CHAPTER VI

Developed turbulence

§6.1 Functional integral representation of stationary distributions.

From a qualitative viewpoint the onset of turbulence, *i.e.* the birth of chaos, is rather well understood, as analyzed in Chap. IV.

Here we shall try to begin an analysis devoted to understanding and organizing the properties of developed turbulence, "at large Reynolds number".

One could think that the only difference is the large dimension of the attracting set A (or of the attracting sets, in the hysteresis cases *c.f.r.* §4.3), and that the statistics that describes motions following random choices of initial data is determined by Ruelle's principle, *c.f.r.* §5.7, at least if initial data are chosen within the basin of the attracting set and with a distribution absolutely continuous with respect to volume.

The main difficulty in considering such viewpoint as satisfactory, even only approximately, is that usually we do not have any idea whatsoever of the nature and location in phase space of the attracting set. Hence it is difficult, if not impossible, to apply the principle as we lack, in a certain sense, the "raw material".

One can begin by trying to study probability distributions, defined on the space of the of velocity fields, and invariant with respect to the evolution defined by NS equations, incompressible to fix the ideas. Such distributions can be candidates to describe the statistics of motions developing on a some attracting set, *i.e.* stationary states of the fluid.

The approach is suggested and justified by the success of equilibrium statistical mechanics where there exists a natural invariant distribution, proportional to the Liouville measure, which in fact describes the statistical and thermodynamic properties of most systems.

For the NS equation the problem is, however, difficult because there are no obvious invariant distributions when the external force that keeps the fluid in motion is constant.

A better situation is met, instead, in cases in which the external force is random; and it is useful to quote briefly the basic formal results, warning immediately that one is (still) unable to go much further than formal discussions.

Consider the equation on $\Omega = [0, L]^3$ with periodic conditions

$$\begin{aligned} &\underline{\dot{u}} + \underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}}\, \underline{u} = \nu \Delta \underline{u} - \underline{\partial} p + \underline{f}(t) \\ &\underline{\partial} \cdot \underline{u} = 0, \qquad \int_\Omega \underline{u}\, d\underline{x} = \underline{0} \end{aligned} \tag{6.1.1}$$

where $t \to \underline{f}(t)$ is a randomly chosen volume force.

We shall suppose that the force $\underline{f}$ has a *Gaussian distribution*, determined, therefore, by its *covariance*.[1] For example if we set

$$\underline{f}(\underline{x}, t) = \sum_{\underline{k} \neq \underline{0}} e^{i \underline{k} \cdot \underline{x}} \underline{f}_{\underline{k}}(t) = \sum_{\underline{k} \neq \underline{0}} \int dk_0\, e^{i\,(\underline{k} \cdot \underline{x} + k_0 t)}\, \underline{f}_{\underline{k}} \tag{6.1.4}$$

an interesting covariance for the variable $f_{\underline{k},\alpha}$, *i.e.* for the component of mode $\underline{k}$ and label $\alpha = 1, 2, 3$, of the forcing (interesting because it is an hypothesis "maximum simplicity") can be

$$\langle\, \overline{f_{\underline{k},\alpha}(t)} f_{\underline{h},\beta}(t') \rangle = \delta_{\underline{h}\,\underline{k}}\, g_{\underline{k},\alpha,\beta}(t - t') \tag{6.1.5}$$

with $g_{\underline{k}}$ proportional to $\gamma \delta_{\alpha\beta} \delta(t - t')$ and given by:

$$g_{\underline{k},\alpha,\beta}(t - t') = \frac{\gamma \delta_{\alpha\beta}}{|\underline{k}|^a} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k_0 (t - t')}\, dk_0 \tag{6.1.6}$$

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[1] A *Gaussian process* is a probability distribution $\overline{P}$ for random choices of functions $\xi \to f_\xi$, defined on a *finite* space of N labels ξ such that every linear functional having the form $\sum_\xi c_\xi f_\xi$ with c_ξ arbitrary constants is a random variable with Gaussian distribution. This simply means that one can find a positive definite symmetric matrix A such that the probability of the infinitesimal volume element $df = \prod_\xi df_\xi$ is

$$\overline{P}(df) \equiv e^{-\frac{1}{2}(Af, f)} \frac{\prod_\xi df_\xi}{(\pi^N \det A^{-1})^{1/2}} \tag{6.1.2}$$

The distribution $\overline{P}$ is therefore uniquely defined by the matrix A or, what is the same, by A^{-1}. Since one can check that $\langle f_\xi f_\eta \rangle \overset{def}{=} \int \overline{P}(df) f_\xi f_\eta \equiv (A^{-1})_{\xi\eta}$ it is convenient to say that the stochastic process is defined by the *covariance matrix* A^{-1}, simply called *covariance*. Note that the average values of the products $\prod_{i=1}^{2n} f_{\xi_i}$ are easily expressible in terms of the covariance ("*Wick theorem*"). This follows by differentiating both sides of the identity

$$\int \overline{P}(df) e^{\sum c_\xi f_\xi} = e^{\frac{1}{2}(A^{-1} c, c)} \tag{6.1.3}$$

with respect to the parameters c_ξ and then setting them equal to zero.

If the space of the labels not is finite, then a gaussian process can be defined in the same way in terms of the covariance. This means that one assigns a positive definite quadratic form denoted $(A^{-1} f, f)$ and one defines the integrals of the products $\prod_{i=1}^{2n} f_{\xi_i}$ by suitable differentiations via (6.1.3).

Note that if $\overline{P}$ is a gaussian process then every linear operator $f \to O(f)$ defines a gaussian random variable.

(having represented $\delta(t-t')$ as an ntegral over an auxiliary variable k_0) for some constants $\gamma > 0, a$ that are called the *intensity* and the *color* of the random force $\underline{f}$. The constant a will be taken ≥ 0 but it could be taken < 0 as well. Indeed the case $a = -2$ is of particular interest as, for instance, argued in [YO86]: it should give the statistical properties of a fluid in thermal equilibrium at temperature T if $\gamma = 2\nu k_B T/\rho$ with k_B being Boltzmann's constant and ρ the fluid density. It is believed that the most relevant case for our purposes is $a = 3$, see [YO86].

Imagining, for simplicity, that the fluid is initially motionless at time $-\Theta$ (and therefore $\underline{u}(-\Theta) = \underline{0}$), we shall have

$$\underline{u}_{\underline{k}}(t) = \\ = \int_{-\Theta}^{t} e^{-\nu \underline{k}^2 (t-\tau)} \Big(\Pi_{\underline{k}} \underline{f}_{\underline{k}}(\tau) - i \sum_{\underline{k}_1 + \underline{k}_2 = \underline{k}} \underset{\sim}{u}_{\underline{k}_1}(\tau) \cdot \underset{\sim}{k}_2 \, \Pi_{\underline{k}} \underline{u}_{\underline{k}_2}(\tau) \Big) \, d\tau \tag{6.1.7}$$

where (*c.f.r.* (2.2.7)) $\Pi_{\underline{k}} \equiv \Pi_{\rm rot}$ denotes the projection on the plane orthogonal to $\underline{k}$ and we have *ignored* the uniqueness problems of the solutions of (6.1.7) (which we do not know how to solve, something that should not be forgotten).

Let us proceed heuristically by asking ourselves which could be the probability distribution of the field $\underline{u}(t)$ for t large and write symbolically the equation (6.1.7) as:

$$u = v + Tu \tag{6.1.8}$$

where v depends *only on the external force* $\underline{f}$ and $Tu \equiv Q(u,u)$ is the nonlinear term in (6.1.7) thought of as a function defined on the fields $u = \underline{u}(\underline{x}, t)$ with zero divergence. Hence, always formally

$$v = (1-T)u, \qquad u = (1-T)^{-1} v \tag{6.1.9}$$

The probability distribution of u is then a distribution μ such that the μ–probability that u is in "an infinitesimal volume element du" is

$$\mu(du) = P_0(dv) \tag{6.1.10}$$

where P_0 is the probability distribution of v (inherited directly from that of $\underline{f}$, assumed known *c.f.r.* (6.1.6)) and dv is the image of the volume element du under the map $(1-T)$: symbolically $dv = (1-T)du$.

Note that in the example (6.1.5) the distribution $P_0(dv)$ is the distribution of $v = \int_{-\Theta}^{t} \Gamma_{t-\tau} * \Pi_{\rm rot} \underline{f}(\tau) d\tau$ (a relation following from first term in the r.h.s. of (6.1.7), where Γ is the periodic heat equation Green's function, cf. (3.3.17)) and it is a Gaussian distribution. Hence if we write, for the purpose of a formal analysis, $P_0(dv) = G(v)dv$ we find

$$\mu(du) = G(v) \frac{dv}{du} du = G((1-T)u) \det \frac{\partial (1-T)u}{\partial u} du \tag{6.1.11}$$

Obviously this formal computation has to be interpreted. A possible way is to discretize "everything": *i.e.* to fix an interval of time t_0, to set $\tau = jt_0$ and to write the integral in (6.1.7) as a finite sum $t_0 \sum_{j=0}^{t/t_0-1}$, and furthermore to truncate also the sum over $\underline{k}_1, \underline{k}_2$ at a cut–off value N: $|\underline{k}| < N$. Here t_0, N are regularization parameters introduced for the purposes of our calculations and that should be let eventually to 0 and ∞, respectively.

If this viewpoint is adopted we see that the jacobian matrix $\frac{\partial(1-T)u}{\partial u}$ has a triangular form with 1 on the diagonal, because the times in the left hand side of the discretized (6.1.7) are all $< t$ (because jt_0 goes up to $t - t_0$ *excluded*).[2]

Hence we obtain from the (6.1.11), *c.f.r.* [Gh60], the *Ghirsanov formula*

$$\mu(du) = G((1-T)u)du \tag{6.1.12}$$

where $P_0(dv) \stackrel{def}{=} G(v)dv$ is the distribution of $v = \int_{-\Theta}^{t} \Gamma_{t-\tau} * \Pi_{\rm rot}\, \underline{f}(\tau)d\tau$.

With the notation $(\underline{u}, \underline{v}) = \int \underline{u}(\underline{x}, t)\cdot \underline{v}(\underline{x}, t)\, d\underline{x}\, dt/2\pi L^3$ that defines a scalar product between *zero divergence* fields $\underline{u}, \underline{v}$ we want to suppose that the external force field $\underline{f}$ is a random force with a gaussian distribution.

Therefore v, being a linear function of $\underline{f}$ *i.e.* $v = \int_{-\Theta}^{t} \Gamma_{t-\tau} * \Pi_{\rm rot}\, \underline{f}(\tau)\, d\tau$, will also have a gaussian distribution and it will be possible to write (formally) its distribution as $G(v)\, dv = \mathcal{N} \exp -(Av, v)/2\, dv$ where $\mathcal{N}$ is a constant of normalization and (Av, v) is a suitable quadratic form. We shall also write $Tu \equiv Q(u, u)$, to remind us that T is a nonlinear operator quadratic in u (*c.f.r.* (6.1.7)). With the latter notations we see that eq. (6.1.12) becomes

$$\begin{aligned}\mu(du) =&\mathcal{N} e^{-\frac12 (Au,u)} e^{-\frac12 (AQ(u,u),Q(u,u))-(Au,u),u)} du \equiv\\ \equiv& P_0(du)\, e^{-\frac12 (AQ(u,u),Q(u,u))-(AQ(u,u),u)}\end{aligned} \tag{6.1.13}$$

and the gaussian process $P_0(dv) = G(v)dv$ has covariance, *c.f.r.* the *first* term of the right hand side of (6.1.7):

$$\begin{aligned}&\int \overline{u}_{\underline{k},\alpha}(t) u_{\underline{h},\beta}(\tau) dP_0 = \int_{-\Theta}^{t} d\vartheta \int_{-\Theta}^{\tau} d\vartheta' e^{-\nu \underline{k}^2 (t-\vartheta)} e^{-\nu \underline{h}^2 (\tau-\vartheta')}\cdot\\ &\cdot \int dP_0\, \delta_{\underline{h}\,\underline{k}}\, (\Pi_{\underline{k},\alpha\alpha'}\, \overline{f}_{\underline{k}\alpha'}(\vartheta))(\Pi_{\underline{h},\beta\beta'} f_{\underline{h}\beta'}(\vartheta')) =\\ &= \delta_{\underline{h}\,\underline{k}} \int_{-\Theta}^{t} \int_{-\Theta}^{\tau} d\vartheta d\vartheta' e^{-\nu \underline{k}^2 (t+\tau-2\vartheta)} \Pi_{\underline{k},\alpha\alpha'} \Pi_{\underline{h},\beta\beta'} \delta_{\alpha'\beta'} \frac{\gamma}{|\underline{k}|^a} \delta(\vartheta - \vartheta') =\end{aligned}$$

[2] One could object that all this ignores that although in the discretization of the integral we wrote it as a sum up to $j = t/t_0 - 1$ we could *equally well* have written it as sum up to $j = t/t_0$. And in this second way the elements on the diagonal *would not have all been* equal to 1. But they would have differed from 1 by a quantity proportional to t_0 that, formally, in the limit $t_0 \to 0$, would vanish, except possibly the term linear in t_0, if the determinant was computed as series in t_0. The linear term in t_0 however *is zero* because the sum on $\underline{k}_1$ and $\underline{k}_2$ escludes always that $\underline{k}_1 = \underline{k}$ or $\underline{k}_2 = \underline{k}$. Such last terms are those that would contribute to the value of the determinant at first order in t_0.

$$= \frac{\gamma}{|\underline{k}|^a}\delta_{\underline{k}\,\underline{h}}\Pi_{\underline{k}\,\alpha\beta}\int_{-\Theta}^{\min(t,\tau)} d\vartheta e^{-\nu\underline{k}^2(t+\tau-2\vartheta)} \equiv \tag{6.1.14}$$
$$\equiv \frac{\gamma}{|\underline{k}|^a}\delta_{\underline{k}\,\underline{h}}\,(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{|\underline{k}|^2})\frac{e^{-\nu\underline{k}^2|t-\tau|}-e^{-\nu\underline{k}^2|t+\tau+2\Theta|}}{\nu\underline{k}^2}$$

with the convention of summation on repeated labels.

The gaussian distribution P_0 depends on Θ and we shall consider the limit P_∞ as $\Theta\to\infty$ of P_0. This will be a (formally) stationary gaussian distribution which should not be confused with the limit as $\Theta\to\infty$ of the non gaussian distribution μ in (6.1.10) or (6.1.13). It is now interesting to study first the process P_∞. More precisely we consider the process $\underline{u}(\underline{x},t)$ for $\Theta\to\infty$.

Such process is defined by its covariance

$$\int \overline{u}_{\underline{k},\alpha}(t)u_{\underline{h},\beta}(\tau)dP_\infty \stackrel{def}{=} \lim_{\Theta\to\infty}\int \overline{u}_{\underline{k},\alpha}(t)u_{\underline{h},\beta}(\tau)dP_0 \tag{6.1.15}$$

and we see that for $\Theta\to\infty$ the distribution converge, in the sense that its covariance $\langle\,\overline{u}_{\underline{k},\alpha}(t)u_{\underline{k},\beta}(\tau)\rangle$ tends to a limit that is interpreted as covariance of the gaussian process (invariant under time translation, obviously) with covariance, *c.f.r.* (6.1.14)

$$\langle\,\overline{u}_{\underline{k}\alpha}(t)u_{\underline{k}\beta}(\tau)\rangle = \frac{\gamma}{|\underline{k}|^a}\delta_{\underline{k}\,\underline{h}}(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{|\underline{k}|^2})\frac{e^{-\nu\underline{k}^2|t-\tau|}}{\nu\underline{k}^2}\equiv$$
$$\equiv \frac{2\gamma}{|\underline{k}|^a}\delta_{\underline{k}\,\underline{h}}(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{|\underline{k}|^2})\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{e^{ik_0(t-\tau)}}{k_0^2+(\nu\underline{k}^2)^2} \tag{6.1.16}$$

as we deduce simply by computing the integral with the method of the residues. Introducing the Fourier transform $u_{\underline{k},\alpha,k_0}$ in time of $u_{\underline{k},\alpha}(t)$ defined by setting $u_{\underline{k},\alpha}(t)=\int_{-\infty}^{\infty}dk_0\,u_{\underline{k},\alpha,k_0}\,e^{-ik_0\,t}$ the (6.1.16) can also be written as an equality between the covariance $\langle\,\overline{u}_{\underline{k},\alpha,k_0}u_{\underline{h},\beta,h_0}\rangle$ and $\gamma\pi^{-1}\delta_{\underline{k}\,\underline{h}}(\delta_{\alpha,\beta}-k_\alpha k_\beta/\underline{k}^2)\,\delta(k_0-h_0)/(k_0^2+(\nu\underline{k}^2)^2)$.

Hence if A is the matrix of the quadratic form (Au,u) defined by

$$\frac{2\pi}{2\gamma}\int_\Omega\int_{-\infty}^{\infty}\frac{d\underline{x}\,dt}{2\pi L^3}(-\Delta)^{a/2}(\partial_t^2+\nu^2\Delta^2)\underline{u}(\underline{x},t)\cdot\underline{u}(\underline{x},t)=(Au,u) \tag{6.1.17}$$

i.e. :

$$A=\frac{\pi}{\gamma}(-\Delta)^{a/2}(\partial_t^2+\nu^2\Delta^2)\equiv\frac{\pi}{\gamma}(-\Delta)^{a/2}(\partial_t-\nu\Delta)(-\partial_t-\nu\Delta) \tag{6.1.18}$$

we see that $P_\infty(du)$ is a Gaussian process defined by the operator A on the space of the *zero divergence velocity fields.*

Turning our attention to the actual velocity field which we have also called u, *c.f.r.* (6.1.13), it follows that the asymptotic distribution of the field u is, *c.f.r.* (6.1.13)

$$\mu(du)=P_\infty(du)\,e^{-\frac{1}{2}(AQ(u,u),Q(u,u))-(AQ(u,u),u}} \tag{6.1.19}$$

that provides us with a first *formal expression* for μ.

The quadratic form $Q(u,u)$ can be expressed, in the limit $\Theta\to\infty$, as

$$\begin{aligned} Q(\underline{u},\underline{u})_{\underline{k}}(t) &= \int_{-\infty}^{t} d\tau e^{-\nu\underline{k}^2(t-\tau)}\,\Pi_{\rm rot}\,\big(\underset{\sim}{u}(\tau)\cdot\underset{\sim}{\partial}\,\underline{u}(\tau)\big)_{\underline{k}} \equiv \\ &\equiv \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty}\frac{dk_0}{\pi}\,\frac{e^{ik_0(t-\tau)}}{ik_0+\nu\underline{k}^2}\,\Pi_{\rm rot}\,\big(\underset{\sim}{u}(\tau)\cdot\underset{\sim}{\partial}\,\underline{u}(\tau)\big)_{\underline{k}} = \\ &= \Big(\frac{1}{(\partial_t-\nu\Delta)}\Pi_{\rm rot}\,\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}\Big)_{\underline{k}} \end{aligned} \tag{6.1.20}$$

because if $\tau>t$ the integral over k_0 in the intermediate term of (6.1.20) vanishes (by integrating on k_0 with the method of the residues); hence

$$\begin{aligned} (AQ(u,u),Q(u,u)) &= \frac{\pi}{\gamma}\Big((-\Delta)^{a/2}(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}),\Pi_{\rm rot}\,(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u})\Big) \\ (AQ(u,u),u) &= \frac{\pi}{\gamma}\Big((-\Delta)^{a/2}(\partial_t-\nu\Delta)(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}),\underline{u}\Big) \end{aligned} \tag{6.1.21}$$

meaning that this is a function on the zero divergence fields $\underline{u}(\underline{x},t)$ defined for $\underline{x}\in\Omega$, $t\in(-\infty,+\infty)$.

The conclusion is that the probability distribution on the space of velocity fields with zero divergence

$$\begin{aligned} &\mu(d\underline{u}) = \mathcal{N}\, e^{-\frac{(2\pi L^3)^{-1}}{2\gamma/\pi}\int d\underline{x}dt\big((-\Delta)^{a/2}(\partial_t^2+\nu^2\Delta^2)u,u\big)}\cdot du\,\cdot \\ &\cdot e^{-\frac{(2\pi L^3)^{-1}}{\gamma/\pi}\int d\underline{x}dt\big[\big(\frac12(-\Delta)^{a/2}\Pi_{\rm rot}\,(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}),(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u})\big)-\big((-\Delta)^{a/2}(-\partial_t-\nu\Delta)(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}),\underline{u}\big)\big]} \end{aligned} \tag{6.1.22}$$

is a formally invariant measure for the NS equation subject to a random force, with covariance in space–time

$$\frac{\gamma/\pi}{(-\Delta)^{a/2}(k_0^2+\nu^2\Delta^2)}\delta_{\underline{k}\,\underline{k}'}(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{|\underline{k}|^2})\,\delta(k_0-k_0') \tag{6.1.23}$$

and the case $a=0$ corresponds to a *white noise*, while the cases $a\neq 0$ will be called *colored noises* of color $a/2$.

If *a is even and positive* we have a very interesting case in which the functional integral (6.1.23) is, from a formal viewpoint, a *local field theory*. In this case the quadratic forms in (6.1.22) indeed are expressed as integrals of products of velocity fields and their derivatives: all of them computed at the *same point* of space–time.[3] This no longer true if $a/2$ is not an integer ≥ 1 (because $(-\Delta)^{a/2-1}$ is not a local operator). Here one should keep in mind that heuristic arguments suggest that a most interesting case should be the odd color $a=3$.

[3] Note that $(-\Delta)^a\Pi_{\rm rot}$ is an ordinary differential operator if a is even ≥ 2 because its Fourier transform is the matrix $|\underline{k}|^a\delta_{\alpha\beta}-k_\alpha k_\beta|\underline{k}|^{a-2}$, $\alpha,\beta=1,2,3$.

Here, as in all formal field theories, it is necessary to clarify what one should exactly understand when considering the functional integral (6.1.22). And this must be done even if we wish to, or are willing to, forget the mathematical check that it *really* defines a probability distribution invariant for the evolution generated by the NS equation (with noise).[4]

Before attempting to set up this ("overwhelming") task, it is good to stop to assess the situation. Precisely we must ask which could be the interest of what we are saying and writing (or perhaps reading).

Why to modify and widen the problem by considering random volume forces, when what interests us is the case of constant volume forces?

First of all it is easy to add a constant force $\underline{g}$ (that we shall suppose with zero divergence (without affecting the generality). It suffices indeed to replace in (6.1.7) $\underline{f}_{\underline{k}}$ with $\underline{f}_{\underline{k}} + \underline{g}_{\underline{k}}$ and proceed, without hesitation, in the same way. We see immediately that in such case the formally invariant distribution, that we denote $\mu_{\underline{g}}(du)$, is written

$$\mu_{\underline{g}}(du) = \mu(du)\, e^{-\nu^{-1}(\Delta^{-1}A\underline{g}, u - Q(u,u))} \tag{6.1.24}$$

where $\mu(du)$ is the (6.1.22): $\mu \equiv \mu_{\underline{0}}$.
Hence an invariant distribution in presence of external force $\underline{g}$ and *absence* of noise can be obtained as the *limit* ($\gamma \to 0$) of a field theory with Lagrangian $\mathcal{L}(\underline{u})$

$$\begin{aligned}
&-\frac{\gamma}{\pi}\mathcal{L}(\underline{u},\underline{\dot{u}}) = \int \frac{d\underline{x}dt}{2\pi L^3}\Big(-\frac{1}{2}|(-\Delta)^{a/4}(\partial_t+\nu\Delta)\underline{u}|^2 -\\
&-\frac{1}{2}|(-\Delta)^{a/4}\Pi_{\rm rot}\,\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}|^2 - \big((-\Delta)^{a/2}(\partial_t-\nu\Delta)(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u})\big)\cdot\underline{u}\Big)+\\
&+\nu^{-1}([(-\Delta)^{-1+a/2}(-\partial_t^2+(\nu\Delta)^2)\underline{g}],\underline{u})-\\
&-\nu^{-1}\big([(-\Delta)^{a/2}(\partial_t-\nu\Delta)\underline{g}],(\underset{\sim}{u}\cdot\underset{\sim}{\partial}u)\big)
\end{aligned} \tag{6.1.25}$$

This limit could be interpreted as a probability distribution concentrated on the fields that minimize the $\mathcal{L}$ in (6.1.25). They are solutions of the differential equations that are obtained by imposing a stationarity condition on the functional (6.1.25). *The color parameter can be chosen arbitrarily in the sense that it will be possible, for any value a, to interpret every limit as a distribution invariant for the evolution of NS.*
It is, however, difficult to see what has been gained.
The extremes of the action built with the lagrangian $\mathcal{L}$ are, formally, invariant measures but they obey to differential equations (if a is even, at least) that must be equivalent to the initial equation of NS, or better to its

[4] Which would lead us back to the theory of exixtence and uniqueness of the solution of the NS equation which we have already seen to be still in dire need of new ideas.

"dual" version as equation of evolution for probability distributions on the space of velocity fields. We shall not write it, but it is known that searching for solutions of this type did not yet turn out to be more generous in terms of results than the search, also quite inconclusive, of solutions of the NS equation itself.

In reality the interest of the above analysis lies, rather, in the possibility of using what has been recently understood in the field theory, after the development of the methods of analysis based on the *renormalization group*, in the case $\gamma > 0$, for instance *c.f.r.* [BG95].

The possibility arises that for $\nu \to 0$ and $\underline{g}$ fixed, *i.e.* as the Reynolds number tends to infinity, the results could become *independent* of the value of γ, at least some of them among those of physical relevance and at least for what concerns the distribution of the components $\underline{u}_{\underline{k}}$ with $\underline{k}$ not too large.

At $\underline{g}$ fixed, $\nu \to 0$ and γ small we see that one studies the problem of what happens at large Reynolds number in presence of a small noise. In a certain sense *we study, therefore, properties of the attractors which are stable with respect to (certain) random perturbations*. Hence we see that, if we succeeded in giving some answer to the question of the existence and properties of the distributions (6.1.24) we would obtain the answer to a problem that perhaps is even the most interesting among those that we can pose.

Until now research in the direction of studying the integral (6.1.24) has not borne relevant or *unambiguous* fruits. But the matter continues to attract the attention of many. Among the attempts we quote here [YO86] where functional integration is not explicitly used because the work relies on the different but equivalent approach of perturbation expansions (we could say via an explicit evaluation of Feynman diagrams, although the diagrams are not even mentioned in the quoted paper): approximations must be made and so far it is not yet clear how to estimate the neglected "marginal terms", *c.f.r.* p. 48 in [YO86]. Hence it is important to try to find on heuristic bases what we could "reasonably" expect from a future theory. Hopefully we shall reach some better understanding within a not too remote future.

The treatment parallel to what has been discussed above in the much simpler case of the *Stokes equation*

$$\begin{aligned} \dot{\underline{u}} &= \nu\Delta\underline{u} - \underline{\partial}p + \underline{g} + \underline{f}(t) \\ \underline{\partial}\cdot\underline{u} = 0, &\qquad \int \underline{u} = \underline{0} \end{aligned} \tag{6.1.26}$$

is left as a problem for this section (see problems below): this is a case which is easy but very interesting and instructive: its simplicity makes useless a formal discussion.

Problems.

[6.1.1]: Study the invariant distributions of the evolution defined by the equation, defined on the circle $x \in [0, 2\pi]$, $\dot{u} = \partial_x^2 u + f(t)$ with $f(t)$ a noise of color $c = a/2$ and average zero and with initial data with zero average, along the lines of what has been discussed above. (*Idea:* Note that the theory is purely Gaussian. Compute the covariance.)

[6.1.2]: Study formally along the lines discussed for the NS equation with random noise the invariant distributions for the evolution defined by the equation on the line $\dot{u} = -u + \partial_x^2 u - u^3 + f(t)$ with $f(t)$ a noise of color $a/2$. Derive the lagrangian of the field theory generated by the applicaton of Ghirsanov's formula (6.1.12) and the lagrangian corresponfing to (6.1.15).

[6.1.3] Formulate the theory of the invariant distributions for the equation of Stokes (6.1.26), along the lines of what discussed above. (*Idea:* Note that the theory is purely Gaussian and compute the covariance of the velocity field distribution).

[6.1.4] Compute, in the case of the problem [6.1.1], the behavior of the function of $\underline{x}, \underline{y}$ defined by $\langle(\underline{u}(\underline{x}, t) - \underline{u}(\underline{y}, t))^2\rangle$ when $|\underline{x} - \underline{y}| \to 0$, as a function of the color of the noise: the average is understood over the stationary distribution for the equations (6.1.26) constructed in the problem [6.1.3]. (*Idea:* Study the covariance of the Gaussian distribution in [6.1.3]).

[6.1.5] As in the problem [6.1.4] but for the function $\langle|\underline{u}(\underline{x}, t) - \underline{u}(\underline{y}, t)|^3\rangle$.

[6.1.6] Using the technique of problem [2.4.7] in §2.4 (Wiener theorem) discuss the class of regularity in $\underline{x}$ at fixed t of the samples of the gaussian process in problem [6.1.3] showing that they are Hölder continuous fields with an exponent that can be bounded below in terms of the color of the noise $a/2$ and that becomes positive for a large enough, say $a \ge a_0$ and estimate a_0. (*Idea:* Remark that the covariance becomes regular if a is large enough and then proceed as in the quoted problem of §2.4 to prove the Hölder continuity, with probability 1 of the Brownian motion.)

Bibliography: The analysis in this section is classical, see for example [An90]. For further developments see, for instance, [YO86]. A quote from the latter reference, p. 47, can be interesting in order to develop some intuition about the philosophy behind the more recent attempts at using the renormalization group ("RNG") to study turbulence statistics (however the quote can be better understood after reading the next section where the notion of inertial range is introduced): "*The RNG method developed here* [in [YO86]] *is based on a number of ideas. First there is the correspondence principle, which can be stated as follows. A turbulent fluid characterized in the inertial range by scaling laws can be described in this inertial range by a corresponding Navier–Stokes equation in which a random force generates velocity fluctuations that obey the scaling of the inertial range of the original unforced system. Second*" ... "*We believe that the results of the RNG fixed–point calculations can be applied to any fluid that demonstrates Kolmogorov–like scale–invariant behavior in some range of wavevectors and frequencies. This situation resembles the theory of critical phenomena in the sense that the critical exponents computed at the fixed points are approximately valid in the vicinity of the critical point where* $(T - T_c)/T_c \ll 1$."

Very recent results on the existence of stationary distributions have been obtained in the 2–dimensional case, [KS00], [BKL00]. These works essentially solve completeley some of the problems posed in this section.

§6.2 Phenomenology of developed turbulence and Kolmogorov laws.

Perhaps the most striking of all properties of three–dimensional turbulence is that decrease of the viscosity, with no change of the conditions of generation of the turbulence, is accompanied by increase of the mean square vorticity (the process of magnification of vorticity by extension of vortex lines then being restrained by viscous diffusion of vorticity), and that in the limit $\nu \to 0$ the rate of energy dissipation is conserved (Batchelor),[Ba69].

Consider a Navier–Stokes fluid in a cube $\Omega = [0, L]^d$, $d = 2, 3$, with periodic boundary conditions and subject to a constant force $\underline{g}$, regular enough so that only its Fourier components $\underline{g}_{\underline{k}}$ with $|\underline{k}| \approx 2\pi/L$ are appreciably different from 0. We say that the system is "forced on the *scale* L of the container", or that the fluid receives energy from the outside "on scale" L. We shall study the stationary state of the fluid as we decrease the value ν of the viscosity.

(A): Energy dissipation: inertial and viscous scales.

The constant external force sustains motion and not only it injects into the fluid an energy ε_ν, in average, per unit time and unit volume, but it also has the effect that the velocity components which are appreciably not zero increase in number while the viscosity ν decreases. Hence the components of the Fourier transform of the velocity field which are appreciably not zero will be those with $|\underline{k}| < k'_\nu$ where k'_ν is a suitable scale (with dimension of an inverse length). The length scale $(k'_\nu)^{-1}$ will be called the *viscous scale*. We recall that our conventions on the Fourier transform are such that $\underline{u}(\underline{x})$ and its transform $\underline{u}_{\underline{k}}$ are related by $\underline{u}(\underline{x}) = \sum_{\underline{k}} e^{i\,\underline{k}\cdot\underline{x}}\,\underline{u}_{\underline{k}}$. We shall also fix the density ρ to be $\rho = 1$.

In the NS–equation appear both the *transport term* also called *inertial term*, $\underset{\sim}{u} \cdot \underset{\sim}{\partial}\,\underline{u}$, and the *friction term*, $\nu\,\Delta\,\underline{u}$, also called *viscous term*: the latter contributes to the $\underline{k}$–mode of the Fourier transform of the equation a quantity of size $\nu\,\underline{k}^2\,|\underline{u}_{\underline{k}}|$; while the inertial term shall contribute to the same Fourier component a quantity of order of magnitude $\simeq |\underline{k}||\underline{u}_{\underline{k}}|^2$ at least. Then the friction terms become dominant over the inertial ones at large $|\underline{k}|$: the cross over takes place at values of $\underline{k}$ where (in average over time) $\nu|\underline{k}|^2\,|\underline{u}_{\underline{k}}| \approx |\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underset{\sim}{u}_{\underline{k}_1} \cdot \underline{k}_2 \Pi_{\underline{k}} \underline{u}_{\underline{k}_2}|$, which is a criterion to determine what we shall call the "*viscous scale*" k'_ν. The transition will not be sharp hence we introduce another scale $k_\nu < k'_\nu$ *of the same order of magnitude* so that for $\underline{k}$ in the buffer region between k_ν and k'_ν the inertial and the viscous terms will be comparable (in average over time). It will be supposed that $\underline{u}_{\underline{k}} \equiv \underline{0}$ for $|\underline{k}| > k'_\nu \xrightarrow[\nu\to 0]{} \infty$. We shall set $g = \max_{\underline{k}} |\underline{g}_{\underline{k}}|$, keeping in mind that the forcing $\underline{g}_{\underline{k}}$ differs from $\underline{0}$ sensibly only if $|\underline{k}|$ is close to its minimum value $|\underline{k}| \simeq 2\pi L^{-1} \stackrel{def}{=} k_0$.

If the external force is fixed the low $|\underline{k}|$ modes are certainly not viscous. In fact we could expect that (in the average over time) $|\underline{u}_{\underline{k}}| \gg \nu|\underline{k}|$ for ν small and $|\underline{k}| \approx k_0$ because the external force will be able to produce on the Fourier components of the velocity on which it acts directly velocities of order $|\underline{g}_{\underline{k}}|L^2/\nu$ (because $\nu\underline{k}^2\underline{u}_{\underline{k}} \approx \underline{g}_{\underline{k}}$ would be the time independent solution in absence of inertial terms and $\underline{g}_{\underline{k}}$ is different from $\underline{0}$, appreciably, only if $|\underline{k}| \approx 2\pi/L$).

However $gL^2\nu^{-1}$ is not the only magnitude with the dimensions of a velocity that can be formed with the parameters of the problem (*i.e.* g, L, ν): another one is $\sqrt{gL}$. It seems from the experimental data that, as ν tends to 0, the stationary state of the fluid dissipates per unit time and volume a quantity $\eta = \int \underline{g}\cdot\underline{u}\,d\underline{x} \simeq L^3 g\,\langle|\underline{u}_{\underline{k}_0}|\rangle$ of energy that behaves as a power, negative or possibly 0, of ν: hence we must think that the average velocity on the macroscopic scale ($\langle|\underline{u}_{\underline{k}}|\rangle$ with $|\underline{k}| \simeq k_0 = 2\pi L^{-1}$) also behaves as a power of ν. If it approaches a limit value as $\nu \to 0$ (*i.e.* the exponent of the power of ν is 0), *c.f.r.* [Ta35] and [Kr75a] p. 306, then such limit must be of order $\sqrt{gL}$, which is the only ν–independent velocity that can be formed with the available parameters.

A theoretical confirmation of the existence of a "saturation value" of the dissipation per unit time, as viscosity tends to zero, has been obtained in some cases: *e.g.* in the case (*different from the above* because there is no volume force) of the NS equation for a fluid between two parallel plates in motion with parallel (but different) velocities: *c.f.r.* [DC92]. A simple (not optimal) technique for estimating the dissipation rate, as $\nu \to 0$, in a forced system with constant volume force, is in problems [6.2.1], [6.2.2].

(B): Digression on the physical meaning of "$\nu \to 0$".

It is convenient to make a short digression on the question of viscosity dependence of various quantities (like the average dissipation) in the limit in which the latter vanishes.

It is clear that it is not possible in experiments, other than numerical, to vary viscosity so that it tends to zero. But experiments on Navier–Stokes fluids (*i.e.* that remain well described by NS equations as the experimental parameters vary) can be used to infer what would happen if ν was as variable a parameter as wished. Let us examine two interesting cases.

Consider a fluid in a cubic container of side L, with horizontal *periodic* conditions but with conditions of adherence to the upper and lower walls. The first is supposed to move with constant velocity U while the second is fixed. This is a model (*"shear flow"*) simpler than that of a fluid between two cylinders one of which is fixed and the other rotating (*"Couette flow"*)

Then the Reynolds number is $R = UL\nu^{-1}$. Setting $\underline{u}(\underline{x},t) = \frac{\nu}{L}\underline{w}(\frac{\underline{x}}{L},\frac{\nu t}{L^2})$, *c.f.r.* §1.3, one checks that the equation for $\underline{w}$ becomes that of a fluid with $\nu = 1, L = 1$ and with shear velocity $U' = R$. Hence we see how a real experiment on a fluid with $\nu = 1, L = 1$ and shear velocity $U' = R \to \infty$, conceivable as a laboratory experiment, can give informations on a fluid in which U is fixed and $\nu \to 0$. The relation between the dissipation per unit time and volume of the two fluids, related by the described rescaling of variables, is simple and it is $\eta = \nu L^{-3}\int|\underset{\sim}{\partial}\underline{u}|^2 d\underline{x} = \nu(\nu L^{-1})^2\eta_0(R)$, if $\eta_0(R)$ is the dissipation of the "rescaled" fluid (*i.e.* with unit volume and viscosity).

Therefore if the dissipation η is independent of ν in the limit $\nu \to 0$ this means that $\eta_0(R) \propto R^3$ for $R \to \infty$: in [DC92] the dissipation η is *bounded* above independently of ν in the above shear flow problem.

A second case is a fluid in a periodic cubic container of side L subject to a force $F\underline{g}(\underline{x}/L)$ with $\underline{g}$ fixed (the dependence via $\underline{x}/L$ means that the force acts on "macroscopic scales"). The rescaling of the variables described in the preceding case leads us to say that the rescaled fluid flows in a container of side $L = 1$, with viscosity $\nu = 1$ and subject to a force $FL^3\nu^{-2}\underline{g}(\underline{x}) \equiv R^2\underline{g}(\underline{x})$, where $R = \sqrt{FL^3\nu^{-2}}$. The relation between the dissipation $\eta_0(R)$ in the rescaled fluid and that in the original fluid is still $\eta = \nu^3L^{-4}\eta_0(R)$, and hence the ν–independence (if verified) corresponds to a proportionality of $\eta_0(R)$ to R^3.

The theory that follows will not be based on any hypothesis on the (average) dissipation per unit time and volume as $\nu \to 0$: the validity of any assumption on this quantity (*e.g.* that it tends to a constant) cannot be taken for granted and it should be analyzed on a case by case basis as we are far from a full understanding of it.

(C): The K41 tridimensional theory.

> *The physical picture is that large–scale motions should carry small eddies about without distorting them. It is not obvious that this need be true, but the idea is certainly intuitively plausible, (Kraichnan)* [Kr64].

Coming back to the theme of this section we note that the presence of the inertial terms causes an appreciable dispersion of energy that, in their absence, would stay confined to the modes on which the external force acts: but there is no reason that the average (in time) amplitude of $|\underline{u}_{\underline{k}}|$ for $\underline{k}$ on scale $k_0 = 2\pi L^{-1}$ is not, in a stationary regime, monotonically increasing as ν decreases, and that at the same time the average size of $|\underline{u}_{\underline{k}}|$ is decreasing to 0 as $\underline{k}$ increases to ∞ (at fixed ν). This, alone, is already sufficient to state that it will be possible to neglect friction forces until down to scales $k_\nu \xrightarrow[\nu\to 0]{} \infty$, with k_ν smaller than the momentum k'_ν of the *viscous* scales (where viscous terms become dominant): $k_0 \ll k_\nu < k'_\nu$. In fact, as noted already the ratio between inertial and friction terms is, dimensionally, $|\underline{u}_{\underline{k}}|/\nu\,|\underline{k}|$.

The scale k_ν will be called *Kolmogorov scale*, the scale k'_ν will be called *viscous scale*, but what said until now *does not, yet, allow* us to determine the size of the scales k_ν, k'_ν. They will be determined below on dimensional grounds after discussing the key hypothesis of the theory, namely the "*homogeneity*" of turbulence, *c.f.r.* (6.2.6).

Hence we get to the point of regarding turbulent motion as well described by the equations (if $k_0 \equiv 2\pi L^{-1}$)

$$\begin{aligned}
\dot{\underline{u}}_{\underline{k}} &= -\, i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underset{\sim}{u}_{\underline{k}_1} \cdot \underset{\sim}{k}_2 \Pi_{\underline{k}} \underline{u}_{\underline{k}_2} + \underline{g}_{\underline{k}} && k_0 \le |\underline{k}| \le k_\nu \\
\dot{\underline{u}}_{\underline{k}} &= -\, \nu \underline{k}^2 \underline{u}_{\underline{k}} - i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underset{\sim}{u}_{\underline{k}_1} \cdot \underset{\sim}{k}_2 \Pi_{\underline{k}} \underline{u}_{\underline{k}_2} && k_\nu < |\underline{k}| < k'_\nu \qquad (6.2.1) \\
\dot{\underline{u}}_{\underline{k}} &= -\, \nu \underline{k}^2 \underline{u}_{\underline{k}} && |\underline{k}| > k'_\nu
\end{aligned}$$

that are read by saying that the motion of a fluid at high Reynolds number, *i.e.* at small viscosity, is described by the Euler equation on the scales of length larger than the Kolmogorov scale but it is, instead, described by the Stokes equation on scales smaller than the viscous scale.

One says that
(1) the scales of length are *inertial* for $k_0 \ge |\underline{k}| \ge k_\nu$ and *viscous* if $|\underline{k}| > k'_\nu$. The intermediate modes $k_\nu < |\underline{k}| < k'_\nu$ can be called *dissipative modes* because it is "in them" that the dissipation of the energy provided by the forcing takes place, and
(2) "turbulence" is due to the inertial modes, while the motion of the dissipative and viscous modes is always "laminar". The motion of the viscous modes is trivial and in fact we can set $\underline{u}_{\underline{k}} = \underline{0}$ for the viscous components, *c.f.r.* (6.2.1).
(3) The "role" of the dissipative modes is to absorb energy from the inertial modes and dissipate it.

The problem is now to determine k_ν, k'_ν. Imagine drawing in the space of the modes[1] $\underline{k}$ the sphere of radius $\kappa < k_\nu$. We can then define, from (6.2.1), the energy E^κ per unit time that "exits from the sphere" (of momenta) of radius κ by setting

$$E^\kappa \stackrel{def}{=} \frac{1}{2}\frac{d}{dt} L^3 \sum_{|\underline{k}|<\kappa} |\underline{u}_{\underline{k}}|^2 = \mathcal{L} - \mathcal{E}_{\kappa,\kappa'} - \mathcal{E}_{\kappa,\kappa',\infty} \qquad (6.2.2)$$

[1] One often confuses the "mode" $\underline{k}$, that is a label with the dimension of an inverse length, and the Fourier transform component $\underline{u}_{\underline{k}}$ that is, instead, a dynamical observable with the dimension of a velocity. I will not make here any distinction, because it is not possible (reasonably) to be worried by the ambiguity that one could fear: "temer si dee di sole quelle cose c' hanno potenza di fare altrui male; dell' altre no, chè non son paurose" (II, 88, *Inferno*), *we should fear only about that which can cause harm, not of that which cannot as it is not scary.*

where $\mathcal{L}$ is the work done by the external force $\underline{g}$ per unit time ($\mathcal{L} = L^3\sum_{\underline{k}}\overline{\underline{g}_{\underline{k}}}\cdot\underline{u}_{\underline{k}}$ equal to $2L^3\mathrm{Re}\,\overline{\underline{g}_{\underline{k}_0}}\cdot\underline{u}_{\underline{k}_0}$ if $\underline{g}$ has only the component $\underline{k}_0$ in its Fourier transform), and

$$\mathcal{E}_{\kappa,\kappa'} = iL^3\sum_{|\underline{k}_3|<\kappa}\ \sum^{*}_{\underline{k}_1+\underline{k}_2+\underline{k}_3=\underline{0}} \underset{\sim}{u}_{\underline{k}_1}\cdot\underset{\sim}{k}_2\,\underline{u}_{\underline{k}_2}\cdot\underline{u}_{\underline{k}_3} \tag{6.2.3}$$

and $\sum^*$ means that $|\underline{k}_2|$ is in the interval $[\kappa,\kappa')$ and $|\underline{k}_1|$ is in $[k_0,\kappa')$. Finally $\mathcal{E}_{\kappa,\kappa',\infty}$ is given by a similar expression with $|\underline{k}_2|$ in $[\kappa',\infty)$ or $|\underline{k}_2|\in[\kappa,\kappa')$ and $|\underline{k}_1|\in[\kappa',\infty)$. The apparently missing terms have zero sum because $\underline{k}_1+\underline{k}_2=-\underline{k}_3$ is orthogonal to $\underline{u}_{\underline{k}_1}$, as already seen several times (*c.f.r.* (3.2.14), for example).

One can read (6.2.2) by saying that the energy on scales $<\kappa$ changes because of the work "done by the external force" (expressed by $\mathcal{L}$), because of the work that "the modes with $|\underline{k}|<\kappa$ perform on those with $|\underline{k}|\in[\kappa,\kappa')$ "directly" (expressed by $\mathcal{E}_{\kappa,\kappa'}$) and because of the work that "the modes with $|\underline{k}|<\kappa$ perform on those with $|\underline{k}|\in[\kappa',\infty)$" (expressed by $\mathcal{E}_{\kappa,\kappa',\infty}$).

Hypothesis *(homogeneous turbulence): The fundamental hypothesis of the Kolmogorov's theory is that $\mathcal{E}_{\kappa,\kappa',\infty}$ is, "in average", zero if κ and $\kappa'=2\kappa$ are $<k_\nu$, at least for the motions in a asymptotic regime which, therefore, develop on an attracting set.*

Hence it really makes sense to say that the energy is dissipated only on scales $|\underline{k}|>k_\nu$, "cascading" without sensible dissipation from the large length scales to the small ones.

The hypothesis, which has to be understood as an asymptotic property as $R\to\infty$, does not have a justification other than its simplicity and elegance (besides being, perhaps, very natural) and, *a posteriori*, its eventual self consistency and, even better, its adherence to experimental observations. We shall see, indeed, that it has remarkable implications that can be tested experimentally.

Alternative hypotheses, apparently also reasonable, are possible and lead to *qualitatively different* results, *c.f.r.* (4.1) and (4.3) in [Kr64], see also §6.3.

Note that the assumption is indedependent on whether the dissipation per unit time and volume tends to a constant as $R\to\infty$ or behaves as a power of ν or R.

A way to read the hypothesis is the following: "in the inertial regime ($|\underline{k}|<k_\nu$) we neglect energy exchanges between scales of momentum which are *not* contiguous" (*locality of the energy cascade*), *c.f.r.* [Kr64], [Ga99d]. Or "the smaller vortices are passively transported by the larger ones that contain them": in this form the hypothesis does not seem well in agreement with the visual experience, at least not with the experience on turbulent motions in which the small vortices are such only in one or two dimensions while in the other, or in the other two, they have dimensions of many orders

of magnitude larger and hence their motion is "distorted" by what happens on very different ("not contiguous") length scales. The phenomenon is sometimes called *intermittency.* Therefore we should expect that the theory developed upon the above homogeneous turbulence hypothesis will need corrections.

On the basis of the above heurisic analysis we shall suppose that $\varepsilon_\nu \equiv \varepsilon \equiv L^{-3}E^\kappa$ is constant for κ in the range $k_0 \ll \kappa < k_\nu$. We must now remark that, for what concerns the average properties of the Fourier components of the velocity field with mode $\underline{k}$ of any order κ with $\kappa \in (k_0, k_\nu)$, it *must be equivalent* to force the system on a scale of length L or on another scale, smaller but still $\gg \kappa^{-1} \gg k_\nu^{-1}$, *provided* the quantity of energy dissipated per unit time and mass is always ε.

It follows that ε must be expressible, on all scales $\kappa < k_\nu$, only in terms of quantities that pertain the scale $l = \kappa^{-1}$ *excluding* viscosity (that does not enter in the first of (6.2.1) that describes the motion of the components on scales $\kappa < k_\nu$). Hence if v_l is a velocity variation characteristic of the scale l we must have, for dimensional reasons

$$\varepsilon \simeq v_l^3 l^{-1} \tag{6.2.4}$$

because this is the only quantity with the dimension of energy dissipation per unit volume and unit time that can be formed with quantities characteristic of the length scale l; equation (6.2.4) must hold, in particular, on scale $\simeq L$.

Let us define v_l precisely as a time average

$$v_l^2 \stackrel{def}{=} \langle \big(|\Delta|^{-1} \int_\Delta (\underline{u}(\underline{x}) - \underline{u}(\underline{x}_0)) d\underline{x}\big)^2 \rangle \tag{6.2.5}$$

where Δ is a small cube arbitrarily centered and side $l \equiv \kappa^{-1}$; and (in the stationary states) assume that

(1) the components of the Fourier transform of the velocity relative to different modes $\underline{k}$ are statistically independent,
(2) the velocity variations in disjoint little cubes Δ are statistically independent,
(3) the variables $\underline{u}_{\underline{k}}$, components of the Fourier transform of $\underline{u}$, are essentially equally distributed if $|\underline{k}|$ has a given order of magnitude $k_\nu \gg |\underline{k}| \gg k_0$.

We then see that the energy "contained in the scale k", *i.e.* the energy of the modes $\underline{k}$ such that $k_0 \ll k < |\underline{k}| < 2k \ll k_\nu$, is approximately $l^3 v_l^2$ times the number of the small cubes of side $l = k^{-1}$ that pave Ω (a number that is of order $(kL)^3$, having made use of the hypothesis (2)).

On the other hand, from hypothesis (1) and with the normalizations in (2.2.2) on the Fourier transforms, we see that $v_l^2 \simeq \langle (\sum_{k<|\underline{k}|<2k} \underline{u}_{\underline{k}})^2 \rangle$ is proportional to the number of modes $\underline{k}$ between k and $2k$ (*i.e.* $\propto (kL)^3$), by the central limit theorem and by assumptions (1) and (3). Hence

$$v_l^2 \sim (kL)^3 \langle |\underline{u}_{\underline{k}}|^2 \rangle \qquad \text{and} \qquad v_l^3 = \varepsilon l, \qquad l = |\underline{k}|^{-1} \tag{6.2.6}$$

if $\langle|\underline{u}_{\underline{k}}|^2\rangle$ is the average quadratic value (with respect to time or to the statistics of the attracting set that describes the asymptotic motion) of a single Fourier component of the velocity, (such average does not depend on $\underline{k}$ for $k < |\underline{k}| < 2k$, by the hypothesis (3)).

Hence the average energy $L^3K(k)dk$ contained between k and $k+dk$ is (if $l = k^{-1}$) is given by

$$L^3K(k)dk = L^3 \sum_{k<|\underline{k}|<k+dk} \langle|\underline{u}_{\underline{k}}|^2\rangle = \sum_{k<|\underline{k}|<k+dk} \frac{v_l^2}{k^3} = \frac{1}{k^3}\big(\frac{\varepsilon}{k}\big)^{2/3}\frac{4\pi k^2\,dk}{(2\pi/L)^3} \tag{6.2.7}$$

because $4\pi k^2\,dk/k_0^3$, $k_0 = 2\pi/L$, is the number of modes between k and $k+dk$. It follows that

$$K(k) = cost\ \varepsilon^{2/3}\,k^{-5/3}, \qquad k_0 \ll k \ll k_\nu \tag{6.2.8}$$

is the *energy density* per unit of $|\underline{k}|$ and per unit of mass: "*law* 5/3 *of Kolmogorov*".[2]

We also say that the energy spectrum (*i.e.* $K(k)$) is concentrated at small k, while the vorticity spectrum (*i.e.* $K(k)k^2$) is concentrated at large k.[3] This has been observed by Taylor in 1938, beginning the chain that led Kolmogorov, in 1941, to formulate the theory that we are explaining (*c.f.r.* [Ba70], p. 112) also called *K41 theory.*

The Reynolds number of the fluid is $R = v_L L\nu^{-1}$: and we can introduce the more general notion of *Reynolds number on scale* l as $R_l \equiv v_l l\nu^{-1}$. Therefore it can be computed by using the basic assumption $v_l^3/l = v_L^3/L$ (*c.f.r.* (6.2.4)), as: $R_l = v_L L\nu^{-1}(l/L)^{4/3} \equiv (l/L)^{4/3}\,R$.

After the above discussion of the implications of the homogeneous turbulence hypothesis *the Kolmogorov scale* k_ν *can be naturally identified as the only length scale that can be formed with* ε *and* ν, or as the scale on which the Reynolds number becomes of order $O(1)$. The latter is the scale $l_\nu \equiv k_\nu^{-1}$ such that $R_{l_\nu} = 1$ or $k_\nu = L^{-1}R^{3/4}$, while the first scale is

$$k_\nu \equiv l_\nu^{-1} = \big(\varepsilon/\nu^3\big)^{1/4} = L^{-1}R^{3/4} \tag{6.2.9}$$

where the last equality follows taking, still, into account (6.2.4): the two definitions lead to the same result. We see that, actually, k_ν tends to ∞ as

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[2] The argument according to which, for dimensional reasons, the energy on scales k, contained between k and $2k$, is $v_l^2L^3$ and hence $L^3K(k)k \sim v_l^2L^3$ leads, combined with (6.2.4), directly to (6.2.8), with less assumptions; it allows us to suppose only independence of the $\underline{u}_{\underline{k}}$'s with $\underline{k}$ relative to different scales. However if we only supposed this we would not understand how it could be that $(\sum_{k<|\underline{k}|<2k}\underline{u}_{\underline{k}})^2$, square of the sum of $\sim (kL)^3$ random variables, could have order $(kL)^3$, as instead it follows from the independence of the $\underline{u}_{\underline{k}}$ with different $\underline{k}$.

[3] Because $K(k)$ is summable for $k\to\infty$, but $k^2K(k)$ is not.

$\nu \to 0$ if ε stays fixed or does not tend to 0 too fast (in any event *it is not at all reasonable* to think that ε tends to 0).

Wishing to express everything in terms of the intensity g of the force (*e.g.* $g = (L^{-3} \int \underline{g}(x)^2 dx)^{1/2}$) acting on the fluid and assuming that as the viscosity decreases the energy dissipated per unit time *reaches finite saturation value*, as it is observed in several experiments on systems forced by boundary forces, one should think in the present case that, for dimensional reasons, the asymptotic value (for $\nu \to 0$) of the dissipation ε must be proportional to $L^{1/2} g^{3/2}$, if L is the side of the container. We mention here that the existence of a finite saturation value for the energy dissipation when friction tends to 0 has far reaching consequences: for instance it led Onsager to propose that also the Euler equations did admit stationary states in which energy remains constant in spite of the action of an external force and of the absence of friction in the equations. *c.f.r.* [Ta35],[On49],[Kr75a],[Ey94],[CWT94]).

Note that $v_L \sim (\varepsilon L)^{1/3}$ and $R = \nu^{-1}(\varepsilon L)^{1/3}$ (and if we accept that the asymptotic dissipation is $\propto L^{1/2} g^{3/2}$ then also $R = (gL)^{1/2} L \nu^{-1}$). It follows that the number of Fourier components with $k < k_\nu$ is of order

$$N_\nu \sim \frac{8\pi}{3} \frac{k_\nu^3}{(2\pi L^{-1})^3} = cost\, R^{9/4} \tag{6.2.10}$$

that gives the order of magnitude of the *apparent number of degrees of freedom*, probably proportional to the *information dimension* of the attractor. See §6.3 for a model simpler than the NS equation, but similar to them, in which the fractal dimension of the attractor turns out to be half of the apparent dimension.

We can ask whether the Kolmogorov law is exactly true already for $k \sim k_0 = 2\pi/L$. In reality we must expect deviations at least on scales comparable to that on the which the force acts (*i.e.* L); because on such scales the details of the structure of the force must be important. Hence we must think that what said on the law 5/3 is true in an interval of scales $k_0' \ll k \ll k_\nu$ with $k_0 \ll k_0'$ and $k_\nu = k_0 R^{3/4}$. This interval of scales is called *inertial domain*, or *inertial field*, and in it we have *homogeneous turbulence*, in which homogeneous universal laws hold for the energy distribution (*c.f.r.* (6.2.8)) and for other quantities (*c.f.r.* problems).

Let us now determine the viscous scale $k_\nu' > k_\nu$ that provides us with a natural *ultraviolet cut–off* for the NS equation: it is interesting here to recall the discussion in §2.2 where the ultraviolet cut–off has been introduced without discussing it physical meaning to formulate an empirical algorithm of solution of the NS equation.

By what said above the value of k_ν' must be determined from the condition that the viscous terms, which become comparable with the inertial ones on scales $k \sim k_\nu$, dominate on the inertial ones for $k > k_\nu'$.

We have seen that the condition is the equality of the averages (in time) $\nu|\underline{k}|^2\,\langle|\underline{u}_{\underline{k}}|\rangle \approx \langle|\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underset{\sim}{u}_{\underline{k}_1}\cdot \underset{\sim}{k}_2\Pi_{\underline{k}}\underline{u}_{\underline{k}_2}|\rangle$, and the sum can be roughly estimated as $|\underline{k}|\langle|\underline{u}_{\underline{k}}|^2\rangle(|\underline{k}|L)^{\frac{3}{2}}$ by using the hypothesis (1), which means that it is a sum of $O((|\underline{k}|L)^3)$ independent random variables with zero average (since $\underline{k}_1$ cannot be parallel to $\underline{k}_2$ by the zero divergence property of $\underline{u}$), and $\langle|\underline{u}_{\underline{k}}|^2\rangle^{1/2}$ can be evaluated as $v_l(|\underline{k}|L)^{-\frac{3}{2}}$ by using (6.2.6) (with $l=k^{-1}$); hence we see that this condition leads to $k'_\nu \sim (\varepsilon\nu^{-3})^{\frac{1}{4}} = k_\nu$. Here it is useful to keep in mind that k_ν was defined, heuristically and independently, on purely dimensional grounds, *c.f.r.* equation (6.2.9).

Hence the scales k'_ν and k_ν have the same order of magnitude, a not surprising consistency check: this means that the arguments given here are not sufficient to determine the ratio k'_ν/k_ν and they only say that it will be ≈ 1; a value of $k'_\nu = 2k_\nu$ (for example) could be acceptable for possible numerical experiments or further theoretical deductions (and the correct value of the ratio would be determined by finding at which magnitude of the ratio the results become independent from value of the ratio itself).

We can formulate a more general hypothesis (*c.f.r.* [Ba70], p. 114) that for *all the scales k, also "close"(enough) to k_0 or "beyond" k_ν*, the spectrum of energy depends only on ε and ν: the hypothesis is that $K(k) = v_c^2 l_c K_{univ}(l_c k)$, with v_c, l_c "characteristic" velocity and length scales only dependent on ε,ν, *i.e.* $l_c = (\nu^3\varepsilon^{-1})^{1/4}$ and $v_c = (\nu\varepsilon)^{1/4}$, and with K_{univ} a suitable universal function of its argument.

This implies that if k is in the inertial domain, in which the (average) energy density spectrum depends only on ε and is a given by a power law, the universal function $K_{univ}(x)$ must be $\propto x^{-5/3}$, because only the combination $v_c^2 l_c l_c^{-5/3} = \varepsilon^{2/3}$ depends only on ε.

The domain of validity of the more general formula should be more extended than that in which $K_{univ} \propto k^{-5/3}$ and the question of which are the scales k_1, k'_0, k_2 that delimit

(a) the inertial domain (k'_0, k_ν) and

(b) the larger domain (k_1, k_2), with $k_0 < k_1 < k'_0 < k_\nu < k_2 < k'_\nu$, in which the more general law holds

is a question which is not soluble within the present framework and it requires a more detailed understanding of the problem.

Particular interest has the question whether it is $k_0/k_1 \xrightarrow[\nu\to0]{} 0$ and/or $k_\nu/k_2 \xrightarrow[\nu\to0]{} 0$: the idea of the existence of the universal function suggests that $k_0/k_1 \to 0$ and k_1 should have order of magnitude such that $l_c k_1 \ll 1$. It seems (without any surprise) that the experiments are compatible *both* with $k_0/k_1 \sim O(1/\log R)$ *and* with $O(1)$.

Finally if the external force acts on scales $k_{in} \gg k_0$ but $k_{in} \ll k_\nu$ we expect to have the law 5/3 for $k > k_{in}$ with a distortion around a $k = k_{in}$. For $k_0 \ll k \ll k_{in}$ we expect equipartition of energy among the modes.

Finally some of the above results can be put in a remarkably rigorous setting: see the discussion in (G) below

(D): Bidimensional theory.

> *I propose to adopt the hypotheses that material lines are extended in two–dimensional turbulence, that there is a cascade process of transfer of mean square vorticity to higher wavenumbers, and that the limiting value of the rate of dissipation of mean–square vorticity as $\nu \to 0$ is nonzero. (Batchelor), [Ba69].*

If $d = 2$ we could repeat an analogous argument: but this time there is also, if $\nu = 0$, the first integral of the vorticity, *enstrophy*, that could be treated in the same way as the energy integral in the 3–dimensional case. This means that we could suppose that, for $\kappa < \kappa'$, the quantity $\mathcal{S}_{\kappa,\kappa',\infty}$, defined by replacing in the l.h.s. of (6.2.2) $|\underline{u}_{\underline{k}}|^2$ with $\underline{k}^2|\underline{u}_{\underline{k}}|^2$ and working out (starting from the NS equations) the r.h.s. to achieve a similar decomposition into three terms as in (6.2.2), vanishes. Hence the enstrophy would be transfered *in a local way* (in the sense discussed in (C)) from the scales of large length to those of small length where it would be dissipated.

The energy and enstrophy dissipations per unit volume and time are $\varepsilon = \nu L^{-3}\int_\Omega |\underline{\partial}\wedge\underline{u}|^2 d\underline{x}$ and $\sigma = \nu L^{-3}\int_\Omega |\Delta u|^2 d\underline{x}$: their dimensions are of a velocity cube over a length and of a velocity cube over a length cube (*i.e.* $[v^3/l]$ and $[v^3/l^3]$) respectively.

The above hypothesis is natural if we imagine that for $\nu \to 0$ the energy and enstrophy dissipation, per unit time and volume, have finite limits denoted ε and σ respectively, and $\varepsilon = 0$.

Taking $\varepsilon = 0$ requires a justification, at least on heuristic grounds. Indeed if $E = \int d\underline{x}\,|\underline{u}(\underline{x})|^2$ denotes the energy and $\omega^2 = \Omega = \int d\underline{x}\,|\underline{\partial}\wedge\underline{u}(\underline{x})|^2$ the enstrophy it is, by the NS equations:

$$\frac{1}{2}\dot{E} = -\nu\omega^2 + \mathcal{L}, \qquad \frac{1}{2}\dot{\Omega} = -\nu L^3 \sum_{\underline{k}} |\underline{k}|^4 |u_{\underline{k}}|^2 + \tilde{\mathcal{L}}$$

with $\mathcal{L}$ defined in (6.2.3), and $\tilde{\mathcal{L}} = L^3 \sum (i\underline{k}\wedge\overline{\underline{g}}_{\underline{k}})\cdot\underline{u}_{\underline{k}}$, that implies $|\tilde{\mathcal{L}}| \le g\omega_0 \le g\omega$ if $g = ||\underline{g}||_2$ and ω_0 is the vorticity on scale $\sim k_0$.

We see that if in the stationary state vorticity is concentrated at large values of $\underline{k}$, *i.e.* for $|\underline{k}| \sim k_\nu =$ where k_ν is the Kolmogorov scale $k_\nu = (\sigma\nu^{-3})^{1/6}$ (which is the only L–independent inverse length that one can form with the parameters σ, ν), then $L^3 \sum_{\underline{k}} |\underline{k}|^4 |u_{\underline{k}}|^2 \simeq k_\nu^2 \omega^2$.

Denoting time averages by $\langle\cdot\rangle$ stationarity implies that the time average $2^{-1}\langle\dot{\Omega}\rangle$ vanishes so that $-\nu k_\nu^2\langle\omega^2\rangle = -\langle\tilde{\mathcal{L}}\rangle < g\langle\omega\rangle$ *i.e.* $\langle\omega\rangle < g/(\nu k_\nu^2)$.

Therefore $\varepsilon = \nu\langle\omega^2\rangle < g^2/\nu k_\nu^4 \xrightarrow[\nu\to 0]{} 0$ because k_ν^4 tends to ∞ as ν^{-2}: we see that for $\nu \to 0$ the system (formally verifying the Euler equation) *conserves* *the energy in the inertial range, namely* $\varepsilon = 0$.

The enstrophy, in the inertial range, is not conserved[4] and one can assume that it "cascades" through the inertial range, at rate σ.[5]

We can easily repeat the dimensional analysis of the K41 theory with enstrophy replacing energy: the dimensions of the various quantities change and the results are somewhat different. If σ denotes the enstrophy that is communicated from the external force to the system per unit time and volume, we see that on scale l it would be $\sigma = v_l^3 l^{-3}$, hence the energy between k and $k+dk$ would be $L^2 K(k)dk = \sum \frac{v_l^2}{k^2}$ and hence, since $v_l = \sigma^{1/3} k^{-1}$, and proceeding as in (6.2.7)

$$K(k) = cost\, \sigma^{2/3}\, k^{-3}, \quad k_0 \ll k \ll k_\nu \tag{6.2.11}$$

that is a scaling law that turns out to be *summable* for k large and hence it means that, in a regime of developed turbulence, *the energy remains concentrated on the large scales* (while enstrophy is distributes on the whole inertial domain and in fact the quantity of enstrophy is asymptotically concentrated on modes with $k \sim k_\nu$ because $k^2 K(k)$ is not summable, for k large, while $K(k)$ is summable).

If $d = 2$ we can, therefore, say that, in the stationary state, enstrophy is concentrated on modes with large k, *i.e.* on the Kolmogorov scale, although the phenomenon is less pronounced than the correspondent phenomenon at $d = 3$ because, if $d = 2$, the enstrophy integral is only logarithmically divergent at large k (while if $d = 3$ it diverges as $k^{4/3}$). Energy remains concentrated on the modes at scales of order $k_0 = 2\pi L^{-1}$.

Note that if $d = 2$ the Reynolds number on scale l is given by $R_l = (\frac{l}{L})^2 R$, *i.e.* it depends from the Reynolds number R via the power 2 instead of 4/3; the *Kolmogorov scale* and the number N_ν of apparent degrees of freedom are, therefore

$$k_\nu = L^{-1} R^{1/2}, \qquad N_\nu = R \tag{6.2.12}$$

respectively.

We can also consider the case in which the external force acts on a scale of momentum $k_{in} \gg k_0$. In this case the stationary distribution of the energy depends on value of k with respect to a k_{in}. For $k > k_{in}$ the (6.2.11) should hold while for $k_0 \ll k \ll k_{in}$ the energy should be equidistributed

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[4] This does not contradict the regularity theorems for the solutions of the Euler equations for $d = 2$ because it is a property of the motions that develop on the attractor for the NS evolution. We lack any real knowledge of this set: which could consist of fields that, although approximable with very regular functions, are rather singular so that one cannot conclude that enstrophy (which is formally conserved if $\nu = 0$) is *really* conserved. This is very similar to what has been said about the energy in the case $d = 3$.

[5] Which, perhaps, tends to become ν–independent if an analogy with the corresponding energy cascade in the $d = 3$ case is correct (which is not clear).

between the modes $k_0 < |\underline{k}| < k_{in}$ and its value should be such that the energy density (here proportional to k) matches with the one in (6.2.11) at $|\underline{k}| = k_{in}$.

(E): Remarks on the K41 theory.

The discussion of Kolmogorov's scaling laws, certainly rather heuristic and questionable under many aspects, may appear not too convincing. Why indeed should we suppose that only if $d = 2$ the enstrophy dominates the cascade at short length scales (k large)? In reality, in absence of viscosity, also at $d = 3$ one has vorticity conservation in the form of Thomson's theorem; and we did not take this into account. It is legitimate to think that, succeeding in taking that into account "correctly", we could obtain different results even in 3 dimensions, (and, more generally, dimension dependent results at all dimensions because an analogue of the theorem of Thomson holds at all dimensions). The difficulty in taking into account these conservation laws does not seem a sufficient reason for not considering them as relevant.

It remains, therefore, to see if at least the experiments agree with the Kolmogorov scaling law. The answer seems strongly positive if $d = 3$, [Ba70]: but since we do not have an idea of how to estimate the corrections, we cannot be really certain of the quality of the result and some doubts stand. More, and very careful, research on the theme is certainly desirable and explains the interest that have raised "alternative" methods, such as the functional method of the §6.1.

Recently, [VW93], experimental evidence accumulated on deviations from the law 5/3, (6.2.8). One of the mechanisms of such deviations, which also allows us to keep the essential part of Kolmogorov ideas, the *multifractality*, will be discussed in the §6.3 in a simplified model.

In the problems below we examine other simple consequences of the Kolmogorov hypothesis that lead to interesting statements on the quantity

$$\langle\prod_{i=1}^{n}(u_{\alpha_i}(\underline{x}_i, T + t_i) - u_{\alpha_i}(\underline{y}_i, T + t'_i))\rangle \tag{6.2.13}$$

where the average is understood as average over the time T; or, assuming that the motion for large times is described by an attracting set with a (ergodic) statistics, the average can be understood with respect to the statistics of the attracting set (assuming T large).

For example we must have, and this is a possible precise definition [5] of v_l, that $\langle(\underline{u}(\underline{x})-\underline{u}(\underline{y}))^2\rangle$ be $cost\, l^{2/3}$ if $|\underline{x}-\underline{y}|=l$ (provided $L\gg l\gg k_\nu^{-1}$). And this is an interesting property of the statistics of the velocity field in the inertial domain.

It implies, indeed, that the velocity field has fractal nature, in a certain sense analogous to that of the *Brownian motion*: the latter provides us with trajectories that are fractal in the sense that the increments of the position as time increases are proportional to the power 1/2 of the increment of time. Fluids in states of homogeneous turbulence seem to provide examples of velocity fields which, for instance, are fractal in space, in the sense that the velocity increments are proportional to the power 1/3 of the spatial variations (and they also seem to provide examples of fields which are fractal in time, *c.f.r.* problems).

A simple heuristic argument is the following. Suppose that $(x-y)k_c\simeq O(1)$, *i.e.* suppose that we look at the velocity field on a small scale of the order of the Kolmogorov length. Then $\underline{u}(\underline{x})-\underline{u}(\underline{y})=\sum_{|\underline{k}|\le k_c}(e^{i\underline{k}\cdot\underline{x}}-e^{i\underline{k}\cdot\underline{y}})\,\underline{u}_{\underline{k}}$. Therefore $|\underline{u}(\underline{x})-\underline{u}(\underline{y})|^2$ has an average value (under the assumption that $\underline{u}_{\underline{k}}$ are random independent variables with variance $\langle|\underline{u}_{\underline{k}}|^2\rangle=v_\ell^2\,(|\underline{k}|L)^{-3}$) given by

$$\begin{aligned}\langle|\underline{u}(\underline{x})-\underline{u}(\underline{y})|^2\rangle &\simeq \sum_{\underline{k}}|e^{i\underline{k}\cdot\underline{x}}-e^{i\underline{k}\cdot\underline{y}}|^2\langle|\underline{u}_{\underline{k}}|\rangle^2\le\\ &\le const\,\frac{L^3}{(2\pi)^3}\int_0^{k_c}k^{2\alpha}|\underline{x}-\underline{y}|^{2\alpha}\frac{v_\ell^2}{(k\,L)^3}\,4\,\pi\,k^2\,dk=\\ &=const\,|\underline{x}-\underline{y}|^{2\alpha}\int_0^{k_c}\frac{d\,k}{k}\,k^{2\alpha}\,(\frac{\varepsilon}{k})^{2/3}\end{aligned}\tag{6.2.14}$$

where α can be taken any number in $(0,1)$ having used, if C_α is a suitable constant and if $k_c|\underline{x}-\underline{y}|\le O(1)$, the bound $|e^{i\underline{k}\cdot(\underline{x}-\underline{y})}-1|\le C_\alpha\,(|\underline{k}|\,|\underline{x}-\underline{y}|)^\alpha$.

So if $\alpha<1/3$ the integral converges and we see that $|\underline{u}(\underline{x})-\underline{u}(\underline{y})|\propto|\underline{x}-\underline{y}|^{1/3}$ uniformly in the Reynolds number provided we look at velocity increments over distances of the order of the Kolmogorov length. If $\alpha\ge1/3$ a uniform bound of "Hölder type" on the velocity increment is not possible along the above lines.

Although this does not prove that a better uniform bound is not actually possible it clearly shows the special role plaid by the value 1/3. And if we try to bound the velocity increment proportionally to $|\underline{x}-\underline{y}|^\alpha$ with $\alpha>1/3$ then we expect to see that the constant grows with k_c as $k_c^{\alpha-1/3}$ or, *c.f.r.* (6.2.9), as the $(\alpha-1/3)/4$ power of the Reynolds number, $R^{(\alpha-1/3)/4}$. The behavior with exponent 1/3 of the Hölder continuity extends, beyond the region $k_\nu|\underline{x}-\underline{y}|=O(1)$, to cover increments over scales into the whole inertial range as discussed in problem [6.2.7], see [LL71].

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[5] Which should be equivalent to (6.2.5).

The interest of such random fields, for probability theory, has been stressed by Taylor in 1935 (*c.f.r.* [Ba70], p.8): it is even increased by another of their properties, theoretically predicted by Taylor in 1938 and observed experimentally later by Steward, 1951 (*c.f.r.* [Ba70], p. 171): it is the *skewness* of the distribution of $\underline{\delta}(\underline{\rho}) = \underline{u}(\underline{x}) - \underline{u}(\underline{x}+\underline{\rho})$. The observable $\underline{\delta}(\underline{\rho})$ has zero *average* in space and in time for each $\underline{\rho}$, but it has third moment *different from zero*, so that it is not a centered random variable (and obviously it is not Gaussian).

Kolmogorov's theory of homogeneous turbulence provides the second non trivial example of fractal properties that become manifest in situations with a direct physical meaning, after Einstein's theory of Brownian motion.

In the next section we shall try to make clear the deep difference with respect to Brownian motion, that allows us to say that the statistics of developed turbulence has *multifractal* character: here this means that the time average of $\delta_n(\underline{\rho}) = |\underline{u}(\underline{x}) - \underline{u}(\underline{x}+\underline{\rho})|^n$ *does not behave* like $\langle\delta_2(\underline{\rho})\rangle^{n/2}$ as $\underline{\rho} \to \underline{0}$, not even in the inertial domain $k_0^{-1} \ll l \ll k_\nu^{-1}$, but rather like $\langle\delta_2(\underline{\rho})\rangle^{\zeta_n/2}$ with $\zeta_n \neq n/2$ a *nonlinear* function of n.

(F): The dissipative Euler equation.

We shall call *dissipative Euler equation* the equation for an incompressible fluid

$$\partial_t \underline{u} + \underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}} \underline{u} = -\underline{\partial} p - \chi \underline{u} + \underline{g}, \qquad \underline{\partial} \cdot \underline{u} = 0 \tag{6.2.15}$$

in a container Ω that is a torus or that has perfect walls so that a *no slip* boundary condition $\underline{u} \cdot \underline{n} = 0$ holds, if $\underline{n}$ is the external normal.

The constant χ will be called *sticky viscosity* and it does not corresponds to a constitutive equation of the type considered in §1.1,§1.2. It rather corresponds to a perfect fluid that flows with friction over a background: the model can be physically interesting mainly in 2–dimensional cases, when one often imagines that the fluid flows on a "rough" surface . But in order that a bidimensional fluid that "flows on a table" be well modeled by (6.2.15) it would be necessary that the thickness of the fluid is less than the Kolmogorov length k_ν^{-1} if ν is its true viscosity: this makes it difficult to find real applications of the equation. Here we consider it only as a mathematical model.

Is it possible to conceive a theory analogous to the theory K41? The following few comments are an attempt at posing the problems rather than at suggesting their solutions.

Let $\eta = \chi L^{-3} \int_\Omega |\underline{u}|^2 d\underline{x}$ be the energy dissipated per unit volume and per unit time. Hence the length scale that can be formed with the quantities η and χ is $l_\chi = (\eta\,\chi^{-3})^{1/2}$, and $k_\chi = l_\chi^{-1} = (\chi^3\eta^{-1})^{1/2}$.

There is a difference between the present case and the NS case: namely the friction is "weaker" at large $\underline{k}$ and therefore the inertial terms dominate, at least dimensionally, at large $\underline{k}$ rather than at small $\underline{k}$. Note also that if η is constant then $k_\chi \xrightarrow[\chi\to 0]{} 0$ "*unlike*" k_ν (which diverges as $\nu \to 0$) in the NS case.

Therefore if k_0 is the momentum scale on which the force $\underline{g}$ acts and if we assume that the energy "cascades" above the forcing scale, see the homogeneous turbulence hypothesis in (C) above, then for $d = 3$

(1) If $k_\chi < k_0$ we should have an inertial domain above k_0. However this would mean that energy is never dissipated which is impossible because the energy is *a priori* bounded. It seems that waiting an infinite time the energy will simply extend further and further in $\underline{k}$–space and if there is an ultraviolet cut–off K_{uv} it would tend to become equipatitioned between all modes with $k_0 < |\underline{k}| < K_{uv}$ with the exception of the energy contained in the modes close to k_0, "directly" forced and which seem likely to hold a finite fraction of the total energy. This situation is the more likely as we expect that $k_\chi \to 0$ as $\chi \to 0$ since it does not seem possible that $\eta \xrightarrow[\chi\to 0]{} 0$ faster than χ, see problem [6.2.1]: therefore eventually $k_\chi < k_0$.
(2) if $k_\chi \gg k_0$ we should have again equipartition (in presence of an ultraviolet cut–off) among the modes above k_χ while in the range between k_0 and k_χ, if large enough, we should have a scaling inertial range with the 5/3–law for the energy distribution.

The $d = 2$ case cannot be discussed similarly: if we assume that, as in the NS case, there is an enstrophy cascade and if σ is the enstrophy dissipated per unit time and volume then σ has the dimension of inverse time cube; since χ has also dimension of inverse time we *cannot form any length scale* with χ and σ. This might mean that in $d = 2$ there is always equipartition (in presence of a cut–off). Note, however, that the basis for an enstrophy cascade assumption is unclear in this case because the argument of (D) cannot be adapted to the present situation.

In all cases we see that it is necessary to impose *a priori* an ultraviolet cut–off K_{uv} on the ED equations so that all this has a meaning. Otherwise, in absence of a cut–off, the system will necessarily tend towards a stationary state in which every mode, but the ones directly forced and their neighbors, has zero energy (*i.e.* equidistribution among infinitely many modes).
The situation recalls the black body problem and, as in that case, we can think that it might be unnecessary to introduce an ultraviolet cut–off, *c.f.r.* [Ga92], because the time scales needed to transfer appreciable amounts of energy to the modes of large $|\underline{k}|$ may increase very rapidly with the value of the momentum $\underline{k}$ itself. An hypothesis which, in the case of the black body, was advanced by Jeans in his attempts of a classical explanation of the black body spectrum. But as in the case of the Jeans' problem the question is very difficult to analyze. Since long times are involved direct numerical tests are not possible. And a possible (if existent) effective ultraviolet cut–off K must be, on dimensional grounds, a momentum formed with the quantities $k_0, k_{in}, \varepsilon, \chi, g$: with such quantities we can form several momenta, $k_0, k_{in}, \sqrt{\chi^2/\varepsilon}, \chi/\sqrt{gL}$, hence K cannot be determined without a detailed

theory.

(G) The Ruelle–Lieb bounds and K41 theory

We conclude this section by quoting certain really remarkable rigorous results due to Ruelle and Lieb which, although not well known as they deserve, are closely related to the K41 theory.

The results that we select concern an estimate of the Kaplan–Yorke dimension of the attractor for the $d = 2, 3$ NS equations or for the $d = 2$ Rayleigh equations. Calling $\varepsilon = \nu(\underset{\sim}{\partial}\underline{u})^2$ the energy dissipation per unit volume and assuming that $\varepsilon \in L_{1+d/2}(\Omega)$ then the Kaplan–Yorke and the Hausdorff dimensions of the attractor associated with any ergodic invariant measure μ for the incompressible NS flow under a smooth constant forcing is bounded by

$$K\, R^{3d/4} \tag{6.2.16}$$

where $R = v_L L \nu^{-1}$ is the Reynolds number defined in terms of the "velocity on the scale L of the forcing" by setting $||\varepsilon||_{L_{1+d/2}} \stackrel{def}{=} v_L^3 L^{-1}$.

The constant K depends on the boundary conditions and can be explicitly estimated to be a constant of order 1. The result, and many technical ideas to obtain it, was proposed for $d = 3$ in [Ru82] (*c.f.r.* equation (2.8) of this reference) where it was proved subject to further assumptions; it was successively fully proved in [Li84] (*c.f.r.* equation (43) of this reference) which also extended it to $d = 2$.[6]

The same idea was then applied in [Ru84] to the analysis of the Rayleigh model for convection, *c.f.r.* §1.5, obtaining a bound on the dimension of the attracting set for the convection model as

$$K\, R\, (1 + R_{Pr}^{-1})\, a^2/H^2 \tag{6.2.17}$$

for large Rayleigh number R ans small Prandtl number R_{Pr}, where a is the spatial period of the flow and H is the height (see §1.5, (1.5.17), and §4.1 for the notations); K is a (dimensionless) constant.

Here we should note the *remarkable agreement* between (6.2.16) and the K41 prediction (6.2.9): in the case in which we can regard ε as constant (as empirically assumed in the K41 theory) (6.2.16) is a precise and rigorous statement identical to (6.2.9): this is one of the achievements of the mathematical theory of fluids.

The shortcoming of the result is that we do not know *a priori* that on the attractor the dissipation per unit volume is bounded in $L_{1+d/2}$ if $d = 3$. We do not know that in $d = 2$ either: in spite opf the fact that we have

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6 In fact what one really needs is a bound on $||\varepsilon||_{L_{1+d/2}}$ which has a finite average with respect to the considered invariant distribution μ.

(*c.f.r.* Chap. III) a rather good existence and regularity theory of the NS equations (with no results, however, imply a bound global in time on the $L_{1+d/2}$ norm of the dissipation ε per unit volume). Furthermore in the $d=2$ case we would like to have a bound in terms of the enstrophy dissipation rate rather than in terms of the energy dissipation rate as it emerges from the analysis in (D) above. Such a bound is possible but in this case we lack *a fortiori* an estimate (global in time) of the enstrophy dissipation on the attracting set.

A rough idea of the arguments in [Ru82], [Li84], [Ru84], is presented in problems [6.2.11] and [6.2.12].

Problems: *Dissipation and attractor dimension as $\nu \to 0$. Kolmogorov's skew correlations.*

We define $\langle X\rangle$ the time average of an observable X as seen on a "typical" motion of the fluid, that develops on an attracting set A on which it is described by a statistics (such that the average with respect to the statistics of X coincides with $\langle X\rangle$). The following problems closely summarize the analysis of [LL71].

[6.2.1]: (*Dissipation rate as $\nu \to 0$*) Consider a NS fluid with viscosity ν in a periodic container of side L and subject to a volume force of density $F\sqrt{\nu}\underline{g}(\underline{x}/L)$ (with $\underline{g}$ admensional and fixed). Then the average dissipation $\langle\eta\rangle$ is bounded uniformly in the Reynolds number $R=\sqrt{F\sqrt{\nu}L^3/\nu^2}$ (*c.f.r.* the discussion at the beginning of the section). Suppose (as always in this section) that the solutions of the NS equation are C^∞. Show that the same argument implies a uniform bound on the dissipation in the case of the ED equation (*i.e.* $\chi\langle E\rangle \le (F||\underline{g}||_2)^2$). (*Idea:* (NS case) If $\underline{u}=\sum_{\underline{k}} e^{i\underline{k}\cdot\underline{x}}$ and $E=L^3\sum_{\underline{k}}|\underline{u}_{\underline{k}}|^2=\int|\underline{u}|^2d\underline{x}$ and $S=L^3\sum_{\underline{k}}\underline{k}^2|\underline{u}_{\underline{k}}|^2=2^{-1}\int|\underline{\partial}\underline{u}|^2d\underline{x}$ one finds, immediately (or *c.f.r.* (3.2.15))

$$\dot{E}/2=-\nu S+\sqrt{\nu}F\int \underline{u}\cdot\underline{g}\,d\underline{x}$$

Hence averaging over time the left hand side, which is a time derivative of a bounded quantity (if $\nu>0$, *c.f.r.* §3.2 for instance) and therefore has zero time average, we get $0=-\nu\langle S\rangle+F\sqrt{\nu}\int\langle\underline{u}\rangle\cdot\underline{g}\,d\underline{x}$. It follows that: $0\le-\nu\langle S\rangle+||\underline{g}||_2\sqrt{\langle E\rangle}\,F\,\sqrt{\nu}$ *i.e.* (noting, as usual, that $E\le k_0^{-2}S$, if $k_0=2\pi/L$) it is $\nu\langle S\rangle\le(F\,||\underline{g}||_2k_0^{-1})^2$.)

[6.2.2]: (*dissipation rate as $\nu\to0$*) If the bound in [6.2.1] was optimal (*i.e.* equality holds) then it not would be true that η is independent of ν in the case considered [6.2.1]. Check this statement. Suppose that in [6.2.1] the force has the form $F\nu^\alpha\underline{g}$ and that for $\alpha=0$ it is $\eta\xrightarrow[\nu\to0]{}\infty$: show that there is a value of α ($\le\frac12$) such that η has an nonzero (upper) limit for $\nu\to0$. (*Idea:* In the present notations the case discussed at the beginning of the section would be written as a fluid subject to a force of the form $F\underline{g}$, *i.e.* $F\nu^\alpha\underline{g}$ with $\alpha=0$ so that the bound of [6.2.1] would be a bound proportional to ν^{-1} which, if optimal, gives a divergent η. The second part is, given the result of [6.2.1], a continuity statement in α.)

[6.2.3]: *(Scaling properties of velocity correlations in the inertial range)* Define

$$V_{\alpha\beta}(\underline{\rho})=\langle u_\alpha(\underline{x})u_\beta(\underline{x}+\underline{r})\rangle,\quad V_{\alpha\beta\gamma}(\underline{\rho})=\langle u_\alpha(\underline{x})u_\beta(\underline{x})u_\gamma(\underline{x}+\underline{r})\rangle$$

and suppose that the length $|\underline{r}|\equiv\rho\ll L$, so that the scale of length ρ is in the inertial domain or viscous, where these tensors are, by hypothesis, rotation and translation invariant functions (of $\underline{x}$). Show that, by this invariance:

$$\begin{aligned}V_{\alpha\beta}(\underline{r}) =&A(\rho)\delta_{\alpha\beta}+B(\rho)\frac{\rho_\alpha\rho_\beta}{\rho^2} \qquad \rho\ll L\\ V_{\alpha\beta\gamma}(\underline{r}) =&C(\rho)\delta_{\alpha\beta}\frac{\rho_\gamma}{\rho}+D(\rho)(\delta_{\alpha\gamma}\frac{\rho_\beta}{\rho}+\delta_{\beta\gamma}\frac{\rho_\alpha}{\rho})+E(r)\frac{\rho_\alpha\rho_\beta\rho_\chi}{\rho^3}\end{aligned}$$

where A, B, C, D, E are suitable functions. (*Idea:* The tensors are the only ones that can be formed with the vector $\underline{r}$ alone).

[6.2.4]: Define the tensors

$$\begin{aligned}B_{\alpha\beta}(\underline{r}) =&\langle(u_\alpha(\underline{x})-u_\alpha(\underline{x}+\underline{r}))(u_\beta(\underline{x})-u_\beta(\underline{x}+\underline{r}))\rangle\\ B_{\alpha\beta\gamma}(\underline{r}) =&\langle(u_\alpha(\underline{x})-u_\alpha(\underline{x}+\underline{r}))(u_\beta(\underline{x})-u_\beta(\underline{x}+\underline{r}))(u_\gamma(\underline{x})-u_\gamma(\underline{x}+\underline{r}))\rangle\end{aligned}$$

and show that if $\rho\ll L$

$$\begin{aligned}B_{\alpha\beta}(\underline{r}) =&2(V_{\alpha\beta}(\underline{r})-V_{\alpha\beta}(\underline{0}))\\ B_{\alpha\beta\gamma}(\underline{r}) =&2(V_{\alpha\beta\gamma}(\underline{r})+V_{\gamma\alpha\beta}(\underline{r})+V_{\beta\gamma\alpha}(\underline{r}))\end{aligned}$$

(*Idea:* If $\rho\ll L$ the results of [6.2.3] hold).

[6.2.5]: Show that if $|\underline{r}|=\rho\ll L$ it is

$$\partial_\gamma V_{\gamma\alpha\beta}(\underline{r})-2\nu\Delta V_{\alpha\beta}(\underline{r})+\langle g_\alpha(\underline{x})u_\beta(\underline{x}+\underline{r})\rangle=0 \tag{!}$$

as a consequence of the law of evolution of the field $\underline{u}$ according to the NS equation. (*Idea:* Note that

$$\begin{aligned}\partial_t V_{\alpha\beta} =&-\langle(u_\gamma(\underline{x})\partial_\gamma u_\alpha(\underline{x}))u_\beta(\underline{x}+\underline{r})\rangle-\langle u_\alpha(\underline{x})(u_\gamma(\underline{x}+\underline{r})\partial_\gamma u_\beta(\underline{x}+\underline{r}))\rangle+\\ &+\nu\langle\Delta u_\alpha(\underline{x})u_\beta(\underline{x}+\underline{r})\rangle+\nu\langle u_\alpha(\underline{x})\Delta u_\beta(\underline{x}+\underline{r})\rangle+\\ &+\langle(-\partial_\alpha p(\underline{x})+g_\alpha(\underline{x}))u_\beta(\underline{x}+\underline{r})\rangle+\langle u_\alpha(\underline{x})(-\partial_\beta p(\underline{x}+\underline{r})+g_\beta(\underline{x}+\underline{r}))\rangle\end{aligned}$$

and the hypothesis of independence of the velocity fields in the various points of the fluid allows us to think the averages over t also as averages over $\underline{x}$ and hence, after some integrations by parts with respect to $\underline{x}$, we obtain the result because we see that the pairs of terms on the various rows are equal; and furthermore $\langle p(\underline{x})u_\alpha(\underline{x}+\underline{r})\rangle$ must be a vector with zero divergence formed with the only vector $\underline{r}$ and it must hence have the form $t(\rho)\rho_\alpha/\rho$ with $\frac{d}{d\rho}\rho^2 t(\rho)=0 \longleftrightarrow t(\rho)=c\rho^{-2}=0$, (because $c=0$ since t must be regular for $\underline{r}\to\underline{0}$). The first member has zero average by the hypothesis of stationarity).

[6.2.6]: Compute the trace of the tensor (identically zero) in [6.2.5], equation (!), and evince that

$$\partial_\gamma V_{\gamma\alpha\alpha}(\underline{r})-\nu\Delta V_{\alpha\alpha}(\underline{r})+\varepsilon=0$$

(*Idea:* Note that $\langle\underline{g}(\underline{x})\cdot\underline{u}(\underline{x})\rangle$ is the power dissipated per unit volume, being the averages on $\underline{x}$ equal to those on t; hence $\langle\underline{g}(\underline{x})\cdot\underline{u}(\underline{x}+\underline{r})\rangle\xrightarrow[\underline{r}\to\underline{0}]{}\varepsilon$).

[6.2.7]: If $\underline{r}$ is in the inertial domain the vector $V_{\gamma\alpha\alpha}(\underline{r})$ must depend only on ε and $\underline{r}$, hence

$$V_{\gamma\alpha\alpha}(\underline{r})=\Gamma\varepsilon r_\gamma \qquad \text{for} \quad k_\nu^{-1}\ll\rho<\ll L$$

where Γ is a suitable (universal) constant. Hence $\partial_\gamma V_{\gamma\alpha\alpha}=3\Gamma\varepsilon$. We deduce, from [6.2.4], that $\Gamma=-1/3$ because in the inertial domain the term $-\nu\Delta V_{\alpha\alpha}(\underline{r})$ is negligible with respect to ε. (*Idea:* To check this note that $\Delta V_{\alpha\alpha}(\underline{r})\equiv\Delta(V_{\alpha\alpha}(\underline{r})-\Delta V_{\alpha\alpha}(\underline{0}))=\frac{1}{2}\Delta B_{\alpha\alpha}(\underline{r})$

(by [6.2.2]) and $B_{\alpha\alpha}(\underline{r}) = cost\,\varepsilon^{2/3}\rho^{2/3}$ in this domain; hence $\Delta V_{\alpha\alpha} \propto \varepsilon^{2/3}\rho^{-4/3}$. Imposing that $\nu\varepsilon^{2/3}\rho^{-4/3} \ll \varepsilon$ one finds $\rho \gg l_\nu$, *c.f.r.* (6.2.9), *i.e.* the condition that ρ is in the inertial domain).

[6.2.8]: *(Skewness in the inertial range)* Check that [6.2.7], *i.e.* $\Gamma = -1/3$, implies immediately that the quantity $B_{rrr} \equiv B_{\alpha\beta\gamma}(\underline{r})\, r_\alpha\, r_\beta\, r_\gamma\, \rho^{-3}$ is $-4\varepsilon\rho/5$ in the inertial domain. This is the result on the skewness of the distribution of the velocity field in the inertial domain (due to Kolmogorov).

[6.2.9]: Consider the tensors of [6.2.3] and verify, by direct computation, that the incompressibility conditions $\partial_\beta V_{\alpha\beta}(\underline{r}) = 0$ and $\partial_\gamma V_{\alpha\beta\gamma}(\underline{r}) = 0$ imply the following relations between the coefficients $A, B, \ldots$:

$$A' + B' + \frac{2B}{\rho} = 0, \quad C' + \frac{2(C+D)}{\rho} = 0, \quad 3C + 2D + E = 0$$

$$B(0) = C(0) = D(0) = E(0) = 0, \quad A(0) = \frac{1}{3}\langle \underline{u}^2\rangle$$

$$A(\rho) - A(0) = -\frac{1}{2\rho}\frac{d}{d\rho}\rho^2 B(\rho), \qquad D(\rho) = -\frac{1}{2\rho}\frac{d}{d\rho}\rho^2 C(\rho)$$

(*Idea:* Perform explicitly the derivatives, then impose the vanishing of the coefficient w of the vector $\partial_\beta V_{\alpha\beta} = w(\rho) r_\alpha$ and of the coefficients of the tensors $\delta_{\alpha\beta}$ and $r_\alpha r_\beta$ in terms of which one expresses $\partial_\gamma V_{\alpha\beta\gamma}$. Rather than $3C + 2D + E = 0$ one will find $\frac{d}{d\rho}\rho^2(3C + 2D + E) = 0$ that, however, implies the preceding relation because the field $\underline{u}$, and hence the its averages, are regular functions for $\rho \to 0$, *i.e.* for ρ smaller than the Kolmogorov scale. For the same reason B, C, D, E must vanish for $\rho = 0$).

[6.2.10]: *(Regularity in the dissipative range)* Imagine now that ρ is in the dissipative domain ($\rho \ll l_\nu$ and all the fields are regular, differentiable functions) and, hence, $B_{\alpha\alpha}(\underline{r}) = a\rho^2$; show that the constant a that, for dimensional reasons, must have the form $a = c\,\varepsilon\nu^{-1}$ is such that $c = 1/15$. (*Idea:* From [6.2.2] deduce $V_{\alpha\beta}(\underline{r}) = V_{\alpha\beta}(\underline{0}) - \frac{1}{2}B_{\alpha\beta}(\underline{r})$; and from the incompressibility condition of [6.2.9], $A' + B' + \frac{2B}{\rho} = 0$ deduce that $\frac{1}{2}B_{\alpha\alpha}(\underline{r}) = -a\rho^2\delta_{\alpha\beta} + \frac{a\rho^2}{2}\frac{\rho_\alpha}{\rho}\frac{\rho_\beta}{\rho}$. Hence differentiating $V_{\alpha\beta}$ one finds:

$$\langle \partial_\beta u_\alpha \partial_\beta u_\alpha\rangle = 15a, \qquad \langle \partial_\beta u_\alpha \partial_\alpha u_\beta\rangle = 0$$

but $\varepsilon = \frac{1}{2}\nu\,\langle(\partial_\alpha u_\beta + \partial_\beta u_\alpha)^2\rangle = \nu\,\langle(\partial_\alpha u_\beta)^2 + \partial_\alpha u_\beta \partial_\beta u_\alpha\rangle = 15a\nu$).

[6.2.11] *(Bounds on the dimension of an attractor) (Ruelle)* Let $\dot{\underline{u}} = \underline{f}(\underline{u})$ be a differential equation with an *a priori* bounded attracting set (*i.e.* such that $||\underline{u}||_{L_2} \le B$ if $\underline{u}$ is on the attracting set). Let $\underline{u}_0$ be a solution and let

$$\dot{\underline{\delta}} = \frac{\partial}{\partial \underset{\sim}{\underline{u}}}\underline{f}(\underline{u}_0)\underset{\sim}{\underline{\delta}} \stackrel{def}{=} H_{\underline{u}_0}\underline{\delta}$$

Suppose that the eigenvalues $\lambda_j^{(\underline{u}_0)}$ of the *Hermitian part* of $H_{\underline{u}_0}$ are ordered in a non decreasing way and that they are such that for some N, *independent of* $\underline{u}_0$, it is $\sum_{j=1}^N \lambda_j^{(\underline{u}_0)} \le 0$. Then, no matter which invariant distribution μ we consider on the attracting set it follows that the attracting set cannot have Lyapunov dimension $> N$ (*c.f.r.* comment (6) to definition 2 in §5.5). (*Idea:* Let $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_N$ be a basis for the first N eigenvalues (assumed distinct for simplicity) then the parallelogram $\underline{e}_1 \wedge \underline{e}_2 \wedge \ldots \wedge \underline{e}_N$ must contract in volume because

$$\frac{d}{dt}\frac{1}{2}\log||\underline{e}_1 \wedge \underline{e}_2 \wedge \ldots \wedge \underline{e}_N||^2 =$$

$$= \sum_{i=1}^N \frac{(\underline{e}_1 \wedge \underline{e}_2 \wedge \ldots \wedge \underline{e}_N,\, \underline{e}_1 \wedge \underline{e}_2 \wedge \ldots \wedge H_{\underline{v}_0}\underline{e}_k \wedge \ldots \wedge \underline{e}_N)}{||\underline{e}_1 \wedge \underline{e}_2 \wedge \ldots \wedge \underline{e}_N||^2} \le \sum_{j=1}^N \lambda_j^{(\underline{v}_0)}$$

see [Ru82].)

[6.2.12] (*estimate of the dimension of the NS attractor (Ruelle–Lieb)*) Consider the 2 or 3 dimensional NS equation in a domain Ω with periodic boundary conditions and side L. Let $\underline{u}_0$ be in an attractive set and consider the linearization operator acting on divergenceless fields $\underline{\delta} \in L_2$ with $\underline{\partial} \cdot \underline{\delta} = 0$ defined by

$$H_{\underline{u}_0}\underline{\delta} \stackrel{def}{=} \nu\,\Delta\,\underline{\delta} - (\underline{u}_0 \cdot \underset{\sim}{\underline{\partial}}\underline{\delta} + \underset{\sim}{\underline{\delta}} \cdot \underset{\sim}{\underline{\partial}}\underline{u}_0) - \underline{\partial}p'$$

where p' is such that the r.h.s. has zero divergence. We order the eigenvalues $\lambda_j^{(\underline{u}_0)}$ of the hermitian part of $H_{\underline{u}_0}$ (as an operator densely defined on $C^\infty(\Omega)$ and extended to a self–adjoint operator via the quadratic form that it defines in $L_2 \cap C^\infty$) by increasing order. Then if for some N, *independent of* $\underline{u}_0$, it is $\sum_{j=1}^N \lambda_j^{(\underline{u}_0)} \le 0$ the Lyapunov dimension of the attracting set does not exceed N. (*Idea:* Remark that the attracting sets only contain fields $\underline{u}_0$ verifying the *a priori* bound $||\underline{u}_0||_{L_2} \le E$ with $E = const\,||\underline{g}||_{L_2} L^2/\nu$, *if* $\int_\Omega |\underline{\partial}\underline{u}|^{2+d}\, d\underline{x} < \infty$ *c.f.r.* (3.2.17). The estimate on the sum $\sum_{j=1}^N \lambda_j^{(\underline{u}_0)}$ is done by bounding this quantity by the corresponding quantity for the Schrödinger operator $\nu\Delta + w(\underline{x})$ where $w(\underline{x})^2 = \nu\,(\underset{\sim}{\underline{\partial}}\underline{u} + \underset{\sim}{\underline{\partial}}\underline{u})^2$ as an ordinary Shrödinger operator (acting on vector fields not necessarily divergenceless). An estimate can be done and it yields for $d = 3$ (6.2.16) [Ru82], [Li84] extendible to $d = 2$ as in (6.2.16), [Li84]: the details are nontrivial and we refer to the original works. The method can be further developed to be applicable to the Rayleigh equations in $d = 2$, *c.f.r.* (6.2.17), see [Ru84].)

Bibliography: The theory K41 is taken from [LL71] and [Ba70]. The bidimensional theory is inspired by [Ba69] where it is exposed in detail, see also [Kr67],[Kr75b]. The reference [DC92] inspires the first two problems, the others are taken from [LL71]. A recent review of the scaling laws in developed turbulence, which includes some of the main new results of the late 1990's is in [Ga99d]. The results in (G) come from the works [Ru82], [Li84], [Ru84].

§6.3 The shell model. Multifractal statistics.

Having seen the heuristic theory of homogeneous turbulence we can try to avoid the study of the functional integrals of section §6.1 by making an hypothesis on the structure of the attracting set A and then choose the invariant distribution according to Ruelle's principle of §5.7.

We have seen that the Kolmogorov theory of homogeneous turbulence supposes that there is no interchange of energy between the *shell* of the modes $\kappa < |\underline{k}| < \kappa'$ and the shell of the modes $\kappa_1 < |\underline{k}| < \kappa_2$, if the shells $[\kappa, \kappa']$ and $[\kappa_1, \kappa_2]$ are separated by at least "an order of magnitude" (for instance $\kappa_1 > 2\kappa'$).

Imagine dividing the modes $\underline{k}$ into shells of different orders of magnitude, *i.e.* so that the "*n-th shell*", or "*shell of scale n*", is defined by

$$\Delta_n : \qquad 2^{n-1}k_0 < |\underline{k}| < 2^n k_0 \tag{6.3.1}$$

where k_0 is the "scale of the container":*i.e.* $k_0 = 2\pi L^{-1}$ is the minimum of the values possible for $|\underline{k}|$ and depends only on the length $L \equiv l_0$ of the side of the container, imagined as a cube Ω with periodic boundary conditions.

The absence of energy interchange between distant shells, or distant scales, will be then *imposed* by writing the evolution equation as the NS equation modified as follows, *c.f.r.* [Kr64], if $\underline{k} \in \Delta_n$:

$$\dot{\underline{u}}_{\underline{k}} = -\nu \underline{k}^2 \underline{u}_{\underline{k}} - i \sum_{\substack{\underline{k}_1+\underline{k}_2=\underline{k} \\ \underline{k}_2 \in \Delta_{n\pm 1}}} \underset{\sim}{u}_{\underline{k}_1} \cdot \underset{\sim}{k}_2 \, \Pi_{\underline{k}} \underline{u}_{\underline{k}_2} + \underline{g}_{\underline{k}} \tag{6.3.2}$$

In this way the absence of direct energy interchange between shells separated (by at least 2 units of scale) is guaranteed *a priori* , and hence by a stronger reason the laws of Kolmogorov should hold.

The model (6.3.2) can be *further simplified* by turning it into a model in which one thinks of replacing the set of approximately 2^{3n} modes of the n-th shell with a *single mode* $\underline{u}_n = (u_{n1}, u_{n2})$ and write, having set $k_n = 2^n k_0$, for $\alpha, \beta, \gamma = 1, 2$

$$\begin{aligned} \dot{u}_{n\alpha} = & -\nu k_n^2 u_{n,\alpha} + k_n \sum_{\beta\gamma} \Big(C^{\alpha}_{\beta\gamma}(n) u_{n-1,\beta} u_{n+1,\gamma} + \\ & + D^{\alpha}_{\beta\gamma}(n) u_{n+2,\beta} u_{n+1,\gamma} + E^{\alpha}_{\beta\gamma}(n) u_{n-2,\beta} u_{n-1,\gamma} \Big) + \underline{g}_n \end{aligned} \tag{6.3.3}$$

where $C^{\alpha}_{\beta\gamma}(n), D^{\alpha}_{\beta\gamma}(n), E^{\alpha}_{\beta\gamma}(n)$, $\alpha, \beta, \gamma = 1, 2$ are constants, symmetric in $\beta \longleftrightarrow \gamma$, defined so that the sum:

$$\begin{aligned} & \sum_{\alpha\beta\gamma;n} k_n \Big(C^{\alpha}_{\beta\gamma}(n) u_{n-1,\beta} u_{n+1,\gamma} + D^{\alpha}_{\beta\gamma}(n) u_{n+2,\beta} u_{n+1,\gamma} + \\ & + E^{\alpha}_{\beta\gamma}(n) u_{n-2,\beta} u_{n-1,\gamma} \Big) u_{n\alpha} \equiv 0 \end{aligned} \tag{6.3.4}$$

that guarantees conservation of the energy $\mathcal{E} = \frac{1}{2} \sum_n |\underline{u}_n|^2$. The function $\underline{g}_n$ is the "external force" and is imagined different from 0 only for the first values of n (for instance only for $n = 4$).

The condition (6.3.4) can be imposed by requiring that

$$\frac{1}{2} C^{\beta}_{\alpha\gamma}(n+1) + \frac{1}{4} D^{\gamma}_{\alpha\beta}(n+2) + E^{\alpha}_{\beta\gamma}(n) = 0 \tag{6.3.5}$$

where, for example, $E(n) \equiv E$, $C(n) \equiv C$ and $D(n) \equiv D$ are tensors suitably fixed. A simple choice is provided by the model of Gledzer, Ohkitani and Yamada, that is called GOY*model.* We set $u_n = u_{n,1} + i u_{n,2}$ and consider the equation:

$$\dot{u}_n = -\nu k_n^2 u_n + i k_n \big(-\frac{1}{4} \, \overline{u}_{n-1} \, \overline{u}_{n+1} + \overline{u}_{n+1} \, \overline{u}_{n+2} - \frac{1}{8} \, \overline{u}_{n-1} \, \overline{u}_{n-2} \big) + g \delta_{n,4} \tag{6.3.6}$$

where we suppose that the components u_n with $n = -1, 0$ vanish by definition, *c.f.r.* [BJPV98] p. 55%56.[1]

The equations (6.3.4) and (6.3.6) have been intensely studied from a numerical point of view: in performing the truncations (a necessary step in any numerical computation) the equations truncated continue to conserve the energy (if $\nu = 0$): provided, if N is the value of the truncation over n, one requires that u_{N+1}, u_{N+2} are interpreted as $\equiv 0$. It becomes then possible to obtain an *a priori* bound on the energy $\mathcal{E} \equiv \frac{1}{2}\sum |u_n|^2$: for instance for (6.3.6) it is

$$\frac{1}{2}\frac{d}{dt}|u_n|^2 = -\nu \sum_n k_n^2 |u_n|^2 + \sum_n \overline{u}_n g_n \tag{6.3.8}$$

valid also for the truncated equations (in which case the sums extend until N).

A second constant of motion is the quantity

$$\mathcal{H} = \sum_n (-1)^n k_n |u_n|^2 \tag{6.3.9}$$

that plays the role of the *elicity*, *c.f.r.* §2.2, of the equations of NS in $d = 3$ although it is unsatisfactory that the terms with n even and those with n odd play a different role.[2]

In this model u_n can be thought of as defining a function with real values

$$u(x) = \sum_{n=1}^{\infty} \Big(u_n e^{i2^n x k_0} + \overline{u}_n e^{-i2^n x k_0} \Big) \tag{6.3.10}$$

and the Kolmogorov analysis can be repeated, obtaining that the energy "on scale $k_n = l_n^{-1} = 2^n k_0$" is $\mathcal{E}_n = l_0^3 v_n^2$ and such that $v_n^3/l_n = \varepsilon$ with ε a parameter that depends by g and is independent on n and ν for $n \gg 1$.

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[1] Another, between the infinite choices that satisfy the (6.3.5), is

$$\dot{u}_n = -\nu k_n^2 u_n + i k_n (4\,\overline{u}_{n-1}\,\overline{u}_{n+1} - 4\,\overline{u}_{n+1}\,\overline{u}_{n+2} - \overline{u}_{n-2}\,\overline{u}_{n-1}) + g_n \tag{6.3.7}$$

Furthermore we could ask why the external force acts on the component $n = 4$. The reason rests probably on the property of the NS equations in 2 dimensions, in which one has (linear) stability of the motion subject to a force acting on only one component, $g_n = \delta_{nn_0} g$, for *all* values of ν, if the component n_0 is that with k_n minimum. Furthermore the "laminar" motion , $u_n = (\,k_{n_0}^2)^{-1} g \delta_{nn_0}$ is, *c.f.r.* problem [4.1.13] and [Ma86], globally attracting for all values of g_{n_0} (or ν). Although this *is not* so for (6.3.6) if $n_0 = 1$, *c.f.r.* problem [6.3.1], nevertheless in numerical experiments one often chooses $n_0 > 3$.

[2] We could remedy this by further complicating the model and introducing 2 complex components for every k_n that play the role of the components of different elicity seen in §2.2, *c.f.r.* [BJPV98]. However it is better to recognize that the GOY model is useful because it can illustrate several behaviors that could appear also in solutions of the NS model but that cannot be taken too seriously as a model with direct physical interest.

Hence $v_n = (\varepsilon k_n^{-1})^{1/3}$ and $R_n = (k_n/k_0)^{4/3}\, R$ if $R = v_0 l_0/\nu$, and therefore the energy on scale k_n, the scale $k_{n_\nu} \equiv l_{n_\nu}^{-1}$ of Kolmogorov[3] and the apparent number of degrees of freedom N_{n_ν} will be, respectively,

$$\mathcal{E}_n = (\varepsilon k_n^{-1})^{2/3}, \quad k_{n_\nu} = (\varepsilon^{1/3}\nu^{-1})^{3/4}, \quad N_\nu = \frac{3}{4}\log R \tag{6.3.11}$$

The average value of the variation of the velocity will be still

$$\langle |u(x) - u(y)|^2 \rangle \propto |x-y|^{2/3} \qquad k_\nu^{-1} \ll |x-y| \ll 1 \tag{6.3.12}$$

For dimensional reasons we could believe that $\varepsilon = (\sqrt{gl})^3 l^{-1}$: in fact it appears likely that also in this model the dissipation ε has a behavior for $\nu \to 0$ that saturates tending to a limit value (*c.f.r.* §6.2, comment to the (6.2.9)).

The advantage of this model is its simplicity, that allows us to perform a more detailed numerical analysis which can serve as benchmark for checking various ideas on which Kolmogorov theory rests. The disadvantage is that it does not allow us to check the relevance of the Thomson law for the theory of Kolmogorov, because such law is not simulated in the GOY model, not even in a simplified form (unlike the energy conservation). Indeed the existence of the integral of motion (6.3.9) is only poorly analogous to a *partial* consequence of the theorem of Thomson.[4]

Furthermore the GOY model *presupposes* the phenomenon of the energy "cascade", *i.e.* that energy is not transmitted directly from one scale to another scale of different order of magnitude: the latter, however, is one of the key points to understand because it repflects an important feature of the structure of the NS equations.

Coming back to the analysis of the GOY shell model we see that, if we can take as valid a representation of Kolmogorov type for the stochastic process describing the statistics of the attracting set, then at large Reynolds number we obtain an interesting example of a statistical distribution that produces, with probability 1, samples $u(x)$ that are Hölder continuous functions with exponent 1/3. Such a field appears to be very interesting also from the point of view mathematical probability because we must expect that it is a nontrivial distribution (for instance *skew*, *c.f.r.* problems of the§6.2).

An interesting question that can be examined is whether the Kolmogorov law could be "violated", for instance if the distribution of the field u could

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3 Defined by $v_{n_\nu} k_{n_\nu} \nu^{-1} = 1$.

4 It would be vane, then, to think of simulating conservation of vorticity by requiring that between the C, D, E a second relation holds implying an analogous conservation, if $\nu = 0$, of $\sum_n k_n^2 |\underline{u}_n|^2$. This model would be indeed analogous to NS in 2 dimensions, in which the vorticity is a scalar quantity. Instead the characteristic of the dimensions > 2 is precisely that the vorticity *not* is a scalar and its conservation, if $\nu = 0$, is only expressed by the theorem of Thomson. We appreciate therefore the *intrinsic* perfidy of the principle of conservation of difficulties. In fact precisely in order not to fall back on a bidimensional problem we shall have to impose that C, D, E *are not* such that they would guarantee that $\sum_n k_n^2 |\underline{u}_n|^2$ is conserved if $\nu = 0$.

deviate from the naive expectation according to which the correlation defined by the average (over the time) of $\delta_p(r) = |u(x) - u(x+r)|^p$ is proportional to $r^{p/3}$, as one thinks in the theory of Kolmogorov (*c.f.r.* the theory of the skewness in the problems of section §6.2, for an example).

In general, denoting $\langle\cdot\rangle$ the time average, we can define ζ_p via

$$\langle|\delta_p(r)|\rangle \propto r^{\zeta_p} \tag{6.3.13}$$

to leading order as $r \to 0$ and postulate, following [BPPV93], p.164, a distribution P_r for the random variable $a = a(u)$ defined (as a function of the sample u of the field) by setting $|u(x) - u(x+r)| = \overline{u}\,(k_0 r)^a$ where $\overline{u}$ is a typical velocity variation. The assumption is that the distribution P_r has, for r small, taking $k_0 = 1$, the form:

$$P_r(a)da = r^{g_r(a)}\rho(a)da \tag{6.3.14}$$

with $g_r(a)$ (and ρ) "slightly" dependent on r; in this case it is

$$\zeta_p = \min_a(pa + g(a)) \tag{6.3.15}$$

if $g(a) = \lim_{r\to 0} g_r(a)$.

It is easy to verify, *c.f.r.* problem [6.3.3], that in the case of the Brownian motion the analogous distribution posed in the form (6.3.14) leads immediately to $\zeta_p \equiv \frac{p}{2}$, *i.e.* the minimum is always (for any p) at $a = \frac{1}{2}$ (which is possible only if $g(a)$ vanishes for $a \neq 1/2$ and is $-\infty$ for $a = 1/2$, *e.g.* $g(a) \sim -\log\delta(a - 1/2)$)).

Because of this one says that the samples of Brownian motion are *simple fractals.* If, instead, ζ_p does not turn out to be linear in p we shall say that the distribution of u has *multifractal* samples, *c.f.r.* [FP84].

And the question that arises is whether the "Kolmogorov fields", *i.e.* the samples of the velocity fields distributed with the stationary distribution of a fluid at large Reynolds number R are, in the limit $R \to \infty$, simple fractals with $\zeta_p = \frac{1}{3}p$ (as the result on the skewness of §6.2 suggests) or whether they are multifractal, as suggested in [FP84].

The study of this problem is very difficult but its analogue in the case of the GOY model has been analyzed from a numerical point of view and the results *suggest* precisely that the distribution *is multifractal.* If this was true also for the statistics of the equation of NS then we should think that the laws of Kolmogorov are a first approximation, *even in the hypothesis of locality of the energy cascade*, *c.f.r.* §6.2, and that in principle they require corrections that bring up the true multifractal structure.

Recent experiments on turbulence that develops as air passes through a grid in a wind tunnel seem to indicate that actually we can observe, also in real fluids, multifractal corrections, *c.f.r.* [VW93].

The first remark towards a more detailed study in the case of the GOY model is that (6.3.6) says that, for any $\varepsilon > 0$ and any real α

$$u_n = 2\varepsilon^{1/3}k_n^{-1/3}e^{i\alpha} \tag{6.3.16}$$

is a "solution" of the (6.3.3) for $n \geq 5$ and $\nu = 0$. This shows that the special role of the exponent 1/3 *is intrinsic*, right from the beginning in equation (6.3.3); and tells us that the law 5/3 represents an "*exact*" solution (and a "*trivial*" one) of the GOY equation.[5] This is so at least in the inertial domain: because it is neither a solution for the first values of n, ($n \leq 4$, where one cannot neglect the forcing g), nor for n too large (where one cannot even neglect the friction terms).

The study can proceed in two directions: by making use of the functional method of the §6.1, or by trying to apply the *principle of Ruelle* of §5.7. The second viewpoint is useful if one succeeds in identifying the attracting set, or at least some invariant probability distributions.

In a sense to make precise, the solution that we can call "Kolmogorov solution" appears as a *fixed point* in phase space for the evolution of the GOY model at zero viscosity. And to make since this *rigorously* true we shall imagine to modify the function g_n in such a way that the (6.3.6) is *exactly* solved by (6.3.16) for $k_0 \geq k_n \geq k_{n_\nu}$, *i.e.* also for the extreme values of n close to 1 or n_ν (therefore g_n must now be thought of as defined in terms of the parameter ε in (6.3.16) via the (6.3.11)).

We can expect that this fixed point is *unstable*. It is then natural to think that its *unstable manifold* generates an attracting set that really attracts motions.

In this scenario the possible multifractality of the statistics of the motion would be due precisely to the extension in phase space of the attracting set, *i.e.* to the fact that it is not the single point (6.3.16).

Let us proceed to rewrite the equation (6.3.16) in dimensionless form: we choose as time unit the time scale associated with the Kolmogorov length l_{n_ν}, (6.3.11) and set:

$$t' = t k_{n_\nu}^{2/3} \varepsilon^{1/3} \tag{6.3.17}$$

One verifies immediately, via (6.3.11), that $k_{n_\nu}^{2/3}\varepsilon^{1/3} \equiv \nu k_{n_\nu}^2$: *i.e.* one dimensionless time unit t' corresponds to a physical unit of time associated with the scale of Kolmogorov, *i.e.* it is the *fastest* (nontrivial) time of the motion.

Changing notation so that the components $n = 0, 1, \ldots$ are counted starting from the Kolmogorov scale downwards, *i.e.* setting $u_n(t) \equiv k_n^{-1/3}\varepsilon^{1/3}\varphi_{n_\nu - n}(t')$, one finds the dimensionless equation

$$\dot\varphi_k + 2^{-2k}\varphi_k = i2^{-2k/3}\big(-\frac{1}{2}\,\overline\varphi_{k-1}\,\overline\varphi_{k+1} + \overline\varphi_{k-2}\,\overline\varphi_{k-1} - \frac{1}{2}\,\overline\varphi_{k+1}\,\overline\varphi_{k+2} + \gamma_k(\alpha)\big) \tag{6.3.18}$$

with $k = 0, 1, \ldots, n_\nu - 1$ and $\gamma_n(\alpha)$ *defined so that* $\varphi_k \equiv e^{i\alpha}$ (*i.e.* the (6.3.16) in the new variables) makes $\dot\varphi_k$ to vanish *exactly* in the (6.3.18).

Note that, as already observed, this last condition *implies* that only the components with n of order 1 and n_ν of $\gamma_n(\alpha)$ are "appreciably" not zero:

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[5] We can say that the (6.3.11) is the law 5/3 for the GOY model because it predicts that the energy contained in the shell $(k_n, 2k_n)$ is proportional to $\varepsilon^{2/3}k_n^{-2/3}$ as it would follow from (6.2.8) integrated between k_n and $2k_n$.

indeed the $\varphi_h = e^{i\alpha}$ already makes exactly zero the sum of the first three terms of the right hand side of (6.3.18) and hence $\gamma_k(\alpha)$ must "only" compensate the term $2^{-2k}\varphi_k$ that tends rapidly a 0 as k increases (which is, hence, present only for k small) and the original force (present only for k near n_ν). In this way the $\varphi_k \equiv e^{i\alpha}$ corresponds to an exact "Kolmogorov solution" (6.3.16).

The necessity of the term $\gamma_k(\alpha)$ and its dependence on α arises only to impose that (6.3.16) makes $\dot\varphi_k$ vanish when inserted in (6.3.18) *also for the extreme values of* k: hence we can say that the *GOY* equation has a "symmetry" that manifests itself through the fact that the existence of a one parameter family of fixed points is only forbidden by the "boundary conditions".

This means that one must imagine that γ_k is fixed, and hence α is fixed, because we must fix once and for all the equation that we want to study. Since γ_k differs from zero only for few extreme values of k one can wonder whether in the limit in which $\nu \to 0$ (and hence $n_\nu \to \infty$) the symmetry is *restored* in the minimal sense that there exist exact solutions of (6.3.18) having essentially the form $e^{i\alpha'}$ for every α' *at least* for $1 \ll n \ll n_\nu$.

From the results of [BPPV93] evidence emerges, both experimental and theoretical, that in reality the rotation *symmetry* of the parameter α is spontaneously broken and *is not* restored, not even in the limit $\nu \to 0$: precisely the attracting set of (6.3.6) is determined by the (6.3.18) with $\alpha = \pi/2$. And the statistics of (6.3.18) with $\alpha \neq \pi/2$ is *also* identical to the statistics of (6.3.6) with $\alpha = \pi/2$.

If one assumes this (experimentally suggested) property one can then remark that (6.3.18) with $\alpha = \pi/2$ is *universal*: it does not depend any more by any parameter. Suppose that the solutions of (6.3.6) have the property that there exists a scale k'_ν above which "nothing interesting happens" so that it will be possible to replace (6.3.6) with the same equations truncated at n'_ν, depending on g (that defines a suitable value of ε). Then it becomes tautological to say that, if the statistics of the motion will be universal and determined from the statistics of (6.3.18).

Setting $\alpha = \pi/2$ the SRB statistics of the field u is related to that of the solution of (6.3.18) through the relation:

$$u_n(t) = 2\varepsilon^{1/3} k_n^{-1/3} \varphi_{n_\nu - n}(t k_{n_\nu}^{2/3} \varepsilon^{1/3}) \tag{6.3.19}$$

and $\varphi_k(t)$ is distributed according to the SRB statistics of the equation (6.3.18) with $\alpha = \pi/2$.

If the solution $\varphi_k \equiv i$ of the (6.3.18) was stable we would have the exact validity of the law 5/3 of Kolmogorov and the statistics $\mu(du)$ should be a Dirac delta concentrated on u given by (6.3.19) with $\alpha = \frac{\pi}{2}$.

It is convenient therefore to examine the question of the stability, setting $\varphi_k = \rho_k e^{i\vartheta_k}$ and writing the equation for ρ_k, ϑ_k and linearizing them near the stationary solution $\varphi_k \equiv i$. The equations for ρ_k, ϑ_k are:

$$\begin{aligned}\dot\rho_k + 2^{-2k}\rho_k =& -2^{-3k/2}\big(\rho_{k-2}\rho_{k-1}\cos\Delta_{k-2}-\\&-\frac12\rho_{k-1}\rho_{k+1}\cos\Delta_{k-1}-\frac12\rho_{k+1}\rho_{k+2}\cos\Delta_k\big)+r_k\\ \dot\vartheta_k + 2^{-2k} =& 2^{-3k/2}\rho_k^{-1}\big(\rho_{k-2}\rho_{k-1}\sin\Delta_{k-2}-\\&-\frac12\rho_{k-1}\rho_{k+1}\sin\Delta_{k-1}-\frac12\rho_{k+1}\rho_{k+2}\sin\Delta_k\big)+\sigma_k\end{aligned}\tag{6.3.20}$$

and $\Delta_k \equiv \vartheta_k + \vartheta_{k+1} + \vartheta_{k+2}$ and r_k, s_k are such that $\rho_k \equiv 1$ and $\vartheta_k \equiv \pi/2$ is an exact stationary solution.

The linearization of the (6.3.20) around to the exact stationary solution gives the following equations for η_k, δ_k, obtained by setting $\Delta_k = 3\pi/2 + \delta_k$ and $\rho_k = 1 + \eta_k$ and developing in series of the increments η_k,

$$\begin{aligned}\dot\eta_k + 2^{-2k}\eta_k =& -2^{-3k/2}\big(\eta_{k-2}+\frac12\eta_{k-1}-\eta_{k+1}-\frac12\eta_{k+2}\big)\\ \dot\delta_k =& +2^{-3k/2}\big(\delta_{k-2}+\frac12\delta_{k-1}-\delta_{k+1}-\frac12\delta_{k+2}\big)\end{aligned}\tag{6.3.21}$$

The *particularity* of the choice $\alpha = \pi/2$ is seen by noting that $\alpha = \pi/2$ is the *only choice* of α in (6.3.18) such that the linearized equations (6.3.21) for η and δ are separated.

It is easy to realize that the matrix M defining the equation for δ is a real traceless matrix: hence it must have *some* eigenvalues with positive real part unless they are all purely imaginary. But the matrix M is the product of a diagonal matrix times a matrix of Toeplitz and is easy see that it admits eigenvalues with nonzero real part, *c.f.r.* [BPPV93].

We deduce the *instability of the Kolmogorov solution* for the GOY model. Thus the problem arises of trying to understand how is the attracting set made and which is the correspondent statistics.

The matrix $M'_{kk'}$ that defines the stability equation for η_k is different from the $-M'_{k'k}$ because of the addition of a diagonal term $2^{-2k}\delta_{kk'}$. We can see that this matrix M' has all the eigenvalues with negative part real. Therefore one gets is led to think that the attracting set for (6.3.18) could be such that on it $\rho_k \stackrel{def}{=} (1+\eta_k)$ can be expressed in terms of the Δ_k. It is natural to suppose, [BPPV93], that in the inertial domain the relation expressing the ρ's in terms of the Δ's is essentially the condition of vanishing of the term in parenthesis in the first of (6.3.20).

We shall write the latter condition, implicitly defiining F, as

$$\rho_{k+1} = \rho_k\, e^{F(\Delta_{k+1},\Delta_k,\Delta_{k-1},\rho_k,\rho_{k-2},\rho_{k-3})}\tag{6.3.22}$$

but on the Δ_k (or the δ_k) the instability does not allow us to make any statement other than these variables will probably have a random distribution on an interval around a $3\pi/2$ (or, respectively, around a 0).

In such case the attracting set, as a geometric locus in the space of the $(\underline{\eta}, \underline{\delta})$ variables, would have (6.3.22) as equation; *and if the parameters* Δ_k *were independently distributed* then the attracting set would have dimension n_ν (*i.e.* half of the dimension $2n_\nu$ of the phase space). Furthermore the (6.3.22) show that the variables ρ_k would have a statistics that is well described by a random process with short memory.

If the eq. (6.3.22) is assumed as the equation for the attractor then the SRB distribution is concentrated on the attractor and it should *therefore* be a probability distribution μ, on the sequence of the phases $\{\delta_k\}$, characterized by Ruelle's variational principle (5.7.4): however the motion on the attractor is controlled by an equation for the δ-phases of which the second of (6.3.21) is not a good approximation away from the unstable fixed point. Therefore it is not possible, with information obtained so far, to apply the principle because it would require determining the Jacobial determinant along the unstable manifold for the evolution on the attractor. Since there is no reason to think that the phases δ_k should assume privileged values it is a reasonable guess to identify the SRB distribution with the uniform independent distribution on the parameters δ_k. From the point of view of the variational principle this is equivalent to assuming that the dynamics preserves the volume in the space of the phases $\{d_k\}$ and that the expansion along the unstable manifold is essentially constant so that the variational principle will give that the SRB distribution just maximizes the entropy and coincides with the distribution attributing a uniform and independent distribution between 0 and $2p$ to each phase d_k.

This seems to give a qualitative explanation of why a violation of the Kolmogorov law due to the instability of the solution $\varphi_k \equiv i$ leads to a multifractal distribution of the u_n.

To clarify the mechanism of this last property consider the simple case in which the Δ_k are *really independently distributed* and the relation (6.3.22) is replaced by the relation $\rho_{k+1} = \rho_k \exp f(\Delta_k)$ and the distribution of each Δ_k is $\pi(\Delta)d\Delta$, with a suitable density function π. Then under this assumption[6]

$$|u_n| = 2k_n^{-1/3}\varepsilon^{1/3}\rho_n = 2\rho_1 k_n^{-1/3}\varepsilon^{1/3}e^{\sum_{j=1}^{n-1} f(\Delta_j)} \tag{6.3.23}$$

and hence, recalling that $k_n = 2^n$:

$$\frac{\langle |u_n|^p\rangle}{\langle |u_1|^p\rangle} = 2^{-np/3}\int \prod_{j=1}^{n-1} \pi(\Delta_j)\, e^{p\sum_j f(\Delta_j)}\, d\Delta_j \tag{6.3.24}$$

We write this as a single integral with a density function $e^{G(p)}$ defined be $e^{G(p)} \equiv \int \pi(x)\,dx\, e^{pf(x)}$, furthermore we set $l_n = k_n^{-1}$ and writing $e^{nG} =$

[6] Which has illustrative character only, because this is obviously not the case if ρ_k is as in (6.3.22) where the variables $\Delta_{j-1}, \Delta_j, \Delta_{j+1}$ influence each other distribution. However a nearest neighbor dependence turns the distribution of Δ_j from "Bernoullian" to "markovian" and it could be equally well treated with minor technical problems to solve.

$2^{nG/\log 2} = \big(l_0/l_n\big)^{G/\log 2}$, we realize that

$$\frac{\langle |u_n|^p\rangle}{\langle |u_1|^p\rangle} = (\frac{l_n}{l_0})^{p/3-G(p)/\log 2} \quad \rightarrow \quad \zeta_p = \frac{1}{3}p - \frac{G(p)}{\log 2} \tag{6.3.25}$$

thus we see a concrete mechanism that can lead to a multifractal distribution, *c.f.r.* [BPPV93].

The application to the GOY model of the functional integration method of §6.1 is also promising, but it has not yet been really considered in the literature. We must note that in the present situation we can apply the method in a somewhat different version: *i.e.* to the *universal* equation (6.3.18). In this way the problem of field theory that one has to consider is a problem of scales decreasing from 1 to 0, *i.e.* it is an "infrared" problem, rather than a ultraviolet problem as in §6.2, and as it would also be in the present case if we proceeded exactly in the same way followed in §6.2. But this is not the appropriate place to attack a problem which is still so little considered in the literature.

Problems.

[6.3.1]: Study the stability of the "laminar" solution $u_n = (g/\nu k_n^2)\,\delta_{nn_0}$ of the (6.3.6) with $n_0 = 1$ and show that becomes unstable for ν small enough. (*Idea:* Check first that, as a consequence of the negative signs in the terms with n odd in (6.3.9) the argument used in problem [4.1.13] in the case of the two dimensional Navier Stokes equation does not apply to the present case.)

[6.3.2]: As [6.3.1] for the equation (6.3.7) in footnote [1].

[6.3.3]: Check that if $t \to \omega(t)$, $t \in [0,\infty)$, is a sample of a Brownian motion then $\langle|\omega(t)-\omega(t')|^{2p}\rangle, \langle|\omega(t)-\omega(t')|^2\rangle^p, |t-t'|^p$ are proportional with t,t'–independent proportionality constant: hence ζ_p in (6.3.13) is $\zeta_p = \frac{p}{2}$ (*Idea:* By definition in the Brownian motion the increments $\delta = \omega(t)-\omega(t')$ have a Gaussian distribution $e^{-\delta^2/2|t-t'|}d\delta\,/\sqrt{2\pi|t-t'|}$.)

Bibliography: The discussion of the attracting set in the GOY model is taken from [BPPV93] where one finds a complete discussion of the theory and of the experimental results on the GOY model.