CHAPTER III

Analytical theories and mathematical aspects

§3.1 Spectral method and local existence, regularity and uniqueness theorems for Euler and Navier–Stokes equations, $d \ge 2$.

One of the most immediate and elementary applications of the spectral method of §2.2 is the *local* existence, regularity and uniqueness theory of the solutions of the Euler and Navier–Stokes equations in arbitrary dimension.

We shall illustrate the theory only in the case in which Ω is a cube with opposite sides identified (*i.e.* "periodic boundary conditions").

Consider initial data $\underline{u}_0(\underline{x})$ and force density $\underline{g}(\underline{x})$ which are *analytic* in $\underline{x}$, hence such that

$$\begin{aligned} \underline{u}^0(\underline{x}) &= \sum_{\underline{k}\neq\underline{0}} \underline{\gamma}^0_{\underline{k}} e^{i\underline{k}\cdot\underline{x}}, \qquad \underline{\gamma}^0_{\underline{k}} = \overline{\underline{\gamma}}^0_{-\underline{k}}, \quad \underline{k}\cdot\underline{\gamma}^0_{\underline{k}} \equiv 0, \quad |\underline{\gamma}^0_{\underline{k}}| \le V e^{-\xi|\underline{k}|} \\ \underline{g}(\underline{x}) &= \sum_{\underline{k}\neq\underline{0}} \underline{g}_{\underline{k}} e^{i\underline{k}\cdot\underline{x}}, \qquad \underline{g}_{\underline{k}} = \overline{\underline{g}}_{-\underline{k}}, \quad \underline{k}\cdot\underline{g}_{\underline{k}} \equiv 0, \quad |\underline{g}_{\underline{k}}| \le G e^{-\xi|\underline{k}|} \end{aligned} \tag{3.1.1}$$

where V, G and ξ are suitable positive constants and $\underline{k}$ is a *nonzero* vector with components integer multiples of $k_0 = 2\pi L^{-1}$ and $|\underline{k}|$ will denote $\sum_i |k_i|$.[1] The averages $\underline{u}_{\underline{0}}$ and $\underline{g}_{\underline{0}}$ are supposed zero, as usual. To simplify calculations we shall assume $\xi \equiv \xi_0/k_0$ with $\xi_0 \le 1$ which, evidently, is not restrictive.

Then the following proposition holds

Proposition *("perturbative local" solution of NS): Consider the NS–equation in d–dimensions (d = 2, 3)*

$$\dot{\underline{u}} = \nu\,\Delta\,\underline{u} - \underset{\sim}{\underline{u}}\cdot\underset{\sim}{\underline{\partial}}\,\underline{u} - \rho^{-1}\,\underline{\partial}\,p + \underline{g}, \qquad \underline{\partial}\cdot\underline{u} = 0 \tag{3.1.2}$$

[1] This is a natural definition when $\underline{k}$ are the Fourier transform mode labels for an analytic function, *c.f.r.* [3.1.1].

with initial datum and force satisfying (3.1.1).
There is $B_0 > 0$ *such that if* $\xi_0 \stackrel{def}{=} k_0\xi \le 1$, $V_c \stackrel{def}{=} \nu L^{-1}$, $T_c \stackrel{def}{=} L^2\nu^{-1}$ *and* $V_0 \stackrel{def}{=} B_0\,(V + G\,T_c)\,\xi_0^{-d-1}$ *then (3.1.2), admits a solution* $\underline{u}(\underline{x},t)$ *analytic in* $\underline{x}$ *and* t *with*

$$|\underline{\gamma}_{\underline{k}}(t)| \le V_0 e^{-\xi_0|\underline{k}|/2k_0}, \qquad \text{for} \quad 0 \le t \le t_0 \stackrel{def}{=} T_c\Big(\frac{V_c}{V_0}\Big)^2 \tag{3.1.3}$$

and the solution is the unique one enjoying the above properties in the time interval $[0, t_0]$.

Remark:

(1) Hence a local existence and regularity theorem holds, and explicit estimates on the duration of the existence and regularity interval are possible: "analytic data evolve remaining analytic at least for a small enough time".
(2) Uniqueness holds even within a much wider class of solutions $t \to \underline{u}(\underline{x},t)$, $p(\underline{x},t)$: for instance it holds for solutions in class $C^1([0,t_0]\times\Omega)$. See problem [3.1.6] below.
(3) Existence regularity and uniqueness can be studied also in other classes of functions. For instance if $\underline{u}^0 \in W^m(\Omega)$ and $m \ge 4$ (so that since $4 > \frac{3}{2}+2$ one has, *c.f.r.* [2.2.20] that $\underline{u}^0 \in C^1(\Omega)$) one can easily show the existence of a time T_m such that the equation (3.1.3) admits a solution in $W^m(\Omega)$ for each $t \in (0, T_m)$: see problems. The analytic case considered here is more involved: but the result is perhaps more interesting, mainly because of the technique that is employed and that illustrates a method often used in other physics problems.

proof: The part concerning uniqueness is absolutely elementary and, as noted, it will show in reality uniqueness among solutions of class $C^1([0,t_0]\times\Omega)$ (hence, *a fortiori*, in the analytic class of the type stated in the proposition). Calling $\underline{u}_1$ and $\underline{u}_2$ two solutions of class $C^1([0,t_0]\times\Omega)$ let $\delta\underline{u} = \underline{u}_1 - \underline{u}_2$ and consider the relation obtained by computing the difference between the two members of the equations that each of the $\underline{u}_i$ verifies, multiplying at the same time both sides by $\delta\underline{u}$ and integrating on Ω. One finds

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\delta\underline{u})^2\,d\underline{x} = -\nu\int_\Omega (\underset{\sim}{\partial}\,\delta\underline{u})^2\,d\underline{x} + \int_\Omega (-(\underset{\sim}{u}_1\cdot\underset{\sim}{\partial}\,\delta\underline{u} - (\delta\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}_2))\cdot\delta\underline{u}\,d\underline{x} \tag{3.1.4}$$

because the pressure term disappears by integration by parts (by using the zero divergence property of the velocity field) and in the same way the first term in the third integral of (3.1.4) also disappears (because it can be written $\int_\Omega -\underset{\sim}{u}_1\cdot\underset{\sim}{\partial}\,(\delta\underline{u})^2\,d\underline{x}$ and it is, therefore, zero for the same reason).

Hence setting $D = \int_\Omega (\delta\underline{u})^2\,d\underline{x}$ one has, for $0 \le t \le t_0$

$$\frac{1}{2}\dot{D} \le D \max_{\underline{x},\,0\le t\le t_0} |\underset{\sim}{\partial}\underline{u}_2(\underline{x},t)| \stackrel{def}{=} D\,M \tag{3.1.5}$$

where M is the indicated maximum ($M < \infty$ because $\underline{u}_2 \in C^1([0,t_0)\times\Omega)$): the (3.1.5) implies $D(t) \le D(0)e^{2Mt}$; hence $D(0) = 0$ yields $D(t) = 0$ for each $0 \le t \le t_0$; *i.e.* one has uniqueness of the solutions of the (3.1.2) verifying (3.1.3).

Note that in (3.1.5) we did not take advantage of the non positivity of the term proportional to the viscosity: hence the just seen uniqueness property remains valid in the case of the Euler equation. However in the remaining discussion we shall make explicitly use of the hypothesis $\nu \ne 0$, *c.f.r.* problem [3.1.6].

We now look at the more interesting question of existence. We shall write (3.1.2) in the spectral form (2.2.10):

$$\dot{\underline{\gamma}}_{\underline{k}} = -\nu\,\underline{k}^2\underline{\gamma}_{\underline{k}} - i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2\,\Pi_{\underline{k}}\underline{\gamma}_{\underline{k}_2} + \underline{g}_{\underline{k}} \tag{3.1.6}$$

where $\Pi_{\underline{k}}\underline{\hat{g}}_{\underline{k}}$ of (2.2.10) is called here $\underline{g}_{\underline{k}}$. Let us rewrite it as

$$\begin{aligned}\gamma_{\underline{k}}(t) =& \underline{\gamma}^0_{\underline{k}}e^{-\nu\underline{k}^2t} + \int_0^t e^{-\nu\,\underline{k}^2(t-\tau)}\Big(\underline{g}_{\underline{k}} -\\ &- i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underset{\sim}{\gamma}_{\underline{k}_1}(\tau)\cdot\underset{\sim}{k}_2\,\Pi_{\underline{k}}\underline{\gamma}_{\underline{k}_2}(\tau)\Big)\,d\tau\end{aligned} \tag{3.1.7}$$

We shall set $\overline{\underline{\gamma}}^0_{\underline{k}}(\tau) \equiv \underline{\gamma}^0_{\underline{k}}e^{-\nu\underline{k}^2\tau} + \underline{g}_{\underline{k}}\,(1-e^{-\nu\,\underline{k}^2\tau})/\nu\,\underline{k}^2$ and we get

$$|\,\overline{\underline{\gamma}}^0_{\underline{k}}(\tau)| < \overline{V}_0 e^{-\xi_0|\underline{k}|/k_0}, \qquad \overline{V}_0 \equiv V + G\,(\nu k_0)^{-2} \tag{3.1.8}$$

where $k_0 = 2\pi/L$, *c.f.r.* (3.1.1).

Imagine solving (3.1.7) recursively, setting $\underline{\gamma}^0_{\underline{k}}(t) \equiv \overline{\gamma}^0_{\underline{k}}(t)$ and

$$\underline{\gamma}^{n+1}_{\underline{k}}(t) = \overline{\underline{\gamma}}^0_{\underline{k}}(t) - i\int_0^t e^{-\nu\,\underline{k}^2(t-\tau)}\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underset{\sim}{\gamma}^n_{\underline{k}_1}(\tau)\cdot\underset{\sim}{k}_2\,\Pi_{\underline{k}}\underline{\gamma}^n_{\underline{k}_2}(\tau)\,d\tau \tag{3.1.9}$$

for $n \ge 0$. Then by "iterating" (3.1.9) once we get

$$\begin{aligned}&\underline{\gamma}^{n+1}_{\underline{k}}(t) = \overline{\underline{\gamma}}^0_{\underline{k}}(t) + \int_0^t e^{-\nu\,\underline{k}^2(t-\tau)}d\tau\,\Big(-i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \overline{\underset{\sim}{\gamma}}^0_{\underline{k}_1}(\tau)\cdot\underset{\sim}{k}_2\,\Pi_{\underline{k}}\overline{\underline{\gamma}}^0_{\underline{k}_2}(\tau)\Big) +\\ &- i\int_0^t e^{-\nu\,\underline{k}^2(t-\tau)}d\tau\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}\int_0^\tau d\tau'\,\Big(-i\sum_{\underline{k}'_1+\underline{k}'_2=\underline{k}_1} e^{-\nu\,\underline{k}_1^2(\tau-\tau')}\\ &\underset{\sim}{\gamma}^{n-1}_{\underline{k}'_1}(\tau')\cdot\underset{\sim}{k}'_2\,\Pi_{\underline{k}_1}\underline{\gamma}^{(n-1)}_{\underline{k}'_2}(\tau')\Big)\cdot\underline{k}_2\Pi_{\underline{k}}\,\overline{\underline{\gamma}}^{(0)}_{\underline{k}_2}(\tau) +\\ &- i\int_0^t e^{-\nu\,\underline{k}^2(t-\tau)}d\tau\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} -i\int_0^\tau e^{-\nu\,\underline{k}_2^2(\tau-\tau')}\sum_{\underline{k}'_1+\underline{k}'_2=\underline{k}_2}\end{aligned}$$

$$\overline{\gamma}^{(0)}_{\underline{k}_1}(\tau')\cdot \underset{\sim}{\underline{k}}_2\Pi_{\underline{k}}\Big(\gamma^{n-1}_{\underline{k}'_1}(\tau')\cdot\underset{\sim}{\underline{k}}'_2\,\Pi_{\underline{k}_2}\overline{\gamma}^{(n-1)}_{\underline{k}'_2}(\tau')\Big)d\tau' \tag{3.1.10}$$
$$-\,i\int_0^t e^{-\nu\,\underline{k}^2(t-\tau)}\,d\tau\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}$$
$$\Big(-i\sum_{\underline{k}'_1+\underline{k}'_2=\underline{k}_1}\int_0^\tau e^{-\nu\,\underline{k}_1^2(\tau-\tau_2)}\big(\gamma^{(n-1)}_{\underline{k}'_1}(\tau_2)\cdot\underset{\sim}{\underline{k}}'_2\,\Pi_{\underline{k}_1}\gamma^{(n-1)}_{\underline{k}'_2}(\tau_2)\big)d\tau_2\Big)\cdot\underline{k}_2$$
$$\Pi_{\underline{k}}\Big(-i\sum_{\underline{k}''_1+\underline{k}''_2=\underline{k}}\int_0^\tau e^{-\nu\,\underline{k}_2^2(\tau-\tau_3)}\gamma^{(n-1)}_{\underline{k}''_1}(\tau_3)\cdot\underset{\sim}{\underline{k}}''_2\,\Pi_{\underline{k}_2}\gamma^{(n-1)}_{\underline{k}''_2}(\tau_3)d\tau_3\Big)$$

for $n\ge 1$.

The (3.1.10) can be further iterated: *clearly one needs a better notation* because the (3.1.10) is a straightforward consequence of (3.1.9) and yet it looks very cumbersome and promises to become even more so, unpractically so, upon further iterations.

Therefore we discuss how to find a simpler representation for $\gamma^n_{\underline{k}}(t)$: it turns out that it can be represented graphically as

$$\gamma^n_{\underline{k}}=\overline{\gamma}^0_{\underline{k}}(t)+\sum_{1\le m<2^n}\ \sum_{\Theta\in\ \text{decorated m-trees}}\text{Val}(\Theta) \tag{3.1.11}$$

where Θ is a "tree" with m internal vertices ("decorated m-tree"), and Val(Θ) is its "*value*" (that will be defined below); furthermore the tree has a root and every branching happens by doubling, *i.e.* into every internal vertex two branches of the tree enter and one exits.

To understand the notation in detail define a *m–tree* as a connected set of $2m+1$ oriented lines, that we call "branches", with no cycles. Let us denote with (v,v') an oriented line drawn on a plane and going from the point v to the point v'.

(a) Given $2m+1$ oriented lines of unit length and fixed a point r on the plane, we copy on the plane a first oriented line (v_0,r) that ends in r. We shall say that r is the *root* of the tree.

(b) Then we copy on the plane two other oriented lines that end in v_0, *i.e.* at the starting point of the already drawn line. Let (v_1,v_0) and (v_2,v_0) be these two lines.

(c) Continue the construction by attaching to some of the initial vertices of the last drawn lines pairs of oriented lines with the end vertex in common. Until (after m steps) a figure Θ is obtained that we shall call *tree* with root r and $2m+1$ branches, or *m–tree* (not decorated).

(d) We shall call *internal vertices* or *nodes* the vertices into which two lines enter (and one exits, necessarily). While the vertex r will be called root, the vertices out of which only a line exits and none enters will be called *final vertices* or *external* and their collection will be denoted by V_f.

(e) Here we shall consider, therefore, only trees with three branches per node, one exiting and two entering: hence the number of nodes is m and it is one unit lower than number of external vertices (root excluded because it *will not be considered a vertex of the tree*). The trees are thought of with the branches entering every node marked, and distinguished, by a label $\delta = 0, 1$; furthermore trees that can be overlapped (labels included) by means of a sequence of rotations of the lines entering the nodes will be considered identical.

In terms of this definition and adding to the trees a certain number of other labels or *decorations* that shall be described in following explanations on how to read the figures, it is possible to reinterpret the (3.1.9) so that it is represented by the following graph

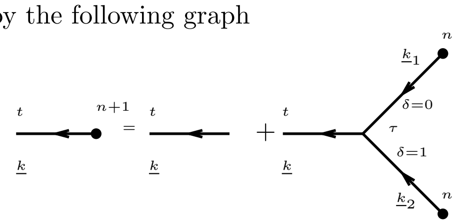

Fig (3.1.1) The representation of (3.1.9).

We read this Fig. (3.1.1) as follows: the l.h.s. represents the l.h.s. of (3.1.9): hence it carries the decorations $t, \underline{k}, n+1, t$ needed to identify it (*i.e.* it is a notation "alternative" to $\underline{\gamma}^{n+1}_{\underline{k}}(t)$). The first graph in the r.h.s. (it is a 0–tree) represents $\overline{\underline{\gamma}}^{0}_{\underline{k}}(t)$, *i.e.* the first term of the r.h.s. of (3.1.9). Note that it does not carry the label $\bullet$ on the final vertex: a reminder that this branch represents a "known" term.

The second term is read by interpreting the line with labels $\delta = 0$ and $\underline{k}_1$, ending in the node carrying the label τ and starting in a final vertex with label $\bullet^n$, as representing (see (3.1.9)) $\underline{\gamma}^{n}_{\underline{k}_1}(\tau) \cdot \underline{k}_2$; the other line (with $\delta = 1$) instead is interpreted as $\Pi_{\underline{k}}\, \underline{\gamma}^{n}_{\underline{k}_2}(\tau)$, where $\underline{k} \equiv \underline{k}_1 + \underline{k}_2$ is the label of the line exiting from the node into which the two lines merge: and this node must be interpreted as the operation $-i \int_0^t d\tau\, e^{-\nu\, \underline{k}^2 (t-\tau)}\cdot$ performed in (3.1.9).

Hence the node represents an integration operation and the label δ distinguishes the two factors with $\underline{\gamma}^n$ in (3.1.9): this is a label made necessary by their not symmetric role. We see that the decorations and the form of the tree identify uniquely the operations to perform: thus they are just an alternative notation to (3.1.9).

Likewise the equation (3.1.10), obtained replacing the functions $\underline{\gamma}^{n}_{\underline{k}}(\tau)$ in (3.1.9) with the expression that is provided by (3.1.9) itself (with n replaced by $n-1$), is susceptible of a graphical interpretation. An attentive examination of (3.1.10) indeed gives us the following representation, see Fig. (3.1.2), consistent with the preceding one, and *manifestly much simpler than* (3.1.10).

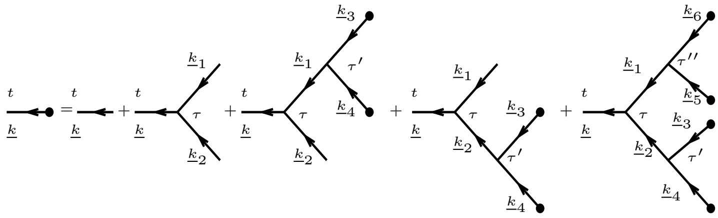

Fig. (3.1.2) The representation of (3.1.10).

where the vertices marked by $\bullet$ should carry the label $n+1$ in the l.h.s. and $n-1$ in the r.h.s., and the lines entering the nodes should carry the label $\delta = 0, 1$ (*not marked in the figure* for simplicity).

We must remark that Fig. (3.1.2) is obtained from Fig. (3.1.1) by simply replacing the final lines (*i.e.* lines with initial vertex in V_f and, therefore, without entering lines) carrying the label $\bullet^n$ by one of the two trees of the r.h.s. of Fig. (3.1.1).

Hence Fig. (3.1.2) can be further "developed" by iterating the construction. At every iteration the trees that represent $\gamma^n_{\underline{k}}(t)$ will have final vertices without $\bullet$ and final vertices still carrying this label that, however, at the k–th step will be $\bullet^{n-k}$. Hence the procedure will stop when $k = n$, provided the vertices with label $\bullet^0$ will be simply drawn *without* this label (as it can be given no other meaning). The graphs that are obtained in this way are trees in the sense defined by (a)%(e) suitably decorated with labels (and with the final branches deprived of the labels $\bullet$).

It is clear that the result of the iteration of the (3.1.9), *i.e.* of the iteration developed until the $\gamma^{(j)}_{\underline{k}}(\tau)$ with $j > 0$ disappear, can be written in the form (3.1.11), as we see already in the case of (3.1.10). For this purpose we shall formally describe the further "decorations" that it is necessary to attach to the trees. The natural decorations, suggested from the above interpretation of (3.1.9) and (3.1.10), are

(1) A label $\underline{k}_\rho$ is attached to every branch ρ of Θ: such vector labels will have to be subjected to the constraint that the sum of the $\underline{k}_\rho$ of the (two) branches ρ entering a node be equal to the label $\underline{k}_\rho$ of the exiting branch. Hence the labels $\underline{k}_\rho$ of the final branches (*i.e.* of those that begin in vertices $v \in V_f$) determine all the labels $\underline{k}_\rho$ of the other branches. We shall set, however, the restriction that the label $\underline{k}_\rho$ of the branch that ends into the root be $\underline{k}$, if we are computing $\gamma_{\underline{k}}(t)$).

We shall call, in analogy with the "*Feynman graphs*" of field theory, the vector $\underline{k}_\rho$ the *momentum flowing* in the line ρ and the momentum of the line that ends in the root will be the *total momentum* of the tree. Note that $\underline{k} = \sum^*_\rho \underline{k}_\rho$ where the sum is restricted only to the final branches (*i.e.* with initial vertex in V_f).

(2) With every node v we associate a time $\tau_v \in [0, t]$ so that the function $v \to \tau_v$ increases as a function of v (consistently with order on the tree,

from the final vertices towards the root). With the root we associate the time t larger than all τ_v.
(3) Finally we recall that, by construction, every branch ρ that precedes a node v carries a label $\delta_\rho = 0, 1$.

If ρ_v is the branch that has v as initial point and v'_ρ, v_ρ denote the final and initial points of the branch ρ, consider the following quantity that we call the *value* Val(Θ) of the decorated tree (Θ)

$$(-i)^m \int (\prod_{v\,\text{nodes}} d\tau_v)(\prod_{\rho\,\text{int}} e^{-\nu\,\underline{k}_\rho^2(\tau_{v'_\rho}-\tau_{v_\rho})}\Pi_{\underline{k}_\rho})(\prod_{v\in V_f} \overline{\underline{\gamma}}^0_{\underline{k}_{\rho_v}}(\tau_v))(\prod_{\rho,\delta_\rho=1} \underline{k}_\rho) \tag{3.1.12}$$

where the first product is over the non trivial vertices, *i.e.* the nodes, of the tree (in number of m); the second product is over the $m-1$ internal branches (*i.e.* over the branches that do not start at a final vertex $v \in V_f$); the third product on $v \in V_f$ concerns the $m+1$ final vertices and the last product is over the m branches with label $\delta_\rho = 1$.
The domain of the integrals on the τ_v is described by the conditions in (2). The labels of the vectors $\underline{k}$ and $\overline{\underline{\gamma}}^0_{\underline{k}}$ and of the tensors $\Pi_{\underline{k}_\rho}$ that project on the planes orthogonal to $\underline{k}_\rho$ must be contracted between each other in a suitable way (that can be inferred by reading (3.1.9), (3.1.10) and that here it is not interesting to make explicit): *then the sum (3.1.11) of the values (given by (3.1.12)) of the decorated m–trees Θ with $m < 2^n$ will give us the value of $\underline{\gamma}^{(n)}_{\underline{k}}(t)$. The sum over all the decorated trees will be exactly the limit for $n \to \infty$ of $\underline{\gamma}^{(n)}(\tau)$.*
In other words (3.1.12) gives a formula, as a development in a series, for the solution of the NS equation, obviously modulo problems of convergence and provided no double counting errors are made. The latter are simply avoided if we impose to consider only the contribution of decorated trees which are "different", considering as equal two decorated trees that can be overlapped by permuting the branches that enter the same node (with the decorations rigidly attached).

Wishing to perform a bound of the convergence of the series for $t \le t_0$, with $t_0 > 0$, we remark that every factor $\overline{\underline{\gamma}}^0_{\underline{k}}$ can be bounded by $\overline{V}_0 e^{-\xi_0|\underline{k}|/k_0}$ using (3.1.8), with $\overline{V}_0 = V + (\nu k_0)^{-2}\,G$.
Writing $e^{-\xi_0|\underline{k}|/k_0} \equiv e^{-\xi_0|\underline{k}|/2k_0}e^{-\xi_0|\underline{k}|/2k_0}$ and recalling that the sum of the $\underline{k}_\rho$ is conserved at every vertex, we see that $\underline{k} = \sum_{v\in V_f} \underline{k}_{\rho_v}$ and hence the contribution to the sum in (3.1.12) due to the trees with m nodes (hence with $m+1$ final vertices and $m-1$ internal branches) is bounded by

$$\begin{aligned}&\overline{V}_0^{n+1} e^{-\xi_0|\underline{k}|/2k_0} \sum_{\underline{k}_\rho} \int (\prod_{v\in\text{nodes}} d\tau_v)\cdot \\ &\cdot(\prod_{v\ni V_f} e^{-\nu\,\underline{k}^2_{\rho_v}(\tau_{v'}-\tau_v)})(\prod_{v\in V_f} e^{-\xi_0|\underline{k}_v|/2k_0})(\prod_{\rho,\,\delta_\rho=1} |\underline{k}_\rho|)\end{aligned} \tag{3.1.13}$$

where the sum only runs over the $\underline{k}_{\rho_v}$'s with ρ_v being a final branch, (the other $\underline{k}_\rho$'s are fixed by the rule of conservation at the nodes). Recall here that the final vertices $v \in V_f$ are *not* nodes (only the internal vertices are nodes).

The integrals on τ_v can be bounded, enlarging to $[0,t]$ their domains of integration and for *all* trees, by

$$\prod_\rho \frac{1-e^{-\nu\,\underline{k}_\rho^2 t}}{\nu\,\underline{k}_\rho^2} \le \prod_\rho t^\varepsilon\,(\nu\,\underline{k}_\rho^2)^{-(1-\varepsilon)} \tag{3.1.14}$$

for $\varepsilon \in [0,1]$ arbitrary;[2] the product in (3.1.14) runs on the branches ρ which are *not final*, *c.f.r.* above.

Hence by selecting $\varepsilon = 1/2$ we see that (3.1.12), relative to a tree with $2m+1$ branches and m nodes, hence, $m+1$ final vertices, is bounded above by

$$e^{-\xi_0|\underline{k}|/2k_0} 2^{4n} (t\nu^{-1})^{n/2}\,\overline{V}_0^{m+1}\Big(\sum_{\underline{k}'}\frac{|\underline{k}'|}{k_0}\,e^{-\xi_0|\underline{k}'|/2k_0}\Big)^{m+1} \tag{3.1.15}$$

because the factors $|\underline{k}_\rho|$ corresponding to the *internal* branches with label $\delta_\rho = 1$ (in (3.1.13)) are compensated by the corresponding $|\underline{k}_\rho|^{-1}$, (that are generated by (3.1.14) with $\varepsilon = 1/2$). The factors $|\underline{k}_\rho|$ relative to the branches with label $\delta_\rho = 1$ but that exit from final vertices instead cannot be compensated in this way (as the factors in (3.1.14) are simply not there, see (3.1.12)) hence we left them, just writing them as $k_0\,|\underline{k}_\rho|/k_0$. We put them together with the bound on the factors $\underline{\gamma}^0_{\underline{k}_\rho}$ associated with the $m+1$ final branches (*i.e.* $\overline{V}_0 e^{-\xi_0|\underline{k}_\rho|/2k_0}$) generating a certain number $m' < m+1$ of factors that are *some* of the factors in the last term in (3.1.15); each of the other $m+1-m'$ could be bounded by the same sum deprived of the term $|\underline{k}'|/k_0$, which we instead prefer to leave, being ≥ 1, to get a simpler bound.

The remaining $|\underline{k}_\rho|^{-1}$, corresponding to the branches with label $\delta_\rho = 0$ are trivially bounded by k_0^{-1}. And the factor 2^{4m} bounds the number of trees with $2m+1$ branches.[3]

We find, therefore

$$|\underline{\gamma}_{\underline{k}}| \le e^{-\xi_0|\underline{k}|/2k_0}\,\overline{V}_0 \sum_{m=0}^\infty \big(\overline{V}_0 t^{1/2}\nu^{-1/2}\big)^m \xi_0^{-(d+1)(m+1)} B^{m+1} 2^{4m} \tag{3.1.16}$$

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[2] Because $(1-e^{-ab})/a$ is bounded by both a^{-1} and by b, for all $a,b>0$.

[3] We see immediately, in fact, that the number of trees with m branches is not larger than the number of paths on the lattice of the positive integers with $2(m-1)$ steps of size 1 (a number bounded by $2^{2(m-1)}$): given a tree think of walking on its branches starting from the root and always choosing the left branch if possible and otherwise coming back until returning to the root (and, by construction, without ever running more than twice on the same branch); we associate with each branch a step forward on the integers lattice, or a step backwards when we could not proceed (choosing the branch on the left) and had to come back. In our case a m–tree contains $2m+1$ branches.

if $\xi_0^{-d-1}B$ is a bound of $\sum_{\underline{k}} \frac{|\underline{k}|}{k_0} e^{-\xi_0|\underline{k}|/2k_0}$ (recall that we are assuming $\xi_0 \le 1$, for simplicity, *c.f.r.* the comment to (3.1.1)).

Hence $t \le \bar t_0 = \nu\,(16B\,\overline{V}_0\xi_0^{-d-1})^{-2}$ is the condition of convergence of the series in (3.1.16). The theorem follows by selecting $t_0 = \bar t_0/4$, hence such that the sum on n is bounded by 2, and $V_0 = B\,\overline{V}_0\xi_0^{-d-1}$.

At the end one can redefine the numerical constants so that the bound (3.1.3) can be expressed in terms of a single numerical constant B_0.

Remarks: If we define the operator $(-\Delta)^\alpha$, with α real , as the operator that multiplies by $|\underline{k}|^{2\alpha}$ the Fourier transform harmonic of mode $\underline{k}$ of a function in $L_2(\Omega)$, then we note that if the friction term $\nu\, D\underline{u}$ in (3.1.2) is replaced by $-\nu(-\Delta)^\alpha$ with $\alpha > 0$ the proposition remains valid, with some obvious modifications. The main point of the proof is indeed the bound (3.1.14) that can be replaced by

$$\frac{|e^{-\nu|\underline{k}|^{2\alpha}t} - 1|}{\nu|\underline{k}|^{2\alpha}} \le t^\varepsilon(\nu|\underline{k}|^{2\alpha})^{-(1-\varepsilon)} \tag{3.1.17}$$

and the argument can be adapted *provided one can choose* $\varepsilon > 0$ *such that* $2(1-\varepsilon)\alpha > 1$, *i.e. if* $\alpha > 1/2$. This restriction is indeed necessary because only in this way we can to eliminate from the bounds the factors $|\underline{k}_\rho|$ due to the final branches with label $\delta_\rho = 1$, that cannot be otherwise dominated.

Hence in a certain sense the friction term in the NS equations is "*larger than needed*", at least for the purpose of being able to guarantee that analytic initial data generate an analytically regular motion, at least for a short enough time: this is the case not only for the "normal viscosity" (friction term $\nu\Delta\underline{u}$) but also for the *ipoviscous NS* with friction term $-\nu|\Delta|^\alpha$ provided $\alpha > \frac{1}{2}$ and (of course) for the *hyperviscous NS* with $\alpha > 1$.

Concerning the incompressible Euler equation, where $\nu = 0$, the method now illustrated does not lead to conclusions, since we cannot show convergence of the sreis. A local theory of the Euler equation is nevertheless possible (and classical) and we shall illustrate it in the following problems.

Problems. *Classical local theory for the Euler and Navier–Stokes equations with periodic conditions.*

Below we suppose that Ω is a cube with side L and with opposite sides identified: this is done for the sake of simplicity as most of the statements hold also in the case of a boundary condition of the type $\underline{u}\cdot\underline{n} = 0$ at a point where the external normal is $\underline{n}$.

[3.1.1]: (*Cauchy's estimate*) If $\underline{x} \to f(\underline{x})$ is analytic and periodic on the torus $[0,L]^d$, then there is a $\xi > 0$ and a holomorphic function F of d complex variables that extends the function f to a vicinity of $[0,L]^d$ in C^d consisting in the points $\underline{x}$ such that $|\mathrm{Im}\, x_i| < \xi$. Furthermore if $|F| \le \Phi$ for $|\mathrm{Im}\, x_i| < \xi$ the Fourier transform of f verifies $|f_{\underline{k}}| \le \Phi\, e^{-\xi|\underline{k}|}$ where $|\underline{k}|$ is defined as $|\underline{k}| = \sum_i |k_i|$. Find the connection between this result and the theory of the Laurent series. (*Idea:* Study first the case $d = 1$. Write the transform $f_{\underline{k}}$ as an integral on the torus, applying the definition, and "deform the integration path" to the lines $\mathrm{Im}\, x_j = \pm\xi_0$, with $\xi_0 < \xi$, depending on the sign of k_j (*i.e.* use the Cauchy theorem on holomorphic functions). The connection with the theory of the Laurent series is made by thinking of the formula giving the Fourier transform as an integral over the variables $\zeta_j = e^{ix_j}$).

[3.1.2]: (*generalized energy identity*) Show that if $\underline{u}$ is a $C^\infty(\Omega)$ velocity field then

$$\sum_{|\underline{m}|=m}\int_\Omega \vec\partial^{\underline{m}}\underline{u}\cdot\vec\partial^{\underline{m}}(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{u})\,d\underline{x} = \sum_{|\underline{m}|=m}\sum_{\substack{\underline{a}\le\underline{m}\\|\underline{a}|>0}}\int_\Omega(\vec\partial^{\underline{m}}\underline{u})\cdot(\vec\partial^{\underline{a}}\underset{\sim}{u})\cdot(\underset{\sim}{\partial}\vec\partial^{\underline{m}-\underline{a}}\underline{u})\,d\underline{x}$$

i.e. the term $\underline{a}=\underline{0}$ is missing because of a *cancellation*. (*Idea:* The term with $\underline{a}=\underline{0}$ is a sum of terms like $\int\underline{w}\cdot(\underset{\sim}{u}\cdot\underset{\sim}{\partial})\,\underline{w}\,d\underline{x}\equiv\frac12\int\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{w}^2\,d\underline{x}$ with $\underline{w}$ suitable and this quantity vanishes (integrate by parts, as usual)).

[3.1.3]: Let $\underline{u}\in C^\infty(\Omega)$ and $|\underline{a}|<m-d/2$ (with $|\underline{a}|=\sum|a_i|$ for $\underline{a}=(a_1,\ldots,a_d)$) then, by [2.2.22], $|\underset{\sim}{\partial}^{\underline{a}}\underset{\sim}{u}(\underline{x})|\le\Gamma L^{-|\underline{a}|}||\underline{u}||_{W^m(\Omega)}$. Deduce that, if $m>d+1$

$$\Big|L^{2m-d+1}\sum_{|\underline{m}|=m}\int_\Omega\underline{\partial}^{\underline{m}}\underline{u}\cdot\underline{\partial}^{\underline{m}}(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u})\,d\underline{x}\Big|\le\Gamma_1||\underline{u}||^3_{W_m(\Omega)},\qquad m>d+1$$

with Γ_1 a suitable constant. (*Idea:* Apply [3.1.1] and the Schwartz inequality to each addend, thereby reducing to estimating $\mathcal{N}=||\vec\partial^{\underline{a}}\underset{\sim}{u}\cdot\underset{\sim}{\partial}\vec\partial^{\underline{m}-\underline{a}}\underline{u}||_{L_2(\Omega)}$ with $|\underline{a}|\ge1$. Note that if $|\underline{a}|<m-d/2$ then the estimate of [2.2.22] implies: $L^{|\underline{a}|}|\vec\partial^{\underline{a}}\underline{u}(\underline{x})|<\Gamma_2||\underline{u}||_{W_m(\Omega)}$, allowing us to estimate from above the first of the two factors in the $\mathcal{N}$; if instead $|\underline{a}|\ge m-d/2$ then $|\underline{m}-\underline{a}|+1\equiv m-|\underline{a}|+1\le\frac{d}{2}+1$ hence if $m>d+1\equiv(\frac{d}{2}+1)+\frac{d}{2}$ (hence $|\underline{m}-\underline{a}|+1<m-\frac{d}{2}$) we find, always because of [2.2.22], that

$$L^{(m-|\underline{a}|)+1)}|\underset{\sim}{\partial}\vec\partial^{\underline{m}-\underline{a}}\underline{u}(\underline{x})|\le\Gamma_3||\underline{u}||_{W_m(\Omega)}$$

since $|\underline{a}|\ge1$: which allows us to get an upper bound on the second factor in $\mathcal{N}$).

[3.1.4]: (*bounds uniform in the regularization parameter for Euler flows*) Show that if $\underline{u}^N$ is a solution of the regularized Euler equations, with vanishing or just conservative external volume force, obtained by an "ultraviolet cut–off" (in the sense of §2.2) at $|\underline{k}|<N$ one has, if $m>d+1$

$$\frac12\frac{d}{dt}||\underline{u}^N||^2_{W_m(\Omega)}\le G_m||\underline{u}^N||^3_{W_m(\Omega)}$$

with G_m independent from N. (*Idea:* This follows from [3.1.3] by differentiating $\underline{m}$ times the Euler equation and by multiplying both sides by $\partial^{\underline{m}}\underline{u}$, summing over $\underline{m}$ with $|\underline{m}|\le m$ and integrating over $\underline{x}$: one must remark that the truncation operations do not interfere with the derivation of the conclusions of [3.1.3]. In the sense that the identity in [3.1.2] remains valid if we replace $\underline{u}$ with $\underline{u}^N\equiv P_N\underline{u}$ where P_N is the orthogonal projection, in $L_2(\Omega)$, on the subspace generated by the plane waves of momentum $|\underline{k}|<N$ and if, furthermore, $(\vec\partial^{\underline{a}}\underset{\sim}{u})\cdot(\underset{\sim}{\partial}\vec\partial^{\underline{m}-\underline{a}}\underline{u})$ is replaced by $P_N\big((\vec\partial^{\underline{a}}\underset{\sim}{u}^N)\cdot(\underset{\sim}{\partial}\vec\partial^{\underline{m}-\underline{a}}\underline{u}^N)\big)$: this is immediately checked through the properties of the scalar product in L_2 and of the Fourier transform).

[3.1.5]: (*uniform smoothness of regularized Euler flows*) Show that if $\underline{u}^0\in C^\infty$ then given $m>d+1$ it is, setting $W_m=W_m(\Omega)$

$$||\underline{u}^N(t)||_{W_m}\le\frac{||\underline{u}^0||_{W_m}}{1-G_m||\underline{u}^0||_{W_m}t},\quad 0\le t<T_m$$

with $T_m = (G_m ||\underline{u}^0||_{W_m})^{-1}$ and if $\underline{u}^N$ is the solution of the regularized equation with cut–off at $|\underline{k}| < N$ and initial datum $\underline{u}^0$ (truncated with the same cut–off).

[3.1.6]: (*uniqueness for smooth Euler flows*) (Graffi) Show that the Euler equation does not admit more than one solution continuous in $\underline{x}, t$, with first derivatives with respect to $\underline{x}$ and t continuous and bounded in the $\underline{x}$ variables and in every finite time interval $[0, T]$, and verifying the same initial datum. (*Idea:* If $\underline{u}^1$ and $\underline{u}^2$ are two solutions $\underline{\delta} \equiv \underline{u}_1 - \underline{u}_2$ and $D(t)$ is the integral of $\underline{\delta}^2$ on Ω it is

$$\frac{d}{dt} D(t) \equiv -\int_\Omega \underline{\delta} \cdot (\underline{\delta} \cdot \underset{\sim}{\partial} \underline{u}_1)\, d\underline{x} - \int_\Omega \underline{\delta} \cdot (\underset{\sim}{u}_2 \cdot \underset{\sim}{\partial})\, \underline{\delta}\, d\underline{x} \le U \int_\Omega \underline{\delta}^2\, d\underline{x} = U\, D(t)$$

because the second term in the intermediate step vanishes because of the usual reasons, if $U = \max |\underset{\sim}{\partial}\, \underset{\sim}{u}_1|$ and if $D(t)$ is defined by the first identity and therefore $D(t) \le D(0) e^{2Ut}$, $t \in [0, T]$, so that *if* $D(0) = 0$ it is $D(t) \equiv 0$).

[3.1.7]: (*local existence and smoothness for Euler flows*) Show, in the context of [3.1.5], that if $\underline{u}^0$ is in $C^\infty(\Omega)$ then $\lim_{N\to\infty} \underline{u}^N(t)$, $t \in [0, T]$ exists and it is a C^m solution of Euler equation for $T < T_{m+1+d/2}$, with T_m defined in [3.1.5], for each $m > 1 + d/2$. (*Idea:* Consider a subsequence $\underline{u}^{N_i}$ uniformly convergent together with its first derivatives. Note there is a uniform estimate of the derivatives up to order $k < m - d/2$ *i.e.*, since $m - d/2 > 1$, certainly for $k = 0, 1$; hence the sequences of both $\underline{u}^N$ and $\underset{\sim}{\partial} \underset{\sim}{u}^N$ are equibounded and equicontinuous so that the Ascoli–Arzelà theorem applies. Note also that the limit of the subsequence is a solution of the Euler equation bounded, and with bounded derivatives, in $\underline{x}$ and t (continuity in t follows from the uniform continuity of the $\underline{u}^N$ in t due to the estimate of the time derivative that one sees from the fact that the $\underline{u}^N$ verify the truncated equation). By [3.1.6] there is only one such solution: hence all subsequences must have the same limit.)

[3.1.8]: (*local continuity with respect to initial data in Euler flows*) Let $\underline{u}_0, \underline{u}'_0 \in C^\infty(\Omega)$ be two initial data for solutions considered in [3.1.7] and suppose that $||\underline{u}_0||_{W_{h+1}}$, $||\underline{u}'_0||_{W_{h+1}} < R$, for a fixed $h > d + 1$; then $||\underline{u}||_{W_h} \le 2||\underline{u}_0||_{W_h}$, $||\underline{u}'||_{W_h} \le 2||\underline{u}'_0||_{W_h}$ and

$$||\underline{u} - \underline{u}'||_{W_h} \le \lambda_{h+1}(R) ||\underline{u}_0 - \underline{u}'_0||_{W_h}, \quad 0 \le t \le \frac{1}{2G_{h+1}R} \equiv \tau_h(R)$$

with G_m introduced in [3.1.4] and $\lambda_h(R)$ suitable. (*Idea:* Let $\underline{\delta} = \underline{u} - \underline{u}'$ and note that if $|\underline{m}| \le h$: hence

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega (\partial^{\underline{m}} \underline{\delta})^2\, d\underline{x} &= -\int_\Omega \partial^{\underline{m}} \underline{\delta}\, \partial^{\underline{m}} (\underset{\sim}{\delta} \cdot \underset{\sim}{\partial}\, \underset{\sim}{u} + \underset{\sim}{u}' \cdot \underset{\sim}{\partial} \underset{\sim}{\delta})\, d\underline{x} \\ &\le C_{|\underline{m}|} R ||\underline{\delta}||^2_{W_h}, \qquad h > d + 1 \end{aligned}$$

because if the cancellation in [3.1.2], *i.e.* because $\partial^{|\underline{m}|+1} \underline{\delta}$ never appears, and because of the argument in [3.1.3]: then $\lambda_{h+1}(R) = \exp(R \sum_{|\underline{m}| \le h} L^{2m-d} C_{\underline{m}} \tau_h(R))$, independently of R.)

[3.1.9]: (*local error estimate for regularized approximations Euler flows*) Likewise show that the solution in [3.1.7] enjoys the property

$$||\underline{u} - \underline{u}^N||_{W_h} \le \lambda_{h+1} ||\underline{u}_0 - \underline{u}_0^N||_{W_h} + \frac{\tilde{C}_p \lambda_{h+1}}{N^{p-2d}} ||\underline{u}_0||^2_{W_{h+p}}$$

with $\tilde{C}_p$ suitable and $p > 2d$. Check that this implies the possibility of a constructive algorithm[4] to approximate the Euler equations solution in the sense of the distance in

[4] By "*constructive*" algorithm we mean an algorithm that produces an approximation of the solution within an *a priori* fixed error ε via a coomputer program that ends in a time that can be *a priori* estimated in terms of ε.

the space W_h valid for times $t \le \tau_{h+p}$, with τ_m defined in [3.1.8] and $p > 2d$. (*Idea:* Write the equations for $\underline{u}$ and $\underline{u}^N$ keeping in mind that $\dot{\underline{u}}^N = -\underline{u}^N \cdot \underline{\partial}\underline{u}^N + (\underline{u}^N \cdot \underline{\partial}\underline{u}^N)^{>N}$ where $f^{>N}$ denotes the part of f obtained as sum of the harmonics of f with $|\underline{k}| > N$. Estimate this "ultraviolet part" by using the result of [3.1.4] (subtracting the equations for $\underline{u}$ and $\underline{u}^N$) obtaining an inequality similar to the one in the hint for [3.1.8] with an additional term).

[3.1.10]: (*summary of classical local results for Euler flows*) The results of the above problems can be summarized by saying that, given $h > d+1$ and $\underline{u}_0, \underline{u}'_0 \in C^\infty(\Omega)$, $||\underline{u}_0||_{W_{h+1}}, ||\underline{u}'_0||_{W_{h+1}} \le R$ there is a local solution of the Euler equation, defined for $0 \le t \le \tau_h(R)$ and such that in this time interval

$$\begin{aligned}
&(1) \qquad ||\underline{u}||_{W_h} \le 2R\\
&(2) \qquad ||\underline{u} - \underline{u}'||_{W_h} \le \lambda_{h+1} ||\underline{u}_0 - \underline{u}'_0||_{W_h}\\
&(3) \qquad ||\underline{u} - \underline{u}^N||_{W_h} \le \lambda_{h+1} ||\underline{u}_0 - \underline{u}_0^N||_{W_h} + \frac{C_p \lambda_{h+1}}{N^{p-2d}} ||\underline{u}_0||^2_{W_{h+p}}
\end{aligned}$$

Since in (3) $||\underline{u}_0||^2_{W_{h+p}}$, with $p > 2d$, appears rather than $||\underline{u}_0||^2_{W_h}$ we see that the approximation error on $\underline{u}$ by $\underline{u}^N$ has not been estimated "constructively" on the *whole* time interval $[0, \tau_h(R)]$ along which one can guarantee *a priori* existence of a solution in W_h and bounded in terms of known quantities: *rather it is estimated in the shorter interval* $[0, \tau_{h+p}(R)]$. Note that, on the basis of what said so far, this would not be possible *even* if we knew an *a priori* estimate of the size $||\underline{u}||_{W_k}$, with k arbitrary, in terms of the properties of $\underline{u}_0$ only. It seems that, for a constructive estimate, bounds on $||\underline{u}^N||_{W_k}$ in terms of $\underline{u}_0$ only are also needed. Show that, indeed, if such bounds existed then a constructive algorithm would be possible, valid for all times for which the considered bounds on of $\underline{u}$ and $\underline{u}_0$ hold. The relevance of this comment is that it points out that *although by abstract arguments* we can show existence and even some *a priori* bound on $||\underline{u}||_{W_k}$, in terms of t and $\underline{u}_0$, *nevertheless a constructive estimate* for $\underline{u}^N$ is not known.

The following problems are devoted to deriving a priori estimates on the solution $\underline{u}$ *of the Euler equations in* $d = 2$ *that shows existence, uniqueness and regularity as well as a priori estimates of the size of* $||\underline{u}||_{W_h}$*, valid at all times and dependent on* $\underline{u}_0$ *only, provided we accept the sinister axiom of choice (whose use we strongly disrecommend). The constructive part of the theory (as also the rest if the axiom of choice is accepted) is due to Wolibner, Judovic and Kato, c.f.r. [Ka67]. In what follows* Ω *is chosen* $\Omega = [0, L]^2$ *with periodic boundary conditions; furthermore, for simplicity, we suppose* $\underline{g} = \underline{0}$*. Finally we shall consider the Euler equation in an arbitrarily prefixed time interval,* $[0, T]$*. Given a function* $f(\underline{x})$ *on* Ω *or an* $f(\underline{x}, t)$ *on* $\Omega \times [0, T]$ *we shall say that it is of of class* $C^h(\Omega)$ *or* $C^{h,k}(\Omega \times [0, T])$ *if it has* h *continuous derivatives in* $\underline{x}$ *and* k *continuous derivatives in* t*; we shall say that it if of class* $C^h(\Omega \times [0, T])$ *if it has* h *continuous derivatives in* $\underline{x}$ *or* t*. We shall set*

$$\begin{aligned}
||f||_h &= \sum_{0 \le |\underline{a}| \le h} \max_{\underline{x} \in \Omega} L^{|\underline{a}|} \, |\partial_{\underline{x}}^{\underline{a}} f(\underline{x})|\\
||f||_{h,k} &= \sum_{\substack{0 \le |\underline{a}| \le h\\ 0 \le \beta \le k}} \max_{(\underline{x}, t) \in \Omega \times [0, T]} L^{|\underline{a}|} T^\beta \, |\partial_{\underline{x}}^{\underline{a}} \partial_t^\beta f(\underline{x}, t)|
\end{aligned}$$

[3.1.11]: (*Euler flows as vorticity transport in 2–dimensions*) If $\omega(\underline{x}, t) = \mathrm{rot}\, \underline{u}(\underline{x}, t) \in C^\infty$ then $\omega(\underline{x}, t)$ is a scalar unction with zero average for each t. Show the equivalence between the solutions of the Euler equations in $C^\infty(\Omega\times[0,T])$ in the form of the (3.1.2) with $\nu=0,\underline{g}=\underline{0}$ and the same equations in the form (2.3.3) $\nu,\gamma=0$:

$$\partial_t\omega+\underline{u}\cdot\underline{\partial}\omega=0,\qquad \underline{u}=-\underline{\partial}^\perp\Delta^{-1}\omega$$
$$\int_\Omega\omega(\underline{x},t)\,d\underline{x}=0,\qquad \omega(\underline{x},0)=\omega_0(\underline{x})$$

where $\omega_0=\operatorname{rot}\underline{u}_0$. Show that the condition $\int_\Omega\omega d\underline{x}=0$ is necessary in order that $\Delta^{-1}\omega$ be meaningful. (*Idea:* The rotation ω of a C^∞ solution of the Euler equations satisfies the above equations, as noted in §1.7. For the converse the only delicate point is checking existence of the pressure p from the remark that the first equation can be rewritten as $\operatorname{rot}(\partial_t\underline{u}+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{u})=0$ as $\underline{\partial}\cdot\underline{u}=\underline{0}$, because the torus Ω is not simply connected. Note that the vanishing of the rotor implies that the circulations of the vector field $\underline{w}\equiv\partial_t\underline{u}+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{u}$ along the torus cycles: $C_1(y)\equiv\{(x,y);\ y=cost\}$ e $C_2(x)=\{(x,y);\ x=cost\}$, defined by $I_1(y)=\oint_{C_1(y)}\underline{w}\cdot d\underline{l}$ and $I_2(x)=\oint_{C_2(x)}\underline{w}\cdot d\underline{l}$, are independent from x and y respectively hence are equal to their averages over these coordinates, *i.e.* $\underline{I}=(I_1,I_2)=L^{-1}\int_\Omega(\partial_t\underline{u}+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{u})\,d\underline{x}$ which we can see to vanish, as a consequence of $\int_\Omega\underline{u}d\underline{x}=\underline{0}$).

[3.1.12]: (*existence of current lines for a smooth voricity field*) Consider the space $\mathcal{M}_0\subset C^\infty$ defined by the

(1) $\zeta\in C^\infty(\Omega\times[0,T])$, (3) $\int_\Omega\zeta(\underline{x},t)\,d\underline{x}=0$

(2) $\zeta(\underline{x},0)=\omega_0(\underline{x})$, (4) $\max_{(\underline{x},t)\in\Omega\times[0,T]}|\zeta(\underline{x},t)|\le||\omega_0||_0$

Show that the vector field $\underline{v}^\zeta(\underline{x},t)=-\underline{\partial}^\perp\Delta^{-1}\zeta(\underline{x},t)$ is of class $C^\infty(\Omega\times[0,T])$. And that the equations for the "current" that as the time s varies passes at time $s=t$ through the point $\underline{x}$

$$\partial_sU^\zeta_{s,t}(\underline{x})=\underline{v}^\zeta(U^\zeta_{s,t}(\underline{x}),s),\qquad U^\zeta_{t,t}(\underline{x})\equiv\underline{x}$$

admit a global solution of class C^∞ in $\underline{x}\in\Omega$, $s,t\in[0,T]$. (*Idea:* The solution is global because Ω has no boundary).

[3.1.13]: (*the vorticity map*) Check that the preceding problem implies that the following operator $\mathcal{R}$, "*vorticity map*" is well defined on $\mathcal{M}_0$

$$\mathcal{R}\zeta(\underline{x},t)=\omega_0(U^\zeta_{0,t}(\underline{x}))$$

and that $\mathcal{R}\mathcal{M}_0\subset\mathcal{M}_0$. (*Idea:* One has to check only property (3): note that the field $\underline{\zeta}$ has zero divergence, hence the transformation $U^\zeta_{0,t}(\underline{x})=\underline{x}'$ conserves the volume and therefore $\int_\Omega\mathcal{R}\zeta(\underline{x},t)\,d\underline{x}\equiv\int_\Omega\omega_0(\underline{x})\,d\underline{x}$).

[3.1.14]: (*fixed point interpretation of Euler flows*) The existence of a point $\omega\in\mathcal{M}_0$ which is a fixed point for $\mathcal{R}$ implies that ω satisfies the Euler equation. Hence we can write the latter equation as $\omega=\mathcal{R}\omega$. (*Idea:* Note that Euler equation says that vorticity is transported by the current hence at time t it has in $\underline{x}$ the value that at time 0 it had in the point $\underline{x}'$ that is transported by the current to $\underline{x}$ in time t: *i.e.* just $U^\omega_{0.t}(\underline{x})$).

[3.1.15]: (*continuous dependence of the velocity field from the vorticity field*) Show that the field $\underline{v}^\zeta=-\partial^\perp\Delta^{-1}\zeta$, see [3.1.12], verifies the following properties

$$||\underline{v}^\zeta(\underline{x},t)||_0\le C_0L||\zeta||_0,\qquad \frac{|\underline{v}^\zeta(\underline{x},t)-\underline{v}^\zeta(\underline{x}',t)|}{(L^{-1}|\underline{x}-\underline{x}'|)\log_+(L|\underline{x}-\underline{x}'|^{-1})}\le C_0L||\zeta||_0$$

where $\log_+ z \equiv \log(e+z)$ (here $\log_+$ can obviously be replaced by any function of z continuous, positive, increasing and asymptotic to $\log z$ for $z \to \infty$). (*Idea:* Note that $\underline{v}^\zeta = -\underline{\partial}^\perp \Delta^{-1}\zeta$ and the operator $\underline{\partial}\Delta^{-1}$ can be computed as an integral operator starting from the Green function of the Laplacian,, *i.e.* $\frac{1}{2\pi}\log|\underline{x}-\underline{y}|^{-1}$, by the method of images, *c.f.r.* [2.3.12]. One gets formulae containing series, over a label $\underline{n}$ with integer components (*sums over the images of* Ω), that are not absolutely convergent

$$\sum_{\underline{n}} \int_\Omega \frac{d\underline{y}}{2\pi}\, \frac{(\underline{x}-\underline{y}+\underline{n}L)^\perp}{|\underline{x}-\underline{y}+\underline{n}L|^2}\, \zeta(\underline{y}) \equiv \sum_{\underline{n}} \int_\Omega d\underline{y}\, \underline{K}(\underline{x}-\underline{y}+\underline{n}L)\, \zeta(\underline{y})$$

but the hypothesis that ζ has zero average allows us to rewrite them as sums of absolutely convergent series and this reduces the problem to that of showing the validity term by term of the estimates (*i.e.* for each integer vector $\underline{n}$). The sum over all the $\underline{n}$ with $|\underline{n}| > 2$ can be trivially bounded and the case that one really has to understand is the case $\underline{n} = \underline{0}$. The latter is also easy for what concerns the first estimate. The $\underline{n} = \underline{0}$ contribution to the second estimate is obtained by writing the difference between the two integrals for $\underline{v}^\zeta(\underline{x},t)$ and $\underline{v}^\zeta(\underline{x}',t)$ as an integral over Ω of the difference of the integrands. The integral over Ω can then be decomposed into the integral over the sphere of radius $2r = 2|\underline{x}-\underline{x}'|$ and center in $\underline{x}$ and in the integral on the complement. The first integral is estimated by introducing the modulus under the integral and bounding separately the two terms: it yields a result proportional to $r||\zeta||_0$ while the second integral will be bounded by Lagrange mean value theorem by estimating the difference $\underline{K}(\underline{x}-\underline{y}) - \underline{K}(\underline{x}'-\underline{y})$ as $r|\underline{y}|^{-2}$, because now $|\underline{y}-\underline{x}|$ and $|\underline{y}-\underline{x}'|$ can be bounded by a constant times $|\underline{y}|$ since $\underline{y}$ is "far" from both $\underline{x}$ and $\underline{x}'$, and the integral over the complement of the small sphere is bounded by $L^{-1} r\, \log L\, r^{-1}$, leading to the second inequality).

[3.1.16]: (*continuous dependence of the flow lines from the vorticity field*) There exists C_0 such that, defining $M_0 = ||\omega||_0$ and $\delta \stackrel{def}{=} e^{-C_0\, M_0\, T}$ for $T > 0$, then the currents generated by the two fields $\underline{v}^\zeta, \underline{v}^{\zeta'}$ with $\zeta, \zeta' \in \mathcal{M}_0$ are such that for all $\underline{x} \in \Omega$ and $s,t \in [0,T]$

$$|U^\zeta_{s,t}(\underline{x}) - U^{\zeta'}_{s,t}(\underline{x})| \le C_0\, M_0\, T\, \Big(||\zeta-\zeta'||_0 M_0^{-1}\Big)^\delta L$$

Hence the $\mathcal{R}$ can be thought of as defined on the closure $\overline{\mathcal{M}}_0$ of $\mathcal{M}_0$ with respect to the metric of the uniform convergence and it can be extended to all continuous vector fields ζ, without any differentiability property, verifying the (2),(3),(4) of problem [3.1.11]. (*Idea:* Setting $\rho_s = L^{-1}(U^\zeta_{s,t}(\underline{x}) - U^{\zeta'}_{s,t}(\underline{x}))$, note that $\rho_t = \underline{0}$; furthermore from problem [3.1.15], by subtracting and adding $\underline{v}^\zeta(U^{\zeta'}_{s,t}(\underline{x}), s)$, there exist constants C_1, C_2, C_0:

$$\begin{aligned}|\partial_s \rho_s| = &|L^{-1}|\underline{v}^\zeta(U^\zeta_{s,t}(\underline{x}),s) - \underline{v}^{\zeta'}(U^{\zeta'}_{s,t}(\underline{x}),s)| \le C_1|\rho_s|\log_+|\rho_s|^{-1} + \\ &+ L^{-1}|\underline{v}^\zeta(U^{\zeta'}_{s,t}(\underline{x}),s) - \underline{v}^{\zeta'}(U^{\zeta'}_{s,t}(\underline{x}),s)| \le \\ &\le C_1|\rho_s|\log_+|\rho_s|^{-1} + C_2||\zeta-\zeta'||_0 \le \\ &\le C_0\Big(|\rho_s|\log_+|\rho_s|^{-1} + ||\zeta-\zeta'||_0\Big)\end{aligned}$$

Since $0 \le s \le T$, by integration it follows that $|\rho_s| \le R$ if R is

$$\int_0^R \frac{d\rho}{\rho\log_+\rho^{-1} + ||\zeta-\zeta'||_0 M_0^{-1}} = C_0 M_0 T$$

which means that R can be taken $R \le K(M_0T)(||\zeta-\zeta'||M_0^{-1})^\delta$ with $K(M_0T)$ a continuous increasing function and $\delta = \exp -M_0C_0T$).

[3.1.17]: (*Hölder continuity of the flow lines*) Show that the current lines verify

$$\frac{|U^\zeta_{s,t}(\underline{x}) - U^\zeta_{s',t'}(\underline{x}')|}{(L^{-1}|\underline{x}-\underline{x}'|)^\delta + (T^{-1}|s-s'|)^\delta + (T^{-1}|t-t'|)^\delta} \le F(M_0T)L$$

where F is an increasing continuous function of its argument; $M_0 \equiv ||\zeta||_0$. (*Idea:* Set $\rho_s = L^{-1}(U^\zeta_{s,t}(\underline{x}) - U^\zeta_{s,t}(\underline{x}'))$, by the inequality in [3.1.15], one finds

$$|\partial_s\rho_s| \le C_0M_0|\rho_s|\log_+|\rho_s|^{-1}, \qquad \rho_t \equiv \underline{x}-\underline{x}'$$

hence $|\rho_s| \le R$, where R is such that

$$\int_{L^{-1}|\underline{x}-\underline{x}'|}^R \frac{d\rho}{\rho\log_+\rho^{-1}} = M_0C_0T \quad\Rightarrow\quad R \le (L^{-1}|\underline{x}-\underline{x}'|)^\delta\, F'(M_0T)$$

where F' is a suitable continuous increasing function and δ is as in the preceding problem. Moreover note that, by the first of the inequalities in [3.1.15]: $L^{-1}|U^\zeta_{s,t}(\underline{x}') - U^\zeta_{s',t}(\underline{x}')| \equiv |L^{-1}\int_s^{s'} \underline{v}^\zeta(U^\zeta_{\sigma,t}(\underline{x}'))\,d\sigma| \le C_0M_0|s-s'|$. Finally if $\underline{x}'' = U^\zeta_{t,t'}(\underline{x}')$ then

$$L^{-1}|U^\zeta_{s',t}(\underline{x}') - U^\zeta_{s',t'}(\underline{x}')| \equiv L^{-1}|U^\zeta_{s',t}(\underline{x}') - U^\zeta_{s',t}(\underline{x}'')| \le (L^{-1}|\underline{x}'-\underline{x}''|)^\delta\, F'(M_0T)$$

by what already seen. But $|\underline{x}'-\underline{x}''| \equiv |\int_t^{t'} \underline{v}^\zeta(U^\zeta_{\sigma,t'}(\underline{x}'),\sigma)\,d\sigma| \le LM_0C_0|t-t'|$ and the conclusion follows).

[3.1.18]: (*the vorticity map regularizes*) Show that the $\mathcal{R}$ transforms $\overline{\mathcal{M}}_0$ into the subspace $\overline{\mathcal{M}}_0^\delta$ of the continuous functions verifying (2),(3),(4) of [3.1.12] and the

$$||\zeta||_\delta \equiv \sup_{\Omega\times[0,T]\times\Omega\times[0,T]} \frac{|\zeta(\underline{x},t) - \zeta(\underline{x}',t')|}{(L^{-1}|\underline{x}-\underline{x}'|)^\delta + (T^{-1}|t-t'|)^\delta} \le M_0F''(M_0T) \qquad (!)$$

for a suitable function F''. Hence the continuous transformation $\mathcal{R}$ transforms the convex set $\overline{\mathcal{M}}_0^\delta$ into itself; furthermore such set is, as usually said, "compact" in the uniform convergence topology because it consists in a set of equicontinuous equibounded functions (Ascoli–Arzelá theorem) hence by using the sad axiom of choice (and its consequence expressed by the Schauder fixed point) we infer the existence of a field $\omega \in \overline{\mathcal{M}}_0^\delta$ such that $\omega = \mathcal{R}\omega$. (*Idea:* All follows from the form of $\mathcal{R}$, from the regularity of ω_0 and from the inequality in [3.1.17]. Furthermore the space of the functions that verify the inequality (*) is closed in the uniform convergence topology).

[3.1.19] (*extension of the regularization property of the vorticity map to higher order derivatives*) The result of [3.1.17] can be generalized, with some patience. If we define for $0 < \gamma < 1$

$$||f||_{k+\gamma} = ||f||_k + \sup_{(\underline{x},t)\in\Omega\times[0,T]} \sum_{|\underline{\alpha}|+\beta=k} \frac{L^{|\underline{\alpha}|}T^\beta|\partial^{\underline{\alpha}}_{\underline{x}}\partial^\beta_t f(\underline{x},t) - \partial^{\underline{\alpha}}_{\underline{x}}\partial^\beta_t f(\underline{x}',t')|}{(L^{-1}|\underline{x}-\underline{x}'|)^\gamma + (T^{-1}|t-t'|)^\gamma}$$

then, if $\gamma+\delta < 1$

$$||\mathcal{R}\zeta||_{k+\gamma+\delta} \le F_k||\zeta||_{k+\gamma}$$

and if $\gamma+\delta > 1$ then: $||\mathcal{R}\zeta||_{k+1} \le F_k||\zeta||_{k+\gamma}$, where F_k are suitable constants depending only from ω_0 and through its first $k+1$ derivatives. In other words $\mathcal{R}$ "regularizes", transforming C^k into $C^{k+\gamma}$.

[3.1.20] (*global existence and smoothness theorem for Euler flows* (Wolibner, Yudovitch, Kato) Show that the problem [3.1.19] implies that ω, the fixed point of $\mathcal{R}$ in $\overline{\mathcal{M}}_0$, is C^∞; and its derivatives can be bounded to order k in terms of the derivatives of order $\le k-\delta+1$ of the function ω_0. (*Idea:* $\mathcal{R}$ "regularizes" hence $\mathcal{R}\omega$ is more regular than ω; but it is equal to ω so that ω is C^∞).

[3.1.21] Check that, sadly enough, all what has been said above is, sadly, not sufficient to allow us to write a computation program that produces as a result the ω within a prefixed approximation ε in the metric of C^k, for any k ($k = 0$ included). Meditate on the event (or disaster) and if possible find a solution: unlike what is often stated, or unless the estimates of this section are substantially improved, the problem of global existence of solutions of the Euler equation is completely open even in 2 dimensions at least if one demands the "constructivity" of the method employed.

Bibliography: The global existence, smoothness and uniqueness theory for Euler flows in 2 dimensions is taken from [Ka67] and is due to Wolibner, Judovitch and Kato.

§3.2 Weak global existence theorems for NS. Autoregularization, existence, regularity and uniqueness for $d = 2$

We shall consider an incompressible fluid enclosed in a cubic region $\Omega \subset R^d$ with side L, with periodic boundary conditions and subject (for simplicity) to a time independent or quasi periodic volume force $\underline{g}$ of class $C^\infty(\Omega)$ exercising a vanishing total force on the fluid ($\int_\Omega \underline{g}\, d\xi = \underline{0}$).

In this case the center of mass of the fluid moves with rectilinear uniform motion. Calling $\underline{v} = \int \underline{u}\, d\xi / L^d$ the baricenter velocity, one can write the Navier–Stokes equations in the frame which "rotates" on the torus Ω uniformly with velocity $\underline{v}$. The galileian coordinate transformation is:

$$\begin{aligned} \xi' &= \xi - \underline{v}t, & \underline{u}'(\xi', t) &= \underline{u}(\xi' + \underline{v}t, t) - \underline{v}, \\ p'(\xi', t) &= p(\xi' + \underline{v}t, t), & \underline{g}'(\xi', t) &= \underline{g}(\xi + \underline{v}t, t) \end{aligned} \tag{3.2.1}$$

so that one sees that $\underline{u}'$ verifies in Ω the equations

$$\begin{aligned} &\dot{\underline{u}} + (\underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}})\underline{u} = -\underline{\partial}\, p + \nu \Delta \underline{u} + \underline{g} \\ &\underline{\partial} \cdot \underline{u} = 0, \qquad \int_\Omega \underline{u}\, d\xi = \underline{0} \end{aligned} \tag{3.2.2}$$

where units are so chosen that density is $\rho = 1$.

Thus if, *as we shall always suppose*, $\int \underline{g}\, d\xi \equiv \underline{0}$ it is not restrictive to assume that $\underline{v} = \underline{0}$ provided we suppose that $\underline{g}$ is quasi periodic in t; in fact the function $\underline{g}(\xi + \underline{v}t, t)$ is quasi periodic in t even if $\underline{g}$ is time independent, because of its periodicity in ξ.

Condition $\underline{\partial} \cdot \underline{u} = 0$ can be regarded as a constraint and we can eliminate it by choosing, as already seen several times (*e.g. c.f.r.* §2.2), suitable coordinates to represent $\underline{u}$. More precisely we shall take as coordinates the coefficients $\underline{\gamma}_{\underline{k}}$ of its Fourier transform that we define, with the conventions of (2.2.2), as

$$\underline{u}(\xi,t)=\sum_{\underline{k}\neq\underline{0}}\underline{\gamma}_{\underline{k}}(t)e^{i\underline{k}\cdot\xi},\qquad \underline{\gamma}_{\underline{k}}\equiv\overline{\underline{\gamma}}_{-\underline{k}},\qquad \underline{\gamma}_{\underline{k}}\cdot\underline{k}\equiv 0 \tag{3.2.3}$$

where $\underline{k}\neq\underline{0}$ expresses the relation $\int\underline{u}\,d\xi\equiv\underline{0}$ (*momentum conservation*), $\underline{\gamma}_{\underline{k}}=\overline{\underline{\gamma}}_{-\underline{k}}$ expresses that $\underline{u}$ is real valued and the condition $\underline{\gamma}_{\underline{k}}\cdot\underline{k}=0$ is equivalent to $\underline{\partial}\cdot\underline{u}=0$. Here, by our periodicity assumption on the sides of Ω, the vector $\underline{k}$ is a vector with components that are integer multiples of $k_0\stackrel{def}{=}2\pi L^{-1}$ and, therefore, $|\underline{k}|\geq k_0$ because $\underline{k}\neq\underline{0}$; the latter property will be often used in what follows. Consistently with (2.2.2) we shall denote:

$$||f||_2^2=\int_\Omega|f(\underline{x})|^2\,d\underline{x},\qquad\text{and}\quad ||\hat{f}||_2^2=\sum_{\underline{k}}|\hat{f}(\underline{k})|^2\;\longleftrightarrow\;||f||_2^2=L^d||\hat{f}||_2^2 \tag{3.2.4}$$

so that the $\underline{\gamma}_{\underline{k}}$ are "proper" *Lagrangian* coordinates, which can be freely assigned, without further constraints, as discussed in §2.2.

Assuming $p,\underline{u}\in C^\infty(\Omega)$ and substituting in (2.2.13) one sees that the NS equation becomes the (2.2.10). To recall the notations of §2.2 (*c.f.r.* (2.2.7)%(2.2.11)) let $\underline{\gamma}_{\underline{k}}=\overline{\underline{\gamma}}_{-\underline{k}}$ and $\underline{k}\cdot\underline{\gamma}_{\underline{k}}=0$ and let $\Pi_{\underline{k}}$ be the orthogonal projection, in R^d, on the plane orthogonal to $\underline{k}$ and let $\Pi^{||}_{\underline{k}}$ be the orthogonal projection, in R^d, on $\underline{k}$:

$$(\Pi_{\underline{k}}\,\underline{w})_i\equiv w_i-k_i\frac{\underline{k}\cdot\underline{w}}{|\underline{k}|^2}\,,\qquad \underline{w}\equiv\frac{\underline{w}\cdot\underline{k}}{\underline{k}^2}\underline{k}+\Pi_{\underline{k}}\,\underline{w}\equiv\Pi^{||}_{\underline{k}}\,\underline{w}+\Pi_{\underline{k}}\,\underline{w} \tag{3.2.5}$$

Set $\underline{\varphi}_{\underline{k}}\stackrel{def}{=}\Pi_{\underline{k}}\,\underline{g}_{\underline{k}}$ and define $t_{\underline{k}}$ so that it is $\underline{g}_{\underline{k}}=-i\underline{k}\,t_{\underline{k}}+\underline{\varphi}_{\underline{k}}$; then (2.2.10) can be written:

$$\begin{aligned}\dot{\underline{\gamma}}_{\underline{k}}=&-\underline{k}^2\nu\,\underline{\gamma}_{\underline{k}}-i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2\,\Pi_{\underline{k}}\,\underline{\gamma}_{\underline{k}_2}+\underline{\varphi}_{\underline{k}}\stackrel{def}{=}\\ \stackrel{def}{=}&-\underline{k}^2\nu\,\underline{\gamma}_{\underline{k}}+N_{\underline{k}}(\underline{\gamma})+\underline{\varphi}_{\underline{k}}\\ p_{\underline{k}}=&-\frac{1}{\underline{k}^2}\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}(\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2)\,(\underline{\gamma}_{\underline{k}_2}\cdot\underline{k})+t_{\underline{k}}\end{aligned} \tag{3.2.6}$$

It will be very useful to realize that *the nonlinear term $N_{\underline{k}}(\underline{\gamma})$ in (3.2.6) is meaningful as soon as $||\underline{u}||_2^2$ is finite*: for instance if $||\underline{u}||_2^2$ is bounded by a constant E_0, for all t. In fact, one has

$$|N_{\underline{k}}(\underline{\gamma})|\leq|\underline{k}|\,||\underline{\gamma}||_2^2 \tag{3.2.7}$$

as it can be seen (from the first sum in (3.2.6)) by remarking that $\underline{\gamma}_{\underline{k}_1} \cdot \underline{k}_2 \equiv \underline{\gamma}_{\underline{k}_1} \cdot \underline{k}$ (because $\underline{k}_2 = \underline{k} - \underline{k}_1$ and $\underline{k}_1 \cdot \underline{\gamma}_{\underline{k}_1} = 0$) and, thence, by applying Schwartz' inequality to the sum $\sum_{\underline{k}_1}$, in (3.2.6), keeping $\underline{k}_2 \equiv \underline{k} - \underline{k}_1$.

Remark: An important property of $N_{\underline{k}}(\underline{\gamma})$ has appeared in the proof of (3.2.7): it is related to the orthogonality of $\underline{k}$ to $\underline{\gamma}_{\underline{k}}$, which allowed us to write *two equivalent expressions* for N:

$$N_{\underline{k}}(\underline{\gamma}) = -i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} (\underline{\gamma}_{\underline{k}_1} \cdot \underset{\sim}{\underline{k}}_2)\, \Pi_{\underline{k}}\, \underline{\gamma}_{\underline{k}_2} \equiv -i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} (\underline{\gamma}_{\underline{k}_1} \cdot \underset{\sim}{\underline{k}})\, \Pi_{\underline{k}}\, \underline{\gamma}_{\underline{k}_2} \tag{3.2.8}$$

showing that the factors $\underset{\sim}{\underline{k}}_2$ and $\underset{\sim}{\underline{k}}$ can be interchanged in this relation: a property that will be used several times in the following.

Therefore the problem consists in solving the first of the equations (3.2.6) subject to the conditions $\underline{\gamma}_{\underline{k}} = \overline{\underline{\gamma}}_{-\underline{k}}$ and to the initial condition $\underline{\gamma}_{\underline{k}}(0) = \underline{\gamma}^0_{\underline{k}}$ (the second equation in (3.2.6) should be regarded just as being the definition of p, as already noted in §2.2, *c.f.r.* (2.2.10)). Such initial condition is automatically imposed if one writes the equations as:

$$\begin{aligned}\underline{\gamma}_{\underline{k}}(t) =& e^{-\nu \underline{k}^2 t} \underline{\gamma}_{\underline{k}}(0) + \int_0^t e^{-\nu \underline{k}^2 (t-\tau)} \cdot \\ &\cdot \Big[\underline{\varphi}_{\underline{k}}(\tau) - i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underline{\gamma}_{\underline{k}_1}(\tau) \cdot \underset{\sim}{\underline{k}}_2 \Pi_{\underline{k}} \underline{\gamma}_{\underline{k}_2}(\tau) \Big] \, d\tau\end{aligned} \tag{3.2.9}$$

(*c.f.r.* (2.2.11)). Let us, therefore, set

1 Definition: *(weak solutions of NS and Euler equations) We shall say that $t \to \underline{u}(\xi, t)$ is a weak solution of the NS equations, (3.2.2), with initial datum $\underline{u}^0(\xi) \in L_2(\Omega)$ if there exists $E_0 > 0$ such that:*
(i) for each $t \ge 0$ it is: $||\underline{u}||_2^2 \le E_0$,
(ii) the functions $t \to \underline{\gamma}_{\underline{k}}(t)$ are continuous in t and verify (3.2.9).
(iii) the above makes sense even if $\nu = 0$, i.e. setting $\nu = 0$ and requiring properties (i), (ii) one gets the definition of weak solution for the Euler equations.

Remarks: We shall see that weak solutions with initial datum $\underline{u}^0 \in L_2(\Omega)$ always exist: however, in general, there might be many weak solutions with the same initial data: *the proof of existence is in fact non constructive.*

(1) If $\underline{u}$ is a weak solution then (3.2.9), (3.2.7) imply that $\underline{\gamma}_{\underline{k}}$ is differentiable almost everywhere[1] in t, and that the derivatives verify the first of

[1] This is the "Torricelli–Barrow" theorem: note that continuity of $\underline{\gamma}_{\underline{k}}(t)$ in t for each $\underline{k}$ *does not necessarily imply* continuity in t of $N_{\underline{k}}(\underline{\gamma})$ so that the "almost everywhere" is here necessary and expresses an important aspect of our lack of understanding on NS.

the (3.2.6). Furthermore $\underline{\gamma}_{\underline{k}}(t)$ is the integral of its derivative or, as one says, it is *absolutely continuous*.

(2) Most of what follows will concern weak solutions with initial datum $\underline{u}^0 \in C^\infty(\Omega)$. One might be interested in studying solutions in which the initial datum $\underline{u}^0$ is less regular than C^∞: many results can be easily extended to the case in which the initial datum is such that $\underset{\sim}{\partial}\underline{u}^0 \in L_2(\Omega)$ (*i.e.*, with the notations of problem [2.2.20] of §2.2, $\underline{u}^0 \in W^1(\Omega)$) or even just such that $\underline{u}^0 \in L_2(\Omega)$. It will simply suffice, as we shall point out whenever appropriate, to follow the analysis under this sole assumption. *However* one should note that the very derivation of the fluid mechanics equations (*c.f.r.* §1.1) *assumes regularity* of the velocity field so that non smooth initial data are of little physical interest or, to say the least, require a physical discussion of their meaning.

(3) However we shall see that *only* if $d = 2$ the initial regularity of the solution can be proved to be maintained at positive times. In fact in such case we shall see that even if the initial datum is only in $L_2(\Omega)$ there will be weak solutions that immediately become "regularized": becoming C^∞ at any positive time, and even analytic as a recent theorem shows (see below).

(4) Solutions may *a priori* exist that, starting from a smooth initial datum, evolve at a later time into singular ones, *i.e.* become non smooth (meaning that they do not have a well defined derivative, of some order). Such solutions can have an interesting physical significance. For instance they might signal cases in which the model of a continuum, based on smoothness of its motion, *becomes self contradictory* and therefore it should no longer be considered valid: the motion should be considered in a less phenomenological way (with respect to the theory of §1.1), ultimately possibly reverting to a model based on the microscopic structure of the fluid.

(5) A consequence of the uniform bound assumed on $||\underline{\gamma}||_2$ and of (3.2.7) is that any weak solution will verify, for all $t_0 \geq 0$

$$
\begin{aligned}
&\underline{\gamma}_{\underline{k}}(t) = e^{-\nu \underline{k}^2 (t-t_0)} \underline{\gamma}_{\underline{k}}(t_0) + \\
&+ \int_{t_0}^{t} e^{-\nu \underline{k}^2 (t-\tau)} \Big[\underline{\varphi}_{\underline{k}}(\tau) - i \sum_{\substack{\underline{k}_1 + \underline{k}_2 = \underline{k} \\ |\underline{k}_i| \leq \infty}} \underline{\gamma}_{\underline{k}_1}(\tau) \cdot \underset{\sim}{\underline{k}}_2 \, \Pi_{\underline{k}} \, \underline{\gamma}_{\underline{k}_2}(\tau) \Big] \, d\tau
\end{aligned}
\tag{3.2.10}
$$

which we call the "*self-consistence property*" of the above weak solutions: *i.e.* the solution, $t \to \underline{\gamma}(t)$, regarded as a function defined in $[t_0, \infty)$ is still a solution of the NS equation with initial datum $\underline{\gamma}(t_0)$.[2]

To prove existence of weak solutions with initial data in $\underline{u}^0 \in C^\infty$ we replace (3.2.6) with a "*regularized*" equation parameterized by a "cut–off"

[2] Since we do not know, in general, uniqueness this property requires checking (immediate in this case). This follows from (3.2.9) , the additivity of the integrals and the conitnuity of $\underline{\gamma}_{\underline{k}}(t)$ for each $\underline{k}$.

parameter that we shall call R:

$$\begin{aligned}\dot{\underline{\gamma}}^R_{\underline{k}} =& -\nu\underline{k}^2\underline{\gamma}^R_{\underline{k}} - i\sum_{\substack{\underline{k}_1+\underline{k}_2=\underline{k}\\ |\underline{k}_i|\le R}} (\underline{\gamma}^R_{\underline{k}_1}\cdot\underline{k}_2)\,\Pi_{\underline{k}}\,\underline{\gamma}^R_{\underline{k}_2} + \underline{\varphi}_{\underline{k}} \equiv\\ \equiv& -\nu\underline{k}^2\underline{\gamma}^R_{\underline{k}} + \underline{N}^R_{\underline{k}}(\underline{\gamma}^R) + \underline{\varphi}_{\underline{k}}, \quad ,\qquad |\underline{k}|\le R\end{aligned}\tag{3.2.11}$$

where $\underline{\gamma}^R_{\underline{k}}(t)\overset{def}{=}0$ for $|\underline{k}|>R$ and $\underline{\gamma}^R_{\underline{k}}(0)\overset{def}{=}\underline{\gamma}^0_{\underline{k}}$ for $|\underline{k}|\le R$. Note that, as in the case of (3.2.7), it is $|\underline{N}^R_{\underline{k}}(\underline{\gamma}^R)|\le|\underline{k}|\,||\underline{\gamma}^R||_2^2$ *independently of* R. Then we begin by showing the following proposition about properties of (3.2.11) which are *independent of the value of the regularization parameter* R:

I. Proposition *(a priori bounds on solutions of regularized NS equations): Suppose* $\underline{u}^0\in C^\infty(\Omega)$*,*
(i) equation (3.2.11) admits a solution, global for $t\ge0$*, with initial datum* $\underline{\gamma}^0_{\underline{k}}$*,* $|\underline{k}|\le R$*, and such solution verifies the a priori estimate:*

$$||\underline{\gamma}^R(t)||_2\equiv\big(\sum_{\underline{k}}|\underline{\gamma}^R_{\underline{k}}|^2\big)^{1/2}\le\max\big(||\underline{\gamma}^0||_2,\frac{||\underline{\varphi}||_2}{\nu k_0^2}\big)\overset{def}{=}\sqrt{E_0L^{-d}}\tag{3.2.12}$$

for all $t\ge0$, $R>0$, *where* E_0 *is defined by (3.2.12); hence if* $\underline{\varphi}\equiv\underline{0}$ *it is* $E_0\equiv||\underline{u}^0||_2^2$*, c.f.r. (3.2.4).*
(ii) Furthermore:

$$\int_0^T d\tau\sum_{\underline{k}}\underline{k}^2|\underline{\gamma}^R_{\underline{k}}(\tau)|^2\le\frac12E_0L^{-d}\nu^{-1}+T\sqrt{E_0L^{-d}}\nu^{-1}||\underline{\varphi}||_2\tag{3.2.13}$$

for all $T>0$ *and* $R>0$.

proof: equation (3.2.11) is an ordinary differential equation and it it is enough to multiply (3.2.11) by $\overline{\underline{\gamma}_{\underline{k}}}\equiv\underline{\gamma}_{-\underline{k}}$ and to sum over $|\underline{k}|\le R$ to find, if $(\underline{f},\underline{h})$ denotes the usual scalar product in $L_2(\Omega)$ of the fields $\underline{f},\underline{h}$,

$$\begin{aligned}\frac12\frac{d}{dt}||\underline{\gamma}^R||_2^2 =& -\nu\sum_{|\underline{k}|\le R}\underline{k}^2|\underline{\gamma}^R_{\underline{k}}|^2+(\underline{\varphi}\cdot\underline{\gamma}^R)-i\sum_{\underline{k}_1+\underline{k}_2+\underline{k}_3=\underline{0}}\underline{\gamma}^R_{\underline{k}_1}\cdot\underline{k}_2\,\underline{\gamma}^R_{\underline{k}_2}\cdot\underline{\gamma}^R_{\underline{k}_3}\\ \equiv& -\nu\sum_{|\underline{k}|\le R}\underline{k}^2|\underline{\gamma}^R_{\underline{k}}|^2+(\underline{\varphi}\cdot\underline{\gamma}^R)-\frac{i}{2}\sum_{\underline{k}_1+\underline{k}_2+\underline{k}_3=0}\underline{\gamma}^R_{\underline{k}_1}\cdot(\underline{k}_2+\underline{k}_3)\,\underline{\gamma}^R_{\underline{k}_2}\cdot\underline{\gamma}^R_{\underline{k}_3}\equiv\\ \equiv& -\nu\sum_{|\underline{k}|\le R}\underline{k}^2|\underline{\gamma}^R_{\underline{k}}|^2+(\underline{\varphi}\cdot\underline{\gamma}^R)\end{aligned}\tag{3.2.14}$$

having used, in the second step, the symmetry between the summation labels $\underline{k}_2$ and $\underline{k}_3$ and, in the third step, the property that $\underline{k}_2+\underline{k}_3$ is parallel to $\underline{k}_1$ while $\underline{\gamma}_{\underline{k}_1}$ is, instead, orthogonal to $\underline{k}_1$.

It follows that the right hand side is ≤ 0 if $\sum_{\underline{k}} |\gamma^R_{\underline{k}}|^2 > E_0 L^{-d}$: therefore equation (3.2.12) holds. Moreover by integrating (3.2.14) we get:

$$\frac{1}{2}(||\underline{\gamma}^R(t)||_2^2 - ||\underline{\gamma}^R(0)||_2^2) \le -\nu \int_0^t \sum_{|\underline{k}|\le R} \underline{k}^2 |\gamma^R_{\underline{k}}|^2 \, d\tau + \int_0^t ||\underline{\varphi}||_2 ||\underline{\gamma}^R||_2 d\tau \tag{3.2.15}$$

which implies:

$$\int_0^t \sum_{|\underline{k}|\le R} \underline{k}^2 |\gamma^R_{\underline{k}}|^2 \, d\tau \le \frac{1}{2} L^{-d} E_0 \nu^{-1} + t\nu^{-1} ||\underline{\varphi}||_2 \sqrt{E_0 L^{-d}} \tag{3.2.16}$$

As a corollary, always assuming that $\underline{u}^0 \in C^\infty(\Omega)$, we get

II. Corollary *(global existence of weak solutions for the NS equations): Consider the regularized equation (3.2.11); then:*
(i) the functions $\underline{\gamma}^R_{\underline{k}}(t)$ are bounded by $\sqrt{E_0 L^{-d}}$ and have first derivative with respect to t that can be bounded above by $\nu \underline{k}^2 \sqrt{E_0 L^{-d}} + |\underline{k}| E_0 L^{-d} + ||\underline{\varphi}||_2$. Hence there is a sequence $R_j \to \infty$ such that the limits $\gamma^{R_j}_{\underline{k}}(t) \to \gamma^\infty_{\underline{k}}(t)$ exist, for all $\underline{k}$, uniformly in every bounded interval inside $t \ge 0$. Each such $\underline{\gamma}^\infty(t)$ will be called a weak limit, as $R \to \infty$, of $\underline{\gamma}^R(t)$.
(ii) Every weak limit $\underline{\gamma}^\infty$ verifies, for all $t \ge 0$

$$\begin{aligned} &||\underline{\gamma}^\infty||_2 \le \sqrt{E_0 L^{-d}} \\ &\int_0^t d\tau \sum_{\underline{k}} \underline{k}^2 |\underline{\gamma}^\infty_{\underline{k}}(\tau)|^2 \le \frac{1}{2} E_0 L^{-d} \nu^{-1} + t\nu^{-1} \sqrt{E_0 L^{-d}} ||\underline{\varphi}||_2 \end{aligned} \tag{3.2.17}$$

It also verifies (3.2.9): therefore it will be a weak solution.

proof: Boundedness of $\dot{\underline{\gamma}}_{\underline{k}}(t)$ immediately follows from (3.2.11) and from the successive remark; hence properties (i) and (ii) follow. Property (iii) is slightly more delicate to check; rewriting (3.2.11) for $|\underline{k}| \le R$ as:

$$\begin{aligned} &\underline{\gamma}^R_{\underline{k}}(t) = e^{-\nu \underline{k}^2 t} \underline{\gamma}^0_{\underline{k}} + \\ &+ \int_0^t e^{-\nu \underline{k}^2 (t-\tau)} \Big[\varphi_{\underline{k}}(\tau) - i \sum_{\substack{\underline{k}_1 + \underline{k}_2 = \underline{k} \\ |\underline{k}_i| \le R}} \underline{\gamma}^R_{\underline{k}_1}(\tau) \cdot \underline{k}_2 \, \Pi_{\underline{k}} \, \underline{\gamma}^R_{\underline{k}_2}(\tau) \Big] \, d\tau \end{aligned} \tag{3.2.18}$$

the problem is to take the limit under the integral sign in (3.2.18). Given that the first *a priori* bound in (3.2.17) guarantees the absolute convergence of the series obtained by taking the term by term limits, the passage to the limit will be possible if we shall show that the series in (3.2.18) is uniformly convergent for $R \to \infty$ (*i.e.* the remainder of its partial sum of order N approaches 0 uniformly in R as $N \to \infty$).

Fix $N > 0$ and recall (3.2.8) and the remark in (3.2.13): note that if $|\underline{k}_1|$ or $|\underline{k}_2|$ are $\ge N/2$ then $|\underline{k}_1| \ge N/2$ and $|\underline{k}_2| \ge k_0 \equiv 2\pi L^{-1}$ or viceversa, hence

$$\int_0^t \sum_{\substack{\underline{k}_1+\underline{k}_2=\underline{k}\\ |\underline{k}_1|+|\underline{k}_2|>N}} \Big|\underline{\gamma}^R_{\underline{k}_1}(\tau)\cdot\underline{k}_2\Big|\Big|\underline{\gamma}^R_{\underline{k}_2}(\tau)\Big| \le \int_0^t |\underline{k}| \sum_{\substack{\underline{k}_1+\underline{k}_2=\underline{k}\\ |\underline{k}_1|+|\underline{k}_2|\ge N}} \Big|\underline{\gamma}^R_{\underline{k}_1}(\tau)\Big|\,\Big|\underline{\gamma}^R_{\underline{k}_2}(\tau)\Big| \le$$

$$\le \frac{2|\underline{k}|}{Nk_0}\int_0^t \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} |\underline{k}_1|\,|\underline{\gamma}^R_{\underline{k}_1}(\tau)|\,|\underline{k}_2|\,|\underline{\gamma}^R_{\underline{k}_2}(\tau)| \le \frac{2|\underline{k}|}{Nk_0}\int_0^t \sum_{\underline{k}} |\underline{k}|^2|\underline{\gamma}_{\underline{k}}(\tau)|^2 \le$$

$$\le \frac{2|\underline{k}|}{Nk_0}\Big(\frac{E_0L^{-d}}{2\nu} + \frac{t}{\nu}||\underline{\varphi}||_2\sqrt{E_0L^{-d}}\Big) \xrightarrow[N\to\infty]{} 0 \tag{3.2.19}$$

where the integrals are performed with respect to $d\tau$ and we must understand that $\underline{\gamma}^R_{\underline{k}_i} \equiv \underline{0}$ if $|\underline{k}_i| \ge R$ or if $|\underline{k}_i| = 0$; in the first step of the second line we multiply and divide by $|\underline{k}_1|\,|\underline{k}_2|$ and bound from below the denominator $|\underline{k}_1|^{-1}|\underline{k}_2|^{-1}$ by $Nk_0/2$; and, finally, the vector $\underline{k}_2$, in the first inequality, is replaced by $\underline{k}$ using (3.2.8) obtaining (iii).

Remarks:
(1) Equations (3.2.9) have therefore a weak solution verifying (3.2.17) and (3.2.10): hence, almost everywhere in t, the first of (3.2.6). Consequently, the NS equations admit a weak solution, independently of the dimension $d \ge 2$ of the space into which the fluid flows, for all initial data $\underline{u}^0 \in C^\infty(\Omega)$ (with $\underline{\partial}\cdot\underline{u}^0 = 0$). Such solution is consistent with itself in the sense (3.2.10).
(2) *(Non smooth cases)*: However the same proof would work if we only assumed that $\underline{u}^0 \in L_2(\Omega)$ (with $\underline{\partial}\cdot\underline{u} = 0$ in the sense of distributions, *i.e.* $\underline{\gamma}^0_{\underline{k}}\cdot\underline{k} = 0$). We note also that the solutions discussed in the corollary have finite total vorticity $S(t) = \sum_{\underline{k}} |\underline{k}|^2|\underline{\gamma}_{\underline{k}}(t)|^2$ for almost all t: by (3.2.17) we do not even have to suppose that $\underline{u}^0 \in W^1(\Omega)$ because (3.2.16) puts a bound on the integral of the square of the $W^1(\Omega)$ norm of the solution which only depends on the L_2 norm of $\underline{u}^0$. Thus the NS equations in dimension $d = 2, 3$ admit a weak solution for all initial data in L_2 and such solution has finite vorticity for almost all times.

The solutions that arise in the above corollary might be not unique and it might be possible to exhibit other weak solutions by other methods. It is therefore convenient to give them a special name to distinguish them from weak solutions developed (later) by following other methods and which might be different. Hence we set the following definition

2 Definition *(C–weak solutions): A weak solution obtained through the limits in (i) of corollary II with initial datum in $\underline{u}^0 \in L_2(\Omega)$, i.e. from the solutions of the cut–off regularized equations (3.2.11), will be called a "C-weak soluion" of the NS equation.*

We shall see that there could be weak solutions different from them, even if $\underline{u}^0 \in W^1(\Omega)$, *c.f.r.* below and §3.3.

Equations (3.2.9) have other remarkable consequences. The most notable is certainly the *autoregularization theorem* valid *for all weak solutions*, (hence in particular for the C-weak solutions of definition 2

III. Proposition *(autoregularization): Let $\underline{u}(t)$ be a weak solution with initial datum $\underline{u}^0 \in C^\infty$. Given $T > 0$ suppose that, for some $\alpha \geq 0$, there exists a constant $C_\alpha > 0$ such that for all $0 \leq t \leq T$ it is $\sup_{\underline{k}} |\underline{k}|^\alpha |\underline{\gamma}_{\underline{k}}(t)| \leq C_\alpha$. Then if $\alpha > d-1$ it will be $C_\beta < \infty$ for all $\beta > 0$: hence $\underline{u} \in C^\infty(\Omega)$.*

Remark: In other words $\underline{\gamma}_{\underline{k}}$ is the Fourier transform of an infinitely smooth solution the NS equation, if it is "just" the Fourier transform of a regular enough solution. Note that if $\underline{u} \in C^\infty(\Omega)$ is a weak solution considered in corollary II then $\underline{u}$ is also C^∞ in t because the time derivatives can be expressed in terms of the $\underline{x}$–derivatives by differentiating suitably many times the equations verified by $\underline{\gamma}_{\underline{k}}$ or $\underline{u}$ (*i.e.* the NS equations).

proof: Given (3.2.9) we can bound the nonlinear term:

$$|N_{\underline{k}}(\underline{\gamma})| \leq \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} |\underline{\gamma}_{\underline{k}_1}|\,|\underline{\gamma}_{\underline{k}_2}|\,|\underline{k}| \leq C_\alpha^2\,|\underline{k}| \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \frac{1}{|\underline{k}_1|^\alpha |\underline{k}_2|^\alpha} \leq$$

$$\leq \begin{cases} \infty & \alpha \leq d/2 \\ |\underline{k}|\,C_\alpha^2\,B_\alpha |\,\underline{k}|^{-(2\alpha-d)} & d/2 < \alpha < d \\ |\underline{k}|\,C_\alpha^2\,B_\alpha\,|\underline{k}|^{-\alpha}\,\log|\underline{k}| & \alpha = d \\ |\underline{k}|\,C_\alpha^2\,B_\alpha\,|\underline{k}|^{-\alpha} & \alpha > d \end{cases} \tag{3.2.20}$$

where all vectors $\underline{k}, \underline{k}_i$ are different from $\underline{0}$.

Since $\int_0^t e^{-\nu \underline{k}^2\,\tau} d\tau \leq 1/\nu\underline{k}^2$ then, if $\alpha > d/2$ and if we take into account (3.2.8), we find an estimate of $\int_0^t N_{\underline{k}}(\underline{\gamma}(\tau))\,d\tau$ as

$$\left| \int_0^t d\tau \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underline{\gamma}_{\underline{k}_1} \cdot \underset{\sim}{k}_2 \Pi_{\underline{k}}\,\underline{\gamma}_{\underline{k}_2} e^{-\underline{k}^2 \nu(t-\tau)} \right| \leq \frac{C_\alpha^2 B_\alpha}{\nu|\underline{k}|} \begin{cases} |\underline{k}|^{-(2\alpha-d)} & \alpha < d \\ |\underline{k}|^{-\alpha}\log|\underline{k}| & \alpha = d \\ |\underline{k}|^{-\alpha} & \alpha > d \end{cases} \tag{3.2.21}$$

Suppose $\underline{u}^0 \in C^\infty(\Omega)$, hence $\sup_{\underline{k}} |\underline{k}|^\beta\,|\underline{\gamma}^0_{\underline{k}}| \stackrel{def}{=} C^0_\beta < \infty$ for all $\beta \geq 0$, and suppose $|\underline{k}|^\alpha\,|\underline{\gamma}_{\underline{k}}(t)| = C_\alpha < \infty$, for some $\alpha > d-1$. Then setting $\eta = \min(\alpha - d + 1, 1) > 0$ we have, therefore, shown by (3.2.21) that there exists $C'_{\alpha+\eta} < \infty$ such that

$$|\underline{\gamma}_{\underline{k}}(t)| \leq \frac{C^0_{\alpha+\eta}}{|\underline{k}|^{\alpha+\eta}} + \frac{C'_{\alpha+\eta}}{|\underline{k}|^{\alpha+\eta}} \tag{3.2.22}$$

where the first term arises by bounding the first and second terms in (3.2.9) by $|\underline{\gamma}^0_{\underline{k}}| + \frac{|\underline{\varphi}_{\underline{k}}}{\nu \underline{k}^2} \leq C^0_{\alpha+\eta}\,|\underline{k}|^{-\alpha-\eta}$: hence the claim in the theorem follows (by indefinitely repeating the argument gaining each time some extra decay in $|\underline{k}|$).

Remarks:

(1) It is important to note that if C_{α_0} is a constant such that $|\underline{k}|^{\alpha_0}|\underline{\gamma}_{\underline{k}}| \le C_{\alpha_0}$ and $\alpha_0 > d-1$, then not only it follows that $|\underline{k}|^{\alpha}|\underline{\gamma}_{\underline{k}}| \le C_\alpha$ for all $\alpha > \alpha_0$ and suitable C_α, but also that the constant C_α can be explicitly bounded in terms of C_{α_0} and of the quantities C^0_α relative to the initial datum and the forcing (*i.e.* $(|\gamma^0_{\underline{k}}| + \frac{|\varphi_{\underline{k}}|}{\nu \underline{k}^2})\,|\underline{k}|^\alpha, \le C^0_\alpha$) by a function of C_{α_0}, C^0_α that we shall call $B_\alpha(\alpha_0, C_{\alpha_0}, C^0_\alpha)$. An explicit bound on B_α can be easily derived from (3.2.20) but we shall not need it now (*c.f.r.* the proof of proposition IX below).
(2) Furthermore the bounds described above hold also for the cut–off regularized equations with a cut–off parameter $R < \infty$: this means that $|\gamma^R_{\underline{k}}||\underline{k}|^{\alpha_0} \le C_{\alpha_0}$ and $\alpha_0 > d-1$, imply $|\gamma^R_{\underline{k}}(t)||\underline{k}|^\alpha \le B_\alpha(\alpha_0, C_{\alpha_0}, C^0_\alpha)$ where the function B *coincides* with that of the non regularized case. We shall see that the last remarks will allow us to bound, in the case $d=2$, the difference between $\underline{\gamma}$ and $\underline{\gamma}^R$.
(3) *(Non smooth cases):* In the above proof the assumption that $\underline{u}^0 \in C^\infty$ has been used only to insure that $\sup |\underline{k}|^\alpha |\underline{\gamma}^0_{\underline{k}}| \stackrel{def}{=} \overline{C}^0_\alpha < \infty$. We have in fact shown a lot more in the above proof. Indeed the first term in (3.2.9) can be alternatively bounded rather than by $\overline{C}^0_{\alpha+\eta}|\underline{k}|^{-\alpha-\eta}$ (which could be ∞ if we only assume that $\underline{u}^0 \in L_2(\Omega)$) by the quantity $\sqrt{L^{-d}E_0}e^{-\nu \underline{k}^2 t}$ which is certainly a finite bound, and a very good one, for $t>0$. This bound, under the only assumption that $\underline{u}^0 \in L_2$ and $|\underline{k}|^\alpha\, |\underline{\gamma}_{\underline{k}}(t)| < C_\alpha$ for some $\alpha > d-1$ (which is a non trivial assumption if $d>2$, see below), implies

$$
\begin{aligned}
|\underline{\gamma}_{\underline{k}}(t)| \le& \sqrt{L^{-d}E_0}\, e^{-\nu \underline{k}^2 t} + \frac{C'_{\alpha+\eta}}{|\underline{k}|^{\alpha+\eta}} + \frac{|\underline{\varphi}_{\underline{k}}|}{\nu \underline{k}^2}\\
|\underline{\gamma}_{\underline{k}}(t)| \le& \frac{k_0^{\alpha+\eta}\sqrt{L^{-d}E_0}\, \max_{x\ge 0} x^{\alpha+\eta}e^{-x^2}}{(k_0^2\nu\, t)^{(\alpha+\eta)/2}\, |\underline{k}|^{\alpha+\eta}} + \frac{C'_{\alpha+\eta}}{|\underline{k}|^{\alpha+\eta}} + \frac{|\underline{\varphi}_{\underline{k}}|}{\nu \underline{k}^2} \le\\
\le& \big(1 + \frac{1}{(k_0^2\nu\, t)^{(\alpha+\eta)/2}}\big)\, \frac{C''_{\alpha+\eta}}{|\underline{k}|^{\alpha+\eta}} \qquad (3.2.23)\\
C''_{\alpha+\eta} =& C'_{\alpha+\eta} + k_0^{\alpha+\eta}\sqrt{L^{-d}E_0}\, \max_{x\ge 0} x^{\alpha+\eta}e^{-x^2} + \max \frac{|\underline{k}|^{\alpha+\eta}||\underline{\varphi}_{\underline{k}}|}{\nu |\underline{k}|^2}
\end{aligned}
$$

Recursively this means, of course, that the solution will be C^∞ for $t>0$: *even* if the initial datum $\underline{u}^0$ is just L_2 and *if*, for some, $\alpha > d-1$ one has the further (in general very nontrivial if $d>2$, see below) information that $\sup_{\underline{k}} |\underline{k}|^\alpha\, |\underline{\gamma}_{\underline{k}}(t)| < \infty$. In this case the quantities $\sup_{\underline{k}} |\underline{k}|^\beta\, |\underline{\gamma}_{\underline{k}}(t)|$ are $< \infty$ for all $\beta \ge 0$ and they can be bounded by a t dependent quantity which diverges as $t \to 0$ as an easily computable inverse power of t, *c.f.r.* (3.2.23).

The above regularization theorems are very useful in the two dimensional case, $d=2$: in such case in fact they imply the following theorem:

IV. Proposition: *(existence and smoothness for NS in dimension 2) If $d = 2$ the C-weak solutions*[3] *with datum $\underline{u}^0 \in L_2(\Omega)$ are velocity fields of class C^∞ in $\underline{x}$ and t, for $t > 0$, and even for $t \geq 0$ when one supposes that $\underline{u}^0$ is of class C^∞ as well.*
Furthermore if $\underline{u}^0 \in W^1(\Omega)$ (i.e. such that $\underset{\sim}{\partial}\underline{u} \in L_2(\Omega)$, c.f.r. problem [2.2.20]) and if one sets: $F_{-1} = \big(\sum_{\underline{k}} |\varphi_{\underline{k}}|^2 |\underline{k}|^2\big)^{1/2}$, one finds that the solutions $\underline{\gamma}^R_{\underline{k}}(t)$ of (3.2.18) for $R \leq \infty$ verify

$$\Big(\sum_{|\underline{k}|\leq R} |\underline{k}|^2 |\underline{\gamma}^R_{\underline{k}}(t)|^2\Big)^{1/2} \leq \tag{3.2.24}$$
$$\leq \max\Big[\big(F_{-1}\nu^{-1}k_0^{-2}\big),\ \Big(\sum_{\underline{k}} \underline{k}^2 |\gamma_{\underline{k}}(0)|^2\Big)^{1/2}\Big] \stackrel{def}{=} \sqrt{S_0}$$

where k_0 is the minimum value that $|\underline{k}|$ can take (i.e. $k_0 = 2\pi L^{-1}$).

proof: Suppose first that $\underline{u}^0 \in W^1(\Omega)$ so that $S_0 < \infty$. Note that:

$$\sum_{|\underline{k}_3|\leq R} N_{\underline{k}_3}(\underline{\gamma}^R) \cdot \overline{\underline{\gamma}}^R_{\underline{k}_3} |\underline{k}_3|^2 \equiv 0, \qquad \text{if } d = 2 \tag{3.2.25}$$

that can be proved by remarking that in this case:

$$\underline{\gamma}_{\underline{k}} = \gamma_{\underline{k}} \frac{\underline{k}^\perp}{|\underline{k}|}, \qquad \gamma_{\underline{k}} = -\,\overline{\gamma}_{-\underline{k}}, \qquad \text{if} \quad \underline{k}^\perp = (k_2, -k_1), \qquad \underline{k} = (k_1, k_2) \tag{3.2.26}$$

with $\gamma_{\underline{k}}$ *scalar* and therefore:

$$\underline{k}_3^2\,(\underline{\gamma}_{\underline{k}_1} \cdot \underset{\sim}{k}_2)(\underline{\gamma}_{\underline{k}_2} \cdot \underline{\gamma}_{\underline{k}_3}) = \frac{(\underline{k}_1^\perp \cdot \underline{k}_2)\,(\underline{k}_2^\perp \cdot \underline{k}_3^\perp)}{|\underline{k}_1|\ |\underline{k}_2|\ |\underline{k}_3|} \underline{k}_3^2\,\gamma_{\underline{k}_1}\gamma_{\underline{k}_2}\gamma_{\underline{k}_3} \tag{3.2.27}$$

Summing over the permutations of $\underline{k}_1, \underline{k}_2, \underline{k}_3$ and using $\underline{k}_1^\perp \cdot \underline{k}_2 = \underline{k}_2^\perp \cdot \underline{k}_3 = \underline{k}_3^\perp \cdot \underline{k}_1$ we find zero (see also (2.2.29), that imply the (3.2.25), for a rapid and interesting way to reach this conclusion). Hence the same argument used in connection with the above *a priori* estimate leads to:

$$\frac{d}{dt}\frac{1}{2} \sum_{|\underline{k}|\leq R} |\underline{k}|^2 |\underline{\gamma}^R_{\underline{k}}|^2 = -\nu \sum_{|\underline{k}|\leq R} |\underline{k}|^4 |\underline{\gamma}^R_{\underline{k}}|^2 + \sum_{\underline{k}} \underline{k}^2 \varphi_{\underline{k}} \underline{\gamma}^R_{\underline{k}} \tag{3.2.28}$$

Hence one gets: $\dot S/2 \leq -\nu k_0^2 S + F_{-1}\sqrt{S}$: *i.e.* $\dot S/2 \leq 0$ if $S > S_0$, with S_0 defined in (3.2.24), so that the inequality (3.2.24) follows.
Thus the weak solutions must verify:

$$S(t) = \sum_{\underline{k}} |\underline{k}|^2 |\underline{\gamma}_{\underline{k}}(t)|^2 \leq S_0, \qquad \forall\, t \geq 0 \tag{3.2.29}$$

[3] See definition 2 and the remarks preceding proposition III.

and this implies, by Schwartz' inequality:

$$\begin{aligned} |N_{\underline{k}}(\underline{\gamma})| \le |\underline{k}| \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} |\underline{\gamma}_{\underline{k}_1}|\,|\underline{\gamma}_{\underline{k}_2}| \le \\ \le |\underline{k}| \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \frac{|\underline{k}_1|\,|\underline{\gamma}_{\underline{k}_1}|\,|\underline{k}_2|\,|\underline{\gamma}_{\underline{k}_2}|}{|\underline{k}_1|\,|\underline{k}_2|} \le \frac{2S_0}{k_0} \end{aligned} \tag{3.2.30}$$

because if $\underline{k}_1 + \underline{k}_2 = \underline{k}$ then at least one among $\underline{k}_1$ and $\underline{k}_2$ has modulus $\ge |\underline{k}|/2$ (and both have modulus $\ge k_0$, being non zero) and therefore we can say that: $|\underline{k}_1||\underline{k}_2| \ge k_0\,|\underline{k}|/2$.

Hence, by (3.2.30),(3.2.9), and if $|\underline{\varphi}_{\underline{k}}| \le G_\alpha |\underline{k}|^{-\alpha}$, we deduce:

$$|\underline{\gamma}^R_{\underline{k}}(t)| \le e^{-\nu \underline{k}^2 t} |\underline{\gamma}^R_{\underline{k}}(0)| + \Big(\frac{G_0}{\nu \underline{k}^2} + \frac{2S_0}{\nu \underline{k}^2\, k_0}\Big)(1 - e^{-\nu \underline{k}^2 t}) \tag{3.2.31}$$

and we see as well that $|\underline{\gamma}^\infty_{\underline{k}}(t) - e^{-\underline{k}^2 \nu t}\underline{\gamma}^0_{\underline{k}}| \le (G_0\nu^{-1} + 2S_0/k_0)/|\underline{k}|^2$, which will be useful later.

Hence there is a constant $C_2(t)$ such that $|\underline{\gamma}_{\underline{k}}(t)| \le C_2(t)/|\underline{k}^2|$ with $C_2(t) = const\,(k_0^2 + (\nu t)^{-1})$ and we *can apply the autoregularization theorem for* $\alpha > d-1$ *(as* $\alpha = 2$ *and* $d - 1 = 1$*) to the evolution considered for* $t > 0$ (using the self-consistency property of corollary II).

Hence the solution (which we shall show, in proposition VI below, to be unique) $\underline{u}^\infty(\underline{x}, t)$ is C^∞ in $\underline{x}$ for each t, and each of its derivative is uniformly bounded in time (in terms of the constants $C_\alpha(t)$ which as $t \to 0$ may diverge if $\underline{u}^0 \in W^1(\Omega)$ but which stay finite if $\underline{u}^0 \in C^\infty(\Omega)$). It follows that, by differentiating with respect to t both sides of the equation (3.2.9), for instance, that $\underline{u}$ in class C^∞ in $\underline{x}, t$, for $t > 0$.

The case $\underline{u}^0 \in L_2(\Omega)$ (when S_0 could be ∞) is immediately reduced to the one just treated. Because the estimate (3.2.17) remains valid for the weak solutions of corollary II even when the datum is just L_2 and it imples that $\underline{u}(t) \in W^1(\Omega)$ for almost all times $t > 0$: therefore it suffices to consider as initial datum the value of the weak solution $\underline{u}(t)$ at such an instant and take into account the self-consistence property of the weak solutions discussed in corollary II, *c.f.r.* (3.2.10).

Remarks:

(1) Note that the derivation just discussed shows that (3.2.31) holds also for $\underline{\gamma}^R_{\underline{k}}(t)$ if $R < \infty$, a property that will be useful below, *c.f.r.* proposition VII below.
(2) One concludes, from the autoregularization property with $\alpha = 2$ and $d = 2$, that initial data in C^∞ evolve into solutions of the Navier–Stokes equation which are of class C^∞ for $t \ge 0$ and one concludes that also data initially in L_2 evolve into solutions that for $t > 0$ are in C^∞. Hence the

analogy with the heat equation and with the Stokes equation (*c.f.r.* §9, (C)) is rather strong if $d = 2$. [4]

A further autoregularization theorem *valid for $d \ge 2$ and for all weak solutions*, becomes evident when one ponders the above proof and it is:

V. Proposition *(finite vorticity implies smoothness in dimension $d = 2, 3$) Suppose that $t \to \underline{\gamma}_{\underline{k}}(t)$ verifies for $t \in [0, T]$ the (3.2.9) or (3.2.11) with $d \le 3$ and initial datum $\underline{u}^0$ of class C^∞. Furthermore, given $T > 0$, suppose that $\sum_{\underline{k}} |\underline{k}|^2 |\underline{\gamma}_{\underline{k}}(t)|^2 \le S_1$ for $0 \le t \le T$. Then for each $\alpha \ge 0$ there exists a constant C_α for which it is: $|\underline{\gamma}_{\underline{k}}(t)| \le C_\alpha |\underline{k}|^{-\alpha}$: i.e. the $\underline{u}$ is in class C^∞ in $\underline{x}$ (hence it is C^∞ also in $t \ge 0$). See the following comment (4) for the case in which $\underline{u} \in L_2(\Omega)$.*

proof: clearly, by the assumptions, the result holds for $\alpha \le 1$ and $C_1 \equiv S_1^{1/2}$. From the expression of the inertial term $N_{\underline{k}}(\underline{\gamma})$ in (3.2.9) we see that (by replacing, as done often in the preceding pages, $\underset{\sim}{k}_2$ with $\underset{\sim}{k}$, before bounding the left hand side term) (3.2.30) holds:

$$|N_{\underline{k}}(\underline{\gamma})| \le \frac{2}{k_0} S_1 \tag{3.2.32}$$

hence the integral in (3.2.21) is bounded by $S_1/(k_0 \, \nu \, \underline{k}^2)$ and we see, from (3.2.9) or (3.2.11), the existence of a constant C_2' for which $|\underline{\gamma}_{\underline{k}}| \le e^{-\nu \underline{k}^2 t} |\underline{\gamma}^0_{\underline{k}}| + C_2' |\underline{k}|^{-2}$. Since $\underline{u}^0 \in C^\infty$ then it is $|\underline{\gamma}_{\underline{k}}| \le C_2 |\underline{k}|^{-2}$ for a conveniently chosen constant C_2. But then, again

$$\begin{aligned} |N_{\underline{k}}(\underline{\gamma})| &\le \sum_{\underline{k}_1 + \underline{k}_2 = \underline{k}} |\gamma_{\underline{k}_1}| \, |\underline{k}_2| \, |\gamma_{\underline{k}_2}| \le \sum_{\underline{k}_1 + \underline{k}_2 = \underline{k}} \frac{|\underline{k}_1| \, |\gamma_{\underline{k}_1}| \, |\underline{k}_2|^2 \, |\gamma_{\underline{k}_2}|}{|\underline{k}_1| \, |\underline{k}_2|} \le \\ &\le C_2 S_1^{1/2} \Big(\sum_{\underline{k}_1 + \underline{k}_2 = \underline{k}} \frac{1}{|\underline{k}_1|^2 \, |\underline{k}_2|^2} \Big)^{1/2} \le C_2 \sqrt{S_1} \frac{2^{1/4} B}{k_0^{1/4} |\underline{k}|^{1/4}} \end{aligned} \tag{3.2.33}$$

because the last sum is bounded by $2^{1/4} (k_0 |\underline{k}|)^{-1/4}$ times the square root of $\big(\sum (|\underline{k}_1| |\underline{k}_2|)^{-(2-1/4)} \big)$; and the latter quantity is bounded via Schwartz's inequality by $B = \sum_{\underline{k}} |\underline{k}|^{-(4-1/2)}$; therefore, proceeding as above, we bound again the integral in (3.2.21) and find $|\underline{\gamma}_{\underline{k}}| < C_{2+1/4} \sqrt{S_1} |\underline{k}|^{-(2+1/4)}$. But $2 + 1/4 > d - 1$ if $d \le 3$, and then the preceding general autoregularization theorem applies.

Remarks: The above result is particularly interesting if $d = 3$ but it also holds if $d = 2$.

---

[4] Although we cannot now prove that data in L_2 become analytic for $t > 0$, but only that they become C^∞, however see propositions VIII and IX below.

(1) Hence every weak solution of the Navier–Stokes equation with smooth initial datum and total vorticity bounded in every finite time interval is in class C^∞ for $t \ge 0$; and the maxima of its derivatives can be explicitly bounded in terms of the initial data and of the vorticity estimate S_1. This happens, for instance, if $d = 2$: *c.f.r.* proposition IV.
(2) The difference between the cases $d = 2$ and $d = 3$ is that if $d = 3$ we do not know how to control *a priori* the quantity $\sum_{\underline{k}} |\underline{k}|^2 |\underline{\gamma}_{\underline{k}}(t)|^2$, *i.e.* we do not know how to prove the existence of a quantity S_1 that bounds it; while if $d = 2$ vorticity conservation guarantees such a property. For $d = 2$ this is the contents of proposition IV.
(3) Note that the theorem holds for (3.2.9) and for (3.2.11) as well and *with the same bounds*, independent of R.
(4) *(non smooth case)*: Looking more closely at the above proof we realize that it provides informations on the cases in which $\underline{\gamma}^0$ is just in W^1 (*i.e.* the quantity denoted S_1 in the proof is initially finite): in such cases (taking advantage of the factor $e^{-\nu \underline{k}^2 t}$ that will multiply $\underline{\gamma}^0$ making it rapidly decreasing in $|\underline{k}|$ for $t > 0$) for each $\alpha \ge 0$ the above proof shows that there exists $C_\alpha(t)$ for which we have: $|\underline{\gamma}_{\underline{k}}(t)| \le C_\alpha(t) |\underline{k}|^{-\alpha}$, and $C_\alpha(t)$ can be chosen continuous for $T \ge t > 0$, *i.e.* excluding $t = 0$. *Hence in* $d = 2, 3$ *weak solutions with bounded vorticity are* C^∞ *for* $t > 0$.

Concerning uniqueness we can prove the following proposition, which is strengthened in the remarks following it

VI. Proposition *(uniqueness of smooth solutions of NS): If* $d \ge 2$ *and if* $\underline{\gamma}^1_{\underline{k}}$ *and* $\underline{\gamma}^2_{\underline{k}}$ *are two (weak) solutions rapidly decreasing for* $|\underline{k}| \to \infty$, *in the sense that both are bounded in a given interval* $0 \le t \le T$ *by* $C_\alpha |\underline{k}|^{-\alpha}$ *for every* $\underline{k}$ *and for some* $\alpha > d - 1$, *then the two solutions coincide if they have the same initial datum* $\underline{u}^0 \in C^\infty(\Omega)$.

proof: By the autoregularization theorem we deduce that then $|\underline{\gamma}^i_{\underline{k}}| \le C_\alpha |\underline{k}|^{-\alpha}$ for every $\alpha > 0$ (and a suitable corresponding constant C_α).
Their difference will verify, if we define $\Delta \equiv ||\underline{\gamma}^1 - \underline{\gamma}^2||_2^2$ and $\Delta_1 \equiv ||\, |(\underline{\gamma}^1 - \underline{\gamma}^2)|\, |\underline{k}|\, ||_2^2$:

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\Delta &\le -\nu\Delta_1 + \sum_{\underline{k}} |\underline{\gamma}^1_{\underline{k}} - \underline{\gamma}^2_{\underline{k}}| \sum_{\underline{k}_1 + \underline{k}_2 = \underline{k}} |\underset{\sim}{\gamma}^1_{\underline{k}_1} \cdot \underset{\sim}{k}_2\, \Pi_{\underline{k}}\, \underline{\gamma}^1_{\underline{k}_2} - \underset{\sim}{\gamma}^2_{\underline{k}_1} \cdot \underset{\sim}{k}_2\, \Pi_{\underline{k}}\, \underline{\gamma}^2_{\underline{k}_2}| \\
&\le -\nu\Delta_1 + \sum_{\underline{k}} |\underline{\gamma}^1_{\underline{k}} - \underline{\gamma}^2_{\underline{k}}| \sum_{\underline{k}_1 + \underline{k}_2 = \underline{k}} |(\underset{\sim}{\gamma}^1_{\underline{k}_1} - \underset{\sim}{\gamma}^2_{\underline{k}_1}) \cdot \underset{\sim}{k}_2\, \Pi_{\underline{k}}\, \underline{\gamma}^1_{\underline{k}_2}| + \\
&+ \sum_{\underline{k}} |\underline{\gamma}^1_{\underline{k}} - \underline{\gamma}^2_{\underline{k}}| \sum_{\underline{k}_1 + \underline{k}_2 = \underline{k}} |(\underset{\sim}{\gamma}^2_{\underline{k}_1} \cdot \underset{\sim}{k}_2\, \Pi_{\underline{k}}\, (\underline{\gamma}^1_{\underline{k}_2} - \underline{\gamma}^2_{\underline{k}_2})| \qquad (3.2.34)
\end{aligned}
$$

and if we replace $\underset{\sim}{k}_2$ with $\underline{k}$ and if we use the regularity property $|\underline{\gamma}^i_{\underline{k}}| \le C_\alpha |\underline{k}|^{-\alpha}$ with, for instance, $\alpha = 4$ (note that under the present hypotheses this inequality holds for all $\alpha > 0$) we find:

$$\frac{1}{2}\dot\Delta \le -\nu\Delta_1 + 2\sum_{\underline{k}}\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \big(|\underline{k}|\,|\gamma^1_{\underline{k}} - \gamma^2_{\underline{k}}|\big)\big(|\gamma^1_{\underline{k}_1} - \gamma^2_{\underline{k}_1}|\big)\frac{C_4}{|\underline{k}_2|^4} \le$$
$$\le -\nu\Delta_1 + 2\sum_{\underline{k}}\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \big(|\underline{k}|^2|\gamma^1_{\underline{k}} - \gamma^2_{\underline{k}}|^2\frac{\varepsilon}{2} + \frac{1}{2\varepsilon}|\gamma^1_{\underline{k}_1} - \gamma^2_{\underline{k}_1}|^2\big)\frac{C_4}{\underline{k}_2^4} \tag{3.2.35}$$

having made use of the inequality $|ab| \le a^2/(2\varepsilon) + \varepsilon b^2/2$ (holding for each $\varepsilon > 0$) and $|\underline{k}| \ge k_0$. Hence, if $B = \sum_{\underline{k}} C_4 |\underline{k}|^{-4}$:

$$\frac{1}{2}\dot\Delta(t) \le -\nu\Delta_1 + \varepsilon B\Delta_1 + B\varepsilon^{-1}\Delta \qquad \text{for every } \varepsilon > 0 \tag{3.2.36}$$

and, by choosing ε so that $\varepsilon B = \nu$, we deduce:

$$\dot\Delta \le 2B^2\nu^{-1}\,\Delta \qquad \Rightarrow \qquad \Delta(t) \le \Delta(0)\,e^{2B^2\nu^{-1}t} \tag{3.2.37}$$

so that $\Delta(0) = 0$ implies $\Delta(t) \equiv 0$, for $t > 0$.

Remarks: The theorem can also be proved directly by noting that $\underline{u}^i(t)$ must be C^∞ by proposition V: hence they must coincide by a uniqueness theorem that can be obtained along the lines followed in the case of the Euler equation in problem [3.1.6]. The above proof is interesting because it leads to the remarks that follow and because it suggests the proof of the following proposition VII.

(1) *(Non smooth cases)*: If we suppose that the two weak solutions correspond to an initial datum $\underline{u}^0 \in L_2(\Omega)$ only and *furthermore* are such that $\lim_{t\to 0} ||\underline{u}^i(t) - \underline{u}^0||_2 = 0$, $i = 1, 2$ then we conclude from the above proof that the two solutions coincide. In fact we shall take as initial time a time $t_0 > 0$ where the solutions have become C^∞ (by proposition III): of course $\Delta(t_0)$ might be $\ne 0$ and to prove uniqueness we still need to check that $\Delta(t_0) \to 0$ as $t_0 \to 0$. This would be implied by $\lim_{t\to 0} ||\underline{u}^i(t) - \underline{u}^0||_2 = 0$, $i = 1, 2$.

However if $\underline{u}(t)$ is a weak solution with datum L_2 it is not known whether $\underline{u}(t)$ in general tends to $\underline{u}^0$ in L_2 as $t \to 0$!

If instead $\underline{u}^0 \in W^1(\Omega)$ and $d = 2$, from the relation $|\gamma_{\underline{k}}(t) - e^{-\underline{k}^2\nu t}\gamma^0_{\underline{k}}| \le (G_0\nu^{-1} + 2S_0/k_0)/|\underline{k}|^2$, *c.f.r.* the comment to (3.2.31), we deduce that $\lim_{t\to 0} ||\underline{u}(t) - \underline{u}^0||_2^2 = 0$ and therefore $\Delta(t_0) \to 0$ as $t_0 \to 0$. Hence *in $d = 2$ the weak solutions with datum in $W^1(\Omega)$ are unique.* And the only case, in $d = 2$, in which weak solutions might be not unique is when $\underline{u}^0 \in L_2(\Omega)$ but $\underline{u}^0 \notin W^1(\Omega)$: a problem which to my knowledge is unsolved in this case.

(2) The argument used in the above proof allows us to find an explicit estimate of the error that is made on the solutions by truncating the Navier Stokes equation. In fact the following *theorem of spectral approximability* holds:

VII. Proposition *(constructive approximations errors estimate for NS solutions in 2 dimensions): Suppose $d = 2$ and that $\underline{u}^0$ is in class C^∞; calling $\Delta_R(t)$ the square of the L_2–norm of the difference between the solution of the regularized equations with a regularization parameter R (see (3.2.11)) and the solution of class C^∞ discussed in the previous proposition, it follows that $\Delta_R(t)$ can be bounded for every integer $q > 0$ by:*

$$\Delta_R(t) \le V_q e^{Mt} R^{-q} \tag{3.2.38}$$

where V_q, M are suitable constants computable in terms of the initial datum but which do not depend on the time t at which the solutions are considered to be evaluated.

Remark: Equation (3.2.38) shows that the truncation method, often also called *spectral method*, provides us with a *truly constructive algorithm* for the solution of the *bidimensional* NS equations. Note that the error estimate is exponentially increasing with the time t: this is a property that, in general, one cannot expect to improve because, as usual in differential equations, data that initially differ by little keep differing more and more as time increases and diverge exponentially, even when they evolve with the same differential equation (furthermore in our case, the truncated equation describing $\underline{\gamma}^R$ is also somewhat different from the not truncated one). This exponential divergence is well established on heuristic and experimental grounds in the case of the NS equations at even moderately large Reynolds number, *c.f.r.* the following Ch. 4,5,6,7.

proof: Suppose for simplicity that there is no external force: $\underline{g} = \underline{0}$.
Equation (3.2.31) holds, as already noted, also for $\underline{\gamma}^R$. Call then C_2 the constant bounding above $|\underline{k}|^2|\underline{\gamma}_{\underline{k}}|$, and note that (3.2.31) shows that a possible choice is:

$$C_2 = C_2^0 + \nu^{-1}G_0 + 2S_0\nu^{-1}k_0^{-1} \tag{3.2.39}$$

where C_p^0 and G_p are upper bounds of $|\underline{k}|^p|\underline{\gamma}^0_{\underline{k}}|$ and of $|\underline{k}|^p|\underline{g}_{\underline{k}}|$, respectively, for each $\underline{k}$.

Let $C_p = B_p(C_2, C_p^0, G_p)$ be the constant, independent of t, such that:

$$|\underline{\gamma}^R_{\underline{k}}|, |\underline{\gamma}_{\underline{k}}| < C_p|\underline{k}|^{-p} \tag{3.2.40}$$

which exists by the autoregularization theorem and is $< \infty$ for all $p \ge 0$; and consider $p \ge 4$.

Then the equation verified by $\underline{\gamma} - \underline{\gamma}^R$ is, for $|\underline{k}| \le R$:

$$\dot{\underline{\gamma}}_{\underline{k}} - \dot{\underline{\gamma}}^R_{\underline{k}} = -\nu\underline{k}^2(\underline{\gamma}_{\underline{k}} - \underline{\gamma}^R_{\underline{k}}) - i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \big(\underset{\sim}{\gamma}_{\underline{k}_1} \cdot \underset{\sim}{k}_2\, \Pi_{\underline{k}}\, \underline{\gamma}_{\underline{k}_2} - \underset{\sim}{\gamma}^R_{\underline{k}_1} \cdot \underset{\sim}{k}_2\, \Pi_{\underline{k}}\, \underline{\gamma}^R_{\underline{k}_2}\big) \tag{3.2.41}$$

where $\gamma^R_{\underline{k}_i} \equiv \underline{0}$ if $|\underline{k}|_i > R$: clearly also for such large values of $|\underline{k}_i|$ the $\underline{\gamma}_{\underline{k}_i}$ is in general not zero.

Multiply scalarly both sides by $\overline{\gamma}_{\underline{k}} - \overline{\gamma}^R_{\underline{k}}$ and summing over $\underline{k}$, we find, with the notations and the procedure followed for the (3.2.36), (3.2.37):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Delta_R \leq & -\nu\Delta^1_R + \varepsilon B\Delta^1_R + \\ & + B\varepsilon^{-1}\Delta_R + \sum_{\underline{k}}\frac{2C_p}{|\underline{k}|^p}\sum_{\substack{|\underline{k}_1| \text{ or } |\underline{k}_2|>R \\ \underline{k}_1+\underline{k}_2=\underline{k}}} |\underline{k}||\underline{\gamma}_{\underline{k}_1}|\,|\underline{\gamma}_{\underline{k}_2}| \leq \\ & \leq B^2\nu^{-1}\Delta_R + \frac{4C_p^3}{(2\pi L^{-1})^{p-1}}\,(\sum_{|\underline{k}_1|>R}\frac{1}{|\underline{k}_1|^p})(\sum_{\underline{k}_2}\frac{1}{|\underline{k}_2|^p}) \leq \\ & \leq B^2\nu^{-1}\Delta_R + \frac{4C_p^3 K_p}{(2\pi L^{-1})^{p-1}R^{p-2}}\end{aligned} \tag{3.2.42}$$

where $\Delta_R(t) = \sum_{\underline{k}}|\underline{\gamma}_{\underline{k}} - \underline{\gamma}^R_{\underline{k}}|^2$ and $\Delta^1_R = \sum_{\underline{k}}|\underline{k}|^2\,|\underline{\gamma}_{\underline{k}} - \underline{\gamma}^R_{\underline{k}}|^2$; and K_p estimates the quantity $(\sum_{\underline{k}}|\underline{k}|^{-p})^2$. Hence:

$$\begin{aligned}\Delta_R(t) \leq & e^{2B^2\nu^{-1}t}\Delta_R(0) + \int_0^t e^{2B^2\nu^{-1}(t-\tau)}\frac{8C_p^3K_p}{(2\pi L^{-1})^{p-1}R^{p-2}}\,d\tau \leq \\ \leq & e^{2B^2\nu^{-1}t}\Delta_R(0) + \frac{8C_p^3K_p}{2(2\pi L^{-1})^{p-1}B^2\nu^{-1}}\frac{1}{R^{p-2}}\end{aligned} \tag{3.2.43}$$

recalling that $B \equiv C_4\sum_{\underline{k}}|\underline{k}|^{-4}$ (see (3.2.36).

But $\Delta_R(0)$ is small for large R: in fact the $\Delta_R(0) \equiv \sum_{|\underline{k}|\geq R}|\underline{\gamma}_{\underline{k}}(0)|^2$ is bounded above by ${C^0_r}^2\sum_{|\underline{k}|\geq R}|\underline{k}|^{-2r}$, if $|\underline{\gamma}_{\underline{k}}(0)|\,|\underline{k}|^r \leq C^0_r$ for all $r \geq 0$; and therefore $\Delta_R(0) \leq {C^0_r}^2 R^{-2r+2}B'_r$ for a suitably chosen B'_r and:

$$\Delta_R(t) \leq e^{2B^2\nu^{-1}t}\big(\frac{B'_r}{R^{2r-2}} + \frac{B"_p}{R^{p-2}}\big) \tag{3.2.44}$$

with a suitable $B"_p$. This yields, in particular, an explicit estimate of the difference between $\underline{\gamma}_{\underline{k}}(t)$ and $\underline{\gamma}^R_{\underline{k}}(t)$ at $\underline{k}$ fixed, $|\underline{k}| \leq R$, and the (3.2.38) follows by choosing the arbitrary parameter r so that $2r - 2 = p - 2$, *i.e.* $r = p/2$.

We conclude by showing that if $d = 2$ the weak solutions with mildly regular initial data become analytic at positive time

Proposition VIII *(an analytic regularity result for NS in $d = 2$, (Mattingly, Sinai)): If $d = 2$ the (unique) solution of the NS equations with initial datum $\underline{u}^0$ and forcing $\underline{g}$ such that $|\underline{\gamma}_{\underline{k}}(0)| < U|\underline{k}|^{-3}, |\underline{g}_{\underline{k}}| < Fe^{-\kappa|\underline{k}|}$ with $U, F, \kappa > 0$, is analytic for $t \in (0, T]$ with T arbitrarily fixed.*

Proof: We follow, and implement in our context, the idea in [MS99]. Consider the regularized equations with cut-off R; by propositions III, IV *for* *all times* it is: $|\gamma^R_{\underline{k}}(t)| \le \frac{C(t)}{|\underline{k}|^3}$ for a non decreasing, finite, function $C(t) > 0$ depending on U but not on R.

Hence, by choosing ε sufficiently small, for $i, j = 1, 2$ (i is the component index, j distinguishes real or imaginary part) we have[5]

$$|\gamma^{(i,j)R}_{\underline{k}}(t)| < 2 \cdot e^{-|\underline{k}|\varepsilon t} \frac{C(T)}{|\underline{k}|^3} \tag{3.2.45}$$

for $|\underline{k}| \le K_0 \equiv 8\nu^{-1} 2^3 C(T)\, 2B$ and $t \in [0, T]$, with $B = \sum_{|\underline{k}|\neq 0} |\underline{k}|^{-3}$. This means that we can take $\varepsilon = (K_0 T)^{-1} \log(U/2C(T))$.

We take, *just for simplicity*, $\underline{g} = \underline{0}$. Then assuming the existence of a time t when the (3.2.45) is violated for some $\overline{k} > K_0$ we define $\overline{t}_R$, with $0 < \overline{t}_R < T$, so that at $t = \overline{t}_R$, "for the first time", $\gamma^{(i,j)R}_{\overline{\underline{k}}}(\overline{t}) = \pm 2 \cdot e^{-|\overline{\underline{k}}|\varepsilon \overline{t}} C(T)\, |\overline{\underline{k}}|^{-3}$ for at least one $\underline{k} = \overline{\underline{k}}$ ($|\underline{k}| > K_0$) and $i, j = 1$ or 2; in the + case, for instance, $\dot{\gamma}^{(i,j)R}_{\underline{k}}$ will be bounded by:

$$\begin{aligned} \dot{\gamma}^{(i,j)R}_{\overline{\underline{k}}}(\overline{t}) &\le -\nu |2\overline{\underline{k}}|^2 \gamma^{(i,j)R}_{\overline{\underline{k}}} + |\overline{\underline{k}}| \sum_{\underline{k}_1 + \underline{k}_2 = \overline{\underline{k}}} |\gamma^R_{\underline{k}_1}| \cdot |\gamma^R_{\underline{k}_2}| \le \\ &\le -\nu |2\overline{\underline{k}}|^2 2 \cdot e^{-|\overline{\underline{k}}|\varepsilon \overline{t}} \frac{C(T)}{|\overline{\underline{k}}|^3} + |\overline{\underline{k}}|\, e^{-|\underline{k}_1|\varepsilon \overline{t}} \cdot e^{-|\underline{k}_2|\varepsilon \overline{t}} \cdot \frac{2C(T)}{(|\underline{k}|/2)^3} C(T) \cdot 4\, B \end{aligned} \tag{3.2.46}$$

because if $\underline{k}_1 + \underline{k}_2 = \overline{\underline{k}}$ then either $|\underline{k}_1| > |\overline{\underline{k}}|/2$ or $|\underline{k}_2| > |\overline{\underline{k}}|/2$.

Therefore the first term in the r.h.s., since $|\overline{\underline{k}}| \le |\underline{k}_1| + |\underline{k}_2|$, will be a quantity which for $|\overline{\underline{k}}| > K_0$ is less than $-\frac{1}{2}\, |\overline{\underline{k}}|\, \nu\, |\gamma^{(i,j)R}_{\overline{\underline{k}}}| < 0$, because K_0 was defined just to make this true.

Hence the derivative of $\gamma^{(i,j)R}_{\underline{k}}$ is less than the "speed of contraction of the boundary" of the region where (3.2.45) is satisfied for all $|\underline{k}| < R$. Therefore for every R, $\overline{t}_R = T$. Since the regularized solution converges *for every* $\underline{k}$, as $R \to \infty$, to the solution of NS equations, the latter solution verifies (3.2.45) *for all* $\underline{k}$. Hence it is analytic.

Remark: (Non smooth case) Hence, if $d = 2$, the weak solutions with data in $L_2(\Omega)$ also become analytic at positive time because by proposition IV they become immediately C^∞.

Of course the size ε of the "strip of analyticity" depends on the prefixed T and tends to 0 as $T \to \infty$: this is quite different with respect to the "analyticity improving" nature of the heat equation or of the Stokes equation. Then one can ask if there are weaker regularity properties that the solution acquires for positive time, say for $t > t_0 > 0$, and that do not depend on T.

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[5] The choice of the factor 2 is arbitrary: a constant > 1 would be equally suitable for our purposes.

A (quite weak) regularization result of this type can be obtained independently of the proposition VIII and it follows immediately from the autoregularization estimates

Proposition IX *(a C^∞–regularity estimate for NS in $d = 2$): Let $\underline{u}^0 \in L_2$, $d = 2$, and let $\underline{u}(t)$ a corresponding C-weak*[6] *solution of the NS equation. There exist two functions $H(t), h(t)$ finite and non increasing for $t > 0$, divergent as $t \to 0$, such that*

$$|\gamma_{\underline{k}}(t)| \le H(t_0) \cdot e^{-(\log|\underline{k}|)^2/h(t_0)}, \qquad \text{for all} \quad t \ge t_0 > 0 \qquad (3.2.47)$$

Proof: (sketch) In fact the constant C_α in proposition III can be estimated (by (3.2.21), and we leave this as a problem) as $C_{\alpha+1} < \alpha! B' + 2^\alpha B''\, C_\alpha$ for some constants B', B'' and $\alpha > 0$. This immediately gives: $C_\alpha \le \alpha!\, 2^{\alpha^2} D$ for some $D > 0$, and consequently (as in deriving (3.2.23)) the estimate

$$|\gamma_{\underline{k}}(t)| \le H(t_0) \cdot e^{-(\log|\underline{k}|)^2\, h(t_0)^{-1}}, \qquad \text{for all} \quad t \ge t_0 > 0 \qquad (3.2.48)$$

where $H(t_0), h(t_0) < +\infty$ if $t_0 > 0$ are constants depending on the initial conditions and on $\nu, \underline{g}$, *but not on time* $t > t_0$.

Noting that $\exp -(\log|\underline{k}|)^2/h$ tends to 0 as $|\underline{k}| \to \infty$ faster than any power, this means that the C-weak solution condidered here (unique at least if $\underline{u}^0 \in W^1(\Omega)$ by proposition VI) has a Fourier transofrm that acquires a C^∞ regularity expressed quantitatively by the parameters H, h in (3.2.47) at any time $t_0 > 0$ and keeps it forever *with the same parameters* (at least).

Problems

[3.2.1]: Estimate the function B_α, described in remark (1) following the proof of proposition III, in the cases $d = 2, \alpha_0 = 2$ and $d = 3, \alpha_0 = 2$ for $\alpha = \alpha_0 + 1, \alpha_0 + 2$.

[3.2.2]: Estimate the constants B_α in (3.2.20).

[3.2.3]: Prove proposition VII without assuming the absence of the external force; determine the $\underline{g}$–dependence of the constants V_q, M.

Bibliography: This section is based on my lecture notes (see [Bo79]) on the work [FP67], which discusses the above matters in a domain with boundary

[6] Recall that we have proved uniqueness of the weak solutions only under the slightly stronger condition $\underline{u}^0 \in W^1(\Omega)$.

and no slip boundary conditions. Proposition VIII is, however, a recent result in [MS99].

§3.3 Regularity: partial results for the NS equation in $d = 3$. The theory of Leray.

The theory of §3.2 is very unsatisfactory in the 3–dimensional case, because it is nonconstructive and lacks a uniqueness theorem.

The consequent ambiguity *only* consists in having been found as a limit of an unspecified subsequence extracted from a family of approximate solutions and therefore one can fear that by selecting different subsequences one obtains different solutions. It also consists in the *possibility* that there are other regularizations which, through a limiting procedure, could lead to yet other solutions.

We shall consider a cubic container Ω with side L and periodic boundary conditions, and we shall fix units so that the density is $\rho = 1$. Therefore there are two scales intrinsically associated with the system: a time scale T_c and a velocity scale V_c:

$$V_c \equiv \frac{\nu}{L}, \qquad T_c \equiv \frac{L^2}{\nu} \tag{3.3.1}$$

where ν is the kinematic viscosity. It will be natural to call them together with the length scale L, the scales "characteristic of the geometry of the system"

(A) Leray's regularization.:

An interesting regularization, different from the one by *cut–off* used so far is *Leray's regularization*.

Let $\underline{x} \to \chi(\underline{x}) \ge 0$ be a C^∞ function defined on Ω and not vanishing in a small neighborhood of the origin and with integral $\int \chi(\underline{x})\, d\underline{x} \equiv 1$: the function $\chi(\underline{x})$ can be regarded as a periodic function on Ω or as a function on R^3 with value 0 outside Ω, as we shall imagine that Ω is centered at the origin, to fix the ideas. For $\lambda \ge 1$ also the function $\chi_\lambda(\underline{x}) \stackrel{def}{=} \lambda^3 \chi(\lambda \underline{x})$ can be regarded as a periodic function on Ω or as a function on R^3: it is an "approximate Dirac's δ–function".

It will be relevant to point out that there is a simple relation between the Fourier transforms of the function χ regarded as defined on Ω and its Fourier transform when it is regarded as a function on R^3: namely if $\hat\chi(\underline{k})$, $\underline{k} \in R^3$ is the Fourier transform of χ as defined on R^3, then the Fourier tranform of χ regarded as a function on Ω is simply $\hat\chi(\underline{k})$ evaluated at $\underline{k} = \underline{n}\, k_0$ with $k_0 = 2\pi L^{-1}$ and $\underline{n}$ is an integer components vector (just write the definitions of the Fourier transforms to check this statement). Note that $\hat\chi(\underline{k})$ decreases faster than any power as $|\underline{k}| \to \infty$.

We shall examine the regularization with (dimensionless) parameter λ of the NS equation "*in the sense of Leray*" defined as

$$\dot{\underline{u}} = \nu\Delta\underline{u} - \langle\underline{u}\rangle_\lambda \cdot \underline{\partial}\,\underline{u} - \underline{\partial}p + \underline{g}, \qquad \underline{\partial}\cdot\underline{u} = 0, \qquad \int_\Omega \underline{u}\,d\underline{x} = \underline{0} \tag{3.3.2}$$

where $\langle\underline{u}\rangle_\lambda \equiv \int_{R^3} \chi_\lambda(\underline{y})\,\underline{u}(\underline{x}+\underline{y})\,d^3\underline{y}$, and $\underline{u}, \underline{g}$ are divergenceless fields. The volume force $\underline{g}$ will be assumed time indpendent (for simplicity).

In (3.3.2) we see that the approximation corresponding to the regularization consists in having the velocity field at a point $\underline{x}$ no longer "transported", as it should, by the velocity field itself as in the Euler and NS equations, but *rather by the average of the velocity* on a region of size of the order of λ^{-1} around $\underline{x}$.

Rewrite (3.3.2) as an equation for the Fourier components $\underline{\gamma}_{\underline{k}} \equiv \overline{\underline{\gamma}}_{-\underline{k}}$ of the velocity field (*c.f.r.* §2.2)

$$\dot{\underline{\gamma}}^\lambda_{\underline{k}} = -\nu\,\underline{k}^2\,\underline{\gamma}^\lambda_{\underline{k}} - i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \hat{\chi}\big(\underline{k}_1\lambda^{-1}\big)\underline{\gamma}^\lambda_{\underline{k}_1}\cdot\underline{k}_2\,\Pi_{\underline{k}}\,\underline{\gamma}^\lambda_{\underline{k}_2} + \underline{g}_{\underline{k}} \tag{3.3.3}$$

where $\hat{\chi}(0) = 1$ e $\underline{k}\cdot\underline{\gamma}^\lambda_{\underline{k}} = 0$, $\underline{\gamma}^\lambda_{\underline{0}} \equiv \underline{g}_{\underline{0}} \equiv \underline{0}$ and the functions $\underline{\gamma}_{\underline{k}}(0) \equiv \underline{\gamma}^0_{\underline{k}}$ e $\underline{g}_{\underline{k}}$ are assigned as Fourier transforms (with the conventions fixed in (2.2.2))

$$\underline{\gamma}^0_{\underline{k}} = L^{-3}\int_\Omega e^{-i\underline{k}\cdot\underline{x}}\,\underline{u}^0(\underline{x})d\,\underline{x}, \qquad \underline{g}_{\underline{k}} = L^{-3}\int_\Omega e^{-i\underline{k}\cdot\underline{x}}\,\underline{g}(\underline{x})d\,\underline{x} \tag{3.3.4}$$

of a datum $\underline{u}^0 \in C^\infty(\Omega)$ and of the intensity of external force $\underline{g} \in C^\infty(\Omega)$. We shall use the convention on the Fourier transform so that $\underline{u}^0 = \sum_{\underline{k}} \underline{\gamma}^0_{\underline{k}} e^{i\underline{k}\cdot\underline{x}}$, hence $\sum_{\underline{k}} |\underline{\gamma}^0_{\underline{k}}|^2 = L^{-3}\int |u^0(\underline{x})|^2\,d\underline{x}$. Both $\underline{u}$ and its Fourier transform $\underline{\gamma}$ have the dimension of a velocity.

Remarks:

(1) The (3.3.3) is similar to the truncation regularizations considered in the previous section, because the large values of $\underline{k}_1$, $|\underline{k}_1| \gg \lambda$, are "*suppressed*" in (3.3.3). Nevertheless (3.3.3) *does not* reduce to the regularization by truncation, not even in the case in which $\hat{\chi}(\underline{k}_1/\lambda)$ is chosen as a characteristic function of the set of $\underline{k}$'s with $|\underline{k}| < \lambda L^{-1}$: indeed with such a choice of $\hat{\chi}$ the vectors $\underline{k}_2$ and $\underline{k}$ in (3.3.3) remain free and only $|\underline{k}_1|$ is forced to be $\le \lambda$. In the equations obtained by truncation (for the NS equation) discussed in the preceding section, instead, also $|\underline{k}_2|$ and $|\underline{k}|$ are constrained to be $\le \lambda$ (λL^{-1} correspond to R of the preceding section).

(2) We have not, however, chosen χ with a $\hat{\chi}$ being a characteristic function of a sphere not just because the corresponding $\chi(\underline{x})$ would have a so slow decrease at ∞ to make improper the integral defining the $\langle\underline{u}\rangle_\lambda(\underline{x})$; but also, and mainly, because the following inequality (fundamental to the theory) would not be generally true

$$|\langle\underline{u}\rangle_\lambda(\underline{x})| \le \max_{\underline{y}\in\Omega} |\underline{u}(\underline{y})| \tag{3.3.5}$$

With our choice of χ as an approximate δ–function the (3.3.5) is correct and simple to check: *but* it is based upon the positivity of the function $\chi(\underline{y})$ and on the fact that its integral over the whole space is equal to 1.

(B) Properties of the regularized equation and new weak solutions.

The theory of (3.3.3) at λ fixed is very simple. The key is that if $||\underline{\gamma}||_2 < \infty$ then, for each $\lambda > 1$ and each sequence $\{\underline{\gamma}_{\underline{k}}\}$ an identity, that we have already seen to be the root of the energy conservation relation, remains true in the form

$$\sum_{\substack{\underline{k}_1+\underline{k}_2+\underline{k}_3=0\\ |\underline{k}_i|\le N}} \hat{\chi}\big(\underline{k}_1\lambda^{-1}\big)\,\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2\,\underline{\gamma}_{\underline{k}_2}\cdot\underline{\gamma}_{\underline{k}_3}=0 \tag{3.3.6}$$

for all integers N, because it is based only on the symmetry of this expression with respect to $\underline{k}_2$ and $\underline{k}_3$ and to the orthogonality between $\underset{\sim}{k}_2+\underset{\sim}{k}_3\equiv-\underset{\sim}{k}_1$ and $\underset{\sim}{\gamma}_{\underline{k}_1}$.

This means that one can envisage the same method of §3.2 to solve the equation (3.3.3) as a limit of solutions $\gamma^{\lambda,N}_{\underline{k}}(t)$ of the cut–off equations (here N is a cut–off parameter whose role is completely different from that of λ as the latter is necessary to make sense of the equations while N is introduced here as a technical tool to establish properties of the regularize equations and it will soon disappear) with $|\underline{k}|\le N$

$$\begin{aligned}\underline{\gamma}^{\lambda,N}_{\underline{k}}(t)=&\,e^{-\nu\underline{k}^2t}\underline{\gamma}^0_{\underline{k}}+\int_0^t e^{-\nu\underline{k}^2(t-\tau)}\Big(\underline{g}_{\underline{k}}-i\sum_{\substack{\underline{k}_1+\underline{k}_2=\underline{k}\\ |\underline{k}_i|\le N}}\hat{\chi}\big(\lambda^{-1}\underline{k}_1\big)\cdot\\ &\cdot\underset{\sim}{\gamma}^{\lambda,N}_{\underline{k}_1}(\tau)\cdot\underset{\sim}{k}_2\,\Pi_{\underline{k}}\gamma^{\lambda,N}_{\underline{k}_2}(\tau)\Big)\,d\tau\end{aligned} \tag{3.3.7}$$

In fact one proceeds as in §3.2 and the approach yields a solution verifying the "same" *a priori* estimates, discussed in (3.2.12) and (3.2.13)

$$\begin{aligned}&||\underset{\sim}{\gamma}^{\lambda,N}(t)||_2^2\le E_0L^{-3},\\ &\sum_{\underline{k}}\int_0^t d\tau\underline{k}^2|\gamma^{\lambda,N}_{\underline{k}}(\tau)|^2\le\frac{1}{2}\,E_0L^{-3}\,\nu^{-1}+t\sqrt{E_0L^{-3}}\,\nu^{-1}\;||\underline{g}||_2\end{aligned} \tag{3.3.8}$$

where $L^3||\underset{\sim}{\gamma}^{\lambda,N}(t)||_2^2=\int|\underline{u}^{\lambda,N}(t)|^2\,d\underline{x}$ and the regularization parameter N plays the same role of R in §3.2 and λ is, for the time being, kept fixed.

By the argument of proposition I of §3.2 the equation (3.3.8), in turn, implies existence of a limit as $N\to\infty$, $\gamma^{\lambda}_{\underline{k}}(t)$, possibly obtained on a subsequence $N_j\to\infty$, verifying the (3.3.7) and (3.3.3).

This time however the autoregolarization is stronger (because the (3.3.3) *is not* the NS equation). Indeed if $|\underline{k}|^\alpha|\gamma^{\lambda,N}_{\underline{k}}(t)| \le C_\alpha$ for all $\underline{k}$ and $|t| \le T$, then $\forall\ |t| \le T$:

$$|\gamma^{\lambda,N}_{\underline{k}}(t)| \le e^{-\nu\underline{k}^2 t}|\gamma^0_{\underline{k}}| + \frac{|g_{\underline{k}}|}{\nu\underline{k}^2} + \int_0^t d\tau\, |\underline{k}|e^{-\nu\underline{k}^2(t-\tau)}. \tag{3.3.9}$$

$$\cdot \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \hat\chi(\lambda^{-1}\underline{k}_1)\frac{C^2_\alpha}{|\underline{k}_1|^\alpha|\underline{k}_2|^\alpha} \le e^{-\nu\underline{k}^2 t}|\gamma^0_{\underline{k}}| + \frac{|g_{\underline{k}}|}{\nu\underline{k}^2} + \frac{C^2_\alpha B(\lambda)}{\nu|\underline{k}|^{\alpha+1}}$$

where $B(\lambda) = 2^{1+\alpha}k_0^{-\alpha}\sum_{\underline{k}}|\underline{k}|^{2\alpha}\hat\chi(\underline{k}\lambda^{-1}) < \infty$, if $k_0 = 2\pi L^{-1}$ is the minimum value of $|\underline{k}|$ (note that, obviously, $B(\lambda) \xrightarrow[\lambda\to\infty]{} \infty$).

Hence if $\underline{u}^0 \in C^\infty$, $\underline{g} \in C^\infty$ we see that there is $C_{\alpha+1} < \infty$ such that $|\gamma^{\lambda,N}_{\underline{k}}(t)| \le C_{\alpha+1}|\underline{k}|^{-\alpha-1}$, $\forall\ 0 \le t \le T$. Hence repeating the argument we see that $|\gamma^{\lambda,N}_{\underline{k}}(t)|\ |\underline{k}|^{\alpha'} \le C_{\alpha'}$, $\forall\ \alpha' \ge \alpha$, $t \le T$.

This means that all limits of convergent subsequences as $N \to \infty$ of $\underline{\gamma}^{\lambda,N}$ are C^∞, because the (3.3.8) trivially guarantees the validity of this estimate for $\alpha = 0$. And we find immediately, as in the case $d = 2$, that such solution of (3.3.3) is unique and therefore $\underline{\gamma}^\lambda = \lim_{N\to\infty}\underline{\gamma}^{\lambda,N}$, without need of considering subsequences (*c.f.r.* (3.2.37)).

But in trying to perform also the limit $\lambda \to \infty$ one risks loss of regularity obtaining only a weak solution, in the same sense of §3.2, as a limit on a suitable subsequence $\lambda_j \to \infty$.

The latter weak solution will still verify, by the same argument used in §3.2, the *a priori* estimates (3.3.8) or (3.2.12), with the same right hand side members, and the NS equation

$$\begin{aligned}\underline{\gamma}_{\underline{k}}(t) =&e^{-\nu\underline{k}^2(t-t_0)}\underline{\gamma}(t_0)_{\underline{k}} + \int_{t_0}^{t_0+t} e^{-\nu\underline{k}^2(t-\tau)}\Big(\underline{g}_{\underline{k}} - i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \\ &\underline{\gamma}_{\underline{k}_1}(\tau)\cdot \underset{\sim}{k}_2\,\Pi_{\underline{k}}\,\underline{\gamma}_{\underline{k}_2}(\tau)\Big)\,d\tau\end{aligned} \tag{3.3.10}$$

where $t_0 \ge 0$, *c.f.r.* (3.2.10).

Remark: Not having established a uniqueness theorem for weak solutions it is not necessarily true that such solutions coincide with the ones discussed in §3.2. Hence it should not be a surprise that *it will be possible* to prove that such new solutions enjoy properties that we *would not know* how to get for the others.

(C) The local bounds of Leray. Uniformity in the regularization parameter.

Let $t_0 \ge 0$ be an arbitrary time; let

$$J_0(t_0) = \sum_{\underline{k}}|\underline{\gamma}^\lambda_{\underline{k}}(t_0)|^2\ ,\quad J_1(t_0) = L^2\sum_{\underline{k}}|\underline{k}|^2|\underline{\gamma}^\lambda_{\underline{k}}(t_0)|^2\ ,\quad G_0 = \sum_{\underline{k}}|\underline{g}_{\underline{k}}|^2 \tag{3.3.11}$$

and introduce

$$D_m(t) = \sup_{\underline{x},|\underline{\alpha}|=m} |L^m \partial_{\underline{x}}^{\underline{\alpha}} \underline{u}^\lambda(\underline{x},t)|, \qquad J_m(t) = L^{2m}\sum_{\underline{k}} |\underline{k}|^{2m} |\underline{\gamma}_{\underline{k}}^\lambda(t)|^2 \tag{3.3.12}$$

noting that the quantities $D.$ have dimension of a velocity while $J.$ of velocity squares and $\sqrt{G_0}$ is an acceleration.

It is useful to define, for the purpose of a clearer formulation of Leray's theory, the *Reynolds' number*. Given the importance of this notion we give a formal definition

1. Definition *(Reynolds number): If $\underline{u}$ is a velocity field in Ω we define the "Reynolds number" R and the "dimensionless strength" R_g of the external force in terms of "geometric" scales of velocity and time, c.f.r. (3.3.1), and of the velocities $V_1 \equiv \sqrt{J_1}$ and $W_0 = T_c\sqrt{G_0}$ which we shall call, respectively, the "velocity variation scale" of the field $\underline{u}$ and the "external force" or "free fall" velocity scale. The definitions are*

$$R \equiv \frac{V_1}{V_c}, \qquad R_g = \frac{W_0}{V_c} \tag{3.3.13}$$

The quantities V_1, J_1, R are properties of the velocity field $\underline{u}$, while R_g is instead a function of the density of external force.

In particular R depends on time t if $\underline{u}$ does; and $R < \infty$ is tantamount to saying $\underline{u} \in W^1(\Omega)$. We show that

I. Proposition *(regularization independent a priori bounds): Let $\underline{u}^0, \underline{g} \in C^\infty(\Omega)$. There exist λ–independent dimensionless constants $F < 1$, F_m, $m = 0, 1, 2\ldots$ such that if $t \to \underline{u}^\lambda(t)$ is a solution of the regularized Navier Stokes equations (3.3.2) which at a time $t_0 \ge 0$ has Reynolds number $R(t_0) < \infty$ then*

$$\begin{aligned} D_m(t) \le& \Big(\frac{V_c R(t_0)}{\sqrt[4]{L^{-2}\nu(t-t_0)}} + W_0\Big)\frac{F_m}{(L^{-2}\nu\,(t-t_0))^{\frac{m}{2}}}, \\ R(t)^2 \le& 8(R(t_0)^2 + R_g^2) \end{aligned} \tag{3.3.14}$$

for all $t \in [t_0, t_0 + T_0]$, with T_0 given by

$$T_0 = F\,\frac{T_c}{R(t_0)^4 + R_g^2 + 1} \tag{3.3.15}$$

with R_g defined in (3.3.13).

Remark: Note that while the velocities D_m depend on λ their bounds, together with the corresponding time of validity, *do not depend on* λ. They are therefore bounded in terms of the Reynolds number at the instant t_0 only. But in general the latter depends on λ, *except when* $t_0 = 0$.

proof: Rewrite (3.3.7) in "position space", *i.e.* for $\underline{u}^\lambda(\underline{x},t)$. We find

$$\begin{aligned} u_j^\lambda(\underline{x},t) =& \int_\Omega d\underline{y}\,\Gamma(\underline{x}-\underline{y},t-t_0)\,u_j^\lambda(\underline{y},t_0)+\\ &+\int_{t_0}^t d\tau\int_\Omega \Gamma(\underline{x}-\underline{y},\tau-t_0)\,g_j(\underline{y},\tau)\,d\underline{y}+\\ &-\sum_{p,h=1}^3\int_{t_0}^t d\tau\int_\Omega d\underline{y}\,\partial_p T_{jh}(\underline{x}-\underline{y},t-\tau)\,\langle u_p^\lambda(\underline{y},\tau)\rangle_\lambda\,u_h^\lambda(\underline{y},\tau)\end{aligned} \tag{3.3.16}$$

where we have set

$$\begin{aligned}\Gamma(\underline{x},t) =&L^{-3}\sum_{\underline{k}} e^{-\nu\underline{k}^2 t}e^{i\underline{k}\cdot\underline{x}},\\ T(\underline{x},t)_{ij} =&L^{-3}\sum_{\underline{k}\ne\underline{0}} e^{-\nu\underline{k}^2 t}\big(\delta_{ij}-\frac{k_ik_j}{|\underline{k}|^2}\big)e^{i\underline{k}\cdot\underline{x}}\end{aligned} \tag{3.3.17}$$

which are Green's functions for the heat equation on the torus Ω of side L.
The following properties of Γ, T for $t>0$ or for $t=0$ will be used

$$\begin{aligned} &(\partial_t-\nu\Delta)\Gamma(\underline{x},t)=0, &&\Gamma(\underline{x},0)=\delta(\underline{x})\\ &(\partial_t-\nu\Delta)T(\underline{x},t)=0, &&T_{jh}(\underline{x},0)=\delta(\underline{x})\delta_{jh}-\partial_j\partial_h G(\underline{x})\end{aligned} \tag{3.3.18}$$

here $\delta(\underline{x})$ is Dirac's delta and $G(\underline{x})$ is the Green's function of the Laplace operator on the torus Ω. The latter can be expressed, via the images method (*c.f.r.* problems [2.3.12]÷[2.3.14]) and [3.1.12], as

$$\begin{aligned} G(\underline{x}) =& -\frac{1}{4\pi}\sum_{\underline{n}}\big\{\frac{1}{|\underline{x}+\underline{n}L|}-\frac{1}{|\underline{n}L|}+\frac{\underline{x}\cdot\underline{n}L}{|\underline{n}L|^3}+\\ &+\frac12\frac{1}{|\underline{n}L|^3}\big(\underline{x}^2-3\frac{(\underline{x}\cdot\underline{n}L)^2}{|\underline{n}L|^2}\big)\big\}=\\ \equiv& \lim_{N\to\infty}-\frac{1}{4\pi}\sum_{|\underline{n}|\le N}\big(\frac{1}{|\underline{x}+\underline{n}L|}-\frac{1}{|\underline{n}L|}\big),\end{aligned} \tag{3.3.19}$$

where $\underline{n}=(n_1,n_2,n_3)$ is an integer components vector.[1]
The Γ, T can also be computed by the method of images (from (3.3.18))

$$\begin{aligned} \Gamma(\underline{x},t) =& \sum_{\underline{n}}\frac{e^{-(\underline{x}+\underline{n}L)^2/4\nu t}}{(4\pi\nu t)^{3/2}}\\ T_{jn}(\underline{x},t) =& \Gamma(\underline{x},t)\delta_{jn}+\partial_j\partial_n\int_\Omega G(\underline{x}-\underline{y})\Gamma(\underline{y},t)\,d\underline{y}\end{aligned} \tag{3.3.20}$$

[1] Note that the first formula in (3.3.19) implies the second because the first is absolutely convergent and therefore can be trivially obtained as the limit for $N\to\infty$ of the sum over $|\underline{n}|\le N$; but if the sum over $\underline{n}$ is restricted to the $\underline{n}$ such that $|\underline{n}|\le N$ then the linear terms in $\underline{x}$ and the quadratic ones as well add up to zero (for every N).

Finally Γ, T verify the following basic inequalities, for $\nu t \le L^2$

$$0 \le \Gamma(\underline{x}, t) \le \frac{C_0}{(\underline{x}^2 + \nu t)^{3/2}} \qquad \int_\Omega \Gamma(\underline{x}, t)\, d\underline{x} \equiv 1 \tag{3.3.21}$$

for a suitable C_0, and if $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, with $\alpha_j \ge 0$ integer, and $|\underline{\alpha}| \stackrel{def}{=} \alpha_1 + \alpha_2 + \alpha_3$, then

$$|\partial^{\underline{\alpha}}_{\underline{x}} \Gamma(\underline{x}, t)| \;,\; |\partial^{\underline{\alpha}}_{\underline{x}} T_{ij}(\underline{x}, t)| \le \begin{cases} C_{|\underline{\alpha}|}\, (\underline{x}^2 + \nu t)^{-(3+|\underline{\alpha}|)/2} & \text{for } \nu t \le L^2 \\ C_{|\underline{\alpha}|} L^{-(3+|\underline{\alpha}|)}\, (L^2/\nu t)^{|\underline{\alpha}|/2} & \text{for } \nu t > L^2 \end{cases} \tag{3.3.22}$$

See problems [3.3.6] and [3.3.9] for a check of the inequalities.

If the derivatives $\partial^{\underline{\alpha}} \Gamma$ and $\partial^{\underline{\alpha}} T$ are considered as convolution operators on $L_2(\Omega)$ which are, therefore, defined by

$$\begin{aligned} (\partial^{\underline{\alpha}} \Gamma * f)(\underline{x}) &= \int_\Omega \partial^{\underline{\alpha}} \Gamma(\underline{x} - \underline{y}, t) f(\underline{y})\, d\underline{y} \\ (\partial^{\underline{\alpha}} \underset{\sim}{\underline{T}} * \underline{f})(\underline{x}) &= \int_\Omega \partial^{\underline{\alpha}} \underset{\sim}{\underline{T}}(\underline{x} - \underline{y}, t) \underline{f}(\underline{y})\, d\underline{y} \end{aligned} \tag{3.3.23}$$

we bound immediately their sizes in $L_2(\Omega)$ by evaluating, via (3.3.17), the "norms" in $L_2(\Omega)$ via the Fourier transforms and we get, if $|\underline{\alpha}| = m$

$$||\partial^{\underline{\alpha}} \Gamma * \cdot||_2 \;,\; ||\partial^{\underline{\alpha}} T * \cdot||_2 \le \sup_{\underline{k}} |\underline{k}|^{|\underline{\alpha}|} e^{-\nu \underline{k}^2 t} \le \frac{B_m}{(\nu t)^{m/2}} \tag{3.3.24}$$

if B_m are suitable dimensionless constants.

Returning to (3.3.17) we recall that, besides the characteristic time $T_c = L^2/\nu$ and the characteristic velocity $V_c = \nu/L$, *c.f.r.* (3.3.13), we associated with the fluid the velocities $V_1 = J_1(t_0)^{1/2}$ and $W_0 = T_c\, G_0^{1/2}$. Then, in terms of these quantities $D_0(t)$, *c.f.r.* (3.3.22), can be estimated starting from (3.3.16) by

$$\begin{aligned} D_0(t) \le& \sum_{\underline{k}} \frac{e^{-\nu \underline{k}^2 (t - t_0)}}{|\underline{k}|} |\underline{k}|\; |\underset{\sim}{\gamma}{}^{\lambda}_{\underline{k}}(t_0)| + \int_{t_0}^t \sum_{\underline{k}} e^{-\nu \underline{k}^2 (\tau - t_0)} |g_{\underline{k}}|\, d\tau + \\ &+ C_1 \int_{t_0}^t d\tau \int_\Omega d\underline{y} \frac{D_0(\tau)^2}{(|\underline{x} - y|^2 + \nu(t - \tau))^2} \le \\ &\le F_0' \big(\frac{V_1}{\sqrt[4]{(t - t_0)/T_c}} + W_0 \big) + F_0'' \int_{t_0}^t \frac{D_0(\tau)^2}{\sqrt{(t - \tau)/T_c}} \frac{d\tau}{T_c V_c} \end{aligned} \tag{3.3.25}$$

where F_0' and F_0'' are suitable dimensionless constants and $\nu(t - t_0) \le L^2$.

Since $D_0(t)$ is the maximum of $\underline{u}^\lambda(\underline{x}, t)$ in $\underline{x}$ and since $\underline{u}^\lambda$ is C^∞ in $\underline{x}, t$, *because it solves the regularized equation*, for $t \ge t_0$, then $D_0(t)$ is continuous in t. Therefore we conclude the validity of the inequality

$$D_0(t) < 2 \big(\frac{V_1}{((t - t_0)/T_c)^{1/4}} + W_0 \big) F_0' \tag{3.3.26}$$

for $t - t_0 > 0$ small enough. This is simply the statement that $D_0(t)$ is continuous in t: *but* $t - t_0$ could have to be extremely smaal and it depends on λ in a, so far, uncontrolled way.

However, for $t < T_c$ the inequality (3.3.26) *will stay valid as long as the upper bound of* $D_0(t)$ *in (3.3.25), evaluated by replacing* $D_0(t)$ *with the bound (3.3.26), stays smaller than the right hand side of (3.3.26) itself*, as a moment of thought reveals.

Remark: the condition $t < T_c$ appears because the inequalities have been derived supposing $t - \tau < T_c$ (*i.e.* $\nu(t - \tau) < L^2$).

Therefore, if $t_0+\tau_0$ is the maximal value of $t > t_0$ up to which the inequality (3.3.26) holds, the (3.3.25) implies that the time τ_0 is not smaller than the maximum $\tau \geq 0$ for which

$$\begin{aligned}&\big((\tau/T_c)^{-1/4}\,V_1 + W_0\big)\,F_0' +\\ &+ 8F"_0 F_0'^2 \int_0^\tau \Big((\frac{T_c}{\tau'})^{1/2}\,V_1^2 + W_0^2\Big)\,(\frac{T_c}{\tau-\tau'})^{1/2}\,\frac{d\tau'}{T_cV_c} \leq \\ &\leq 2((\tau/T_c)^{-1/4}\,V_1 + W_0)\,F_0'\end{aligned} \tag{3.3.27}$$

Note in fact that the l.h.s. is an estimate of the r.h.s. of (3.3.25) obtained from it by replacing $D_0(t)$ by (3.2.29).

The (3.3.27) is, taking into account the τ–independence of the integral $\pi = \int_0^\tau [\tau'(\tau-\tau')]^{-1/2}\,d\tau'$, a consequence of

$$8F_0'F_0''\pi\frac{(V_1^2 + (\tau/T_c)^{1/2}W_0^2)V_c^{-1}}{V_1 + (\tau/T_c)^{1/4}W_0} \leq \frac{1}{(\tau/T_c)^{1/4}} \tag{3.3.28}$$

so that, setting $F' = 1/(8\pi F_0'F_0'')$ and $x \equiv (\tau/T_c)^{1/4}$, the condition is

$$x < F'\frac{V_c(V_1 + xW_0)}{V_1^2 + x^2W_0^2} \tag{3.3.29}$$

which, by $\frac{a+b}{c+d} \geq \min(\frac{a}{c}, \frac{b}{d}$ is implied by $x < F'\min\big(V_c/V_1,\, V_c/(xW_0)\big)$.

For simplicity we shall impose the validity of the last condition (cf. remark above) and of the $t < T_c$ by assuming the simpler (but more restrictive)

$$\tau \leq T_0 \stackrel{def}{=} F\,\frac{T_c}{\big(\frac{V_1}{V_c}\big)^4 + \big(\frac{W_0}{V_c}\big)^2 + 1} \equiv F\,\frac{T_c}{R^4 + R_g^2 + 1} \tag{3.3.30}$$

for F small enough, which has a conveniently simple form (for our purposes) in terms of dimensionless quantities.

Remark: The main feature of the definition of the estimate T_0 for τ_0 is of being a function of the Reynolds number R which is *independent of the* *cut–off parameter* λ. Of course the Reynolds number in general depends on λ unless $t_0 = 0$, as already noted).

A further consequence of the (3.3.16) and (3.3.24), that is obtained in a similar way to the one followed in deriving (3.3.25), is

$$J_1(t)^{1/2} \le J_1(t_0)^{1/2} + T_c G_0^{1/2} + \int_{t_0}^{t} D_0(\tau) \frac{B_1}{\sqrt{\nu(t-\tau)}} J_1(\tau)^{1/2}\, d\tau \quad (3.3.31)$$

This can also be bounded by the same kind of reasoning just presented for the bound on $D_0(t)$ in deriving (3.3.26), (3.3.27) and (3.3.30). One gets

$$J_1(t)^{1/2} \le 2(J_1(t_0)^{1/2} + T_c G_0^{1/2}), \qquad t \le T_0 \quad (3.3.32)$$

under the same condition for τ_0, (3.3.30), with a possibly different constant $\tilde F$ in place of F.

Hence we have obtained the existence of suitable dimensionless constants $\tilde F, \hat F$ such that

$$\begin{aligned} &\text{if} \qquad t \le T_0 = T_c \min\big(1, \frac{\tilde F}{R^4 + R_g^2}\big) \qquad \text{then :}\\ &D_0(t) \le \hat F \Big(V_1 \big(\frac{t-t_0}{T_c}\big)^{-1/4} + W_0 \Big) \stackrel{def}{=} d(t-t_0) \\ &J_1(t) \le 8\big(J_1(t_0) + T_c^2 G_0\big) \stackrel{def}{=} j(t_0) \end{aligned} \quad (3.3.33)$$

We remark, again, the λ–independence of the right hand side and that the function $d(\varepsilon)$ is decreasing with ε.

Thus the proposition is proved in the case $m = 0$. The $m > 0$ cases are treated analogously, or they follow as special cases from propositions IV,V,VI below.

(D) Local existence and regularity. Leray's local theorem.

Proposition I yields the following corollary

II. Proposition *(local existence, regulatity and uniqueness, (Leray)): There is a dimensionless constant $F > 0$ such that if $\underline{u}^0 \in L_2(\Omega)$ is a velocity field with Reynolds number $R < \infty$ (which is equivalent to $\underline{u}^0 \in W^1(\Omega)$) and if the dimensionless strength of the external force is R_g (in the sense of definition 1, (3.3.13)) then there is a weak solution with initial datum $\underline{u}^0$ at $t = t_0$, of the NS equations which verifies the following properties*

$$\begin{aligned} &(1) \quad \underline{u} \in C^\infty((t_0, t_0 + T_0) \times \Omega), \qquad T_0 = F \frac{T_c}{R^4 + R_g^2 + 1}\\ &(2) \quad ||\underline{u}(t) - \underline{u}^0||_2 \xrightarrow[t \to t_0]{} 0 \\ &(3) \quad \int_0^{T_0} D_0(\tau)^2\, d\tau < \infty \end{aligned} \quad (3.3.34)$$

where $D_0(t)$ *is the maximum* $\max_{\underline{x}} |\underline{u}(\underline{x},t)|$. *Two weak solutions enjoying properties 1,2,3 necessarily coincide. Finally the solution can be obtained via a constructive algorithm.*

Remarks: (1) We recall that "weak solution" means (*c.f.r.* definition 1 in §3.2) a function $t \to \underline{u}(\underline{x},t)$ with finite $L_2(\Omega)$–norm (*i.e.* finite kinetic energy) verifying a uniform estimate in every finite time interval and making (3.3.10) an identity, as well as (3.2.6), almost everywhere in $t > t_0$.
(2) If there is no force all Leray's solutions will become eventually smooth, see problem [3.3.4]. These are examples of several results of Leray on global existence (see also problem [3.3.5]). The conclusion is that we miss an existence and uniqueness theorem under *general* initial data.

proof: Setting $t_0 = 0$ property (1) follows immediately from the estimates (3.3.14), from the autoregularization property of proposition V of §3.2 and from the remark (3) to the latter proposition.
Furthermore from (3.3.7), and $\int_0^t e^{-\nu\underline{k}^2\tau}\,d\tau \le t^\varepsilon/(\nu\underline{k}^2)^{1-\varepsilon}$ for each $\varepsilon \in [0,1]$, we deduce for all $\lambda \ge 0$

$$\begin{aligned}|\underline{\gamma}^\lambda_{\underline{k}}(t) - e^{-\nu\underline{k}^2 t}\underline{\gamma}^0_{\underline{k}}| \le \frac{|\underline{g}_{\underline{k}}|t^\varepsilon}{(\nu\underline{k}^2)^{1-\varepsilon}}+\\ + \frac{t^\varepsilon}{(\nu\underline{k}^2)^{1-\varepsilon}}\frac{\sqrt{E_0L^{-3}}}{L}\;2\big(J_1(0)^{1/2} + T_cG_0^{1/2}\big)\end{aligned} \tag{3.3.35}$$

where the second term is obtained from (3.3.7) by bounding the sum proportionally to $||\underline{\gamma}||_2 \cdot ||\underline{k}\,\underline{\gamma}||_2$ (the $||\cdot||_2$ is defined in (3.2.4)), via the energy estimate (3.2.12) and the estimate (3.3.33). This shows (if $\varepsilon > 0$ is chosen so that $(1-\varepsilon)\cdot 4 > 3$) that

$$||\underline{\gamma}^\lambda(t) - \underline{\gamma}(0)||_2 \le \big(\sum_{\underline{k}} |\underline{\gamma}^0_{\underline{k}}|^2(1 - e^{-\nu\underline{k}^2 t})^2\big)^{1/2} + O(t^\varepsilon) \xrightarrow[t\to 0]{} 0 \tag{3.3.36}$$

hence $||\underline{\gamma}^\infty(t) - \underline{\gamma}(0)||_2 \le \lim_{\lambda_j\to\infty} ||\underline{\gamma}^{\lambda_j}(t) - \underline{\gamma}^0(0)||_2 \xrightarrow[t\to 0]{} 0$ and property (2) of (3.3.34) follows.
Property (3) is implied by the second of (3.3.33), saying that $D_0(\tau)$ diverges at most as $\tau^{-1/4}$ as $\tau \to 0$.
It remains to check uniqueness. Indeed, given two solutions $\underline{u}^1$ e $\underline{u}^2$ verifying (3.3.34), let $\Delta = ||\underline{\gamma}^1 - \underline{\gamma}^2||_2^2$ and $\Delta_1 = |||\underline{k}|\,(\underline{\gamma}^1 - \underline{\gamma}^2)||_2)$. Proceeding as usual, *c.f.r.* proposition VI of §3.2, we then get

$$\begin{aligned}\frac{d}{dt}\frac{\Delta}{2} &= -\nu\Delta_1 - \sum_{\underline{k}_1+\underline{k}_2+\underline{k}_3=0} i(\underset{\sim}{\gamma}^1_{\underline{k}_1}\cdot \underset{\sim}{k}_2\underline{\gamma}^1_{\underline{k}_2} - \underset{\sim}{\gamma}^2_{\underline{k}_1}\cdot \underset{\sim}{k}_2\underline{\gamma}^2_{\underline{k}_2})\cdot(\underline{\gamma}^1_{\underline{k}_3} - \underline{\gamma}^2_{\underline{k}_3}) =\\ &= -\nu\Delta_1 + \sum i(\underset{\sim}{\gamma}^1_{\underline{k}_1} - \underset{\sim}{\gamma}^2_{\underline{k}_1})\cdot\underset{\sim}{k}_3\underline{\gamma}^1_{k_2}\cdot(\underline{\gamma}^1_{\underline{k}_3} - \underline{\gamma}^2_{\underline{k}_3})+\\ &+ \sum i\underset{\sim}{\gamma}^2_{\underline{k}_1}\cdot\underset{\sim}{k}_2(\underline{\gamma}^1_{\underline{k}_2} - \underline{\gamma}^2_{\underline{k}_2})\cdot(\underline{\gamma}^1_{\underline{k}_3} - \underline{\gamma}^2_{\underline{k}_3}) =\end{aligned} \tag{3.3.37}$$

$$
\begin{aligned}
&= -\nu\Delta_1 + L^{-d}\int(\underline{u}^1-\underline{u}^2)\cdot[\underset{\sim}{u}^1\underset{\sim}{\partial}\cdot(\underset{\sim}{u}^1-\underset{\sim}{u}^2)]+\\
&+L^{-d}\int[\underset{\sim}{u}^2\cdot\underset{\sim}{\partial}(\underline{u}^1-\underline{u}^2)]\cdot(\underline{u}^1-\underline{u}^2)\ \le\ -\nu\Delta_1+\\
&+2D_0(t)\sqrt{\Delta_1}\sqrt{\Delta}\ \le\nu\Delta\max_{y>0}\big(-y^2+2\nu^{-1}D_0(t)y\big)\le\Delta\,\nu^{-1}D_0^2(t)
\end{aligned}
$$

hence

$$
\Delta(t)\le\Delta(0)\,e^{2\nu^{-1}\int_0^t D_0(\tau)^2\,d\tau} \tag{3.3.38}
$$

thus, since $\Delta(0)=0$, we see that the third of the (3.3.34) implies $\Delta(t)\equiv 0$, for $t\le T_0$.[2]

(E) Exceptionality of singularities. Global Leray's theorem. Leray's solutions.

2. Definition *(L-weak solutions): Consider solutions $\underline{u}^\lambda(t)$ of the Leray's regularized equation with C^∞ initial datum $\underline{u}(0)=\underline{u}^0$ and for all times t. Let $\lambda_j\to\infty$ be a sequence such that $\underline{\gamma}^{\lambda_j}_{\underline{k}}(t)$ converges, for all t and $\underline{k}$, to a weak solution $\gamma^\infty_{\underline{k}}(t)$ with Fourier transform $\underline{u}^\infty(\underline{x},t)$. Such "Leray's solutions" may be distinct from the C-weak solutions of §3.2: hence we shall call them "L-weak solutions".*

Keeping in mind (3.3.11) and the deinition 1 following it, let

$$
R^2(t)V_c^2\overset{def}{=}J_1(t)=\liminf_{j\to\infty}\,L^2\sum_{\underline{k}}|\underline{k}|^2|\underline{\gamma}^{\lambda_j}_{\underline{k}}(t)|^2 \tag{3.3.39}
$$

Note that all quantites do depend also on the sequence $\{\lambda_j\}$. Then the second of (3.3.8) gives

$$
\begin{aligned}
&\int_0^T R(t)^2\,\frac{dt}{T}\le\liminf_{j\to\infty}\int_0^T\ V_c^{-2}L^2\sum_{\underline{k}}|\underline{k}|^2|\underline{\gamma}^{\lambda_j}_{\underline{k}}(t)|^2\,\frac{dt}{T}\le\\
&\le\Big(\frac{L^2}{T}E_0L^{-3}\nu^{-1}+L^2\,\nu^{-1}\sqrt{E_0L^{-3}}\,||\underline{g}||_2\Big)V_c^{-2}<\infty
\end{aligned} \tag{3.3.40}
$$

hence $R(t)^2<\infty$ almost everywhere.
Let $\mathcal{E}_n=\{t|R(t)^2<n\}$ and set,*c.f.r.* (3.3.15)

$$
\tau_n=F\,\frac{T_c}{n^2+R_g^2+1} \tag{3.3.41}
$$

---

[2] Strictly speaking, *c.f.r.* definition 2, one gets $\Delta(t)\le\Delta(t_0)\exp 2\nu^{-1}\int_{t_0}^t D_0(\tau)^2\,d\tau$ for $t_0>0$ because $\underline{u}^i\in C^\infty$ for $t>t_0$; then it follows from (2) in (3.3.34) that also $\underline{u}^i(t_0)$ tend to the same limit as $t_0\to 0$, namely to $\underline{u}(0)$, in L_2: so that $\Delta(t_0)$ also tends to $\Delta(0)=0$ as $t_0\to 0$ and (3.3.38) follows.

The set $\cup_n \mathcal{E}_n$ has a zero measure complement. Call $t_1^{(n)}, t_2^{(n)}, \ldots$ a denumerable family of times in $\mathcal{E}_n$ such that the set $\tilde{\mathcal{E}}_n$:

$$\tilde{\mathcal{E}}_n \equiv \cup_{j=1}^{\infty} [t_j^{(n)}, t_j^{(n)} + \tau_n) \quad \text{coincides with} \qquad \cup_{t \in \mathcal{E}_n} [t, t + \tau_n) \tag{3.3.42}$$

and it is not difficult to see, by abstract thinking (*i.e.* no estimates are needed), that such a family exists, and the open set $\mathcal{E}_n^0 \stackrel{def}{=} \tilde{\mathcal{E}}_n \supset \cup_{t \in \mathcal{E}_n} (t, t + \tau_n)$ is contained in $\tilde{\mathcal{E}}_n$.

As a consequence of propositions I and II we see that $u^\infty(\underline{x}, t)$ is C^∞ on the set $\mathcal{E}_n^0$ and there it verifies, together with all its derivatives, estimates depending only on n and upon the distance of t from $\partial \mathcal{E}_n^0$.

Since $\cup_n \mathcal{E}_n^0$ differs by an at most denumerable set [3] from the set $\cup_n \tilde{\mathcal{E}}_n$ and $\cup_n \tilde{\mathcal{E}}_n \supset \cup_n \mathcal{E}_n$ it follows that $\cup_n \mathcal{E}_n^0$ is open and has a zero measure complement.

Therefore

III. Proposition *(L-weak solutions can only have sporadic singularities (Leray)): There exist weak solutions of the 3–dimensional NS equation which are C^∞ on an open set with a zero measure complement. All Leray's solutions enjoy this property.*

This is, in essence, the high point of Leray's theory. A few more substantially stronger properties have been very recently obtained: we shall analyze them in the following sections §3.4,§3.5.

(F) Characterization of the singularities. Leray–Serrin theorem.

A further corollary, where $\nu = 1$ for simplicity, is the following theorem (*Serrin*).

IV. Proposition *(velocity is unbounded near singularities): Let $\underline{u}(\underline{x}, t)$ be a weak solution of the NS equation verifying $||\underline{u}||_2^2 < E_0$ for $t > 0$ and*

$$\begin{aligned} \underline{u}(\underline{x}, t) = & \int_\Omega \Gamma(\underline{x} - \underline{y}, t)\, \underline{u}^0(\underline{y})\, d\underline{y} + \int_0^t d\tau \int_\Omega \Gamma(\underline{x} - \underline{y}, t - \tau)\, \underline{g}(\underline{y})\, d\underline{y} + \\ & - \int_0^t d\tau \int_\Omega \vec{\partial} \underset{\approx}{\underline{T}}\, (\underline{x} - \underline{y}, t - \tau)\, \vec{u}(\underline{y}, \tau)\, \underset{\sim}{\underline{u}}\, (\underline{y}, \tau)\, d\underline{y} \end{aligned} \tag{3.3.43}$$

Given $t_0 > 0$ suppose that $|\underline{u}(\underline{x}, t)| \le M$, $(\underline{x}, t) \in U_\rho(\underline{x}_0, t_0) \equiv$ sphere of radius $\rho < t_0$ around $(\underline{x}_0, t_0)$, for some $M < \infty$: then $\underline{u} \in C^\infty(U_{\rho/2}(\underline{x}_0, t_0))$.

Remarks:

(1) This means that the only way a singularity can manifest itself, in a weak solution (in the sense (3.3.43)) of the NS equations, is through a divergence

[3] Since $\mathcal{E}_n^0$ is defined in terms of open intervals $(t, t + \tau_n)$ rather than semiclosed $[t, t + \tau_n)$ the set $\mathcal{E}_n^0$ might fail to contain the points $\cup_{n,j} t_j^{(n)}$.

of the velocity field itself. For instance it is impossible to have a singular derivative *without* having the velocity itself unbounded. Hence, if $d \ge 3$ velocity discontinuities are impossible (and even less so shock waves), for instance. Naturally if $\underline{u}(\underline{x}, t)$ is modified on a set of points $(\underline{x}, t)$ with zero measure it remains a weak solution (because the Fourier transform, in terms of which the notion of weak solution is defined, does not change), hence the condition $|\underline{u}(\underline{x}, t)| \le M$ *for each* $(\underline{x}, t) \in U_\lambda(\underline{x}_0, t_0)$ can be replaced by the condition: *for almost all* $(\underline{x}, t) \in U_\rho(\underline{x}_0, t_0)$.
(2) It will become clear that the above result is not strong enough to overcome the difficulties of a local theory of regularity of the L–weak solutions. Therefore one looks for other results of the same type and it would be desirable to have results concerning regularity implied by *a priori* informations on the vorticity. We have already seen that bounded total vorticity implies regularity (*c.f.r.* §3.2 proposition 5): however it is very difficult to go really beyond; hence it is interesting to note that also other properties of the vorticity may imply regularity. A striking result in this direction, although insufficient for concluding regularity (if true at all) of L–weak solutions, is the Constantin–Fefferman theorem that we describe without proof in proposition VII in (G) below.

proof: This is a consequence of a few remarkable properties of the integrals, often called with the generic name of "*heat kernels*",

$$\begin{aligned} V(\underline{x}, t) =& \mathcal{P}' F \equiv \int_\Omega P(\underline{x} - \underline{y}, t - \tau) F(\underline{y}, \tau)\, d\underline{y} \\ V(\underline{x}, t) =& \mathcal{P} F \equiv \int_0^t \int_\Omega \partial_j P(\underline{x} - \underline{y}, t - \tau) F(\underline{y}, \tau)\, d\tau\, d\underline{y} \end{aligned} \tag{3.3.44}$$

where $P = \Gamma$ or $P = T$, see (3.3.20), as operators on a function F which is bounded by a constant M. The properties below can be regarded as establishing the analogue of the autoregularization property directly "in position space". They are expressed by the following propositions

V. Proposition *(regularization induced by heat kernels): There is a suitable function* K^0_t*, decreasing in* t *(hence bounded for* $t > 0$*) such that*

$$|V(\underline{x}, t)| + \frac{|V(\underline{x}, t) - V(\underline{x}', t)|}{(L^{-1}|\underline{x} - \underline{x}'|)^{1/2}} \le M\, K^0_t \qquad \forall\ \underline{x}, \underline{x}' \tag{3.3.45}$$

In other terms the operators in (3.3.44) transform bounded functions into Hölder continuous functions with exponent $1/2$.

See problems [3.3.10], [3.3.11] below for a proof, and furthermore

VI. Proposition*(stronger regularization induced by heat kernels): Likewise if* F *verifies (3.3.45) (with* F *in place of* V*) then* $\mathcal{P}F$ *and* $\mathcal{P}'F$ *are differentiable in* $\underline{x}$ *and, for a suitable* t*–decreasing (hence finite for* $t > 0$*)* K^1_t*, have derivatives bounded by*

$$|\partial_i V(\underline{x}, t)| \le K^1_t M \tag{3.3.46}$$

i.e. the operator in (3.3.44) transforms Hölder continuous functions into differentiable ones.

A proof is described in problem [3.3.12] below.

Accepting for the time being (3.3.45), (3.3.46), *c.f.r.* problems [3.3.11], [3.3.12], we can complete the proof of proposition IV.

Let $\chi = \chi_{\underline{x}_0,t_0}(\underline{y},\tau) \le 1$ be a C^∞ function vanishing outside the ball U_ρ and equal to 1 inside the ball $U_{3\rho/4}$. Since Γ, T are C^∞ if $\underline{\xi}^2 + \tau > 0$ the exspression (3.3.43) yields a C^∞ function for $(\underline{x},t) \in U_{\rho/2}$ if $\underline{u}$ is everywhere replaced, in the r.h.s., by $(1 - \chi_{\underline{x}_0,t_0})\underline{u}$.

Thus it remains to analyze (3.3.43) with $\underline{u}$ replaced, in the r.h.s., by $\chi_{\underline{x}_0,t_0}\,\underline{u}$. But the function $\chi\,\underline{u}$ is, by assumption, bounded by M so that by propositions V, VI (3.3.43) is differentiable in $\underline{x}$. Hence $\underline{u}$ is differentiable in $\underline{x}$, and still by (3.3.43) we see that $\underset{\sim}{\partial}\,\underline{u}$ is given by expressions similar to (3.3.43) as the new $\underline{x}$ derivatives of T, Γ can be "transferred", by integration by parts, on the functions $\underline{u}$. Hence by the same argument $\underset{\sim}{\partial}\,\underline{u}$ is differentiable in $\underline{x}$. Repeating the argument we see that $\underline{u}$ is infinitely differentiable in $U_{\rho/2}$. Knowing differentiability in $\underline{x}$ and differentiating (3.3.43) with respect to t one can integrate by parts and "transfer" the time derivatives acting on the kernels Γ, T into $\underline{x}$–derivatives of the functions $\underline{u}$ (which exist by the argument above) because $\partial_t\Gamma = \nu\,\Delta\Gamma$ and $\partial_t T = \nu\,\Delta\,T$. Hence $\underline{u}$ is C^∞ in t as well for $(\underline{x},t) \in U_{\rho/2}$.

(G) Vorticity orientation uncertainty at singularities.

It is interesting, see remark (2) in (F) above, to quote the following "vorticity based" regularity proposition (Constantin and Fefferman, *c.f.r.* [CF93]):

Proposition VII *(vorticity orientation is uncertain at singularities):*
Suppose that $\underline{u}(t)$ is a L–weak solution of the NS equations with periodic boundary conditions in a cubic container Ω with C^∞ external force $\underline{g}$ and initial datum $\underline{u}^0 \in C^\infty$. Let $\underline{\omega} = \underline{\partial} \wedge \underline{u}$ be its vorticity field. Let $\underline{u}^{(N)}$ be the sequence of approximating solutions with Leray regularization parameter N; let their vorticity be $\underline{\omega}^{(N)}$ and $\underline{\xi}^{(N)} = \underline{\omega}^{(N)}/|\underline{\omega}^{(N)}|$ be the its "direction". Suppose that there exist constants $X, \rho > 0$ such that if both $|\underline{\omega}^{(N)}(\underline{x},t)| > X$ and $|\underline{\omega}^{(N)}(\underline{x}+\underline{y},t)| > X$ then the projection of $\underline{\xi}^{(N)}(\underline{x}+\underline{y},t)$ on the plane orthogonal to $\underline{\xi}^{(N)}(\underline{x},t)$ is bounded above by $|\underline{y}|/\rho$ for all $0 \le t \le T$, where T is a positive time, and for all pairs $\underline{x},\underline{y}$, uniformly in the cut–off parameter N.
Then the solution is of class C^∞ in the time interval $[0,T]$.

Hence the "NS *solution is smooth unless in the (smooth) approximations vorticity at $\underline{x}$ changes wildly direction*" as $\underline{x}$ varies. This result is remarkable because it gives a regularity property under conditions involving the vorticity. The result does not apply to C-weak solutions (*c.f.r.* (3.2.11)).

As it will appear clearly in the following sections, particularly in §3.5, what is really missing in the theory of regularity of NS solutions is a proper way of taking into account that vorticity is transported (one also says "advected") by the fluid flow. Even the strongest regularity result, the CKN theorem, *c.f.r.* §3.5, gives detailed local information (unlike the above theorem that relies on an assumption that involves vorticities at all points at a given instant) but *without* referring to the validity of the Thomson theorem at zero voscosity.

(H) Large containers.

The theory just developed is dimensionally satisfactory only if the initial velocity field $\underline{u}_0$ and the density of force $\underline{g}$ are regular on a length scale L equal to that of the container.

Sometimes, however, one considers situations in which the length scale characteristic of the initial data, be it r, and of the forces is different from that of the container: *e.g.* "a vortex in the sea", see problem [3.3.8] below. In such cases it will be important to keep the roles of the two scales distinct.

Suppose that data and forces are "on scale" r: mathematically this means that data and forces are appreciably nonzero on a small sphere of radius r where their properties are simply described by parameters $\tilde{v}$ and $\tilde{g}$ that fix the order of magnitude. A precise way to formulate the latter property is to assume that $\underline{u}^0$ and $\underline{g}$ are analytic functions of $\underline{x}$ with holomorphy domain $|\mathrm{Im}\, x_i| < r$ and are bounded in this complex polistrip by $\tilde{v}$ and $\tilde{g}$, respectively, and furthermore that they are negligible ("small") for $|\mathrm{Re}\, x_i| > r$, (*c.f.r.* the final note in §1.2).

Then the relevant norms for the formulation of the theorems can be estimated as follows

$$\begin{aligned}
&\int_\Omega d\underline{x}\, |\underset{\sim}{\partial} \underline{u}^0(\underline{x})|^2 \sim \tilde{v}^2 r,\\
&\int_0^t d\tau \int_\Omega |\Gamma(\underline{x}-\underline{y}, t-\tau)\underline{g}(\underline{y})|\, d\underline{y} \sim \frac{\tilde{g} r^2}{\nu}
\end{aligned} \tag{3.3.47}$$

where the second quantity is what becomes of the W_0 in (3.3.26) in the proof of Leray's's bounds.[4] Hence the relevant quantities in the formulation of the results of Leray's theorem are

$$\begin{aligned}
&V_c = \frac{\nu}{L}, \quad T_c = \frac{L^2}{\nu}, \quad V_1 \sim \tilde{v}(\frac{r}{L})^{1/2}, \quad W_0 \sim \tilde{g}\frac{r^2}{\nu}\\
&R_g^2 \sim \tilde{g}\frac{r^2 L}{\nu^2}, \qquad R \sim \frac{\tilde{v}(Lr)^{1/2}}{\nu},
\end{aligned} \tag{3.3.48}$$

It is therefore useful to introduce "local" scales of velocity $\tilde{V}_c$ and time $\tilde{T}_c$ as well as the Reynolds number $\tilde{R}$ and the dimensionless strength number,

[4] To estimate (3.3.47) one can bound Γ with (3.3.21) and use the property of "smallness" of $\underline{g}$ outside the ball of radius r.

$\tilde R_g$

$$\tilde V_c = \frac{\nu}{r}, \qquad \tilde T_c = \frac{r^2}{\nu}, \qquad \tilde R = \frac{\tilde v r}{\nu}, \qquad \tilde R_g = \frac{\tilde g r^3}{\nu^2} \tag{3.3.49}$$

It follows that the characteristic time scale T_0 of the local solution of Leray and the corresponding velocity estimate $D_0(t)$ are

$$\begin{aligned} T_0 =& \bar F T_c\,(R^4 + R_g^2)^{-1} = \bar F \tilde T_c\,(\tilde R^4 + \tilde R_g^2)^{-1},\\ D_0(t) \le& F\tilde v\,\big((r^2\,(\nu(t-t_0))^{-1})^{1/4} + \tilde g r^2\,\nu^{-1}\big) \end{aligned} \tag{3.3.50}$$

which, as we should have expected, are *independent* from L and essentially the same that we would find if the container had side size $\sim r$.

Problems: *Further results in Leray's theory*

[3.3.1]: Check that there would be difficulties in showing that the Leray's solutions with initial data $\underline u^0 \in W^1(\Omega)$ (*i.e.* $J_1(0) < \infty$) coincide with the C-weak solutions of §3.2 with the same initial data. Show that they would coincide if the C-weak solutions verified property (2) in (3.3.34), *i.e.* took the initial value with continuity in $L_2(\Omega)$ as $t \to 0$. (*Idea:* it is not known whether the two notions coincide.)

[3.3.2]: (Leray's) Check that the technique used to obtain (3.3.25) can be adapted to show that if $E(t) = L^3 \sum_{\underline k} |\gamma_{\underline k}(t)|^2$ then there are B_1, B_2 such that

$$D_0(t) \le B_1\Big(D_0(0) \wedge \frac{V_1}{\sqrt[4]{t/T_c}} + W_0\Big) + B_2 \int_0^t \Big(\frac{D_0(\tau)^2}{\sqrt{(t-\tau)/T_c}} \wedge^* \frac{L^{-3}E(\tau)}{((t-\tau)/T_c)^2}\Big)\frac{d\tau}{T_c V_c} \tag{3.3.51}$$

where $a \wedge^* b$ means a if $\nu(t-\tau) \le L^2$ and $\min(a,b)$ otherwise. (*Idea:* Note that $\int_\Omega |\Gamma(\underline x, t)|\,d\underline x \le B_1$ for a suitable B_1 and for each t, by (3.3.21) and, furthermore, if $t - \tau \ge T_c$, by the Schwartz inequality and (3.3.22):

$$\sup_{\underline y} |\partial_{\underline y} \underline T(\underline y, t-\tau)| \int_\Omega d\underline y |\,\langle \underline u(\underline y, \tau)\rangle_\lambda|\,|\underline u(\underline y, \tau)| \le \frac{L^{-4}E(\tau)}{((t-\tau)/T_c)^{\frac12}} C_1 \tag{3.3.52}$$

where C_1 is a suitable constant. Furthermore

$$\begin{aligned} &\sum_{\underline k} e^{-\nu \underline k^2 t} |\underline\gamma_{\underline k}(0)| \le \sum_{\underline k} \frac{e^{-\nu \underline k^2 t}}{|\underline k|} |\underline k| |\underline\gamma_{\underline k}(0)| \le \\ &\le (\frac{J_1(0)}{L^{-1}})^{1/2} (\sum_{\underline k} \frac{e^{-\nu \underline k^2 t}}{\underline k^2})^{1/2} \le B \frac{V_1}{(t/T_c)^{1/4}} \end{aligned} \tag{3.3.53}$$

for a suitable B. Hence (3.3.51) follows.)

[3.3.3]: (*small initial data and global solutions for NS in $d = 3$, (Leray)*) Let $V_0 = D_0(0) + W_0$ and check that if $V_0/V_c < \delta$ is small enough then the solution exists for all times and $D_0(t)$ can be bounded proportionally to V_0. (*Idea:* If $E(t) = L^3 \sum_{\underline k} |\underline\gamma_{\underline k}(t)|^2$ the *a priori* energy estimate holds also in this case, see the comment preceding (3.3.10), and it will be $L^{-3}E(\tau) \le B_3 V_0^2$ for all τ (*c.f.r.* (3.2.12)). Then by the inequality in problem [3.3.2] it will be $D_0(t) \le 2B_1 V_0$ as long as it is

$$B_2 \int_0^t \frac{4B_1^2\,(1 + t/T_c) V_0^2}{\sqrt{(t-\tau)/T_c}} \wedge^* \frac{B_3 V_0^2}{((t-\tau)/T_c)^2} \frac{d\tau}{T_c V_c} < B_1 V_0 \sqrt{1 + t/T_c} \tag{3.3.54}$$

where B_2 is the constant of [3.3.2], *i.e.* dividing the integral between 0 and $t-T_c$ and between $t-T_c$ and t, as long as

$$\frac{V_0^2 B_2}{V_c}\Big(\int_0^{t-T_c}\frac{B_3}{((t-\tau)/T_c)^{\frac12}}\frac{d\tau}{T_c}+\int_{t-T_c}^t\frac{4B_1^2\sqrt{1+t/T_c}}{\sqrt{(t-\tau)/T_c}}\frac{d\tau}{T_c}\Big)<B_1V_0\sqrt{1+t/T_c} \tag{3.3.55}$$

where the first term should be omitted if $t\le T_c$ while, in this case, the second should be the integral over $[0,t]$.

Therefore there is a $B<\infty$ majorizing the sum of the integrals (*for every* t): and if $V_0<V_cB_1/(BB_2)$ (which implies $\delta\le B_1/(B\,B_2)$) the $D_0(t)\le 2B_1V_0\sqrt{1+t/T_c}$ will hold for all times: apply at this point the Leray's–Serrin theorem).

[3.3.4] (Leray): Supposing $\underline g=\underline 0$ show that if $\underline u^0\in L_2(\Omega)$ is an initial datum for the NS equations then there exists a time $\bar T$ large enough such that (all) the L-weak solutions with $\underline u^0$ as initial datum are of class C^∞ for $t>\bar T$. And $\bar T$ can be chosen

$$\bar T=B\,T_c\log_+\Big(E(0)L^{-3}/V_c^2\Big)^{-1} \tag{3.3.56}$$

where B is a suitable constant and $\log_+ x$ denotes the maximum betwen 1 and $\log x$. (*Idea:* Since $\underline g=\underline 0$ we have, as in (3.2.14) with $\underline\varphi=\underline 0$, $\dot E\le-\nu k_0^2E$ and $E(t)\le E(0)e^{-\nu k_0^2t}$, $k_0=2\pi L^{-1}$. Furthermore (3.2.13) give for all $T\ge0$ and using the definition (3.3.13)

$$\int_T^{T+T_c}\frac{d\tau}{T_c}\,J_1(\tau)\le\frac12\,\frac{E(0)L^{-1}}{\nu T_c}\,e^{-\nu k_0^2T}\stackrel{def}{=}V_c^2r_T^2 \tag{3.3.57}$$

Hence if $\tilde T$ is large enough and $T>\tilde T$ it is (*c.f.r.* (3.3.15))

$$T_c\,F\,r_T^{-4}>3T_c \tag{3.3.58}$$

and in every interval $[T-2T_c,T-T_c]$ of length T_c there will be a time t_0 where $J_1(t_0)<V_c^2r_T^2$, *i.e.* where the Reynolds' number $R(t_0)$ will be smaller than r_T, to the right of which the solution is bounded by (3.3.14). In particular, keeping in mind the definitions (3.3.1) and equation (3.3.14), it will be bounded in the interval $[T,T+T_c]$ by $D_0(t)\le V_c\,F_0\,r_{\tilde T}$ and by the arbitrariness of T the bound on $D_0(t)$ holds for all $t\ge\tilde T+2T_c$. Hence the regularized solution is bounded uniformly in the regularization parameter and for $t\ge\tilde T+2T_c$ provided (3.3.58) holds for $T>\tilde T=(4\nu k_0^2)^{-1}\log(\frac3F\big(E(0)L^{-1}/2\nu T_cV_c^2\big)^{-2})$, *i.e.* recalling the definition (3.3.1) we get (3.3.56) by proposition IV.)

[3.3.5]: (*a second global result*(Leray)) Supposing $\underline g=\underline 0$ show that if, for p large enough,

$$\Big(\frac{(L^{-3}E(0))^{1/2}}{V_c}\Big)^{p-3}\Big(\int_\Omega\Big|\frac{\underline u(\underline x,0)}{V_c}\Big|^p\frac{d\underline x}{L^3}\Big)^{1/p}<\varepsilon$$

and ε is small enough then there exists a unique C^∞ solution defined for all times. (*Idea:* Similar to problem [3.3.2].)

[3.3.6]: Consider an approximate δ–function as in (A) $N^d\chi(N\underline x)$ and define $\langle\underline u\rangle_N(\underline x)$ as $\int_{R^d}N^d\chi(N\underline y)\underline u(\underline x+\underline y)\,d\underline y$, interpreting $\underline u$ as extended to the entire space R^d by defining it a speriodic with period L in all the d coordinate directions. We call *Euler equation with Leray's regularization* the equation

$$\dot{\underline u}+\langle\underline u\rangle_N\cdot\underline\partial\,\underline u=-\underline\partial p+\underline g,\qquad \underline\partial\cdot\underline u=0$$

Adapt the analysis of the problems of §3.1 to check that if $\underline{u}^N$ is a solution of the regularized (with Leray's regularization) Euler equation then, for $\underline{g} = \underline{0}$

$$\frac{1}{2}\frac{d}{dt}||\underline{u}^N||^2_{W_m} \le G_m ||\underline{u}^N||^3_{W_m}$$

with G_m independent from N. (*Idea:* This is the "same" as the corresponding inequality for the Euler equation, see (3.1.2), (3.1 .3). Differentiate $\underline{m}$ times Euler equation and multiply both sides by $\partial^{\underline{m}}\underline{u}$; sum over $\underline{m}$ with $|\underline{m}| \le m$ and integrate over $\underline{x}$. One should note that the conclusions of [3.3.6] (and [3.3.2]) hold also if $\underset{\sim}{u}$ is replaced by $\langle \underline{u} \rangle_N$ because the differentiations commute with the averaging operation and the modulus of an average is majorized by the average of the moduli *because* χ is nonnegative).

[3.3.7]: Combine the analysis of the problems in §3.1 with the ideas of this section and with [3.3.6] to derive an *a priori* estimate on the kinetic energy $E(t)$, of $\underline{u}^N$, which is independent of N (so that $E(t)^{1/2} \le E(0)^{1/2} + ||\underline{g}||_{L_2} t$). And to show that the Leray regularized Euler equation of [3.3.6] has a solution C^∞ in $t, \underline{x}$ for each initial datum $\underline{u}^0 \in C^\infty$,

[3.3.8]: Consider a vortex in R^3, regarded as motion of water in normal conditions with velocity

$$\underline{u} = \frac{1}{2}\underline{\omega}(\underline{r}) \wedge \underline{r}, \qquad \underline{\omega}(\underline{r}) = \Gamma e^{-\underline{r}^2/r_0^2}\,\underline{k}$$

where r_0 is a length scale and $\underline{k}$ is the unit vector of the z axis; Γ is an inverse time scale. Suppose that no volume force acts on the fluid. Compute:
(1) The Reynolds number (in the sense of Leray's, (3.3.13), and in the sense discussed for "large containers" in (H) above).
(2) Assuming that the vortex has radius $r = 1.\,m$ with $\Gamma = 1.\,s^{-1}$ estimate the time of existence of the local Leray's solution (refer to the theory of large containers in (H)).
(3) Estimate how large should the time scale Γ^{-1} (*c.f.r.* [3.3.3]) be to be sure of the existence of a global solution, given that the length scale is $r = 1.m$?
(*Idea:* Compute the constants of Leray's theory and apply it, via the extension to large containers. The kinematical viscosity of water is $\nu = 0.01cm^2/sec$ and its density is $1.g/cm^3$ (normal conditions 4^oC, $1\,atm$).

[3.3.9]: Check that $\Gamma(\underline{x}, t)$ defined in (3.3.17) is:

$$\Gamma(\underline{x}, t) = \frac{e^{-\underline{x}^2/4t\nu}}{\sqrt{4\pi\nu t}^3} + \sum_{\underline{n}\ne 0} \frac{e^{-(\underline{x}+\underline{n}L)^2/4t\nu}}{\sqrt{4\pi\nu t}^3} \tag{3.3.59}$$

and show that, therefore, it suffices to prove the (3.3.22) for $|\underline{x}|$ and $|t|$ small. Note that it is necessary to check only the first term, the others are C^∞ corrections (*c.f.r.* problem [2.3.12])). (*Idea:* By (3.3.18) Γ is the heat equation kernel (on the torus) and the first term in (3.3.59) is the heat equation kernel in R^3; all terms are regular near the origin except the first.)

[3.3.10]: *(properties of heat kernels)* Let, in this and in the following problems, $\nu = 1$ for simplicity. Note that (3.3.22) can be studied for $\nu t < L^2$ by assuming alternatively that $x^2 \le t$ and $x^2 > t$ and showing that in the first case it is: $|\partial^{\underline{\alpha}} P| \le C_{\underline{\alpha}} (\nu t)^{-(3+|\underline{\alpha}|)/2}$, where $P = \Gamma$ or $P = T$; and in the second: $|\partial^{\underline{\alpha}} P| \le C_{\underline{\alpha}}\, (\underline{x}^2)^{-(3+|\underline{\alpha}|)/2}$, if $C_{\underline{\alpha}}$ is a suitable constant. For $\nu t > L^2$ the r.h.s. is replaced by $C_{\underline{\alpha}} L^{-(3+|\underline{\alpha}|)} (L^2/\nu t)^{|\alpha|/2}$.
(*Idea:* In the case of Γ this is a simple direct check. So we can assume that the (3.3.22) holds for Γ. Note that the second term in T (*c.f.r.* (3.3.20)) is not proportional to Γ; it can, however, be thought of as the convolution product between the derivatives $\partial_i\partial_j G$ of the Green's function of the Laplace operator and Γ, *c.f.r.* (3.3.22). Then study

$$\partial_i\partial_j G * \Gamma(\underline{x}, t) \equiv \partial_i\partial_j \int_\Omega G(\underline{x} - \underline{y})\Gamma(\underline{y}, t)\, d\underline{y} \tag{3.3.60}$$

with $|\underline{x}|$, $|t|$ small and consider only the term $\underline{n}=0$ in (3.3.59), because the others contribute a correction of class C^∞ in $\underline{x}$. Again we can replace $G(\underline{x})$ with $-1/4\pi|\underline{x}|$ since $G(\underline{x})\equiv-\frac{1}{4\pi|\underline{x}|}+\gamma(\underline{x})$ and $\gamma(\underline{x})$ is of class C^∞ for $|\underline{x}|$ small and with derivatives trivially bounded proportionally to $L^{-3-|\underline{\alpha}|}$, *c.f.r.* problems [2.3.12]÷[2.3.14].

Hence, if $t\ge\underline{x}^2$ and if the constants $C^j, C_{|\underline{\alpha}|+2}$ are suitably chosen we use the bound (3.2.22) for Γ

$$\begin{aligned}&|\partial^{\underline{\alpha}}\partial_i\partial_j\int\frac{1}{|\underline{x}-\underline{y}|}\frac{e^{-\underline{y}^2/4t}}{\sqrt{4\pi t}^3}\,d\underline{y}|\le C_{|\underline{\alpha}|+2}\int\frac{1}{|\underline{x}-\underline{y}|}\frac{d\underline{y}}{(\underline{y}^2+t)^{(5+|\underline{\alpha}|)/2}}\\&\le C_{|\underline{\alpha}|+2}\Big(\int_{|\underline{y}|<2|\underline{x}|}\cdot+\int_{|\underline{y}|>2|\underline{x}|}\cdot\Big)\le C^1\frac{1}{t^{(5+|\underline{\alpha}|)/2}}|\underline{x}|^2+\int_{|\underline{y}|>2|\underline{x}|}\cdot\le\\&\le C^2\Big(\frac{1}{t^{(3+|\underline{\alpha}|)/2}}+\int\frac{d^3\underline{y}}{|\underline{y}|(\underline{y}^2+t)^{(5+|\underline{\alpha}|)/2)}}\Big)\le\frac{C^3}{t^{(3+|\underline{\alpha}|)/2}}\end{aligned}$$

If instead $t<\underline{x}^2$ divide the integral in the part $|\underline{y}|<|\underline{x}|/8$ and in the part with $|\underline{y}|>|\underline{x}|/8$. The second part can be bounded as needed (change variable $\underline{y}'=\frac{\underline{y}}{|\underline{x}|}$ to get $\le|\underline{x}|^{-(3+|\underline{\alpha}|)}\int\frac{d\underline{y}'}{|\underline{e}-\underline{y}'|}|\underline{y}'|^{-(5+|\underline{\alpha}|)}$, if $\underline{e}$ is the unit vector $\underline{x}/|\underline{x}|$). Therefore it remains to study: $\int_{|\underline{y}|<|\underline{x}|/8}\frac{d\underline{y}}{|\underline{x}-\underline{y}|}\partial^{\underline{\alpha}+2}\frac{e^{-y^2/4t}}{\sqrt{4\pi t}^3}$. Note that the integral can be performed by parts generating various terms, on the boundary of the sphere $|\underline{y}|=|\underline{x}|/8$, which can be majorized by their maximum (on the sphere). One obtains quantities proportional to expressions like

$$\begin{aligned}&\int_{|\underline{y}|=|\underline{x}|/8}d\sigma\partial^{p-1}\frac{1}{|\underline{x}-\underline{y}|}\partial^{|\underline{\alpha}|+2-p}(\frac{e^{-y^2/4t}}{\sqrt{4\pi t}^3})\le\\&\le\text{const}\,\frac{|\underline{x}|^2}{|x|^p}\frac{1}{(x^2+t)^{(3+|\underline{\alpha}|+2-p)/2}}\le\text{const}|\underline{x}|^{-(|\underline{\alpha}|+3)}\end{aligned}$$

where we denote, generically, by $\partial^{|\underline{\alpha}|+2-p}$ and ∂^p a derivative with respect to the components of $\underline{y}$ and of order $|\underline{\alpha}|+2-p$ or of order p).

Furthermore one has to consider the volume integral. This involves a sum of quantities bounded by $\text{const}|\underline{x}-\underline{y}|^{-|\underline{\alpha}|-3}$ *i.e.* by $\text{const}\,|\underline{x}|^{-|\underline{\alpha}|-3}$ (as $|\underline{y}|\ll|\underline{x}|$) multiplied by $e^{-\underline{y}^2/4t}/(4\pi t)^{3/2}$. Hence by using that the integral of the heat kernel, as $\underline{y}$ varies in the whole space, has value 1, we find that the part with $t<\underline{x}^2$ is also bounded by $\text{const}|\underline{x}|^{-|\underline{\alpha}|-3}$. Hence the (3.3.22) follow).

[3.3.11]: (*proof of proposition V*) Referring to the first of (3.3.22) consider the second of (3.3.44) and prove that, if $|F|\le M$, the (3.3.45) holds. An analogous analysis holds for the first of the (3.3.44). (*Idea:* For instance let $P=\Gamma$ and note: $|V(\underline{x},t)-V(\underline{x}',t)|\le\int_0^t\int_\Omega|\partial_j\Gamma(\underline{x}-\underline{y},t-\tau)-\partial_j\Gamma(\underline{x}'-\underline{y},t-\tau)|M\,d\underline{y}\,d\tau$. Decompose the integral into the sum of the integrals extended to the domain $|\underline{y}-(\underline{x}+\underline{x}')/2|<|\underline{x}-\underline{x}'|$ and to its complement. The first integral is bounded by majorizing the modulus of the difference by the sum of the moduli and making use of (3.3.22) to bound each of the two terms (performing first the integral over τ and then that on $\underline{y}$).

If B is a suitable constant we get: $\le 2MC_1\int_0^t\int_{|\underline{y}|<2|\underline{x}-\underline{x}'|}(\underline{y}^2+t-\tau)^{-2}\,d\underline{y}\,d\tau\le 2MC_1B|\underline{x}-\underline{x}'|$. On the other hand the integral over $|\underline{y}-(\underline{x}+\underline{x}')/2|\ge|\underline{x}-\underline{x}'|$ is simply bounded by applying Taylor formula to the integrand majorizing it by $\int_0^1d\sigma\,|\underline{\partial}\partial_j\Gamma(\underline{x}+\sigma(\underline{x}'-\underline{x})-y,t-\tau)|\,|\underline{x}'-\underline{x}|$ and applying then the (3.3.22) to estimate the expression with the gradient as $\le MC_2|\underline{x}'-\underline{x}|\int_{|\underline{y}|>|\underline{x}'-\underline{x}|/2}\int_0^t \frac{d\underline{y}\,d\tau}{(\underline{y}^2+t-\tau)^{5/2}}$. We get

$$\le MC_2|\underline{x}'-\underline{x}|\cdot B\int_{|\underline{x}'-\underline{x}|/2}^{L}\frac{dy}{y^3}\le MC_2B'|\underline{x}-\underline{x}'|\log\frac{2L}{|\underline{x}-\underline{x}'|}$$

if B, B' are suitable constants; and this also shows what we wanted in the case $P = T$ (because the proof depends only on (3.3.22)) and it estimates K_t^0, which we thus see that can even be chosen *independent of* t, furthermore the Hölder exponent $\frac{1}{2}$ in Proposition 5 can be replaced by any η with $\eta < 1$.)

[3.3.12]: (*proof of proposition VI*) suppose that F is Hölder continuous with exponent $\alpha > 0$, *i.e.* suppose that, for a fixed $0 < \alpha \le 1$,

$$|F(\underline{x},t)| + \frac{|F(\underline{x},t)-F(\underline{x}',t)|}{(L^{-1}|\underline{x}-\underline{x}'|)^{\alpha}} \le M_\alpha \tag{3.3.61}$$

Then show that the function V in the second of (3.3.44) is differentiable if $P \equiv \Gamma$ and verifies (3.3.46), (the case $P = T$ will be identical because we shall use only (3.3.22)). Analogously deduce the validity of (3.3.46) for the first of the (3.3.44). (*Idea:* Differentiability can be studied by evaluating the differential ratio in the direction of the unit vector $\underline{e}$

$$\Delta \equiv \frac{V(\underline{x}+\varepsilon\underline{e},t)-V(\underline{x},t)}{\varepsilon} = \int_0^t\int_\Omega \frac{1}{\varepsilon}(\partial_j\Gamma(\underline{x}+\varepsilon\underline{e}-\underline{y},t-\tau)-\partial_j\Gamma(\underline{x}-\underline{y},t-\tau))F(\underline{y},\tau)$$

and *noting that if* $F(\underline{y},\tau)$ *was replaced by* $F(\underline{x},t)$ *the second member would vanish*, by integration by parts. Hence

$$\Delta \equiv \int_0^t d\tau\int_\Omega d\underline{y}\frac{1}{\varepsilon}(\partial_j\Gamma(\underline{x}+\varepsilon\underline{e}-\underline{y},t-\tau)-\partial_j\Gamma(\underline{x}-\underline{y},t-\tau))(F(\underline{y},\tau)-F(\underline{x},\tau))$$

Divide the integral into the part over $|\underline{y}-\underline{x}| < \varepsilon^\beta$, $\beta < 1$ and in the remainder choosing β so that $\beta(1+\alpha) > 1$, if α is the Hölder continuity exponent. The contribution to Δ of the term with $|\underline{y}-\underline{x}| < \varepsilon^\beta$ is then estimated by

$$\Delta \le \frac{2}{\varepsilon}\int_{|\underline{y}-\underline{x}|<2\varepsilon^\beta}\frac{C_2}{(|\underline{x}-\underline{y}|^2+(t-\tau))^2}M_\alpha|\underline{y}-\underline{x}|^\alpha\,d\underline{y}\,d\tau \le \bar{C}_2M_\alpha\varepsilon^{-1}\varepsilon^{(1+\alpha)\beta}\xrightarrow[\varepsilon\to0]{}0$$

with suitable $\bar{C}_2$, having separately bounded the $\partial_j\Gamma$, via (3.3.22).

While the contribution to the integral coming from $|\underline{y}-\underline{x}| > \varepsilon^\beta$ can be estimated by using the Taylor formula to write the integral as

$$\sum_i\int_0^1 d\sigma\,d\tau\int_{|\underline{y}-\underline{x}|\ge\varepsilon^\beta} d\underline{y}\,\partial_{ij}\Gamma(\underline{x}-\underline{y}+\sigma\,\varepsilon\,\underline{e},\,t-\tau)\,(F(\underline{y},\tau)-F(\underline{x},\tau))\,e_i \tag{3.3.62}$$

And note that the integrand is bounded, for all the ε, $\underline{e}$ (since $|\underline{x}-\underline{y}| \ge \varepsilon^\beta \gg \varepsilon$,because $\beta < 1$ and by using again the (3.3.21)), by: $(\frac{1}{2}|\underline{x}-\underline{y}|^2+t-\tau)^{-5/2}M_\alpha|\underline{y}-\underline{x}|^\alpha$, *i.e.* by an ε–independent function whose integral is

$$\int_0^t d\tau\int_\Omega\frac{M_\alpha|\underline{y}-\underline{x}|^\alpha\,d\underline{y}}{(\frac{1}{2}(\underline{x}-\underline{y})^2+t-\tau)^{5/2}} \le C\int_\Omega M_\alpha|\underline{x}-\underline{y}|^{-3+\alpha}\,d\underline{y} < \overline{C}_\alpha M_\alpha L^\alpha < \infty$$

hence the integral that expresses the differential ratio is uniformly convergent in ε and, therefore, it is possible to take the limit in (3.3.62) under the integral sign to evaluate the limit as $\varepsilon \to 0$. Thus V is differentiable and its derivative is given by $\partial_i V(\underline{x}, t) = \int d\tau \int_\Omega d\underline{y}\, \partial_{ij}\Gamma(\underline{x} - \underline{y}, t - \tau)(F(\underline{y}, \tau) - F(\underline{x}, \tau))$ and $|\partial_i V(\underline{x}, t)| \le \overline{C}_\alpha M_\alpha L^\alpha$, (note that the integral is convergent and bounded as described by the preceding majorizations).

Bibliography: Leray's theory, exposed in this section, is taken from [Le34]. Proposition VII is taken from [CF93] where it is discussed for bounded vorticity solutions in R^3 rather than in a cubic container with periodic boundary conditions.

§3.4 Fractal dimension of singularities of the Navier–Stokes equation, $d = 3$.

Here we ask which could be the structure of the possible set of the singularity points of the solutions of the Navier–Stokes equation in $d = 3$.

We have already seen in §3.3 that the set of times at which a singularity is possible has zero measure (on the time axis).

Obviously sets of zero measure can be quite structured and even large in other senses. One can think to the example of the Cantor set which is non denumerable and obtained from an interval I by deleting an open concentric subinterval of length 1/3 that of I and then repeating recursively this operation on each of the remaining intervals (called n–th generation intervals after n steps); or one can think to the set of rational points which is dense.

(A) Dimension and measure of Hausdorff.

An interesting geometric characteristic of the size of a set is given by the Hausdorff dimension and by the Hausdorff measure, *c.f.r.* [DS60], p.174.

1. Definition *(Hausdorff α–measure): The Hausdorff α-measure of a set A contained in a metric space M is defined by considering for each $\delta > 0$ all coverings $\mathcal{C}_\delta$ of A by closed sets F with diameter $0 < d(F) \le \delta$ and setting*

$$\mu_\alpha(A) = \lim_{\delta\to 0} \inf_{\mathcal{C}_\delta} \sum_{F\in\mathcal{C}_\delta} d(F)^\alpha \tag{3.4.1}$$

Remarks:

(1) The limit over δ exists because the quantity $\inf_{\mathcal{C}_\delta} \ldots$ is monotonic non-decreasing.

(2) It is possible to show that the function defined on the sets A of M by $A \to \mu_\alpha(A)$ is completely additive on the smallest family of sets containing all closed sets and invariant with respect to the operations of complementation

and countable union (which is called the *σ-algebra Σ of the Borel sets of M*), *c.f.r.* [DS60].

One checks immediately that given $A \in \Sigma$ there is a α_c such that

$$\begin{aligned} \mu_\alpha(A) =& \infty \qquad \text{if} \quad \alpha < \alpha_c \\ \mu_\alpha(A) =& 0 \qquad \text{if} \quad \alpha > \alpha_c \end{aligned} \tag{3.4.2}$$

and it is therefore natural to set up the following definition

2. Definition *(Hausdorff measure and Hausdorff dimension): Given a Borel set $A \subset R^d$ the quantity α_c, (3.4.2), is called* Hausdorff dimension *of A, while $\mu_{\alpha_c}(A)$ is called* Hausdorff measure *of A.*

It is not difficult to check that

(1) Denumerable sets in $[0, 1]$ have zero Hausdorff dimension and measure.
(2) Hausdorff dimension of n-dimensional regular surfaces in R^d is n and, furthermore, the Hausdorff measure of their Borel subsets defines on the surface a measure μ_{α_c} that is equivalent to the area measure μ: namely there is a $\rho(x)$ such that $\mu_{\alpha_c}(dx) = \rho(x)\,\mu(dx)$.
(3) The Cantor set, defined also as the set of all numbers in $[0, 1]$ which in the representation in base 3 do not contain the digit 1, has Hausdorff dimension

$$\alpha_c = \log_3 2 \tag{3.4.3}$$

Indeed with 2^n disjoint segments with size 3^{-n}, uniquely determined (the n–th generation segments), one covers the whole set C; hence

$$\mu_{\alpha,\delta} \stackrel{def}{=} \inf_{\mathcal{C}_\delta} \sum_{F \in \mathcal{C}_\delta} d(F)^\alpha \le 1 \qquad \text{if} \quad \alpha = \alpha_0 = \log_3 2 \tag{3.4.4}$$

and $\mu_{\alpha_0}(C) \le 1$: *i.e.* $\mu_\alpha(C) = 0$ if $\alpha > \alpha_0$. Furthermore, *c.f.r.* problem [3.4.3] below, if $\alpha < \alpha_0$ one checks that the covering $\mathcal{C}^0$ realizing the smallest value of $\sum_{F\in\mathcal{C}_\delta} d(F)^\alpha$ with $\delta = 3^{-n}$ is precisely the just considered one consisting in the 2^n intervals of length 3^{-n} of the n–th generation and the value of the sum on such covering diverges for $n \to \infty$. Hence $\mu_\alpha(C) = \infty$ if $\alpha < \alpha_0$ so that $\alpha_0 \equiv \alpha_c$ and $\mu_{\alpha_c}(C) = 1$.

(B) Hausdorff dimension of singular times in the Navier–Stokes solutions $(d = 3)$.

We now attempt to estimate the Hausdorff dimension of the sets of times $t \le T < \infty$ at which appear singularities of a given weak solution of Leray, *i.e.* a solution of the type discussed in §3.3, definition 2, (E). Here T is an *a priori* arbitrarily prefixed time.

For simplicity we assume that the density of volume force vanishes and we recall that in §3.3 we have shown that if at time t_0 it is $J_1(t_0) =$

$L^{-1}\int(\partial\underline{u})^2 d\underline{x} < \infty$, *i.e.* if the Reynolds number $R(t_0) = J_1(t_0)^{1/2}/V_c \equiv V_1/V_c$, *c.f.r.* (3.3.13), is $< +\infty$, then the solution stays regular in a time interval $(t_0, t_0+\tau]$ with (see proposition II in §3.3):

$$\tau = F\frac{T_c}{R(t_0)^4 + R_g^2 + 1} \tag{3.4.5}$$

From this it will follow, see below, that there are $A > 0, \gamma > 0$ such that if

$$\liminf_{\sigma\to 0}\Big(\frac{\sigma}{T_c}\Big)^{\gamma}\int_{t-\sigma}^{t}\frac{d\vartheta}{\sigma}R^2(\vartheta) < A \tag{3.4.6}$$

then $\tau > \sigma$ and the solution is regular in an interval that contains t so that the instant t is an instant at which the solution is regular. Here, as in the following, we could fix $\gamma = 1/2$: but γ is left arbitrary in order to make clearer why the choice $\gamma = 1/2$ is the "best".

We first show that, indeed, from (3.4.6) we deduce the existence of a sequence $\sigma_i \to 0$ such that

$$\int_{t-\sigma_i}^{t}\frac{d\vartheta}{\sigma_i}\,R^2(\vartheta) < A\Big(\frac{\sigma_i}{T_c}\Big)^{-\gamma} \tag{3.4.7}$$

therefore, the l.h.s. being a time average, there must exist $\vartheta_{0i} \in (t-\sigma_i, t)$ such that

$$R^2(\vartheta_{0i}) < A\Big(\frac{\sigma_i}{T_c}\Big)^{-\gamma} \tag{3.4.8}$$

and then the solution is regular in the interval $(\vartheta_{0i}, \vartheta_{0i}+\tau_i)$ with length τ_i at least

$$\tau_i = FT_c\frac{(\sigma_i/T_c)^{2\gamma}}{A^2 + R_g^2(\sigma_i/T_c)^{2\gamma}} > \sigma_i \tag{3.4.9}$$

provided $\gamma \le 1/2$, and σ_i is small enough and if A is small enough: if $\gamma = \frac{1}{2}$ we take $A^2 = \frac{1}{2}F$, for instance. Under these conditions the size of the regularity interval is longer than σ_i and *therefore it contains t itself.*

It follows that, if t is in the set S of the times at which a singularity is present, it must be

$$\liminf_{\sigma\to 0}\Big(\frac{\sigma}{T_c}\Big)^{\gamma}\int_{t-\sigma}^{t}\frac{d\vartheta}{\sigma}\,R^2(\vartheta) \ge A \qquad \text{if } t\in S \tag{3.4.10}$$

i.e. every singularity point is covered by a family of infinitely many intervals F with diameters σ *arbitrarily small* and satisfying

$$\int_{t-\sigma}^{t} d\vartheta\, R^2(\vartheta) \ge \frac{A}{2}\sigma\Big(\frac{\sigma}{T_c}\Big)^{-\gamma} \tag{3.4.11}$$

From Vitali's covering theorem (*c.f.r.* problem [3.4.1]) it follows that, given $\delta > 0$, one can find a denumerable family of intervals $F_1, F_2, \ldots$, with $F_i =$

$(t_i - \sigma_i, t_i)$, pairwise disjoint and verifying the (3.4.11) and $\sigma_i < \delta/4$, such that the intervals $5F_i \stackrel{def}{=} (t_i - 7\sigma_i/2, t_i + 5\sigma_i/2)$ (obtained by dilating the intervals F_i by a factor 5 about their center) cover S

$$S \subset \cup_i 5F_i \tag{3.4.12}$$

Consider therefore the covering $\mathcal{C}$ generated by the sets $5F_i$ and compute the sum in (3.4.1) with $\alpha = 1 - \gamma$:

$$\begin{aligned} &\sum_i (5\sigma_i)\Big(\frac{5\sigma_i}{T_c}\Big)^{-\gamma} = 5^{1-\gamma}\sum_i \sigma_i\Big(\frac{\sigma_i}{T_c}\Big)^{-\gamma} < \\ &< \frac{2\,5^{1-\gamma}}{A}\sum_i\int_{F_i} d\vartheta\, R^2(\vartheta) \le \frac{2\,5^{1-\gamma}}{A}\int_0^T d\vartheta\, R^2(\vartheta) < \infty \end{aligned} \tag{3.4.13}$$

where we have made use of the *a priori* estimates on vorticity derived in (3.3.8), and we must recall that $\gamma \le 1/2$ is a necessary condition in order that what has been derived be valid (*c.f.r.* comment to (3.4.9)).

Hence it is clear that for each $\alpha \ge 1/2$ it is $\mu_\alpha(S) < \infty$ (pick, in fact, $\alpha = 1 - \gamma$, with $\gamma \le 1/2$) hence the Hausdorff dimension of S is $\alpha_c \le 1/2$. Obviously the choice that gives the best regularity result (with the informations that we gathered) is precisely $\gamma = 1/2$.

Moreover one can check that $\mu_{1/2}(S) = 0$: indeed we know that S has zero measure, hence there is an open set $G \supset S$ with measure smaller than a prefixed ε. And we can choose the intervals F_i considered above so that they also verify $F_i \subset G$: hence we can replace the integral in the right hand side of (3.4.13) with the integral over G hence, since the integrand is summable, we shall find that the value of the integral can be supposed as small as wished, so that $\mu_{1/2}(S) = 0$.

(C) Hausdorff dimension in space–time of the solutions of NS, ($d = 3$).

The problem of which is the Hausdorff dimension of the points $(\underline{x}, t) \in \Omega \times [0, T]$ which are singularity points for the Leray's solutions is quite different.

Indeed, *a priori* , it could even happen that, at one of the times $t \in S$ where the solution has a singularity as a function of time, *all* points $(\underline{x}, t)$, with $\underline{x} \in \Omega$, are singularity points and therefore the set S_0 of the singularity points thought of as a set in space–time could have dimension 3 (and perhaps even 3.5 if we take into account the dimension of the singular times discussed in (B) above).

With some optimism one can think that a version "local in space" of Leray's theorem, see proposition II of §3.3 which is "only local in time", holds. Under the influence of such wishful thought we then examine what we can expect as an estimate of the Hausdorff dimension of S_0.

The notions of characteristic time T_c, of characteristic velocities V_c, V_1, W_0, of characteristic acceleration $\sqrt{G_0}$, of characteristic size of the forcing R_g, introduced in §3.3 and playing a major role in the development of the theory of L–weak solutions of Leray, were "global" notions in the sense that they were associated with properties of the whole fluid in Ω and were not associated with any particular point or subregion of Ω.

These notions can be "localized", *i.e.* given a different meaning at different points of Ω, in a rather naive way: namely given $\underline{x} \in \Omega$ and a length scale r we can imagine that the whole fluid consists of the part that is contained in a ball $S(\underline{x}, r)$ of radius r around $\underline{x}$ and then define the various characteristic scales by replacing the container size L by r and the container Ω with $S(\underline{x}, r)$. Thus regarding a small fluid volume as in some way similar, apart from obvious scaling, to a large one is the natural idea behind the following definitions and the guesses that they inspire: it is an idea that already proved fruitful in the discussion of "large containers" in §3.3, (G).

Consider a time ϑ and define the *local Reynolds number* on scale r at time ϑ as the ratio between a quantity characterizing the velocity variation on scale r, near $\underline{x}$, and a characteristic velocity V_{cr} associated with the geometric dimension r, *c.f.r.* (3.3.1), (3.3.13). It will be the ratio between $V_{1r} \stackrel{def}{=} (r^{-1} \int_{s(\underline{x},r)} (\underset{\sim}{\partial \underline{u}})^2 d\underline{\xi})^{1/2}$ and $V_{cr} = \nu r^{-1}$ namely

$$R_r^2(\vartheta) \stackrel{def}{=} \big(\frac{V_{1r}}{V_{cr}}\big)^2 = \frac{r}{\nu^2} \int_{S(\underline{x},r)} (\underset{\sim}{\partial \underline{u}}(\underline{\xi}, \vartheta))^2 d\underline{\xi} < \infty \tag{3.4.14}$$

where $S(\underline{x}, r)$ is the sphere of radius r and center $\underline{x}$.

Likewise, in analogy with (3.3.13), we can define the "local" strength of the forcing by "localizing" the definition in (3.3.13). Let $G_{0r} = r^{-3} \int_{S(\underline{x},r)} |\underline{g}(\underline{x})|^2 \, d\underline{x}$ be a local acceleration scale; let W_{cr} be the corresponding local velocity scale $W_{0r} = T_{cr} \sqrt{G_{0r}}$ with $T_{cr} = r^2 \nu^{-1}$, see (3.3.1), (3.3.13), and let the dimensionless strength R_{gr} of the forcing be

$$R_{gr} \stackrel{def}{=} \frac{W_{0r}}{V_{cr}} = \frac{T_{cr} \sqrt{G_{0r}}}{V_{cr}} = \frac{r^3}{\nu^2} \Big(r^{-3} \int_{S(\underline{x},r)} |\underline{g}(\underline{x})|^2 d\underline{x} \Big)^{1/2} \tag{3.4.15}$$

hence since we take $\underline{g}$ to be smooth it is $R_{gr}^2 \le C \tilde{g}^2 r^6 \nu^{-4}$ where C is a geometric constant and $\tilde{g}$ is an estimate of the maximum of the density of volume force $\underline{g}$.

Guess: *A "local version" of the Leray theorem in question, see Proposition II in Sect. §3.3 and remark (2) following it, "could" say that, under the condition (3.4.14) the solution is regular in the space–time region*

$$\underline{x} \in S(\underline{x}, r) \qquad \text{and} \qquad t \in (\vartheta, \vartheta + T_{cr} \min(1, \frac{F}{R_r^4 + R_{gr}^2})) \tag{3.4.16}$$

Obviously such a statement is very strong, far from proven in the analysis that we performed until now, and somewhat surprising: in fact since the fluid is incompressible sound waves propagate at infinite speed and therefore one can fear that a singularity that at time t is at a position $\underline{\xi}$ far from $\underline{x}$ could arrive, in an arbitrarily short time, near $\underline{x}$, even though no singularity was present near $\underline{x}$ at time t.

On the other hand it does not seem absurd that (3.4.14) or something similar to it implies (by the temporarily assumed validity of a space–local version of Leray theorem) regularity in $\underline{x}$ for a short successive time because it is also difficult that the wave associated with the singularity far away from $\underline{x}$ does not dissolve right away because of the friction. After all the theorem of Leray–Serrin, *c.f.r.* §3.3, excludes the possibility of real shock waves in an incompressible NS fluid.

Assuming, *still temporarily*, that (3.4.14) implies regularity of the solution in the vicinity of the points $(\underline{x}, t)$ with $t \in (\vartheta, \vartheta + FT_{cr}(R_r^4(\vartheta) + R_{gr}^2)^{-1})$ see (3.4.16) above, then we can use the ideas already used to estimate the time-singularities measure.

Indeed if regularity of $\underline{u}$ holds in the vicinity of $\underline{x}, \vartheta$ it follows (just from the regularity of $\underline{u}$) that $\lim_{r\to 0} \frac{\nu}{r^2} \int_{t-r^2/\nu}^{t} d\vartheta R_r^2(\vartheta) = 0$ (because $R_r^2(\vartheta)$ would have size $O(r^4)$).

Viceversa if the implication of (3.4.14) on regularity in the space–time set specified in (3.4.16) around $\underline{x}$ and in the time intervals $(\vartheta, \vartheta - r^2\nu^{-1})$ is accepted (keep in mind, however, that it is a property that we are considering for the sake of establishing some intuition about what should be attempted in the coming analysis), we could argue as follows.

Suppose *knowing that*

$$\limsup_{r\to 0} \frac{\nu}{r^2} \int_{t-r^2/\nu}^{t} d\vartheta R_r^2(\vartheta) < \varepsilon \tag{3.4.17}$$

for some $\varepsilon > 0$. Then a sequence $r_i \to 0$ would exist such that

$$\frac{\nu}{r_i^2} \int_{t-r_i^2/\nu}^{t} d\vartheta\, R_{r_i}^2(\vartheta) \,<\, \varepsilon \tag{3.4.18}$$

Hence, the latter expression being a time average, a time $\vartheta_i \in (t, t - r_i^2\nu^{-1})$ would exist where $R(\vartheta_i)^2 < \varepsilon$ and also, by the regularity of the external force $\underline{g}$, $R_{gr}^2 \le C^2\tilde{g}^2 r_i^6 \nu^{-4} < \varepsilon^2$ if r_i is small enough.

So that regularity would follow in the vicinity of

$$\underline{x} \times (\vartheta_i, \vartheta_i + \frac{r_i^2}{\nu}) \tag{3.4.19}$$

hence in $(\underline{x}, t)$, *provided* $2^{-1}F\varepsilon^{-2} > 1$ as a consequence of the "guess" above.

It would follow that the set S_0 of the space–time singularity points could be covered by sets $C_r = S(\underline{x}, r) \times (t - r^2\nu^{-1}, t]$ with r arbitrarily small[1] and

[1] Note that in order that this be true it suffices to require the validity of (3.4.17) with the lower limit only: here we require the upper limit because, as we shall see in the following, the (3.4.17) is, in the latter more restrictive form, closer to the property that one can really prove.

will be such that

$$\frac{1}{r\nu}\int_{t-r^2\nu^{-1}}^{t} d\vartheta \int_{S(\underline{x},r)} d\underline{x}\,(\partial \underset{\sim}{u})^2 > \varepsilon \tag{3.4.20}$$

which is the negation of the property in (3.4.17).

Again by a covering theorem of Vitali (*c.f.r.* problems [3.4.1],[3.4.2]), we could find a family F_i of sets of the form $F_i = S(\underline{x}_i, r_i) \times (t_i - \nu^{-1} r_i^2, t_i]$ pairwise disjoint and such that the sets $5F_i$= set of points $(\underline{x}', t')$ such that $|\underline{x}' - \underline{x}_i| < 5r_i$ and $|t' - t_i - \nu^{-1}\frac{1}{2}r_i^2| < \nu^{-1}(5r_i)^2$ covers the singularity set S_0.[2] One could then estimate the sum in (3.4.1) for such a covering, by using that the sets F_i are pairwise disjoint and that $5F_i$ has diameter, if $\max r_i$ is small enough, not larger than $11r_i$:

$$\sum_i (11r_i) \le \frac{11}{\nu\varepsilon}\sum_i \int_{F_i} (\partial \underset{\sim}{u})^2 d\underline{\xi} dt \le \frac{11}{\nu\varepsilon}\int_0^T \int_\Omega (\partial \underset{\sim}{u})^2 d\underline{\xi} dt < \infty \tag{3.4.21}$$

i.e. the 1-measure of Hausdorff $\mu_1(S_0)$ would be $< \infty$ hence the Hausdorff dimension of S_0 would be ≤ 1.

Since S_0 has zero measure, being contained in $\Omega \times S$ where S is the set of times at which a singularity occurs somewhere, see (3.4.10), it follows (still from the covering theorems) that in fact it is possible to choose the sets F_i so that their union U is contained into an open set G which differs from S_0 by a set of measure that exceeds by as little as desired that of S_0, (which is zero); one follows the same method used above in the analysis of the time–singularity. Hence we can replace the last integral in (3.4.21) with an integral extended to the union U of the $F_i's$: the latter integral can be made as small as wished by letting the measure of G to 0. It follows that not only the Hausdorff dimension of S_0 is ≤ 1, but also the $\mu_1(S_0) = 0$.

Remark: One could in this way exclude that the set S_0 of the space–time singularities contains a regular curve: singularities, *if existent*, cannot move along trajectories (like flow lines) otherwise the dimension of S would be $1 > 1/2$) nor they can be distributed, at fixed time, along lines and, hence, in a sense they must appear isolates and immediately disappear (always assuming their real existence).

Going back to the basic assumption behind the above wishful[3] reasoning, we realize that the condition (3.4.16) can be replaced, for the purposes of the argument discussed above and to conclude that the Hausdorff 1–measure of S_0 vanishes, by the *weaker statement* in (3.4.17) with ε small enough, that implies regularity in $(\underline{x}, t)$.

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[2] Here the constant 5, as well as the other numerical constants that we meet below like 5, 11 have no importance for our purposes and are just simple constants for which the estimates work.

[3] Being based on the guess above.

Or, with an obvious modification of the argument discussed above, it would suffice that the regularity in $(\underline{x}, t)$ was implied by a relation similar to (3.4.17) in which the cylinder $S(\underline{x}, r) \times (t - r^2\nu^{-1}, t]$ is replaced by a similar cylinder with $(\underline{x}, t)$ in its interior, *i.e.* if the regularity was implied by a relation of the type

$$\begin{aligned} &\limsup_{r\to 0} \frac{\nu}{r^2} \int_{t-r^2/2\nu}^{t+r^2/2\nu} R_r(\vartheta)^2 < \varepsilon, \qquad \text{or} \\ &\limsup_{r\to 0} r^{-1} \int_{t-r^2/2\nu}^{t+r^2/2\nu} \int_{S(\underline{x},r)} \frac{d\vartheta}{\nu}\, d\underline{\xi}\, (\underline{\partial} \underline{u})^2 < \varepsilon \end{aligned} \tag{3.4.22}$$

with ε small enough.

The latter property can be actually proved to hold [CKN82]: it will be discussed in detail in §3.5.

Remark: A conjecture (much debated and that I favor) that is behind all our discussions is that *if the initial datum* $\underline{u}^0$ *is in* $C^\infty(\Omega)$ *then there exists a solution to the equation of Navier Stokes that is of class* C^∞ *in* $(\underline{x}, t)$", *i.e.* $S_0 = \emptyset$!

The problem is, still, open: counterexamples to the conjecture are not known (*i.e.* singular weak solutions with initial data and external force of class C^∞) but the matter is much debated and different alternative conjectures are possible (*c.f.r.* [PS87]).

In this respect one should keep in mind that if $d \geq 4$ it is possible to show that *not all* smooth initial data evolve into regular solutions: counterexamples to smoothness can indeed be constructed, *c.f.r.* [Sc77].

Problems.

[3.4.1]: (*covering theorem*, (Vitali)) Let S be an arbitrary set inside a sphere of R^n. Consider a *covering* of S with little open balls with the *Vitali property*: *i.e.* such that every point of S is contained in a family of open balls of the covering whose radii have a zero greatest lower bound. Giben $\eta > 0$ show that if $\lambda > 1$ is large enough it is possible to find a denumerable family $F_1, F_2, \ldots$ of pairwise disjoint balls of the covering with diameter $< \eta$ such that $\cup_i \lambda F_i \supset S$ where λF_i denotes the ball with the same center of F_i and radius λ times longer. Furthermore λ can be chosen independent of S, see also [3.4.2]. (*Idea:* Let $\mathcal{F}$ be the covering and let $a = max_{\mathcal{F}}\, diam(F)$. Define $a_k = a2^{-k}$ and let $\mathcal{F}_1$ be a *maximal* family of *pairwise disjoint* ball of $\mathcal{F}$ with radii $\geq a2^{-1}$ and $< a$. Likewise let $\mathcal{F}_2$ be a maximal set of balls of $\mathcal{F}$ with radii between $a2^{-2}$ and $a2^{-1}$ pairwise disjoint between themselves and with the ones of the family $\mathcal{F}_1$. Inductively we define $\mathcal{F}_1, \ldots, \mathcal{F}_k, \ldots$. It is now important to note that if $x \notin \cup_k \mathcal{F}_k$ it must be: $distance(x, \mathcal{F}_k) < \lambda a 2^{-k}$ for some k, if λ is large enough. If indeed δ is the radius of a ball S_δ containing x and if $a2^{-k_0} \leq \delta < a2^{-k_0+1}$ then the point of S_δ farthest away from x is at most at distance $\leq 2\delta < 4a2^{-k_0}$; and if, therefore, it was $d(x, \mathcal{F}_{k_0}) \geq 4a2^{-k_0}$ we would find that the set $\mathcal{F}_{k_0}$ could be made larger by adding to it S_δ, against the maximality supposed for $\mathcal{F}_{k_0}$.)

[3.4.2]: Show that if the balls in [3.4.1] are replaced by the *parabolic cylinders* which are Cartesian products of a radius r ball in the first k coordinates and one of radius r^α, with $\alpha \geq 1$ in the $n - k$ remaining ones, then the result of problem [3.4.1] still holds if one interprets λF_i as the parabolic cylinder obtaine by applying to F_i a homothety, with respect to the center of F_i, of scale λ on the first k coordinates and λ^2 on the others. Check that if $\alpha = 2$ the value $\lambda = 5$ is sufficient.

[3.4.3]: Check that the Hausdorff dimension of the Cantor set C is $\log_3 2$, *c.f.r.* (3.4.3). *Idea:* It remains to see, given (3.4.4), that if $\alpha < \alpha_0$ then $\mu_\alpha(C) = \infty$. If $\delta = 3^{-n}$ the covering $\mathcal{C}_n$ of C with the n–th generation intervals is "the best" among those with sets of diameter $\le 3^{-n}$ because another covering could be refined by deleting from each if its intervals the points that are out of the n–th generation intervals. Furthermore the inequality $1 < 2 \cdot 3^{-\alpha}$ for $\alpha < \log_3 2$ shows that it will not be convenient to further subdivide the intervals of $\mathcal{C}_n$ for the purpose of diminishing the sum $\sum |F_i|^\alpha$. Hence for $\delta = 3^{-n}$ the minimum value of the sum is $2^n 3^{-n\alpha} \xrightarrow[n\to\infty]{} \infty$.

Bibliography: See [DS60], vol. I, p. 174, (Hausdorff dimension and measure); [Ka76] p. 74, (Vitali covering); see also [DS60]; [Sc77],[CKN82] (fractal dimension of the singularities).

§3.5 Local homogeneity and regularity. CKN theory.

The theory of space–time singularities, *i.e.* the proofs of the statements that have been heuristically discussed in §3.4, will be partly based upon simple *kinematic inequalities*, which therefore have little to do with the Navier–Stokes equation, and partly they will be based on the local energy conservation which follows as a consequence of the Navier–Stokes equations.

We suppose that the volume Ω is a 3-dimensional torus (*i.e.* we assume "periodic" boundary conditions) and that the initial datum is C^∞.

(A) Energy balance for weak solutions.

Energy conservation for the regularized equations (3.3.2) says that the kinetic energy variation in a given volume element Δ of the fluid, in a time interval $[t_0, t_1]$, plus the energy dissipated therein by friction, equals the sum of the kinetic energy that in the time interval $t \in [t_0, t_1]$ enters in the volume element plus the work performed by the pressure forces (on the element boundary) plus the work of the volume forces (if any), *c.f.r.* (1.1.17). The analytic form of this relation is simply obtained by multiplying both sides of the first of the (3.3.2) by $\underline{u}$ and integrating on the volume element Δ and over the time interval $[t_0, t_1]$.

The relation that one gets can be generalized to the case in which the volume element has a time dependent shape. And an even more general relation can be obtained by multiplying both sides of (3.3.2) by $\varphi(\underline{x}, t)\underline{u}(\underline{x}, t)$ where φ is a $C^\infty(\Omega \times (0, s])$ function with $\varphi(\underline{x}, t)$ zero for t near 0 (here s is a positive parameter).

The preceding cases are obtained as limiting cases of choices of φ in the limit in which it becomes the characteristic function of the space–time volume element $\Delta \times [t_0, t_1]$. Making use of a regular function $\varphi(\underline{x}, t)$ is useful, particularly in the rather "desperate" situation in which we are when using the theory of Leray, in which the "solutions" $\underline{u}$ (obtained by removing, in (3.3.2), the regularization) are only weak solutions and, therefore, the relations that are obtained can be interpreted as valid only after suitable integrations by parts that allow us to avoid introducing derivatives of $\underline{u}$ (whose existence is not guaranteed by the theory) at the "expense" of differentiating the "test function" φ.

We shall assume the absence of volume forces: this is a simplicity assumption as the extension of the theory to cases with time independent smooth (*e.g.* C^∞) volume forces is trivial.

Performing analytically the computation of the energy balance, described above in words, in the case of the regularized equation (3.3.2) and via a few integrations by parts[1] we get the following relation

$$\begin{aligned}&\frac{1}{2}\int_\Omega d\underline{\xi}|\underline{u}(\underline{\xi},s)|^2\varphi(\underline{\xi},s)+\nu\int_0^s dt\int_\Omega \varphi(\underline{\xi},t)|\underset{\sim}{\underline{\partial}}\underline{u}(\underline{\xi},t)|^2 d\underline{x}=\\&=\int_0^s\int_\Omega\Big[\frac{1}{2}(\varphi_t+\nu\,\Delta\varphi)|\underline{u}|^2+|\underline{u}|^2\langle\underset{\sim}{u}\rangle_\lambda\cdot\underset{\sim}{\partial}\varphi+p\,\underline{u}\cdot\underline{\partial}\varphi\Big]\,dt\,d\underline{\xi}\end{aligned}\tag{3.5.1}$$

where $\varphi_t\equiv\partial_t\varphi$ and $\underline{u}=\underline{u}^\lambda$ is in fact depending also on the regularization parameter λ; here p is the pressure $p=-\sum_{ij}\Delta^{-1}\partial_i\partial_j(u_iu_j)$.

Suppose that the solution of (3.3.2) with fixed initial datum $\underline{u}_0$ converges, for $\lambda\to\infty$, to a "Leray solution" $\underline{u}$, described in (E) and definition 2 of §3.3, possibly over a subsequence $\lambda_n\to\infty$.

The (3.5.1) implies, see below, that (any, in case of non uniqueness) Leray solution $\underline{u}$ verifies the *energy inequality*:

$$\begin{aligned}&\frac{1}{2}\int_\Omega|\underline{u}(\underline{\xi},s)|^2\varphi(\underline{\xi},s)\,d\underline{\xi}+\nu\int_{t\le s}\int_\Omega\varphi(\underline{\xi},t)|\underset{\sim}{\underline{\partial}}\underline{u}(\underline{\xi},t)|^2 d\underline{\xi}dt\le\\&\le\int_{t\le s}\int_\Omega\Big[\frac{1}{2}(\varphi_t+\nu\,\Delta\varphi)|\underline{u}|^2+|\underline{u}|^2\underset{\sim}{u}\cdot\underset{\sim}{\partial}\varphi+p\,\underset{\sim}{u}\cdot\underset{\sim}{\partial}\varphi\Big]\,d\underline{\xi}\,dt\end{aligned}\tag{3.5.2}$$

where the pressure p is given by $p=-\sum_{ij}\Delta^{-1}\partial_i\partial_j(u_iu_j)\equiv-\Delta^{-1}\underset{\sim}{\partial}\,\underline{\partial}(\underline{u}\,\underset{\sim}{u})$ $\equiv-\Delta^{-1}(\underset{\sim}{\partial}\,\underline{u})^2$.

Remarks:

(1) It is important to note that in this relation one might expect the equal sign =: as we shall see the fact that we cannot do better than just obtaining an inequality means that the limit necessary to reach a Leray solution can introduce a "spurious dissipation" that we are simply unable to understand on the basis of what we know (today) about the Leray solutions.

(2) The above "strange" phenomenon reflects our inability to develop a complete theory of the Navier–Stokes equation, and it is possible to conjecture that no other dissipation can take place and that a (yet to come) complete theory of the equations could show this. Hence we should take

[1] As discussed in §3.3 the solutions of (3.3.2) are $C^\infty(\Omega\times[0,\infty))$ so that there is no need to justify integrating by parts.

the inequality sign in (3.5.2) as one more manifestation of the inadequacy of the Leray's solution.

(B) General Sobolev's inequalities and further a priori bounds.

The proof of (3.5.2) and of the other inequalities that we shall quote and use in this section is elementary and based, *c.f.r.* problem [3.5.15] below, on a few general "kinematic inequalities" that we now list (all of them will be used in this section although to check (3.5.2) only (S) and (CZ) are employed)

(P) *Poincarè inequality:*

$$\int_{B_r} d\underline{x}\,|f-F|^\alpha \le C^P_\alpha\, r^{3-2\alpha}\Big(\int_{B_r} d\underline{x}\,|\underline{\partial} f|\Big)^\alpha, \qquad 1\le\alpha\le\frac32 \tag{3.5.3}$$

where F is the average of f on the ball B_r with radius r and C^P_α is a suitable constant. We shall denote (3.5.3) by (P).

(S) *Sobolev inequality:*

$$\begin{aligned}\int_{B_r}|\underline{u}|^q\,d\underline{x}\le C^S_q\Big[\Big(\int_{B_r}(\underset{\sim}{\underline{\partial u}})^2\,d\underline{x}\Big)^a\cdot\Big(\int_{B_r}|\underline{u}|^2\,d\underline{x}\Big)^{q/2-a}+\\ +\,r^{-2a}\Big(\int_{B_r}|\underline{u}|^2\,d\underline{x}\Big)^{q/2}\Big]\end{aligned} \tag{3.5.4}$$

if $2\le q\le 6,\quad a=\frac34(q-2)$, where B_r is a ball of radius r and the integrals are performed with respect to $d\underline{x}$. The C^S_q is a suitable constant; the second term of the right hand side can be omitted if $\underline{u}$ has zero average over B_r. We shall denote (3.5.4) by (S), [So63].

(CZ) *Calderon–Zygmund inequality:*

$$\int_\Omega |\sum_{i,j}(\Delta^{-1}\partial_i\partial_j)(u_iu_j)|^q d\underline{\xi}\le C^L_q\int_\Omega|\underline{u}|^{2q}d\underline{\xi}\,,\quad 1<q<\infty \tag{3.5.5}$$

which we shall denote (CZ): here Ω is the torus of side L and C^L_q is a suitable constant, [St93].

(H) *Hölder inequality*:

$$\Big|\int f_1f_2\ldots f_n\Big|\le\prod_{i=1}^n\Big(\int|f_i|^{p_i}\Big)^{\frac1{p_i}},\qquad \sum_{i=1}^n\frac1{p_i}=1 \tag{3.5.6}$$

which we shall denote (H): the integrals are performed over an arbitrary domain with respect to an arbitrary measure.

Remarks:

The (H) are a trivial extension of the Schwartz-Hölder inequalities; while (S) and (P) (mainly in the cases, important in what follows, $q = 6$ and $\alpha = \frac{3}{2}$) and (CZ) are less elementary and we refer to the literature, footnote at p.43 [So63], [St93], and p. 213, 219 of [LL01].

A proof of (3.5.2), given the (3.5.3)÷(3.5.6), is illustrated in problem [3.5.1]. Another important consequence of the inequalities is

I *Proposition: Let $\underline{u}$ be a Leray solution verifying (therefore) the a priori bounds in (3.3.8): $\int_\Omega |\underline{u}(\underline{x},t)|^2 d\underline{x} \le E_0$ and $\int_0^T dt \int_\Omega |\underset{\sim}{\partial}\underline{u}(\underline{x},t)|^2\, d\underline{x} \le E_0\nu^{-1}$ then*

$$\int_0^T dt \int_\Omega d\underline{x}\, |\underline{u}|^{10/3} + \int_0^T dt \int_\Omega d\underline{x}\, |p|^{5/3} \le C\nu^{-1} E_0^{5/3} \tag{3.5.7}$$

where C can be chosen $C^S_{\frac{10}{3}}\,(1 + C^L_{\frac{5}{3}})$.

proof: Apply (S) with $q = \frac{10}{3}$ and $a = 1$:

$$\begin{aligned}\int_\Omega |\underline{u}|^{\frac{10}{3}}\, d\underline{x} \le C^S_{\frac{10}{3}} \Big(\int_\Omega (\underset{\sim}{\partial}\underline{u})^2\, d\underline{x}\Big)^1 \cdot \Big(\int_\Omega \underline{u}^2\, d\underline{x}\Big)^{\frac{5}{3}-1} \le \\ \le C^S_{\frac{10}{3}} E_0^{\frac{2}{3}} \int_\Omega |\underset{\sim}{\partial}\underline{u}|^2\end{aligned} \tag{3.5.8}$$

hence integrating over t between 0 and T using also the second *a priori* estimate, we find

$$\int_0^T dt \int_\Omega |\underline{u}|^{\frac{10}{3}}\, d\underline{x} \le C^S_{\frac{10}{3}} E_0^{\frac{2}{3}} \int_0^T dt \int_\Omega d\underline{x}\, (\underset{\sim}{\partial}\underline{u})^2 \le C^S_{\frac{10}{3}} \nu^{-1} E_0^{1+\frac{2}{3}} \tag{3.5.9}$$

while the (CZ) yields: $\int_\Omega d\underline{x}\, |p|^{\frac{5}{3}} \le C^L_{\frac{5}{3}} \int_\Omega d\underline{x}\, |\underline{u}|^{\frac{10}{3}}$ which, integrated over t and combined with (3.5.9), gives the announced result.

(C) Pseudo Navier Stokes velocity–pressure pairs. Scaling operators.

As already mentioned the CKN theory will not fully use that $\underline{u}$ verifies the Navier–Stokes equation: in order to better realize this (unpleasant) property it is convenient to define separately the only properties of the Leray solutions that are really needed to develop the theory, *i.e.* to obtain an estimate of the fractal dimension of the space–time singularities set S_0. This leads to the following notion

1. Definition *(pseudo NS velocity field): Let $t \to (\underline{u}(\cdot,t), p(\cdot,t))$ be a function with values in the space of zero average square integrable "velocity" and "pressure" fields on Ω. Suppose that for each $\varphi \in C^\infty(\Omega \times (0,T])$ con*

$\varphi(\underline{x},t)$ vanishing for t near zero the following properties hold. For each $T<\infty$ and $s\le T$:

$$\begin{aligned}
&(a)\quad \int_\Omega \underline{u}\,d\underline{x}=\underline{0},\qquad \underline{\partial}\cdot\underline{u}=\underline{0},\qquad p=-\sum_{i,j}\partial_i\partial_j\Delta^{-1}(u_iu_j)\\
&b)\quad \int_0^T dt\int_\Omega d\underline{x}\,|\underline{u}|^{10/3}+\int_0^T dt\int_\Omega d\underline{x}\,|p|^{5/3}<\infty \qquad (3.5.10)\\
&(c)\quad \frac12\int_\Omega d\underline{x}\,|\underline{u}(\underline{x},s)|^2\varphi(\underline{x},s)+\nu\int_{t\le s}\int_\Omega \varphi(x,t)|\underset{\sim}{\underline{\partial u}}|^2 d\underline{x}dt\le\\
&\qquad\le\int_{t\le s}\int_\Omega\Big[\frac12(\varphi_t+\nu\,\Delta\varphi)|\underline{u}|^2+\frac12|\underline{u}|^2\underset{\sim}{u}\cdot\underset{\sim}{\partial}\varphi+p\,\underset{\sim}{u}\cdot\underset{\sim}{\partial}\varphi\Big]\,d\underline{x}\,dt
\end{aligned}$$

Then we shall say that the pair $(\underline{u},p)$ is a pseudo NS velocity and pressure pair. The singularity set of $(\underline{u},p)$ will be defined as the set S_0 of the points $(\underline{x},t)\in\Omega\times[0,T]$ that do not admit a vicinity U where $|\underline{u}|$ is bounded.[2]

The remaining part of this section will concern the general properties of the pseudo NS pairs and their regularity at a given point $(\underline{x},t)$: it will not have more to do with the velocity and pressure fields that solve the Navier–Stokes equations. It is indeed easy to convince oneself that the (3.5.10), in spite of the arbitrariness of φ *are not equivalent*, not even formally, to the Navier–Stokes equations, and they pose on $\underline{u},p$ restrictions far less severe. We should not be surprised, therefore, if it turned out possible to exhibit pseudo NS pairs that really have singularities on "large sets" of space–time. In a way it is already surprising that the pseudo NS fields verify the regularity properties discussed below.

The analysis of the latter properties (of pseudo NS fields) is based on the reciprocal relations between certain quantities that we shall call "*dimensionless operators*" relative to the space–time point $(\underline{x}_0,t_0)$

2 Definition: *(dimensionless "operators" for NS) Let $(\underline{x}_0,t_0)\in\Omega\times(0,\infty)$ and suppose*

$$\begin{aligned}
\Delta_r(t_0)=&\{t|\,|t-t_0|<r^2\nu^{-1}\}\\
B_r(\underline{x}_0)=&\{\underline{\xi}\,|\,|\underline{\xi}-\underline{x}_0|<r\}\equiv B_r\\
Q_r(\underline{x}_0,t_0)=&\{(\underline{\xi},\vartheta)\,|\,|\underline{\xi}-\underline{x}_0|<r,\,|\vartheta-t_0|<r^2\nu^{-1}\}=\\
=&\Delta_r(t_0)\times B_r(\underline{x}_0)\equiv Q_r
\end{aligned}\qquad(3.5.11)$$

define:[3]

[2] Here we mean bounded outside a set of zero measure in U or, as one says, *essentially bounded* because it is clear that, being $\underline{u},p$ in $L_2(\Omega)$, they are defined up to a set of zero measure and it would not make sense to ask that they are bounded everywhere without specifying which realization of the functions we take.

[3] If $r\ge L/2$ this is interpreted as $B_r\equiv\Omega$.

(i) "dimensionless kinetic energy operator" on scale r:

$$A(r) = \frac{1}{\nu^2 r} \sup_{|t-t_0|\le \nu^{-1}r^2} \int_{B_r} |\underline{u}(\underline{\xi},t)|^2 \, d\underline{\xi} \tag{3.5.12}$$

and we say that the dimension of A is 1 : this refers to the factor r^{-1} that is used to make the integral dimensionless.
(ii) "local Reynolds number" on scale r:

$$\delta(r) = \frac{1}{\nu r} \int_{Q_r} d\vartheta d\underline{\xi} \, |\underset{\sim}{\partial} \, \underline{u}|^2 \tag{3.5.13}$$

and we say that the dimension of δ is 1 : this refers to the factor r^{-1} that is used to make the integral dimensionless.
(iii) "dimensionless energy flux" on scale r:

$$G(r) = \frac{1}{\nu^2 r^2} \int_{Q_r} d\vartheta d\underline{\xi} \, |\underline{u}|^3 \tag{3.5.14}$$

and we say that the dimension of G is 2: this refers to the factor r^{-2} that is used to make the integral dimensionless.
(iv) "dimensionless pressure power" forces on scale r:

$$J(r) = \frac{1}{\nu^2 r^2} \int_{Q_r} d\underline{\xi} d\vartheta \, |\underline{u}| \, |p| \tag{3.5.15}$$

and we say that the dimension of J is 2 : this refers to the factor r^{-2} that is used to make the integral dimensionless.
(v) "dimensionless non locality" on scale r:

$$K(r) = \frac{r^{-13/4}}{\nu^{3/2}} \int_{\Delta_r} d\vartheta \Big(\int_{B_r} |p| \, d\underline{\xi} \Big)^{5/4} \tag{3.5.16}$$

and we say that the dimension of K is $13/4$: this refers to the factor $r^{-13/4}$ that is used to make the integral dimensionless.
(vi) "dimensionless intensity" on scale r:

$$S(r) = \nu^{-7/3} r^{-5/3} \int_{Q_r} (|\underline{u}|^{10/3} + |p|^{5/3}) d\vartheta d\underline{\xi} \tag{3.5.17}$$

where the pressure is always defined in terms of $\underline{u}$ by the expression $p = -\sum_{i,j=1}^{3} \partial_i \partial_j \Delta^{-1}(u_i u_j)$. And we say that the dimension of A is $5/3$: this refers to the factor $r^{-5/3}$ that is used to make the integral dimensionless.

Remarks:

(1) The $A(r), \ldots$ *are not* operators in the common sense of functional analysis. Their name is due to their analogy with the quantities that appear in

problems that are studied with the methods of the "renormalization group" (which, also, are not operators in the common sense of the words). Perhaps a more appropriate name could be "dimensionless observables": but we shall call them operators to stress the analogy of what follows with the methods of the renormalization group.

(2) The $A(r), G(r), J(r), K(r), S(r)$ are in fact estimates of the quantities that their name evokes. We omit the qualifier "estimate" when referring to them for brevity.

(3) The interest of (i)÷(iv) becomes manifest if we note that the energy inequality (3.5.10) can be expressed in terms of such quantities if φ is suitably chosen. Indeed let

$$\varphi = \chi(\underline{x}, t)\, \frac{\exp -\big(\frac{(\underline{x}-\underline{x}_0)^2}{4(\nu(t_0-t)+2r^2)}\big)}{(4\pi\nu(t-t_0)+8\pi r^2)^{3/2}} \tag{3.5.18}$$

where $\chi(\underline{x}, t)$ is C^∞ and has value 1 if $(\underline{x}, t) \in Q_{r/2}$ and 0 if $(\underline{x}, t) \not\in Q_r$. Then there exists a constant $C > 0$ such that

$$\begin{aligned} &|\varphi| < \frac{C}{r^3}, \quad |\underline{\partial}\varphi| < \frac{C}{r^4}, \qquad |\partial_t\varphi + \nu\Delta\varphi| < \frac{C}{\nu^{-1}r^5}, \qquad \text{everywhere}\\ &|\varphi| > \frac{1}{Cr^3}, \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad (\underline{x}, t) \ni Q_{r/2} \end{aligned} \tag{3.5.19}$$

Hence (3.5.10) implies

$$\frac{\nu^2}{Cr^2}\big(A(\tfrac{r}{2}) + \delta(\tfrac{r}{2})\big) \le C\,\big(\frac{1}{\nu^{-1}r^5}\int_{Q_r} |\underline{u}|^2 + \frac{1}{r^4}\int_{Q_r} |\underline{u}|^3 + \frac{1}{r^4}\int_{Q_r} |\underline{u}||p|\big) \tag{3.5.20}$$

and, since $\int_{Q_r} |\underline{u}|^2 \le C\,(\int_{Q_r} |\underline{u}|^3)^{2/3}(\nu^{-1}r^5)^{1/3}$ with a suitable C, it follows that for some $\tilde{C}$

$$A(\tfrac{r}{2}) + \delta(\tfrac{r}{2}) \le \tilde{C}\, \big(G(r)^{2/3} + G(r) + J(r)\big) \tag{3.5.21}$$

(4) Note that the operator $\delta(r)$ is an average of the "local Reynolds' number" of §3.4, see (3.4.14), (3.4.22), which has therefore dimension 1 in the above sense.

(5) The operator (v) appears if one tries to bound $J(\frac{r}{2})$ in terms of $A(r) + \delta(r)$: such an estimate is indeed possible and it leads to the following *local Scheffer theorem*

(D) The theorems of Scheffer and of Caffarelli–Kohn–Nirenberg.

We can state the strongest results known (in general and to date) about the regularity of the weak solutions of Navier Stokes equations (which however hold also for the pseudo Navier Stokes velocity–pressure pairs).

II Theorem *(upper bound on the dimension of the sporadic set of singular times for NS, (Scheffer)): There are two constants $\varepsilon_s, C > 0$ such that if*

$G(r) + J(r) + K(r) < \varepsilon_s$ for a certain value of r, then $\underline{u}$ is bounded in $Q_{\frac{r}{2}}(\underline{x}_0, t_0)$:

$$|\underline{u}(\underline{x}, t)| \le C\frac{\varepsilon_s^{1/3}}{r}, \quad (\underline{x}, t) \in Q_{\frac{r}{2}}(\underline{x}_0, t_0), \quad \text{almost everywhere} \tag{3.5.22}$$

having set $\nu = 1$.

Remarks: (1) *c.f.r.* problems [3.5.5]÷[3.5.11] for a guide to the proof.
(2) This theorem can be conveniently combined, for the purpose of checking its hypotheses, with the inequality: $J(r) + G(r) + K(r) \le C\,\big(S(r)^{9/10} + S(r)^{3/4}\big)$, which follows immediately from inequality (H) and from the definitions of the operators, with a suitable C.
(3) In other words *if the operator $S(r)$ is small enough then $(\underline{x}_0, t_0)$ is a regular point.*
(4) This implies, with the *a priori* bound (3.5.7), and by a repetition of the analysis in §3.4, with the $S(r)$ playing the role of(3.4.22), that the fractal dimension of the space–time singularities set is $\le 5/3$. In fact an *a priori* estimate on the global value of an operator with dimension α implies that the Hausdorff' measure of the set of points around which the operator is large does not exceed α: in §3.4 the dimension 1 operator $\delta(r)$, *i.e.* (3.5.13) or (3.4.22), was used together with the *a priori* vorticity estimate (3.3.8) (*c.f.r.* (3.4.21)) obtaining a Hausdorff's dimension bound 1; here the operator $S(r)$ has dimension 5/3 and therefore together with the *a priori* bound (3.5.7) it yields and estimate $\le 5/3$ for the Hausdorff dimension of the singularity set. This also justifies the introduction of the operator $S(r)$.

It is now easy, in terms of the just defined operators, to illustrate the strategy of the proof of the following CKN theorem (due to Caffarelli, Kohn, Nirenberg, [CKN82]) which by the arguments in §3.4 (*c.f.r.* (3.4.22)) implies in turn that the fractal dimension of the space time singularities set S_0 for a pseudo NS field is ≤ 1 and that its 1–measure of Hausdorff $\mu_1(S_0)$ vanishes.

III Theorem: *(upper bound on the Hausdorff dimension of the sporadic singular points in space-time ("CKN theorem")) There is a constant ε_{ckn} such that if $(\underline{u}, p)$ is a pseudo NS pair of velocity and pressure fields and*

$$\limsup_{r\to 0} \frac{1}{\nu r}\int_{Q_r(\underline{x}_0, t_0)} |\underset{\sim}{\partial}\underline{u}(\underline{x}', t')|^2\, d\underline{x}' dt' \equiv \limsup_{r\to 0}\, \delta(r) \,<\, \varepsilon_{ckn} \tag{3.5.23}$$

then $\underline{u}(\underline{x}', t')$, $p(\underline{x}', t')$ are C^∞ in the vicinity of $(\underline{x}_0, t_0)$.[4]

For fixed $(\underline{x}_0, t_0)$, consider the "sequence of length scales": $r_n \equiv L2^n$, with $n = 0, -1, -2, \ldots$. We shall set $\alpha_n \equiv A(r_n)$, $\kappa_n = K_n^{8/5}$, $j_n = J_n$, $g_n - G_n^{2/3}$,

[4] This means that near $(\underline{x}, t)$ the functions $\underline{u}(\underline{x}', t'), p(\underline{x}', t')$ coincide with C^∞ functions apart form a set of zero measure (recall that the pseudo NS fields are defined as fields in $L_2(\Omega)$).

$\delta_n = \delta(r_n)$ which is a natural definition as it will shortly appear. And define $\underline{X}_n \equiv (\alpha_n, \kappa_n, j_n, g_n) \in R_+^4$. Then the proof of this theorem is based on a bound that allows us to estimate the size of $\underline{X}_n$, defined as the sum of its components, in terms of the size of $\underline{X}_{n+p}$ provided the Reynolds number δ_{n+p} on scale $n+p$ is $\le \delta$.

The inequality will have the form, if $p > 0$ and $0 < \delta < 1$,

$$\underline{X}_n \le \mathcal{B}_p(\underline{X}_{n+p}; \delta) \tag{3.5.24}$$

where $\mathcal{B}_p(\cdot;\delta)$ is a map of the whole R_+^4 into itself and the inequality has to be understood "component wise", *i.e.* in the sense that each component of the l.h.s. is bounded by the corresponding component of the r.h.s. We call $|\underline{X}|$ the sum of the components of $\underline{X} \in R_+^4$.

The map $\mathcal{B}_p(\cdot;\delta)$, which to some readers will appear as strongly related to the "*beta function*" for the "running couplings" of the "renormalization group approaches",[5] will enjoy the following property

IV *Proposition: Suppose that p is large enough; given* $\rho > 0$ *there exists* $\delta_p(\rho) > 0$ *such that if* $\delta < \delta_p(\rho)$ *then the iterates of the map* $\mathcal{B}_p(\cdot;\delta)$ *contract any given ball in* R_+^4*, within a finite number of iterations, into the ball of radius* ρ*: i.e.* $|\mathcal{B}_p^k(\underline{X};\delta)| < \rho$ *for all large k's.*

Assuming the above proposition theorem III follows:

proof of theorem III: Let $\rho = \varepsilon_s$, *c.f.r.* theorem II, and let p be so large that proposition IV holds. We set $\varepsilon_{ckn} = \delta_p(\varepsilon_s)$ and it will be, by the assumption (3.5.23), that $\delta_n < \varepsilon_{ckn}$ for all $n \le n_0$ for a suitable n_0 (recall that the scale labels n are negative).

Therefore it follows that $|\mathcal{B}_p^k(\underline{X}_{n_0};\varepsilon_{ckn})| < \varepsilon_s$ for some k. Therefore by the theorem II we conclude that $(\underline{x}_0, t_0)$ is a regularity point.

(E) Proof that the renormalization map contracts.

Proposition IV follows immediately from the following general "Sobolev inequalities"

(1) *"Kinematic inequalities": i.e.* inequalities depending only on the fact that $\underline{u}$ is a divergence zero, average zero and is in $L_2(\Omega)$ and $p = -\Delta^{-1}(\underset{\sim}{\partial}\, \underline{u})^2$

$$\begin{aligned} J_n &\le C\,(2^{-p/5} A_{n+p}^{1/5} G_n^{1/5} K_{n+p}^{4/5} + 2^{2p} A_{n+p}^{1/2} \delta_{n+p}) \\ K_n &\le C\,(2^{-p/2} K_{n+p} + 2^{5p/4} A_{n+p}^{5/8} \delta_{n+p}^{5/8}) \\ G_n^{2/3} &\le C\,(2^{-2p} A_{n+p} + 2^{2p} A_{n+p}^{1/2} \delta_{n+p}^{1/2}) \end{aligned} \tag{3.5.25}$$

[5] As it relates properties of operators on a scale to those on a different scale. Note, however, that the couplings on scale n, *i.e.* the components of $\underline{X}_n$, provide information on those of X_{n+p} rather than on those of $\underline{X}_{n-p}$ as usual in the renormalization group methods, see [BG95].

where C denotes a suitable constant (*independent on the particular pseudo NS field*). The proof of the inequalities (3.5.25) is not difficult, assuming the (S,H,CZ,P) inequalities above, and it is illustrated in the problems [3.5.1], [3.5.2], [3.5.3].

(2) *"Dynamical inequality: i.e.* an inequality based on the energy inequality (c) in (3.5.10) which implies, quite easily, the following "*dynamic inequality*"[6]

$$A_n \le C\,(2^p G_{n+p}^{2/3} + 2^p A_{n+p}\delta_{n+p} + 2^p J_{n+p}) \tag{3.5.26}$$

whose proof is illustrated in problem [3.5.4].

proof of proposition IV: Assume the above inequalities (3.5.25), (3.5.26) and setting $\alpha_n = A_n, \kappa_n = K_n^{8/5}, j_n = J_n, g_n = G_n^{2/3}$, $\delta_{n+p} = \delta$ and, as above, $\underline{X}_n = (\alpha_n, \kappa_n, j_n, g_n)$. The r.h.s. of the inequalities defines the map $\mathcal{B}_p(\underline{X};\delta)$.

If one stares long enough at them one realizes that the contraction property of the proposition is an immediate consequence of

(1) The exponents to which $\varepsilon = 2^{-p}$ is raised in the various terms are either positive or not; in the latter cases the inverse power of ε is always appearing multiplied by a power of δ_{n+p} which we can take so small to compensate for the size of ε to any negative power, *except in the one case corresponding to the last term in* (3.5.26) where we see ε^{-1} without any compensating δ_{n+p}.

(2) Furthermore the sum of the powers of the components of $\underline{X}_n$ in each term of the inequalities is *always* ≤ 1: this means that the inequalities are "almost linear" and a linear map that "bounds" $\mathcal{B}_p$ exists and it is described by a matrix with small entries *except one off–diagonal element.* The iterates of the matrix therefore contract unless the large matrix element "is ill placed" in the matrix: and one easily sees that it is not.

A formal argument can be deviced in many ways: we present one in which several choices appear that are quite arbitrary and that the reader can replace with alternatives. In a way one should really try to see why a formal argument is not necessary.

The relation (3.5.26) can be "iterated" by using the expressions (3.5.25) for G_{n+p}, J_{n+p} and then the first of (3.5.25) to express $G_{n+p}^{1/5}$ in terms of A_{n+2p} with n replaced by $n+p$:

$$\begin{aligned}\alpha_n \le\; & C\,(2^{-p}\alpha_{n+2p} + 2^{3p}\delta_{n+2p}^{1/2}\alpha_{n+2p}^{1/2} +\\ & + 2^{p/5}(\alpha_{n+2p}\kappa_{n+2p})^{1/2} + 2^{7p/5}\delta_{n+2p}\alpha_{n+2p}^{7/20}\kappa_{n+2p}^{1/2} +\\ & + 2^{3p}\delta_{n+2p}\alpha_{n+2p})\end{aligned} \tag{3.5.27}$$

[6] We call it "dynamic" because it follows from the energy inequality, *i.e.* from the equations of motion.

It is convenient to take advantage of the simple inequalities $(ab)^{\frac{1}{2}} \le za + z^{-1}b$ and $a^x \le 1 + a$ for $a, b, z, x > 0$, $x \le 1$.

The (3.5.27) can be turned into a relation between α_n and $\alpha_{n+p}, \kappa_{n+p}$ by replacing p by $\frac{1}{2}p$. Furthermore, in the relation between α_n and $\alpha_{n+p}, \kappa_{n+p}$ obtained after the latter replacement, we choose $z = 2^{-p/5}$ to disentangle $2^{p/10}(\alpha_{n+p}\kappa_{n+p})^{1/2}$ we obtain recurrent (generous) estimates for α_n, κ_2

$$\begin{aligned} \alpha_n &\le C\,(2^{-p/10}\alpha_{n+p} + 2^{3p/10}\kappa_{n+p} + \xi^{\alpha}_{n+p})\\ \kappa_n &\le C\,(2^{-4p/5}\kappa_{n+p} + \xi^{\kappa}_{n+p})\\ \xi^{\alpha}_{n+p} &\stackrel{def}{=} 2^{3p}\delta_{n+p}(\alpha_{n+p} + \kappa_{n+p} + 1)\\ \xi^{\kappa}_{n+p} &\stackrel{def}{=} 2^{3p}\delta_{n+p}\alpha_{n+p} \end{aligned} \tag{3.5.28}$$

We fix p once and for all such that $2^{-p/10}C < \frac{1}{3}$.

Then if $C2^{3p}\delta_n$ is small enough, *i.e.* if δ_n is small enough, say for $\delta_n < \overline{\delta}$ *for all* $|n| \ge \overline{n}$, the matrix $M = C\begin{pmatrix} 2^{-p/10} + 2^{3p}\delta_{n+p} & 2^{3p/10} + 2^{3p}\delta_{n+p} \\ 0 & 2^{-4p/5} + 2^{3p}\delta_{n+p}\end{pmatrix}$ will have the two eigenvalues $< \frac{1}{2}$ and iteration of (3.5.27) will contract any ball in the plane α, κ to the ball of radius $2\,\overline{\delta}$.

If α_n, κ_n are bounded by a constant $\overline{\delta}$ for all $|n|$ large enough the (3.5.25) show that also g_n, j_n are going to be eventually bounded proportionally to $\overline{\delta}$.

Hence by imposing that δ is so small that $|\underline{X}_n| = |\alpha_n| + \kappa_n + j_n + g_n < \rho$ we see that proposition IV holds.

Problems. The CKN theory.

In the following problems we shall set $\nu = 1$, with no loss of generality, thus fixing the units so that time is a square length. The symbols $(\underline{u}, p)$ will denote a pseudo NS field, according to definition 1 in (C). Moreover, for notational simplicity, we shall set $A_\rho \equiv A(\rho)$, $G_\rho \equiv G(\rho), \ldots$, and sometimes we shall write $A_{r_n}, G_{r_n} \ldots$ as $A_n, \ldots$ with an abuse that should not generate ambiguities. The validity of the (3.5.10) for Leray's solution is checked in problem [3.5.15], at the end of the problems section, to stress that the theorems of Scheffer and CKN concern pseudo NS velocity–pressure fields: however it is independent of the first 14 problems.

[3.5.1]: Let $\rho = r_{n+p}$ and $r = r_n$, with $r_n = L2^n$, *c.f.r.* lines following (3.5.23), and apply (S),(3.5.4), with $q = 3$ and $a = \frac{3}{4}$, to the field $\underline{u}$, at t fixed in Δ_r and using definition 2 deduce

$$\begin{aligned} \int_{B_r} |\underline{u}|^3 d\underline{x} \le C_3^S [\Big(\int_{B_r} |\underset{\sim}{\partial} \underline{u}|^2\, d\underline{x}\Big)^{\frac{3}{4}} \Big(\int_{B_r} |\underline{u}|^2\, d\underline{x}\Big)^{\frac{3}{4}} + r^{-3/2}\Big(\int_{B_r} |\underline{u}|^2\Big)^{3/2}] \le\\ \le C_3^S [\rho^{3/4} A_\rho^{3/4} \Big(\int_{B_r} |\underset{\sim}{\partial} \underline{u}|^2 d\underline{x}\Big)^{3/4} + r^{-3/2}\Big(\int_{B_r} |\underline{u}|^2\Big)^{3/2}] \end{aligned}$$

Infer from the above the third of (3.5.25). (*Idea:* Let $\overline{|\underline{u}|^2_\rho}$ be the average of $\underline{u}^2$ on the ball B_ρ; apply the inequality (P), with $\alpha = 1$, to show that there is $C > 0$ such that

$$\begin{aligned}&\int_{B_r} d\underline{x}\,|\underline{u}|^2 \le \Big(\int_{B_\rho} d\underline{x}\,\Big||\underline{u}|^2 - \overline{|\underline{u}|^2_\rho}\Big|\Big) + \overline{|\underline{u}|^2_\rho}\int_{B_r} d\underline{x} \le\\ &\le C\rho\int_{B_\rho} d\underline{x}\,|\underline{u}||\underset{\sim}{\partial}\underline{u}| + C\Big(\frac{r}{\rho}\Big)^3\int_{B_\rho} d\underline{x}\,|\underline{u}|^2 \le C\rho^{3/2}A_\rho^{1/2}\Big(\int_{B_\rho} d\underline{x}\,|\underset{\sim}{\partial}\underline{u}|^2\Big)^{1/2} +\\ &+ C\Big(\frac{r}{\rho}\Big)^3\rho A_\rho\end{aligned}$$

where the dependence from $t \in \Delta_r$ is not explicitly indicated; hence

$$\int_{B_r} d\underline{x}\,|\underline{u}|^3 \le C\,(r\rho^{-1})^3 A_\rho^{3/2} + C\,(\rho^{3/4} + \rho^{9/4}r^{-3/2})A_\rho^{3/4}\Big(\int_{B_\rho} d\underline{x}\,|\underset{\sim}{\partial}\underline{u}|^2\Big)^{3/4}$$

then integrate both sides with respect to $t \in \Delta_r$ and apply (H) and definition 2.)

[3.5.2]: Let $\varphi \le 1$ be a non negative C^∞ function with value 1 if $|\underline{x}| \le 3\rho/4$ and 0 if $|\underline{x}| > 4\rho/5$; we suppose that it has the "scaling" form $\varphi = \varphi_1(\underline{x}/\rho$ with $\varphi_1 \ge 0$ a C^∞ function fixed once and for all. Let B_ρ be the ball centered at $\underline{x}$ with radius ρ; and note that, if $\rho = r_{n+p}$ and $r = r_n$, pressure can be written, at each time (without explicitly exhibiting the time dependence), as $p(\underline{x}) = p'(\underline{x}) + p''(\underline{x})$ with

$$\begin{aligned}p'(\underline{x}) =&\frac{1}{4\pi}\int_{B_\rho}\frac{1}{|\underline{x}-\underline{y}|}p(y)\Delta\varphi(\underline{y})\,d\underline{y} + \frac{1}{2\pi}\int_{B_\rho}\frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^3}\cdot\underline{\partial}\varphi(\underline{y})\,p(\underline{y})\,d\underline{y}\\ p''(\underline{x}) =&\frac{1}{4\pi}\int_{B_\rho}\frac{1}{|\underline{x}-\underline{y}|}\varphi(\underline{y})\,(\underset{\sim}{\partial}\underline{u}(\underline{y}))\cdot(\underset{\sim}{\partial}\underline{u}(\underline{y}))\,d\underline{y}\end{aligned}$$

if $|\underline{x}| < 3\rho/4$; and also $|p'(\underline{x})| \le C\rho^{-3}\int_{B_\rho} d\underline{y}\,|p(\underline{y})|$ and all functions are evaluated at a fixed $t \in \Delta_r$. Deduce from this remark the first of the (3.5.25). (*Idea:* First note the identity $p = -(4\pi)^{-1}\int_{B_\rho}|\underline{x}-\underline{y}|^{-1}\,\Delta\,(\varphi\,p)$ for $\underline{x} \in B_r$ because if $\underline{x} \in B_{3\rho/4}$ it is $\varphi\,p \equiv p$. Then note the identity $\Delta\,(\varphi\,p) = p\,\Delta\,\varphi + 2\underline{\partial}\,p\cdot\underline{\partial}\varphi + \varphi\,\Delta\,p$ and since $\Delta\,p = -\underline{\partial}\cdot(\underset{\sim}{u}\cdot\underline{\partial}\,\underline{u}) = -(\underline{\partial}\,\underset{\sim}{u})\cdot(\underline{\partial}\,\underline{u})$: the second of the latter relations generates p'' while $p\Delta\,\varphi$ combines with the contribution from $2\underline{\partial}\,p\cdot\underline{\partial}\varphi$, after integrating the latter by parts, and generates the two contributions to p'.
From the expression for p'' we see that

$$\begin{aligned}\int_{B_r} d\underline{x}\,|p''(\underline{x})|^2 \le &\int_{B_\rho\times B_\rho} d\underline{y}\,d\underline{y}'\,|\underset{\sim}{\partial}\underline{u}(\underline{y})|^2|\underset{\sim}{\partial}\underline{u}(\underline{y}')|^2\int_{B_r} d\underline{x}\frac{1}{|\underline{x}-\underline{y}||\underline{x}-\underline{y}'|} \le\\ &\le C\rho(\int_{B_\rho} d\underline{y}\,|\underset{\sim}{\partial}\underline{u}(\underline{y})|^2)^2\end{aligned}\qquad (!)$$

The part with p' is more interesting: since its integral expression above contains inside the integral kernels apparently singular at $\underline{x} = \underline{y}$ like $|\underline{x}-\underline{y}|^{-1}\,\Delta\,\varphi$ and $|\underline{x}-\underline{y}|^{-1}\,\underline{\partial}\,\varphi$ one notes that this is not true because the derivatives of φ vanish if $\underline{y} \in B_{3\rho/4}$ (where $\varphi \equiv 1$) so that $|\underline{x}-\underline{y}|^{-k}$ can be bounded "dimensionally" by ρ^{-k} in the whole region $B_\rho/B_{3\rho/4}$ for all $k \ge 0$ (this remark also shows why one should think p as sum of p' and p'').
Thus replacing the (apparently) singular kernels with their dimensional bounds we get

$$\int_{B_r} d\underline{x}\,|\underline{u}||p'| \le \frac{C}{\rho^3}\Big(\int_{B_r} d\underline{x}\,|\underline{u}|\Big)\cdot\Big(\int_{B_\rho} d\underline{x}\,|p|\Big)$$

which can be bounded by using inequality (H) as

$$\begin{aligned}
&\le \frac{C}{\rho^3}\Big(\int_{B_r} d\underline{x}\,|\underline{u}|^{2/5}\cdot|\underline{u}|^{3/5}\cdot 1\Big)\cdot\Big(\int_{B_\rho} d\underline{x}\,|p|\Big)\le\\
&\le \frac{C}{\rho^3}\Big(\int_{B_r} d\underline{x}\,|\underline{u}|^{2}\Big)^{1/5}\cdot\Big(\int_{B_r} d\underline{x}\,|\underline{u}|^{3}\Big)^{1/5}(r^3)^{3/5}\cdot\int_{B_\rho} d\underline{x}\,|p|\le\\
&\le \frac{C\,r^{9/5}}{\rho^3}\,(\rho A_\rho)^{1/5}\Big(\int_{B_r} d\underline{x}\,|\underline{u}|^{3}\Big)^{1/5}\cdot\Big(\int_{B_\rho} d\underline{x}\,|p|\Big)
\end{aligned}$$

where all functions depend on $\underline{x}$ (besides t) and then, integrating over $t\in\Delta_r$ and dividing by r^2 one finds, for a suitable $C>0$:

$$\frac{1}{r^2}\int_{Q_r} dt d\underline{x}\,|\underline{u}||p'|\le C\,(\frac{r}{\rho})^{1/5}G_r^{1/5}K_\rho^{4/5}A_\rho^{1/5}$$

that is combined with $\int_{B_r} d\underline{x}|\underline{u}||p''|\le(\int_{B_r} d\underline{x}\,|\underline{u}|^2)^{1/2}(\int_{B_r} d\underline{x}\,|p''|^2)^{1/2}$ which, integrating over time, dividing by ρ^2 and using inequality (!) for $\int_{B_r} d\underline{x}\,|p''|^2$ yields: $r^{-2}\int_{Q_r} dt d\underline{x}\,|\underline{u}||p''|\le C(\rho r^{-1})^2A_\rho^{1/2}\delta_\rho)$.

[3.5.3] In the context of the hint and notations for p of the preceding problem check that $\int_{B_r} d\underline{x}\,|p'|\le C(r\rho^{-1})^3\int_{B_\rho} d\underline{x}|p|$. Integrate over t the power 5/4 of this inequality, rendered adimensional by dividing it by $r^{13/4}$; one gets: $r^{-13/4}\int_{\Delta_r}(\int|p'|)^{5/4}\le C(r\rho^{-1})^{1/2}K_\rho$, which yields the first term of the second inequality in (3.5.25). Complete the derivation of the second of (3.5.25). (*Idea:* Note that $p''(\underline{x},t)$ can be written, in the interior of B_r, as $p''=\tilde p+\hat p$ con:

$$\tilde p(\underline{x})=-\frac{1}{4\pi}\int_{B_\rho}\frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^3}\varphi(\underline{y})\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}\,d\underline{y},\qquad \hat p(\underline{x})=-\frac{1}{4\pi}\int_{B_\rho}\frac{\underline{\partial}\varphi(\underline{y})\cdot(\underset{\sim}{u}\cdot\underset{\sim}{\partial})\underline{u}}{|\underline{x}-\underline{y}|}\,d\underline{y}$$

(always at fixed t and not declaring explicitly the t–dependence). Hence by using $|\underline{x}-\underline{y}|>\rho/4$, for $\underline{x}\in B_r$ and $\underline{y}\in B_\rho/B_{3\rho/4}$, *i.e.* for $\underline{y}$ in the part of B_ρ where $\underline{\partial}\,\varphi\ne\underline{0}$) we find

$$\begin{aligned}
\int_{B_r}|\tilde p|\,d\underline{x}&\le C\int_{B_\rho}d\underline{y}\Big(\int_{B_r}\frac{d\underline{x}}{|\underline{x}-\underline{y}|^2}\,|\underline{u}(\underline{y})|\,|\underset{\sim}{\partial}\,\underset{\sim}{u}(\underline{y})|\Big)\le\\
&\le Cr\Big(\int_{B_\rho}|\underline{u}|^2\Big)^{1/2}\Big(\int_{B_\rho}|\underline{\partial}\underline{u}|^2\Big)^{1/2}\le Cr\rho^{1/2}A_\rho^{1/2}\Big(\int_{B_\rho}|\underline{\partial}\underline{u}|^2\Big)^{1/2}\\
\int_{B_r}|\hat p|\,d\underline{x}&\le C\frac{r^3}{\rho^2}\int_{B_\rho}|\underline{u}||\underline{\partial}\underline{u}|\le C\,r\rho^{1/2}A_\rho^{1/2}\Big(\int_{B_\rho}|\underline{\partial}\underline{u}|^2\Big)^{1/2}
\end{aligned}$$

and $\Big(\int_{B_r}|p''|\Big)^{5/4}$ is bounded by raising the right hand sides of the last inequalities to the power 5/4 and integrating over t, and finally applying inequality (H) to generate the integral $\Big(\int_{Q_\rho}|\underline{\partial}\underline{u}|^2\Big)^{5/8}$).

[3.5.4]: Deduce that (3.5.26) holds for a pseudo–NS field $(\underline{u},p)$, *c.f.r.* definition 1. (*Idea:* Let $\varphi(\underline{x},t)$ be a C^∞ function which is 1 on $Q_{\rho/2}$ and 0 outside Q_ρ; it is: $0\le\varphi(\underline{x},t)\le 1$,

$|\underline{\partial}\varphi| \le \frac{C}{\rho}$, $|\Delta\varphi + \partial_t\varphi| \le \frac{C}{\rho^2}$, if we suppose that φ has the form $\varphi(\underline{x}, t) = \varphi_2(\frac{x}{\rho}, \frac{t}{\rho^2}) \ge 0$ for some φ_2 suitably fixed and smooth. Then, by applying the third of (3.5.10) and using the notations of the preceding problems, if $\bar t \in \Delta_{\rho/2}(t_0)$, it is

$$\begin{aligned}
&\int_{B_r\times\{\bar t\}} |\underline{u}(\underline{x}, t)|^2 d\underline{x} \le \frac{C}{\rho^2}\int_{Q_\rho} dt d\underline{x}\, |\underline{u}|^2 + \int_{Q_\rho} dt d\underline{x}\, (|\underline{u}|^2 + 2p)\, \underline{u}\cdot\underline{\partial}\varphi \le \\
&\le \frac{C}{\rho^2}\int_{Q_\rho} dt d\underline{x}\, |\underline{u}|^2 + \Big|\int_{Q_\rho} dt d\underline{x}\, (|\underline{u}|^2 - \overline{|\underline{u}|^2_\rho})\underline{u}\cdot\underline{\partial}\varphi\Big| + 2\int_{Q_\rho} dt d\underline{x}\, p\, \underline{u}\cdot\underline{\partial}\varphi \le \\
&\le \frac{C}{\rho^{1/3}}\Big(\int_{Q_\rho} dt d\underline{x}\, |\underline{u}|^3\Big)^{2/3} + \Big|\int_{Q_\rho} dt d\underline{x}\, \Big(|\underline{u}|^2 - \overline{|\underline{u}|^2_\rho}\Big)\, \underline{u}\cdot\underline{\partial}\varphi\Big| + \frac{2C}{\rho}\int_{B_\rho} dt d\underline{x}\, |p||\underline{u}| \le \\
&\le C\rho G_\rho^{2/3} + C\rho J_\rho + \rho\Big|\frac{1}{\rho}\int_{Q_\rho} dt d\underline{x}\, (|\underline{u}|^2 - \overline{|\underline{u}|^2_\rho})\, \underline{u}\cdot\underline{\partial}\varphi\Big| \qquad (*)
\end{aligned}$$

We now use the following inequality, at t fixed and with the integrals over $d\underline{x}$

$$\begin{aligned}
&\frac{1}{\rho}\Big|\int_{B_\rho} d\underline{x}\, (|\underline{u}|^2 - \overline{|\underline{u}|^2_\rho})\underline{u}\cdot\underline{\partial}\varphi\Big| \le \frac{C}{\rho^2}\int_{B_\rho} d\underline{x}\, |\underline{u}|\, \Big||\underline{u}|^2 - \overline{|\underline{u}|^2_\rho}\Big| \le \\
&\le \frac{C}{\rho^2}\Big(\int_{B_\rho} d\underline{x}\, |\underline{u}|^3\Big)^{1/3}\Big(\int_{B_\rho} |\underline{u}^2 - \overline{|\underline{u}|^2_\rho}|^{3/2}\Big)^{2/3}
\end{aligned}$$

and we also take into account inequality (P) with $f = \underline{u}^2$ and $\alpha = 3/2$ which yields (always at t fixed and with integrals over $d\underline{x}$):

$$\Big(\int_{B_\rho} \Big|\underline{u}^2 - \overline{|\underline{u}|^2_\rho}\Big|^{3/2}\Big)^{2/3} \le C\Big(\int_{B_\rho} |\underline{u}||\underline{\partial}\underline{u}|\Big)$$

then we see that

$$\begin{aligned}
&\int_{B_\rho} \Big||\underline{u}|^2 - \overline{|\underline{u}|^2_\rho}\Big|\, |\underline{u}|\, |\underline{\partial}\varphi| \le \frac{C}{\rho}\Big(\int_{B_\rho} |\underline{u}|^3\Big)^{1/3}\Big(\int_{B_\rho} |\underline{u}||\underline{\partial}\underline{u}|\Big) \le \\
&\le \frac{C}{\rho}\Big(\int_{B_\rho} |\underline{u}|^3\Big)^{1/3}\Big(\int_{B_\rho} |\underline{u}|^2\Big)^{1/2}\Big(\int_{B_\rho} |\underline{\partial}\underline{u}|^2\Big)^{1/2} \le \\
&\le \frac{C}{\rho}\rho^{1/2}A_\rho^{1/2}\Big(\int_{B_\rho} |\underline{u}|^3\Big)^{1/3}\cdot\Big(\int_{B_\rho} |\underline{\partial}\underline{u}|^2\Big)^{1/2}\cdot 1
\end{aligned}$$

Integrating over t and applying (H) with exponents $3, 2, 6$, respectively, on the last three factors of the right hand side we get

$$\frac{1}{\rho^2}\int_{Q_\rho} |\underline{u}|\Big|\, |\underline{u}|^2 - \overline{|\underline{u}|^2_\rho}\Big| \le C A_\rho^{1/2} G_\rho^{1/3}\delta_\rho^{1/2} \le C\,(G_\rho^{2/3} + A_\rho\delta_\rho)$$

and placing this in the first of the preceding inequalities (*) we obtain the desired result).

The following problems provide a guide to the proof of theorem II. Below we replace, unless explicitly stated the sets B_r, Q_r, Δ_r introduced in definition 2, in (C) above, and employed in the previous problems with B^0_r, Δ^0_r, Q^0_r with $B^0_r = \{\underline{x}|\, |\underline{x} - \underline{x}_0| < r\}$,

$\Delta^0_r = \{t \,|\, t_0 > t > t - r^2\}$, $Q^0_r = \{(\underline{x}, t) | \, |\underline{x} - \underline{x}_0| < r, \, t_0 > t > t - r^2\} = B^0_r \times \Delta^0_r$. *Likewise we shall set* $B^0_{r_n} = B^0_n, \Delta^0_{r_n} = \Delta^0_n, Q^0_{r_n} = Q^0_n$ *and we shall define new operators* A, δ, G, J, K, S *by the same expressions in (3.5.11)%(3.5.17) in (C) above but with the just defined new meaning of the integration domains. However, to avoid confusion, we shall call them* $A^0, \delta^0, \ldots$ *with a superscript* 0 *added.*

[3.5.5]: With the above conventions check the following inequalities

$$A^0_n \le C A^0_{n+1}, \qquad G^0_n \le C {G^0}_{n+1}, \qquad G^0_n \le C\,({A^0_n}^{3/2} + {A^0_n}^{3/4} {\delta^0_n}^{3/4})$$

(*Idea:* The first two are trivial consequences of the fact that the integration domains of the right hand sides are larger than those of the left hand sides, and the radii of the balls differ only by a factor 2 so that C can be chosen 2 in the first inequality and 4 in the second. The third inequality follows from (S) with $a = \frac{3}{4}$, $q = 3$:

$$\int_{B^0_r} |\underline{u}|^3 \le C \Big[\Big(\int_{B^0_r} |\underset{\sim}{\partial} \underline{u}|^2 \Big)^{3/4} \Big(\int_{B^0_r} |\underline{u}|^2 \Big)^{3/4} + r^{-3/2} \Big(\int_{B^0_r} |\underline{u}|^2 \Big)^{3/2} \Big] \le$$
$$\le C \Big[r^{3/4} {A^0_r}^{3/4} \Big(\int_{B^0_r} |\underset{\sim}{\partial} \underline{u}|^2 \Big)^{3/4} + {A^0_r}^{3/2} \Big]$$

where the integrals are over $d\underline{x}$ at t fixed; and integrating over t we estimate G^0_r by applying (H) to the last integral over t.)

[3.5.6]: Let $n_0 = n + p$ and $Q^0_n = \{(\underline{x}, t) | \, |\underline{x} - \underline{x}_0| < r_n, \, t_0 > t > t - r_n^2\} \stackrel{def}{=} B^0_n \times \Delta^0_n$ consider the function:

$$\varphi_n(\underline{x}, t) = \frac{\exp(-(\underline{x} - \underline{x}_0)^2 / 4(r_n^2 + t_0 - t))}{(4\pi(r_n^2 + t_0 - t))^{3/2}}, \qquad (\underline{x}, t) \in Q^0_{n_0}$$

and a function $\chi_{n_0}(\underline{x}, t) = 1$ on $Q^0_{n_0 - 1}$ and 0 outside $Q^0_{n_0}$, for instance choosing,, a function which has the form $\chi_{n_0}(\underline{x}, t) = \tilde{\varphi}(r_{n_0}^{-1} \underline{x}, r_{n_0}^{-1/2} t) \ge 0$, with $\tilde{\varphi}$ a C^∞ function fixed once and for all. Then write (3.5.10) using $\varphi = \varphi_n \chi_{n_0}$ and deduce the inequality

$$\frac{A^0_n + \delta^0_n}{r_n^2} \le C \Big[r_{n+p}^{-2} {G^0}^{2/3}_{n+p} + \sum_{k=n+1}^{n+p} r_k^{-2} G^0_k + r_{n+p}^{-2} {J^0}_{n+p} + \sum_{k=n+1}^{n+p-1} r_k^{-2} L_k \Big] \qquad (@)$$

where $L_k = r_k^{-2} \int_{Q^0_k} d\underline{x}\, dt\, |\underline{u}| |p - \overline{p^k}|$ with $\overline{p^k}$ equals the average of p on the ball B^0_k; for each $p > 0$. (*Idea:* Consider the function φ and note that $\varphi \ge (C r_n^3)^{-1}$ in Q^0_n, which allows us to estimate *from below* the left hand side term in (3.5.10), with $(C r_n^2)^{-1} (A^0_n + {\delta^0}_n)$. Moreover one checks that

$$|\varphi| \le \frac{C}{r_m^3}, \qquad |\underset{\sim}{\partial} \varphi| \le \frac{C}{r_m^4}, \quad n \le m \le n + p \equiv n_0, \quad \text{in} \quad Q^0_{m+1} / Q^0_m$$
$$|\partial_t \varphi + \Delta \varphi| \le \frac{C}{r_{n_0}^5} \qquad\qquad \text{in} \quad Q^0_{n_0}$$

and the second relation follows noting that $\partial_t \varphi + \Delta \varphi \equiv 0$ in the "dangerous places", *i.e.* $\chi = 1$, because φ is a solution of the heat equation (backward in time). Hence the first term in the right hand side of (3.5.10) can be bounded *from above* by

$$\int_{Q^0_{n_0}} |\underline{u}|^2 |\partial_t \varphi_n + \Delta \varphi_n| \le \frac{C}{r_{n_0}^5} \int_{Q^0_{n_0}} |\underline{u}|^2 \le \frac{C}{r_{n_0}^5} \Big(\int_{Q^0_{n_0}} |\underline{u}|^3 \Big)^{2/3} r_{n_0}^{5/3} \le \frac{C}{r_{n_0}^2} {G^0}^{2/3}_{n_0}$$

getting the first term in the r.h.s. of (@).

Using here the scaling properties of the function φ the second term is bounded by

$$\int_{Q^0_{n_0}} |\underline{u}|^3|\underline{\partial}\varphi_n| \le \frac{C}{r_n^4}\int_{Q^0_{n+1}} |\underline{u}|^3 + \sum_{k=n+2}^{n_0}\frac{C}{r_k^4}\int_{Q^0_k/Q^0_{k-1}} |\underline{u}|^3 \le$$
$$\le \sum_{k=n+1}^{n_0}\frac{C}{r_k^4}\int_{Q^0_k}|\underline{u}|^3 \le C\sum_{k=n+1}^{n_0}\frac{G^0_k}{r_k^2}$$

Calling the third term (*c.f.r.* (3.5.1)) Z we see that it is bounded by

$$Z \le \Big|\int_{Q^0_{n_0}} p\,\underline{u}\cdot\underline{\partial}\,\chi_{n_0}\varphi_n\Big| \le \Big|\int_{Q^0_{n+1}} p\,\underline{u}\cdot\underline{\partial}\,\chi_{n+1}\varphi_n\Big| +$$
$$+\sum_{k=n+2}^{n_0}\Big|\int_{Q^0_k} p\,\underline{u}\cdot\underline{\partial}\,(\chi_k-\chi_{k-1})\varphi_n\Big| \le \Big|\int_{Q^0_{n+1}} (p-\overline{p^{n+1}})\,\underline{u}\cdot\underline{\partial}\,\chi_{n+1}\varphi_n\Big| +$$
$$+\sum_{k=n+2}^{n_0-1}\Big|\int_{Q^0_k}(p-\overline{p^k})\,\underline{u}\cdot\underline{\partial}\,(\chi_k-\chi_{k-1})\varphi_n\Big| + \int_{Q^0_{n_0}} |\underline{u}|\,|p|\,|\underline{\partial}(\chi_{n_0}-\chi_{n_0-1})\varphi_n$$

where $\overline{p^m}$ denotes the average of p over B^0_m (which only depends on t): the possibility of replacing p by $p-\overline{p}$ in the integrals is simply due to the fact that the 0 divergence of $\underline{u}$ allows us to ad to p *any* constant because, by integration by parts, it will contribute 0 to the value of the integral.

From the last inequality it follows

$$Z \le \sum_{k=n+1}^{n_0-1}\frac{C}{r_k^4}\int_{Q^0_k}|p-\overline{p^k}|\,|\underline{u}| + J^0_{n_0}r_{n_0}^{-2} = \sum_{k=n+1}^{n_0-1}\frac{C}{r_k^2}L_k + J^0_{n_0}r_{n_0}^{-2}$$

then sum the above estimates.)

[3.5.7] If $\underline{x}_0$ the center of Ω the function $\chi_{n_0}p$ can be regarded, if $n_0<-1$, as defined on the whole R^3 and zero outside the torus Ω. Then if Δ is the Laplace operator *on the whole* R^3 note that the expression of p in terms of $\underline{u}$ (*c.f.r.* (a) of (3.5.10)) implies that in $Q^0_{n_0}$:

$$\chi_{n_0}p = \Delta^{-1}\Delta\chi_{n_0}p \equiv \Delta^{-1}\Big(p\Delta\chi_{n_0} + 2(\underline{\partial}\chi_{n_0})\cdot(\underline{\partial}p) - \chi_{n_0}\underset{\sim}{\partial}\underline{\partial}\cdot(\underset{\sim}{u}\underline{u})\Big)$$

Check that this expression can be rewritten, for $n<n_0$, as

$$-\underset{\sim}{\partial}\underline{\partial}\Delta^{-1}(\chi_{n_0}\underset{\sim}{u}\underline{u}\vartheta_{n+1}) - \underset{\sim}{\partial}\underline{\partial}\Delta^{-1}(\chi_{n_0}\underset{\sim}{u}\underline{u}(1-\vartheta_{n+1}))-$$
$$-2\underline{\partial}\Delta^{-1}(\underset{\sim}{\partial}\chi_{n_0}\underset{\sim}{u}\underline{u}) - \Delta^{-1}((\underset{\sim}{\partial}\underline{\partial}\chi_{n_0})\underline{u}\underset{\sim}{u})-$$
$$-\Delta^{-1}((\Delta\chi_{n_0})p) - 2\underline{\partial}\Delta^{-1}(\underline{\partial}\chi_{n_0}\,p)$$

where ϑ_k is the characteristic function of ${B^0}_k$.

[3.5.8] In the context of the previous problem check that in $Q^0_{n_0}$ it is $p = p_1+p_2+p_3+p_4$ with

$$p_1 = -\,(\underset{\sim}{\partial}\underset{\sim}{\partial}\Delta^{-1})\cdot\big(\chi_{n_0}\vartheta_{n+1}\underline{u}\,\underset{\sim}{\underline{u}}\big),\qquad p_2 = -\frac{1}{4\pi}\int_{B^0{}_{n_0}/B^0{}_{n+1}}\big(\underset{\sim}{\partial\partial}\frac{1}{|\underline{x}-\underline{y}|}\big)\cdot\chi_{n_0}\underline{u}\,\underset{\sim}{\underline{u}}$$

$$p_3 = -\frac{1}{2\pi}\int_{B^0_{n_0}}\frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^3}(\underset{\sim}{\partial}\chi_{n_0})\underline{u}\,\underset{\sim}{\underline{u}} + \frac{1}{4\pi}\int_{B^0_{n_0}}\frac{1}{|\underline{x}-\underline{y}|}(\underset{\sim}{\partial\partial}\chi_{n_0})\underline{u}\,\underline{u}$$

$$p_4 = \frac{1}{4\pi}\int\frac{1}{|\underline{x}-\underline{y}|}p(\underline{y})\Delta\chi_{n_0} + \frac{2}{4\pi}\int p(\underline{y})\frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^3}\cdot\underline{\partial}\chi_{n_0}$$

where $n < n_0$ and the integrals are over $\underline{y}$ at t fixed, and the functions in the left hand side are evaluated in $\underline{x}, t$.

[3.5.9]: Consider the quantity L_n introduced in [3.5.6] and show that, setting $n_0 = n+p$, $p>0$, it is

$$L_n \le C\Big[\Big(\frac{r_{n+1}}{r_{n_0}}\Big)^{7/5}A^0_{n+1}{}^{1/5}G^0_{n+1}{}^{1/5}K^0_{n_0}{}^{4/5} + \Big(\frac{r_{n+1}}{r_{n_0}}\Big)^{5/3}G^0_{n+1}{}^{1/3}G^0_{n_0}{}^{2/3} +$$

$$+\,G^0_{n+1} + r^3_{n+1}G^0_{n+1}{}^{1/3}\sum_{k=n+1}^{n_0} r_k^{-3}A^0_k\Big]$$

(*Idea:* Refer to [3.5.8] to bound L_n by: $\sum_{i=1}^4 r_n^{-2}\int_{Q^0_n}|\underline{u}||p_i-\overline{p_i^n}|$ where $\overline{p_i^n}$ is the average of p_i over B^0_n; and estimate separately the four terms. for the first it is not necessary to subtract the average and the difference $|p_1-\overline{p_1^n}|$ can be divided into the sum of the absolute values each of which contributes equally to the final estimate which is obtained via the (CZ), and the (H)

$$\int_{B^0_{n+1}}|p_1-\overline{p_1}||\underline{u}| \le 2\Big(\int_{B^0_{n+1}}|p_1|^{3/2}\Big)^{2/3}\Big(\int_{B^0_{n+1}}|\underline{u}|^3\Big)^{1/3}\le C\int_{B^0_{n+1}}|\underline{u}|^3$$

and the contribution of p_1 at L_n is bounded, therefore, by CG^0_{n+1}: note that this would not be true with p instead of p_1 because in the right hand side there would be $\int_\Omega|p|^{3/2}$ rather than $\int_{B^0_{n+1}}|p|^{3/2}$, because the (CZ) is a "nonlocal" inequality. The term with p_2 is bounded as

$$\int_{\Delta^0_n}\int_{B^0_n}|p_2-\overline{p_2^n}||\underline{u}|\le\int_{\Delta^0_n}\int_{B^0_n}|\underline{u}|\,r_n\max|\underline{\partial}p_2|\le$$

$$\le r_n\Big(\int_{Q^0_n}\frac{|\underline{u}|^3}{r_n^2}\Big)^{1/3}r_n^{2/3}r_n^{10/3}\max_{Q^0_n}|\underline{\partial}p_2|\le$$

$$\le r_n^5G^0_n{}^{1/3}\sum_{m=n+1}^{n_0-1}\max_{t\in\Delta^0_m}\int_{B^0_{m+1}/B^0_m}\frac{|\underline{u}|^2}{r_m^4} = r_n^5G^0_n{}^{1/3}\sum_{m=n+1}^{n_0}\frac{A^0_m}{r_m^3}$$

Analogously the term with p_3 is bounded by using $|\underline{\partial}p_3|\le Cr_{n_0}^{-4}\int_{B^0_{n_0}}|\underline{u}|^2$ which is majorized by $Cr_{n_0}^{-3}(\int_{B^0_{n_0}}|\underline{u}|^3)^{2/3}$ obtaining

$$\frac{1}{r_n^2}\int_{Q^0_n}|\underline{u}||p_3-\overline{p_3^n}|\le\frac{C}{r_n^2}r_{n_0}^{-3}\int_{\Delta^0_n}[(\int_{B^0_n}|\underline{u}|^3)^{2/3}r_n\int_{B^0_n}|\underline{u}|]\le$$

$$\le\frac{C}{r_n^2}r_n^3r_{n_0}^{-3}\int_{\Delta^0_n}(\int_{B^0_n}|\underline{u}|^3)^{2/3}(\int_{B^0_n}|\underline{u}|^3)^{1/3}\le$$

$$\le\frac{C}{r_n^2}(\frac{r_n}{r_{n_0}})^3r_{n_0}^{4/3}r_n^{2/3}G^0_{n_0}{}^{2/3}G^0_n{}^{1/3} = C(\frac{r_n}{r_{n_0}})^{5/3}G^0_{n_0}{}^{2/3}G^0_n{}^{1/3}$$

Finally the term with p_4 is bounded (taking into account that the derivatives $\Delta\chi_n, \underline{\partial}\chi_n$ vanish where the kernels become bigger than what suggested by their dimension) by noting that

$$\int_{B_n^0} |p_4 - \overline{p_4^n}||\underline{u}| \le C r_n \int_{B_n^0} |\underline{u}| \max_{B_n^0} |\underline{\partial} p_4| \le C r_n \Big(\int_{B_n^0} |\underline{u}|\Big)\Big(\int_{B_{n_0}^0} \frac{|p|}{r_{n_0}^4}\Big)$$

Denoting with $\tilde K_{n_0}^0$ the operator $K_{n_0}^0$ without the factor $r_{n_0}^{-13/4}$ which makes it dimensionless, and introducing, similarly, $\tilde A_n^0, \tilde G_n^0$) we obtain the following chain of inequalities, using repeatedly (H)

$$\begin{aligned}
&\frac{1}{r_n^2}\int_{Q_n^0} |p_4 - \overline{p_4^n}||\underline{u}| \le \frac{C}{r_n^2} r_n \Big(\int_{\Delta_n^0}\Big(\int_{B_{n_0}^0}\frac{|p|}{r_{n_0}^4}\Big)^{5/4}\Big)^{4/5}\Big(\int_{\Delta_n^0}\Big(\int_{B_n^0}|\underline{u}|\Big)^5\Big)^{1/5} \le\\
&\le \frac{C}{r_n^2}\frac{r_n}{r_{n_0}^4}\tilde K_{n_0}^{0\,4/5}\Big(\int_{\Delta_n^0}\Big(\int_{B_n^0}|\underline{u}|^{2/5}|\underline{u}|^{3/5}\cdot 1\Big)5\Big)^{1/5} \le\\
&\le \frac{C}{r_n^2}\frac{r_n}{r_{n_0}^4}\tilde K_{n_0}^{0\,4/5}\Big(\int_{B_n^0}|\underline{u}|^2\Big)^{1/5}\Big(\int_{Q_n^0}|\underline{u}|^3\Big)^{1/5} r_n^{9/5} \le\\
&\le \frac{C}{r_n^2}\frac{r_n}{r_{n_0}^4} r_n^{12/5}\tilde K_{n_0}^{0\,4/5}\tilde A_n^{0\,1/5}\tilde G_n^{0\,1/5} \le C\Big(\frac{r_n}{r_{n_0}}\Big)^{7/5} A_n^{0\,1/5} G_n^{0\,1/5} K_{n_0}^{0\,4/5}
\end{aligned}$$

Finally use the inequalities of [3.5.5] and combine the estimates above on the terms p_j, $j = 1, .., 4$.)

[3.5.10] Let $T_n = A_n^0 + \delta_n^0$; combine inequalities of [3.5.6] and [3.5.9], and [3.5.5] to deduce

$$\begin{aligned}
T_n \le 2^{2n}\Big(& 2^{-2n_0}\varepsilon + \sum_{k=n+1}^{n_0} 2^{-2k} T_k^{3/2} + 2^{-2n_0}\varepsilon + 2^{-7n_0/5}\varepsilon \sum_{k=n+1}^{n_0} 2^{-3k/5} T_k^{1/2} +\\
&+ \varepsilon 2^{-5n_0/3}\sum_{k=n_0+2}^{n_0} 2^{-k/3} T_k^{1/2} + \sum_{k=n+1}^{n_0} 2^{-2k} T_k^{3/2} + \sum_{k=n+1}^{n_0-1} 2^k T_k^{1/2} \sum_{q=k}^{n_0} 2^{-3q} T_q\Big)\\
&\varepsilon \equiv C \max(G_{n_0}^{0\,2/3}, K_{n_0}^{0\,4/5}, J^0{}_{n_0})
\end{aligned}$$

and show that, by induction, if ε is small enough then $r_n^{-2} T_n \le \varepsilon^{2/3} r_{n_0}^{-2}$.

[3.5.11]: If $G(r_0) + J(r_0) + K(r_0) < \varepsilon_s$ with ε_s small enough, then given $(\underline{x}', t') \in Q_{r_0/4}(\underline{x}_0, t_0)$, show that if one calls $G_r^0, J_r^0, K_r^0, A_r^0, \delta_r^0$ the operators associated with $Q_r^0(\underline{x}', t')$ then

$$\limsup_{n\to\infty} \frac{1}{r_n^2} A_n^0 \le C \frac{\varepsilon_s^{2/3}}{r_0^2}$$

for a suitable constant C. (*Idea:* Note that $Q^0_{r_0/4}(\underline{x}', t') \subset Q_{r_0}(\underline{x}_0, t_0)$ hence $G^0_{r_0/4}, J^0_{r_0/4}, \ldots$ are bounded by a constant, ($\le 4^2$), times the respective $G(r_0), J(r_0)\ldots$. Then apply the result of [3.5.10]).

[3.5.12]: Check that the result of [3.5.11] implies theorem II. (*Idea:* Indeed

$$\frac{1}{r_n^2} A_n^0 \ge \frac{1}{r_n^3}\int_{B_n^0} |\underline{u}(\underline{x}, t')|^2 d\underline{x} \xrightarrow[n\to-\infty]{} \frac{4\pi}{3}\,|\underline{u}(\underline{x}', t')|^2$$

where B^0_n is the ball centered at $\underline{x}'$, *for almost all* the points $(\underline{x},', t') \in Q^0_{r_0/4}$; hence $|\underline{u}(\underline{x}', t'))|$ is bounded in $Q^0_{r_0/4}$ and one can apply proposition IV §3.3).

[3.5.13]: Let f be a function with zero average over B^0_r. Since $f(\underline{x}) = f(\underline{y}) + \int_0^1 ds\, \underline{\partial} f(\underline{y} + (\underline{x} - \underline{y})s) \cdot (\underline{x} - \underline{y})$ for each $\underline{y} \in B^0{}_r$, averaging this identity over $\underline{y}$ one gets

$$f(\underline{x}) = \int_{B^0_r} \frac{d\underline{y}}{|B^0_r|} \int_0^1 ds\, \underline{\partial} f(\underline{y} + (\underline{x} - \underline{y})s) \cdot (\underline{x} - \underline{y})$$

Assuming $\alpha = 1$ integer prove (P). (*Idea:* Change variables as $\underline{y} \to \underline{z} = \underline{y} + (\underline{x} - \underline{y})s$ so that for α integer

$$\int_{B^0_r} |f(\underline{x})|^\alpha \frac{d\underline{x}}{|B^0_r|} \equiv \int_{B^0_r} \frac{d\underline{x}}{|B^0_r|} \Big| \int_0^1 \int_{B^0_r} \frac{d\underline{z}}{|B^0_r|} \frac{ds}{(1-s)^3} \underline{\partial} f(\underline{z}) \cdot (\underline{z} - \underline{x}) \Big|^\alpha$$

where the integration domain of $\underline{z}$ depends from $\underline{x}$ and s, and it is contained in the ball with radius $2(1-s)r$ around $\underline{x}$. The integral can then be bounded by

$$\int \frac{d\underline{z}_1}{|B^0_r|} \frac{ds_1}{1-s_1} \cdots \frac{d\underline{z}_\alpha}{|B^0_r|} \frac{ds_\alpha}{(1-s_\alpha)^3} (2r)^\alpha |\underline{\partial} f(\underline{z}_1)| \ldots |\underline{\partial} f(\underline{z}_\alpha)| \int \frac{d\underline{x}}{|B^0_r|}$$

where $\underline{x}$ varies in a domain with $|\underline{x} - \underline{z}_i| \le 2(1-s_i)r$ for each i. Hence the integral over $\frac{d\underline{x}}{|B^0_r|}$ is bounded by $8(1-s_i)^3$ for each i. Performing a geometric average of such bounds (over α terms)

$$\int_{B^0_r} |f(\underline{x})|^\alpha \frac{d\underline{x}}{|B^0_r|} \le 2^{\alpha+3} r^\alpha \prod_{i=1}^\alpha \int \frac{d\underline{z}_i\, ds_i}{|B^0_r|(1-s_i)} ||\underline{\partial} f(\underline{z}_i)|\, (1-s_i)^{3/\alpha} \le$$
$$\le 2^{\alpha+3} r^\alpha \Big(\int_{B^0_r} |\underline{\partial} f(\underline{z})| \frac{d\underline{z}}{|B^0_r|} \Big)^\alpha \cdot \Big(\int_0^1 \frac{ds}{(1-s)^{3-3/\alpha}} \Big)^\alpha$$

getting (P) and an explicit estimate of the constant C^P_α.)

[3.5.14]: Differentiate twice with respect to α^{-1} and check the convexity of $\alpha^{-1} \to ||f||_\alpha \equiv (\int |f(\underline{x})|^\alpha\, d\underline{x}/|B^0_r|)^{1/\alpha}$. Use this to get (P) for each $1 \le \alpha < \alpha_0$ if it holds for $1, \alpha_0$. (*Idea:* Since (P) can be written $||f||_\alpha \le C_\alpha\, (\int |\underline{\partial} f|\, d\underline{x}/r^2)$ then if $\alpha^{-1} = \vartheta \alpha_0^{-1} + (1-\vartheta)$ with α_0 integer it follows that C_α can be taken $C_\alpha = \vartheta C_{\alpha_0} + (1-\vartheta)\, C_{\alpha_0+1}$).

[3.5.15]: Consider a sequence $\underline{u}^\lambda$ of solutions of the Leray regularized equations , *c.f.r.* §3.3, (D), which converges *weakly* (*i.e.* for each Fourier component) to a Leray solution. By construction the $\underline{u}^\lambda, \underline{u}$ verify the *a priori* bounds in (3.3.8) and (hence) (3.5.7). Deduce that $\underline{u}$ verifies the (3.5.10). (*Idea:* Only (c) has to be proved. Note that if $\underline{u}^\lambda \to \underline{u}^0$ weakly, then the left hand side of (3.5.1) is semi continuous hence the value computed with $\underline{u}^0$ is not larger than the limit of the right hand side in (3.5.1). On the other hand the right hand side of (3.5.1) is *continuous* in the limit $l \to \infty$. Indeed given $N > 0$ weak convergence implies

$$\lim_{\lambda\to\infty} \int_0^{T_0} dt \int_\Omega |\underline{u}^\lambda - \underline{u}^0|^2\, d\underline{x} \equiv \int_0^{T_0} dt \sum_{0<|\underline{k}|} |\underline{\gamma}^\lambda_{\underline{k}}(t) - \underline{\gamma}^0_{\underline{k}}(t)|^2 \le$$
$$\le \lim_{\lambda\to\infty} \sum_{0<|\underline{k}|<N} \int_0^{T_0} dt |\underline{\gamma}^\lambda_{\underline{k}}(t) - \underline{\gamma}^0_{\underline{k}}(t)|^2 + \sum_{|\underline{k}|\ge N} \int_0^{T_0} dt \frac{|\underline{k}|^2}{N^2} |\underline{\gamma}^\lambda_{\underline{k}}(t) - \underline{\gamma}^0_{\underline{k}}(t)|^2 \le$$

$$\le \lim_{\lambda\to\infty}\sum_{0<|\underline{k}|<N}\int_0^{T_0} dt|\gamma^\lambda_{\underline{k}}(t)-\gamma^0_{\underline{k}}(t)|^2+\frac{1}{N^2}\int_0^{T_0} dt\int_\Omega |\underline{\partial}(\underline{u}^\lambda-\underline{u}^0)|^2=$$

$$=\lim_{\lambda\to\infty}\frac{1}{N^2}\int_0^{T_0} dt\int_\Omega |\underline{\partial}(\underline{u}^\lambda-\underline{u}^0)|^2\le\frac{2E_0\nu^{-1}}{N^2}$$

using the *a priori* bound in (3.3.8) (with zero force) and component wise convergence of $\gamma^\lambda_{\underline{k}}(t)$ at $\gamma^0_{\underline{k}}(t)$. Hence $\int_0^{T_0}\int_\Omega |\underline{u}^\lambda-\underline{u}^0|^2\to 0$ showing the convergence of the first two terms of the right hand side of (3.5.1) to the corresponding terms of (c) in (3.5.10).

Apply, next, the inequality (S), (3.5.4), with $q=3$, $a=\frac{3}{4}$, $\frac{q}{2}-a=\frac{3}{4}$, and again by the *a priori* bounds in (3.3.8) we get

$$\int_0^{T_0} dt\int_\Omega |\underline{u}^\lambda-\underline{u}^0|^3\,d\underline{x}\le C\int_0^{T_0} dt\,||\underline{\partial}(\underline{u}^\lambda-\underline{u}^0)||_2^{3/2}\,||\underline{u}^\lambda-\underline{u}^0||_2^{3/2}\le$$
$$\le C\Big(\int_0^{T_0} dt\,||\underline{\partial}(\underline{u}^\lambda-\underline{u}^0)||_2^2\Big)^{3/4}\Big(\int_0^{T_0} dt\,||\underline{u}^\lambda-\underline{u}^0||_2^6\Big)^{1/4}\le$$
$$\le C(2E_0\nu^{-1})^{3/4}(2\sqrt{E_0})\int_0^{T_0} dt\,||\underline{u}^\lambda-\underline{u}^0||_2^2\xrightarrow[\lambda\to\infty]{}0$$

showing continuity of the third term in the second member of (3.5.10). Finally, and analogously, if we recall that $p^\lambda=-\Delta^{-1}\sum_{ij}\partial_i\partial_j(u^\lambda_i u^\lambda_j)$ and if we apply the inequalities (CZ) and (H), we get

$$\int_0^{T_0} dt\int_\Omega d\underline{x}|p^\lambda\underline{u}^\lambda-p^0\underline{u}^0|\le\int\int|p^\lambda-p^0|\,|\underline{u}^\lambda|+\int\int|p^0|\,|\underline{u}^\lambda-\underline{u}^0|\le$$
$$\Big(\int\int|p^\lambda-p^0|^{3/2}\Big)^{2/3}\Big(\int\int|\underline{u}^\lambda|^3\Big)^{1/3}+\Big(\int\int|p^0|^{3/2}\Big)^{2/3}\Big(\int\int|\underline{u}^\lambda-\underline{u}^0|^3\Big)^{1/3}$$

where the last integral tends to zero by the previous relation while the first, via (CZ), will be such that $\int_0^{T_0}\int_\Omega|p^\lambda-p^0|^{3/2}\le\Big(\int\int|\underline{u}^\lambda-\underline{u}^0|^3\Big)^{2/3}\xrightarrow[\lambda\to\infty]{}0$ proving the continuity o the fourth term in the right hand side of (c) in (3.5.10). Hence the right hand side is continuous in the considered limit).

Bibliography: [CKN82], [Ga93].