi M(\theta)} \alpha_{n_0}^{-k}(\boldsymbol{\omega}) \exp\left( 4M(\theta) \sum_{n=n_0+1}^{\infty} \frac{1}{2^n} \log \frac{1}{\alpha_n(\boldsymbol{\omega})} \right),
\end{aligned}
$$

for arbitrary $n_0 \in \mathbb{Z}_+$. The last sum converges by Assumption 1, so that one can choose $n_0$ such that

$$
|\mathrm{Val}(\theta;\varepsilon,c)| \le C_0^k D_0^k \alpha_{n_0}^{-k}(\boldsymbol{\omega}) \mathrm{e}^{-\xi' M(\theta)},
$$

with $\xi' = \xi/2$. This is enough to prove the lemma. ∎

**Lemma 3.3** *For any self-energy cluster $T \in \mathfrak{R}_n$ such that $\Xi_n(x_T) \neq 0$, one has $M(T) \ge 2^{n-1}$ and $\mathfrak{N}_p(T) \le 2^{-(p-2)} M(T)$ for all $p \le n$.*

*Proof.* We first prove the bound $M(T) \ge 2^{n-1}$ for $T \in \mathfrak{R}_n$ such that $\Xi_n(x_T) \neq 0$. By construction any $T \in \mathfrak{R}_n$ has at least one line $\ell$ with scale $n_\ell = n$. If $\ell \notin \mathcal{P}_T$ then $\ell$ is the root line of a tree $\theta$ such that $\mathfrak{N}_n(\theta) \le 2^{-(n-2)} M(\theta)$ by Lemma 3.1, so that $1 \le \mathfrak{N}_n(\theta) \le 2^{-(n-2)} M(T)$, which yields the bound. If all lines with scale $n$ are along $\mathcal{P}_T$ then call $\ell$ that which is closest to $\ell_T^2$: by construction $\ell_T^2$ and $\ell$ are the entering line and the exiting line, respectively, of a cluster $T' \subset T$, and $|\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_T^2}| \le M(T')$. Moreover one has $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|, |\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell_T^2}| \le \alpha_{n-1}(\boldsymbol{\omega})/4$, hence $M(T) \ge M(T') \ge |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_T^2}| \ge 2^{n-1}$.

Now we prove that for $T \in \mathfrak{R}_n$ such that $\Xi_n(x_T) \neq 0$ one has $\mathfrak{N}_p(T) \le 2^{-(p-2)} M(T)$ for all $p \le n$. More generally we prove the bound for the elements of a wider class of graphs. We say that a subset $\widetilde{T}$ of a tree belongs to the class $\mathfrak{S}_{n,p}$ if $\widetilde{T}$ has one exiting line $\ell_{\widetilde{T}}^1$ and one entering line $\ell_{\widetilde{T}}^2$, both with scale $\ge p$, and all lines $\ell$ in $\widetilde{T}'$ have scale $n_\ell \le n$. Then we prove the bound $\mathfrak{N}_p(\widetilde{T}) \le 2^{-(p-2)} M(\widetilde{T})$ for all elements $\widetilde{T}$ of the class $\mathfrak{S}_{n,p}$. The proof is by induction on the number of nodes. Given a subset $\widetilde{T}$, let $\ell_1, \ldots, \ell_m$ the lines on scale $\ge p$ closest to $\ell_{\widetilde{T}}^1$. If $m = 0$ then the bound follows easily. Also the case in which all lines do not belong to the path $\mathcal{P}_{\widetilde{T}}$ can be easily discussed by relying on Lemma 3.1. If at least one line, say $\ell_1$, is along the path $\mathcal{P}_{\widetilde{T}}$, then one has

$$
\mathfrak{N}_p(\widetilde{T}) \le 1 + \mathfrak{N}_p(\widetilde{T}') + \mathfrak{N}_p(\theta_2) + \ldots + \mathfrak{N}_p(\theta_m),
$$

where $\theta_i$, $i = 2, \ldots, m$, is the tree with root line $\ell_i$, while $\widetilde{T}'$ is a subset with the same properties as $\widetilde{T}$, i.e. inside the the same class $\mathfrak{S}_{n,p}$, but with $N(\widetilde{T}') < N(\widetilde{T})$. Hence, by the inductive hypothesis, one has $\mathfrak{N}_p(\widetilde{T}') \le 2^{-(p-2)} M(\widetilde{T}')$. Then the assertion follows once more. To conclude the proof simply note that if $T \in \mathfrak{R}_n$ then $T \in \mathfrak{S}_{n,p}$ for all $p \le n$. ∎

**Lemma 3.4** *Assume the propagators $G^{[p]}(x;\varepsilon,c)$ are differentiable in $x$ and there exist constants $C_0$ and $C_1$ such that $|G^{[p]}(x;\varepsilon,c)| \le C_0/\alpha_p(\boldsymbol{\omega})$ and $|\partial_x G^{[p]}(x;\varepsilon,c)| \le C_1/\alpha_p^3(\boldsymbol{\omega})$ for all $p < n$. Then there exists $\varepsilon_0 > 0$ such that, for all $c \in B_r(c_0)$ and all $|\varepsilon| < \varepsilon_0$, the function $x \mapsto M^{[n]}(x;\varepsilon,c)$ is differentiable, and one has*

$$
\left| M^{[n]}(x;\varepsilon,c) \right|, \left| \partial_x M^{[n]}(x;\varepsilon,c) \right| \le D_1 |\varepsilon|^2 \mathrm{e}^{-D_2 2^n},
$$

*for some positive constants $D_1$ and $D_2$.*

*Proof.* By proceeding as in the proof of Lemma 3.2, one finds

$$|\mathrm{Val}(T,x;\varepsilon,c)| \leq C_0^k D_0^k \mathrm{e}^{-\xi M(T)} \alpha_{n_0}^{-k}(\boldsymbol{\omega}) \exp\left(4M(T)\sum_{n=n_0+1}^{\infty}\frac{1}{2^n}\log\frac{1}{\alpha_n(\boldsymbol{\omega})}\right),$$

with $n_0$ chosen as in the proof of Lemma 3.2. Then one can use Lemma 3.3 to bound $M(T)$, and the observation that any self-energy cluster $T$ has at least two nodes to obtain the factor $\varepsilon^2$. This proves the bound on $M^{[n]}(x;\varepsilon,c)$.

To obtain the bound on $\partial_x M^{[n]}(x;\varepsilon,c)$ simply note that

$$\partial_x M^{[n]}(x;\varepsilon,c) = \sum_{T\in\mathfrak{R}_n}\Big(\prod_{v\in E(T)\cup V(T)} F_v\Big)\sum_{\ell\in\mathcal{P}_T}\partial_x G_\ell\Big(\prod_{\ell'\in L(\theta)\setminus\{\ell\}} G_{\ell'}\Big),$$

where $\partial_x G_\ell$ can be bounded as $|\partial_x G_\ell| \leq C_1/\alpha_{n_\ell}^3(\boldsymbol{\omega})$ by hypothesis. ∎

**Lemma 3.5** *Assume there exists a constant $C_0$ such that $|G^{[p]}(x;\varepsilon,c)| \leq C_0/\alpha_p(\boldsymbol{\omega})$ for all $p<n$. Then one has $(\mathcal{M}^{[p]}(x;\varepsilon,c))^* = \mathcal{M}^{[p]}(-x;\varepsilon,c)$ for all $p\leq n$.*

*Proof.* The proof is by induction on $p$. First of all note that if $(\mathcal{M}^{[p]}(x;\varepsilon,c))^* = \mathcal{M}^{[p]}(-x;\varepsilon,c)$ then $(G^{[p]}(x;\varepsilon,c))^* = G^{[p]}(-x;\varepsilon,c)$ by (3.7a). Moreover one has $F_v^* = F_v$ for all internal nodes $v\in V(T)$ and $F_v^* = f_{\boldsymbol{\nu}_v}^* = f_{-\boldsymbol{\nu}_v}$ for all end nodes $v\in E(T)$.

Let $T$ a self-energy cluster contributing to $\mathcal{M}^{[p]}(x;\varepsilon,c)$ – see (3.7b) – for $p\leq n$; then $T\in\mathfrak{R}_q$ for some $q\leq p$. Together with $T\in\mathfrak{R}_q$ consider also the self-energy cluster $T'\in\mathfrak{R}_q$ obtained from $T$ by changing the signs of the mode labels of all the end nodes $v\in E(T)$. Note that there is a one-to-one correspondence between the self-energy clusters $T$ and $T'$. The node factors corresponding to the end nodes $v\in E(T')$ become $f_{-\boldsymbol{\nu}_v}$, and, if we revert the momentum of the entering line $\ell_{T'}^2$, the momenta of all the lines $\ell\in L(T')$ also change sign, that is $\boldsymbol{\nu}_\ell$ is replaced with $-\boldsymbol{\nu}_\ell$ for all $\ell\in L(T')$.

The definition (3.8) and the inductive hypothesis yield $(\mathrm{Val}(T,x;\varepsilon,c))^* = \mathrm{Val}(T',-x;\varepsilon,c)$ for all $q\leq p$ and all $T\in\mathfrak{R}_q$. Then (3.7b) implies the assertion. ∎

**Lemma 3.6** *For all $n\in\mathbb{Z}_+$ the function $x\mapsto\mathcal{M}^{[n]}(x;\varepsilon,c))$ is differentiable and one has $|\mathrm{i}x(1+\mathrm{i}\varepsilon x)-\mathcal{M}^{[n]}(x;\varepsilon,c)|\geq|x|/2$ for all $c\in B_r(c_0)$ and all $\varepsilon$ small enough.*

*Proof.* The proof is by induction on $n$. Assume that the functions $x\mapsto\mathcal{M}^{[p]}(x;\varepsilon,c)$ are differentiable and one has $|\mathrm{i}x(1+\mathrm{i}\varepsilon x)-\mathcal{M}^{[p]}(x;\varepsilon,c)|\geq|x|/2$ for all $p<n$. One can easily verify that then also the propagators $G^{[p]}(x;\varepsilon,c)$ are differentiable and satisfy the bounds $|\partial_x G^{[p]}(x;\varepsilon,c)|\leq C_1/\alpha_p^3(\boldsymbol{\omega})$ for all $p\leq n$ and for some positive constant $C_1$. Indeed one has

$$\partial_x G^{[p]}(x;\varepsilon,c) = \frac{\partial_x\Psi_p(x)}{\mathrm{i}x(1+\mathrm{i}\varepsilon x)-\mathcal{M}^{[p-1]}(x;\varepsilon,c)} - \frac{\Psi_p(x)\left(\mathrm{i}-2\varepsilon x-\partial_x\mathcal{M}^{[p-1]}(x;\varepsilon,c)\right)}{(\mathrm{i}x(1+\mathrm{i}\varepsilon x)-\mathcal{M}^{[p-1]}(x;\varepsilon,c))^2},$$

where

$$\begin{aligned}\partial_x\Psi_p(x) &= \sum_{j=0}^{p-1}\chi_0(|x|)\dots\partial_x\chi_j(|x|)\dots\psi_p(|x|)+\chi_0(|x|)\dots\chi_{p-1}(|x|)\partial_x\psi_n(|x|)\\ &\leq C\sum_{j=0}^{p}\alpha_j^{-1}(\boldsymbol{\omega})\leq Cp\alpha_p^{-1}(\boldsymbol{\omega})\end{aligned}$$

for some constant $C$, and

$$\begin{aligned}\partial_x \mathcal{M}^{[p-1]}(x;\varepsilon,c) &= \sum_{j=0}^{p-1}\left(\sum_{i=0}^{j}\chi_0(|x|)\dots\partial_x\chi_i(|x|)\dots\chi_j(|x|)M^{[j]}(x;\varepsilon,c)+\Xi_j(x)\partial_x M^{[j]}(x;\varepsilon,c)\right)\\ &\le C|\varepsilon|^2\sum_{j=0}^{p-1}\mathrm{e}^{-D_2 2^j}\left(j\alpha_j^{-1}(\boldsymbol{\omega})+1\right)\le C'|\varepsilon|^2p^2\alpha_p^{-1}(\boldsymbol{\omega}),\end{aligned}$$

for some constants $C, C'$.

Then we can apply Lemma 3.4 to conclude that $\mathcal{M}^{[n]}(x;\varepsilon,c)$ is differentiable and its derivative with respect to $x$ is accordingly bounded. Therefore

$$\mathrm{i}x(1+\mathrm{i}\varepsilon x)-\mathcal{M}^{[n]}(x;\varepsilon,c)=\mathrm{i}x(1+\mathrm{i}\varepsilon x)-\mathcal{M}^{[n]}(0;\varepsilon,c)-\left(\mathcal{M}^{[n]}(x;\varepsilon,c)-\mathcal{M}^{[n]}(0;\varepsilon,c)\right),$$

where $\mathcal{M}^{[n]}(0;\varepsilon,c)$ is real by Lemma 3.6, and

$$\left|\mathcal{M}^{[n]}(x;\varepsilon,c))-\mathcal{M}^{[n]}(0;\varepsilon,c)\right|\le C|\varepsilon|^2|x|,$$

for some constant $C$, by Lemma 3.4. ■

**Lemma 3.7** *Then there exists $\varepsilon_0>0$ such that, for all $c\in B_r(c_0)$ and all $|\varepsilon|<\varepsilon_0$, the function $c+\overline{X}(\boldsymbol{\omega}t;\varepsilon,c)$ solves (2.3).*

*Proof.* We have to prove that the coefficients $\overline{X}_{\boldsymbol{\nu}}$, defined abstractly through (3.10), solve the first equation in (2.2), i.e.

$$(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\,(1+\mathrm{i}\varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\nu})\,\overline{X}_{\boldsymbol{\nu}}+\varepsilon[g\circ(c+\overline{X})]_{\boldsymbol{\nu}}=\varepsilon f_{\boldsymbol{\nu}},\qquad \boldsymbol{\nu}\neq\mathbf{0}.$$

Set $D_n(x;\varepsilon,c)=\mathrm{i}x\,(1+\mathrm{i}\varepsilon x)-\mathcal{M}^{[n]}(x;\varepsilon,c)$, so that $G^{[n]}(x;\varepsilon,c)=\Psi_n(x)/D_n(x;\varepsilon,c)$, and $G(x)=1/(\mathrm{i}x)(1+\mathrm{i}\varepsilon x)$. Write also

$$\overline{X}_{\boldsymbol{\nu}}=\sum_{n=0}^{\infty}\overline{X}_{\boldsymbol{\nu},n},\qquad \overline{X}_{\boldsymbol{\nu},n}=\sum_{k=1}^{\infty}\varepsilon^k\sum_{\theta\in\mathfrak{T}_{k,\boldsymbol{\nu},n}}\mathrm{Val}(\theta;\varepsilon,c),$$

where $\mathfrak{T}_{k,\boldsymbol{\nu},n}$ is the subset of $\mathfrak{T}_{k,\boldsymbol{\nu}}$ of the renormalised trees with root line with scale $n$.

If we define

$$\Omega(\boldsymbol{\nu},\varepsilon,c)=G(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\left[\varepsilon f-\varepsilon g(c+\overline{X}(\cdot;\varepsilon,c)\right]_{\boldsymbol{\nu}},\tag{3.13}$$

then we have to prove that $\Omega(\boldsymbol{\nu},\varepsilon,c)=\overline{X}_{\boldsymbol{\nu}}$.

By setting

$$\Psi_{j,n}(x)=\chi_j(|x|)\dots\chi_{n-1}(|x|)\psi_n(|x|),\quad n>j,\qquad\qquad \Psi_{n,n}(x)=\psi_n(|x|),\tag{3.14}$$

note that

$$\Psi_{0,n}(x)=\Psi_n(x),\qquad\qquad \sum_{n=j}^{\infty}\Psi_{j,n}(x)=1\quad\forall j\ge 0.\tag{3.15}$$

Then, by using the last identity in (3.15) with $j = 0$, we can rewrite (3.13) as

$$\Omega(\boldsymbol{\nu}, \varepsilon, c) = G(\boldsymbol{\omega} \cdot \boldsymbol{\nu}) \sum_{n=0}^{\infty} D_n(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c)\, G^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \left[ \varepsilon f - \varepsilon g(c + \overline{X}(\cdot; \varepsilon, c)) \right]_{\boldsymbol{\nu}}, \qquad (3.16)$$

where we can expand

$$\begin{aligned} G^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \left[ \varepsilon f_{\boldsymbol{\nu}} - \varepsilon g(c + \overline{X}(\cdot; \varepsilon, c)) \right]_{\boldsymbol{\nu}} &= \sum_{k=1}^{\infty} \varepsilon^k \sum_{\theta \in \mathfrak{T}_{k,\boldsymbol{\nu},n}} \mathrm{Val}(\theta; \varepsilon, c) \\ &+ G^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \sum_{p=n}^{\infty} \sum_{j=0}^{n-1} M^{[j]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \sum_{k=1}^{\infty} \varepsilon^k \sum_{\theta \in \mathfrak{T}_{k,\boldsymbol{\nu},p}} \mathrm{Val}(\theta; \varepsilon, c) \\ &+ G^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \sum_{p=0}^{n-1} \sum_{j=0}^{p-1} M^{[j]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \sum_{k=1}^{\infty} \varepsilon^k \sum_{\theta \in \mathfrak{T}_{k,\boldsymbol{\nu},p}} \mathrm{Val}(\theta; \varepsilon, c), \end{aligned}$$

where the sum in the second line is present only if $n \geq 1$ and the sum in the third line is present only if $n \geq 2$. Therefore we obtain

$$\begin{aligned} \Omega(\boldsymbol{\nu}, \varepsilon, c) = G(\boldsymbol{\omega} \cdot \boldsymbol{\nu}) &\sum_{n=0}^{\infty} D_n(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \overline{X}_{\boldsymbol{\nu},n} \\ &+ G(\boldsymbol{\omega} \cdot \boldsymbol{\nu}) \sum_{n=1}^{\infty} \Psi_n(x) \sum_{p=n}^{\infty} \sum_{j=0}^{n-1} M^{[j]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \overline{X}_{\boldsymbol{\nu},p} \\ &+ G(\boldsymbol{\omega} \cdot \boldsymbol{\nu}) \sum_{n=2}^{\infty} \Psi_n(x) \sum_{p=0}^{n-1} \sum_{j=0}^{p-1} M^{[j]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \overline{X}_{\boldsymbol{\nu},p}. \end{aligned}$$

The second and third lines, summed together, give

$$G(\boldsymbol{\omega} \cdot \boldsymbol{\nu}) \sum_{n=1}^{\infty} \overline{X}_{\boldsymbol{\nu},n} \sum_{j=0}^{n-1} M^{[j]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \sum_{p=j+1}^{\infty} \Psi_p(x), \qquad \text{where} \qquad \sum_{p=j+1}^{\infty} \Psi_p(x) = \Xi_j(x),$$

where we have written $\Psi_p = \Xi_j \Psi_{j+1,p}$ and used (3.14) to obtain the last equality, so that (3.16) gives

$$\Omega(\boldsymbol{\nu}, \varepsilon, c) = G(\boldsymbol{\omega} \cdot \boldsymbol{\nu}) \sum_{n=0}^{\infty} \Big( D_n(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) + \mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, c) \Big) \overline{X}_{\boldsymbol{\nu},n} = \sum_{n=0}^{\infty} \overline{X}_{\boldsymbol{\nu},n} = \overline{X}_{\boldsymbol{\nu}},$$

which proves the assertion. ∎

**Lemma 3.8** *The function $\overline{X}(\boldsymbol{\psi}; \varepsilon, c)$ is $C^{\infty}$ in $\varepsilon$ and $c$ for $\varepsilon$ and $c - c_0$ small enough.*

*Proof.* The previous results imply that $\overline{X}(\cdot; \varepsilon, \cdot)$ is a well defined function of $\varepsilon$ for $\varepsilon$ small enough. By looking at the tree expansion (3.9) for the coefficients $\overline{X}_{\boldsymbol{\nu}}^{[k]}$ of $\overline{X}(\boldsymbol{\psi}; \varepsilon, c)$, one sees that the function depends on $\varepsilon$ through the factors $\varepsilon^k$ in (3.10) and through the propagators $G_\ell$. The first dependence is trivial, and poses no obstacle in differentiating. Also the dependence through the propagators can be easily handled thanks to Lemma 3.6, which allows to bound from below the denominators. In particular for all $m \geq 0$ one finds

$$\left|\partial_\varepsilon^m G^{[p]}(x;\varepsilon,c)\right| \leq K_m/\alpha_p^m(\omega)$$

for suitable constants $K_m$. Smoothness in $c$ can be discussed in a similar way, by using analyticity of the force $g$ and again Lemma 3.6. ■

# 4 The implicit function equation

We are left with the implicit function equation (2.4), which can be trivially solved under Assumption 2. If we define

$$\Gamma(\varepsilon,c) = [g(c + X(\cdot;\varepsilon,c)]_{\mathbf{0}} - f_{\mathbf{0}}, \tag{4.1}$$

then the following result holds.

**Lemma 4.1** *There exists a neighbourhood $U \times V$ of $(\varepsilon,c) = (0,c_0)$ such that for all $\varepsilon \in U$ there is at least one value $c = c(\varepsilon) \in V$, depending continuously on $\varepsilon$, for which one has $\Gamma(\varepsilon,c(\varepsilon)) = 0$.*

*Proof.* Since $\Gamma(0,c) = g(c) - f_{\mathbf{0}}$, Assumption 2 implies that

$$\frac{\mathrm{d}^k}{\mathrm{d}c^k}\Gamma(0,c_0) = 0 \quad \text{for} \quad k = 0,1,\ldots,\mathfrak{n}-1 \quad \text{and} \quad \Gamma_0 = \frac{\mathrm{d}^{\mathfrak{n}}}{\mathrm{d}c^{\mathfrak{n}}}\Gamma(0,c_0) \neq 0. \tag{4.2}$$

Set $\sigma_0 = \mathrm{sign}(\Gamma_0)$ so that $\sigma_0\Gamma_0 > 0$. By continuity there are neighbourhoods $U$ and $V = [V_-,V_+]$ of $\varepsilon = 0$ and $c = c_0$, respectively, such that for all $\varepsilon \in U$ one has $\sigma_0\Gamma(\varepsilon,c) > 0$ for $c = V_+$ and $\sigma_0\Gamma(\varepsilon,c) < 0$ for $c = V_-$. Therefore, there exists a continuous curve $c = c(\varepsilon)$ such that $\Gamma(\varepsilon,c(\varepsilon)) = 0$. ■

By collecting together the results of the previous sections and Lemma 4.1, Theorem 2.1 follows.

# 5 Zeroes of even order

In this section we prove the following result, which, together with Theorem 2.1, implies Theorem 2.2.

**Lemma 5.1** *Under Assumption 1 for the ordinary differential equation (1.1), assume also that $c_0$ is a zero of even order of (1.1). Then there is no quasi-periodic solution reducing to $c_0$ when $\varepsilon$ tends to 0.*

*Proof.* The analysis of Section 3 shows that a solution of the range equation (2.3) can be proved to exist under the only Assumption 1. Moreover such a solution is $C^\infty$ in both $\varepsilon$ and $c$ (cf. Lemma 3.8). Then, we study the bifurcation equation (2.4) in the case $c_0$ is a zero of even order of (1.4).

If we write $c = c_0 + \zeta$ and expand the function $g \circ (c + X)$ around $c = c_0$, then (2.4) gives

$$[g(c + X(\cdot;\varepsilon, c)]_{\mathbf{0}} - f_{\mathbf{0}} = g_0\langle(\zeta + X)^{\mathfrak{n}}\rangle + \langle O\,(\zeta + X)^{\mathfrak{n}+1}\rangle = 0, \tag{5.1}$$

where $\mathfrak{n}!g_0 = \mathrm{d}^{\mathfrak{n}}g/\mathrm{d}x^{\mathfrak{n}}(c_0) \neq 0$ and $\langle\cdot\rangle$ denotes as usual the Fourier component with label $\boldsymbol{\nu} = \mathbf{0}$. If $\varepsilon = O(\zeta)$ then we have $|g_0|\langle(\zeta + X)^{\mathfrak{n}}\rangle \geq C_1\varepsilon^{\mathfrak{n}}$ for some positive constant $C_1$, because $\mathfrak{n}$ is even, and $O\,(\zeta + X)^{\mathfrak{n}+1} = O(\varepsilon^{\mathfrak{n}+1})$, so that (5.1) cannot be satisfied for $\varepsilon$ small enough.

If $\varepsilon = o(\zeta)$, then

$$\langle(\zeta + X)^{\mathfrak{n}}\rangle = \sum_{k=0}^{\mathfrak{n}} \binom{\mathfrak{n}}{k} \zeta^k\langle X^{\mathfrak{n}-k}\rangle = \zeta^{\mathfrak{n}} + o(\zeta^{\mathfrak{n}}) \tag{5.2}$$

for $\varepsilon$ small enough. On the other hand $O\,(\zeta + X)^{\mathfrak{n}+1} = O(\zeta^{\mathfrak{n}+1})$, and hence once more there is no solution to (5.1) because of (5.2). The case $\zeta = o(\varepsilon)$ can be discussed in a similar way. ■

# 6 Conclusions and open problems

The analysis of the previous sections shows that under Assumptions 1 and 2 the system described by the ordinary differential equation (1.1) admits a response solution. Under some mild conditions on $g$ one can prove that such a solution describes a (local) attractor [1]. It would be interesting to investigate whether the same result can be obtained by only making Assumption 2 on $g$ and requiring $\mathrm{d}^{\mathfrak{n}}g/\mathrm{d}x^{\mathfrak{n}}(c_0) > 0$. Even more interesting would be to understand whether the same scenario persists after removing Assumption 1 on $\boldsymbol{\omega}$. The analysis of [1] shows that, if there is a quasi-periodic solution of the form considered in Theorem 2.1 exists, then it is an attractor (under some conditions on $g$), but if $\boldsymbol{\omega}$ does not satisfy any Diophantine condition, such as the Bryuno condition, then the small divisor problem can not be handled, and it is very unlikely that the dynamics can be conjugated to the unperturbed one.

The analysis in Section 5 shows that, if $c_0$ is a zero of even order $\mathfrak{n}$ for the equation (1.4), then no quasi-periodic solution of the form considered in Theorem 2.1 exists. A natural question in that case is, how the dynamics evolves in time, and what kind of attractors arise.

Furthermore, Theorem 2.1 states that for all $\varepsilon$ small enough there is a value $c(\varepsilon)$ for the average of $x(t)$, such that the solution exists, but provides nothing more than continuity about the dependence of $c(\varepsilon)$ on $\varepsilon$. Thus, another question which should deserve further investigation is, if under some further assumption one can prove some stronger regularity property for the function $c(\varepsilon)$ – note that analyticity fails to hold even in the case of periodic forcings [7]. In this direction, the results of [5] could provide a possible guideline (even if in this case the implicit function equation to be studied is no longer analytic), not only to prove smoothness but also to provide an algorithm to explicitly construct the function $c(\varepsilon)$. Of course, under the Assumption 2 on $\boldsymbol{\omega}$, independently of the conditions on $g$, we have no hope to prove Borel summability [14] in $\varepsilon$ at the origin. Indeed, this should require a much stronger Diophantine condition on $\boldsymbol{\omega}$ [6, 8].

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