SEMINAR 5

THE STRUCTURE OF RENORMALIZATION THEORY: RENORMALIZATION, FORM FACTORS AND RESUMMATIONS IN SCALAR FIELD THEORY

Giovanni GALLAVOTTI*

Mathematics Department, Rutgers Hill Center
New Brunswick, NJ 08903, USA

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* Supported in part by NSF Grant MCS-8108814 (A03) and Dipartimento di Matematica, IIa Università di Roma. This work partially covers three lectures given at the 1984 School of Les Houches.

K. Osterwalder and R. Stora, eds.
Les Houches, Session XLIII, 1984
Phénomènes critiques, systèmes aléatoires, théories de jauge/
Critical phenomena, random systems, gauge theories
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1. Introduction and notations

In this paper I shall describe a recent approach to renormalization theory leading to a description of the renormalized series for the effective potential and for the Schwinger functions in terms of a family of "form factors" containing an important resummation of "infinitely many ordinary Feynman diagrams".

The ideas exposed in this paper overlap partially with Polchinskii (1984); they were developed by Gallavotti and Nicoló (1984), extending previous partial results and techniques of these authors and collaborators. For extensive references see Gallavotti (1984).

Once the form factors are known, any perturbation calculation reduces essentially to a calculation of nonrenormalized quantities.

To set up the notation, I reformulate from the beginning the renormalization problem for scalar fields.

Let φ be a Gaussian random field on a torus Λ of dimension d and side L (i.e., we consider a finite system with periodic boundary conditions). The covariance ("propagator") of φ is assumed to be the Λ-periodized version of the function

$$\bar{C}(x-y)=\frac{1}{(2\pi)^d}\int \mathrm{d}^d p\ \frac{\mathrm{e}^{ip(x-y)}}{1+p^2}\ , \tag{1.1}$$

i.e.

$$C(x-y)=\sum_{\boldsymbol{n}\in Z^d}\bar{C}(x-y+\boldsymbol{n}L)\ . \tag{1.2}$$

The samples of the field φ are very irregular, being only distributions which are locally in the Sobolev space H_r, $r<-(d-2)/2$. Furthermore, their values are independently distributed only if one compares points at a distance O(1) apart; i.e., the local fluctuations causing the roughness of the field samples cannot be regarded as independently distributed.

It is therefore convenient to think of such an "irregular object" as a sum of many "regular objects":

$$\varphi_x=\sum_{j=0}^{\infty}\varphi_x^{(j)}\ . \tag{1.3}$$

Suppose that the $\varphi^{(j)}$ are independent Gaussian fields satisfying (1.3) in the sense that the r.h.s. and the l.h.s. have the same distribution and

suppose that the covariances $C^{(j)}$ of $\varphi^{(j)}$ are given by (1.2) with $\bar{C}$ replaced by

$$\bar{C}^{(j)}(x-y) = \frac{1}{(2\pi)^d} \int \gamma^{-2j} \hat{F}_j(\gamma^{-2j} p^2)\, e^{ip(x-y)}\, d^d p\,, \tag{1.4}$$

where $\gamma > 1$ is some constant "scale factor".

Suppose also that the Fourier transform of $\hat{F}_j$, denoted F_j, satisfies, for some K, κ, $\varepsilon > 0$:

$$\begin{aligned} &|\partial^{2s} F_j(x)| \leqslant K\, e^{-\kappa|x|}, \quad s < n\,, \\ &|\partial^{2s} F_j(x) - \partial^{2s} F_j(0)| \leqslant K|x|^{\varepsilon}, \quad s = n\,. \end{aligned} \tag{1.5}$$

Then one says that (1.3) is a "$C^{(n)}$-multiscale decomposition of the field φ into regular fields". Similarly, one can introduce the "C^{∞}-multiscale decompositions" of φ if (1.5) is replaced by the condition that there are constant $K_{s,\kappa}$ such that

$$|\partial^{2s} F_j(x)| \leqslant K_{s,\kappa}(1+|x|)^{-\kappa}\,, \quad \forall j \geqslant 0, \forall s \geqslant 0, \forall \kappa \geqslant 0\,. \tag{1.6}$$

The identity (1.3) means, in terms of the $\hat{F}_j$, that

$$\frac{1}{1+p^2} = \sum_{j=0}^{\infty} \gamma^{-2j} \hat{F}_j(\gamma^{-2j} p^2)\,. \tag{1.7}$$

Inequalities (1.5) imply that the samples of $\varphi^{(j)}$ are "living on scale γ^{-j}"; in fact, they essentially say that

$$|\partial^s \varphi_x^{(j)}| \leqslant B_j \gamma^{js} \gamma^{(d-2)/2}\,, \quad s \leqslant n,\ j = 0, 1, \ldots, \tag{1.8}$$

holds with probability bounded from below, at fixed Λ, by

$$\prod_{j=0}^{\infty} (1 - e^{-AB_j^2})^{\gamma^{jd}}\,, \tag{1.9}$$

for some $A > 0$ depending only on Λ but not on j. Actually the Λ dependence should simply be

$$\prod_{j=0}^{\infty} (1 - e^{-A_1 B_j^2})^{\gamma^{jd}|\Lambda|}\,, \tag{1.10}$$

if A_1 is the value of A for Λ with side $L = 1$. Bounds like (1.10) actually hold with some extra assumptions on $\hat{F}_j$, as explained in Benfatto et al. (1980).

Furthermore, besides satisfying (1.8), (1.9), the fields $\varphi^{(j)}$ of a $C^{(n)}$-multiscale decomposition possess the property of taking roughly independent values on points x, y with $|x - y| > O(\gamma^{-j}\kappa^{-1})$. Thus one says that the fields $\varphi^{(j)}$ of a $C^{(n)}$-multiscale decomposition are "smooth and regular" on the same scale γ^{-j}.

The decomposition (1.3) will be called "scale-invariant" or "scaling" if $\hat{F}_j(p^2)$ and $\partial^{2s}F_j(x)$, $s \leqslant n$, converge exponentially fast to their $j = \infty$ limits, uniformly in p, x, s. When this happens it is clear that we can regard the fields as almost identically distributed up to scalings

$$\varphi^{(j)}_{\gamma^{-j}x}\gamma^{\gamma^{-j(d-2)/2}} \approx \varphi^{(q)}_{\gamma^{-q}x}\gamma^{-q(d-2)/2} \tag{1.11}$$

in the sense of distributions.

It is very simple to produce examples of multiscale decompositions. The simplest is

$$\hat{F}_j(p^2) = \frac{\gamma^2 - 1}{(1 + p^2)(\gamma^2 + p^2)}\ , \tag{1.12}$$

introduced in Benfatto et al. (1980). This exactly scaling decomposition can be extended to higher dimensions and to higher regularity by the following scaling (although not exactly) decompositions

$$\hat{F}_j(p^2) = (n + d/2)\int_1^{\gamma^2} \frac{(\eta - \gamma^{-2j})^{n+d/2-1}}{(\eta + p^2)^{n+d/2+1}}\,d\eta, \quad d = \text{even},$$

$$\hat{F}_j(p^2) = (n + (d-1)/2)\int_1^{\gamma^2} \frac{(\eta - \gamma^{-2j})^{n+(d-1)/2-1}}{(\eta + p^2)^{n+(d-1)/2+1}}\,d\eta\ , \quad d = \text{odd}, \tag{1.13}$$

giving scale-invariant $C^{(n)}$-multiscale decompositions.

Examples of scale-invariant $C^{(\infty)}$-decompositions are provided by the identities

$$\frac{1}{1 + p^2} = \frac{\chi_0(p)}{1 + p^2} + \sum_{j=1}^{\infty} \frac{\chi(\gamma^{-j}p)}{1 + p^2}\ , \tag{1.14}$$

if χ_0, χ are C^∞-functions with compact support and χ vanishes near the origin, while

$$\frac{1}{1+p^2} = \frac{e^{-p^2}}{1+p^2} + \sum_{j=1}^{\infty} \frac{e^{-\gamma^{-2j}p^2} - e^{-\gamma^{-2j+2}p^2}}{1+p^2} \tag{1.15}$$

or

$$\frac{1}{1+p^2} = \frac{e^{-|p|}}{1+p^2} + \sum_{j=1}^{\infty} \frac{e^{-\gamma^{-j}|p|} - e^{-\gamma^{-j+1}|p|}}{1+p^2} \tag{1.16}$$

provide very smooth decompositions (e.g., (1.16) is even "analytic") which *fail* to satisfy the fast decay property in (1.6). Therefore the latter decompositions are not multiscale decompositions in our sense and will be banished from this paper.

Other examples, similar to (1.13) but "better", can be found in Gallavotti (1984).

2. Perturbations of Gaussian fields and renormalization

Field theory deals with the problem of studying probability measures of the form

$$Z^{-1}\left\{\exp \sum_{\alpha=1}^{t} \lambda_N^{(\alpha)} I_N^{(\alpha)}(\varphi^{(\leq N)})\right\} \prod_{j=0}^{N} P(\mathrm{d}\varphi^{(i)}), \tag{2.1}$$

where $\varphi^{(0)}, \varphi^{(1)}, \ldots$, are the components of a multiscale decomposition of the field φ with covariance (1.2), $\lambda_N^{(\alpha)}$ are "bare coupling constants", $\varphi^{(\leq N)}$ denotes the "field with ultraviolet cut-off N" defined by

$$\varphi_x^{(\leq N)} = \sum_{j=0}^{N} \varphi_x^{(i)}, \tag{2.2}$$

and $I_N^{(1)}, \ldots, I_N^{(t)}$ are a basis in a linear space $\mathcal{I}_N$ which we shall call the "space of the interactions"; Z is a normalization factor.

$$I_N^{(\alpha)}(\varphi^{(\leq N)}) = \int_\Lambda v^{(\alpha)}(\varphi_x^{(\leq N)}, \partial\varphi_x^{(\leq N)}, \ldots)\,\mathrm{d}x, \tag{2.3}$$

the interactions are called "local"; relativistic field theory deals only with such interactions.

In order that (2.3) makes sense, one has to assume that the multiscale decomposition is of class $C^{(n)}$ with n so large that the integrand in (2.3) makes sense at least with probability 1, with respect to the $\varphi^{(\leq N)}$-distribution.

To avoid having too much freedom in changing

$$V = \sum_{\alpha=1}^{t} \lambda_N^{(\alpha)} I_N^{(\alpha)}(\varphi^{(\leq N)}) \tag{2.4}$$

as N varies, I shall introduce the restriction

$$\int P(\mathrm{d}\varphi^{(k+1)})\, I_{k+1}^{(\alpha)}(\varphi^{(\leq k+1)}) \equiv \mathscr{E}_{k+1}(I_{k+1}^{(\alpha)}) = I_k^{(\alpha)}(\varphi^{(\leq k)})\,, \tag{2.5}$$

where $\mathscr{E}_j(\cdot) \int P(\mathrm{d}\varphi^{(j)})\cdot$.

Mathematical language calls (2.5) a "martingale condition", i.e., the $I_k^{(\alpha)}$ form a martingale of random variables. In physics literature (2.5) is identified with the notion of "Wick ordering".

The specification of the spaces of the interactions $\mathscr{I}_k$ on each scale $k = 0, 1, \ldots$, with the Wick ordering condition (2.5), defines a "model" and the problem of its analysis is that of finding if there are sequences $\lambda_N^{(\alpha)}$ such that (2.1) admits a limit as $N \to \infty$.

For instance, the φ_4^4-model has $t = 4$ and

$$\begin{aligned}
I_N^{(1)}(\varphi^{(\leq N)}) &= \int_\Lambda :\varphi_x^{(\leq N)4}: \mathrm{d}x\,,\\
I_N^{(2)}(\varphi^{(\leq N)}) &= \int_\Lambda :(\partial\varphi_x^{(\leq N)})^2: \mathrm{d}x\,,\\
I_N^{(3)}(\varphi^{(\leq N)}) &= \int_\Lambda :\varphi_x^{(\leq N)2}: \mathrm{d}x\,, \qquad I_N^{(4)}(\varphi^{(\leq N)}) = \int_\Lambda 1\, \mathrm{d}x\,,
\end{aligned} \tag{2.6}$$

where the colons denote the Wick ordering

$$:x^n: = \sqrt{2\mathscr{E}(x^2)}^n H_n(x/\sqrt{2\mathscr{E}(x^2)})\,, \quad n \geqslant 0\,, \tag{2.7}$$

for any random variable x, if $\mathscr{E}$ is the expectation over its distribution, and H_n is the nth Hermite polynomial [using as generating functions the $\exp(-\frac{1}{4}\alpha^2 + \alpha x) = \Sigma_{n=0}^{\infty} (\alpha^n/n!) H_n(x)$].

An effective approach to the theory of the limit of (2.1) is through the notion of "effective potential" on scale γ^{-k}. It is defined simply by

$$\exp V^{(k)}(\varphi^{(\leqslant k)}) = \int \exp V(\varphi^{(\leqslant k)}) P(\mathrm{d}\varphi^{(k+1)}) \cdots P(\mathrm{d}\varphi^{(N)})$$

$$\equiv \mathscr{E}_{k+1} \mathscr{E}_{k+2} \cdots \mathscr{E}_N (\exp V) \,. \tag{2.8}$$

One then requires that $V^{(k)}(\varphi^{(\leqslant k)})$ converges to a limit as $N \to \infty$, at fixed k, at least with probability 1. Actually one wishes to have stronger results, namely

$$V^{(k)} = \sum \int W(x_1, \ldots, x_p, y_1, \ldots, y_q, n_1, \ldots, n_p, m_1, \ldots, m_q)$$
$$\times \prod_i (\varphi_{x_i}^{(\leqslant k)})^{n_i} \prod_j (\partial \varphi_{y_j}^{(\leqslant k)})^{m_j} \prod_h (\partial^2 \varphi_{z_h}^{(\leqslant k)})^{r_h} \cdots$$
$$\times \mathrm{d}x_1 \cdots \mathrm{d}y_1 \cdots \mathrm{d}z_1 \cdots \,, \tag{2.9}$$

with kernels W having strong decay properties as $|x_i - x_j|$, $|x_i - y_j|, \ldots \to \infty$ on the scale γ^{-k} and with the field monomial in (2.9) involving only a fixed number of derivatives.

It is clear that starting from (2.8) and expanding the exponential in powers of V and then each of the resulting expressions in powers of $\boldsymbol{\lambda}_N$ one can find a formal power series expression of $V^{(k)}$ in powers of $\boldsymbol{\lambda}_N$.

The "renormalization" theory problem is to find t formal power series

$$\boldsymbol{\lambda}_N = \boldsymbol{\lambda} + \sum_{|\boldsymbol{m}| \geqslant 2} l(\boldsymbol{m}; N) \boldsymbol{\lambda}^{\boldsymbol{m}}, \quad \boldsymbol{m} \in \mathrm{Z}_+^t \,, \tag{2.10}$$

in t parameters $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(t)})$ so that $V^{(k)}(\varphi^{(\leqslant k)})$ admits a formal expansion in powers of $\boldsymbol{\lambda}$ with coefficients admitting a limit as $N \to \infty$, with probability 1 with respect to the distribution of $\varphi^{(\leqslant k)}$, e.g. if $\varphi^{(j)}$ satisfies (1.8) (see also (1.9)).

The first problem is to fix an algorithm by which the coefficients of the series (2.10), called the "counterterms", can be constructed if their existence is assumed from the beginning. It is, however, clear that the solution of the problem cannot be unique. In fact, given one formal power series (2.10) with the good property, i.e. such that the power series for $V^{(k)}$ have coefficients converging to limits as $N \to \infty$, "(perturbative) ultraviolet stability", then one can construct other formal power series by setting $\boldsymbol{\lambda} = \boldsymbol{\lambda}' + \boldsymbol{g}(\boldsymbol{\lambda}')$ with $\boldsymbol{g}(\boldsymbol{\lambda}')$ being a formal power series starting with second-order terms. Substituting in (2.10) $\boldsymbol{\lambda}' + \boldsymbol{g}(\boldsymbol{\lambda}')$ for $\boldsymbol{\lambda}$

and reorganizing the result as a power series in $\boldsymbol{\lambda}'$ one finds a new series like (2.10), in $\boldsymbol{\lambda}'$, with coefficients $l'(\boldsymbol{m}; N) \neq l(\boldsymbol{m}; N)$, which also leads to an ultraviolet stable perturbation theory.

In the following I shall make use of this ambiguity in the definition of the coefficient of (2.10) in order to simplify the expressions of the effective potentials and their analysis. This will be done by using special rules for defining $l(\boldsymbol{m}; N)$ which provide a rather substantial limitation of the ambiguities (although they still leave some arbitrariness).

3. Formal perturbation expansion and its graphical interpretation

Consider the series (2.10) and define its jth-order part

$$\mu_j^{(\alpha)} = \sum_{|\boldsymbol{m}|=j} l^{(\alpha)}(\boldsymbol{m}; N)\boldsymbol{\lambda}^{\boldsymbol{m}}, \quad \alpha = 1, \ldots, t; \tag{3.1}$$

this polynomial will be called the "jth-order counterterm" of type α. The jth-order interaction will be defined as

$$V_j = \sum_{\alpha=1}^{t} \mu_j^{(\alpha)} I_N^{(\alpha)}(\varphi^{(\leq N)}). \tag{3.2}$$

With the above notations the interaction with cut-off N becomes

$$V = V_1 + V_2 + V_3 + \cdots. \tag{3.3}$$

The effective potential can be calculated by using the notion of "truncated expectation". If $x_1, \ldots, x_p$ are p arbitrary random variables, one sets

$$\mathscr{E}^{T}(x_1, \ldots, x_p) = \frac{\partial^p}{\partial\omega_1 \cdots \partial\omega_p} \log \mathscr{E}(\exp \omega_1 x_1 + \cdots + \omega_p x_p)\Big|_{\boldsymbol{\omega}=\mathbf{0}}, \tag{3.4}$$

where $\mathscr{E}$ denotes the integration with respect to the distribution of the random variables and the "truncated expectation", $\mathscr{E}^{T}$, is defined by the form (3.4), multilinear and symmetric in $x_1, \ldots, x_p$.

Since

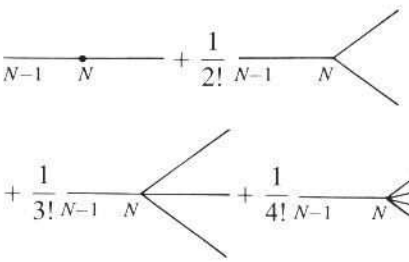
$$\mathscr{E}^{T}(\overbrace{x, \ldots, x}^{p \text{ times}}) = \frac{\partial^p}{\partial\omega^p} \log \mathscr{E}(\exp \omega x)\Big|_{\omega=0}, \tag{3.5}$$

it is formally clear that

$$\exp V_{N-1} = \mathscr{E}_N(\mathrm{e}^V) \equiv \int \mathrm{e}^V P(\mathrm{d}\varphi^{(N)}) = \exp \sum_{p=1}^{\infty} \frac{\mathscr{E}_N^{\mathrm{T}}(V, \ldots, V)}{p!} \,. \tag{3.6}$$

A convenient graphical representation of (3.5) is

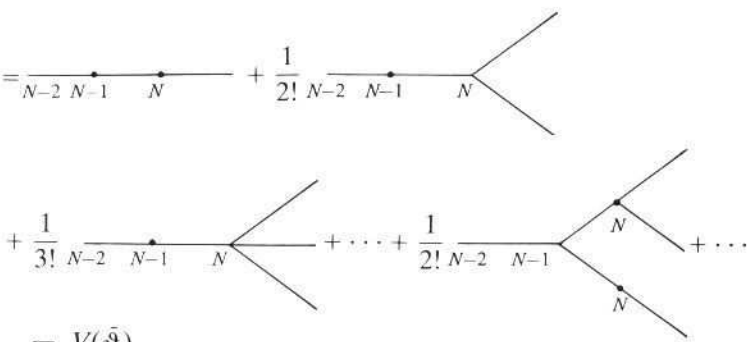
$$V^{(N-1)} =$$



(3.7)

The calculation of $V^{(N-2)}$ can be performed again in the same manner, and one finds

$$V^{(N-2)} =$$



$$\equiv \sum_{\tilde{\vartheta}} \frac{V(\tilde{\vartheta})}{n(\tilde{\vartheta})} \,, \tag{3.8}$$

where the sum runs over all trees $\tilde{\vartheta}$, i.e. partially ordered sets "with no loops", which have one smallest vertex r, called the "root", out of which emerges only one branch; the elements of the tree should be such that if one climbs the tree from the root r to one of the top vertices one meets exactly as many points as necessary to label them from $N-2$ to $N+1$; the root receives label $N-2$ and the top vertices receive label $N+1$, not marked in the above figures. The number $n(\tilde{\vartheta})$ is a suitable

combinatorial factor. (3.8) follows by expressing $V^{(N-2)}$ in terms of $V^{(N-1)}$ via a formula similar to (3.6) and then expressing $V^{(N-1)}$ via (3.7) and using the multilinearity and symmetry of the truncated expectations.

It is then easy to prove the induction that

$$V^{(k)} = \sum_{\bar\vartheta} \frac{V(\bar\vartheta)}{n(\vartheta)} , \tag{3.9}$$

where $\bar\vartheta$ are the trees which have one smallest vertex r, the "root", out of which emerges only one branch and with as many elements as needed to be able to label r by k and all the other elements of $\bar\vartheta$ by integers so that along any totally ordered path going from r to a top vertex one meets exactly $N-k$ vertices with labels increasing from $k+1$ to $N+1$. Before explaining how to constuct $V(\bar\vartheta)$ and $n(\bar\vartheta)$, it is convenient to observe that any tree $\bar\vartheta$ of the above type can be considered as a pair of a tree ϑ which has inner vertices, if any, which are all "nontrivial", i.e. where ϑ actually bifurcates, and a set $\boldsymbol{h} = \{h_v\}_{v\in\vartheta}$ of labels appended to each vertex of ϑ so that $h_r = k$, i.e. the root has index k, and each top vertex v has label $h_v = N+1$, and, furthermore, if $v' < v$ then $h_{v'} < h_v$. Out of the pair $(\vartheta, \boldsymbol{h})$ one reconstructs the above more complicated trees $\bar\vartheta$ simply by inserting between any pair of nearest-neighbor vertices $v' < v$ of ϑ exactly $h_v - h_{v'} - 1$ new "trivial" vertices, i.e. inner vertices where no bifurcation occurs.

Then (3.9) can be rewritten

$$V^{(k)} = \sum_{\vartheta} \sum_{\boldsymbol{h}} \frac{V(\vartheta, \boldsymbol{h})}{n(\vartheta)} , \tag{3.10}$$

and now ϑ is an arbitrary tree whose vertices v can be labeled by integers h_v so that $h_r = k$, $h_v = N+1$ if v is an endpoint, $h_{v'} < h_v$ if $v' < v$; from now on by tree we mean a partially ordered set with no loops, one minimal element r (root), a finite number $\nu(\vartheta)$ of endpoint and with inner vertices v such that at each of them a nontrivial bifurcation takes place; furthermore, at the root no bifurcation takes place (the trees should have a "trunk").

The number $n(\vartheta)$ is defined as follows: If ϑ has no inner vertices, $\vartheta \equiv \overline{}$, then $n(\vartheta) = 1$. Otherwise ϑ bifurcates at the first inner vertex v into s trees $\bar\vartheta_1, \ldots, \bar\vartheta_s$. Regarding a tree as a partially ordered set, in general not all the ϑ_j's will be distinct. We call $\bar\vartheta_1, \ldots, \bar\vartheta_p$ the p

distinct tree species in $(\vartheta_1, \ldots, \vartheta_s)$ and call $q_1, q_2, \ldots, q_p$ their multiplicities in $\vartheta_1, \ldots, \vartheta_s$; of course $s = q_1 + \cdots + q_p$. Then we set

$$n(\vartheta) = \prod_{i=1}^{p} \left(q_i! \prod_i n(\bar{\vartheta}_i)^{q_i} \right), \tag{3.11}$$

which allows us to define $n(\vartheta)$ inductively.

It remains to explain the meaning of $V(\vartheta, h)$. Again its definition is inductive: If $\vartheta = \overline{\qquad}_k$, then

$$V(\vartheta, \boldsymbol{h}) = \mathscr{E}_{k+1} \cdots \mathscr{E}_N(V) \equiv \sum_{\alpha=1}^{t} \lambda_N^{(\alpha)} I_k^{(\alpha)}(\varphi^{(\leq k)}), \tag{3.12}$$

where the Wick ordering property has been used. Otherwise, ϑ has a first inner vertex v_0 where ϑ bifurcates into $\vartheta_1, \ldots, \vartheta_s$, which inherit from ϑ the frequency labels. Then one puts

$$V(\vartheta, \boldsymbol{h}) = \mathscr{E}_{k+1} \cdots \mathscr{E}_{h_{v_0}-1} \mathscr{E}^{\mathrm{T}}_{h_{v_0}}(V(\vartheta_1, \boldsymbol{h}_1), \ldots, V(\vartheta_s, \boldsymbol{h}_s)), \tag{3.13}$$

if $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_s$ are the frequency labels inherited from $\boldsymbol{h}$. (3.13) and (3.12) provide an inductive definition of $V(\vartheta, \boldsymbol{h})$.

One can also say that $(\vartheta, \boldsymbol{h})$ represents the following operations:

(1) each inner vertex v bearing a label h_v represents $\mathscr{E}^{\mathrm{T}}_{h_v}$;

(2) each branch joining two vertices v', v with $v' < v$ represents

$$\mathscr{E}_{h_{v'}+1} \cdots \mathscr{E}_{h_v-1} \equiv \mathscr{E}_{(h_{v'}, h_v)};$$

(3) each endpoint of ϑ represents $V(\varphi^{(\leq N)})$.

It is convenient to stress some combinatorial aspects and to make a precise tree drawing convention. The trees ϑ above are topological objects so that

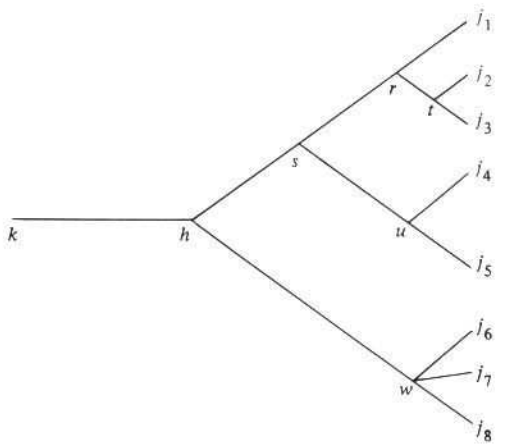
and we imagine to have selected, once and for all, one representative for each species, which will be the one that we draw and which should always be the same graph.

The labels $\boldsymbol{h}$ are imagined to be laid on the representative graph of each species; one should realize that in this way the graphs

will *not* be equal if the decorations are taken into account (although, of course, they give the same contribution to the effective potential).

It is also important to realize that there are "few" trees with n endpoints (i.e. few partial-order relations with one minimal element and n maximal ones); e.g., their number can be bounded by $n \cdot 2^{4n}$ (because the total number of branches is the total number of inner vertices plus the number of endpoints and, hence, it is bounded by twice the number of endpoints; then one observes that one can associate with each tree a one-dimensional walk on a mesh-one lattice with twice as many steps as there are branches in the tree).

The above formula (3.10) provides us with an expansion of $V^{(k)}$ in a power series of the bare couplings: the trees with n endpoints contribute precisely a polynomial of order n, in $\lambda_N^{(\alpha)}$, to $V^{(k)}$. If one wishes to express $V^{(k)}$ in a power series of the renormalized couplings $\boldsymbol{\lambda}$ see (2.10), one has to think of each V as a sum like (3.3). Then, by the multilinearity of the truncated expectation it is clear that the expansion of $V^{(k)}$ in powers of $\boldsymbol{\lambda}$ can be described in terms of trees with extra labels j attached to each endpoint, as in the example



The meaning to be attached to such more decorated trees $(\vartheta, \boldsymbol{j}, \boldsymbol{h})$ is easily understood if one thinks of computing $V(\vartheta, \boldsymbol{j}, \boldsymbol{h})$ by the same sequence of truncated expectations that would be used to evaluate $V(\vartheta)$, but with the V corresponding to the ith endpoint replaced with $\sum_{\alpha=1}^{t} \mu_{j_i}^{(\alpha)} I_N^{(\alpha)}(\varphi^{(\leq N)})$, i.e. by the j_ith order of V. So the expansion of $V^{(k)}$ in terms of the powers of the renormalized becomes

$$V^{(k)} = \sum_{\vartheta} \sum_{\boldsymbol{j}} \sum_{\boldsymbol{h}} \frac{V(\vartheta, \boldsymbol{j}, \boldsymbol{h})}{n(\vartheta)} \,. \tag{3.14}$$

A formal proof of (3.10) is very easy by induction based on the multilinearity of the truncated expectation and on (3.13). (3.14) is an easy corollary of (3.10) and, again, of the multilinearity of $\mathscr{E}^{T}$.

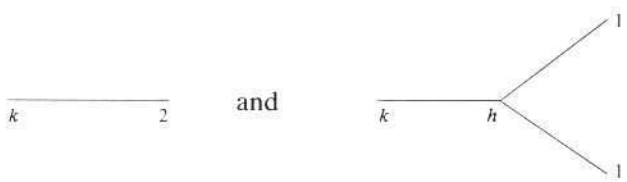
4. Recursive definition of the counterterms

The counterterms can and will be defined, whenever ambiguities may arise, in such a way that the renormalized $V^{(k)}$ admits a simple representation in terms of trees.

The key objects are what we shall call "localization operators on scale k", which we denote by $\mathscr{L}_k$, and the "dressed trees", which are trees carrying an index R over every inner vertex to recall that, instead of $\mathscr{E}_{h_v}^{T}$, the inner vertex v with frequency index h_v represents $(1 - \mathscr{L}_{h_v})\mathscr{E}_{h_v}^{T}$, where the "localization operator" $\mathscr{L}_h$ is an operator mapping functions of $\varphi^{(\leq h)}$ into "interactions", i.e. is an operator with range in $\mathscr{I}_h$.

A second key object will be the "form factors", which are sequences $\boldsymbol{\lambda}(h) = (\lambda^{(1)}(h), \ldots, \lambda^{(t)}(h))$ of formal power series in the renormalized constants $\boldsymbol{\lambda}$; they will be introduced below.

To see how one can set up the search and the definition of the counterterms, one can start by looking at the second-order effective potential; it receives contributions only from the trees



given, respectively [see (3.10)–(3.13), (3.1), (3.2)], by

$$\sum_{\alpha=1}^{t} \mu_2^{(\alpha)} I_k^{(\alpha)} \quad \text{and} \quad \frac{1}{2} \sum_{\alpha_1,\alpha_2} \lambda^{(\alpha_1)} \lambda^{(\alpha_2)} \mathscr{E}_{(k,h)} \mathscr{E}_h^{\mathrm{T}}(I_h^{(\alpha_1)}, I_h^{(\alpha_2)}) . \tag{4.1}$$

Now we discard the "trivial case" in which the second of (4.1) summed over h between $k+1$ and N is "ultraviolet stable", i.e. converges to a limit as $N \to \infty$ for all smooth $\varphi^{(\le k)}$.

The only way the theory can be "renormalizable" is that the above mentioned divergence arises because the sum over h of the second term in (4.1) has a too large component on $\mathscr{I}_k$. Therefore we fix a special value of k, called k_0, and define $\mu_2^{(\alpha)}$ to be exactly as large as necessary to compensate the orthogonal projection π_{k_0} of

$$\frac{1}{2} \sum_{\alpha_1,\alpha_2} \lambda^{(\alpha_1)} \lambda^{(\alpha_2)} \sum_{h>k_0}^{N} \mathscr{E}_{(k_0,h)}(I_h^{(\alpha_1)}, I_h^{(\alpha_2)}) \tag{4.2}$$

on the space $\mathscr{I}_{k_0}$ [with respect to the scalar product in $L_2(P(\mathrm{d}\varphi^{(\le k_0)}))$].

If

$$\pi_{k_0}(F) = \sum_{\alpha=1}^{t} \delta_{k_0}^{(\alpha)}(F) I_{k_0}^{(\alpha)} , \tag{4.3}$$

where $\delta_{k_0}^{(\alpha)}$ are suitable linear functionals, this means that

$$\mu_N^{(\alpha)} = -\frac{1}{2} \sum_{h>k_0}^{N} \sum_{\alpha_1,\alpha_2} \lambda^{(\alpha_1)} \lambda^{(\alpha_2)} \delta_{k_0}^{(\alpha)} (\mathscr{E}_{(k_0,h)} \mathscr{E}_h^{\mathrm{T}}(I_h^{(\alpha_1)}, I_h^{(\alpha_2)})) \tag{4.4}$$

and ultraviolet stability of the second-order part of $V^{(k_0)}$ follows if it is indeed true that the divergent part of (4.2) is its projection on $\mathscr{I}_{k_0}$.

Assuming the latter property, it does not yet follow that the second-order part of $V^{(k)}$, $k > k_0$, is also finite. To discuss this question one introduces the operators $\mathscr{L}_k$ with range in $\mathscr{I}_k$ defined as

$$\mathscr{L}_k F = \sum_{\alpha=1}^{t} \delta_{k_0}^{(\alpha)} (\mathscr{E}_{(k_0,k)} F) I_k^{(\alpha)} \tag{4.5}$$

and $\mathscr{L}_{k_0} \equiv \pi_{k_0}$. Then, if

$$(1 - \mathscr{L}_k) \frac{1}{2} \sum_{\alpha_1,\alpha_2} \lambda^{(\alpha_1)} \lambda^{(\alpha_2)} \sum_{h=k+1}^{N} \mathscr{E}_{(k,h)} \mathscr{E}_h^{\mathrm{T}}(I_h^{(\alpha_1)}, I_h^{(\alpha_2)}) \tag{4.6}$$

converges to a limit as $N \to \infty$, we see that $V^{(k)}$, $k \geqslant k_0$, will be ultraviolet stable to second order because the other second-order part of $V^{(k)}$ is

$$\sum_{\alpha=1}^{t} \mu_N^{(\alpha)} I_k^{(\alpha)} + \mathscr{L}_k \left(\frac{1}{2} \sum_{\alpha_1,\alpha_2} \sum_{h>k}^{N} \lambda^{(\alpha_1)} \lambda^{(\alpha_2)} \mathscr{E}_{(k,h)} \mathscr{E}_h^{\mathrm{T}}(I_h^{(\alpha_1)}, I_h^{(\alpha_2)}) \right)$$

$$= \sum_{\alpha=1}^{t} I_k^{(\alpha)} \left\{ \mu_N^{(\alpha)} + \delta_{k_0}^{(\alpha)} \left(\frac{1}{2} \sum_{\alpha_1,\alpha_2} \sum_{h>k_0}^{n} \lambda^{(\alpha_1)} \lambda^{(\alpha_2)} \mathscr{E}_{(k_0,h)} \mathscr{E}_h^{\mathrm{T}}(I_h^{(\alpha_1)}, I_h^{(\alpha_2)}) \right) \right\}$$

$$= \sum_{\alpha=1}^{t} I_k^{(\alpha)} \left\{ -\delta_{k_0}^{(\alpha)} \left(\frac{1}{2} \sum_{\alpha_1,\alpha_2} \sum_{h=k_0}^{k} \lambda^{(\alpha_1)} \lambda^{(\alpha_2)} \mathscr{E}_{(k_0,h)} \mathscr{E}_h^{\mathrm{T}}(I_h^{(\alpha_1)}, I_h^{(\alpha_2)}) \right) \right\}, \tag{4.7}$$

where in the last relation we have used (4.4); so eq. (4.7) is N-independent and therefore ultraviolet stable.

The ultraviolet stability on scales $k < k_0$ follows from that on scale k_0 provided we assume that the second-order part of the effective potential $V^{(k_0)}$ is uniform in $\cap_{q>1} L_q(P(\mathrm{d}\varphi^{(\leqslant k)}))$, as will be the case in our applications; in fact, one can compute $V^{(k)}$ for $k < k_0$ directly from $V^{(k_0)}$ without any reference to N.

Of course the scale k_0 is rather arbitrary. It seems better to avoid choosing one special value for k_0 by the following formal trick: we shall think that

$$\varphi_x^{(\leqslant N)} = \varphi_x^{(-1)} + \varphi_x^{(0)} + \cdots + \varphi_x^{(N)}, \tag{4.8}$$

where $\varphi_x^{(-1)}$ is an auxiliary field, is independently distributed from the others and has covariance $C^{(-1)}$:

$$C_{xy}^{(-1)} = \eta \left(1 + \eta' \cos \frac{2\pi}{L} \sum_{i=1}^{d} (x_i - y_i) \right), \tag{4.9}$$

where $\eta, \eta' > 0$ are two parameters which will be eventually put equal to zero, so that $\varphi^{(-1)} \equiv 0$.

Then we shall choose $k_0 = -1$ so that the "subtraction scale k_0" is $k_0 = -1$; no physically meaningful scale is preferred in this way and one obtains a more symmetric theory.

Repeating the above discussion in the case $k_0 = -1$ we assume:

(1) If $\mathscr{L}_k$ is defined via (4.5) with $k_0 = -1$, then

$$(1-\mathscr{L}_k)\sum_{h>k}^{N}\mathscr{E}_{(k,h)}\mathscr{E}_h^{\mathrm{T}}(I_h^{(\alpha_1)},I_h^{(\alpha_2)})$$

$$\equiv\sum_{h>k}^{N}\mathscr{E}_{(k,h)}(1-\mathscr{L}_h)\mathscr{E}_h^{\mathrm{T}}(I_h^{(\alpha_1)},I_h^{(\alpha_2)}) \tag{4.10}$$

(where the basic identity $\mathscr{L}_k\mathscr{E}_{(k,h)}\equiv\mathscr{E}_{(k,h)}\mathscr{L}_h$ is used) converges to a limit as $N\to\infty$ and can be bounded uniformly (in N) by $D\|\varphi^{(\le k)}\|^{\delta}_{C^{(n)}}$, if

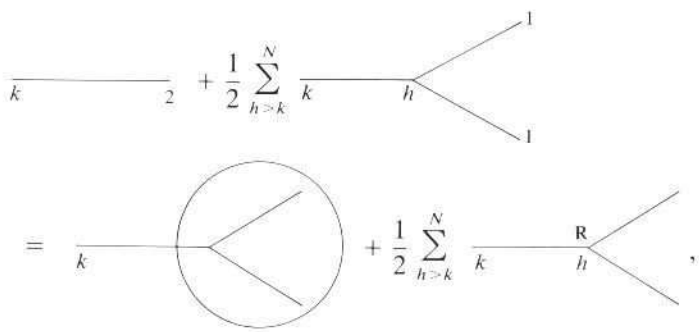
$$\|\varphi^{(\le k)}\|_{C^{(n)}}=\sup_x\gamma^{-k(d-2)/2}\sum_{j=0}^{n}\gamma^{-jk}|\partial^j\varphi_x^{(\le k)}| \tag{4.11}$$

for some $D,\delta>0$, if n is the order of the regularization used. (4.11) implies that (4.10) is in $\cap_{q>1}L_q(P(\mathrm{d}\varphi^{(\le k)}))$.

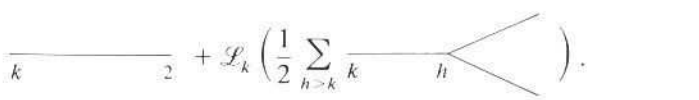
(2) If $\delta^{(\alpha)}\equiv\delta_{-1}^{(\alpha)}$ are the linear functionals associated with the orthogonal projection π_{-1} on $\mathscr{J}_{-1}$,

$$\mu_2^{(\alpha)}=-\frac{1}{2}\sum_{\alpha_1,\alpha_2}\tfrac{1}{2}\delta^{(\alpha)}\Big(\sum_{h=0}^{N}\mathscr{E}_{(-1,h)}\mathscr{E}_h^{\mathrm{T}}(I_h^{(\alpha_1)},I_h^{(\alpha_2)})\Big). \tag{4.12}$$

It is possible to introduce a natural notation for the second-order contribution to $V^{(k)}$:



where in this graphical identity the R over the vertex with label means that in evaluating the expectations symbolized by the tree, one replaces $\mathscr{E}_h^{\mathrm{T}}$ by $(1-\mathscr{L}_h)\mathscr{E}_h^{\mathrm{T}}$ and the first term represents



At this point one can make the following inductive ansatz: If $V = V_1 + \cdots + V_p$, then the effective potential on scale k is represented by trees "partially dressed to order p" defined as follows.

If n_v is the number of endpoints of a tree that can be reached by climbing it, starting from v, we consider all vertices v with $n_v \leqslant p$ (if $p = 1$ only the endpoints qualify for consideration). Then on each such v we put either a label R (for "renormalized") or we enclose in a frame the part of the tree between the predecessor v' of v and the endpoints reachable from v. Then we append frequency labels only on the vertices of the partially dressed tree ϑ which are outside the frames, obtaining a "partially dressed tree ϑ with frequency labels $\boldsymbol{h}$". The tree $\tau\vartheta$ obtained by deleting all the frames and their contents will be called a "trimmed tree"; $\tau\vartheta$ is partially dressed because some labels R may be present above some of its vertices, Then the ansatz is, if $n(\vartheta) \equiv n(\tau\vartheta)$:

The counterterms can be defined so that the effective potential $V^{(k)}$ corresponding to $V_1 + \cdots + V_p$ is given by

$$V^{(k)} = \sum_{\substack{\vartheta \\ \text{dressed to} \\ \text{order } p}} \sum_{\boldsymbol{h}, h_r = k} \frac{V(\vartheta, h)}{n(\vartheta)} , \tag{4.13}$$

where the vertices of ϑ which are outside all frames and bear frequency index h have the meaning

$$\begin{aligned} (1 - \mathscr{L}_h)\mathscr{E}_h^{\mathrm{T}} \quad & \text{if they have a superscript R, or} \\ \mathscr{E}_h^{\mathrm{T}} \quad & \text{if they do not have a superscript R,} \end{aligned} \tag{4.14}$$

while the end lines entering an external frame, after leaving a vertex with frequency index k, have the meaning

$$\sum_{\alpha=1}^{t} \sum_{m_1, \ldots, m_t} L^{(\alpha)}(\boldsymbol{m}, k; \sigma) \boldsymbol{\lambda}^{\boldsymbol{m}} I_k^{(\alpha)} , \tag{4.15}$$

where $L^{(\alpha)}(\boldsymbol{m}, k, \sigma)$ are some, possibly N-dependent, coefficients which depend on the tree σ enclosed in the frame; note that σ will have all its vertices dressed except the first inner vertex (which was dressed by the deleted enclosing frame).

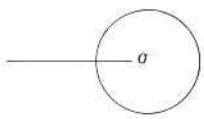
It is easy to check by induction that the above ansatz, clearly valid for

$p = 1$, works also for $p+1$ if the $(p+1)$th-order counterterms are defined as below. Let ϑ be a tree dressed to order p but with $p+1$ endpoints. Let $\boldsymbol{h}$ be frequency labels for ϑ with $h_r = -1$. We define the counterterms $\overline{}^{-1}\quad{}^{p+1}$ as

$$\overline{}^{-1}\quad{}^{p+1} + \sum_{\substack{\vartheta \\ \text{dressed to} \\ \text{order } p \\ \text{and with} \\ p+1 \text{ vertices}}} \sum_{h_{v_0}=0}^{N} \sum_{\boldsymbol{h}'} \mathscr{L}_{-1} \frac{V(\vartheta, \boldsymbol{h})}{n(\vartheta)} \equiv 0 , \tag{4.16}$$

where v_0 is the first inner vertex of θ (necessarily undressed) and $\boldsymbol{h} = (h_{v_0}, \boldsymbol{h}')$, with $\boldsymbol{h}'$ being the frequencies of the other vertices of θ.

Then it follows easily that the inductive assumption holds for $p+1$ if a tree like



with σ dressed to order p and with $p+1$ endpoints is defined so that (4.13) becomes

$$-\sum_{h_{v_0}=0}^{k} \sum_{\boldsymbol{h}'} \sum_{\alpha=1}^{t} \frac{\delta^{(\alpha)}(V(\sigma, \boldsymbol{h}))}{n(\sigma)} I_k^{(\alpha)} , \tag{4.17}$$

which defines $L^{(\alpha)}(\boldsymbol{m}, k, \sigma)$ inductively.

If we suppose that the coefficients $L^{(\alpha)}(\boldsymbol{m}, k, \sigma)$ are uniformly bounded in N (trivial for $p = 1$), and that the contributions to the effective potential due to the trees of order $p+1$ but dressed only to order p are ultraviolet stable after the application of the operator $(1 - \mathscr{L}_k)$ ("renormalizability assumption"), in the sense that for all N

$$\left| \sum_{\boldsymbol{h}} (1 - \mathscr{L}_k) V(\sigma, \boldsymbol{h}) \right| \leq D_\sigma \| \varphi^{(\leq k)} \|_{C^{(n)}}^{\delta_\sigma} \tag{4.18}$$

for some $D_\sigma, \delta_\sigma > 0$, see (4.11), then we see that (4.17) is uniformly bounded in N.

Therefore, after defining V_p for all p it follows that

$$V^{(k)} = \sum_{\substack{\vartheta \\ \text{dressed}}} \sum_{\boldsymbol{h}} \frac{V(\vartheta, \boldsymbol{h})}{n(\vartheta)} . \tag{4.19}$$

By our renormalizability assumption, $V^{(k)}$ will be finite to all orders in the perturbation theory in the renormalized couplings, uniformly in N, and its nth order will admit a bound like (4.9) for some $D, \delta > 0$ as a function of $\varphi^{(\leq k)}$ [which implies that it is in $\cap_{q>1} L_q(P(\mathrm{d}\varphi^{(\leq k)}))$].

5. Form factors and resummations

A dressed tree will in general be rather complicated to evaluate. However, it should be stressed that the real problem is to evaluate only trimmed dressed trees: if two dressed trees ϑ_1, ϑ_2 have the same trimmed part $\tau\vartheta_1 = \tau\vartheta_2 = \bar{\vartheta}$, then $(\vartheta_1, \boldsymbol{h}_1), (\vartheta_2, \boldsymbol{h}_1)$ provide the same contributions to the effective potential up to multiplicative factors which depend on $\boldsymbol{h}$ and on the tree shapes which are enclosed in the external frames.

It is therefore natural to collect together all the dressed trees with the same trimmed part; thus one obtains the following representation for the effective potential (recall that $n(\vartheta') \equiv n(\tau\vartheta')$):

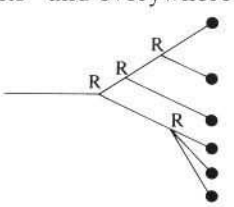
$$V^{(k)} = \sum_{\substack{\vartheta \\ \text{trimmed tree}}} \sum_{\boldsymbol{h}} \sum_{\vartheta' : \tau\vartheta' = \vartheta} \frac{V(\vartheta', \boldsymbol{h})}{n(\vartheta)} . \tag{5.1}$$

It is therefore convenient to define the "form factors" $\boldsymbol{\lambda}(k) = (\lambda^{(\alpha)}(k))_{\alpha=1}^{t}$, $k = -1, 0, \ldots$, so that $\boldsymbol{\lambda}(-1) \equiv \boldsymbol{\lambda}$ and, if $\boldsymbol{m} = (m_1, \ldots, m_t)$, $\boldsymbol{\lambda}^{\boldsymbol{m}} = \lambda^{(1)m_1} \cdots \lambda^{(t)m_t}$:

$$\sum_{\alpha=1}^{t} \lambda^{(\alpha)}(k) I_k^{(\alpha)} = \sum_{\alpha=1}^{t} \lambda^{(\alpha)} I_k^{(\alpha)} + \sum_{\substack{\sigma \\ \text{nontrivial}}} \quad k \text{———} \sigma$$

$$\equiv \sum_{\alpha, \boldsymbol{m}} L^{(\alpha)}(\boldsymbol{m}; k) \boldsymbol{\lambda}^{\boldsymbol{m}} I_k^{(\alpha)} , \tag{5.2}$$

where σ are all the possible trees with the first inner vertex not dressed (but existing; the first term on the r.h.s. corresponds to the case of $\overline{\qquad}\, k$, i.e. of a tree without inner vertices). If we now introduce trees ϑ without any frame but with endpoints bearing "heavy dots" or "fruits" and everywhere else dressed by R's, it is clear that (5.1) can be



rewritten in terms of these new "renormalized trees" as

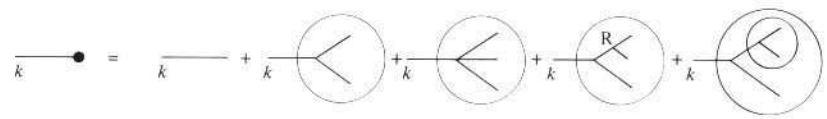
$$V^{(k)} = \sum_{\substack{\vartheta \\ \text{renormalized}}} \sum_{\boldsymbol{h}} \frac{V^{\mathrm{R}}(\vartheta, \boldsymbol{h})}{n(\vartheta)}\,, \tag{5.3}$$

where the rule of evaluation of $V^{\mathrm{R}}(\vartheta, \boldsymbol{h})$ is the same as that of $V(\vartheta, \boldsymbol{h})$, i.e. of the case of a tree without heavy dots at the endpoints, but the interpretation of the end lines

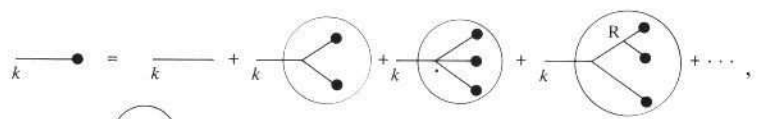
will be

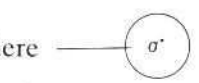
$$\sum_{\alpha=1}^{t} \lambda^{(\alpha)}(h) I_h^{(\alpha)} \quad \text{instead of} \quad \sum_{\alpha=1}^{t} \lambda^{(\alpha)} I_h^{(\alpha)}\,. \tag{5.4}$$

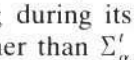
This means that the calculation of $V^{(k)}$ as a power series in the renormalized coupling constants is basically the same as its calculation as a power series in the bare constants once the coefficients of the formal power series for the form factors are known. Their calculation, by (4.1), is in turn reducible to that of the bare expansion of $V^{(k)}$. It is remarkable that (5.2) immediately implies that $\boldsymbol{\lambda}(k)$ satisfies an interesting equation. It is deduced immediately on a graphical basis starting from



which just expresses (5.2), and which can manifestly be rewritten as



where 

, with $\sigma\cdot$ being a nontrivial tree σ dressed completely except on the first nontrivial vertex and *without* any frames and with heavy dots on the endpoints, is defined simply as the similar term in (5.2) interpreting during its calculation the end lines 

as $\Sigma_{\alpha=1}^{t} \lambda^{(\alpha)}(h) I_h^{(\alpha)}$ rather than $\Sigma_{\alpha=1}^{t} \lambda^{(\alpha)} I_h^{(\alpha)}$.

It is easy to write explicitly the above equation for $\boldsymbol{\lambda}(k)$. For each tree σ with no frames and p (non-heavy) endpoints, we define the coefficients (see (4.16)) B_0 so that

$$\sum_{\alpha_1,\ldots,\alpha_p} \lambda^{(\alpha_1)} \cdots \lambda^{(\alpha_p)} B_0^{(\alpha)}(\boldsymbol{h}; \alpha_1, \ldots, \alpha_p; \sigma) = -\frac{\delta^{(\alpha)}(V(\sigma; \boldsymbol{h}))}{n(\sigma)} . \tag{5.5}$$

Then one defines, given $p > 1$, $k \geqslant 0$:

$$B^{(\alpha)}(k; h_1, \ldots, h_p; \alpha_1, \ldots, \alpha_p) = \sum_{\substack{\sigma \\ \text{with } p \text{ endpoints}}} \sum_{\substack{\boldsymbol{h};\, h_r = -1,\, h_0 = k \\ h_{v_1} = h_1, \ldots, h_{v_p} = h_p}} B_0^{(\alpha)}(\boldsymbol{h}; \alpha_1, \ldots, \alpha_p; \sigma) , \tag{5.6}$$

where v_i is the vertex to which the ith endpoint is directly attached by a tree branch, and h_0 is the label to the first nontrivial vertex of σ.

The equation for $\boldsymbol{\lambda}(k)$ in the limit $N \to \infty$ is then (simply see (4.1))

$$\lambda^{(\alpha)}(k) = \lambda^{(\alpha)} + \sum_{p=2}^{\infty} \sum_{h_0=0}^{k} \sum_{h_1, \ldots, h_p \geqslant h_0} \sum_{\alpha_1, \ldots, \alpha_p} B^{(\alpha)}(k; \boldsymbol{h}; \boldsymbol{\alpha}) \times \prod_{j=1}^{p} \lambda^{(\alpha_j)}(h_j) . \tag{5.7}$$

It is important to remark that the coefficients $B^{(\alpha)}(k; \boldsymbol{h}; \boldsymbol{\alpha})$ become N-independent if $N > p$; in fact, we see from (5.6) that the sum over $\boldsymbol{h}$ runs over a finite set and the B_0 are N-independent as long as all the trees σ can be labeled by labels $\boldsymbol{h}$ increasing along the tree branches but *all* $\leqslant N$ (i.e. as long as σ does not contain a chain from the root to the endpoints with more than N vertices; this cannot, in fact, happen if the number of endpoints is $\leqslant N$).

Eq. (5.7) is very interesting because if one writes it as

$$\hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{\lambda}}_0 + \mathscr{B}\hat{\boldsymbol{\lambda}} , \tag{5.8}$$

where $\hat{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}(k))_{k=0}^{\infty}$, $\hat{\boldsymbol{\lambda}}_0 \equiv (\boldsymbol{\lambda})_{k=0}^{\infty}$ and if one defines

$$\hat{\boldsymbol{\lambda}}_{n+1} = \hat{\boldsymbol{\lambda}}_0 + \mathscr{B}\hat{\boldsymbol{\lambda}}_n , \tag{5.9}$$

one sees that $\hat{\boldsymbol{\lambda}}_n$ is a formal power series in $\boldsymbol{\lambda}$ and its pth-order coefficients no longer change for $n > p$. Hence the knowledge of the coefficients B in (5.6) provides us, via a simple algorithm, with the coefficients of the form factors as a formal power series in the renormalized constants.

In other words the coefficients of eq. (5.7) can be viewed as a good way to store the information necessary to perform renormalization calculations.

But a better use of eq. (5.7) is to try to go beyond perturbation theory. The series (5.2) for $\boldsymbol{\lambda}(k)$ is very likely, in the applications, to be nonconvergent; one therefore has to give a summation rule for it. A possible summation rule is to say that, by definition, $\boldsymbol{\lambda}(k)$ is a solution to (5.7).

Therefore, it is interesting to find on which domain (5.7) can be reasonably defined. An important question in this direction is to estimate the size of $\boldsymbol{\lambda}(k)$.

The following results are proved in Gallavotti and Nicoló (1984), see also Gallavotti (1984).

If $I^{(1)}$, $I^{(2)}$, $I^{(3)}$, $I^{(4)}$ are as in (2.6) and if

$$\lambda_k^{(\alpha)} = \gamma^{-\sigma(\alpha)(k+1)} \lambda^{(\alpha)}(k), \quad \alpha = 1, 2, 3, 4, \tag{5.10}$$

with $\sigma(1) = 0 = \sigma(2)$, $\sigma(3) = 2$, $\sigma(4) = 4$, the above "dimensionless form factors" satisfy, for all $k \geq -1$:

$$\lambda_k^{(\alpha)} = \lambda_{k+1}^{(\alpha)} \gamma^{\sigma(\alpha)} + \sum_{p \geq 2} \sum_{\substack{\sigma = \text{tree} \\ \text{with } p \text{ endpoints}}} \sum_{\boldsymbol{h}, \boldsymbol{\alpha}} \beta_\sigma^{(\alpha)}(k+1; \boldsymbol{h}; \boldsymbol{\alpha}) \prod_{i=1}^{p} \lambda_{h_i}^{(\alpha_i)},$$

$$\lambda_{-1}^{(\alpha)} \equiv \lambda^{(\alpha)}, \tag{5.11}$$

obtained by writing (5.7) for $\boldsymbol{\lambda}(k+1)$ and $\boldsymbol{\lambda}(k)$, subtracting them and using (5.10); the β's are trivially related to the B_0's of (5.6).

(1) Then there exist $\eta, \kappa > 0$ such that for all $n \geq 0$:

$$|\boldsymbol{\beta}_\sigma(k; \boldsymbol{h}; \boldsymbol{\alpha})| \leq \nu(p) \eta^{p-1} \prod_{\substack{v \text{ inner} \\ \text{vertex of } \sigma}} \gamma^{-\kappa(h_v - h_{v1})}, \tag{5.12}$$

where $\nu(p)$ is the number of Feynman graphs with p vertices out of which 4 lines or 2 lines emerge.

(2) Let ϑ be a trimmed tree with p endpoints and let $\boldsymbol{h}$ be a set of frequency labels for ϑ with $h_r = k$. We suppose ϑ to be completely dressed and denote by $V_0(\vartheta; \boldsymbol{h})$ the contribution of $(\vartheta, \boldsymbol{h})$ to the effective potential on scale k with the endpoint couplings $\lambda^{(\alpha)}$ set equal to 1 (the subscript 0 is added to recall this). Then $V_0(\vartheta, \boldsymbol{h})$ is a sum of homogeneous polynomials in $\varphi^{(\leq k)}$ and its derivatives up to third order are of the form

$$\int W^{(\vartheta,\boldsymbol{h})}_{\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\rho},\boldsymbol{\sigma}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) : \prod_i (\varphi^{(\leq k)}_{x_i})^{\nu_i} \prod_j (\partial \varphi^{(\leq k)}_{y_j})^{\mu_j} \times \prod_m (\partial^2 \varphi^{(\leq k)}_{z_m})^{\rho_m} \prod_q (\partial^3 \varphi^{(\leq k)}_{w_q})^{\sigma_q} : , \tag{5.13}$$

where $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{u_1})$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{u_2})$, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_{u_3})$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{u_4})$ and where $\boldsymbol{x} = (x_1, \ldots, x_{u_1})$, $\boldsymbol{y} = (y_1, \ldots, y_{u_2})$, $\boldsymbol{z} = (z_1, \ldots, z_{u_3})$, $\boldsymbol{w} = (w_1, \ldots, w_{u_4})$. Then if $u = u_1 + u_2 + u_3 + u_4$, $D = \Sigma \nu_i + \Sigma \mu_j + \Sigma \rho_m + \Sigma \sigma_q$ and if $\Delta_1, \Delta_2, \ldots, \Delta_u$ are u cubes with side γ^{-k}, extracted out of a pavement Q_k of Λ built with such cubes, one can bound

$$\int_{\Delta_1 \times \cdots \times \Delta_u} |W^{(\vartheta,\boldsymbol{h})}_{\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\rho},\boldsymbol{\sigma}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})| \left| : \prod \varphi^{\nu} \prod (\partial \varphi)^{\mu} \prod (\partial^2 \varphi)^{\rho} \prod (\partial^3 \varphi)^{\sigma} : \right| \tag{5.14}$$

for all $k \geq 0$ as

$$e^{-\kappa \gamma^k d(\Delta_1, \ldots, \Delta_u)} \mathcal{N}(D) \nu(p) \beta^{p-1} \left(\prod_{\substack{v \in \vartheta \\ v \text{ inner vertex}}} \gamma^{-(h_v - h_{v'})} \right) \times \| \varphi^{(\leq k)} \|_{C^{(3)}} , \tag{5.15}$$

where $\mathcal{N}(D)$ is a suitable function of D, and $\nu(p)$ is as in (1) above.

(3) The contribution of order r to the form factors has coefficients $\boldsymbol{L}(\boldsymbol{m}; k) \equiv \Sigma_\sigma \boldsymbol{L}(\boldsymbol{m}, k; \sigma)$ (see (4.15), (5.2)), such that:

$$\sum_{|\boldsymbol{m}| = n} |L^{(\alpha)}(\boldsymbol{m}, k)| \leq A^{n-1} (n-1)! \sum_{j=0}^{n-1} \frac{(\beta k)^j}{j!} , \tag{5.16}$$

where A, β are conveniently defined.

(4) Combining (5.16), (5.15) one can verify that they imply that the effective potential on scale k can be written, at a fixed order p in the renormalized couplings, as a sum of terms like (5.13) with $W^{(\theta,h)}$ replaced by a suitable kernel $W^{(p)}$; it follows from (5.16), (5.15) that

$$\int_{\Delta_1\times\cdots\times\Delta_p} |W^{(p)}_{\nu,\mu,\rho,\sigma}(x, y, z, w)| \left|:\prod \varphi^{\nu}_{x}\cdots:\right| \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w$$
$$\leqslant \mathrm{e}^{-\chi\gamma^k \mathrm{d}(\Delta_1,\ldots,\Delta_u)} \mathcal{N}(D) p!\beta^{p-1}k^{p-1}\|\varphi^{(\geqslant k)}\|^{D}_{C^{(1)}} \max_{\alpha} |\lambda^{(\alpha)}|^p , \tag{5.17}$$

where χ, β, $\mathcal{N}(D)$ are suitably chosen constants.

(5) Finally (for the reader familiar with the evaluation techniques of $\mathscr{E}^{T}_{h}$), if one considers only "planar Feynman graphs" in the concrete evaluation of $(1-\mathscr{L}_h)\mathscr{E}^{T}_{h}$, one finds that $\nu(p)$ in (5.12), (5.15) are replaced by C^p for some $C>0$ (while in (5.17), (5.15) it is $\nu(p)\simeq p!$).

Therefore if one studies the planar theory and if one assumes that

$$\sup_{k,\alpha}|\lambda^{(\alpha)}_k| = \|\hat{\boldsymbol{\lambda}}\|_0 \tag{5.18}$$

is small enough, then (5.11) can be written as

$$\lambda^{(\alpha)}_{k+1} = \gamma^{-\sigma(\alpha)}\lambda^{(\alpha)}_k + (B\hat{\boldsymbol{\lambda}})^{(\alpha)}_k , \quad k\geqslant 0 ,$$
$$\lambda^{(\alpha)}_{-1} \equiv \lambda^{(\alpha)} ,$$

and B is a well-defined analytic function in the vicinity of the origin in the Banach space of the $\hat{\boldsymbol{\lambda}}$'s satisfying $\|\hat{\boldsymbol{\lambda}}\|_0<\infty$. Using the bound (5.12) with $\nu(p)$ replaced by C^p it is easy to establish on an abstract basis that (5.15) admits a solution $\hat{\boldsymbol{\lambda}}$ if

$$\lambda^{(\alpha)}_k = \gamma^{\sigma(\alpha)}\lambda_{k+1} + (B_2\hat{\boldsymbol{\lambda}})^{(\alpha)}_k \tag{5.20}$$

admits a solution with $\|\hat{\boldsymbol{\lambda}}\|_0$ small enough and with $\|\hat{\boldsymbol{\lambda}}\|_1 = \sup_{\alpha,k}(1+k)|\lambda^{(\alpha)}_k| = C_0<\infty$, where C_0 is a suitable constant (simply related to η, κ in (5.12) and to the constant C just introduced); in (5.16) B_2 is just defined by the $p=2$ term in (5.11).

The B_2 operator is very simple to compute and (5.20) is

$$\lambda^{(\alpha)}_k = \gamma^{\sigma(\alpha)}\lambda^{(\alpha)}_k + \sum_{\alpha_1,\alpha_2} \beta^{(\alpha)}_{\alpha_1\alpha_2}(1+\xi^{(\alpha)}_{\alpha_1\alpha_2}(k))\lambda^{(\alpha_1)}_{k+1}\lambda^{(\alpha_2)}_{k+1} , \tag{5.21}$$

where ξ denotes a quantity tending to zero exponentially fast as $k \to \infty$ and $\beta^{(\alpha)}_{\alpha_1 \alpha_2}$ are easily computable constants.

In particular, it is found for $\alpha = 1$ that

$$\lambda^{(1)}_k = \lambda^{(1)}_{k+1} - \beta^{(1)}_{11}(1 + \xi^{(1)}_{11}(k))\lambda^{(1)}_{k+1}\lambda^{(1)}_{k+1}\,, \tag{5.22}$$

with $\beta^{(1)}_{11} > 0$. Neglecting $\boldsymbol{\xi}^{(k)}$ it is possible to show that (5.22) admits a one-parameter family of solutions behaving like

$$\lambda^{(1)}_k \simeq \frac{\lambda}{1 - \beta^{(1)}_{11} k\lambda}\,, \quad k = 0, 1, \ldots, \qquad \lambda^{(2)}_k \simeq \text{const. } \lambda^{(1)2}_k\,,$$

$$\lambda^{(3)}_k \simeq \text{const. } \lambda^{(1)2}_k\,, \qquad \lambda^{(4)}_k = \text{const. } \lambda^{(1)2}_k\,, \tag{5.23}$$

where $\lambda < 0$ is a small enough parameter and (5.23) holds for all k.

It follows that (5.19) also admits a solution with the same properties, which, therefore, can be made as small as we like for $\lambda \to 0$.

Combining this result with item (5) above, one immediately finds that if the form factors are *defined* to be the just discussed solution of (5.19), then the effective potential $V^{(k)}$ can be written in the form (5.17), with $W^{\text{planar}}_{\nu,\mu,\rho,\sigma}(x, y, z, w)$ replacing $W^{(p)}$, given by a *convergent* power series in $\lambda^{(\alpha)}(k)$ satisfying (see (5.17))

$$\int_{\Delta_1 \times \cdots \times \Delta_p} |W^{\text{planar}}_{\nu,\mu,\rho,\sigma}(x, y, z, w)| \left| :\prod \varphi^{\nu} \prod \partial\varphi^{\mu} \cdots : \right| \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w$$

$$\leqslant \mathcal{N}(D) \| \hat{\boldsymbol{\lambda}} \|^p \| \varphi^{(\leqslant k)} \|^D_{C^{(3)}} \mathrm{e}^{-\chi\gamma^k \mathrm{d}(\Delta_1, \ldots, \Delta_p)} \beta^p\,. \tag{5.24}$$

If one can show, as seems reasonable and feasible, that the just mentioned solution to (5.19), $(\boldsymbol{\lambda}_k)^{\infty}_0 = \hat{\boldsymbol{\lambda}}$, has the "same analyticity properties" as the r.h.s. of (5.23), and, more precisely,

$$\frac{1}{s!} \left| \frac{\partial^s}{\partial \lambda^s} \lambda^{(\alpha)}_{(k)} \right| \leqslant E^s K^s s!, \quad \forall\, \mathrm{Re}\ \lambda < 0, |\lambda| \text{ small} \tag{5.25}$$

(which implies that the $\lambda^{(\alpha)}(k)$ are Borel summable in the λ variable), then a simple argument of abstract complex analysis shows that exponential decay implied by (5.12) allows us to infer that also the contribution of degree D in the fields to $V^{(k)}$ is the Borel sum of its perturbative series. The same conclusion would apply to the Schwinger functions of the planar theory.

For a more detailed exposition of the simple arguments leading to the bounds discussed above, see Gallavotti (1984), sects. 16–22, where also other material is reviewed, or see the original paper by Gallavotti and Nicoló (1984).

Acknowledgment

I am grateful to G. Felder and F. Nicoló for many discussions and suggestions.

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