Brjuno numbers and dynamical systems

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1 Introduction

We shall consider some elementary analytic dynamical systems widely studied in literature, which are perturbations of integrable (linear) ones. A classical problem is to study the conditions for the perturbed system to be analytically conjugated to the linear one, that is the conditions for the existence of an analytic change of coordinates which maps the perturbed system to the linear one. We shall see that such conditions can be naturally expressed in terms of the Diophantine properties of a suitable parameter.

1.1 Brjuno function and Brjuno numbers

For $\omega \in \mathbb{R} \setminus \mathbb{Q}$ define the *Brjuno function* as the solution of the functional equation

$$\begin{cases} B(\omega+1) = B(\omega), \\ B(\omega) = -\log\omega + \omega B(1/\omega), \qquad \text{if } \omega \in (0,1), \end{cases} \tag{1}$$

and call ω a *Brjuno number* if $B(\omega) < \infty$. We shall be interested essentially in numbers $\omega \in \mathcal{B} = \{\omega \in \mathbb{R} \setminus \mathbb{Q} \cap (0,1) \mid B(\omega) < \infty\}$.

Note that, if we denote with $\{p_n/q_n\}_{n=0}^{\infty}$ the sequence of convergents of ω and define

$$B_1(\omega) = \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n}, \tag{2}$$

there exists a constant C such that $|B(\omega) - B_1(\omega)| < C$ for all $\omega \in \mathcal{B}$ [20].

1.2 Some dynamical systems

Siegel's problem. Consider the holomorphic diffeomorphism [19]

$$z' = f(z), \qquad f(z) = \lambda z + O(z^2), \tag{3}$$

with $\lambda = e^{2\pi i\omega}$. We shall write $f \in G_\lambda$, where G is the group of germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$ and $G_\lambda = \{f \in G \mid f'(0) = \lambda\}$.

Define also $R_\lambda(z) = \lambda z$. We say that $f \in G_\lambda$ is *linearizable* if there exists $h \equiv h_f \in G_1$ such that $f \circ h = h \circ R_\lambda$: in such a case the diffeomorphism (3) is *conjugated* to its linear part R_λ: if we set $\alpha' = \lambda\alpha$ then $z' = h(\alpha') = h \circ R_\lambda(\alpha) = f \circ h(\alpha) = f(h(\alpha))$. The problem of finding conditions under which f is linearizable is usually referred to as *Siegel's problem*. The following result holds.

Theorem 1. *If $\omega \in \mathcal{B}$ and $\lambda = e^{2\pi i\omega}$ then $f \in G_\lambda$ is linearizable.*

This follows from a stronger result of Yoccoz. If $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$ call S_λ the topological space of germs of holomorphic diffeomorphisms $f : \mathbb{D} \to \mathbb{C}$ such that $f(0) = 0$, $f'(0) = \lambda$ and f is univalent on $\mathbb{D}$, and, for $\lambda = e^{2\pi i\omega}$, set $r(\omega) = \inf_{f\in S_\lambda} r(f, \omega)$, if $r(f, \omega)$ is the radius of convergence of h_f for $f \in G_\lambda$: then Theorem 1 is implied from the following one [20].

Theorem 2. *There exists a constant C such that $|\log r(\omega) + B(\omega)| < C$ for all $\omega \in \mathcal{B}$. Moreover if $f(z)$ is the* quadratic polynomial $P_\lambda(z) = \lambda z(1 - z/2)$, *for all $\eta > 0$ there exists a constant C_η such that $-B(\omega) - C < \log r(P_\lambda, \omega) < -(1-\eta)B(\omega) + C_\eta$ for a universal constant C and for all $\omega \in \mathcal{B}$.*

Yoccoz also proved that if P_λ is linearizable then every germ $f \in G_\lambda$ is also linearizable [20].

Standard map, semistandard map and generalizations. Consider the dynamical system

$$T_\varepsilon\colon \begin{cases} x' = x + y + \varepsilon f(x), \\ y' = y + \varepsilon f(x), \end{cases} \tag{4}$$

where f is a zero-average analytic function and ε is a small parameter. If f is real for real x, then we shall study T_ε as a map from the cylinder to itself, $T_\varepsilon : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, otherwise we shall consider T_ε as a map $T_\varepsilon : \mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C} \to \mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}$. As particular cases we consider the *standard map* and the *semistandard map* [9, 16], obtained by choosing $f(x) = s(x) \equiv \sin x$ and $f(x) = e(x) \equiv \exp(ix)/2i$, respectively. In the case of the semistandard map the parameter ε can be eliminated, but we prefer to keep it in order to make easier the comparison with the standard map.

We look for solutions of the form

$$\begin{cases} x = \alpha + u(\alpha, \varepsilon), \\ y = 2\pi\omega + v(\alpha, \varepsilon), \end{cases} \tag{5}$$

such that in the variable α the dynamics is a trivial rotation $\alpha' = \alpha + 2\pi\omega$, with rotation number ω. We see immediately that we can express $v(\alpha, \varepsilon)$ in terms of $u(\alpha, \varepsilon)$ as $v(\alpha, \varepsilon) = u(\alpha, \varepsilon) - u(\alpha - 2\pi\omega, \varepsilon)$, while $u(\alpha, \varepsilon)$ solves the functional equation

$$u(\alpha + 2\pi\omega, \varepsilon) + u(\alpha - 2\pi\omega, \varepsilon) - 2u(\alpha, \varepsilon) = \varepsilon f\left(\alpha + u(\alpha, \varepsilon)\right). \tag{6}$$

We say that T_ε admits an *invariant curve* with rotation number ω if there exists an analytic function $u(\alpha, \varepsilon)$ such that (5) holds with $\alpha \to \alpha + 2\pi\omega$; the function $u(\alpha, \varepsilon)$ is called the *conjugating function.* The following result holds.

Theorem 3. *If $\omega \in \mathcal{B}$ and ε is small enough then T_ε admits an analytic invariant curve with rotation number ω.*

A more quantitative statement is the following. Given $\omega \in \mathcal{B}$ there exists $r(\omega, f) \in \mathbb{R}^+$ such that there exists a solution of (4) of the form (5) with u (and hence v) analytic in ε for $|\varepsilon| < r(\omega, f)$: $r(\omega, f)$ is the radius of convergence of the conjugating function for fixed ω and f (see also (8) below).

For the semistandard and the standard maps the result above can be strengthened. In fact in such cases, if we set $\rho_0(\omega) = r(e, \omega)$ and $\rho(\omega) = r(s, \omega)$, one can prove the following results.

Theorem 4. *There exist a universal constant C_0 such that $|\log \rho_0(\omega) + 2B(\omega)| < C_0$ for all $\omega \in \mathcal{B}$.*

Theorem 5. *There exist a universal constant C such that $|\log \rho(\omega) + 2B(\omega)| < C$ for all $\omega \in \mathcal{B}$.*

The bounds of Theorem 4 were proved by Davie [10], who also proved that one has $\rho(\omega) \leq \rho_0(\omega)$, which implies the upper bound in Theorem 5. In an unpublished paper [11] Davie also showed, by using renormalization group methods, that for all $\eta > 0$ there exists a constant C_η such that one has $\log \rho(\omega) \geq -(1-\eta)B(\omega) - C_\eta$ for $\omega \in \mathcal{B}$. This is not enough to obtain the lower bound in Theorem 5, which was proved by Berretti and Gentile [4].

Here we want to describe a technique which is suitable for proving the lower bounds considered so far. In particular we shall see that it is very easy to obtain the lower bounds for Siegel's problem and the semistandard map, while the case of the standard map turns out to be more difficult. Such a difficulty is mainly due to the optimality of the bound: if we confined ourselves to show that $\rho(\omega)$ (or even $r(\omega, f)$) is strictly positive for $\omega \in \mathcal{B}$, then the proof would be much easier.

2 Tree formalism

We use a technique which shows strong analogies with the renormalization group method in quantum field theory. Such a technique has been introduced by Eliasson [12] and Gallavotti [13] to prove existence of KAM tori for quasi-integrable Hamiltonian systems, and then developed and extended in other papers (we refer to the bibliography for some reviews [15, 14, 6]).

Let us consider first the dynamical system (4). The conjugating function $u(\alpha, \varepsilon)$ admits a formal expansion – the *Lindstedt series* – of the form

$$u(\alpha,\varepsilon)=\sum_{\nu\in\mathbb{Z}}u_\nu(\varepsilon)\,e^{i\nu\alpha}=\sum_{k\ge1}u^{(k)}(\alpha)\,\varepsilon^k=\sum_{k\ge1}\sum_{\nu\in\mathbb{Z}}u_\nu^{(k)}\,e^{i\nu\alpha}\,\varepsilon^k. \tag{7}$$

For the standard map, at order k in ε, the Fourier expansion in α contains only frequencies $|\nu|\le k$, while for the semistandard map one has $\nu=k$. The *radius of convergence* of the Lindstedt series is naturally defined as

$$r(\omega,f)=\inf_{\alpha\in\mathbb{T}}\left(\limsup_{k\to\infty}\left|u^{(k)}(\alpha)\right|^{1/k}\right)^{-1}. \tag{8}$$

By inserting (7) into the functional equation (6), and equating the Taylor-Fourier coefficients we obtain a recursion relation for the coefficients of the Lindstedt series: one has $u_\nu^{(1)}=g(\omega\nu)f_\nu$ and, for $k\ge2$,

$$u_\nu^{(k)}=g(\omega\nu)\sum_{m\ge1}\frac{1}{m!}\sum_{\substack{k_1+\ldots+k_m=k-1\\ \nu_0+\nu_1+\ldots+\nu_m=\nu}}f_{\nu_0}\,(i\nu_0)^m\prod_{j=1}^m u_{\nu_j}^{(k_j)}, \tag{9}$$

with $f_{\nu_0}=-i\nu_0\delta_{|\nu_0|,1}/2$ for the standard map and $f_{\nu_0}=-i\nu_0\delta_{\nu_0,1}/2$ for the semistandard map, and

$$g(\omega\nu)=\frac{1}{\gamma(\omega\nu)},\qquad \gamma(\omega\nu)=2\left[\cos(2\pi\omega\nu)-1\right]. \tag{10}$$

The denominators $\gamma(\omega\nu)$ can become arbitrarily small for ω irrational: this is a manifestation of the famous *small divisors* problem.

By iterating the recursion relation (9), which corresponds to apply to each $u_{\nu_j}^{(k_j)}$ the same decomposition used for $u_\nu^{(k)}$ itself, we see that at the end the coefficients $u_\nu^{(k)}$ can be written as sums of contributions which are represented in terms of tree graphs (or simply *trees*); see for instance Figure 1.

A tree ϑ is constructed in the following way. Consider a family of $k-1$ *lines* arranged to connect a set of k points called *nodes* (so that no loops arise), and add an extra line connecting one of the nodes to another point called the *root*. Each line carries an arrow pointing toward the root: this induces a partial ordering relation on the nodes (and lines), with the lowest nodes drawn to the right and the root to the left. Given two nodes u_1 and u_2, we write $u_2\preceq u_1$ if u_1 is along the path of lines connecting u_2 to the root r of the tree (they can coincide: we say that $u_2\prec u_1$ if they do not).

If a line connects a node u to the right to a point u' to the left we say that the line comes out from u and enters u'. For each node u there are only one line coming out from u and $m_u\ge0$ entering ones; as there is a one-to-one correspondence between nodes and lines, we can associate to each node u a line ℓ_u coming out from it. By construction there is only one line entering the root: we shall call it the *root line*. Note that each line ℓ_u can be considered the root line of the subtree consisting of the nodes satisfying $w\preceq u$ and of the lines connecting them plus the line connecting u to u', which will be the root

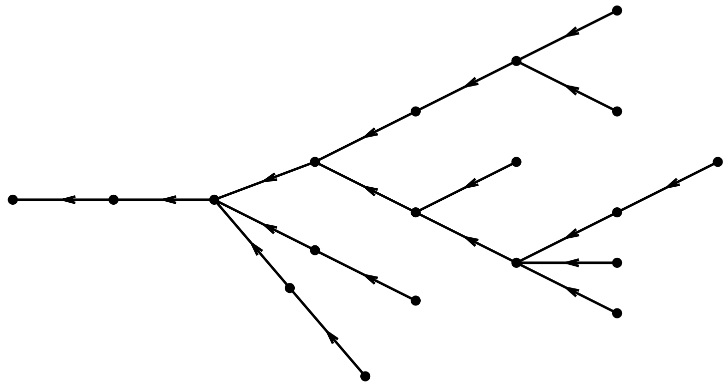

Fig. 1. A tree of order 18. The labels are not shown. All the arrows point toward the point to the extreme left (root). All the other points are called nodes.

of such subtree. The *order* of the tree is defined as the number k of nodes of the tree.

To each node $u \in \vartheta$ we associate a *mode label* $\nu_u \in \mathbb{Z} \setminus \{0\}$, which is a Fourier label of the function $f(x)$; define the *momentum* flowing through the line ℓ_u as $\nu_{\ell_u} = \sum_{w \preceq u} \nu_w$.

Let us denote by $\Theta^0_{k,\nu}$ the set of all trees of order k with momentum ν flowing through the root line, and by $V(\vartheta)$ and $\Lambda(\vartheta)$, respectively, the set of nodes and the set of lines of the tree ϑ. To each node $u \in V(\vartheta)$ we associate a *node factor* $F_v = (i\nu_u)^{m_u+1}/(m_u!2)$, while to each line $\ell \in \Lambda(\vartheta)$ we associate a *propagator* $G_\ell = g(\omega\nu_\ell)$, where $g(\omega\nu)$ is defined in (10).

Given a tree $\vartheta \in \Theta^0_{k,\nu}$ define a map $\mathrm{Val} : \Theta^0_{k,\nu} \to \overline{\mathbb{R}}$ as

$$\mathrm{Val}(\vartheta) = \Big(\prod_{u \in V(\vartheta)} F_v \Big) \Big(\prod_{\ell \in \Lambda(\vartheta)} G_\ell \Big); \tag{11}$$

if $\Theta^0_{k,\nu} = \emptyset$ we interpret $\mathrm{Val}(\vartheta) = 0$. We call $\mathrm{Val}(\vartheta)$ the *value* of the tree ϑ. Then the following result holds, establishing a link between the conjugating function and the trees.

Lemma 1. *One can take* $u_0^{(k)} = 0$ *for all* $k \geq 1$. *Then* $\nu_\ell \neq 0$ $\forall \ell \in \Lambda(\vartheta)$ *and one has* $u_\nu^{(k)} = \sum_{\vartheta \in \Theta^0_{k,\nu}} \mathrm{Val}(\vartheta)$ *for all* $k \geq 1$ *and for all* $\nu \neq 0$.

The (easy) proof can be carried out by induction. We say that two trees are equivalent if they can be transformed into each other by continuously deforming the lines without crossing. The sum over $\Theta^0_{k,\nu}$ is meant as a sum over all the trees which are not equivalent; that this is the correct way to count the trees follows from the fact that it keeps trace of the combinatorial factors naturally arising from the Taylor expansion (7) when we iterate the graphical construction generated by (9). The number of elements of $\Theta^0_{k,\nu}$ is bounded by 2^{2k} in the case of the semistandard map and by $2^k 2^{2k}$ in the case of the standard map. So we are left with the problem of proving that the series (7) converges.

In the case of Siegel's problem we consider the formal expansion

$$h(\alpha) = \alpha + \sum_{k=2}^{\infty} h^{(k)} \alpha^k \equiv \sum_{k=1}^{\infty} h^{(k)} \alpha^k, \tag{12}$$

so that, by writing $f(z) = \lambda z + \sum_{k=2}^{\infty} f_k z^k$ and inserting (12) into (3), we obtain $h^{(1)} = 1$ and, for $k \geq 2$,

$$h^{(k)} = g(\omega k) \sum_{m=2}^{k} f_m \sum_{k_1+\ldots+k_m=k} \prod_{j=1}^{m} h^{(k_j)}, \tag{13}$$

with

$$g(\omega k) = \frac{1}{\lambda^k - \lambda} = \frac{1}{e^{2\pi i \omega k} - e^{2\pi i \omega}}. \tag{14}$$

Therefore also $h^{(k)}$ admits a graphic representation. The trees are defined as before, but with different labels. First of all for each node u one can have either $m_u = 0$ or $m_u \geq 2$, where m_u is the number of lines entering u; one sets $\nu_u = 1$ when $m_u = 0$ and $\nu_u = 0$ when $m_u \geq 2$. The order k of a tree ϑ is defined as $k = \sum_{u \in V(\vartheta)} \nu_u$, while the momentum is defined as before; hence $k = \nu_{\ell_0}$, if ℓ_0 is the root line. To each node $u \in V(\vartheta)$ we associate a node factor f_{m_u}, and to each line ℓ we associate a propagator G_ℓ, with $G_\ell = g(\omega \nu_\ell)$ if $\nu_\ell \geq 2$ and $G_\ell = 1$ if $\nu_\ell = 1$.

By defining $\mathrm{Val}(\vartheta)$ as in (11), with the new definition of the node factors and of the propagators, the following result holds.

Lemma 2. *One has formally* $h^{(k)} = \sum_{\vartheta \in \Theta^0_{k,k}} \mathrm{Val}(\vartheta, \omega)$ *for all* $k \geq 1$.

Again we have to face the problem of proving the convergence of the formal expansion. As the case of Siegel's problem can be essentially dealt with as for the semistandard map (as far as the lower bound on the radius of convergence is concerned) we shall concentrate henceforth on the system (4).

3 Renormalization group and multiscale decomposition

To control the product of the propagators in (11) one needs a multiscale analysis which can be pursued as follows. We say that a line ℓ has scale n if $\|\omega \nu_\ell\| = \min_{p \in \mathbb{Z}} |\omega \nu_\ell - p|$ is approximately equal to $1/q_{n+1}$. A more formal statement can be obtained by introducing a C^∞ partition of unity. Let $\chi(x)$ a C^∞ non-increasing compact-support function defined on $\mathbb{R}^+$, such that

$$\chi(x) = \begin{cases} 1 & \text{for } x \leq 1, \\ 0 & \text{for } x \geq 2, \end{cases} \tag{15}$$

and define $\chi_0(x) = 1 - \chi(96q_1 x)$ and $\chi_n(x) = \chi(96q_n x) - \chi(96q_{n+1}x)$ for each $n \in \mathbb{N}$; then for each line ℓ set

$$G_\ell = g(\omega\nu_\ell) = \sum_{n=0}^{\infty} G_\ell^{(n)}, \qquad G_\ell^{(n)} = \chi_n(\|\omega\nu_\ell\|)\, g(\omega\nu_\ell), \tag{16}$$

and call $G_\ell^{(n)}$ the *propagator on scale* n. [To deal with the semistandard map one could simply to introduce a sharp partition of unity (through step functions); however when discussing the standard map one has to develop to second order the propagators, and this explains why we need a smooth function in (16).]

Given a tree ϑ, we can associate to each line ℓ of ϑ a scale label n_ℓ, using the multiscale decomposition (16) and singling out the summand with $n = n_\ell$. We shall call n_ℓ the *scale label* of the line ℓ, and we shall say that the line ℓ is on scale n_ℓ.

Note that if a line ℓ has momentum ν_ℓ and scale n_ℓ, then

$$\frac{1}{96q_{n_\ell+1}} \le \|\omega\nu_\ell\| \le \frac{1}{48q_{n_\ell}}, \tag{17}$$

provided that one has $\chi_{n_\ell}(||\omega\nu_\ell||) \neq 0$. Note also that given a line ℓ at most only two summands in (16) are really non-vanishing.

Therefore $u_\nu^{(k)}$ can be rewritten as

$$\begin{aligned} u_\nu^{(k)} &= \sum_{\vartheta \in \Theta_{k,\nu}} \mathrm{Val}(\vartheta), \\ \mathrm{Val}(\vartheta) &= \Big(\prod_{u \in V(\vartheta)} F_v\Big)\Big(\prod_{\ell \in \Lambda(\vartheta)} G_\ell^{(n_\ell)}\Big), \end{aligned} \tag{18}$$

where $\Theta_{k,\nu}$ is the set of all trees of order k and with $\nu_{\ell_0} = \nu$ carrying also scale labels (in addition to the node labels); in the case of the semistandard map the number of elements in $\Theta_{k,\nu}$ is bounded by $2^k 2^{2k}$, while in the case of the standard map the number of elements in $\Theta_{k,k}$ is bounded by $2^k 2^k 2^{2k}$.

Given a tree ϑ, a *cluster* T of ϑ on scale n is a maximal connected set of lines of lines on scale $\le n$ with at least one line on scale n; see Figure 2. We shall say that such lines are internal to T, and write $\ell \in \Lambda(T)$. A node u is called *internal* to T, and we write $u \in V(T)$, if at least one of the lines entering or coming out from it is in T. Each cluster has an arbitrary number $m_T \ge 0$ of entering lines but only one or zero line comes out from it; we shall call *external* to T the lines entering or coming out from T (and which are all on scale $> n$). We shall denote with n_T the scale of the cluster T, and with k_T the number of nodes in T.

Note that there is an inclusion relation between clusters: the innermost ones are those with highest scale, while the outermost ones are on the lowest scale. The aim of introducing the clusters is to characterize the lines of the

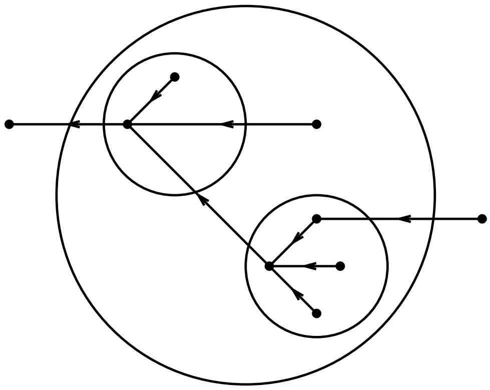

Fig. 2. A tree with its clusters. One should imagine that the tree itself is a cluster including all the other clusters. All the clusters T with $m_T = 1$ are resonances if also the conditions (2) and (3) of the definition of resonance are satisfied.

trees on the basis of the sizes of the corresponding propagators: the lines which are contained inside the outermost clusters correspond to the propagators with the smallest small divisors, and so on.

If we confine ourselves to the semistandard map the notion of cluster is sufficient to prove the lower bound of Theorem 4: this will be discussed in Section 4. On the contrary in the case of the standard map additional problems arise, due to the fact that for each node $u \in V(\vartheta)$ one has $\nu_u = \pm 1$, whereas $\nu_u = 1$ for the semistandard map. To single out the cases which can give problems, we have to introduce the notion of resonance; see Figure 2.

A cluster T will be called a *resonance* with *resonance-scale* n if the following three properties are verified: (1) $m_T = 1$, (2) $\sum_{u \in V(T)} \nu_u = 0$ and (3) $k_T < q_n$, where n is minimum between the scales of the external lines of T.

The condition (1) means that T has only one entering line, which, by the condition (2) must have the same momentum of the exiting one (so that the scales of the two lines can differ at most by one unit). Of course resonances can not arise in the case of the semistandard map.

Then, because of the presence of resonances, in the case of the standard map there can be some trees in which there can be accumulation of small divisors equal to each other, but, when the values of all trees in $\Theta_{k,\nu}$ are summed together, if we group the trees into suitable classes then some remarkable cancellation mechanisms intervene between them, and the overall contribution still admits a bound of the same kind of that of the semistandard map: we shall be more precise in Section 5.

With the notion of resonance given above we are able to prove the lower bound in Theorem 5 with 4 instead of 2 in front of the Brjuno function; see Section 5. In order to obtain 2 a more careful analysis is needed: in particular the cancellation mechanisms have to be extended to a larger class of trees; we shall give some ideas about the proof at the end of Section 5.

4 Lower bound for the semistandard map

Let $\{q_n\}_{n=0}^{\infty}$ be the denominators of the convergents of ω. Then one has [18]

$$\frac{1}{2q_{n+1}} < \|\omega q_n\| < \frac{1}{q_n}, \tag{19a}$$

$$\|\omega\nu\| > \|\omega q_n\| \qquad \forall|\nu| < q_{n+1},\ |\nu| \neq q_n. \tag{19b}$$

Let us denote by $N_n(\vartheta)$ the number of lines $\ell \in \Lambda(\vartheta)$ with scale $n_\ell = n$. Then we want to prove that, in the case of the semistandard map one has

$$N_n(\vartheta) \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}}, \tag{20}$$

which immediately implies the lower bound of Theorem 4. In fact (20), inserted into (18), gives

$$\begin{aligned} |\mathrm{Val}(\vartheta)| &\leq \left(\frac{1}{2}\right)^k \prod_{n=0}^{\infty} \left(c q_{n+1}^2\right)^{N_n(\vartheta)} \\ &\leq \left(\frac{c}{2}\right)^k \exp\left[2k \sum_{n=0}^{\infty} \left(\frac{\log q_{n+1}}{q_n} + \frac{8 \log q_{n+1}}{q_{n+1}}\right)\right], \end{aligned} \tag{21}$$

where, in the first line, $1/2$ is a bound on the node factor F_v, while cq_{n+1}^2 is a bound on the propagator of a line on scale n, with c a constant. Then $|u_k^{(k)}| \leq 2^{3k}\left(ce^{16D}/2\right)^k e^{2B_1(\omega)k}$, where D is a universal constant bounding $\sum_{n=1}^{\infty} q_n^{-1} \log q_n$ for any irrational ω with convergents $\{p_n/q_n\}$, so that, by using the definition (8), the lower bound $\log \rho_0(\omega) + 2B_1(\omega) > -C$ follows for some constant C.

So it remains to check the bound (20). One can prove inductively on the order k the following result.

Lemma 3. *One has*

$$\begin{cases} N_n(\vartheta) = 0, & \text{if } k < q_n, \\ N_n(\vartheta) \leq 2k/q_n - 1, & \text{if } k \geq q_n, \end{cases} \tag{22}$$

for all $n \geq 0$ *and for all* $\vartheta \in \Theta_{k,k}$.

The first bound in (22) is immediately implied by the property (19). Roughly speaking the idea behind the proof of the second bound is the following. First of all note that the propagators G_ℓ are large for $\|\omega\nu_\ell\|$ small. However, even if the quantities $\|\omega\nu_\ell\|$ can become very small for ν_ℓ large enough, they cannot "accumulate" too much. In fact once a line ℓ_1 on scale n (i.e. with momentum ν_{ℓ_1} such that $\|\omega\nu_{\ell_1}\|$ is of order $1/q_{n+1}$) has been obtained, in order to have again a line ℓ_2 on the same scale along the path connecting

ℓ_1 to the root, one needs many nodes between the two lines (i.e. many nodes preceding ℓ_2 and following ℓ_1), as each node contributes a mode label 1 to the momentum ν_{ℓ_2} and to have $\|\omega(\nu_{\ell_2} - \nu_{\ell_1})\| \leq \|\omega\nu_{\ell_1}\| + \|\omega\nu_{\ell_2}\| = O(1/q_{n+1})$ requires $\nu_{\ell_2} - \nu_{\ell_1}$ to be large enough, by (19); but $\nu_{\ell_2} - \nu_{\ell_1}$ is exactly the number of nodes between ℓ_1 and ℓ_2. Therefore the number of lines on a fixed scale n can not grow indefinitely.

Unfortunately the bound (22) is not enough to obtain (20) because of the factor 2 instead of 1. This is not only a technical problem: it is not difficult to provide explicit examples of trees for which the bound (22) with 1 instead of 2 is false. In fact what we can prove is that Lemma 3 can be improved into the following one.

Lemma 4. *If $q_{n+1} > 4q_n$ one has*

$$\begin{cases} N_n(\vartheta) = 0, & \text{if } k < q_n, \\ N_n(\vartheta) \leq k/q_n, & \text{if } q_n \leq k < q_{n+1}/4, \\ N_n(\vartheta) \leq k/q_n + 8k/q_{n+1} - 1, & \text{if } k \geq q_{n+1}/4, \end{cases} \tag{23}$$

for all $n \geq 0$ and for all $\vartheta \in \Theta_{k,k}$.

Again the proof is by induction; the details can be found in literature [6]. A basic result in order to deduce Lemma 4 is the following one [10].

Lemma 5. *Given $\nu \in \mathbb{Z}$ such that $\|\omega\nu\| \leq 1/4q_n$, then (1) either $\nu = 0$ or $|\nu| \geq q_n$, and (2) either $|\nu| \geq q_{n+1}/4$ or $\nu \in q_n\mathbb{Z}$.*

The proof is elementary, but the result plays an essential role in proving Lemma 4; in particular it allows to distinguish between the two latter bounds of (23) by treating in a different way the cases $k < q_{n+1}/4$ and $k \geq q_{n+1}/4$.

5 Lower bound for the standard map

In the case of the standard map, once a line ℓ_1 on a large scale n has appeared, another line ℓ_2 on the same scale can be easily obtained once more along the path connecting ℓ_1 to the root, without going much further along the tree. For instance it is enough to have between ℓ_1 and ℓ_2 two nodes u_1 and u_2 with $\nu_{u_1} = -\nu_{u_2} = 1$, in order to have $\nu_{\ell_2} = \nu_{u_1} + \nu_{u_2} + \nu_{\ell_1} = \nu_{\ell_1}$, so that the line connecting u_1 to u_2 forms a resonance T with external lines ℓ_1 and ℓ_2 (as all the conditions of the definition are satisfied) and the line ℓ_2 coming out from T can have the same scale n; see Figure 3. So we can prove a bound like the one of Lemma 3, but with $N_n(\vartheta)$ replaced by $N_n^*(\vartheta)$, if the latter denotes the number of lines $\ell \in \Lambda(\vartheta)$ with scale $n_\ell = n$ which do not come out from any resonance (we shall call them *non-resonant lines*).

The key remark to deal with the resonances is that, when summing the tree values over all possible trees as in (18), all terms containing the same

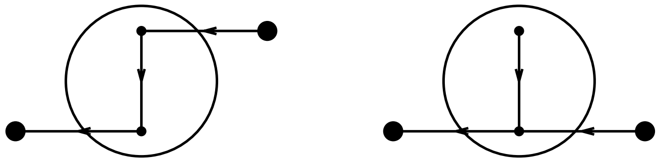

Fig. 3. Trees obtained by shifting the entering line of a resonance T with two nodes u_1 and u_2 with opposite mode labels. The black balls represent the remaining parts of the tree and the labels are not shown.

resonance cancel almost exactly: more precisely, for any resonance T, the values of the trees containing that resonance cancel to order 2 in $\|\omega\nu_{\ell_1}\|$, if ℓ_1 is the line entering T. [To exploit the cancellations we have to derive twice the propagators: here the smoothness of the compact support functions comes in.] The criterion to single out the trees between which the cancellation operates is the following: given a tree ϑ with a resonance T, consider the class $\mathcal{F}_T(\vartheta)$ of all trees obtained by detaching the line entering T and re-attaching it to all the nodes inside T, and for each of such trees consider also the tree obtained by reverting the sign of the mode labels of the nodes contained in T (i.e. by replacing each $\nu_u = 1$ with $\nu_u = -1$ and vice versa); see for an example Figure 3.

But a second order cancellation produces a factor proportional to $\|\omega\nu_{\ell_1}\|^2$ which is exactly of the same size of the propagator of the line ℓ_2 (recall that $\nu_{\ell_2} = \nu_{\ell_1}$). In other words, given a line ℓ_1 on a very large scale n, it is possible to create another line ℓ_2 with the same scale adding only a few nodes (for instance 2 in the example above), but in such a way a resonance arises, and the cancellation mechanism described above produces a gain factor which compensates the propagator of the newly added line ℓ_2: this means that also in the case of the standard map there cannot be any accumulation of small divisors.

The conclusion is that the propagators corresponding to the non-resonant lines can be controlled as in the case of the semistandard map (through a minor extension of Lemma 3), while the propagators corresponding to the lines coming out from some resonance (the *resonant* lines) are in fact compensated by the gain factors produced by the cancellation mechanism. This is enough to prove the convergence of the perturbative series (hence the existence of the corresponding invariant curve), but produces again the extra factor 2 for the lower bound of the radius of convergence.

The first thing one can think about is to try to extend also Lemma 4 to cover the case of the non-resonant lines for the standard map. But this does not work: with the previous definition of resonance the bounds (23) fail to be satisfied. Then one can try to generalize the definition of resonance by enlarging the class of graphs to use in order to exploit the cancellations.

The condition (2) will be certainly retained, as it is needed for the cancellation mechanism to work. On the contrary we eliminate the condition $m_T = 1$:

we allow any number of entering lines (see Figure 4). Note that such conditions are suggested from the study of the conjugating function for complex values of the rotation number ω; see Section 6.

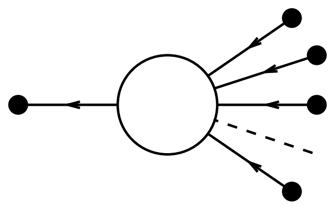

Fig. 4. A resonance, according to the new definition, can have an arbitrary number of entering lines. The black balls represent the remaining parts of the tree and the structure of the cluster is not shown.

The definition of resonance which is obtained at the end is rather involved, and we give it here only for completeness. Given a cluster T denote with n_T^i the minimum of the scales of the lines entering T (if any) and with n_T^o the scale of the line exiting T (if any). A cluster T of ϑ, with one exiting line and at least one entering line, will be called a *resonance* with *resonance-scale* $n = \min\{n_T^i, n_T^o\}$, if (1) $\sum_{u \in V(T)} \nu_u = 0$, (2) all the lines entering V are on the same scale except at most one which can be on a higher scale, (3) $n_T^i \le n_T^o$ if $m_T \ge 2$, and $|n_T^i - n_T^o| \le 1$ for $m_T = 1$, (4) $k_T < q_n$, (5) $m_T = 1$ if $q_{n+1} \le 4q_n$, (6) if $q_{n+1} > 4q_n$ and $m_T \ge 2$, denoting by k_0 the sum of the orders of the subtrees of order $< q_{n+1}/4$ entering T, either there is only one subtree of order $k_1 \ge q_{n+1}/4$ entering T and $k_0 < q_{n+1}/8$, or there is no such subtree and $k_0 < q_{n+1}/4$.

Of course we shall not give here a proof how to extend Lemma 4 to the number of non-resonant lines (with the new definition of resonance), for which we refer to the literature [4]. An important remark is however that if we try to prove a bound like (23) for $N_n^*(\vartheta)$ we have to restrict the definition of non-resonant lines, in order to eliminate some cases which we are not able to control (of course all such cases are not present in the case of the semistandard map because of the condition (1) in the definition of resonance). What is left out at the end is exactly the class of graphs verifying all the conditions (1)÷(6) listed above.

6 Extensions, conclusions and open problems

The analysis of the previous sections extends to more general functions f in (4), and analyticity of the conjugating function can be proved for any analytic f, at least as far as we are not looking for optimal bounds. This means that the proof of Theorem 3 can be easily obtained for any analytic function f in

(4), but no analogous of Theorems 4 and 5 is known for functions f different from those of the semistandard and the standard maps.

Some light can be shed on the problem by studying the conjugating function for $\omega \in \mathbb{C}$. Suppose for instance that, in the case of the standard map, a bound like that of Theorem 5 still holds for complex rotation numbers. One can take ω of the form $\omega = p/q + i\eta$, with $\eta \in \mathbb{R}$ tending to zero (more generally one can consider rotation numbers tending to a rational value along any path of the complex plane non-tangentially to the real axis): since the Brjuno function $B(p/q + i\eta)$ diverges as $q^{-1} \log |\eta|^{-1}$ [17] one can expect that $\rho(p/q + i\eta)$ goes to zero as $|\eta|^{2/q}$. In fact this can be proved, and one can also prove that, by rescaling $\varepsilon \to (2\pi\eta)^{2/q}\varepsilon$ and letting η go to zero, then the conjugating function admits a limit function which can be easily expressed in terms of elliptic functions [2]. An analogous result has been also proved for the semistandard map and Siegel's problem [7].

The advantage to study the conjugating function for complex rotation number is that it is by far easier. Then one can ask how the radius of convergence $r(\omega, f)$ behaves for rotation numbers of the form $\omega = p/q + i\eta$ when more general functions f are taken into account. What is found [3, 5] is that the scaling properties of the radius of convergence strongly depend on the perturbation: one finds the that $r(p/q + i\eta, f)$ goes to zero as $|\eta|^{2/q(f)}$ for an integer $q(f)$ which can be obtained from the solution of a suitable Diophantine problem. Generically one has $q(f) = 1$: in particular this means that the standard map is not generic in this respect. Another consequence is that a bound like those of Theorems 4 and 5 can not hold for any analytic function in general, not even by replacing the factor 2 with another integer or real number. It is an open problem to see if, given a function f in (4), there exists a suitable function, generalizing the Brjuno function, in terms of which one can bound the radius of convergence.

Coming back to the the standard map, another interesting problem is to see what happens by keeping real the parameter ε. In other words one can study until which real value of $\varepsilon_c(\omega)$ the invariant curve with rotation number ω exists for the standard map; of course one has $\varepsilon_c(\omega) \geq \rho(\omega)$ and numerical evidence [1] shows that one can have $\varepsilon_c(\omega) > \rho(\omega)$. Note that a problem of this can kind does not arise for the semistandard map and Siegel's problem, where the analyticity domains (in ε and in α, respectively) are just disks.

The value $\varepsilon_c(\omega)$ is called the *critical function*: the problem of determining $\varepsilon_c(\omega)$ is much difficult, and it can not be studied with the techniques described so far (which work well for ε inside the analyticity domain) There is some numerical (and even some partial theoretical) results on the subject [8], which seem to suggest that if a bound analogous to that of Theorem 5 holds, with the factor 2 replaced by some other number β, then β is likely to be 1. But the situation is not so clear and the problem is still open. Note also that this is a problem for which it is very difficult to obtain results from numerical investigations, as Brjuno numbers (which are not necessarily Diophantine) are very hard to deal with from a numerical point of view.

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