CLASSICAL MECHANICS AND RENORMALIZATION GROUP

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ABSTRACT

The theory of Kolmogorov-Arnold-Moser (KAM) is discussed in detail from the point of view of the "renormalization group approach". Similarly we discuss some aspects of the problem of the existence of universal structures in the chaotic transition. The quasi-periodic Schroedinger equation in one dimension is discussed as a special case.

§1. Perturbations of integrable systems

A classical hamiltonian system is said to be integrable on a region W of phase space if:

i) the region W can be mapped via a canonical map $\mathscr{C}$ onto a region of the form $V \times T^\ell$ where $V \subset R^\ell$ is an open set and T^ℓ is the ℓ-dimensional torus.

ii) if the map $\mathscr{C}$ is such that

$$(\underline{p},\underline{q}) = \mathscr{C}(\underline{A},\underline{\phi}) \tag{1.1}$$

then the hamiltonian H, thought as a function h of the $(\underline{A},\underline{\phi})$ variables, becomes a function of the A's alone

$$H(\mathscr{C}(\underline{A},\underline{\phi})) = h(\underline{A}) \tag{1.2}$$

A hamiltonian system with hamiltonian H considered on a phase space region W will be denoted (H,W); in what follows I shall only consider hamiltonian systems and canonical maps of class C^∞ or analytic.

The notion of canonical map used here will be that usually called "action preserving completely canonical maps" [1] (§3.12 and p. 240): namely a map $\mathcal{C}$, (of class C^∞ at least with inverse of the same regularity), between W and W' is such a map if and only if there is a C^∞-function Φ defined on the graph $G(\mathcal{C})$

$$G(\mathcal{C}) = \{\underline{p},\underline{q},\underline{p}',\underline{q}' \mid (\underline{p},\underline{q}) = \mathcal{C}(\underline{p}',\underline{q}'),\ (\underline{p},\underline{q},\underline{p}',\underline{q}') \in W \times W'\} \tag{1.2}$$

such that

$$\underline{p} \cdot d\underline{q} = \underline{p}' \cdot d\underline{q}' + d\Phi \qquad \text{on} \quad G(\mathcal{C}) \tag{1.3}$$

If $\mathcal{C}$ is canonical in the above sense and if λ is a closed curve in W' then

$$A(\lambda) = \oint_{\mathcal{C}(\lambda)} \underline{p} \cdot d\underline{q} = \oint_{\lambda} \underline{p}' \cdot d\underline{q}' \tag{1.4}$$

More generally if (H,W), (H',W') are two C^∞ hamiltonian systems one defines the notion of relative integrability: H' is H-integrable on W if there exists a canonical map $\mathcal{C}: W \leftrightarrow W'$ and a C^∞-function F such that

$$H'(\mathcal{C}^{-1}(\underline{p},\underline{q})) = F(H(\underline{p},\underline{q}),\ A_2(\underline{p},\underline{q}),\ldots,A_s(\underline{p},\underline{q})) \tag{1.5}$$

where $A_2,\ldots,A_s$ is a maximal set of prime integrals in involution with each other and F is also supposed to be invertible with respect to its first argument.

The preceding notion of integrability is a particular case of the general relative integrability, just introduced, and it will be referred to as "integrability by quadratures"; in terms of the relative integrability it can be described as follows: a system (H,W) is integrable by quadratures if (H,W) is integrable with respect to a "free rotators' system". In fact, if (1.2) holds, then in the $(\underline{A},\underline{\phi})$ variables the system moves as

$$\begin{aligned} \dot{\underline{A}} &= \underline{0} \\ \dot{\underline{\phi}} &= \frac{\partial h}{\partial \underline{A}}(\underline{A}) \equiv \underline{\omega}(\underline{A}) \end{aligned} \tag{1.7}$$

i.e. each angle rotates uniformly.

In this connection it is useful to recall a theorem "of Arnold-Liouville" which, under some technical assumptions of invertibility of certain changes of coordinates, states that a necessary and sufficient condition of integrability by quadratures of (H,W) is the existence of ℓ independent (in the sense of the rank of the Jacobian matrix) prime integrals in involution $I_1 \equiv H, I_2,\ldots,I_\ell$ defined on W and such that the surfaces $I_1 = i_1,\ldots,I_\ell = i_\ell$ are compact and their union is W itself, (ℓ being the number of degrees of freedom), [2] (see p 269), (see also appendix A).

Since the Greek times it was firmly believed that the most general motion of a system would be described by equations (1.7) ("motion by epicycles") and only recently it has been realized by the physicists that this is not the case (starting late in the 1800's with Boltzmann and Poincaré).

The surprise has been particularly strong in the case of the theory of systems of the form

$$H_\varepsilon(\underline{A},\underline{\phi}) = h(\underline{A}) + \varepsilon f(\underline{A},\underline{\phi}) \tag{1.8}$$

with $(A,\phi) \in V \times T^\ell$, V being a sphere (say). This is a system that one would naively believe to move as the imperturbed rotators. In this case one would have expected the existence of an analytic family of canonical maps $\mathcal{C}_\varepsilon$, defined on $W_\varepsilon = V_\varepsilon \times T^\ell$, $V_\varepsilon \subset V$, and of functions h_ε on $V_\varepsilon \subset V$ such that

1) $\mathcal{C}_\varepsilon: W_\varepsilon \to W$; $\mathcal{C}_\varepsilon$-(identity) $\xrightarrow[\varepsilon\to 0]{} 0$

$h_\varepsilon: V_\varepsilon \to R$; $h_\varepsilon - h \xrightarrow[\varepsilon\to 0]{} 0$

with $\mathcal{C}_\varepsilon, h_\varepsilon$ analytic in $\varepsilon,\underline{A},\underline{\phi}$.

2) $\text{Vol } V_\varepsilon / \text{vol } V \xrightarrow[\varepsilon\to 0]{} 1$ (1.9)

3) $H_\varepsilon(\mathcal{C}_\varepsilon(\underline{A}',\underline{\phi}')) \equiv h_\varepsilon(\underline{A}')$ on W_ε

4) $\max_{\underline{A} \in \partial V_\varepsilon} d(\underline{A}',\partial V) \xrightarrow[\varepsilon\to 0]{} 0$

the last property means that V and V_ε differ only because of boundary effects (a difference to be obviously expected).

However, there are easy counter examples:

a) Let $h(\underline{A}) = A_1$ and consider

$$A_1 + \varepsilon f(A_2,\ldots,A_\ell,\phi_2,\ldots,\phi_\ell) \tag{1.10}$$

In this case it is obvious that on a time scale ε^{-1} the evolution of the $A_2,\ldots\phi_2,\ldots$ variables is entirely determined by the perturbation of f so, if one takes f to be such that the motion governed by the hamiltonian f are not quasi-periodic, the perturbed system is not integrable.

This case is a very pathological one and is called a "fully resonant case" because all the motions of the unperturbed system are periodic. More generally the latter situation is realized every time the unperturbed hamiltonian has the form $\underline{\omega}\cdot\underline{A}$ with $\omega \in R^\ell$ having components with rational ratios. Then a situation like the one in the above example arises provided the f depends only on suitable linear combinations of the coordinates

Exercises: 1) show that if ω_i/ω_j is rational then by a linear canonical change of coordinates one can change $\underline{\omega}\cdot\underline{A}$ into $\Omega A'_1$ for some Ω.

2) extend the above problem to the case where there exists $\underline{r} \in Z^\ell$, $\underline{r} \neq \underline{0}$, such that $\underline{\omega}\cdot\underline{r} = 0$ but $\omega_1 r_1 + \omega_2 r_2 \neq 0$ for all $(r_1,r_2) \in Z^2$.

3) extend the counter-example in (1.10) to the case when the unperturbed system is the linear system in problem 2).

b) Another type of counter-example to (1.9) is obtained by considering

$$\omega_1 A_1 + \omega_2 A_2 + \varepsilon(A_2 + f(\phi_1,\phi_2)) \qquad |A_1|,|A_2| < 1 \tag{1.11}$$

This system is not integrable in the sense (1.9) even when $\underline{\omega}$ is such that for some C, $\alpha > 0$

$$|\omega_1 r_1 + \omega_2 r_2|^{-1} < C(|r_1|+|r_2|)^\alpha \qquad \forall\, \underline{r} \in Z^2,\ \underline{r} \neq \underline{0} \tag{1.12}$$

(i.e. "$\underline{\omega}$ is (C,α)-non-resonant"), provided the Fourier transform $\hat{f}_{r_1 r_2}$ of f never vanishes for $\underline{r} \neq 0$. This is seen by solving explicitly the (trivial) equations of motion and checking that not all the motions are quasi-periodic, at least for the set of values of ε, of full measure, for which $\omega/1+\varepsilon$ is rational (exercise).

c) A classic counter-example due to Poincaré [3] is the system

$$H_\varepsilon(\underline{A},\underline{\phi}) = \frac{1}{2}(A_1^2+A_2^2) + \varepsilon\, f(A_1,A_2,\phi_1,\phi_2) \qquad |A_j| < 1 \tag{1.13}$$

with $\hat{f}_{r_1 r_2}(A_1,A_2) \neq 0$, $\forall (r_1,r_2) \neq \underline{0}$ ($f_{\underline{r}}(\underline{A})$ denotes the Fourier's transform of $f(\underline{A},\underline{\phi}) \in C^\infty(V\times T^\ell)$, for $\underline{r} \in Z^\ell$).

If one had integrability in the sense (1.9) one could find $\mathcal{C}_\varepsilon$ and call Φ_ε a generating function for it so that if $(\underline{A},\underline{\phi}) = \mathcal{C}_\varepsilon(\underline{A}',\underline{\phi}')$ it is

$$\underline{A} = \underline{A}' + \frac{\partial \Phi_\varepsilon}{\partial \underline{\phi}}(\underline{A}',\underline{\phi})$$

$$\underline{\phi}' = \underline{\phi} + \frac{\partial \Phi_\varepsilon}{\partial \underline{A}'}(\underline{A}',\underline{\phi})$$

and Φ_ε would be defined for ε small and $|A_j| < 1/2$, say, and $\Phi = \varepsilon\phi + \varepsilon^2\phi' + \dots$. Therefore substituting into the "Hamilton-Jacobi equation" $H_\varepsilon(\mathcal{C}_\varepsilon(\underline{A}',\underline{\phi}')) \equiv h_\varepsilon(\underline{A}')$, see (1.9), and developing both sides in powers of ε one would obtain

$$\frac{1}{2}\underline{A}'^2 + \varepsilon(\underline{A}'\cdot\frac{\partial\Phi}{\partial\underline{\phi}}(\underline{A}',\phi) + f(\underline{A}',\underline{\phi})) + O(\varepsilon^2) = \frac{1}{2}\underline{A}'^2 + \varepsilon h'(\underline{A}') + O(\varepsilon^2)$$

i.e.

$$\underline{A}'\cdot\frac{\partial\Phi}{\partial\underline{\phi}}(\underline{A}',\underline{\phi}) = -f(\underline{A}',\underline{\phi}) + h'(\underline{A}') \tag{1.15}$$

which, by integrating both sides with respect to $\underline{\phi}$, implies

$$h'(\underline{A}') = \frac{1}{(2\pi)^\ell}\int_{T^\ell} f(\underline{A}',\underline{\phi})d\phi \equiv \hat{f}_{\underline{0}}(A') \tag{1.16}$$

where $\hat{f}_{\underline{r}}(\underline{A}')$ denotes the Fourier transform of f, for $\underline{r} \in Z^\ell$. Then by taking the Fourier transform of both sides one finds

$$i\,\underline{A}'\cdot\underline{r}\,\hat{\Phi}_{\underline{r}}(\underline{A}') = -\hat{f}_{\underline{r}}(\underline{A}') \qquad \underline{r} \neq \underline{0} \tag{1.17}$$

which is impossible because $\hat{f}_{\underline{r}}(\underline{A}')$ never vanishes while near every $\underline{A}'_0$ there exist values of $\underline{A}'$ such that $\underline{A}'\cdot\underline{r} = 0$ for some $\underline{r} \neq \underline{0}$!

The above counter-examples' ideas can be combined in various ways to provide other strange examples. However, they basically exhaust the set of all the possible pathologies.

The following theorem illustrates the above statement [4,5].

Theorem: If the unperturbed system h associated with $H_\varepsilon(\underline{A},\underline{\phi}) = h(\underline{A}) + \varepsilon\, f(\underline{A},\underline{\phi})$ verifies suitable "non-resonance conditions" one can define $\mathcal{C}_\varepsilon$, h_ε of class C^∞ on $V \times T^\ell$ or V, respectively, and $C > 0$, so that

1) $\mathcal{C}_\varepsilon$-(identity) $= 0(\varepsilon)$, $h_\varepsilon - h = 0(\varepsilon)$

2) there exists $V_\varepsilon \subset V$ where*

$$H_\varepsilon(\mathcal{C}_\varepsilon(\underline{A}',\underline{\phi}')) \overset{C^\infty}{=} h_\varepsilon(\underline{A}') \qquad \forall(\underline{A}',\underline{\phi}') \in V_\varepsilon \times T^\ell \tag{1.18}$$

3) vol V_ε/vol $V \longrightarrow_{\varepsilon\to 0} 1$

4) $V_\varepsilon \supset \{\underline{A}' \mid |\frac{\partial h_\varepsilon}{\partial \underline{A}}(\underline{A}') \cdot \underline{r}|^{-1} < C_\varepsilon |\underline{r}|^\ell , \quad \forall \underline{r} \in Z^\ell, \underline{r} \neq \underline{0}\}$

and $C_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

There is a variety non-resonance condition under which the above theorem holds. One condition ("an isochrony" or "twist" condition)

$$\det \frac{\partial^2 h}{\partial \underline{A} \partial \underline{A}}(\underline{A}) \neq 0 \qquad \text{on } V \tag{1.19}$$

and it is certainly the simplest. Another condition, valid when $h(\underline{A}) = \underline{\omega} \cdot \underline{A}$, $\underline{\omega} \in R^\ell$, is the existence of $C, \alpha > 0$ such that

$$|\underline{\omega} \cdot \underline{r}|^{-1} < C|\underline{r}|^\alpha \qquad \forall \underline{r} \neq \underline{0} \tag{1.20}$$

and

$$\det \frac{\partial^2 \overline{f}}{\partial \underline{A} \partial \underline{A}}(\underline{A}) \neq 0 \qquad \text{on } V \tag{1.21}$$

where $\overline{f}$ is the average of f over the $\underline{\phi}$-variables.

But there are other conditions which one can impose: the cases studied so far all have the common feature of putting conditions allowing manipulations turning the problem into a problem in which (1.19) is verified. The most brilliant example is provided by the deep results on the three-body problem in [6].

The problem solved by the KAM theorem has a lot of connections with other problems, some of which are of a non-mechanical nature.[7]

*If f,g are two C^∞-functions and S is a closed set one says that $f \overset{C^\infty}{=} g$ on S, or $f = g$ on S in the C^∞-sense, if all their derivatives coincide on S.

§2 Integrability, Wick Ordering and Quasi-periodic Schroedinger Equation

Here we provide an example of a problem (and a related application) which arises in the context of the theory of the systems integrable by quadratures but which is of a somewhat different nature compared to the problem studied by the KAM theorem.

In this problem the connection with the theory of renormalization appears in a different light compared to the one related to the KAM theorem and in these notes I would like to discuss both in some detail.

A problem related to the one of finding non-resonance conditions which make the conclusions of the theorem of §1 valid is that of finding sufficient conditions insuring that H_ε is integrable by quadratures, see (1.8).

Typically, given f, one looks for an operator associating with f a new perturbation $\delta_\varepsilon f$ such that

$$H_\varepsilon(\underline{A},\underline{\phi}) = h(\underline{A}) + \varepsilon f(\underline{A},\underline{\phi}) + \delta_\varepsilon f(\underline{A},\underline{\phi}) \tag{2.1}$$

is integrable with respect to h itself on a large subset W_ε of the phase space (large in the sense of $(1.9)_4$). One has to think of the perturbations integrable in the above sense with respect to the unperturbed system as forming a "manifold" M_1 in the space of the hamiltonians. Then the operation of replacing the original hamiltonian by

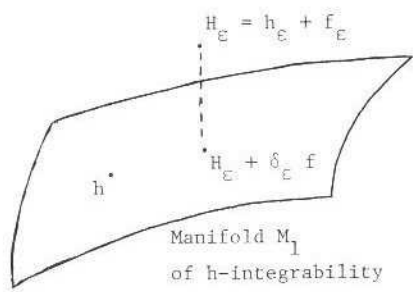


Fig. 2

$H_\varepsilon + \delta_\varepsilon f$ is a sort of "projection" of f on M_1 .

In the applications the "counterterm" $\delta_\varepsilon f$ is not arbitrary (obviously one could always take $\delta_\varepsilon f = -\varepsilon f$!) but its form is restricted by the application itself, e.g. $\delta_\varepsilon f(\underline{A},\underline{\phi})$ can be restricted to be $\underline{\phi}$-independent (in this case the operation of replacing H_ε by $H_\varepsilon + \delta_\varepsilon f$ is called "Wick ordering").

The best example where the above problem arises is in an apparently unrelated problem, i.e. in the theory of the Schroedinger equation in a quasi-periodic potential. Consider the equation

$$-\ddot{q} + \varepsilon V(\underline{\omega}t)q = E\,q \qquad t \in R \tag{2.2}$$

where $\underline{\phi} \to V(\underline{\phi}) \geq 0$ is analytic on T^ℓ and $\underline{\omega} \in R^\ell$ verifies (1.20) ("Diophantine condition"), and $E > 0$ is a parameter. This is the Schroedinger equation with quasi-periodic potential $V(\underline{\omega}t)$ on the line. The abscissa on the line, which is usually called x, is called here t.

One interprets the above system as a mechanical system on $R^{1+\ell} \times (R\times T^\ell)$ with the hamiltonian

$$H(p,\underline{B},q,\underline{\phi}) = \frac{p^2}{2} + (E - \varepsilon V(\underline{\phi}))\,\frac{q^2}{2} + \underline{B}\cdot\underline{\omega} \tag{2.3}$$

where $(p,\underline{B},q,\underline{\phi}) \in R^{1+\ell} \times (R\times T^\ell)$ are canonical variables.

Any solution of (2.2) describes a motion of (2.3) with $\underline{\phi}(0) = \underline{0}$ and vice versa.

For the coming discussion it is useful to change variables replacing (p,q) with the action angle variables of the harmonic oscillator $p^2+Eq^2/2$, which we call (A,θ): $\underline{B}' \equiv \underline{B}$, $\underline{\phi}' \equiv \underline{\phi}$ and

$$A = \frac{1}{2}\,\frac{1}{\sqrt{E}}\,(p^2+Eq^2) \qquad p = \sqrt{2A\sqrt{E}}\,\cos\theta$$
$$\theta = \operatorname{arccotg}\frac{q\sqrt{E}}{\sqrt{p^2+Eq^2}} \qquad q = \frac{\sqrt{2A\sqrt{E}}}{\sqrt{E}}\,\sin\theta \tag{2.4}$$

(i.e. $\sqrt{2A\sqrt{E}},\theta$) are the polar coordinates of the point (p, Eq)).

Then (2.3) becomes

$$\sqrt{E}\,A + \underline{B}\cdot\underline{\omega} - \frac{\varepsilon}{\sqrt{E}}\,A\,V(\underline{\phi})\,\sin^2\theta \tag{2.5}$$

and since the map (2.4) is canonical the solutions of (2.5) with $\underline{\phi}(0) = \underline{0}$ and the solutions of the quasi-periodic Schroedinger

equation (2.2) are in one-to-one correspondence (easily explicitly written by using (2.4)).

We now consider the problem of finding the values of E for which the Eq. (2.2) admits a quasi-periodic, hence bounded, solution. It is the problem of finding values of E for which the quasi-periodic Schroedinger equation admits "Block waves". This problem is closely related to that of finding the continuous spectrum of the Schroedinger operator with potential $V(\underline{\omega}t)$.

One can clearly find a subset of such values of E by imposing that (2.5) is integrable by quadratures. A way to impose integrability of (2.5) is the following. Let λ be a fixed real number and let $\sqrt{E} = \lambda+a$, $\eta = \varepsilon/(\lambda+a)$ and rewrite (2.5) as

$$\lambda A + \underline{\omega}\cdot\underline{B} - \eta A V(\underline{\phi})\,\sin^2\theta + aA \tag{2.6}$$

Then we can try to impose integrability by fixing conveniently a as a function of η, λ, if this is possible. So in the above context we are led to the problem of determining whether a counterterm of the form $\delta\eta = Aa$ turns the system $\lambda\underline{A} + \underline{\omega}\cdot\underline{B} - \eta A V(\phi)\sin^2\theta$ into an integrable system on, say, the whole phase space $R^{\ell+1} \times T^{\ell+1}$, for η small at least.

If λ is chosen so that for some $\alpha > 0$, $c > 0$

$$|\lambda\mu + \underline{\omega}\cdot\underline{r}|^{-1} \leq C(|\mu|+|\underline{r}|)^\alpha \tag{2.7}$$

$\forall(\mu,\underline{r}) \in Z^{\ell+1}$, $(\mu,\underline{r}) \neq \underline{0}$, then it turns out, as we shall show, that the above choice of a is possible for η small and $a(\eta,\lambda)$ is analytic in η for η small so that

$$a(\eta,\lambda) = \alpha(\lambda)\eta + \beta(\lambda)\eta^2 + \ldots \qquad , \alpha(\lambda) \neq 0 \tag{2.8}$$

This means that the original Schroedinger equation has the value

$$E = (\lambda + a_\varepsilon)^2 \tag{2.9}$$

for which there exist Bloch waves, provided a_ε verifies

$$a_\varepsilon = a(\frac{\varepsilon}{\lambda+a_\varepsilon}\,,\lambda) \tag{2.10}$$

which always has a solution for small ε (because $\alpha(\lambda) \neq 0$: actually $\alpha(\lambda) = \{\text{average value of } V(\phi)\cos^2\theta\} \neq 0$, from $V > 0$).

The above-mentioned result is a consequence of the following theorem of which we shall present a proof in these notes [8,9].

Theorem: Suppose that $\underline{\omega} \in R^\ell$ verifies a diophantine condition, (1.20), and let $\underline{f}$ be an R^ℓ-valued analytic function on T^ℓ. Then if ε is small enough there is an analytic function $\varepsilon \to \underline{a}(\varepsilon)$ such that the hamiltonian system described on $R^\ell \times T^\ell$ by

$$\tilde{H}_\varepsilon(A,\phi) = \underline{\omega}\cdot\underline{A} + \varepsilon\underline{A}\cdot f(\phi) + \underline{A}\cdot\underline{a}(\varepsilon) \qquad (2.11)$$

is integrable by quadratures on the whole phase space.

Actually, the above theorem is a slight generalization of [8,9] and is proved in a somewhat different spirit, along the lines of [10]; the only other known result about counterterms is [10], the following

Theorem: Consider $\underline{\omega} \in R^\ell$ verifying a diophantine condition, (1.20) and let f be analytic on $V_r \times T^\ell$ with $V_r = \{\underline{A} | \underline{A} \in R^\ell, |A_j| < r\} \subset R^\ell$. Consider

$$H_\varepsilon(\underline{A},\underline{\phi}) = \underline{A}\cdot\underline{\omega} + \varepsilon f(\underline{A},\underline{\phi}) \qquad (2.1\ )$$

then there is at most one analytic function $N(\underline{A},\varepsilon)$, analytic in $A \in V_r$ and ε for ε small such that $H_\varepsilon - N$ is canonically conjugate to $\underline{\omega}\cdot\underline{A}$ in $V_{r'} \times T^\ell$, $r' < r$, for ε small enough.

Furthermore, there exists a sequence of functions $N_k(A)$ analytic in $\underline{A} \in V_r$, $k = 1,2,\ldots$, such that N exists if and only if the following series

$$\sum_{k=1}^{\infty} \varepsilon^k N_k(\underline{A}) \qquad (2.13)$$

converges for small ε. Its sum is then N.

The above result shows that, sometimes, the operator $\delta_\varepsilon f$ is uniquely defined. (Exercise: Compute explicitly the series $\Sigma_k N_k \varepsilon^k$ in the case $\sqrt{2}\,A_1 + A_2 + \varepsilon(A_2 + f(\phi_1,\phi_2))$).

§3. Existence of Quasi-periodic motions

The KAM theorem admits many proofs. I shall present a proof formulated in the language of the renormalization group, of a slightly weaker version ("Kolmogorov theorem") in which one only shows the existence of a quasi-periodic motion with a given frequency spectrum.

I shall only consider an analytic hamiltonian $h_0(\underline{A})$ with h_0 holomorphic in

$$S_{\rho_0} = \{\underline{A} | \underline{A} \in C^\ell,\ |A_j| < \rho_0, \forall_j\} \qquad (3.1)$$

with an analytic perturbation f_0, analytic on $V_\rho \times T^\ell$ where

$$V_\rho = \{\underline{A}\ |A \in R^\ell,\ |A_j| < \rho\} \qquad (3.2)$$

The analyticity of f_0 will be imposed by demanding that $(\underline{A},\underline{\phi}) \to f_0(\underline{A},\underline{\phi})$ regarded as a function of $A \in V$ and of $\underline{z} = (z_1,\ldots,z_\ell) \equiv (e^{i\phi_1},\ldots,e^{i\phi_\ell})$ is holomorphic in a set of the form

$$C(\rho_0,\xi_0) = S_{\rho_0} \times C(\xi_0) \subset C^{2\ell} \qquad (3.3)$$

where S_{ρ_0} is defined in (3.1) and

$$C(\xi_0) = \{\underline{z} | \underline{z} \in C^\ell,\ e^{-\xi_0} < |z_j| < e^{\xi_0}\} \qquad (3.4)$$

Let us introduce a few parameters measuring the size of various quantities

$$\|\underline{W}\| = \sup_j |W_j|\ , \quad |\underline{W}| = \sum_{j=1}^{p} |W_j| \qquad \forall \underline{W} \in C^p \qquad (3.5)$$

and

$$E_0 = \left\|\frac{\partial h_0}{\partial \underline{A}}\right\|_{\rho_0} \equiv \sup_{A \in S_{\rho_0}} \left\|\frac{\partial h_0}{\partial \underline{A}}(A)\right\| \qquad (3.6)$$

and E_0^{-1} "measures the time scale" on which the unperturbed (or "free") system evolves. The quantity (in computing norms we regard a matrix $\ell\times\ell$ as a vector in C^p, $p = \ell^2$ and use (3.5)):

$$\eta_0 = \left\|\left(\frac{\partial^2 h_0}{\partial \underline{A}\partial \underline{A}}\right)^{-1}\right\|_{\rho_0} \qquad (3.7)$$

will be called "resonance mobility" or as "twist parameter" and η_0^{-1} measures how fast the frequencies change as $\underline{A}$ changes, hence how fast $\underline{\omega}(\underline{A})$ moves away from a resonant value as $\underline{A}$ varies. The quantity

$$\varepsilon_0 = \|\frac{\partial f_0}{\partial \underline{A}}\|_{\rho_0,\xi_0} + \frac{1}{\rho_0}\|\frac{\partial f_0}{\partial \underline{\phi}}\|_{\rho_0,\xi_0} \qquad (3.8)$$

where $\partial/\partial\phi_j \equiv iz_j\partial/\partial z_j$, is related to the time scale on which the perturbation starts revealing itself, which is clearly ε_0^{-1}. Then the following theorem holds.

Theorem: Let h_0 be holomorphic on S_{ρ_0} and suppose that $\underline{\omega}_0 = \partial h_0/\partial \underline{A}\,(\underline{0})$ verifies a non-resonance condition for some $C_0 > 0$

$$|\underline{\omega}_0 \cdot \underline{r}|^{-1} < C_0 |\underline{r}|^{\ell} \qquad \forall \underline{r} \in Z^{\ell}\,,\ |\underline{r}| \neq 0 \qquad (3.9)$$

Let f_0 be a perturbation holomorphic in $S_{\rho_0} \times C(\xi_0)$. Consider the hamiltonian system on $V \times T^{\ell}$ with hamiltonian

$$H_0(\underline{A},\underline{\phi}) = h_0(\underline{A}) + f_0(\underline{A},\underline{\phi}) \qquad (3.10)$$

Assume, for simplicity, $\varepsilon_0 C_0$, ε_0/E_0, $\zeta_0 < 1$. There exist two functions $\underline{\alpha}$, $\underline{\beta}$ holomorphic on $C(\xi_0/2)$ such that the torus $\mathcal{T}$ with parametric equations

$$\begin{cases} \underline{A} = \underline{\alpha}(\underline{\phi}') \\ \underline{\phi} = \underline{\phi}' + \underline{\beta}(\underline{\phi}') \end{cases} \qquad (3.11)$$

is invariant, for ε_0 small, in the flow generated by (3.10) and, actually, this flow is described on σ by $\underline{\phi}' \to \underline{\phi}' + \underline{\omega}_0 t$.

Furthermore $\underline{\alpha}$, $\underline{\beta}$ tend to zero as $\varepsilon_0 \to 0$ and ε_0 small means that there are suitable constants depending only on ℓ, denoted G, a_1, a_2, ..., $G < 1/2$, such that ε_0 should verify

$$\varepsilon_0 C_0 < G(E_0 C_0)^{-a_1}(E_0\eta_0\rho_0^{-1})^{-a_2}\xi_0^{a_3} \qquad (3.13)$$

In other words, in a non-resonant situation ($C_0 < +\infty$, $\eta_0 < +\infty$) a non-resonant torus is not destroyed by a small perturbation in the sense that in its vicinity there is another torus which is invariant for the perturbed flow and is run with a quasi-periodic motion with the same frequencies (i.e. the "same spectrum") as the unperturbed one.

Proof: Introduce a convenient sequence of parameters δ_k, $k = 0, 1, \ldots$, which will be used in the estimates; for instance (so that $\sum_{k=0}^{\infty} 5\,\delta_k = \xi_0/2 > 0$)

$$\delta_k = \xi_0/5\pi(1+k)^2 \qquad (3.13)$$

Define the following "renormalization transformation", acting on the pair (h_0,f_0) analytic on $S_{\rho_0} \times C(\xi_0)$ and mapping it in (h_1,f_1) analytic on $S_{\rho_0} \times C(\xi_0-\delta_0)$ via the following steps.

1) Truncate the Fourier's series of f_0 in two parts

$$f_0 = f_0^{[\le N_0]} + f_0^{[>N_0]} \qquad (3.14)$$

$$f^{[>N_0]}(\underline{A},\underline{z}) = \sum_{\substack{|\underline{r}|>N_0 \\ \underline{r}\in Z^{\ell}}} \hat{f}_{\underline{r}}(\underline{A})\,\underline{z}^{\underline{r}}$$

where $\underline{z}^{\underline{r}} \equiv z_1^{r_1}\ldots z_{\ell}^{r_{\ell}}$, chosing N_0 so that $f^{[>N_0]}$ is (ε_0^2) i.e. for some B_1

$$\|\frac{\partial f_0^{[>N_0]}}{\partial \underline{A}}\|_{\rho_0,\xi_0-\delta_0} + \frac{1}{\rho_0}\|\frac{\partial f_0^{[>N_0]}}{\partial \underline{\phi}}\|_{\rho_0,\xi_0-\delta_0} \le B_1 C_0 \varepsilon_0^2 \qquad (3.15)$$

Since the (3.8) implies bounds on the Fourier coefficients $\partial\hat{f}_{0\underline{r}}/\partial\underline{A}\,(\underline{A})$, $|\underline{r}|\hat{f}_{0\underline{r}}(\underline{A})$, as

$$\|\frac{\partial \hat{f}_{0\underline{r}}}{\partial \underline{A}}\| \le \varepsilon_0\, e^{-\xi_0|\underline{r}|}\,,\quad \|\underline{r}\,\hat{f}_{0\underline{r}}(\underline{A})\|_{\rho_0} \le \varepsilon_0\rho_0\, e^{-\xi_0|\underline{r}|} \qquad (3.16)$$

it is easy to see that one can take

$$N_0 = \delta_0^{-1}\log(C_0\varepsilon_0\delta_0^{-\ell})^{-1} \qquad (3.17)$$

which if inserted in the l.h.s. of (3.15) provides the bound appearing in (3.16). The number (3.17) will be called the "ultra violet cut-off".

2) Define for $\|\underline{A}'\| < \tilde{\rho} < \rho_0/2$

$$\Phi_0(\underline{A}',\underline{z}) = \sum_{0<|\underline{r}|\le N} \frac{\hat{f}_{\underline{r}}(\underline{A}')}{-i\underline{\omega}_0(\underline{A}')\cdot\underline{r}}\, \underline{z}^{\underline{r}} \qquad (3.18)$$

where $\tilde{\rho}$ is chosen so small that $\underline{\omega}(\underline{A}')\cdot\underline{r}$ stays away from zero for all $|\underline{r}| \le N_0$, i.e.

$$|\underline{\omega}_0(\underline{A}')\cdot r|^{-1} \le 2\, C_0 |\underline{r}|^{\ell} \qquad \|\underline{A}'\| < \tilde{\rho} \qquad (3.19)$$

Using obvious dimensional estimates* on the derivatives of h_0 we find, using $\underline{\omega}(\underline{0}) = \underline{\omega}_0$ and (3.9)

$$|\underline{\omega}_0(\underline{A}')\cdot\underline{r}|^{-1} \equiv |(\underline{\omega}_0 + (\underline{\omega}(A') - \underline{\omega}_0))\cdot\underline{r}|^{-1} \le$$

$$\le |\underline{\omega}_0\cdot\underline{r}|^{-1} \left|1 - \frac{|(\underline{\omega}_0(\underline{A}')-\underline{\omega}_0(\underline{0})\cdot\underline{r}|}{|\underline{\omega}_0\cdot\underline{r}|}\right|^{-1} \le \qquad (3.20)$$

$$\le C_0 |\underline{r}|^{\ell} \left|1 - \ell C_0 N_0^{\ell+1} \frac{2E_0}{\rho_0}\tilde{\rho}\right|^{-1}$$

which shows that one can take

$$\tilde{\rho} = \rho_0/2(2\ell C_0 E_0 N_0^{\ell+1}) \qquad (3.21)$$

Then we can easily estimate, using (3.16), (3.19), the function Φ_0

$$\|\Phi_0\|_{\tilde{\rho},\xi_0-\delta_0} \le B_2 C_0 \xi_0 \rho_0 \delta_0^{-2\ell+1} \qquad (3.22)$$

hence by dimensional estimates, and by (3.21)

$$\left\|\frac{\partial\Phi_0}{\partial\underline{A}}\right\|_{\tilde{\rho}/2,\xi_0-2\delta_0} + \frac{2}{\tilde{\rho}} \left\|\frac{\partial\Phi_0}{\partial\phi}\right\|_{\tilde{\rho}/2,\xi_0-2\delta_0} \le$$

$$\le B_3' C_0 \varepsilon_0 \delta_0^{-2\ell} \rho_0/\tilde{\rho} = B_3 C_0 \varepsilon_0 C_0 E_0 N_0^{\ell+1} \delta_0^{-2\ell} \qquad (3.23)$$

for a suitably chosen B_3', B_3. (Exercise: find explicit estimates for B_2, B_3).

*By dimensional estimate we mean the estimate of a derivative of a holomorphic function in a point by the supremum of the function in a domain divided by the distance of the point to the boundary i.e.

$$\left\|\frac{\partial^2 h_0}{\partial\underline{A}\partial\underline{A}}\right\|_{\tilde{\rho}} \le \frac{1}{\rho_0-\tilde{\rho}} \left\|\frac{\partial h_0}{\partial\underline{A}}\right\|_{\rho_0} \le 2\, E_0/\rho_0 \quad \text{if} \quad \tilde{\rho}<\rho_0/2\,, \text{ for instance.}$$

3) Define on $C(\tilde{\rho}/2,\xi_0-2\delta_0)$ the canonical maps, inverse of each other, generated by Φ_0 via

$$\underline{A} = \underline{A}' + \frac{\partial\Phi_0}{\partial\underline{\phi}}(\underline{A}',\underline{\phi}) \qquad (\underline{A},\underline{\phi}) = \mathcal{C}(\underline{A}',\underline{\phi}')$$

$$\Rightarrow \qquad (3.24)$$

$$\underline{\phi}' = \underline{\phi} + \frac{\partial\Phi_0}{\partial\underline{A}'}(\underline{A}',\underline{\phi}) \qquad (\underline{A}',\underline{\phi}') = \mathcal{C}'(\underline{A},\underline{\phi})$$

The possibility of this can be imposed by requiring that ε_0 and, hence, Φ_0 and its derivatives be small. By using a simple implicit functions' theorem one finds that, under the condition that $\partial^2\Phi_0/\partial\underline{A}\partial\underline{A}$ is small compared to 1 and $\partial\Phi_0/\partial\underline{A}$, $\tilde{\rho}^{-1}\partial\Phi_0/\partial\underline{\phi}$ are small compared to δ_0,* the maps $\mathcal{C}, \mathcal{C}'$ can be defined with the following properties

a) $\mathcal{C}, \mathcal{C}' : C(\tilde{\rho}/4,\xi_0-3\delta_0) \to C(\tilde{\rho}/2,\xi_0-2\delta_0)$

b) $\mathcal{C}, \mathcal{C}' = \mathcal{C}'\mathcal{C}$ = identity on $C(\tilde{\rho}/8,\xi_0-4\delta_0)$ (3.25)

c) $\mathcal{C}(C(\tilde{\rho}/32,\ \xi_0-5\delta_0)) \supset C(\tilde{\rho}/64,\xi_0-5\delta_0)$ **

and $(\underline{A},\underline{z}) = \mathcal{C}(\underline{A}',\underline{z}')$, $(\underline{A}',\underline{z}') = \mathcal{C}'(\underline{A},\underline{z})$ are defined as

$$A = \underline{A}' + \Xi(\underline{A}',\underline{\phi}') \quad , \quad \underline{A}' = \underline{A} + \Xi'(\underline{A},\underline{\phi})$$
$$\underline{\phi} = \underline{\phi}' + \Delta(\underline{A}',\underline{\phi}') \quad , \quad \underline{\phi}' = \underline{\phi} + \Delta'(\underline{A},\underline{\phi}) \qquad (3.26)$$

with

$$\Xi(\underline{A}',\underline{\phi}') = \frac{\partial\Phi_0}{\partial\underline{\phi}}(\underline{A}',\underline{\phi}) = -\Xi'(\underline{A},\underline{\phi})$$
$$\Delta(\underline{A}',\underline{\phi}') = -\frac{\partial\Phi_0}{\partial\underline{A}'}(\underline{A}',\phi) = -\Delta'(\underline{A},\underline{\phi}) \qquad (3.27)$$

so that, by (3.25)

$$\|\Xi\|_{\tilde{\rho}/8,\xi_0-4\delta_0}, \|\Xi'\|_{\tilde{\rho}/8,\xi_0-4\delta_0} \le \left\|\frac{\partial\Phi_0}{\partial\underline{\phi}}\right\|_{\tilde{\rho}/2,\xi_0-2\delta_0}$$
$$\qquad (3.28)$$
$$\|\Delta\|_{\tilde{\rho}/8,\xi_0-4\delta_0}, \|\Delta'\|_{\tilde{\rho}/8,\xi_0-4\delta_0} \le \left\|\frac{\partial\Phi_0}{\partial\underline{A}'}\right\|_{\tilde{\rho}/2,\xi_0-2\delta_0}$$

The mentioned smallness condition to be imposed on ε_0 to insure the existence of $\mathcal{C}, \mathcal{C}'$ with the above properties can be easily

*The first such condition will insure local invertibility while the second will guarantee global extension of the local invertibility and will allow control of the domains.

**32,64,5,6 are here quite arbitrary and this choice will be useful later.

deduced by estimating $\partial^2\Phi_0/\partial\underline{A}\partial\underline{\phi}$, $\partial\Phi_0/\partial\underline{A}$, $\partial\Phi_0/\partial\underline{\phi}$ dimensionally, starting from (3.22). Not surprisingly the three smallness conditions basically coincide and can be imposed through a single condition like

$$B_4\varepsilon_0 C_0 E_0 C_0 \; N_0^{\ell+1} \; \delta_0^{-2\ell-1} < 1 \tag{3.29}$$

The fact that (3.29) is sufficient, for B_4 large, to guarantee (3.25) ÷ (3.28) is easily proved (see [1], §3.11, p. 490, and appendix N).

4) Define

$$\begin{aligned} \tilde{h}_1(\underline{A}') &= h_0(\underline{A}') + \bar{f}_0(\underline{A}') \\ \tilde{f}_1(\underline{A}',\underline{\phi}') &= H_0(\mathcal{C}(\underline{A}',\underline{\phi}')) - \tilde{h}_1(\underline{A}') \end{aligned} \tag{3.30}$$

and $\tilde{h}_1,\tilde{f}_1$ are holomorphic at least on $C(\tilde{\rho}/8,\xi_0-4\delta_0)$, by the properties of $\mathcal{C}$: and the motions starting in the $\mathcal{C}$-image of this set can be described in the $(\underline{A}',\underline{\phi})$ - coordinates as hamiltonian motions with hamiltonian $\tilde{H} = \tilde{h} + \tilde{f}$, at least as long as they do not leave the set $\mathcal{C}(C(\tilde{\rho}/8,\xi_0-4\delta_0)$.

Then we look for a point $\underline{A}' \in V_{\tilde{\rho}/8}$ which would move with the right spectrum $\underline{\omega}_0$ if $\tilde{f}_1$ were zero (by construction $\tilde{f}_1$ is, in any case, much smaller than f_0 as we shall see) i.e. we look for a solution of

$$\underline{\omega}_0(\underline{A}') + \frac{\partial\bar{f}_0}{\partial\underline{A}'}(\underline{A}') = \underline{\omega}_0 \tag{3.31}$$

where $\bar{f}_0$ denotes the average of f_0 on T^ℓ .

The solution to this equation can be found by recalling that $\underline{\omega}_0 = \underline{\omega}(\underline{0})$ and rewriting it using the Taylor expansion and calling $M(\underline{A}) = \partial^2 h_0/\partial\underline{A}\partial\underline{A} = \partial\underline{\omega}_0/\partial\underline{A}$

$$\frac{1}{2}M(\underline{0})\underline{A}' + [\underline{\omega}_0(\underline{A}') - \underline{\omega}_0(\underline{0}) - \frac{1}{2}M(\underline{0})\underline{A}' + \frac{\partial\bar{f}_0}{\partial\underline{A}'}(\underline{A}')] = \underline{0}$$

i.e. as

$$\begin{aligned} &\underline{A}' + \underline{n}(\underline{A}') = \underline{0} \\ &\underline{n}(\underline{A}') = M(\underline{0})^{-1}\{\frac{\partial f}{\partial\underline{A}},(\underline{A}') + \int_0^1 d\tau\int_0^\tau d\theta\,\frac{d^2}{d\theta^2}\,\underline{\omega}_0(\theta\underline{A}') \end{aligned} \tag{3.32}$$

so that for a suitable $B_5 > 2$ and $\forall\beta < \rho/2$ it is, by a dimensional estimate

$$\|\underline{n}\|_\beta \le B_5\eta_0(\varepsilon_0 + (\frac{\beta}{\rho_0})^2 E_0) \tag{3.33}$$

Hence, if $\beta = \rho_0(\varepsilon_0/E_0)^{1/2}$ it is

$$\|\underline{n}\|_\beta \le 2B_5\,\eta_0\,\varepsilon_0 \tag{3.34}$$

By an implicit function theorem it can be easily seen (see [1], p. 490) that if $\|\underline{n}\|_\beta$ is much smaller than β the (3.32) has a solution $\underline{A}'_0$ verifying, of course, $|\underline{A}'_0| \le \|\underline{n}\|_\beta << \beta$. So if B'_6 is a suitably large constant and $B'_6\,\|\underline{n}\|_\beta\,\beta^{-1} < 1$, i.e. with the above value of β and by (3.34) if

$$B_6(\eta_0 E_0\rho_0^{-1})\sqrt{\frac{\varepsilon_0}{E_0}} < 1 \tag{3.35}$$

there is a solution $\underline{A}'_0$ to (3.31) verifying

$$\|\underline{A}'_0\| \le 2B_5\,\varepsilon_0\,\eta_0 \tag{3.36}$$

Since we want that $\underline{A}'_0$ be well inside the region where $\hat{h}_1$ is defined, e.g. $\|\underline{A}'_0\| < \tilde{\rho}/16$, we impose $2B_5\,\varepsilon_0\eta_0 < \tilde{\rho}/16$. Recalling the (3.21) we see that the latter condition as well as the (3.35) and (3.29) can be imposed simultaneously by requiring the single inequality

$$B_7(\eta_0 E_0\rho_0^{-1})^2\;\varepsilon_0 C_0 E_0 C_0 N_0^{\ell+1}\delta_0^{-2\ell-1} < 1 \tag{3.37}$$

(where we use $\varepsilon_0/E_0 = \varepsilon_0 C_0/E_0 C_0 \le \varepsilon_0 C_0$ because (3.9) implies $C_0 E_0 \ge 1$).

So, if (3.37) holds we can draw around $\underline{A}'_0$ a polidisk of radius $\tilde{\rho}/16$, at least, contained inside $S_{\tilde{\rho}/8}$.

5) Let $\lambda_0^{-1} = \tilde{\rho}/32\;\rho_0 << 1$, and let

$$\begin{aligned} &\underline{\phi}_1 = \underline{\phi}' \\ &\underline{A}_1 = (\underline{A}'-\underline{A}'_0)\lambda_0 \qquad \text{for } \|\underline{A}'-\underline{A}'_0\| < \tilde{\rho}/32 \end{aligned} \tag{3.38}$$

Then the variable $\underline{A}_1$ varies again in S_{ρ_0} as $\underline{A}'$ varies. We finally define

$$\begin{aligned} h_1(\underline{A}_1) &= \lambda_0(\tilde{h}_1(\underline{A}'_0 + \lambda_0^{-1}\underline{A}_1) - \tilde{h}_1(\underline{A}'_0)) \\ f_1(\underline{A}_1,\underline{\phi}_1) &= \lambda_0\tilde{f}_1(\underline{A}'_0 + \lambda_0^{-1}\underline{A}_1,\underline{\phi}_1) \end{aligned} \tag{3.39}$$

for $(\underline{A}_1,z_1) \in C(\rho_0,\xi_0-5\delta_0)$, and on this set the above functions are holomorphic.

This is the final step in the construction of the renormalization transformation $\mathcal{T}_0$. Its domain consists in the functions h_0,f_0 verifying (3.37).

It is now very easy, although a little long, to check, using dimensional estimates and (3.16), (3.25), that if (3.37) holds, then

$$E_1 \le E_0 + \varepsilon_0 \quad , \quad \eta_1 \le \lambda_0\eta_0(1+B_8\varepsilon_0\eta_0\rho_0^{-1})$$

$$\varepsilon_1 \le B_9 C_0\varepsilon_0^2 (C_0E_0)^2 N_0^{\ell+1}\delta_0^{-b_1} \quad , \quad \xi_1 = \xi_0 - 5\delta_0, \ \rho_1 \equiv \rho_0 \tag{3.40}$$

with B_8, B_9, b_1 suitably chosen. For details see for instance [1], §5.12, pp 510-512.

The above construction can be iterated to build $\mathcal{T}_1, \mathcal{T}_2, \ldots$. The construction is identical to that of $\mathcal{T}_0$ with δ_j replacing δ_0 . The above analysis implies that if we define

$$\begin{cases} \bar{E}_n = (\bar{E}_{n-1} + \bar{\varepsilon}_{n-1}) \\ \bar{\eta}_n = \bar{\lambda}_{n-1}\bar{\eta}_{n-1}\,(1 + B_8\,\bar{\varepsilon}_{n-1}\eta_{n-1}\rho_0^{-1}) \qquad n \ge 1 \\ \bar{\varepsilon}_n = B_9 C_0 \bar{\varepsilon}_{n-1}^2\,(C_0\bar{E}_{n-1})^2\,\bar{N}_n^{\ell+1}\delta_n^{-b_1} \\ \bar{\xi}_n = \bar{\xi}_{n-1} - 5\,\delta_{n-1} \quad , \quad \bar{\rho}_n = \rho_0 \end{cases} \tag{3.41}$$

with $\bar{N}_n = \delta_n^{-1}\log(C_0\varepsilon_n\delta_n^{-\ell})^{-1}$, $\bar{\lambda}_n = 32\cdot 4\ell C_0\bar{E}_n\bar{N}_n^{\ell+1}$ and for $n = 0$ the barred quantities are identified with the unbarred ones, and if

$$B_7(\bar{\eta}_n\bar{E}_n\rho_0^{-1})^2\bar{\varepsilon}_n C_0\bar{E}_n C_0\bar{N}_n^{\ell+1}\delta_n^{-2\ell-1} < 1 \tag{3.42}$$

it will result, $\forall\, n \ge 1$

$$E_n \le \bar{E}_n, \ \varepsilon_n \le \bar{\varepsilon}_n \ , \ \eta_n \le \bar{\eta}_n \ , \ \zeta_n = \bar{\xi}_n \ , \ \rho_n = \rho_0 \tag{3.43}$$

It is easy to check by induction that if ε_0/E_0 is small enough the (3.41) imply the (3.42) for all n and

$$(C_0\varepsilon_0)^{2^n} \le C_0\bar{\varepsilon}_n \le (C_0\varepsilon_0)^{(3/2)^n}$$

$$\bar{\eta}_n \le \eta_0(\bar{E}_0C_0)^n(\xi_0^{-1}\log(C_0\varepsilon_0)^{-1})^{2n}\,2^{2n^2(\ell+1)}$$

$$\bar{E}_n \le 2E_0 \tag{3.44}$$

and "small enough" means a condition like the (3.12). For a complete discussion of the implication of (3.44) from (3.41), (3.42) see for instance [1], p 512-515. However, the discussion by induction is so simple that it should be worked out as an exercise. A more careful analysis of (3.41), (3.42) easily leads to the conclusion that in (3.12) $a_1 > 2$, $a_2 > 2$, $a_3 > 2(\ell+1)$ are allowed choices. It is very instructive to try to find an explicit value of G fixed a_1, a_2, a_3 or even to try to optimize all the above bounds. There seems to be no really satisfactory analysis of such bounds.

To summarize, we have found that under the condition (3.12)

$$\mathcal{T}_n\mathcal{T}_{n-1}\ldots\mathcal{T}_0(h_0, f_0) \to (\underline{\omega}_0\cdot\underline{A}, 0) \tag{3.45}$$

uniformly in the compact sets of $C(\rho_0, \xi_0/2)$ (just use a dimensional estimate on $\partial^2 h_n/\partial A\partial A$ to see that $h_n \to \underline{\omega}_0\cdot A$ via Vitali's theorem on sequences of homomorphic functions; $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ implies that $f_n \xrightarrow[n\to\infty]{} 0$. Actually, it only implies that f_n tends to a constant. However, in the proof of (3.41) one finds first a bound on f_n itself (rather than directly a bound on its derivatives) and then by a dimensional estimate the bound ε_n on the derivatives of ε_n is found. It turns out that even the bound on f_n goes to zero. Note, in any case, that for the purposes of the coming discussion replacing 0 by a constant would be irrelevant, see [1], (5.12.69)).

We can read (3.45) as saying that for ε_0 small enough the chain of renormalization transformations drives the system "to a trivial fixed point", i.e. to the harmonic oscillator.

To summarize, we have defined a sequence $\bar{\mathcal{E}}_0, \bar{\mathcal{E}}_1, \ldots$ of maps

$$\bar{\mathcal{E}}_j(\underline{A}_{j+1}, \underline{\phi}_{j+1}) \equiv \mathcal{E}_j(\underline{A}'_j + \lambda_j^{-1}\underline{A}_{j+1}, \underline{\phi}_{j+1}) \tag{3.46}$$

which, although not canonical, map solutions of the hamiltonian equations relative to H_{j+1} on $V_\rho \times T^\ell$ into solutions relative to H_j on $V_\rho \times T^\ell$.

By construction the maps $\mathcal{C}_j$ are close to the identity, see (3.28), (3.44), in their holomorphy domains, within

$$\mathcal{O}((C_0\varepsilon_0)^{(3/2)^n})$$

while the holomorphy domain itself shrinks only by $\bar{\lambda}_j^{-1}$ in the action variables (and almost nothing in the angle variables) which by (3.41) behaves $\simeq \eta_j^{-1}$ i.e. goes to zero at a rate which, even if raised to an arbitrary power, is much slower than

$$\mathcal{O}((C_0\varepsilon_0)^{(3/2)^n}) \quad .$$

This immediately implies, by a dimensional argument, that

$$\bar{\mathcal{C}} = \lim_{n\to\infty} \bar{\mathcal{C}}_0 \ldots \bar{\mathcal{C}}_n \tag{3.47}$$

exists and, by Vitali's theorem on converging sequences of holomorphic functions, is holomorphic on $S_{\rho_0} \times C(\xi_\infty)$, $\xi_\infty = \xi_0/2$.

Furthermore, $\bar{\mathcal{C}}(\underline{A}',\underline{\phi}') = (\underline{A},\underline{\phi})$ is close to the identity in the $\underline{\phi}'$-variable. However, on $S_{\rho_0} \times C(\xi_\infty)$

$$\frac{\partial \bar{\mathcal{C}}(\underline{A}',\underline{\phi}')}{\partial \underline{A}'} \equiv \underline{0} \tag{3.48}$$

because by construction this derivative is proportional to

$$\prod_{j=1}^{\infty} \lambda_j^{-1} \equiv 0 \quad .$$

This means that $\bar{\mathcal{C}}$ is $\underline{A}'$-independent.

Finally, $\bar{\mathcal{C}}$ still maps solutions of the hamiltonian equations with hamiltonian $H_\infty(\underline{A}',\underline{\phi}') \equiv \underline{\omega}_0 \cdot \underline{A}'$ on $V_{\rho_0} \times T^\ell$ into solutions relative to the hamiltonian H_0 on $V_{\rho_0} \times T^\ell$.

Since $\bar{\mathcal{C}}$ is independent on $\underline{A}'$ and close to the identity in $\underline{\phi}'$ we may write it as $(\underline{A},\underline{\phi}) = \bar{\mathcal{C}}(\underline{A}',\underline{\phi}')$ with

$$\begin{aligned} \underline{A} &= \underline{\alpha}(\underline{\phi}') \\ \underline{\phi} &= \underline{\phi}' + \underline{\beta}(\underline{\phi}') \end{aligned} \tag{3.49}$$

The functions $\underline{\alpha},\underline{\beta}$ are holomorphic on $C(\xi_0/2)$ and small of $\mathcal{O}(C_0\varepsilon_0)$ and the motion $(\underline{A}',\underline{\phi}') \to (\underline{A}',\underline{\phi}' + \underline{\omega}_0 t)$ becomes in the $(\underline{A},\underline{\phi})$ variables, via (3.49) a solution to the equation of motion with hamiltonian H_0 lying on a torus with parametric equations (3.49) and the proof is complete. For more details on the above simple implications of (3.44) see [1], pp 515 - 517, (where the rescalings of the domains by λ_j are not done but, as it is equivalent , one deals with shrinking domains and essentially constant η_j's).

§4. Comments and Generalizations

A) The preceding discussion is very close to the simplest versions of the renormalization group applications. The annoying feature is that we do not iterate the same map $\mathcal{T}$ but we apply recursively $\mathcal{T}_n \ldots \mathcal{T}_0$ and $\mathcal{T}_i$ depends on i via the a priori given constants δ_i.

This can be manifestly avoided if instead of defining δ_j a priori we define them through h_j, f_j ; a possible definition could be

$$\delta_j = \xi_j/[\log(\log(C_0\varepsilon_j)^{-1})]^2 \tag{4.1}$$

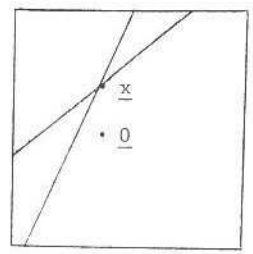
This would lead essentially to the same proof with $\mathcal{T}^{n+1}$ replacing $\mathcal{T}_n\mathcal{T}_{n-1}\ldots\mathcal{T}_0$.

B) It is nice to give a picture of the proof of the preceding theorem.

Define the "surfaces of resonance of order n" by considering the surfaces

$$\frac{\partial h_n}{\partial \underline{A}}(\underline{A}) \cdot \underline{r} = 0 \ , \quad 0 < |\underline{r}| \le N_n \tag{4.2}$$

The position of such surfaces depends on how non-linear h_n is (recall that $h_n(\underline{A}) \to \underline{\omega}_0 \cdot \underline{A}$). They are roughly planes in $\underline{A}$-space going through the point $\underline{x}$:



$$\underline{x} = -M_n^{-1}\underline{\omega}_0 \tag{4.3}$$

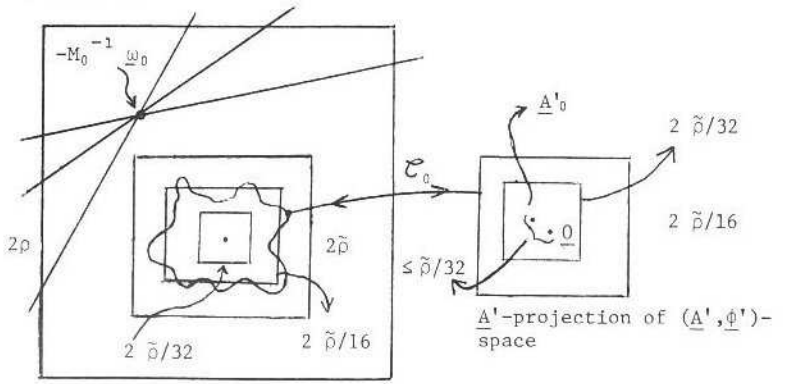
The larger n is the further from the origin is $\underline{x}$ and the less is the number of such surfaces crossing a given neighborhood of the origin. Note, however, that this is so only because $N_n < \infty$ and $N_n \to \infty$ much slower than M_n^{-1} (as $\simeq \varepsilon_n C_0$ compared to $\simeq \log(\varepsilon_n C_0)^{-1}$).

Then we can think of the renormalization transformation σ_0 as follows.

1) Draw the resonance surfaces of order 0 and find a non-resonant vicinity $V_{\tilde\rho}$ of $\underline{0}$.

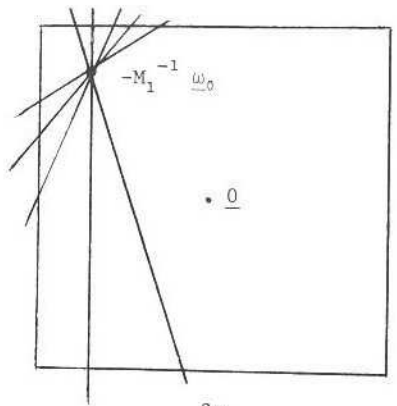
2) Map $V_{\tilde\rho/8} \times T^\ell$ canonically, with a map close to the identity, into $V_{\rho_0} \times T^\ell$ over a set containing $V_{\tilde\rho/16} \times T^\ell$ and locate A'_0 where $\tilde h_1$ has spectrum $\underline{\omega}_0$.

3) Take $\underline{A}'_0$ as origin and rescale the $(\underline{A}'-\underline{A}'_0)$ coordinates so that $V_{\tilde\rho/32} \times T^\ell$ becomes $V_{\rho_0} \times T^\ell$ again and draw the resonances of h_1 of index $\underline{r}$, $0 < |\underline{r}| \le N_1$, i.e. the "resonances or order 1".



$\underline{A}'$-projection of $(\underline{A}',\underline{\phi}')$-space

$\underline{A}$ projection of $(\underline{A},\underline{\phi})$-space



$\lambda_0(\underline{A}'-\underline{A}'_0) \equiv \underline{A}_1$ - projection in $(\lambda_0(\underline{A}'-\underline{A}'_0),\underline{\phi}')$ space

4) Iterate (after a while $-M_n^{-1}\ \underline{\omega}_0$ gets out of the picture but some resonance lines still may cross it, closer and closer to $\underline{0}$ because $\tilde\rho = \rho_0/4\ell C_0 N_n^{\ell+1}\ E_n \underset{n\to\infty}{\longrightarrow} 0$.

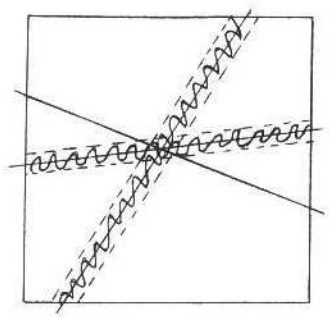
C) Another important comment will be about the approach necessary to prove the full KAM theorem quoted in §2.

In this case we cannot, of course, define a family of renormalization transformations conjugating our system with a given linear oscillator, since every quasiperiodic motion will have its own frequency spectrum.

We shall define our renormalization transformations on triples (h,f,V) where V is an open region. The transformations will still be defined in terms of a sequence of numbers $\delta_n = \xi_0/5\pi(1+n)^2$, $C_n = C_0(1+n)^2$ (this time also C varies with n). We then define $\mathcal{J}_0$ as follows. Draw inside the phase space the resonances of order $\le N_0$ (the picture deals with the special case $h_0(\underline{A}) = \underline{A}^2/2$, i.e. the surfaces $\partial h_0/\partial\underline{A}\ (\underline{A}) \cdot \underline{r} = 0$, for $0 < |\underline{r}| \le N_0$. Around them draw corridors of width $\tilde\rho_0$, so large that, $\forall\ 0 < |\underline{r}| \le N_0$

$$|\underline{\omega}_0(\underline{A}) \cdot \underline{r}|^{-1} < C_0 |\underline{r}|^{\ell+1} \tag{4.4}$$

for $\underline{A}$ outside the corridors.



Calling $V_1 = V_0/\{\text{corridors}\}$ we proceed as before to build (h_1,f_1,V_1). This time, however, we do not rescale and we simply set $h_1 \equiv \tilde h_1, f_1 \equiv \tilde f_1$, and iterate.

At the end one will be left with a set $V_\infty \subset V$ with a dense open complement but with very large measure for C_0 large, because

the widths of the corridors are of $O(|\underline{r}|^{-\ell-1})C_0^{-1}$ (as it can be seen from an argument similar to the one following (3.20), (see for details, [11]) hence summable. At the end we define C_0 in terms of ε_0 so that $C_0\varepsilon_0 \xrightarrow[\varepsilon_0\to 0]{} 0$ and $C_0 \xrightarrow[\varepsilon_0\to 0]{} \infty$ to obtain the final result vol V_∞/vol $V_0 \xrightarrow[\varepsilon_0\to 0]{} 1$.

§5. The Chaotic Transition's Problem and a Related Problem

From the discussion of the preceding lectures it emerges clearly that in some sense each invariant torus has to be regarded as sitting in a neighborhood of phase space where the hamiltonian looks like a perturbation of

$$\bar{h}(\underline{A}) = \underline{\omega}_0 \cdot \underline{A} \tag{5.1}$$

with $\underline{\omega}_0 \in R^\ell$ verifying the non-resonance condition (3.9).

We call $\underline{\omega}_0$ the spectral vector of a given torus and we imagine to fix $\underline{\omega}_0$ and to let the perturbation size ε , in the original hamiltonian (1.8), grow. Very close to the selected invariant torus the system will look , in suitable coordinates, as described by a hamiltonian

$$H_\varepsilon(\underline{A},\underline{\phi}) = \underline{\omega}_0 \cdot \underline{A} + \sigma\tilde{f}(\underline{A},\underline{\phi},\sigma) \tag{5.2}$$

where $\sigma \equiv \sigma(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$ and $\underline{A},\underline{\phi}$ have to be identified as the rescaled coordinates found in the proof in §3,i.e. with $\underline{A}_n,\underline{\phi}_n$ with n quite large.

As ε grows a critical value ε_c may be reached when for $\varepsilon > \varepsilon_c$ there will be no longer an invariant torus, with spectral vector $\underline{\omega}_0$, continuously merging with the family of tori, with given spectral vector, defined for $\varepsilon < \varepsilon_c$.

To study the phenomenon of disappearance of the torus with spectral vector $\underline{\omega}_0$ there are several routes. A possible one is the following. Fixed $\underline{\omega}_0$ consider the interval where the invariant torus exists and is analytic. Let for $\varepsilon \in [0,\varepsilon_c]$

$$\begin{aligned} \underline{A} &= \underline{\alpha}_\varepsilon(\underline{\phi}') \\ \underline{\phi} &= \underline{\phi}' + \underline{\beta}_\varepsilon(\underline{\phi}') \end{aligned} \tag{5.3}$$

be its parametric equations, see (3.11). Of course we have only proved in §3 that (5.3) makes sense for ε small. Here, however, we are proceeding in a heuristic, non-rigorous, way.

To analyze a case slightly simpler, but already very non-trivial, suppose $\ell = 2$. Consider the circle

$$\begin{aligned} \underline{A} &= \underline{\alpha}_\varepsilon(\phi'_1,0) \\ \underline{\phi} &= (\phi'_1,0) + \underline{\beta}_\varepsilon(\phi'_1,0) \end{aligned} \tag{5.4}$$

and the map generated on it by the evolution looked at integer multiples of $2\pi\,\omega_2^{-1}$

$$\begin{aligned} \underline{A} &\to \underline{\alpha}_\varepsilon(\phi'_1 + 2\pi\,\frac{\omega_1}{\omega_2}\,,\ 0) \\ \underline{\phi} &\to (\phi'_1 + 2\pi\,\frac{\omega_1}{\omega_2},\ 0) + \beta_\varepsilon(\phi'_1 + 2\pi\,\frac{\omega_1}{\omega_2},\ 0) \end{aligned} \tag{5.5}$$

We describe this map by introducing the arc length on the circle (5.4), which we call θ and normalize so that the circle's length is 2π , and then rewrite the map (5.5) as the circle's map

$$\theta \to \theta + g_\varepsilon(\theta) \tag{5.6}$$

Of course, by construction, this map is smoothly conjugated to the perfect rotation of the circle, with rotation number $r = \omega_1/\omega_2$, for ε small. As long as the torus exists and is smooth it is clear that the above map of the circle must be smoothly conjugated to a perfect rotation (the conjugation being again given by the map $\theta \leftrightarrow \phi'_1$).

If for some value ε_c such smooth conjugation with the circle's rotation breaks down, it will necessarily mean that the torus (5.3) becomes non-smooth or disappears or more generally the equations (5.3) become singular in $\underline{\phi}'$ and no longer describe a diffeomorphism of T^ℓ into the phase space $V \times T^\ell$.

The smooth conjugability with the circle's rotation can disappear either because g_ε develops a singularity at $\varepsilon = \varepsilon_c$ or because $\theta \to \theta' = g_\varepsilon(\theta) + \theta$ becomes non-invertible as a map of the circle (without any singularity developing in ε), or because of other reasons.

The above discussion shows well the relevance for the theory of the disappearance of the quasi-periodic motions ("chaotic transitions") of the theory of the maps of the circle depending on one parameter ε and having a fixed rotation number.

The study of maps of the form

$$\theta \to \theta + 2\pi\, r(\varepsilon) - \varepsilon \sin\theta \tag{5.7}$$

with $r(\varepsilon)$ so chosen that the above transformation keeps a fixed rotation number $r \equiv r(0)$ ("standard map of the unit circle") has been pursued in considerable detail [12], and it is treated in the parallel lectures. It illustrates what can happen when the conjugacy of the motion with the perfect rotation is broken because $\theta \to \theta + g_\varepsilon(\theta)$ develops at $\varepsilon = \varepsilon_c$ an inflection point becoming non-invertible.

Note also the strong analogy between the problem of finding a function $r(\varepsilon)$ such that (5.7) has a fixed rotation number as ε grows and the problem posed in §1,2 of finding counterterms $\delta_\varepsilon f$ which make (2.1) conjugated canonically with the hamiltonian $\underline{\omega} \cdot \underline{A}$ independently on ε.

This remark leads to the following problem. Imagine that we are given a one-parameter hamiltonian system, with $\ell = 2$, of the form

$$\underline{\omega}_0 \cdot \underline{A} + \varepsilon f(\underline{A},\underline{\phi},\varepsilon) \tag{5.8}$$

which for ε small is canonically conjugated to $\underline{\omega}_0 \cdot \underline{A}$ on a large phase space region $W_\varepsilon \subset V_\rho \times T^\ell$ (i.e. $\partial W \sim \partial(\overline{V}_\rho \times T^\ell)$).

This problem certainly arises in some applications. In §2 we have explained this in connection with the theory of the quasi-periodic Schroedinger equation, when we fix our attention on a Block wave described by $\underline{\omega}_\theta = (\lambda,\underline{\omega})$.

Then it might happen that, as ε grows through ε_c, the integrability is lost in the whole phase space leading to a phenomenon like a phase transition which, by the remarks related to (5.7), could also be relevant to the theory of the chaotic transition. In the next section we discuss in some detail the theory of systems like (5.8) which are canonically conjugated to $\underline{\omega}_0 \cdot \underline{A}$ for all $0 \le \varepsilon < \varepsilon_c$.

In the final section we point out some problems which in our opinion arise naturally if one tries to study what happens at ε_c in the above systems, using the ideas of the renormalization group.

§6. Renormalization and Isochronous Systems

The purpose of this lecture is to define a renormalization transformation acting on the one parameter hamiltonians of the form

$$H_\varepsilon(\underline{A},\underline{\phi}) = \underline{\omega}_0 \cdot \underline{A} + f(\underline{A},\underline{\phi},\varepsilon) \tag{6.1}$$

with f analytic in ε at $\varepsilon = 0$ and divisible by ε, and $\underline{\omega}_0$ verifying (3.9).

The renormalization transformation $\mathcal{T}$ should be designed so that $\forall \varepsilon \in [0,\bar{\varepsilon}]$ it is $\mathcal{T}^n H_\varepsilon \xrightarrow[n\to\infty]{} \underline{\omega}_0 \cdot \underline{A}$ if H_ε is canonically conjugate to $\underline{\omega}_0 \cdot A$ on a set $W_\varepsilon \supset V_\rho \times T^\ell$ for $\varepsilon \in [0,\bar{\varepsilon})$ where $\rho > 0$ is some given radius.

The renormalization transformation is defined as follows

1) Suppose that f is divisible by ε^k, $2^{n-1} \le k < 2^n$, $n = 1,2,\ldots$, and define the "truncation" operation on the formal power series as usual*. Then define

$$f^{[<2^n]} \equiv \sum_{k=1}^{2^n-1} f_k \varepsilon^k$$
$$\bar{f}(\underline{A}) = \frac{1}{(2\pi)^\ell} \int_{T^\ell} f(\underline{A},\underline{\phi})\, d\underline{\phi} \tag{6.2}$$

Consider

$$\tilde{H}_\varepsilon(\underline{A},\underline{\phi}) = \underline{\omega}_0 \cdot \underline{A} + f(\underline{A},\underline{\phi},\varepsilon) - \bar{f}^{[<2^n]}(\underline{A},\varepsilon) \tag{6.3}$$

and let Φ be such that

$$\underline{\omega}_0 \cdot \frac{\partial \Phi}{\partial \phi}(\underline{A}',\underline{\phi}) + f^{[<2^n]}(\underline{A},\underline{\phi},\varepsilon) - \bar{f}^{[<2^n]}(\underline{A},\varepsilon) = 0 \tag{6.4}$$

If f is holomorphic in the region

$$C(\rho,\xi,\theta) \equiv S_\rho \times C(\xi) \times S^1_\theta \tag{6.5}$$

*i.e. if $g = \sum_{k=0}^{\infty} \varepsilon^k g_k$ one defines $g^{[<n]} \equiv \sum_{k=0}^{n-1} \varepsilon^k g_k$, and $g^{[\ge n]} = g - g^{[<n]}$.

in the variables $(\underline{A},\underline{\phi},\varepsilon)$, where $S^1_0 = \{\varepsilon | \varepsilon \in C, |\varepsilon| < \theta\}$, then Φ is holomorphic in $C(\rho,\xi,\theta)$ as it follows immediately from the bound

$$n_\delta(\Phi) \equiv \|\Phi\|_{\rho e^{-\delta},\xi-\delta,e^{-\delta}} \le B_1 \delta^{-2\ell} C_0 \rho \|f\|_{\rho,\xi,\theta} \equiv$$

$$\equiv B_1 \delta^{-2\ell} C_0 \rho\, n_0(f) \tag{6.6}$$

where $\|f\|_{\rho,\xi,\theta} = \sup_{C(\rho,\xi,\theta)} |f(\underline{A},\underline{z},\varepsilon)|$ and $n_\beta(f)$ is defined implicitly in (6.6). The proof of (6.6) is the same as that of (3.22) using the estimates analogous to (3.16). This time ρ does not become as small as $\tilde{\rho}$ because in solving (6.4) no small denominators appear because ω_0 does not depend on $\underline{A}$ and verifies the diophantine condition (3.9) with a non-resonance constant C_0.

As in the proof of §3 it is possible to use Φ as a generating function of two canonical maps $\mathcal{C}, \mathcal{C}'$ analytic in $\underline{A},\underline{\phi},\varepsilon$

$$\begin{aligned} &\mathcal{C},\mathcal{C}' : C(\rho e^{-3\delta},\xi-3\delta,\theta e^{-\delta}) \to C(\rho e^{-2\delta},\xi-2\delta) \\ &\mathcal{C},\mathcal{C}' : C(\rho e^{-4\delta},\xi-4\delta,\theta e^{-\delta}) \to C(\rho e^{-3\delta},\xi-3\delta) \end{aligned} \tag{6.7}$$

such that $\mathcal{C}\,\mathcal{C}' = \mathcal{C}'\,\mathcal{C}$ = identity in $C(\rho e^{-4\delta},\xi-4\delta)$, $\forall \varepsilon$, $|\varepsilon| < \theta e^{-4\delta}$. The maps $\mathcal{C},\mathcal{C}'$ will have the form (3.26) and verify (3.27) with the consequent bounds of the type (3.28). Of course this will be possible under the condition that Φ is small which, as in §3, will take the form, if $\xi < 1$

$$B_2\, n(\Phi) \rho^{-1} \delta^{-2} < 1 \tag{6.8}$$

Therefore, if $\lambda = e^{4\delta}$, we define, in analogy with §3, on $C(\rho,\xi-4\delta,\theta e^{-2\delta})$

$$H^1_\varepsilon(\underline{A}',\underline{\phi}') = \lambda \tilde{H}_\varepsilon(\mathcal{C}(\lambda^{-1}\underline{A}',\underline{\phi}')) \equiv \underline{\omega}_0 \cdot \underline{A}' + \lambda f_1(\lambda^{-1}\underline{A}',\underline{\phi}',\varepsilon) \tag{6.9}$$

which is holomorphic on $C(\rho,\xi-4\delta,\theta e^{-2\delta})$ and f_1 turns out, by construction, to be divisible by ε^{2n}.

Our map $\mathcal{T}_\delta$ will be the map

$$\underline{\omega}_0 \cdot \underline{A} + f(\underline{A},\underline{\phi},\varepsilon) \to \underline{\omega}_0 \cdot \underline{A}' + \lambda f_1(\lambda \underline{A}',\underline{\phi}',\varepsilon) \tag{6.10}$$

with $\lambda = \exp 4\delta$, and its domain of definition will be the set of the f's verifying (6.8).

Let $\delta_k = \xi/5\pi(1+k)^2$, $k = 0,1,\ldots$, then for all ε, $|\varepsilon| < \bar{\theta} < \theta$ for some $\bar{\theta}$ small enough,

$$\mathcal{T}_{\delta_n} \ldots \mathcal{T}_{\delta_0} H \to \underline{\omega}_0 \cdot \underline{A} \tag{6.11}$$

and the maps

$$\bar{\mathcal{C}}_i : (\underline{A}_{i+1},\underline{\phi}_{i+1}) \to (\underline{A}_i,\underline{\phi}_i) = \mathcal{C}_i(\lambda_i^{-1}\underline{A}_{i+1},\underline{\phi}_{i+1}) \tag{6.12}$$

where $\mathcal{C}_i$ is the canonical map generated in the construction of $\mathcal{T}_{\delta_i}$, can be decomposed to yield a map

$$\bar{\mathcal{C}} = \lim_n \bar{\mathcal{C}}_n \ldots \bar{\mathcal{C}}_0 \tag{6.13}$$

holomorphic in $C(\rho,\xi/2,\bar{\theta})$.

The map $\bar{\mathcal{C}}$ will not be canonical but, by construction, has a jacobian determinant equal to $(\prod_{i=0}^{\infty} \lambda_i^{-1})^\ell$; also by the construction it maps solutions of the hamiltonian equations on $V_\rho \times T^\ell$ with the hamiltonian $\underline{\omega}_0 \cdot \underline{A}$ into solutions of the hamiltonian equations in $\bar{\mathcal{C}}(V_\rho \times T^\ell) \subset V_\rho \times T^\ell$, with hamiltonian

$$\underline{\omega}_0 \cdot \underline{A} + f(\underline{A},\underline{\phi},\varepsilon) - \sum_{k=0}^{\infty} \lambda_k \bar{f}_k(\lambda_k^{-1}(\lambda_k^{-1}\underline{A}_k,\varepsilon)^{[<2^{k+1}]} \tag{6.14}$$

where $(\underline{A}_k,\underline{\phi}_k) = \bar{\mathcal{C}}_{k-1}(\underline{A}_{k-1},\underline{\phi}_{k-1})$, $\underline{A} \equiv \underline{A}_0$, $\underline{\phi} \equiv \underline{\phi}_0$, $\lambda_k = \exp -4\delta_k$ and $\bar{f}_1,\bar{f}_2,\ldots$ are constructed recursively in building the maps $\mathcal{T}_{\delta_0},\mathcal{T}_{\delta_1},\ldots$ as the "counterterms" necessary to define the solution Φ of the equation corresponding to (6.4).

In this case the main difference with respect to the case treated in §3 is that $\prod_{k=0}^{\infty} \lambda_k^{-1} > 0$. This means that there is very little loss of analyticity in defining Φ. The phenomenon is due to the systematic subtraction procedure used at each step, which eliminates the resonance problems.

The fact that $\prod_{k=0}^{\infty} \lambda_k < +\infty$ also implies that the rescalings performed at each step could be avoided. In fact the maps $\mathcal{C}_i$ can be composed on the set $V_{\rho e^{-\Sigma\delta_i}} \times T^\ell$, for $|\varepsilon| < \bar{\theta}$,

and yield a canonical map

$$\mathcal{C}_\infty = \lim_n \mathcal{C}_n \dots \mathcal{C}_0 \tag{6.15}$$

mapping solutions of the hamiltonian equations with hamiltonian $\underline{\omega}_0 \cdot \underline{A}$ into solutions relative to

$$\underline{\omega}_0 \cdot \underline{A} + f(\underline{A},\underline{\phi},\varepsilon) - \sum_{k=0}^{\infty} \overline{f}_k(\underline{A}_k,\varepsilon)^{[<2^{k+1}]} \tag{6.16}$$

if we define recursively $(\underline{A}_k,\underline{\phi}_k)$ such that $(\underline{A}_0,\underline{\phi}_0) \equiv (\underline{A},\underline{\phi})$, $(\underline{A}_k,\underline{\phi}_k) = \mathcal{C}_k(\underline{A}_{k+1},\underline{\phi}_{k+1})$.

Simple arguments on formal power series show that a necessary and sufficient condition for having $\overline{f}_k \equiv 0$, hence for integrability by quadratures of $\omega_0 \cdot A + f(\underline{A},\underline{\phi},\varepsilon)$ and its canonical conjugability to $\underline{\omega}_0 \cdot \underline{A}$ is that the equation

$$\underline{\omega}_0 \cdot (\underline{A}' + \frac{\partial\Phi}{\partial\underline{\phi}}(\underline{A}',\underline{\phi})) + f(\underline{A}' + \frac{\partial\Phi}{\partial\underline{\phi}}(\underline{A}',\underline{\phi}),\underline{\phi},\varepsilon) = \underline{\omega}_0 \cdot \underline{A}' \tag{6.17}$$

with

$$f = \sum_{k=1}^{\infty} \varepsilon^k f^{(k)}(\underline{A},\underline{\phi}) \; ; \; \Phi(\underline{A},\underline{\phi}) = \sum_{k=1}^{\infty} \varepsilon^k \Phi^{(k)}(\underline{A},\underline{\phi}) \; , \quad \overline{\Phi}^{(k)}(\underline{A}) \equiv 0 \tag{6.18}$$

admits a solution in the sense of the formal power series. It is easy to check that this implies infinitely many conditions on f e.g. for k = 1 it implies

$$\overline{f}^{(1)}(\underline{A}) \equiv 0 \tag{6.19}$$

For k = 2 it implies that the equation for $\Phi^{(2)}$

$$\underline{\omega}_0 \cdot \frac{\partial\Phi^{(2)}}{\partial\underline{\phi}}(\underline{A}',\underline{\phi}) + [\frac{\partial f^{(1)}}{\partial\underline{A}'}(\underline{A}',\underline{\phi}) \cdot \frac{\partial\Phi^{(1)}}{\partial\underline{\phi}}(\underline{A}',\underline{\phi}) + f^{(2)}(\underline{A}',\underline{\phi})] = 0 \tag{6.20}$$

should have a solution $\Phi^{(2)}$, if

$$\Phi^{(1)}(\underline{A}',\underline{\phi}) = \sum_{\underline{r}\neq 0} \hat{f}_{\underline{r}}^{(1)}(\underline{A}')\, e^{i\underline{r}\cdot\underline{\phi}}(-i\underline{\omega}_0 \cdot \underline{r})^{-1} \tag{6.21}$$

and this means that the average over $\underline{\phi}$ of the function in square brackets in (6.20) vanishes

$$\sum_{\underline{r}\neq 0} (\frac{\partial \hat{f}_{\underline{r}}^{(1)}}{\partial\underline{A}'}(\underline{A}') \cdot \frac{i\underline{r}}{-i\underline{\omega}_0\cdot\underline{r}}) \hat{f}_{-\underline{r}}^{(1)}(\underline{A}') = 0 \tag{6.22}$$

etc.

So we arrive at the following theorem.

Theorem: Consider the hamiltonian system with hamiltonian

$$H_\varepsilon = \underline{\omega}_0 \cdot \underline{A} + f(\underline{A},\underline{\phi},\varepsilon) \tag{6.23}$$

and analytic on $V_\rho \times T^\ell$ and in ε near zero. Suppose that the (6.17) is formally soluble in the sense of the power series. Then it is in fact soluble for ε small enough with a Φ given by a convergent power series in ε .

The above theorem is due to Russmann [13], it was found also in the context of renormalization theory [10]. Its formulation in terms of our analysis is more natural as follows.

Theorem: Under the assumptions above, suppose that the hamiltonian system (6.23) can be conjugated canonically to $\underline{\omega}_0 \cdot \underline{A}'$ on large sets of phase space for ε small* . Then the maps $\mathcal{J}_{\delta_i}$ are such that

$$\mathcal{J}_{\delta_n} \dots \mathcal{J}_{\delta_0} H_\varepsilon \xrightarrow[n\to\infty]{} \underline{\omega}_0 \cdot \underline{A}' \tag{6.24}$$

It would of course be nicer to deal with a single transformation rather than with a sequence of them. This can be easily done by defining a map $\mathcal{J}$ in the space of the analytic one-parameter families of hamiltonians of the form (6.23) as $\mathcal{J}_\delta$ with δ directly defined in terms of f . For instance we could take

$$\delta_f = \xi/(\log[-\log C_0 \|f\|_{\rho,\xi,\theta}\, \rho^{-2} + e^2])^2 \tag{6.25}$$

Let us now derive the following corollary of the above ideas (it is basically equivalent to a theorem of Dinaburg-Sinai on the quasi-periodic Schroedinger equation [8],[9]).

*i.e. given $\bar{\rho} < \rho$, $\bar{\xi} < \xi$, $\exists\vartheta > 0$ such that $\underline{\omega}_0 \cdot \underline{A}'$ on $V_{\bar{\rho}} \times T^\ell$ is canonically conjugated to (6.23) by a canonical map $\mathcal{C}$ holomorphic on $C(\bar{\rho},\bar{\xi},\bar{\theta})$, close to the identity within $\theta(\varepsilon)$.

Corollary: Let $\underline{f}$ be holomorphic in $C(\xi_0)$ with values in C^ℓ, real on T^ℓ. Consider

$$H_0(\underline{A},\underline{\phi},\varepsilon) = \underline{\omega}_0 \cdot \underline{A} + \varepsilon \underline{A} \cdot \underline{f}(\underline{\phi}) \tag{6.26}$$

There exist, for ε small enough, a "counterterm" of the form $\underline{A} \cdot \underline{a}(\varepsilon)$ such that

$$\underline{\omega}_0 \cdot \underline{A} + \varepsilon \underline{A} \cdot \underline{f}(\underline{\phi}) - \underline{A} \cdot \underline{a}(\varepsilon) \tag{6.27}$$

is canonically conjugate to $\underline{\omega}_0 \cdot A$ on the whole space $R^\ell \times T^\ell$.

Proof: We apply a recursive algorithm. Imagine labeling its steps by an index $n = 1,2,\ldots$, so that after the n-th step we shall have built a canonical transformation $\mathcal{C}^{(n-1)}$ changing

$$\underline{\omega}_0 \cdot \underline{A} + \varepsilon \underline{A} \cdot \underline{f}(\underline{\phi}) - \underline{A} \cdot \sum_{k=1}^{2^n-1} \varepsilon^k \underline{a}_k \tag{6.28}$$

into

$$H_n(\underline{A}_n,\underline{\phi}_n) = \underline{\omega}_0 \cdot \underline{A}_n + \underline{A}_n \cdot f_n(\underline{\phi}_n,\varepsilon) \tag{6.29}$$

where $(\underline{A},\underline{\phi}) \equiv (\underline{A}_0,\underline{\phi}_0) = \mathcal{C}^{(n-1)}(\underline{A}_n,\underline{\phi}_n)$ and f_n is divisible by ε^{2^n}, holomorphic in $\underline{\phi}_n,\varepsilon$ on $C(\xi_n, \theta_n) \equiv C(\xi_n) \times S^1_{\theta_n}$ and $\underline{a}_1,\ldots,\underline{a}_{2^n-1}$ are also defined inductively together with ξ_n, θ_n; furthermore $\mathcal{C}^{(n-2)^{-1}}\mathcal{C}^{(n-1)} \equiv \{\text{variation of } \mathcal{C}^{(n-1)} \text{ with respect to } \mathcal{C}^{(n-2)}\} \equiv \mathcal{C}_{n-1}$ is holomorphic on $C(\xi_n, \theta_n)$ and maps $C^\ell \times C(\xi_n)$ into $C^\ell \times C(\xi_{n-1})$ as

$$\begin{aligned} \underline{A}_{n-1} &= (1 + J_{n-1}(\underline{\phi}_{n-1}))\underline{A}_n \\ \underline{\phi}_n &= \underline{\phi}_{n-1} + \underline{\phi}_{n-1}(\underline{\phi}_{n-1}) \end{aligned} \tag{6.30}$$

where $\underline{\phi}_{n-1}(\underline{\phi}_{n-1}) \cdot \underline{A}_n \equiv \phi(\underline{A}_n,\underline{\phi}_{n-1})$ is the generating function of $\mathcal{C}_{n-1}$, and $J_{n-1}(\underline{\phi}) \equiv (\partial\underline{\phi}_{n-1}/\partial\underline{\phi})^T$.

Given $\underline{f}_n$ the function $\underline{\phi}_n$ and the $\underline{a}_j$, $2^n \le j < 2^{n+1} - 1$, are determined as follows.

1) Define

$$\begin{aligned} \bar{\underline{a}}_n(\varepsilon) &\equiv \bar{\underline{a}}_{2^n} \varepsilon^{2^n} + \ldots + \bar{\underline{a}}_{2^{n+1}-1} \varepsilon^{2^{n+1}-1} = \\ &= \int \frac{d\underline{\phi}}{(2\pi)^\ell} [\underline{f}_n(\underline{A}_n,\underline{\phi}_n,\varepsilon)]^{[<2^{n+1}-1]} \end{aligned} \tag{6.31}$$

where $[<p]$ is the series trunctation operation considered above, (6.2).

2) Next consider the matrices $J_0(\underline{\phi}_0),\ldots, J_{n-1}(\underline{\phi}_{n-1})$ and note that

$$\underline{A} = (1+J_0(\underline{\phi}_0))^{-1}\ldots(1+J_{n-1}(\underline{\phi}_{n-1}))^{-1}\underline{A}_n \equiv M_n(\underline{\phi}_n)\underline{A}_n \tag{6.32}$$

so that if $\bar{M}_n(\varepsilon)$ denotes the average of $M_n(\underline{\phi})$ over $\underline{\phi}$ and if

$$[(\bar{M}_n(\varepsilon))^{-1}\bar{\underline{a}}_n(\varepsilon)]^{[<2^{n+1}-1]} = \underline{a}_n(\varepsilon) \tag{6.33}$$

defines $\underline{a}_n(\varepsilon) = \underline{a}_{2^n}\varepsilon^{2^n} + \ldots + \underline{a}_{2^{n+1}}\varepsilon^{2^{n+1}-1}$, we see that the function

$$\begin{aligned} \underline{A}_n \cdot \tilde{\underline{f}}_n(\underline{\phi}_n) &= \underline{A}_n \cdot \underline{f}_n(\underline{\phi}_n,\varepsilon) - \underline{A}_0 \cdot \underline{a}_n(\varepsilon) \equiv \\ &\equiv \underline{A}_n \cdot \underline{f}_n(\underline{\phi}_n,\varepsilon) - \underline{a}_n(\varepsilon) \cdot M_n(\underline{\phi}_n)\underline{A}_n \end{aligned} \tag{6.34}$$

has zero average over $\underline{\phi}_n$ to order $\varepsilon^{2^{n+1}-1}$.

3) Hence we can define

$$\underline{\phi}_n(\underline{\phi}_n) = \left[\sum_{r\neq 0} \tilde{f}_{nr} \frac{e^{i\underline{r}\cdot\underline{\phi}_n}}{-i\underline{\omega}_0\cdot\underline{r}}\right]^{[<2^{n+1}-1]} \tag{6.35}$$

and iterate the procedure.

To discuss the bounds necessary in order that the above construction is really possible let

$$\varepsilon_n = \|\underline{f}\|_{\xi_n,\theta_n}, \quad \eta_n = \|J_{n-1}\|_{\xi_n,\mathcal{D}_n} \tag{6.36}$$

From (6.31) we find, choosing $\delta_k = \xi_0/5\pi(1+k)^2$, $\rho \geq 0$

$$\|\bar{a}_{n+1}\|_{\theta_n e^{-\delta_n}} \leq B_1 \varepsilon_n \delta_n^{-1} \tag{6.37}$$

using bounds similar to (3.16), and from (6.32)

$$\|M_n\|_{\varepsilon_n-\delta_n,\ \theta_n e^{-\delta_n}} \leq \prod_{i=1}^{n-1} (1+B_2\eta_i) \tag{6.38}$$

having also used in (6.37), (6.38) dimensional bounds. So that by (6.33)

$$\|\underline{a}_{n+1}\|_{\hat{\theta} e^{-\delta_n}} \leq B_3 \delta_n^{-3} \varepsilon_n \prod_{j=1}^{n} (1+B_2\eta_j) \tag{6.39}$$

and by (6.34), (6.35)

$$\|\tilde{\underline{f}}_n\|_{\xi_n-\delta_n,\ \theta_n e^{-\delta_n}} \leq \varepsilon_n + B_2 \delta_n^{-3} \varepsilon_n \prod_{j=1}^{n} (1+B_2\eta_j)^2 \tag{6.40}$$

$$\|\Phi_n\|_{\xi_n-2\delta_n,\ \theta_n e^{-\delta_n}} \leq B_4 C_0 \varepsilon_n \delta_n^{-3} \prod_{j=1}^{n} (1+B_2\eta_j)^2\ \delta_n^{-2\ell-1} \tag{6.41}$$

Therefore, if $\|\Phi\|_n$ is small enough, by the implicit functions' theorems applied several times, the function $\underline{A}'_{n+1} \cdot \underline{\Phi}_n(\underline{\phi}_n)$ generates canonical maps $\mathcal{C}_n'$, $\mathcal{C}_n$ such that

$$\begin{aligned} \mathcal{C}_n', \mathcal{C}_n : C(\xi_n-3\delta_n,\ \theta_n e^{-\delta_n}) &\to C(\xi_n-2\delta_n) \\ C(\xi_n-4\delta_n,\ \theta_n e^{-\delta_n}) &\to C(\xi_n-3\delta_n) \end{aligned} \tag{6.42}$$

of the form (6.30). The smallness condition is that $\partial\Phi_n/\partial\underline{\phi}$ be small compared to δ_n. One easily finds that the precise condition has the form.

$$B_5 C_0 \varepsilon_n \delta_n^{-4+2\ell} \prod_{j=1}^{n} (1+B_2\eta_j)^2 < 1 \tag{6.43}$$

for a suitable B_5 .

Then from (6.41) it follows

$$\eta_{n+1} \equiv \|J_n\|_{\xi_n-4\delta_n,\ \theta_n e^{-\delta_n}} \leq B_6 C_0 \varepsilon_n \delta_n^{-4-2\ell} \prod_{j=1}^{n} (1+B_2\eta_j)^2 \tag{6.44}$$

Finally, with some patience and using the same methods quoted in §3 to estimate ε_{n+1} in terms of ε_n we find that, if $\xi_{n+1} = \xi_n - 5\delta_n$, $\theta_{n+1} = \theta_n e^{-2\delta_n}$

$$\|\underline{f}_{n+1}\|_{\xi_{n+1},\ \theta_{n+1}} \equiv \varepsilon_{n+1} \leq B_7(C_0\varepsilon_n^2 + \varepsilon_n e^{-2^{n+1}\delta_n})\delta_n^{-b} \tag{6.45}$$

where B_7, $b > 0$ are suitably chosen, provided (6.43) holds.

To summarize, if (6.43) holds

$$\begin{aligned} &\xi_{n+1} = \xi_n - 5\delta_n \qquad \theta_{n+1} = \theta_n e^{-2\delta_n} \\ &\varepsilon_{n+1} \leq B_7(C_0\varepsilon_n^2 + \varepsilon_n e^{-2^{n+1}\delta_n})\delta_n^{-b_1} \\ &\eta_{n+1} \leq C_0\varepsilon_n\delta_n^{-b_2} \prod_{j=1}^{n} (1+B_2\eta_j)^2 \end{aligned} \tag{6.46}$$

where the B's and the b's are suitable constants (depending only on ℓ).

Proceeding as in §3 one easily deduces from (6.46) that if ε_0 is small enough then η_n , $\varepsilon_n \to 0$ very fast. In turn, similarly in §3, this easily implies the existence of the limit

$$\mathcal{C} = \lim_{n\to\infty} \mathcal{C}^{(n)} \equiv \lim_{n\to\infty} \mathcal{C}_n \ldots \mathcal{C}_0 \tag{6.47}$$

and its holomorphy on $C^\ell \times C(\xi_0/2) \times C(\theta_\infty)$ where $\theta_\infty = \theta_0 \exp -2 \sum_{n=0}^{\infty} \delta_j$. The (6.39) and the rapidity at which δ_n , $\eta_n \to 0$ as $n \to \infty$ also imply that the series for $\mathcal{C}(\varepsilon)$ converges for $|\varepsilon| < \theta_\infty$ and, by construction,

$$\underline{\omega}_0 \cdot \underline{A} + \varepsilon \underline{A} \cdot \underline{f}(\underline{\phi}) - \underline{A} \cdot \underline{a}(\varepsilon) \tag{6.48}$$

is conjugated by $\mathcal{C}$ to $\underline{\omega}_0 \cdot \underline{A}$.

With some more work one could prove that $a(\varepsilon)$ can be defined over a large set of $\underline{\omega}_0$'s and "depends smoothly on them". The model for such a proof is provided by [14], [12], [5], [11]; precisely.

Theorem: There is a C^∞-function $\underline{\omega}_0 \to \underline{a}(\underline{\omega}_0, \varepsilon)$ such that when $\underline{\omega}_0$ varies in the set of the $\underline{\omega}_0 \in R^\ell$ verifying, for $\underline{r} \in Z^\ell$

$$|\underline{\omega}_0 \underline{r}|^{-1} < C_0 |\underline{r}|^\ell \qquad \forall\, 0 < |\underline{r}| \tag{6.49}$$

the hamiltonian

$$\underline{\omega}_0 \cdot \underline{A} + \varepsilon A \cdot \underline{f}(\underline{\phi}) - \underline{A} \cdot \underline{a}(\varepsilon, \underline{\omega}_0) \tag{6.50}$$

is analytically canonically conjugated with $\underline{\omega}_0 \cdot \underline{A}$ on the whole phase space.

See also [15].

A corollary to the last two theorems are the results on the quasi-periodic Schroedinger equation and its Bloch waves, discussed in §2.

§7 Remarks and Problems on the Chaotic Transition in Isochronous Systems

We consider a one-parameter family of systems like (6.1)

$$H_\varepsilon = \underline{\omega}_0 \cdot \underline{A} + f(\underline{A}, \underline{\phi}, \varepsilon) \qquad \text{on} \quad V_\rho \times T^\ell \tag{7.1}$$

which is supposed to be analytically conjugated to the system $\underline{\omega}_0 \cdot \underline{A}$ for ε small enough, in the sense of the footnote to p. 78. The vector $\underline{\omega}_0$ verifies (3.9).

Then the analysis of §6 shows that if we act on H_ε with the transformations $\mathcal{T}$ defined by (6.10), (6.25) and if ε is small enough

$$\mathcal{T}^n H_\varepsilon \to \underline{\omega}_0 \cdot \underline{A} \qquad \text{on} \quad V_{\bar\rho} \times T^\ell \tag{7.2}$$

for $\bar\rho < \rho$.

We now investigate the possibility that for some value of $\varepsilon = \varepsilon_c$ the (7.2) might fail but still $\mathcal{T}^n H_{\varepsilon_c} \to \underline{\omega}_0 \cdot \underline{A} + f^*(\underline{A}, \underline{\phi})$ for some f^*. It is clear that, if this happens, f^* must verify some equation of "fixed point type". The analysis that follows is a heuristic analysis aiming at finding equations for f^*.

To understand which could be this equation, recall the definition of $\mathcal{T}$ on $\mathcal{T}^n H_\varepsilon$.

1) Decompose $f_n(\underline{A}, \underline{\phi}, \varepsilon)$

$$f_n(\underline{A}, \underline{\phi}, \varepsilon) = f_n^-(\underline{A}, \underline{\phi}, \varepsilon) + f_n^+(\underline{A}, \underline{\phi}, \varepsilon) \tag{7.3}$$

where $f_n^-(\underline{A}, \underline{\phi}, \varepsilon) = f_n(\underline{A}, \underline{\phi}, \varepsilon)^{[<2^{n+1}]}$, where $[<p]$ denotes as in §6 the power series truncation operation (with respect to ε).

2) By the integrability hypothesis, see §6, $\bar f^- \equiv 0$ and we do not have to perform any subtraction. Define

$$\phi^-(\underline{A}', \underline{\phi}, \varepsilon) = \sum_{\underline{r} \neq \underline{0}} \frac{\hat f_{\underline{r}}^-(\underline{A}) e^{i\underline{r}\cdot\underline{\phi}}}{-i\underline{\omega}_0 \cdot \underline{r}} \tag{7.4}$$

and generate with it a canonical transformation on a suitable $V_{\tilde\rho} \times T^\ell$, via

$$\begin{aligned} \underline{A} &= \underline{A}' + \frac{\partial \phi^-}{\partial \underline{\phi}}(\underline{A}', \underline{\phi}, \varepsilon) \\ \underline{\phi}' &= \underline{\phi} + \frac{\partial \phi^-}{\partial \underline{A}'}(\underline{A}', \underline{\phi}, \varepsilon) \end{aligned} \tag{7.5}$$

3) Then define $\lambda = \rho/\tilde{\rho}$ and

$$\begin{aligned}\mathcal{T}^{n+1}H_\varepsilon &= \lambda\underline{\omega}_0 \cdot (\lambda^{-1}\underline{A}' + \frac{\partial\Phi^-}{\partial\underline{\phi}}(\lambda^{-1}\underline{A}',\underline{\phi},\varepsilon)) + \\ &+ \lambda f^-(\lambda^{-1}\underline{A}' + \frac{\partial\Phi^-}{\partial\underline{\phi}}(\lambda^{-1}\underline{A}',\underline{\phi},\varepsilon),\underline{\phi}) + \\ &+ \lambda f^+(\lambda^{-1}\underline{A}' + \frac{\partial\Phi^-}{\partial\underline{\phi}}(\lambda^{-1}\underline{A}',\underline{\phi},\varepsilon),\underline{\phi},\varepsilon) = \\ &= \underline{\omega}_0 \cdot \underline{A}' + f_{n+1}(\underline{A}',\underline{\phi}',\varepsilon)\end{aligned} \tag{7.6}$$

So we see that the $\mathcal{T}^n H_\varepsilon$ will converge to a limit as $n \to \infty$ for $\varepsilon = \varepsilon_c$ if, for $\varepsilon = \varepsilon_c$, $f_n(\underline{A},\underline{\phi},\varepsilon) \to f^*(\underline{A},\underline{\phi})$. This can be imposed by requiring the (stronger) condition that $f^+ \to f^*_+$, $f^- \to f^*_-$. In this case $f^*_\pm$ have to be related by

$$\begin{aligned}&\lambda f^*_-(\lambda^{-1}\underline{A}' + \frac{\partial\Phi^-}{\partial\underline{\phi}}(\lambda^{-1}\underline{A}',\underline{\phi}),\underline{\phi}) + \lambda f^*_+(\lambda^{-1}\underline{A}' + \frac{\partial\Phi^-}{\partial\underline{\phi}}(\lambda^{-1}\underline{A}',\underline{\phi}),\underline{\phi}) \\ &- \lambda f^*_-(A',\phi,\varepsilon) = f^*_-(\underline{A}',\underline{\phi} + \frac{\partial\Phi^-}{\partial\underline{A}'}(\lambda^{-1}\underline{A}',\phi)) + f^*_+(\underline{A}',\underline{\phi} + \frac{\partial\Phi^-}{\partial\underline{A}'}(\lambda^{-1}\underline{A}',\underline{\phi}'))\end{aligned}$$

$$\Phi^-(\underline{A}',\underline{\phi}) = \sum_{\underline{r}\neq\underline{0}} (\hat{f}^*_-(\underline{A}'))_{\underline{r}}\, e^{i\underline{r}\cdot\underline{\phi}}(-i\underline{\omega}_0\cdot\underline{r})^{-1} \quad \text{and}$$

$$\overline{f^*_-} \equiv 0 \tag{7.7}$$

for some $\lambda > 1$.

The above equation looks quite difficult to study in general. From the discussions of §6 on renormalization and Wick ordering one gets, however, the idea of inquiring into the existence of special solutions of the form

$$\begin{aligned}f^*_-(\underline{A}',\underline{\phi}) &= \underline{A}'\cdot\underline{f}^*(\underline{\phi}) \\ f^*_+(\underline{A}',\underline{\phi}) &= \underline{A}'\cdot\underline{a}^*\end{aligned} \tag{7.8}$$

Then, if $\underline{\Phi}^*$ is so defined that

$$\underline{\omega}_0 \cdot \frac{\partial(\underline{A}'\cdot\underline{\Phi}^*(\underline{\phi}))}{\partial\underline{\phi}} + \underline{A}'\cdot\underline{f}^*(\underline{\phi}) = 0 \tag{7.9}$$

the above fixed point equation becomes

$$\begin{aligned}(\underline{\omega}_0+\underline{a}^*)\cdot\frac{\partial(\underline{A}'\cdot\underline{\Phi}^*)}{\partial\underline{\phi}}(\underline{\phi}) + (\underline{A}' + \frac{\partial(\underline{A}'\cdot\underline{\Phi}^*)}{\partial\underline{\phi}}(\underline{\phi}))\cdot\underline{f}^*(\underline{\phi}) = \\ = \underline{A}'\cdot\underline{f}^*(\underline{\phi}+\underline{\Phi}^*(\underline{\phi}))\end{aligned} \tag{7.10}$$

This equation looks better but it is still unclear whether it does admit non-trivial solutions. If one looks for solutions "with separated variables" i.e. of the form

$$\Phi^*(\underline{\phi}) = (\Phi^*_1(\phi_1),\ldots,\Phi^*_\ell(\phi_\ell)) \tag{7.11}$$

One then finds for each component an equation like

$$\begin{aligned}&(\omega + a^*)\frac{\partial\Phi^*}{\partial\phi}(\phi) + (1+\frac{\partial\Phi^*}{\partial\phi}(\phi))\, f^*(\phi) = f^*(\phi+\Phi^*(\phi)) \\ &\overline{f^*} = 0\,, \qquad \omega_0\frac{\partial\Phi^*}{\partial\phi}(\phi) + f^*(\phi) = 0\end{aligned} \tag{7.12}$$

which, if we set $g^* = f^*/\omega_0$, $\alpha^* = \alpha^*/\omega_0$, becomes[†]

$$\begin{aligned}&\alpha^*\, g^*(\phi) + g^*(\phi)^2 + g^*(\phi+\int_0^\phi g^*(\phi')d\phi') = 0 \\ &\overline{g^*} = 0\end{aligned} \tag{7.13}$$

It is unclear whether this equation in (α^*,g^*) has a non-trivial solution. This seems quite probable. Even less clear is the existence of a non-trivial solution to (7.10) with non-separated variables (note that (7.10) is a system of ℓ equations because of the arbitrariness of $\underline{A}'$)

The latter question should be relevant for the analysis of the chaotic transition in systems with fixed spectrum. In fact if (7.10) admitted non-trivial solutions, they could provide a key to the theory of the disappearance of the quasi-periodic motions which for $\varepsilon < \varepsilon_c$, fill the phase space, in a way similar to that proposed in [12], [16].

I hope that this matter will be investigated further, possibly trying to find a solution to (7.10) by the numerical methods [17], [18], [19], [16].

[†] Assuming that $\Phi^*(0) = 0$, which is not restrictive.

Appendix A. Arnold-Liouville Theorem

The fact that $I_1,\dots,I_\ell$ have vanishing Poisson brackets has two important consequences.

i) On the surface $\underline{I} = \underline{i}$ one can define ℓ independent vector fields which act on functions via the Poisson bracket with the $I_1,\dots,I_\ell$

$$L_j f = \{I_j, f\} \tag{A.1}$$

The assumption insures that such fields commute. Since they are smooth on the smooth compact surfact $T(\underline{I})$ this implies that $T(\underline{I})$ is a torus.

ii) Consider the relations $I_j(\underline{p},\underline{q}) = I_j$ and imagine that they can be inverted as $\underline{p} = \underline{\alpha}(\underline{I},\underline{q})$. Then $\underline{p}\cdot d\underline{q} \equiv \underline{\alpha}(\underline{I},\underline{q})\cdot d\underline{q}$ is a differential form on $T(\underline{I})$ at fixed $\underline{I}$. Its local integrability conditions coincide, as a simple calculation shows ([1], p. 360), with the condition that $\{I_i, I_j\} = 0$.

So $T(\underline{I})$ are "lagrangian manifolds", i.e. manifolds on which $\underline{p}\cdot d\underline{q}$ is a locally integrable form.

Let $\underline{p} = \underline{\alpha}(\underline{\theta})$, $\underline{q} = \underline{\beta}(\underline{\theta})$ be a parametric representation of the torus $T(\underline{I})$. We consider the ℓ closed curves on $T(\underline{I})$

$$\lambda_j = \{\underline{p},\underline{q} \mid \underline{\theta} = (\theta_1,\dots,\theta_\ell),\ \theta_k = 0 \ \text{ for } \ k \neq j\} \tag{A.2}$$

Then let

$$A_j = \frac{1}{2\pi} \oint_{\lambda_i} \underline{p}\cdot d\underline{q} \tag{A.3}$$

and suppose that the $\underline{I}$'s can be expressed in terms of the $\underline{A}$'s.

Define the generating function

$$\Phi(\underline{A},\underline{q}) = \int_0^{\underline{q}} \underline{p}\cdot d\underline{q} \tag{A.4}$$

where the integral is along a line on $T(\underline{I})$ from the $(\underline{p},\underline{q})$ corresponding to $\underline{\theta} = \underline{0}$ (assuming that such a point is chosen to be smoothly dependent on $\underline{I}$) and built by segments along the axes. Then let

$$\begin{aligned} \underline{\phi} &= \frac{\partial \Phi}{\partial \underline{A}}\,(\underline{A},\underline{q}) \\ \underline{p} &= \frac{\partial \Phi}{\partial \underline{q}}\,(\underline{A},\underline{q}) \end{aligned} \tag{A.5}$$

and (A.4) shows that $\underline{p}$ has the right interpretation.

Suppose that (A.5) can be invented smoothly into a map $(\underline{p},\underline{q}) \leftrightarrow (\underline{A},\underline{\phi})$. It is clear that we have to check, in order to insure this smooth invertability that the $\underline{\phi}$'s admit the interpretation of angles. A necessary condition is that if $\underline{q}$ is thought once as the point $\underline{\theta} = \underline{0}$ and once as the point $\underline{\theta}_{2\pi} = (2\pi, 0, \dots, 0)$, obtained by moving on the circle λ_1, the variable ϕ_1 varies by 2π. This in fact follows immediately because in the second case

$$\Phi(\underline{A},\underline{q}) = \oint_{\lambda_1} \underline{p}\cdot d\underline{q} = 2\pi A_1$$

So if the map (A.5) can define canonical transformation it will map $T(\underline{I})$ onto $V \times T^\ell$ where V is the domain of variability of the $\underline{A}$-functions. Clearly $I_1 \equiv H$ will have the form $H = h(\underline{A})$ and the system will have been integrated by quadratures (actually the quadratures are the ℓ one-dimensional integrals (A.4)).

As an application of the above method of construction of the action angle variables we shall show in Appendx B that any one-dimensional periodic Schroedinger equation gives rise to a hamiltonian system (as in §2) integrable by quadratures.

The above procedure does not necessarily work in the sense that not all the assumptions made in the derivation need to be satisfied even when action-angle variables do exist (and, therefore, have to be built in a different way). The simplest example of such a situation is provided by a one-dimensional system with a hamiltonian $H(p,q)$ such that the curves $H(p,q) = E$ contain a horizontal segment, $\forall\, E > 0$, (Exercise).

Appendix B. Integration by quadratures of the hamiltonian system associated with the periodic Schroedinger equation

We study here the hamiltonian (2.5) which is "equilvalent" to

$$-\ddot{q} = (E - \varepsilon V(\phi_0 + \omega t))q \tag{B.1}$$

in the sense that the map $(p,q,B,\phi) \leftrightarrow (A,\ B,\phi)$

$$\begin{aligned} A &= (p^2 + Eq^2)/2\sqrt{E} \\ \cos\theta &= p/\sqrt{2\sqrt{E}\,A}\ , \quad \sin\theta = q\,\sqrt{E}/\sqrt{2\sqrt{E}\,A} \end{aligned} \tag{B.2}$$

maps solutions of (B.1) into solutions of the Hamiltonian

equations with hamiltonian, $V \in C^{\infty}(T^1)$

$$H = A\sqrt{E} + \omega B - \frac{\varepsilon}{\sqrt{E}} V(\phi)\, A \sin^2\theta \tag{B.3}$$

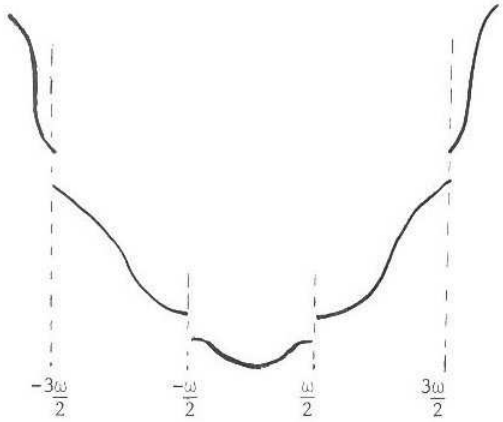
Let $k_0 \in [-\frac{\pi}{\omega}, \frac{\pi}{\omega}]$; one defines a Bloch wave with wave number k_0, a solution of (B.1) of the form

$$q(t) = e^{ik_0 t}\, q_0(\omega t+\phi_0) \tag{B.4}$$

where $q_0 \in C^{\infty}(T^1)$. The equation for q_0 is

$$(k_0^2 - E)q_0 - 2ik_0\,\dot{q}_0 + \varepsilon V(\omega t+\phi_0)q_0 - \omega^2\ddot{q} = 0 \tag{B.5}$$

The (B.5) is a self-adjoint equation on $L_2(T^1)$ with discrete spectrum. Fixed k_0 we can find a sequence $E_n(k_0)$, $n = 0,1\ldots$, of eigenvalues for (B.5). The parametrization can be made so that $E_n(k_0)$ are, for each n, continuous functions of k_0. So one has the following picture:



draw $E_0(k_0)$ over the interval $k_0 \in [-\frac{\omega}{2},\frac{\omega}{2}]$. Every value E has double multiplicity because there are two independent Bloch waves with the values of the wave number equal to $\pm k_0$. Then we draw $E_1(k_0)$ over the interval $(\frac{\omega}{2},\frac{3\omega}{2})$ and $E_2(k_0)$ over the interval $(\frac{\omega}{2},\frac{3\omega}{2})$ and $E_2(k_0)$ over the interval $(-\frac{3\omega}{2},-\frac{\omega}{2})$ instead of drawing them again on $(-\omega,\omega)$. In this way the multiplicity is

automatically taken into account. Furthermore, $E(k_0)/k_0^2 \xrightarrow[k\to\infty]{}$ constant where $E(k_0)$ is the function whose graph has been constructed in the above picture. It is well known that $E(k_0)$ has discontinuities which can possibly exist only at the points $k_0 = \pm(2k+1)\frac{\omega}{2}$, $k = 0,1\ldots$.

Let $E = E(k_0)$. We claim that with such a value of E the (B.3) is integrable by quadratures if k_0 is different from $\pm(2k+1)\frac{\omega}{2}$, $k = 0,1,\ldots$.

The change of variables (B.2) and the correspondence between solutions of (B.1) and (B.3) suggests the introduction of

$$\begin{aligned} Q(\alpha_1,\alpha_2) &= \mathrm{Re}\, e^{i\alpha_1}\, q_{k_0}(\alpha_2) \\ P(\alpha_1,\alpha_2) &= \mathrm{Re}(ik_0\, e^{i\alpha_1}\, q_{k_0}(\alpha_2) + e^{i\alpha_1}\,\omega\, q'_{k_0}(\alpha_2)) \\ a(\alpha_1,\alpha_2) &= \frac{P^2 + EQ^2}{2\sqrt{E}} \\ \Gamma(\alpha_1,\alpha_2) &= \frac{\varepsilon}{2} V'(\alpha_2)Q^2(\alpha_1,\alpha_2) \end{aligned} \tag{B.6}$$

where the prime denotes differentiation.

Then, $\forall\, \rho \in R, \phi_0, \alpha_0 \in T^1$ the following functions

$$\begin{aligned} q(t) &= \rho\, Q(\alpha_0+k_0 t, \phi_0+\omega t) \\ \phi(t) &= \phi_0 + \omega t \end{aligned} \tag{B.7}$$

as $(\rho,\alpha_0,\phi_0) \in R \times T^2$ vary we see that we have found a solution to the hamiltonian equations with initial data

$$\begin{aligned} p &= \rho\, \underline{P}(\alpha_0,\phi_0) & q &= \rho Q(\alpha_0,\phi_0) \\ \phi &= \phi_0 & B &= B_0 \end{aligned} \tag{B.8}$$

provided $B(t)$ is defined so that

$$\begin{aligned} \dot{B} &= \frac{\rho^2}{2}\,\varepsilon\, V'(\phi_0+\omega t)\, Q^2(\alpha_0+k_0 t, \phi_0+\omega_0 t) = \\ &= \rho^2\, \Gamma(\alpha_0+k_0 t, \phi_0+\omega t) \end{aligned} \tag{B.9}$$

i.e. if $|k_0 r_1 + \omega r_2|^{-1} \leq C(|r_1|+r_2|)^{\alpha}$, $\forall\ \underline{r} \in z^2$, $\underline{r} \neq \underline{0}$

$$B(t) = B_0 + \rho^2 \Gamma_{00} t + \rho^2 \sum_{\underline{r}\neq\underline{0}} \Gamma_{\underline{r}}\, e^{i(r_1\alpha_1 + r_2\alpha_2)}\, \frac{e^{i(r_1 k_0 + r_2\omega)t} - 1}{i(r_1 k_0 + r_2\omega)} \tag{B.10}$$

where $\Gamma_{\underline{r}}$ are the Fourier's coefficients of Γ .

However, writing $Q(\alpha_1,\alpha_2) = (e^{i\alpha_1} q_0(\alpha_2) + e^{-i\alpha_1}\, \overline{q}_0(\alpha_2))/2$

it is easy to realize that Γ_{00} is proportional to

$$\int_0^{2\pi} V'(\alpha)\, |q_0(\alpha)|^2 d\alpha \tag{B.11}$$

which vanishes as it follows by considering (B.5), multiplying it by $\overline{q}_0'$ and taking the real part of the result and integrating over a period. So $\Gamma_{00} = 0$.

Therefore, the (B.10) with $\Gamma_{00} = 0$ tells us that setting $\gamma_{\underline{r}} = \Gamma_{\underline{r}}/-i(k_0 r_1 + \omega r_2)$

$$\mathcal{B} = B + \rho^2 \sum_{\underline{r}\neq\underline{0}} \gamma_{\underline{r}}\, e^{i\underline{\alpha}\cdot\underline{r}} \tag{B.12}$$

is a prime integral for H, at least for the values of k_0 such that $\underline{\omega}_0 = (k_0,\omega)$ verifies a diaphantine condition.

For such values we can find action angle variables easily. We have that the surfaces H = constant and ρ = constant (or = constant)* are manifestly tori, see (B.7), of which we know explicit parametric equations

$$p = \rho\, P(\alpha_1,\alpha_2) \qquad q = \rho\, Q(\alpha_1,\alpha_2) \tag{B.13}$$

$$\phi = \alpha_2 \qquad B = \mathcal{B} - \rho^2 \sum_{r\neq 0} \gamma_{\underline{r}}\, e^{i\underline{r}\cdot\underline{\alpha}} \equiv -\rho^2 g(\alpha_1,\alpha_2)$$

So that we can define action angle variables as, (see A.3)

* H,ρ, $\mathcal{B}$ are constants of the motion

$$A_1 = \rho^2 \int_0^{2\pi} P(\alpha,0)\, \frac{\partial Q}{\partial\alpha}(\alpha,0)\, \frac{d\alpha}{2\pi} + \\ + 2\pi\mathcal{B} - \rho^2 \sum_{r_2\neq 0} \gamma_{0 r_2} \tag{B.14}$$

$$A_2 = \rho^2 \int_0^{2\pi} P(0,\beta)\, \frac{\partial Q}{\partial\beta}(0,\beta)\, \frac{d\beta}{2\pi} + 2\pi\mathcal{B} - \rho^2 \sum_{r_1\neq 0} \gamma_{r,0}$$

while the conjugate angles can be computed by first finding an expression for $\mathcal{B}$, ρ^2 in terms of A_1,A_2 , which amounts to solving a linear 2×2 system, see (B.14), and then using the generating function

$$\Phi(A,q,\phi) = \int_0^{\alpha_1} [\rho^2 P(\alpha,0)\, \frac{\partial Q}{\partial\alpha}(\alpha,0) + \mathcal{B} - \rho^2 g(\alpha,0)]d\alpha + \\ + \int_0^{\alpha_2} [\rho^2 P(\alpha_1,\beta)\, \frac{\partial Q}{\partial\alpha}(\alpha_1,\beta) + \mathcal{B} - \rho^2 g(\alpha_1,\beta)]d\beta \tag{B.14}$$

with ρ^2 and $\mathcal{B}$ expressed in terms of A_1,A_2 .

Another family of solutions can be obtained, likewise, by choosing the imaginary part instead of the real part in (B.6).

It remains to study the values of k_0 such that $\underline{\omega}_0 = (k_0,\omega)$ does not verify a diophantine condition. Clearly the problem here is only that of the convergence of the series in (B.10). From the first and last of (B.6) we see that $\Gamma_{\underline{r}} \equiv 0$ unless $r_1 = \pm 2,0$. Since $\Gamma_{0,0} = 0$ we see that no small denominators occur in the addends of (B.10) with $r_1 = 0$. The only problem can occur when $-2k_0 + r\omega$, or when $2k_0 + r\omega$, vanish for some integer r. Since we only consider $k_0 \in [-\frac{\omega}{2},\frac{\omega}{2}]$ this can happen only for r = 1 or r = -1, i.e. for k_0 at the extremes of the interval ("Brilloin zone").

Clearly this means that (B.3) is integrable for all the values of E in the interior of the "bands", i.e. for $E \in \{E' | E' = E_n(k_0)$ for some n and for $k_0 \in (-\frac{\omega}{2},\frac{\omega}{2})\}$. This set is a union of intervals which can be easily visualized from the above picture.

Exercise: find explicitly H in terms of A_1,A_2.

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The main result of this reference was actually essentially proved earlier by H. Rüssmann. However, the difference in notation and spirit might make it worthwhile for the reader to consult also my paper after the original reference of Rüssmann, see [13] below.

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