Stability theory and KAM

Guido Gentile

Dipartimento di Matematica, Università di Roma Tre, Roma, I-00146

Introduction

A Hamiltonian system is a dynamical system whose equations of motions can be written in terms of a scalar function, called the Hamiltonian of the system: if one uses coordinates $(\mathbf{p}, \mathbf{q})$ in a domain (phase-space) $\mathcal{D} \subset \mathbb{R}^{2N}$, where N is the minimum number of independent variables one needs to identify a configuration of the system (degrees of freedom), there is a function $\mathcal{H}(\mathbf{p}, \mathbf{q})$ such that $\dot{\mathbf{p}} = -\partial\mathcal{H}/\partial\mathbf{q}$ and $\dot{\mathbf{q}} = \partial\mathcal{H}/\partial\mathbf{p}$. An integrable (Hamiltonian) system is a Hamiltonian system which, in suitable coordinates $(\mathbf{A}, \boldsymbol{\alpha}) \in \mathcal{A} \times \mathbb{T}^N$, where $\mathcal{A}$ is an open subset of $\mathbb{R}^N$ and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the standard torus, can be described by a Hamiltonian $\mathcal{H}_0(\mathbf{A})$, i.e. depending only on $\mathbf{A}$. The coordinates $(\mathbf{A}, \boldsymbol{\alpha})$ are called action-angle variables. In such a case the dynamics is trivial: any initial condition $(\mathbf{A}_0, \boldsymbol{\alpha}_0)$ evolves in such a way that the action variables are constants of motion (i.e. $\mathbf{A}(t) = \mathbf{A}_0$ for all $t \in \mathbb{R}$), while the angles grow linearly in time according to the low $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}_0 + \boldsymbol{\omega} t$, where $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{A}_0) \equiv \partial_{\mathbf{A}}\mathcal{H}_0(\mathbf{A}_0)$ is called the rotation (or frequency) vector. An integrable system can be thought of as a collection of decoupled (that is independent) rotators: the entire phase space $\mathcal{A} \times \mathbb{T}^N$ is foliated into invariant tori and all motions are quasi-periodic. Integrable systems are stable, in the sense that nearby initial conditions separate at most linearly in time (in particular the actions do not separate at all): mathematically this is expressed by the fact that all the Lyapunov exponents are non-positive.

An example of an integrable system is any one-dimensional conservative mechanical system, in any region of phase-space in which motions are bounded. By increasing the number of degrees of freedom, exhibiting nontrivial integrable systems can become a difficult task. The problem of studying the effects of even small Hamiltonian perturbations on integrable systems and of understanding if the latter remain stable, in the aforementioned sense, was considered by Poincaré to be the fundamental problem of dynamics. For a long time it was commonly thought that all motions could be reduced to superpositions of periodic motions, hence to quasi-periodic motions, but at the end of XIX$^{\text{th}}$ century it was realized by Boltzmann and Poincaré that such a picture was too naive, and that in reality more complicated motions were possible. As a consequence of this it became a widespread belief that, even when starting from an integrable system, the introduction of an arbitrarily small perturbation would break integrability.

This belief was straightened by the work of Poincaré (1898), who showed that the series describing the solution in a perturbation theory approach are in general divergent. The source of divergences of the perturbation series is the presence of small divisors, that is of denominators of the kind of $\boldsymbol{\omega} \cdot \boldsymbol{\nu}$, where $\boldsymbol{\omega}$ is the rotation vector that should characterize the invariant torus (if existent) and $\boldsymbol{\nu}$ is any integer vector. Despite this, however, perturbation series (known as Lindstedt series), continued to be extensively used by astronomers in problems of celestial mechanics, such as the study of planetary motions, for the reason that they provided predictions in good agreement with the observations. But the feeling that the underlying mathematical tools were unsatisfactory persisted.

In fact the well-known Fermi-Pasta-Ulam numerical experiment (1955) was originally conceived in the spirit of confirming that integrability would in general be easily lost. Consider a chain with N harmonic oscillators, with, say, periodic boundary conditions, coupled with cubic and quartic two-body potentials, so that the Hamiltonian is

$$\mathcal{H}(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + W(q_{i+1} - q_i), \qquad W(x) = \frac{1}{2}x^2 + \frac{\alpha}{3}x^3 + \frac{\beta}{4}x^4, \qquad [1]$$

for α, β real parameters and $(\mathbf{p},\mathbf{q}) \in \mathbb{R}^N \times \mathbb{R}^N$. One can introduce new variables such that the Hamiltonian, for $\alpha = \beta = 0$, can be written as

$$\mathcal{H}_0(\mathbf{A}) = \sum_{i=1}^{N} \frac{1}{2} \left(P_k^2 + \omega_k Q_k^2 \right) = \boldsymbol{\omega} \cdot \mathbf{A}, \tag{2}$$

for a suitable rotation vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ (an explicit computation gives $\omega_k = 2\sin(k\pi/N)$).

Imagine we take an initial condition in which all the energy is confined to a few modes, i.e. $A_k \neq 0$ at $t = 0$ only for a few values of k. For $\alpha = \beta = 0$ the system is integrable, so that $A_k(t) = 0$ for all $t \in \mathbb{R}$ and for all k such that $A_k(0) = 0$. If the system ceases to be integrable when the perturbation is switched on, the energy is likely to start to be shared among the various modes, and after a long enough time is elapsed, an equidistribution of the energy among all modes (thermalization) might be expected. At least this behaviour was expected by Fermi, Pasta and Ulam, but it was not what they found numerically: on the contrary all the energy seemed to remain associated with the modes close the few initially excited ones.

At about the same time, Kolmogorov (1954) published a breakthrough paper going exactly in the opposite direction: if one perturbes an integrable system, under some mild conditions on the integrable part, most of the tori are preserved, although slightly deformed. A more precise statement is the following.

Theorem 1. *Let an N-degree-of-freedom Hamiltonian system be described by an analytic Hamiltonian of the form*

$$\mathcal{H}(\mathbf{A}, \boldsymbol{\alpha}) = \mathcal{H}_0(\mathbf{A}) + \varepsilon f(\mathbf{A}, \boldsymbol{\alpha}), \tag{3}$$

with ε a real parameter (perturbation parameter), f a 2π-periodic function of each angle variable (potential or perturbation) and $\mathcal{H}_0(\mathbf{A})$ satisfying the non-degeneracy condition $\det \partial_{\mathbf{A}}^2 \mathcal{H}_0(\mathbf{A}) \neq 0$ *(anisochrony condition). If $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{A}) \equiv \partial_{\mathbf{A}} \mathcal{H}_0(\mathbf{A})$ is fixed to satisfy the Diophantine condition*

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| > \frac{C_0}{|\boldsymbol{\nu}|^\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^N \setminus \mathbf{0}, \tag{4}$$

for some constants $C_0 > 0$ and $\tau > N-1$ (here $|\boldsymbol{\nu}| = |\nu_1| + \ldots + |\nu_N|$ and $\cdot$ denotes the standard inner product: $\boldsymbol{\omega} \cdot \boldsymbol{\nu} = \omega_1\nu_1 + \ldots + \omega_N\nu_N$), then there is an invariant torus with rotation vector $\boldsymbol{\omega}$ for ε small enough, say for ε smaller than some value ε_0 depending on C_0 and τ (and on the function f).

By saying that there is an invariant torus with rotation vector $\boldsymbol{\omega}$, one means that there is an invariant surface in phase space on which, in suitable coordinates, the dynamics is like the one in the unperturbed case, and the conjugation, that is the change of variables which leads to such coordinates, is analytic in the angles variables and in the perturbation parameter. One also says that the torus of an integrable system ($\varepsilon = 0$) is preserved (or even persists) under a small perturbation.

Note that, a posteriori, this proves convergence of the perturbation series: however a direct check of convergence was performed only recently by Eliasson (1988). Kolmogorov's proof was based on a completely different idea, that is performing iteratively a sequence of canonical transformations (which are changes of coordinates preserving the Hamiltonian structure of the equations of motion) such that at each step the size of the perturbation is reduced. Of course, on the basis of Poincaré's result, this iterative procedure cannot work for all initial conditions (for instance when $\boldsymbol{\omega}$ does not satisfy [4]). The key point in Kolmogorov's scheme is to fix the rotation vector $\boldsymbol{\omega}$ of the torus one is looking for, in such a way that the small divisors are controlled through the Diophantine condition [4] and the exponentially fast convergence of the algorithm.

New proofs and extensions of Kolmogorov's theorem were given later by Arnol′d and Moser (1962), hence the acronym KAM to denote such a theorem. Arnol′d gave a more detailed (and slightly different) proof compared to the original one by Kolmogorov, and applied the result to the planar three-body problem, so showing that physical applications of the theorem were possible, while Moser proposed a modified method using a technique introduced by Nash (which approximates smooth functions with analytical ones) to deal with the case of systems with finite smoothness.

For fixed small enough ε the surviving invariant tori cover a large portion of the phase space, called the Kolmogorov set; the relative measure of the region of phase space which is not filled by such tori tends to zero at least as $\sqrt{\varepsilon}$ for $\varepsilon \to 0$. A system described by a Hamiltonian like [3] is then called a quasi-integrable Hamiltonian system.

The excluded region of phase space corresponds to the unperturbed tori which are destroyed by the perturbation: the rotation vectors of such tori are close to a resonance, that is to a value $\boldsymbol{\omega}$ such that $\boldsymbol{\omega} \cdot \boldsymbol{\nu} = 0$ for some integer vector $\boldsymbol{\nu}$, and these are exactly the vectors which do not satisfy the Diophantine condition [4] for any value C_0. A subset of phase space of this kind is called a resonance region.

At first sight this would seem to provide an explanation for the results found by Fermi, Pasta and Ulam, but this is not quite the case. First, the threshold value ε_0 depends on N, and goes to zero very fast as $N \to \infty$ (in general as $N!^{-\alpha}$ for some $\alpha > 0$); however the results of the numerical experiments apparently were insensitive to the number N of oscillators. Second, the KAM theorem deals with maximal tori, i.e. tori characterized by rotation vectors which have as many components as the number of degrees of freedom, while the rotation vectors of the numerical quasi-periodic solutions seem to involve just a small number of components.

Finally, as an extra problem, the validity of the non-degeneracy condition for the unperturbed Hamiltonian is violated, because the unperturbed Hamiltonian is linear in the action variables (one says that the Hamiltonian is isochronous). Recently Rink (2001), by continuing the work by Nishida (1971), showed that in the Fermi-Pasta-Ulam problem it is possible to perform a canonical change of coordinates such that in the new variables the Hamiltonian becomes anisochronous: one uses part of the perturbation to remove isochrony. But the other two obstacles remain.

Lower-dimensional tori

A natural question is what happens to the invariant tori corresponding to rotation vectors which are not rationally independent, that is vectors satisfying n resonance conditions, such as $\boldsymbol{\omega} \cdot \boldsymbol{\nu}_i = 0$ for n independent vectors $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_n$, with $1 \leq n \leq N-2$ (the case $n = N-1$ corresponds to periodic orbits and is comparatively easy); for instance one can take $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n, 0, \ldots, 0)$, and by a suitable linear change of coordinates one can always make the reduction to a case of this kind. In particular one can ask if a result analogous to the KAM theorem holds for these tori. Such a problem for the model [3] has not been studied very widely in the literature. What has usually been considered is a system of n rotators coupled with a system with $s = N - n$ degrees of freedom near an equilibrium point: then one calls normal coordinates the coordinates describing the latter, and the role of the parameter ε is played by the size of the normal coordinates (if their initial conditions are chosen near the equilibrium point). In the absence of perturbation one has either hyperbolic or elliptic or, more generally, mixed tori, according to the nature of the equilibrium points: one refers to these tori as lower-dimensional tori, as they represent n-dimensional invariant surfaces in a system with N degrees of freedom. Then one can study the preservation of such tori.

One can prove that in such a case, in suitable coordinates, n angles rotate with frequencies $\omega_1, \ldots, \omega_n$, respectively, while the remaining $N - n$ angles have to be fixed close to some values corresponding to the extremal points of a the function obtained by averaging the potential over the rotating angles.

The case of hyperbolic tori is easier, as in the case of elliptic tori one has to exclude some values of ε to avoid some further resonance conditions between the rotation vector $\boldsymbol{\omega}$ and the normal frequencies λ_k (that is the eigenvalues of the linearized system corresponding to the normal coordinates), known as the first and second Mel′nikov conditions:

$$
\begin{aligned}
&|\boldsymbol{\omega} \cdot \boldsymbol{\nu} \pm \lambda_k| > \frac{C_0}{|\boldsymbol{\nu}|^\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^N \setminus \mathbf{0}, \quad \forall 1 \leq k \leq s, \\
&|\boldsymbol{\omega} \cdot \boldsymbol{\nu} \pm \lambda_k \pm \lambda_{k'}| > \frac{C_0}{|\boldsymbol{\nu}|^\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^N \setminus \mathbf{0}, \quad \forall 1 \leq k, k' \leq s.
\end{aligned}
\tag{5}
$$

Such conditions appear, with the values of the normal frequencies slightly modified by terms depending on ε, at each iterative step, and at the end only for values of ε belonging to some Cantor set one can have elliptic lower-dimensional tori.

The second Mel′nikov conditions are not really necessary, and in fact they can be relaxed as Bourgain (1997) has showed; this is an important fact, as it allows degenerate normal frequencies, which were forbidden in the previous works by Eliasson (1988) and Pöschel (1989).

Similar results also apply in the case of lower-dimensional tori for the model [3], which represents sort of a degenerate situation, as the normal frequencies vanish for $\varepsilon = 0$. Again one has to use part of the perturbation to remove the complete degeneracy of normal frequencies.

Quasi-periodic solutions in PDE

For explaining the Fermi-Pasta-Ulam experiment, one has to deal with systems with arbitrarily many degrees of freedom. Hence it is natural to investigate systems which have ab initio infinitely many degrees of freedom, such as the nonlinear wave equation, $u_{tt} - u_{xx} + V(x)u = \varphi(u)$, the nonlinear Schrödinger equation, $\mathrm{i}u_t - u_{xx} + V(x)u = \varphi(u)$, the nonlinear Korteweg-de Vries equation $u_t + u_{xxx} - 6u_x u = \varphi(u)$, and other systems of nonlinear partial differential equations (PDE); the continuum limit of the Fermi-Pasta-Ulam model gives indeed a nonlinear Korteweg-de Vries equation, as shown by Zabuski and Kruskal (1965). Here $(t, x) \in \mathbb{R} \times [0, \pi]^d$, if d is the space dimension, and either periodic ($u(0,t) = u(\pi,t)$) or Dirichlet ($u(0,t) = u(\pi,t) = 0$) boundary conditions can be considered; $\varphi(u)$ is a function analytic in u and starting from orders strictly higher than one, while $V(x)$ is an analytic function of x, depending on extra parameters $\xi_1, \ldots, \xi_n$. Such a function is introduced essentially for technical reasons, as we shall see that the eigenvalues λ_k of the Sturm-Liouville operator $-\partial_x^2 + V(x)$ have to satisfy some Diophantine conditions. If we set $V(x) = \mu \in \mathbb{R}$ in the nonlinear wave equation we obtain the Klein-Gordon equation, which, in the particular case $\mu = 0$, reduces to the string equation. Again the role of the perturbation parameter is played by the size of the solution itself.

Small-amplitude periodic and quasi-periodic solutions for PDE systems have been extensively studied, among others, by Kuksin, Wayne, Craig, Pöschel and Bourgain. Results for this kind of system read as follows. Consider for concreteness the one-dimensional nonlinear wave equation with Dirichlet boundary conditions and with $\varphi(u) = u^3 + O(u^5)$. When the nonlinear function $\varphi(u)$ is absent any solution of the linear wave equation $u_{tt} - u_{xx} + V(x)u = 0$ is a superposition of either finitely or infinitely many periodic solutions with frequencies λ_k determined by the function $V(x)$. Let $u_0(\boldsymbol{\omega} t, x)$ be a quasi-periodic solution of the linear wave equation with rotation vector $\boldsymbol{\omega} \in \mathbb{R}^n$, where $\omega_k = \lambda_{m_k}$, for some n-ple $\{m_1, \ldots, m_n\}$. Then for ε small enough there exists a subset Ξ_ε of the space of parameters with large Lebesgue measure (more precisely with complementary Lebesgue measure which tends to zero when $\varepsilon \to 0$) such that for all $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \Xi_\varepsilon$ there is a solution $u_\varepsilon(t, x)$ of the nonlinear wave equation and a rotation vector $\boldsymbol{\omega}_\varepsilon$ satisfying the conditions

$$\begin{aligned} &\left| u_\varepsilon(t, x) - \sqrt{\varepsilon} u_0(\boldsymbol{\omega}_\varepsilon t, x) \right| \leq C\varepsilon, \\ &|\boldsymbol{\omega}_\varepsilon - \boldsymbol{\omega}| < C\varepsilon, \end{aligned} \qquad [6]$$

for some positive constant C.

The case $n = 1$ (periodic solutions) is not as easy as the finite-dimensional case, because there are infinitely many normal frequencies, so that there are small divisor problems which for finite-dimensional systems appears only for $n \geq 2$.

For the nonlinear wave and Schrödinger equations, if $n \geq 1$ one can take $V(x) = \mu$, but one needs $\mu \neq 0$; for $n > 1$ one can take $V(x) = \mu$ as one can perform a preliminary transformation leading to an equation in which a function depending on parameters naturally appears. For $n = 1$ the case $\mu = 0$ has been very recently solved by Gentile, Mastropietro and Procesi (2004).

Statements for more general situations can be also obtained, while extensions to space dimensions $d \geq 2$ are not trivial and have been obtained only in recent years by Bourgain (1998). The above result also holds if the number of components of the rotation vector is less than the number of parameters: one uses such parameters as one needs to impose some Diophantine conditions like [5], now for all the frequencies $\lambda_k = \omega_k$, $k \notin \{m_1, \ldots, m_n\}$. Again the second Mel′nikov conditions were shown by Bourgain to be unnecessary, and this an essential ingredient for the higher-dimensional case.

Even if systems of the kind considered above have been widely studied, they remain different in a deep way from a discrete system like the chain of oscillators [1] for N large enough (also in the limit $N \to \infty$), so that the results which have been found for PDE systems do not really provide an explanation for the numerical findings.

Also in the case of lower-dimensional tori for finite-dimensional systems the main problem is that, even if such tori exist, it is not clear what relevance they can have for the dynamics (but we shall see later a case in which hyperbolic tori play a role). An important feature of maximal tori is that they fill most of phase space, a property which certainly does not hold for lower-dimensional tori, which lie outside the Kolmogorov set.

In the Fermi-Pasta-Ulam experiment one considers initial conditions close to lower-dimensional tori; hence an interesting problem is to study their stability, that is how fast the trajectories starting from such initial conditions drift away.

Arnol′d diffusion and Nekhoroshev's theorem

Let us come back to the maximal tori. For $N = 2$ the preservation of most of the invariant tori prevents the possibility of diffusion in phase space: the tori represent two-dimensional surfaces in a three-dimensional space (as dynamics occur on the level surfaces of the energy in a four-dimensional space), so that if an initial condition is trapped in a gap between two tori the corresponding trajectory remains confined forever between them. The situation is quite different for $N \geq 3$: in such a case the tori do not represent a topological obstruction to diffusion any more.

That mechanisms of diffusion are really possible was shown by Arnol′d (1963). Because of the perturbation, lower-dimensional hyperbolic tori appear inside the resonance regions, with their the stable and unstable manifolds (whiskers). It can happen that the unstable and stable manifolds of the same torus intersect with a non-vanishing angle (homoclinic angle); as a consequence also the angles between the stable and unstable manifolds of nearby tori (heteroclinic angles) can be different from zero, and one can find a set of hyperbolic lower-dimensional tori such that the unstable manifold of each of them intersects the stable manifold of the torus next to it: one says that such tori form a transition chain of heteroclinic connections. Then there can be trajectories moving along such connections, producing at the end a drift of order 1 (in ε) in the action variables. Such a phenomenon is refered to as Arnol′d diffusion.

Of course, diffusing trajectories have to be located in the region of phase space where there are no invariant tori (hence a very small region when ε is small), but an important consequence is that, unlike what happens in the unperturbed case, not all motions are stable: in particular the action variables can change by a large amount over long times.

Providing interesting examples of Hamiltonian systems in which Arnol′d diffusion can occur is not so easy: in fact for the diffusion to really occur one needs a lower bound on the homoclinic angles, and to evaluate these angles can be difficult. For instance Arnol′d's original example (1963), which describes a system near a resonance region, is a two-parameter system given by

$$\frac{1}{2}\left(A_1^2 + A_2^2\right) + A_3 + \mu\left(\cos\alpha_1 - 1\right) + \varepsilon\mu\left(\cos\alpha_1 - 1\right)\left(\sin\alpha_2 + \cos\alpha_3\right), \qquad [7]$$

and the angles can be proved to be bounded from below only by assuming that the perturbation parameter ε is exponentially small with respect to the other parameter μ, which in turns implies a situation not really convincing from a physical point of view. More generally for all the examples which are discussed in literature the relation with physics (as the d'Alembert problem on the possibility for a planet to change the inclination of the precession cone) is not obvious.

So the question naturally arises how fast can such a mechanism of diffusion be, and how relevant is it for practical purposes. A first answer is provided by Nekhoroshev's theorem (1977), which states the following result.

Theorem 2. *Suppose we have an N-degree-of-freedom quasi-integrable Hamiltonian system, where the unperturbed Hamiltonian satisfies some condition like convexity (or a weaker one, known as steepness, which is rather involved to state in a concise way); for concreteness we can consider a function $\mathcal{H}_0(\mathbf{A})$ in [2] which is quadratic in $\mathbf{A}$. Then there are two positive constants a and b such that for times t up to $O(\exp(\varepsilon^{-b}))$ the variations of the action variables cannot be larger than $O(\varepsilon^a)$.*

The constants a and b depend on N, and they tend to zero when $N \to \infty$; Lochak and Neĭshtadt (1992) and Pöschel (1993) found estimates $a = b = 1/2N$, which are probably in general optimal. Nekhoroshev's theorem is usually stated in the form above, but it provides more information than that explicitly written: the trajectories, when trapped into a resonance region, drift away and come close to some invariant torus, henceforth they behave like quasi-periodic motions, up to very small corrections, for a long time, until they enter some other resonance region, and so on. Of course for initial conditions on some invariant torus KAM theorem applies, but the new result is about initial conditions which do not belong to any tori.

Nekhoroshev's theorem gives a lower bound for the diffusion time, that is the time required for a drift of order 1 to occur in the action variables. But of course also an upper bound would be desirable. The diffusion times are related to the amplitude of the homoclinic angles, which are very small (and difficult to estimate as stated before). The strongest results in this direction have been obtained with variational methods, for instance by Bessi, Bernard, Berti and Bolle: at best one finds for the diffusion time an estimate $O(\mu^{-1}\log\mu^{-1})$, if μ is the amplitude of the homoclinic angles (which in turn are exponentially small in some power of ε, as one can expect as a consequence of Nekhoroshev's theorem).

Then one can imagine that also the results of the Fermi-Pasta-Ulam experiment can be interpreted in the light of Nekhoroshev's theorem. The solutions one finds numerically certainly do not correspond to maximal tori, but one could make the guess that they could be solutions which appear to be quasi-periodic for long but finite times (for instance, moving near some lower-dimensional torus determined by the initial conditions), and that if one really insists on observing the time evolution for a very long time then deviations from quasi-periodic behaviour could be detected. This is an appealing interpretation, and the most recent numerical results make it plausible: Galgani and Giorgilli (2002) have found numerically that the energy, even if initially confined to the lower modes, tend to be shared among all the other modes, and the higher are the modes the longer is the time needed for the energy to flow to them. Of course this does not settle the problem, as there is still the issue of the large number of degrees of freedom; furthermore for large N the spacing between the frequencies is small, and they become almost degenerate. Hence the problem still has to be considered as open.

Stability versus chaos

The main problem in applying the KAM theorem seems to be related to the small value of the threshold ε_0 which is required. In general, when the size of the perturbation parameter grows too much, the region of phase space filled with invariant tori decreases (or even disappears), and chaotic motions appear. By the latter one generally means motions which are highly sensitive to the initial conditions: a small variation of the initial conditions produces a catastrophic variation in the corresponding trajectories (this is due to the appearance of strictly positive Lyapunov exponents).

A natural question is then how such a result as the KAM theorem is meaningful in physical situations: in other words for which systems the KAM theorem can really apply.

One of the main motivations to study this kind of problem was to explain astronomical observations and to study the stability of the Solar system. In order to apply the KAM theorem to the Solar system, one has to interpret the gravitational forces between the planets as perturbations of a collection of several decoupled two-body systems (each planet with the Sun). One can write the masses of the planets as εm_i, and ε plays the role of the perturbation parameter. The corresponding Hamiltonian (after suitable reductions and scalings) is

$$\sum_{i=1}^{N} \frac{p_i^2}{2\mu_i} - \sum_{i=1}^{N} \frac{m_i m_0}{|q_i|} + \varepsilon \sum_{1 \leq i < j \leq N} \frac{p_i \cdot p_j}{m_0} + \varepsilon \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|q_i - q_j|}, \tag{8}$$

where $i = 0$ corresponds to the Sun, while $i = 1, \ldots, N$ correspond to the planets (hence $N = 9$), m_0 is the mass of the sun, and $\varepsilon\mu_i$ are the reduced masses ($\mu_i^{-1} = m_i^{-1} + \varepsilon m_0^{-1}$); here the inner product in $p_i \cdot p_j$ is in $\mathbb{R}^3$ and the norm $|\cdot|$ is the Euclidean one.

A first difficulty is that the Solar system is a properly degenerate system, that is the unperturbed Hamiltonian does not depend on all the action variables. But such a degeneracy can be removed by performing a canonical change of coordinates which produces a new Hamiltonian in which the integrable part contains new terms of order ε depending on all action variables and is non-degenerate, while the perturbation becomes of order ε^2: the angle variables corresponding to the actions not originally appearing in the unperturbed Hamiltonian are called the slow variables, while the other ones are called the fast variables.

However, a naive implementation of the KAM theorem, in general, even for simplified but still realistic systems, would provide preposterously small value of the threshold ε_0. The problem could be just a computational one: in principle a very refined estimate of the threshold could give a better value, so that it is very difficult to decide analytically if the real values of the planetary masses allow the Solar system to fall inside the regime of applicability of the KAM theorem. Results in this direction have been obtained, but only for special situations: for instance if one considers the restricted planar circular three-body problem (which provides a simplified description of the system Sun + Jupiter + asteroid), Chierchia and Celletti found analytical bounds on the perturbation parameters comparable with the physical values. Of course this is not conclusive at all for the general situation in which all planets (with their satellites and the asteroids) are considered together; in particular does not shed light on the problem of the stability of the entire Solar system.

On the contrary extensive numerical simulations performed by Laskar (starting from 1989) seem to suggest that the Solar system is unstable. Deflections from the current orbits could be produced, to such an extent that collisions between planets could not be avoided: Mercury could collide with Venus and be ejected from the Solar system. An important issue is to consider the times over which such phenomena can occur. Laskar's numerical simulations show that such times are less than the estimated age of the Solar system, and that one can make accurate predictions for the planetary motions only for a finite amount of time (about 100 millions of years). Furthermore, assumed partial instability of the Solar system has also been used by Laskar (2004) to explain some observed phenomena like evolution of the obliquity (which is the angle between equator and orbital plane) of some planets. Of course these simulations have been carried out with several approximations, as that of averaging over the fast variables, which allows one to use a large integration step in the numerical integration of the equations of motion for the resulting system. This is the so-called secular system introduced by Lagrange: instead of the fast motion of the planets, one describes the slow deformations of the planetary orbits (imagining the planets as regions of mass spread along their obits).

See also

KAM theory and celestial Mechanics. Diagrammatic techniques in perturbation theory. Perturbations of integrable Hamiltonian systems. Integrable systems and differential geometry. Mechanics and stability results. Stability and instability theory of Hamiltonian systems. Stationary solutions of PDEs and heterocline/homocline connections of dynamical systems. Weakly coupled oscillators.

Further Reading

Arnol′d V I, Kozlov V V, Neĭshtadt A I (1988), *Dynamical systems III*, Encyclopedia of Mathematical Sciences 3, Berlin: Springer-Verlag.

Bourgain J (1998) Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations, *Annals of Mathematics* 148: 363–439.

Ford J (1992) The Fermi-Pasta-Ulam problem: paradox turns discovery, *Phys. Rep.* 213: 271–310.

Gallavotti G (1986) Quasi-integrable mechanical systems, *Phénomènes critiques, systèmes aléatoires, théories de jauge* (Les Houches, 1984), 539–624, Amsterdam: North-Holland.

Kuksin S B (1993) *Nearly integrable infinite-dimensional Hamiltonian systems*, Lecture Notes in Mathematics, 1556, Berlin: Springer-Verlag.

Laskar J (2004) Chaos in the Solar System, *Proceedings of the International Conference on Theoretical Physics TH2002* (Paris, 2002), Eds. D. Iagolnitzer, V. Rivasseau, J. Zinn-Justin, Basel: Birkhäuser.

Moser J (1973) *Stable and random motions in dynamical systems*, Annals of Matematical Studies, Princeton: Princeton University Press.