# Some aspects of the renormalization problems in statistical mechanics and field theory

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RIASSUNTO. — In questo lavoro vengono introdotti e studiati alcuni processi stocastici non gaussiani la cui teoria presenta difficoltà assai simili a quelle degli analoghi processi che si incontrano nella teoria costruttiva dei campi in due e tre dimensioni spazio-temporali.

La natura dei modelli qui considerati consente una trattazione dei problemi di rinormalizzazione assai semplice, concettualmente, e basata su idee e metodi della Meccanica Statistica classica.

## 1. INTRODUCTION

We shall discuss two types of models, introduced below, the " Bernoulli hierarchical fields " and the "Markov hierarchical fields ".

The first class of models presents only ultraviolet divergences and they are well suited to illustrate the mechanism that works to eliminate the ultraviolet divergences of field theory ($d = 1, 2$). The second class presents both ultraviolet and infrared problems which can be treated, in some cases, by using standard techniques developed in Statistical Mechanics to understand the thermodynamic limit of the classical rarefied gases.

The mathematical mechanism that underlies the cancellation of the divergences, both ultraviolet and infrared, appears to be extremely simple and natural in such models which, in our opinion, contain most of the real difficulties of the corresponding dimension field theory (cfr. the comments in sec. 4).

The techniques used below are, of course, inspired by those of references [1, 2, 23, 3, 4, 5, 6, 7, 8], and we hope that what follows provides a non trivial simplification of their work if it is regarded as a method of analysis of the models considered here.

The definition of the models stems out of the following remarks on the " free euclidean field " which is defined as the gaussian random field $\varphi$ over $R^{d+1}$ with covariance given by the kernel of the operator, on $L_2(R^d)$

$$C = (I - \Delta)^{-1} \tag{1.1}$$

where $\Delta$ is the Laplace operator on $R^{d+1}$.

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If $d \geq 1$, $C_{\xi\xi} = +\infty$, $\xi \in R^{d+1}$, and the field $\varphi$ is not normalizable. One can, however, introduce the fields $\varphi^{[N]}$, [1, 2], with covariance:

$$C^{[N]} = (2^{2N} - \Delta)^{-1} - (2^{2(N+1)} - \Delta)^{-1} \tag{1.2}$$

such fields are normalizable since, for some $\gamma > 0$, $C^{[N]}_{\xi\xi} = \gamma 2^{(d-1)N}$: define for $\xi \in R^d$

$$z^{(N)}_\xi = (2\gamma 2^{(d-1)N})^{-\frac{1}{2}} \varphi^{[N]}_\xi \tag{1.3}$$

and represent

$$\varphi_\xi = \sum_{N=0}^{\infty} (2\gamma 2^{(d-1)N})^{\frac{1}{2}} z^{(N)}_\xi. \tag{1.4}$$

The random fields $z^{(N)}$ have a covariance which can he easily seen to have the following properties:

i) it decays exponentially on a scale $2^{-N}$.

ii) it is almost constant in cubes of size $2^{-N}$.

iii) the fields $z^{(N)}$ and $z^{(M)}$ are the same modulo a change in the length scale:

$$\frac{1}{2\gamma 2^{(d-1)N}} C^{[N]}_{\xi\eta} = \frac{1}{2\gamma} C^{[0]}_{2^N\xi\, 2^N\eta} \qquad N = 0, 1, \cdots. \tag{1.5}$$

This suggests introducing a new type of free field via the following procedures.

Let $Q_0$ be a pavement of $R^{d+1}$ by non overlapping cubic (open) tesserae.

Define a gaussian random field on $R^{d+1}$ by assigning a covariance $\overline{C}^{[0]}_{\xi\eta}$ which decays exponentially as $|\xi - \eta| \to \infty$ and which is constant as $\xi$ varies over the tesserae $\Delta \in Q_0$ and $\eta$ varies in a tesserae $\Delta' \in Q_0$.

Denote $z^{(0)}_\xi$ this random field (notice that $z^{(0)}_\xi$ is constant as $\xi$ varies over a $\Delta \in Q_0$) and define $z^{(N)}_\xi$ as the random field with covariance

$$\overline{C}^{[N]}_{\xi\eta} = \overline{C}^{[0]}_{2^N\xi, 2^N\xi} \qquad N = 0, 1, \cdots. \tag{1.6}$$

It is clear that the gaussian field over $R^{d+1}$ defined by (1.4) using the new random variables $z^{(N)}$ has the three basic features listed above.

We shall only consider two special choices which correspond to the simplest fields $z^{(0)}$.

The first choice (Bernoulli hierarchical fields) corresponds to choosing $C^{[0]}_{\xi\eta} \equiv 0$ when $\xi, \eta$ do not belong to the same $\Delta \in Q_0$).

The second choice, Markov hierarchical fields, corresponds to choosing $\overline{C}^{[0]}_{\tilde{\xi}\tilde{\eta}}$ = (covariance of the gaussian Ising model on the lattice $\tilde{Z}$ of the centres of the tesserae of $Q_0$), $\tilde{\xi}, \tilde{\eta} \in \tilde{Z}$.

We shall then study the $\varphi^4$ theory with respect to the above free fields.

It will turn out that the theory presented below is a rigorous version of the theory of the renormalizationg roup (or, better, of some aspects of it) in the sense of Wilson [23]. The Fourier transform being here replaced by

the " cell decomposition " of the field associated with the pavement $Q_0$ and the representation (1.4).

The Bernoulli field is particularly interesting since the recursion formula introduced in [23] holds rigorously for this model.

## 2. THE ULTRAVIOLET PROBLEM. DEFINITIONS

Let $R^{d+1}$ be the $(d+1)$-dimensional euclidean space and let $(Q_N)_{N=0}^{\infty}$ be a sequence of pavements of $R^{d+1}$ with non overlapping cubic tesserae. The tesserae $\Delta \in Q_N$ have sides with length $l_N$.

We shall suppose that the pavements are compatible: i.e. each tessera $\Delta \in Q_N$ is exactly paved by those of $Q_{N+1}$ contained in it.

For simplicity of notation we shall eventually choose $l_N = 2^{-N}$, which turns out, also, to be the most difficult case.

We define for each $\Delta \in Q_i$, $i = 0, 1, \cdots$, a gaussian random variable distributed as:

$$(\exp - z_\Delta^2)\, dz_\Delta / \sqrt{\pi} \tag{2.1}$$

and these variables are regarded as independent.

Then we assign a sequence $(\gamma_N)_{N=0}^{\infty}$ of positive numbers and define a gaussian random field, " the free Bernoulli hierarchical field ", as the random field over $R^{d+1}$ given by:

$$\varphi_\xi = \sum_{i=0}^{\infty} \sum_{\Delta \in Q_i} \sqrt{2\gamma_i}\, \chi_\Delta(\xi) \qquad \xi \in R^{d+1} \tag{2.2}$$

where $\chi_\Delta$ is the characteristic function of $\Delta$. We also define:

$$\varphi_\xi^{[N]} = \sum_{\Delta \in Q_N} \sqrt{2\gamma_N}\, z_\Delta\, \chi_\Delta(\xi) \tag{2.3}$$

$$\varphi_\xi^{[\leq N]} = \sum_{i=0}^{N} \sum_{\Delta \in Q_i} \sqrt{2\gamma_i}\, z_\Delta\, \chi_\Delta(\xi) \tag{2.4}$$

called, respectively, the " N-th frequency field " and the " $\leq$ N-frequency field ".

The symbols $P_0$, $\hat{P}_0^{(N)}$, $P_0^{(N)}$ will, respectively, denote the probability distribution of $\varphi$, $\varphi^{[N]}$, $\varphi^{[\leq N]}$.

Clearly $\varphi_\xi$ and $\varphi_\eta$ are independent if $\xi$ and $\eta$ do not belong to the same tessera of $Q_0$.

The " interaction " that we wish to introduce among the above fields is given by:

$$V_0^{(N)}(\varphi, \lambda) = \frac{\lambda}{4!} \int_I : \varphi_\xi^{[\leq N]4} : d\xi \tag{2.5}$$

where I is a finite region of $R^{d+1}$ which consists of a finite union of tesserae of $Q_0$, and:

$$(2.6) \qquad : \varphi_\xi^{[\leq N]4} : = \left(\sqrt{2\, C_{\xi\xi}^{[\leq N]}}\right)^4 H_4 \left(\frac{\varphi_\xi^{[\leq N]}}{\sqrt{2\, C_{\xi\xi}^{[\leq N]}}}\right)$$

where $H_4$ is the fourth Hermite polynomial and:

$$(2.7) \qquad C_{\xi\eta}^{[\leq N]} = \int \varphi_\xi^{[\leq N]}\, \varphi_\eta^{[\leq N]}\, P_0^{(N)}(d\varphi) \equiv [\varphi_\xi^{[\leq N]}\, \varphi_\eta^{[\leq N]}] .$$

More generally, if $x$ is a random variable with covariance $C = E(x^2)$, one puts:

$$(2.8) \qquad : x^n : = (2\,C)^{\frac{n}{2}} H_n \left(\frac{x}{(2\,C)^{\frac{1}{2}}}\right)$$

where $H_n$ is the $n^{\text{th}}$ Hermite polynomial: $H_0 = 1$, $H_1 = x$, $H_2 = x - (1/2)$, $H_3 = x^3 - (3/2)\,x$, $H_4 = x^4 - 3\,x^2 + (3/4)$.

To imitate the characteristic divergences of field theory we shall choose, if $d = l$:

$$(2.9) \qquad \gamma_i = \log (l_{i-1}/l_i) \quad i \geq 1 \quad , \quad \gamma_0 = \log l_0^{-1}$$

or, if $d > 1$:

$$(2.10) \qquad \gamma_i = l_i^{1-d} - l_{i-1}^{1-d} \quad i \geq 1 \quad , \quad \gamma_0 = l_0^{1-d}.$$

To make the above gaussian fields resemble as much as possible the "Euclidean free field", i.e. the gaussian field with covariance:

$$(2.11) \qquad C_{\xi\eta}^{(\text{true})} = (\text{cost}) \int_{-\infty}^{+\infty} (1 + K^2)^{-1} \exp iK(\xi - \eta)\, d^{d+1} K$$

one easily realizes that the ratio $l_{i-1}/l_i$ has to be kept as small as possible: hence the eventual choice $l_{i-1}/l_i = 2$.

In the $d = 1$ case we shall study the following problem: do they exist $E_-$ and $E_+$ such that, for all N:

$$(2.12) \qquad \exp E_- |I| \leq \int \exp V_0^{(N)}\, P_0(d\varphi) \leq \exp E_+ |I|$$

where $|I|$ = volume of the set I?

Because of the independence of the fields relative to different tesserae of $Q_0$ it is enough to study this question in the case $I \in Q_0$, i.e. $|I| = 1$.

In the $d = 2$ case we shall consider the same question but we shall replace $V_0^{(N)}$ by:

$$(2.13) \qquad V^{(N)} = V_0^{(N)} - \frac{1}{2!}\, [V_0^{(N)2}]_{(2)} - \frac{1}{2!}\, [V_0^{(N)2}]_0 - \frac{1}{3!}\, [V_0^{(N)3}]_0$$

where

$$(2.14)\qquad \frac{1}{2!}\,[V_0^{(N)2}]_{(0)} = \frac{1}{2}\int V_0^{(N)2}\,P_0^{(N)}(dz) = \frac{\lambda^2}{2\cdot 4!}\int_{I\times I} C_{\xi\eta}^{[\le N]4}\,d\xi\,d\eta$$

$$(2.15)\qquad \frac{1}{3!}\,[V_0^{(N)3}]_{(0)} = \frac{1}{3!}\int V_0^{(N)3}\,P_0^{(N)}(dz) =$$

$$= -\frac{\lambda^3\,3^3}{4^3\cdot 6}\int_{I\times I\times I} C_{\xi\eta}^{[\le N]2}\,C_{\eta\zeta}^{[\le N]2}\,C_{\zeta\xi}^{[\le N]2}\,d\xi\,d\eta\,d\zeta$$

$$(2.16)\qquad \frac{1}{2!}\,[V_0^{(N)2}]_{(2)} = \frac{\lambda^2}{2\cdot 6}\int_{I\times I} C_{\xi\eta}^{[\le N]3} : \varphi_\xi^{[\le N]2} : d\xi\,d\eta.$$

The choice of the " counterterms " comes from a formal perturbation analysis which turns out to be identical to the usual renormalization theory [9, 10] because the Bernoulli hierarchical field with: $\varphi^4$ : interaction originates Feynman diagrams which present the same divergences of the ordinary: $\varphi^4$ : interaction theory over the free Euclidean field.

## 3. THE ONE DIMENSIONAL BERNOULLI HIERARCHICAL STRING

This case is the simplest.

The lower bound theory is totally trivial: the Jensen convexity inequality says that:

$$(3.1)\qquad \int \exp V_0^{(N)}\,P_0(dz) \ge \exp \int V_0^{(N)}\,P_0(dz) \equiv 1$$

by the orthogonality properties of the Hermite polynomials.

This remark, however, is too general and does not provide a key for the search of an upper bound, neither it can be generalized to the $d = 2$ models: in this last case it would give in fact:

$$(3.2)\qquad \int \exp V^{(N)}\,P_0(dz) \ge \exp -\frac{1}{2}\,[V_0^{(N)2}]_{(0)} - \frac{1}{6}\,[V_0^{(N)3}]_{(0)}$$

and the r.h.s. of this expression approaches zero as N increases, as it is easily checked from the expressions for $[V_0^{(N)2}]_{(0)}$ and $[V_0^{(N)3}]_{(0)}$ given in the last section.

Our first step will be, therefore, to find a lower bound with an alternative method which can be generalized to $d = 2$ as well as adapted to the derivation of the upper bounds.

We start by remarking that $\varphi_\xi^{[\le N]}$ is constant over the tesserae of $Q_N$ (see (2.3)).

It is convenient to normalize $\varphi_\xi^{[\leq N]}$ and introduce the variables:

$$X_\Delta = \frac{\varphi_\xi^{[\leq k]}}{(2\, C_{\xi\xi}^{[\leq k]})^{\frac{1}{2}}} \qquad \text{for} \quad \xi \in \Delta \in Q_k \tag{3.3}$$

which can be expressed in terms of the $z$'s: if $\Delta \in Q_k$ and $\Delta \subset \Delta_i$, $i = 1, 2, \cdots \cdots, k-1$, then:

$$X_\Delta = \frac{\sqrt{2\gamma_k}\, z_\Delta + \sum_{i=0}^{k-1} \sqrt{2\gamma_i}\, z_{\Delta_i}}{\sqrt{2 \sum_{i=0}^{k} \gamma_i}} = \frac{z_\Delta + \sqrt{\Gamma_k}\, X_{\Delta_{k-1}}}{\sqrt{1+\Gamma_k}} \tag{3.4}$$

where

$$\Gamma_k = \sum_{i=0}^{k-1} \frac{\gamma_i}{\gamma_k}. \tag{3.5}$$

Using the above remarks and definitions the function $V_0^{(N)}$ can be conveniently rewritten:

$$V_0^{(N)} = -\lambda \sigma_N \sum_{\substack{\Delta \in Q_N \\ \Delta \subset I}} (1 + \Gamma_N)^2\, H_4(X_\Delta) \tag{3.6}$$

where

$$\sigma_N = \frac{l_N^2\, \gamma_N^2}{3!}. \tag{3.7}$$

Let $B \geq 3$ be a constant to be fixed later and put:

$$B_N = B_0\, N^3,\ N \geq 1 \qquad \text{and} \quad B_0 = B\,(1 + (\log \lambda)^2). \tag{3.8}$$

where the features of $B_0$ to be retained are that $B_0 \geq 3$ and $B_0$ diverges as $\lambda$ approaches zero faster than $(\log \lambda)$.

Furthermore let us denote $\chi$ (something) the characteristic function of the set of fields where the "something" is verified and denote:

$$\chi_B^{(N)} = \prod_{i=0}^{N} \prod_{\substack{\Delta \in z_i \\ \Delta \subset I}} \chi\,(|\, X_\Delta \,| < B_i) \tag{3.9}$$

Then, taking $l_N = 2^{-N}$, so that $\Gamma_N = N$:

$$\int \exp V_0^{(N)}\, P_0^{(N)}(dz) \geq \int (\exp V_0^{(N)})\, \chi_B^{(N)}\, P_0^{(N)}(dz) = \tag{3.10}$$

$$= \int \chi_B^{(N-1)}\, P_0^{(N-1)}(dz) \cdot$$

$$\cdot \prod_{\substack{\Delta \in Q_{N-1} \\ \Delta \subset I}} \left( \int \chi \left( \left| \frac{z + \sqrt{N}\, X_\Delta}{\sqrt{1+N}} \right| < B_N \right) e^{-\lambda \sigma_N (1+N)^2 H_4\left(\frac{z+\sqrt{N} X_\Delta}{\sqrt{1+N}}\right)} \frac{e^{-z^2} dz}{\sqrt{\pi}} \right)^4 .$$

We can evaluate the integral in parenthesis by an asymptotic expansion in $\lambda$. Before doing this it is useful to remark that:

$$(3.11)\qquad \int \chi\left(\left|\frac{z+\sqrt{N}\,X}{\sqrt{1+N}}\right| < B_N\right) e^{-\lambda(1+N)^2\sigma_N H_4\left(\frac{z+\sqrt{N}X}{\sqrt{1+N}}\right)} e^{-z^2}\,\frac{dz}{\sqrt{\pi}} =$$

$$= \int_{-\infty}^{+\infty} e^{-\lambda(1+N)^2\sigma_N H_4\left(\frac{z+\sqrt{N}X}{\sqrt{1+N}}\right)} e^{-z^2}\,\frac{dz}{\sqrt{\pi}} - O_1(1)\int\limits_{\frac{|z+\sqrt{N}X|}{\sqrt{1+N}} > B_N} e^{-z^2}\,dz$$

because $H_4(y) > 0$ for $|y| > 3$. Here as well as in the sequel we shall adopt the convention that the symbol $O_\alpha(s)$, where $\alpha$ is some index, will denote a function for which there exists a constant $C_\alpha$ such that:

$$(3.12)\qquad |O_\alpha(s)| \leq C_\alpha |s|.$$

Hence the above integral can be written, developing the exponential to second order by the Taylor expansion, if $|X| < B_{N-1}$:

$$(3.13)\qquad 1 - \lambda\sigma_N (1+N)^2 \int H_4\left(\frac{z+\sqrt{N}\,X}{\sqrt{1+N}}\right) e^{-z^2}\,\frac{dz}{\sqrt{\pi}} +$$

$$+ O_2(\lambda^2 B_0^8 N^{20} \sigma_N^2 e^{\frac{3}{4}\lambda\sigma_N(1+N)^2}) + O_1'(1)\exp -\frac{5}{2} B_0^2 N^3 =$$

$$= 1 - \lambda\sigma_N N^2 H_4(X) + O_2'(\lambda^2 B_0^8 N^{20} \sigma_N^2 e^{\frac{3}{4}\lambda\sigma_N(1+N)^2})$$

which is identical to:

$$(3.14)\qquad \exp O_3(\lambda^2 B_0^8 e^{\bar{c}\lambda} N^{20} \sigma_N^2)\cdot \exp -\lambda\sigma_N N^2 H_4(X)$$

where $\bar{c} = 1 + \max_N 3/2\,(1+N)^2\,\sigma_N$.

Since $4\sigma_N N^2 = \sigma_{N-1}(1+(N-1))^2$ we realize that is has been shown:

$$(3.15)\qquad \int \exp V_0^{(N)} \chi_B^{(N)} P_0^{(N)}(dz) \geq (\exp -C_3 \lambda^2 B_0^8 e^{\bar{c}\lambda} N^{20} \sigma_N^2 \gamma_N^{-2} |I|)\cdot$$

$$\cdot \int \exp V_0^{(N-1)} \chi_B^{(N-1)} P_0^{(N-1)}(dz)$$

therefore it follows, inductively, that we can take:

$$(3.16)\qquad E_- = -C_3 \lambda^2 B_0^8 e^{\bar{c}\lambda} \sum_{N=0}^{\infty} N^{20} \sigma_N^2 \gamma_N^{-2}.$$

This method for the estimation of the lower bound can be easily adapted to obtain an estimate for $E_+$.

It is useful to remark that the above calculation also prove the existence of $C_4$ such that, if $|X| < B_{N-1}$:

$$(3.17)\quad \int \exp -\lambda (1+N)^2 \sigma_N H_4\left(\frac{z+\sqrt{N}X}{\sqrt{1+N}}\right) \chi\left(\left|\frac{z+\sqrt{N}X}{\sqrt{1+N}}\right| < B_N\right) e^{-z^2}\frac{dz}{\sqrt{\pi}} \leq$$

$$\leq (\exp C_4 \lambda^2 B_0^8 e^{\bar{c}\lambda} N^{20} \sigma_N^2)\,(\exp -\lambda N^2 \sigma_N H_4(X)) \int \chi\left(\left|\frac{z+\sqrt{N}X}{\sqrt{1+N}}\right| < B_N\right) e^{-z^2}\frac{dz}{\sqrt{\pi}}.$$

With this in mind we look for the upper bound as follows; if $Q_i \cap I$ means the set of the $\Delta$'s in $Q_i$ contained in I, we can write:

$$(3.18)\quad 1 \equiv \sum_{\substack{D_i \subset Q_i \cap I \\ i=0,1,\cdots,N}} \prod_{i=0}^{N} \left\{ \prod_{\Delta \in D_i} \chi(|X_\Delta| > B_i) \prod_{\substack{\Delta \notin D_i \\ \Delta \in Q_i \cap I}} \chi(|X_\Delta| < B_i) \right\}$$

and

$$(3.19)\quad \int \exp V_0^{(N)} P_0^{(N)}(dz) \equiv \sum_{\substack{D_i \subset Q_i \cap I \\ i=0,1,\cdots,N}}$$

$$\int \prod_{i=0}^{N} \left\{ \prod_{\Delta \in D_i} \chi(|X_\Delta| > B_i) \prod_{\substack{\Delta \notin D_i \\ \Delta \subset Q_i \cap I}} \chi(|X_\Delta| < B_i) \right\} \exp V_0^{(N)} P_0^{(N)}(dz)$$

Consider a single term of this expansion and call:

$$(3.20)\quad \bar{\chi}^{(k)} = \prod_{i=0}^{k} \left\{ \prod_{\Delta \in D_i} \chi(|X_\Delta| > B_i) \prod_{\substack{\Delta \notin D_i \\ \Delta \in Q_i \cap I}} \chi(|X_\Delta| < B_i) \right\}$$

then the integral to be studied is:

$$(3.21)\quad \int \bar{\chi}^{(N-1)} P_0^{(N-1)}(dz) \cdot$$

$$\cdot \int \left\{ \prod_{\Delta \in D_N} (\exp -\lambda\sigma_N (1+N)^2 H_4(X_\Delta))\, \chi(|X_\Delta| > B_N)\, e^{-z_\Delta^2} \frac{dz_\Delta}{\sqrt{\pi}} \right\} \cdot$$

$$\cdot \int \left\{ \prod_{\substack{\Delta \notin D_N \\ \Delta \in Q_N \cap I}} (\exp -\lambda\sigma_N (1+N)^2 H_4(X_\Delta))\, \chi(|X_\Delta| < B_N)\, e^{-z_\Delta^2} \frac{dz_\Delta}{\sqrt{\pi}} \right\}.$$

We shall first replate $H_4(X_\Delta)$ by zero for all the $\Delta$'s in $D_N$ such that $\Delta \subset \Delta' \in Q_{N-1}$ and $|X_{\Delta'}| > B_{N-1}$, otherwise if $\Delta \in D_N$, $|X_{\Delta'}| < B_{N-1}$ we replace it by $\frac{N^2}{(N+1)^2} H_4(X_{\Delta'})$: in fact this is possible, for the purpose of obtaining an upper bound, because $H_4(y) > 0$ and increases with $y$ for $|y| > 3$.

We consider, next, those $\Delta$'s such that $|X_\Delta| < B_N$ but $|X_{\Delta'}| > B_{N-1}$ where $\Delta' \in Q_{N-1}$, $\Delta' \supset \Delta$. In this case we wish to replace, to obtain an upper bound, $H_4(X_\Delta)$ by zero. To show that this is possible define:

$$(3.22)\quad J(X) = \int e^{-\lambda\sigma_N (1+N)^2 H_4\left(\frac{z+\sqrt{N}X}{\sqrt{1+N}}\right)} \chi\left(\left|\frac{z+\sqrt{N}X}{\sqrt{1+N}}\right| < B_N\right) e^{-z^2}\frac{dz}{\sqrt{\pi}}$$

and show that, if $|X| > B_{N-1}$, $J(X) < p_X$ where:

$$P_X = \int \chi\left(\left|\frac{z+\sqrt{N}\,X}{\sqrt{1+N}}\right| < B_N\right) e^{-z^2}\,\frac{dz}{\sqrt{\pi}} \tag{3.23}$$

provided the constant B is chosen large enough.

In fact, setting:

$$P = \int \chi\left(\left|\frac{z+\sqrt{N}\,X}{\sqrt{1+N}}\right| < \frac{7}{2}\right) e^{-z^2}\,\frac{dz}{\sqrt{\pi}}$$

we see that $J(X) < p_X$ if :

$$P \leq P_X\,\frac{1-\exp-\lambda\sigma_N(1+N)^2 H_4\left(\frac{7}{2}\right)}{\exp\frac{3}{2}\lambda\sigma_N(1+N)^2-\exp-\lambda\sigma_N(1+N)^2 H_4\left(\frac{7}{2}\right)}$$

and this can be easily seen to be true if B is chosen large enough.

Finally, if $|X_\Delta| < B_N$ and $|X_{\Delta'}| < B_{N-1}$, $\Delta \subset \Delta'$, $\Delta' \in Q_{N-1}$, we can apply the iterative method described for obtaining the lower bound and this shows (cfr. Remark at p. 30) that in this situation we can replace to obtain an upper estimate, $-\sigma_N(1+N)^2 H_4(X_\Delta)$ in (3.21) by:

$$-\lambda\sigma_N N^2 H_4(X_{\Delta'}) + C_4\lambda^2 B_0^8 e^{\varepsilon\lambda} N^{20}\sigma_N^2 \tag{3.26}$$

and this proves the existence of $C_4$ such that:

$$\int \exp V_0^{(N)} P_0^{(N)}(dz) \leq (\exp C_4\lambda^2 B_0^8 e^{\varepsilon\lambda} N^{20}\sigma_N^2 l_N^{-2}|I|)\cdot \tag{3.27}$$

$$\sum_{\substack{D_0,\cdots,D_N\\ D_i\subset Q_i\cap I\\ i=0,1,\cdots,N}} \int \overline{\chi}^{(N)} \exp -\lambda \sum_{\Delta'\in Q_{N-1}} \sigma^*_{N-1} N^2 H_4(X_{\Delta'})\, P_0^{(N)}(dz)$$

where $\sigma^*_{N-1}$ is zero if $|X_{\Delta'}| > B_{N-1}$ or it is $\sigma_{N-1}$ if $|X_{\Delta'}| < B_{N-1}$.

The integrand in the above expression is no longer dependent on the $N^{th}$-frequency variables via the argument of the exponential and the sum over the $D_N$ can now be easily performed to get:

$$\int \exp V_0^{(N)} P_0^{(N)}(dz) \leq \exp C_4\lambda^2 B_0^8 e^{\varepsilon\lambda} N^{20}\sigma_N^2 l_N^{-2}|I| \tag{3.28}$$

$$\sum_{\substack{D_0,\cdots,D_{N-1}\\ D_i\subset Q_i\cap I\\ i=0,\cdots,N-1}} \int \overline{\chi}^{(N-1)} \exp -\lambda \sum_{\Delta'\in Q_{N-1}} \sigma^*_{N-1} N^2 H_4(X_{\Delta'})\, P_0^{(N-1)}(dz)$$

which can be iterated.

Calling $\varepsilon_N$ the argument of the exponential in front of the sum in the r.h.s.: of (3.28) and proceeding as before we obtain in fact

$$\int \exp V_0^{(N)} P_0^{(N)} \leq \exp(\varepsilon_N + \varepsilon_{N-1}) \sum_{D_0,\cdots,D_{N-2},D_{N-1}} \tag{3.29}$$

$$\int \overline{\chi}^{(N-1)} \left( \exp -\lambda \sum_{\Delta \in Q_{N-2}} \sigma^*_{N-2}(\Delta)\, \rho(\Delta)\, (N-1)^2\, H_4(X_\Delta) \right) P^{(N-1)}(dz)$$

where $\sigma^*_{N-2}(\Delta) = 0$ if $\Delta \in D_{N-2}$, $\sigma^*_{N-2}(\Delta) = \sigma_{N-2}$ otherwise, and $\rho(\Delta) =$ (fraction of the volume of $\Delta$ occupied by the tesserae of $R_{N-1}$ which are not in $D_{N-1}$).

To continue the iteration we wish to replace $\rho(\Delta)$ by 1 if $\Delta \notin D_{N-2}$.

For this purpose call $D^{ex}_{N-1} =$ (set of the tesserae of $D_{N-1}$ not contained in tesserae of $D_{N-2}$) and $n^{ex}_{N-1} =$ (number of tesserae in $D^{ex}_{N-1}$). Then the r.h.s. of (3.29) can be majorized by

$$\exp(\varepsilon_N + \varepsilon_{N-1}) \sum_{D_0 \cdots D_{N-2}} \sum_{D^{ex}_{N-1}} \exp \lambda \sigma_{N-1} N^2 H_4(B_{N-1})\, n^{ex}_{N-1} \cdot \tag{3.30}$$

$$\int \overline{\chi}^{(N-2)} \prod_{\Delta \in D^{ex}_{N-1}} \chi(|X_\Delta| > B_{N-1}) \exp -\lambda \sum_{\Delta \in Q_{N-2}} \sigma^*_{N-2}(\Delta)\, (N-1)^2\, H_4(X_\Delta)\, P_0^{(N-1)}(dz) \cdot$$

Then, since $|X_\Delta| > B_{N-1}$, $\Delta \in D^{ex}_{N-1}$ implies that $|z_\Delta| = |\sqrt{N}\, X_\Delta - \sqrt{N-1}\, X_{\Delta'}| \geq \sqrt{N}\, B_{N-1} - \sqrt{N-1}\, B_{N-2} \geq 3/2\, B_0\, N^{5/2}$ if $\Delta' \supset \Delta$, $\Delta' \in Q_{N-2}$, it follows:

$$[(3.30)] \leq \exp(\varepsilon_N + \varepsilon_{N-1}) \sum_{D_0 \cdots D_{N-2}} \sum_{D^{ex}_{N-1}} \exp(\lambda \sigma_{N-1} N^2 B^4_{N-1} - B_0^2 N^5)\, n^{ex}_{N-1} \cdot \tag{3.31}$$

$$\int \overline{\chi}^{(N-2)} \exp -\lambda \sum_{\Delta \in Q_{N-2}} \sigma^*_{N-2}(\Delta)\, (N-1)^2\, H_4(X_\Delta)\, P_0^{(N-1)}(dz) \leq$$

$$\leq \exp(\varepsilon_N + \varepsilon_{N-1}) \cdot (1 + \exp(\lambda \sigma_{N-1} N^2 B^4_{N-1} - B_0^2 N^5))^{2^{(N-1)|I|}} \cdot$$

$$\cdot \sum_{D_0 \cdots D_{N-2}} \int \overline{\chi}^{(N-2)} \exp -\lambda \sum_{\Delta \in Q_{N-2}} \sigma^*_{N-2}(\Delta)\, (N-1)^2\, H_4(X_\Delta)\, P_0^{(N-2)}(dz).$$

It is now clean that we can iterate this procedure and that we can take

$$E_+ = C_4 \lambda^2 B_0^8 e^{\varepsilon\lambda} \sum_{N=0}^{\infty} N^2 \sigma_N^2\, 2^{2N} + \tag{3.32}$$

$$+ \sum_{N=0}^{\infty} 2^{2(N-1)} \log(1 + \exp(\lambda \sigma_{N-1} N^2 B^4_{N-1} - B_0^2 N^5))$$

i.e. there exists $\overline{C}_4$ such that we can take

$$E_+ = \overline{C}_4 \lambda^2 B_0^8 e^{\varepsilon\lambda}. \tag{3.33}$$

By expanding to second order, rather than to first order as we just did, one could find that the upper and lower bounds could be taken proportional to $\lambda^2$ (rather than to $\lambda^2 (\log \lambda)^{16}$) for small values of $\lambda$.

## 4. THE TWO DIMENSIONAL BERNOULLI HIERARCHICAL MEMBRANE

With the same notations of the one dimensional case we put:

$$l_N = 2^{-N} \quad , \quad \gamma_0 = l_0^{-1} \quad , \quad \gamma_N = l_N^{-1} - l_{N-1}^{-1} \qquad \text{if } N \geq 1$$

$$\Gamma_N = \sum_{i=0}^{N-1} \frac{\gamma_i}{\gamma_N} \equiv 1 \quad , \quad \sigma_N = \frac{l_N^3 \gamma_N^2}{3!} \tag{4.1}$$

$$X_\Delta = \frac{z_\Delta + \sum_{i=0}^{N-1} \sqrt{\frac{\gamma_i}{\gamma_N}}\, z_{\Delta_i}}{\sqrt{1 + \sum_{i=0}^{N-1} \frac{\gamma_i}{\gamma_N}}} = \frac{z_\Delta + X_{\Delta'}}{\sqrt{2}} \qquad \Delta \in Q_N \, , \, \Delta' \in Q_{N-1} \, , \, \Delta \subset \Delta' .$$

We shall rewrite $V^{(N)}$ defined in (2.13) as

$$V^{(N)} = -\lambda \sigma_N \sum_{\Delta \in Q_N} 4\, H_4(X_\Delta) - \lambda^2 \sigma_N^2 \sum_{\Delta \in Q_N} \mu_N\, 2\, H_2(X_\Delta) - \tag{4.2}$$
$$- \lambda^2 \sigma_N^2 \sum_{\Delta \in Q_N} \nu_N + \lambda^3 \sigma_N^3 \sum_{\Delta \in Q_N} \nu'_N .$$

Before continuing it is useful to remark the following relations:

$$\sigma_N^2 \nu_N - \left(\frac{l_N}{l_{N-1}}\right)^3 \sigma_{N-1}^2 \nu_{N-1} = \frac{3}{4} \sigma_N^2 \delta_4(N) \tag{4.3}$$

where, in general, $\delta_k(N) = (1 + \Gamma_N)^k - \Gamma_N^k$ and

$$\sigma_N^2 \mu_N \frac{\gamma_{N-1}}{\gamma_N} - \left(\frac{l_N}{l_{N-1}}\right)^3 \sigma_{N-1}^2 \mu_{N-1} = 6 \sigma_N^2 \delta_3(N) \frac{\gamma_{N-1}}{\gamma_N} \tag{4.4}$$

$$\sigma_N^3 \nu'_N - \left(\frac{l_N}{l_{N-1}}\right)^3 \sigma_{N-1}^3 \nu'_{N-1} = \tag{4.5}$$
$$= \frac{3^3}{3!} \sigma_N^3 \left\{ \delta_6(N) + 4 \delta_2(N) \left[ \nu_{N-1} \left(\frac{l_N}{l_{N-1}}\right)^3 - \frac{3}{4} \Gamma_N^4 \right] \right\}$$

which, are easily checked to be valid, without the above special assumptions on the values of $l_N$ and $\gamma_N$.

Let us proceed to the estimation of a lower bound to the integral in (2.12) which, as remarked in sec. 2, is no longer obtainable as a consequence of the Jenssen's convexity inequality, as in the $d = 1$ case.

Define $B_N$ and $\chi_B^{(N)}$ as in the preceding section (cfr. p. 8); the arbitrary constant B which appears in the definition of the $B_N$ will be fixed along the lines of the $d = 1$ case, when we shall study the upper bound.

Consider the integral:

$$(4.6)\qquad Y(X) = \int \frac{dz}{\sqrt{\pi}}\, e^{-z^2} \cdot \chi\left(\left|\frac{z+X}{\sqrt{2}}\right| < B_N\right) \cdot$$

$$\cdot \exp\left\{-\lambda\sigma_N\, 4\, H_4\left(\frac{z+X}{\sqrt{2}}\right) - \lambda^2\,\sigma_N^2\,\mu_N\, 2\, H_2\left(\frac{z+X}{\sqrt{2}}\right) - \lambda^2\,\sigma_N^2\,\nu_N + \lambda^3\,\sigma_N^3\,\nu_N'\right\}$$

assuming $|X| < B_{N-1}$.

We wish to proceed as in the $d = 1$ case, iteratively. However since:

$$(4.7)\qquad \sum_{N=0}^{\infty} \sigma_N^3\, l_N^{-3} = +\infty \qquad \sum_{N=0}^{\infty} \sigma_N^4\, l_N^{-3} < +\infty$$

we have to evaluate the integral to third order exactly.

This can be easily done with the help of the orthogonality of the Hermite polynomials which allows to deduce the following relations:

$$(4.8)^{I}\qquad \int e^{-z^2}\,\frac{dz}{\sqrt{\pi}}\,(1+\Gamma)^2\, H_4\left(\frac{z+\sqrt{\Gamma}\,X}{\sqrt{1+\Gamma}}\right) = \Gamma^2\, H_4(X)$$

$$(4.8)^{II}\qquad \int e^{-z^2}\,\frac{dz}{\sqrt{\pi}}\,(1+\Gamma)\, H_2\left(\frac{z+\sqrt{\Gamma}\,X}{\sqrt{1+\Gamma}}\right) = \Gamma\, H_2(X)$$

and, if $\delta_k = (1+\Gamma)^k - \Gamma^k$:

$$(4.8)^{III}\qquad \int e^{-z^2}\,\frac{dz}{\sqrt{\pi}}\left((1+\Gamma)^2\, H_4\left(\frac{z+\sqrt{\Gamma}\,X}{\sqrt{1+\Gamma}}\right)\right)^2 =$$

$$= 2\left[\frac{3}{4}\,\delta_4 + 6\,\delta_3\,\Gamma H_2(X) + 9\,\delta_2\,\Gamma^2\, H_4(X) + 4\,\delta_1\,\Gamma^3\, H_6(X) + (\Gamma^2\, H_\Delta(X))^2\right]$$

$$(4.8)^{IV}\qquad \int e^{-z^2}\,\frac{dz}{\sqrt{\pi}}\,(1+\Gamma)^3\, H_4\left(\frac{z+\sqrt{\Gamma}\,X}{\sqrt{1+\Gamma}}\right) H_2\left(\frac{z+\sqrt{\Gamma}\,X}{\sqrt{1+\Gamma}}\right) =$$

$$= 4\,\delta_1\,\Gamma^2\, H_4(X) + 3\,\delta_2\,\Gamma H_2(X) + \Gamma^3\, H_4(X)\, H_2(X)$$

$$(4.8)^{V}\qquad \int e^{-z^2}\,\frac{dz}{\sqrt{\pi}}\left((1+\Gamma)^2\, H_4\left(\frac{z+\sqrt{\Gamma}\,X}{\sqrt{1+\Gamma}}\right)\right)^3 =$$

$$= (\Gamma H_4(X))^3 + 3^3\,\delta_6 + \sum_{k=1}^{5} A_{2k}\,\delta_{6-k}\,\Gamma^k\, H_{2k}(X)$$

where the numbers $A_2, A_4, A_6, A_8, A_{10}$ are defined by the development

$$(4.9)\qquad (H_4(X))^3 = H_{12}(X) + 3^3 + \sum_{k=1}^{5} A_{2k}\, H_{2k}(X).$$

It then follows, after some algebra and making use of $|X| < B_{N-1}$ and of the relations among $\mu, \nu, \nu'$ given by (4.3), (4.4), (4.5):

$$
\begin{aligned}
(4.10)\qquad Y(X) = 1 &- (\lambda\sigma_N H_4 + \lambda^2 \sigma_N^2 \mu_N H_2 + \lambda^2 \sigma_N^2 \nu_N - \lambda^3 \sigma_N^3 \nu'_N) + \\
&+ \frac{1}{2}\left(\frac{3}{2}\lambda^2 \sigma_N^2 \delta_4 + 2\cdot 6\lambda^2 \sigma_N^2 \delta_3 H_2 + 2\cdot 9\lambda^2 \sigma_N^2 \delta_2 H_4 + \right. \\
&+ 2\cdot 4\lambda^2 \sigma_N^2 \delta_1 H_6 + \lambda^2 \sigma_N^2 H_4^2 + 2\lambda^3 \sigma_N^3 \mu_N H_4 H_2 + \\
&\left. + 2\cdot 4\lambda^3 \sigma_N^3 \mu_N \delta_1 H_4 + 2\cdot 3\lambda^3 \sigma_N^3 \mu_N \delta_2 H_2 + 2\lambda^3 \sigma_N^3 \nu_N H_4\right) - \\
&- \frac{1}{3!}\left(\lambda^3 \sigma_N^3 H_4^3 + 3^3 \delta_6 \sigma_N^3 + \sum_{k=1}^{5} \lambda^3 \sigma_N^3 \delta_{6-k} A_{2k} H_{2k}\right) + \\
&+ O_5(\lambda^4 \sigma_N^4 B_0^{16} N^{52} e^{\bar{c}\lambda^3})
\end{aligned}
$$

where $\bar{c} = 1 + \max(\sigma_N + \sigma_N^2(\mu_N + \nu_N) + \sigma_N^3 \nu'_N)$ and $H_i \equiv H_i(X)$.

To express this result in terms of $V^{(N-1)}$ + corrections define $\lambda_N = \lambda$ $\lambda_k = \lambda_{k+1}(1 - 9\sigma_{k+1}\delta_2 \lambda_{k+1})$, for $0 \le k \le N-1$.

Then some more algebra leads to:

$$
\begin{aligned}
(4.11)\qquad Y(X) = \exp O_6(\lambda^4 e^{\bar{c}\lambda^3} B_0^{16} \sigma_N^4 N^{52}) \cdot \\
\exp 2^{-3}[\lambda_{N-1}\sigma_{N-1} 4 H_4(X) + \lambda_{N-1}^2 \sigma_{N-1}^2 \mu_{N-1} 2 H_2(X) + \\
+ \lambda_{N-1}^2 \sigma_{N-1}^2 \nu_{N-1} - \lambda_{N-1}^3 \sigma_{N-1}^3 \nu'_{N-1}] \cdot \left\{1 + 4\sum_{k=1}^{5} \lambda_N^3 \sigma_N^3 \bar{A}_{2k}(N) H_{2k}(X) + \right. \\
\left. + 4\lambda_N^2 \sigma_N^2 H_6(X)\right\}
\end{aligned}
$$

where the coefficients $\bar{A}_{2k}(N)$ verify:

$$
(4.12)\qquad |\bar{A}_{2k}(N)| \le C'_7(1 + N).
$$

The above relations are what we need to set up a recursive scheme.

Notice, in fact, that if $h_0(\lambda)$ is defined as the smallest integer such that for all $h \ge h_0(\lambda)$ one has $(1 - 9\lambda\delta_2\sigma_h) > 1/2$ then: $0 < \lambda_k \le \lambda$ and:

$$
(4.13)\qquad \lambda_k \ge \lambda \prod_{h \ge h_0(\lambda)} (1 - 9\lambda\delta_2\sigma_h) \qquad \forall k \ge h_0(\lambda)
$$

also notice that $h_0(\lambda) = 0$ for $\lambda$ small enough, while $h_0(\lambda) \propto \log\lambda$ as $\lambda \to \infty$.

It is natural to formulate the following inductive hypothesis: for suitably

chosen constants $C_7$, $C_8$ and for $k < \bar{k}$:

$$(4.14)\qquad \int (\exp V^{(N)}(\varphi, \lambda))\, \chi_B^{(N)}\, P_0^{(N)}(dz) \geq$$

$$\geq \exp - O_8 \left( |I| \sum_{i=0}^{k-1} \lambda^4 B_0^{16} e^{i\lambda^3} (N-i)^{52} \sigma_{N-i}^4 l_{N-i}^{-3} \right) \cdot$$

$$\int (\exp V^{(N-k)}(\varphi, \lambda_{N-k}))\, \chi_B^{(N-k)}\, P_0^{(N-k)}(dz) \cdot$$

$$\cdot \prod_{\Delta \in Q_{N-k}} \left( 1 + 8 \sum_{i=0}^{k-1} \left[ 4\, \sigma_{N-i}^2 \lambda_{N-i}^2 H_6(X_\Delta) + \right.\right.$$

$$\left.\left. + 4^{k-i} \lambda^3 \sigma_{N-1}^3 \sum_{s=1}^{5} \bar{A}_{2s}(N-i, k) H_{2s}(X_\Delta) \right] \right)$$

and

$$(4.15)\qquad |\bar{A}_{2s}(N-i, k)| \leq C_7 (1 + (N-i)) .$$

Using the technique explained to prove the truth of the hypothesis for $k = 1$, it is simple to check that the inductive hypothesis is true for $k = \bar{k}$ provided $N - \bar{k} \geq k_0(\lambda)$ where $k_0(\lambda)$ is the smallest among the numbers $\tilde{k} \geq h_0(\lambda)$ such that if $k \geq \tilde{k}$ then:

$$(4.16)\qquad 8 \cdot 4\, B_k^6 \lambda^2 \sigma_k^2 + 8 \cdot 5 \cdot 6 \cdot C_7 \lambda^3 k B_k^{10} \sigma_k^3 < 2 \cdot 8 \cdot 4\, B_k^6 \lambda^2 \sigma_k^2 < \tfrac{1}{2}$$

this restriction on $k$ is necessary to insure the positivity of the factors of the product appearing in the recursive hypothesis. Notice that $k_0(\lambda) = 0$ for $\lambda$ small enough.

After $N - k_0(\lambda)$ iterations we get an integral which can be estimated by the minimum value of the integrand and one finds that one can take $E_-$ as:

$$(4.17)\qquad E_- = -C_8 \sum_{N \geq k_0(\lambda)} \lambda^4 B_0^{16} e^{i\lambda^3} N^{52} \sigma_N^4 l_N^{-3} -$$

$$- 8^{k_0(\lambda)} k_0(\lambda) \log 2 - \lambda k_0(\lambda)\, 4^{k_0(\lambda)} B_{k_0(\lambda)}^4 - \lambda^2 k_0(\lambda)\, 2^{k_0(\lambda)} (B_{k_0(\lambda)}^2 \mu_{k_0(\lambda)} + \nu_{k_0(\lambda)}).$$

The reader can check that the coupling constant renormalization (i.e. the change of $\lambda_k$ as $k$ varies in the inductive argument) is not really necessary: one could obtain a similar iteration scheme with $\lambda_k \equiv \lambda$ at the price of keeping a term proportional to $\sigma_{N-k}^2 H_4(X_\Delta)$ togheter with the one proportional to $\sigma_{N-k}^2 H_6(X_\Delta)$ in the formulation of the inductive hypothesis. The hypothesis would still be true with some minor modifications of the algebra: this second approach becomes the simplest in the case, treated later, of the two dimensional Markov membrane and we shall use it in that occasion. A term proportional to $\sigma_{N-k}^2 H_2(X_\Delta)$ would however ruin the iteration scheme: this is, of course, the reason of the necessity of the counterterm $[V_0^{(N)2}]_{(2)}$.

The upper bound can be obtained by slightly modifying the $d = 1$, argument.

One has to be more careful because in that case the linearity of $V_0^{(N)}$ in $\lambda$ was implicitely used and it is no longer true for $V^{(N)}$.

Let us analyze the expression:

$$(4.18)\qquad \int (\exp V^{(N)})\, \bar{\chi}^{(N)}\, P_0^{(N)}(dz) \equiv \int (\exp V^{(N)}) \cdot$$

$$\prod_{i=0}^{N} \left( \prod_{\Delta \in D_i} \chi(|X_\Delta| > B_i) \prod_{\substack{\Delta \notin D_i \\ \Delta \subset I}} \chi(|X_\Delta| < B_i) \right) P_0^{(N)}(dz)$$

where we are using the same notations of the $d = 1$ case.

Exactly the same arguments used in the $d = 1$ case show that if the constant B is chosen large enough the above expression is majorized by:

$$(4.19)\qquad \exp O_9 (\lambda^4 e^{\varepsilon\lambda^3} B_N^{16} N^4 l_N^{-3}) \,|I|\cdot \int \bar{\chi}^{(N)}\, P_0^{(N)}(dz) \cdot$$

$$\cdot \prod_{\substack{\Delta \in Q_{N-1} \\ \Delta \notin D_{N-1}}} \Big\{ \exp [\lambda_{\cdot -1}\, \sigma_{N-1}\, 4 H_4(X_\Delta) +$$

$$+ \lambda_{N-1}^2 \sigma_{N-1}^2 \mu_{N-1}\, 2 H_2(X_\Delta) + \lambda_{N-1}^2 \sigma_{N-1}^2 \nu_{N-1} \lambda_{N-1}^3 \sigma_{N-1}^3 \nu'_{N-1}] \cdot$$

$$\cdot \Big[ 1 + 8 (4 \sigma_{(N)}^2 \lambda^2 H_6(X_\Delta) + \lambda^3 \sigma_N^3 \sum_{s=1}^{5} \bar{A}_{2s}(N, 1) H_{2s}(X_\Delta) \Big] \Big\}$$

and the sum over the $D_N$ can be immediately executed to yield the same expression with $\bar{\chi}^{(N)} P_0^{(N)}(dz)$ replaced by $\bar{\chi}^{(N-1)} P_0^{(N-1)}(dz)$.

To continue the procedure let us perform, in fact, one more step; proceeding as in the former step we now get:

$$\exp |I|\, O_9 (\lambda^4 e^{\varepsilon\lambda^3} (\sigma_N^4 N^4 B_N^{16} l_N^{-3} + \sigma_{N-1}^4 (N-1)^4 B_{N-1}^{16} l_{N-1}^{-3})) \cdot$$

$$\int \bar{\chi}^{(N-1)} P_0^{(N-1)}(dz) \cdot \prod_{\substack{\Delta \in Q_{N-2} \\ \Delta \notin D_{N-2}}} \Big\{ \exp - \rho(\Delta) [\lambda_{N-2}\, \sigma_{N-2}\, 4 H_4(X_\Delta) +$$

$$+ \lambda_{N-2}^2 \sigma_{N-2}^2 \mu_{N-2}\, 2 H_2(X_\Delta) + \lambda_{N-2}^2 \sigma_{N-2}^2 \nu_{N-2} \lambda_{N-2}^3 \sigma_{N-2}^2 \nu'_{N-2}] \cdot$$

$$\cdot \Big[ 1 + 8 \rho(\Delta) (4 \sigma_N^2 \lambda^2 H_6(X_\Delta) + 4 \sigma_{N-1}^2 \lambda_{N-1}^2 H_6(X_\Delta) +$$

$$+ 4^2 \lambda^3 \sigma_N^3 \sum_{i=1}^{5} \bar{A}_{2s}(N, 2) H_{2s}(X_\Delta) + 4 \lambda^3 \sigma_{N-1}^3 \sum_{i=1}^{5} \bar{A}_{2s}(N-1, 2) H_{2s}(X_\Delta) \Big] \Big\}$$

where $\rho(\Delta) =$ fraction of tesserae of $Q_{N-1} \setminus D_{N-1}$ which are contained in $\Delta$. To continue the iteration one would have liked to have $\rho(\Delta) \equiv 1$.

This replacement is possible exactly for the same reasons discussed in the $d = 1$ case (introducing an extra error of the same form of the one found (in that case) as long as $N > k_0(\lambda)$.

The result is that, for small values of $\lambda$, one can take

$$E_+ = C_{12}\,\lambda^4\,(\log\lambda)^{32} \tag{4.21}$$

where $C_{12}$ is a suitable positive constant.

The fact that the upper and lower bounds depend on $\lambda$ as $\lambda^4\,(\log\lambda)^{32}$ can be commented as we did comment the bounds found in the $d = 1$ case, at the end of sec. 2.

## 5. The infrared problem and the Markov hierarchical fields. Definitions

A conceptually simple and natural generalization of the Bernuolli hierarchical model are the Markov hierarchical models.

In these models $z_\Delta$ , $z_{\Delta'}$ are still assumed independent if $\Delta \in Q_N$ , $\Delta' \in Q_N$, $N \neq N'$; the variables $(z_\Delta)_{\Delta \in Q_N}$ are assumed to be gaussian variables with formal density:

$$\text{const}\,\exp -\frac{\beta}{2}\left(\sum_{(\Delta,\Delta')}(z_\Delta - z_{\Delta'})^2 + \alpha^2\sum_\Delta z_\Delta^2\right) \tag{5.1}$$

where $\sum_{(\Delta,\Delta')}$ runs over nearest neighbour pairs of tesserae of $Q_N$ and $\beta$ is so chosen that, given $\alpha > 0$ , $[z_\Delta^2] = \frac{1}{2}$ .

More precisely, if $\xi_\Delta =$ centre of $\Delta$, the variables $z_\Delta$ are gaussian randomi variables with covariance:

$$\overline{C}^{[N]}_{\Delta\Delta'} = \frac{1}{\beta}\,\frac{1}{(2\pi)^{d+1}}\int_{-\pi}^{\pi}\frac{e^{ik(\xi_\Delta - \xi_{\Delta'})l_N^{-1}}}{\alpha^2 + 2\sum_{i=1}^{d+1}(1-\cos k_i)}\,\mathrm{d}^{d+1}k\,. \tag{5.1}$$

We shall fix:

$$l_N = \alpha\omega^{-1}\,2^{-N} \qquad N = 0\,,\,1\,,\cdots \tag{5.3}$$

where $\omega$ is an arbitrary positive number, and define:

$$\gamma_0 = \beta\,l_0^{1-d} \quad , \quad \gamma^N = \beta\,(l_N^{1-d} - l_{N-1}^{1-d}) \qquad N = 1\,,\,2\,,\cdots \tag{5.4}$$

for $d > 1$ , or, for $d = 1$ , $\gamma_N \equiv \beta$.

The free parameter $\alpha$ shall be fixed so that, defining: $\varphi_\xi$ , $\varphi_\xi^{[\leq N]}$ , $\varphi_\xi^{[N]}$ as in (2.2), (2.3), (2.4), the covariance of the fields $\varphi_\xi$ resembles as much as possible to (2.1).

There is no real " best choice " of $\alpha$ but one could argue that if $\alpha$ is given, then the divergence fo $C_{\xi\eta}$ at the origin (i.e. for $|\xi - \eta| = x$ small) is like $x^{-1}$, $(d = 2)$, while the behaviour at infinity is, $(d = 2)$, like $(x^{-1}\exp - -\omega x)\,\beta^{-1}$; this would suggest to take $\alpha$ such that $\beta = 1$. On the other hand

the covariance of the above N-th frequency Markov hierarchical fields is "constant" over a scale $\alpha\omega^{-1}2^{-N}$ while the N-th frequency fields in the ordinary field theory can be regarded as constant on a scale $\ll \omega 2^{-N}$ and one would therefore like to take $\alpha$ small compared to 1.

In any case the techniques explained in the forthcoming sections apply only if $\alpha$ is large enough (but still $o(1)$ independently on $\omega$ and $\lambda$).

This is perhaps the real shortcoming of this work: we would like to say that this restriction on $\alpha$ is purely technical and that it could be eliminated by suitably modifying some details: however we feel that this limitation cannot be eliminated without changing the method.

We hope to be able to study the problems of the ultraviolet and infrared stability for arbitrary values of $\alpha$ with other more probabilistic techniques.

As before the probability distributions of $\varphi$, $\varphi^{[\leq N]}$, $\varphi^{[N]}$ will be denoted respectively $P_0$, $P_0^{(N)}$, $\hat{P}_0^{(N)}$.

We shall define $V_0^{(N)}(\varphi, \lambda)$ by (2.5) and the parameters $\sigma_N, \Gamma_N$ and the variables $X_\Delta$ are defined by (3.7), (3.5), (3.4) if $d = 1$ or by (4.1) if $d = 2$ where now $C_{\xi\eta}^{[\leq N]}$ denotes the new covariance of the field $\varphi_\xi^{[\leq N]}$:

$$C_{\xi\eta}^{[\leq N]} = \int P_0^{(N)}(d\varepsilon)\, \varphi_\xi^{[\leq N]} \varphi_\eta^{[\leq N]} \tag{5.5}$$

and I is a finite union of tesserae of $Q_0$.

If $d = 2$ we define $V^{(N)}(\varphi; \lambda)$ as in (2.13) where the various terms are defined by the formulae (2.14), (2.15), (2.16) with the new meaning for the symbol $C_{\xi\eta}^{[\leq N]}$.

It is clear that this problem cannot be trivially reduced to the case $|I| = 1$: in other words the Markov hierarchical model presents non trivial infrared problems.

It is perhaps useful to stress that what we are calling here "infrared problems" or "infrared divergences" are not those which are usually called with the same name and which are associated to zero mass particles. We could define models of hierarchical type which imitate this second type of divergences by just allowing the consideration of a family of compatible pavements without setting a limitation to the maximum size of the tesserae. We shall not consider such models, here; nevertheless we wish to keep the name of "infrared problems" to denote also any class of problems which arise for systems of large size.

The techniques used to solve our infrared problem are typical of Statistical Mechanics.

The ultraviolet problem, solved in the preceding sections in the Bernoulli case, should be thought as analogous to the H-stability problem of Statistical Mechanics which, as is well known, cannot be solved on general grounds: each model presents its own difficulties and is treated in a particular way (e.g. short range hard core systems, short range without hard core systems, Coulomb hard core systems, Coulomb systems, magnetic systems are studied, as far as H-stability is concerned, by ad hoc methods).

The infrared problem, in the case under consideration, corresponds to the thermodynamic limit, which, in Statistical Mechanics, can be studied under fairly general hypotheses.

The novelty of field theory is that there is no minimum scale of length; this means that already in finite volume one has to deal with a thermodynamic limit problem since the system contains subsystems which can be regarded as infinitely extended (actually there are infinitely many such systems! i.e. the $N^{th}$-frequency fields with $l_N^{-1}$ very large compared to $|I|^{1/d+1}$).

The treatment of the infrared problem, in spite of its apparent complexity, can be easily carried out [6, 7, 8, 11] by using well known methods of the theory of the rarefied gases at least if $\alpha$ is large enough (independently of the values of $\lambda$ and $\omega$) in the $d = 1$ or $d = 2$ cases.

The technique of the " rarefied polymers " needed here and resumed in a form suitable for our aims in the next section was developed in this form by Gruber and Kunz in a paper which is, perhaps, not known as widely as it would deserve [12] and, independently, by Glimm, Jaffe, Spencer in the context of field theory [6] as it appears from [11].

## 6. Spins and polymers

The reader should use this section only as a reference section to read the next one. The main result is Corollary 3.

Let $Z_1$ be the $(d + 1)$-dimensional lattice with unit step; we shall think of $Z_1$ as the set of the centres of the tesserae of a pavement of $R^{d+1}$ which we call Q.

Let M be a finite cube, with centre at the origin of $R^{d+1}$, exactly paved by Q.

To each $\xi \in M \cap Z_1$ we associate a random variable $S_\xi$ and we assume that these random variables are distributed with density:

$$(6.1) \qquad (\text{const}) \left[ \exp - \sum_{f \in \mathcal{F}_0 \cap M} \overline{H}_f(S_f) - \alpha^2 \sum_{\xi \in M} S_\xi^2 \right] \prod_{\xi \in M} \psi_\xi(S_\xi)$$

where $\mathcal{F}_0$ is the set of pairs of nearest neighbour tesserae contained in Q, and $\mathcal{F}_0 \cap M$ denotes the set of the elements of $\mathcal{F}_0$ whose cubes are contained in M; $\alpha > 0$ is a parameter, $(\psi_\xi)_{\xi \in M}$ is a family of arbitrary functions (measurable) with values in $[0, 1]$; furthermore, if $f = (\Delta, \Delta')$ and if $\xi_\Delta$ is the centre of $\Delta$, we denote:

$$(6.2) \qquad \overline{H}_f(S_f) = \varepsilon (S_{\xi_\Delta} - S_{\xi_{\Delta'}})^2$$

and we shall occasionally write $f = (\xi_\Delta, \xi_{\Delta'})$; here $\varepsilon > 0$ is a parameter.

We call $P_M$ the distribution of the $(S_\xi)_{\xi \in M}$ variables.

Given another family $(g_\xi)_{\xi \in M}$ of (measurable) functions of one variable we shall analyze expressions of the form:

$$\int \exp - \theta \sum_{\xi \in M} g_\xi (S_\xi) \, P_M (dS) \tag{6.3}$$

where $\theta$ is a positive parameter.

Let $\mathscr{F}$ be the family obtained by adding to the $\mathscr{F}_0$ just defined the family consisting of the single cubes $\Delta$ in Q. By $\mathscr{F} \cap M$ we mean the set of those elements of $\mathscr{F}$ whose cubes are in M and, as before if $\Delta \in Q$, $f = \Delta$, we shall sometimes write $f = \xi_\Delta$.

We put, if $f = \Delta \in \mathscr{F} \cap M$:

$$H_f (S_f) = \theta \, g_{\xi_\Delta} (S_{\xi_\Delta}) \tag{6.4}$$

and we write the above expectation in terms of the quantities:

$$Z_\theta = \int \left( \prod_{f \in \mathscr{F} \cap M} \exp - H_f (S_f) \right) \prod_{\xi \in M \cap Z_1} (\psi_\xi (S_\xi) \, e^{-\alpha^2 S_\xi^2} \, dS_\xi) \tag{6.5}$$

as $Z_\theta / Z_0$.

To show the simple structure of this formula, [12], write:

$$Z_\theta = \int \prod_{\xi \in M \cap Z_1} (\psi_\xi (S_\xi) \, e^{-\alpha^2 S_\xi^2} \, dS_\xi) \prod_{f \in \mathscr{F} \cap M} [1 + (e^{-H_f (S_f)} - 1)] =$$

$$= \sum_{F \subset \mathscr{F} \cap M} \int \left( \prod_{\xi \in M \cap Z_1} \psi_\xi (S_\xi) \, e^{-\alpha^2 S_\xi^2} \, dS_\xi \right) \prod_{f \in F} (e^{-H_f (S_f)} - 1) \tag{6.6}$$

where the sum runs over the subsets of $\mathscr{F} \cap M$.

Consider a set $F \subset \mathscr{F} \cap M$, $F = (f_1, f_2, \cdots, f_n)$; we say that a subset $R \subset F$, $R = (f_1', \cdots, f_{n'}')$, is a "connected component" of F if, given any two cubes $\Delta$, $\Delta'$ appearing in some of the sets of cubes which form R, there is a chain $(f_1'', \cdots, f_{n''}'') \subset R$ such that $\Delta \in f_1''$, $\Delta' \in f_{n''}''$ and $f_i''$ has a cube in common with $f_{i+1}''$ for $i = 1, \cdots, n'' - 1$ and, of course, R is a maximal subset of F with the described properties.

Call $\mathscr{R}$ the family of subsets of $\mathscr{F}$ which can appear as connected components of $F \subset \mathscr{F}$. As before $\mathscr{R} \cap M$ will the set of those elements of $\mathscr{R}$ whose cubes are contained in M.

There is a one-to-one correspondence between the subsets $(R_1, \cdots, R_n)$ such that $R_i \cap R_j = \varnothing$ if $i \neq j$, (i.e. such that none of the cubes appearing in the $R_i$ appears also in the $R_j$, $i \neq j$): and the subsets $F \subset \mathscr{F}$.

The above formula for $Z_\theta$ can therefore be rewritten as:

$$Z_\theta = \prod_{\xi \in M} \left( \int \psi_\xi (S) \, e^{-\alpha^2 S^2} \, dS \right) \cdot \sum_{\substack{R_1, \cdots, R_n \in \mathscr{R} \cap M \\ R_i \cap R_j = \varnothing, \, i \neq j}} \prod_{i=1}^{n} J_\theta (R_i) \tag{6.7}$$

where, if $R \in \mathscr{R} \cap M$ :

$$J_\theta(R) = \frac{\int \prod_{f \in R} (e^{-H_f(S_f)} - 1) \prod_{\xi \in R} (e^{-\alpha^2 S_\xi^2} \psi_\xi(S_\xi)\, dS_\xi)}{\int \prod_{\xi \in R} e^{-\alpha^2 S_\xi^2} \psi_\xi(S_\xi)\, dS_\xi} \tag{6.8}$$

whre $\xi \in R$ means that $\xi$ is the centre of one of the cubes forming at least one of the sets of cubes in R.

Notice that if $R = (f_1, f_2, \cdots, f_p, \bar{f}_1, \bar{f}_2, \cdots, \bar{f}_q)$, where the first $p$ $f$'s consist of just one cube and the remaining $q$ consist of two neighbouring cubes, we have:

$$| J_\theta(R) | \leq (\hat{b} b_1)^p (\hat{b} b_2)^q \tag{6.9}$$

where:

$$\hat{b} = \sup_\xi \frac{\int e^{-\frac{\alpha^2}{2} S^2} \psi_\xi(S)\, dS}{\int e^{-\alpha^2 S^2} \psi_\xi(S)\, dS} \tag{6.10}$$

$$b_1 = \sup_{\xi, S_\xi} \sup_{0 \leq \tau \leq 1} \left( \exp - \frac{\alpha^2}{2 \bar{\nu}_0} S_\xi^2 \right) | \exp - \tau\theta g_\xi(S_\xi) - 1 | \tag{6.11}$$

$$b_2 = \sup_{f \in \mathscr{F}_0 \cap M} \sup_{S_f, \tau \in [0,1]} \left( \exp - \frac{\alpha}{2 \bar{\nu}_0} (S_\xi^2 + S_\eta^2) \right) | \exp - \tau H_f(S_f) - 1 | \tag{6.12}$$

if $\bar{\nu}_0 = 2(d+1) + 1$ , $f \equiv (\xi, \eta)$.

We define, if $R \in \mathscr{R}$ is described as above:

$$| R | = p + q . \tag{6.13}$$

The above formulae show that $Z_\theta$ can be thought as the partition function of a gas of polymers [12].

Let us formulate some results of the theory of hard polymers [12].

Let $\mathscr{N}$ be the family of all the sets $\Gamma = (R_1, R_2, \cdots, R_n)$ of elements of $\mathscr{R}$ such that the union of the cubes in the R's form a connected set in $R^{d+1}$. Notice that we are considering here sets of overlapping polymers, even allowing two identical polymers to be part of the same $\Gamma$: hence the elements of $\mathscr{N}$ are not, in general, configurations of polymers, i.e. they cannot be regarded as subsets of $\mathscr{F}$.

The following basic lemma holds [12]:

LEMMA. *There exists a function* $\varphi^T$ *defined on* $\mathscr{N}$ *with real values, such that*:

i) $\varphi^T$ *is translationally invariant*

ii) *there is a constant* $z_0 \in [0, 1]$ *such that if* $|\Gamma| = \sum_{i=1}^{n} |R_i|$ *when* $\Gamma = (R_1, R_2, \cdots, R_n)$:

$$\sum_{\substack{\Gamma \ni 0 \\ \mathrm{diam}(\Gamma) > l}} z_0^{|\Gamma|} |\varphi^T(\Gamma)| < 2^{-l} \tag{6.14}$$

*where* diam $(\Gamma)$ = *diameter of the set consisting in the union of all the cubes appearing in the polymers which constitute* $\Gamma$.

iii) *if on* $\mathscr{R}$ *is defined a real valued function* $R \to z(R)$ *such that* $|z(R)| \leq z_0^{|R|}$, $\forall R \in \mathscr{R}$, *and if we set*

$$z(\Gamma) = \prod_{R \in \Gamma} z(R) \qquad \Gamma \in \mathscr{N} \tag{6.15}$$

*then*:

$$\sum_{\substack{R_1, \cdots, R_n \in \mathscr{R} \cap M \\ R_i \cap R_j = \emptyset, i \neq j}}^{n} \prod_{i=1} z(R_i) = \exp \sum_{\Gamma \subset M} z(\Gamma) \varphi^T(\Gamma) . \tag{6.16}$$

This nice theorem, or variants of it, have been repeatedly used in Statistical Mechanics [14, 15, 16, 17] and its proof is based on a idea of Ruelle [13]; it was recognized and proved in the above form by Gruber and Kunz [12].

The $\varphi^T$ are purely combinatorial numbers and an expression for them can be easily derived from the statement iii) of the above lemma or it can be found, for instance, in [16] (p. 180 (4.21)) where a different family $\mathscr{R}$ is considered (the formula, however, does not depend on the family $\mathscr{R}$ of the considered polymers) and the numbers $\mu$ appearing there should be thought as equal to zero.

If one looks at the proof of the lemma [12], one realizes that the statements i) and iii) are purely algebraic and that the heart of the theorem is contained in ii).

The above results can be used to obtain the following corollary:

COROLLARY 1. *Assume that* $b\hat{b}_1$, $b\hat{b}_2 < z_0$, *if* $g_\xi \equiv 0$ *for* $\xi \notin \Lambda \subset M$:

$$\int (\exp - \theta \sum_{\xi \in \Lambda} g_\xi) P_M(dS) = \exp \sum_{\substack{\Gamma \subset M \cap \mathscr{N} \\ \Gamma \cap \Lambda \neq \emptyset}} (\zeta_\theta(\Gamma) - \zeta_0(\Gamma)) \varphi^T(\Gamma) \tag{6.17}$$

*and the limit of this expression as* $M \to R^{d+1}$ *is obtained by eliminating the, restriction* $\Gamma \subset M \cap \mathscr{N}$ *in the last sum.*

This result is very useful to analyze the $\theta$-dependence of:

$$\int \left(\exp - \theta \sum_{\xi \in \Lambda} g_\xi\right) P_M(dS). \tag{6.18}$$

Notice first that from the explicit expressions for $J_\theta(R)$ it follows, developing the $(\exp - H_f(S_f) - 1)$ with the Taylor formula and setting:

$$D_k = \sup_{\xi \in M \cap z_1} \sup_{S, 0 \leq \tau \leq 1} e^{-\frac{\alpha^2}{2\bar{\nu}_0} S^2} |g_\xi(S)|^k e^{-\tau \theta g_\xi(S)} \tag{6.19}$$

$$E_k = \sum_{j=0}^{k+1} (j!)^{-1} D_j \tag{6.20}$$

that if $R = (f_1, \cdots, f_p, \bar{f}_1, \cdots, \bar{f}_q)$:

$$X_\theta(R) \equiv J_\theta(R) - J_0(R) = \tag{6.21}$$

$$= \sum_{h=p}^{k} (-\theta)^h \pi_h(R) (E_k)^h (b_2 \hat{b})^q \hat{b}^p + (-\theta)^{k+1} \pi_{k+1}(R) (E_k)^{k+1} (b_2 \hat{b})^q \hat{b}^p (1 - \theta E_k)^{-1}$$

and $\pi_h(R)$ are $\theta$-independent for $0 \leq h \leq k$; $|\pi_h(R)| \leq 1$, $0 \leq h \leq k+1$.

This last relation easily implies that:

$$J_\theta(\Gamma) - J_0(\Gamma) = \sum_{i=\bar{p}}^{k} (-\theta)^i \pi_i'(\Gamma) (4 E_k)^i (b_2 \hat{b})^{\bar{q}} \hat{b}^{\bar{p}} + \tag{6.22}$$

$$+ (-\theta)^{k+1} \pi_{k+1}^1(\Gamma) (4 E_k (1 - \theta E_k)^{-1})^{k+1} (1 - 4 E_k (1 - \theta E_k)^{-1})^{-1}) b_2 \hat{b})^{\bar{q}} \hat{b}^{\bar{p}}$$

where $\pi_h'$ are $\theta$-independent for $0 \leq h \leq k$ and $|\pi_h'| \leq 1$ for $0 \leq h \leq k+1$ and $\bar{p} = \sum_{i=1}^{n} p_i$, $\bar{q} = \sum_{i=1}^{n} q_i$ if $\Gamma = (R_1, \cdots, R_n)$ and $R_i$ contains $p_i$ elements of $\mathscr{F}$ consisting of just one cube and $q_i$ consisting of two cubes, $i = 1, 2, \cdots, n$, as it follows from the expression:

$$\prod_{i=1}^{n} J_0(R_i) - \prod_{i=0}^{n} J_0(R_i) = \sum_{L \subset \{1,2,\cdots,n\}} \left( \prod_{i \notin L} J_0(R_i) \right) \left( \prod_{i \in L} X_\theta(R_i) \right) \tag{6.23}$$

We have thus proven the following Corollary:

COROLLARY 2. *Suppose that*:

$$b_2 \hat{b} < \tfrac{1}{2} z_0, \quad \frac{4 \theta E_k}{1 - \theta E_k} \hat{b} < \tfrac{1}{2} z_0 \tag{6.24}$$

*then*:

$$\int \exp - \theta \sum_{\xi \in \Lambda} g_\xi(S_\xi) P_M(dS) = \exp \sum_{i=1}^{k} \theta^i \frac{\delta(i)}{i!} + \theta^{k+1} \frac{\delta(\theta, k+1)}{(k+1)!} \tag{6.25}$$

*where* $\delta(i), \cdots, \delta(k)$ *are* $\theta$*-independent quantities and*:

$$|\delta(i)| \leq i! (E_k)^i |\Lambda| \qquad i = 1, 2, \cdots, k \tag{6.26}$$

$$|\delta(\theta, k+1)| \leq (k+1)! (E_k)^{k+1} |\Lambda|. \tag{6.27}$$

A slightly more accurate calculation would show that, if there are constants $\tilde{D}$ , $\rho$ such that $|D_j| \leqq \tilde{D}^j (j!)^\rho$, then there is a constant $\tilde{\delta}$ such that:

$$|\delta(j)| \leq j!\,(j!)^\rho\,\tilde{\delta}^j\,|\Lambda| \tag{6.28}$$

which is a well known result of formal perturbation theory.

Let us now observe that if one defines the $n^{\text{th}}$-order "cumulant" of a generic random variable G as:

$$\langle G^n \rangle^T = \frac{d^n}{d\theta^n} \log \langle \exp G\theta \rangle\,|_{\theta=0} \tag{6.29}$$

where $\langle \cdot \rangle$ means average with respect to the distribution of G; then the above corollary can be rephrazed as:

COROLLARY 3. *The coefficients* $\delta(1), \cdots, \delta(k)$ *appearing in the above Corollary 2 are the cumulants of the random variable* $G = -\sum_{\xi \in \Lambda} g_\theta$ *with respect to the distribution* $P_M$.

Hence:

$$\delta(1) = \langle G \rangle\,;\ \delta(2) = \langle G^2 \rangle - \langle G \rangle^2\,;\ \delta(3) = \langle G^3 \rangle - 3\,\langle G^2 \rangle\,\langle G \rangle + 2\langle G \rangle^3 \tag{6.30}$$

Another interesting implication of the above theorems is the possibility of comparing the expectations of $\exp -\theta \sum_{\xi \in \Lambda} g_\xi$ with respect to the probability distributions $P_M$ obtained by choosing $\psi_\xi \equiv 1$ or $\psi_\xi(s) = \chi(|s| < B)$.

Denoting $\tilde{P}_{(\infty)}$ and $\tilde{P}_{(B)}$ such distributions and denoting $J_\theta^{(\infty)}$ and $J_\theta^{(B)}$ the polymer's activities computed with respect to $\tilde{P}_{(\infty)}$ and $\tilde{P}_{(B)}$ one finds:

$$\frac{\langle \exp -\theta \sum_{\xi \in \Lambda} g_\xi \rangle_{\tilde{P}_{(B)}}}{\langle \exp -\theta \sum_{\xi \in \Lambda} g_\xi \rangle_{\tilde{P}_{(\infty)}}} = \tag{6.31}$$

$$= \exp \sum_{\substack{\Gamma \cap \Lambda \neq \emptyset \\ \Gamma \in \mathcal{N} \cap M}} \left( J_\theta^{(B)}(\Gamma) - J_\theta^{(\infty)}(\Gamma) \right) \varphi^T(\Gamma).$$

A simple calculation shows that under the hypotheses of Corollary 2, the relations discussed so far in this section imply:

COROLLARY 4. *Suppose that*

$$\hat{b} b_2 < \frac{z_0}{2} 4 \quad , \quad \theta\, E_k\,(1 - \theta\, E_k)^{-1}\,\hat{b} < \frac{z_0}{2} \tag{6.32}$$

*then there exist constants* $C_{14}$ , $C_{15}$ *such that if* $B > 1$ *(say)*:

$$\left| \left( \frac{\langle \exp -\theta \sum_{\xi \in \Lambda} g_\xi \rangle_{\hat{P}_{(B)}}}{\langle \exp -\theta \sum_{\xi \in \Lambda} g_\xi \rangle_{\hat{P}_{(\infty)}}} \right)^{1/|\Lambda|} - 1 \right| = \exp O_{14}\,(\exp -\alpha^2\, C_{15}\, B^2) \tag{6.33}$$

*Proof.* It is enough to stress the main point in the proof. First one remarks that there are constants $C'_{15}$, $C_{15}$ such that when $|\Gamma| < \exp + C_{15}\,\alpha^2\,B^2$ it follows:

$$|\,J_\theta^{(B)}(\Gamma) - J_\theta^{(\infty)}(\Gamma)\,| \leq C'_{15}\,e^{-\alpha^2 C_{15} B^2}\,z_0^{|\Gamma|}. \tag{6.34}$$

which is a consequence of the definitions of the activities. Otherwise, if $\Gamma$ does not verify the above limitation one has, always:

$$|\,J_\theta^{(B)}(\Gamma)\,|\,,\ |\,J_\theta^{(\infty)}(\Gamma)\,| < z_0^{|\Gamma|} \tag{6.35}$$

These two remarks and property ii) of the Lemma easily imply our corollary.

Another interesting consequence of the above theory is the following corollary whose proof is left to the reader:

COROLLARY 5. *There exists a constant* $C_{16}$ *and another constant* $C_{17}$ *such that if*

$$\hat{b} b_2 < \frac{z_0}{2} \quad , \quad 4\,\theta E_k\,(1 - \theta E_k)^{-1}\,\hat{b} < \frac{z_0}{2} \tag{6.36}$$

*and if* $B > 1$ *(say), then*:

$$J(\Lambda, \Lambda') \equiv \int \prod_{\xi \in \Lambda} \chi(|S_\xi| > B) \prod_{\xi \in \Lambda' \setminus \Lambda} \chi(|S_\xi| < B)\,\tilde{P}_\infty(dS) \leq \tag{6.37}$$

$$\leq (\exp - \alpha^2 C_{16} B^2 |\Lambda|)\,(1 - \exp - \alpha^2 C_{16} B^2)^{|\Lambda'| - |\Lambda|}$$

*and*

$$J(\Lambda, \Lambda') \geq (\exp - \alpha^2 C_{17} B^2 |\Lambda|)\,(1 - \exp - \alpha^2 C_{17} B^2)^{|\Lambda'| - |\Lambda|} \tag{6.38}$$

*which holds for all pairs* $\Lambda$, $\Lambda'$, $\Lambda \subset \Lambda' \subset M$.

We conclude this section by stressing that the interest (and the non triviality) of the results of this section consists in their independence on the size of M; actually if one defines the limit distribution, as $M \to R^{d+1}$, of $P_M$ in a suitable sense, all the above results hold if $P_M$ is replaced by its limit as $M \to R^{d+1}$. Whatever reasonable definition one takes to define the above limit of $P_M$ one finds that it exists if $\psi_\xi \equiv 1$ for all but finitely many $\xi \in Z_1$.

## 7. THE MARKOV HIERARCHICAL STRINGS AND MEMBRANES

We use the notations of sec. 2, 3, 4 and suppose $\lambda < 1/2$, $\omega = 1$. The arguments that follow could be easily modified to work for arbitrary values of $\lambda$ and $\omega$ at the expense of having to deal with more cumbersome formulae; we shall not discuss this point.

In the course of the argument the parameter $\alpha$ will be fixed to be larger than a certain value $\alpha_0$ defined as a number such that for all $\alpha \geq \alpha_0$ :

$$(7.1) \qquad b_2 = \max_{s_1, s_2} e^{-\frac{\beta\alpha^2}{2\bar{\nu}_0}} (1 - e^{-\frac{1}{2}\beta(s_1^2 - s_2)^2}) \equiv O(\alpha^{-2}) < \frac{\tilde{z}_0}{2^2}$$

where $\bar{\nu}_0 = 2(d+1)+1$ and $\tilde{z}_0 < z_0$ is the numerical constant appearing in the Lemma of the next section.

The value of $\alpha_0$, if explicitely computed, turns out to be large (though $\lambda$ and $\omega$ independent), too large to allow stating that the hierarchical models associated to such values of $\alpha$ really imitate the euclidean free field (cfr. comment at p. 38).

Consider first the lower bound estimate in the $d = 1$ case. We have:

$$(7.2) \qquad \int \exp V_0^{(N)} P_0^{(N)}(dz) \geq \int \exp V_0^{(N)} \chi_B^{(N)} P_0^{(N)}(dz) =$$

$$= \int \chi_B^{(N-1)} P_0^{(N-1)}(dz) \int \hat{P}_0^{(N)}(dz) \prod_{\Delta \in Q_N \cap I} \chi(|X_\Delta| < B_N) \exp -\lambda\sigma_N (1+N)^2 H_4(X_\Delta).$$

We proceed as in the Bernoulli case using the corollaries of sec. 5 to evaluate the integrals which, in the Bernoulli case could be easily estimated by explicit calculations.

Let:

$$(7.3) \qquad g_\Delta(z_\Delta) = H_4(X_\Delta) = H_4\left(\frac{z_\Delta + \sqrt{N}\, X_{\Delta'}}{\sqrt{1+N}}\right)$$

$$\psi_\Delta(z_\Delta) = \chi(|X_\Delta| < B_N) \qquad \theta = \lambda\sigma_N (1+N)^2$$

then there exist $C_{18}, C_{19}$ such that:

$$\hat{b} \leq \sqrt{2} \quad , \quad \hat{b} b_2 < \frac{z_0}{2},$$

$$(7.4) \qquad D_l \leq C_{18} B_{N-1}^{4l} \qquad l = 1, 2,$$

$$b_1 \leq E_1 \leq C_{19} B_{N-1}^8$$

using the symbols of sec. 5 and recalling that we have to consider only the cases $|X_{\Delta'}| < B_{N-1}$, $|X_\Delta| < B_{N-1}$, and that one has $\sigma_N B_N^8 \to 0$ as $N \to \infty$.

Hence if $\lambda$ is small enough all the assumptions of the corollaries of the previous section are verified and we can write the integral over $\hat{P}_0^{(N)}(dz)$ as:

$$(7.5) \qquad \left(\int \hat{P}_0^{(N)}(dz) \exp -\lambda\sigma_N (1+N)^2 \sum_{\Delta \in Q_N \cap I} H_4(X_\Delta)\right) \exp |I| l_N^{-2} O_{14}(e^{-C_{15} B_0^2 N^2})$$

using Corollary 4, sec. 5. Hence: using Corollary 3 of sec. 5:

$$\int \exp V_0^{(N)}\, \chi_B^{(N)}\, P_0^{(N)}(dz) \geq \Big(\exp -|I|\, C_{14}\, e^{-C_{15} B_0^2 N^2 l_N^{-2}} \int P_0^{(N-1)}(dz)\, \chi_B^{(N-1)} (\exp -\lambda\sigma_N (1+N)^2 \cdot \cdot \sum_{\Delta \in Q_N \cap I} \langle H_4(X_\Delta)\rangle_{\hat{P}_0^{(N)}} - \lambda^2 \sigma_N^2 (1+N)^2 E_1^2\, 2\, l_N^{-2} |I|\Big). \tag{7.6}$$

Since:

$$(\sigma_N (1+N))^2 \sum_{\Delta \in Q_N \cap I} \langle H_4(X_\Delta)\rangle_{\hat{P}_0^{(N)}} \equiv 4\, \sigma_N N^2 \sum_{\Delta' \in Q_{N-1} \cap I} H_4(X_{\Delta'}) \tag{7.7}$$

we can continue inductively and we find the result that there is $C_{20}$ such that if $\lambda$ is small enough one can take:

$$E_- = -C_{20}\, \lambda^2 (\log \lambda)^{16} |I|. \tag{7.8}$$

Let us remark that, as in the Bernoulli case, the interest of the above calculation lies only in the method, which can be used to find an upper bound along the same lines of the Bernoulli case using the Corollaries 4,5 of the preceding section to estimate the integrals that previously were explicitely treated. The argument based on the Jenssen's convexity inequality would trivially give the lower bound $E_- = 0$.

The next step is to find a lower bound in the $d = 2$ case.

We take:

$$\psi_\Delta(z_\Delta) = \chi(|X_\Delta| < B_N) \qquad \theta = \lambda\sigma_N \tag{7.9}$$

$$g_\Delta(z_\Delta) = 4\, H_4\left(\frac{z_\Delta + X_{\Delta'}}{\sqrt{2}}\right) + \lambda\sigma_N\, \mu_N(\Delta)\, 2\, H_2\left(\frac{z_\Delta + X_{\Delta'}}{\sqrt{2}}\right) + \tag{7.10}$$

$$+ \lambda\sigma_N\, \nu_N(\Delta) - \lambda^3 \sigma_N^2\, \nu'_N(\Delta) \qquad \Delta \in Q_N \cap I$$

the $\mu_N(\Delta)$, $\nu_N(\Delta)$, $\nu'_N(\Delta)$ are no longer $\Delta$-independent, as in the Bernoulli case, but they depend (only slightly if $\alpha$ is large) on the position of $\Delta$ in I:

$$\nu_N(\Delta) = \frac{3}{4} l_N^{-6} \gamma_N^{-4} \int_{\Delta \times I} d\xi\, d\eta\, C_{\xi\eta}^{[\leq N]4} \tag{7.11}$$

$$\nu'_N(\Delta) = \frac{3^3}{3!} l_N^{-9} \gamma_N^{-6} \int_{\Delta \times I \times I} d\xi\, d\eta\, d\zeta\, C_{\xi\eta}^{[\leq N]2}\, C_{\eta\zeta}^{[\leq N]2}\, C_{\zeta\xi}^{[\leq N]2} \tag{7.12}$$

$$\mu'_N(\Delta) = 2 \cdot 6\, \gamma_N^{-4} l_N^{-6} \int_{\Delta \times I} d\xi\, d\eta\, C_{\xi\eta}^{[\leq N]3}. \tag{7.13}$$

Simple considerations show that there exists $C_{21}$ such that for all $N \geq 0$;

$$\nu_N(\Delta) \leq C_{21} \qquad \nu'_N(\Delta), \mu_N(\Delta) \leq C_{21}(1+N) \tag{7.14}$$

which, after some meditation, can be seen to rightly correspond to the expected linear and, respectively, logaritmic divergences of the $d = 2$ theories.

Furthermore there are constants $C_{22}, C_{23}$ such that if $|X_\Delta| < B_{N-1}$, $\Delta \in Q_{N-1}$:

$$\begin{aligned} &\hat{b} \leq \sqrt{4} \quad , \qquad \hat{b} b_2 < \frac{z_0}{2}, \\ &D_2 \leq C_{22} N^l B_{N-1}^{4l} \qquad\qquad l = 1, 2, 3, 4, \\ &E_3 \leq C_{23} N^{52} (\log \lambda)^{32} \end{aligned} \tag{7.15}$$

therefore, if $\lambda$ is small enough:

$$4 E_3 \theta (1 - E_3 \theta) < \frac{z_0}{2}. \tag{7.16}$$

Hence we can apply the corollaries of sec. 5, as in the former $d = 1$ case. Assuming $\lambda$ small enough the result is:

$$\begin{aligned} &\int \exp V^{(N)} P_0^{(N)}(dz) \geq \int (\exp V^{(N)}) \chi_B^{(N)} P_0^{(N)}(dz) \geq \\ &\geq (\exp -|I| C_{24} e^{-C_{25} B_2^0 N^2} l_N^{-3}) \cdot \int \chi_B^{(N-1)} P_0^{(N-1)}(dz) \cdot \\ &\exp\left[\langle V^{(N)}\rangle + \frac{\langle V^{(N)2}\rangle - \langle V^{(N)}\rangle^2}{2!} + \frac{\langle V^{(N)3}\rangle - 3\langle V^{(N)2}\rangle\langle V^{(N)}\rangle + 2\langle V^{(N)}\rangle^3}{3!}\right] \cdot \\ &\exp O_{26}(\lambda^4 \sigma_N^4 N^{52} (\log \lambda)^{32} l_N^{-3}) |I| \end{aligned} \tag{7.17}$$

which immediately follows from the Corollaries 2, 3, 4 of sec. 5, if the symbols $\langle \cdot \rangle$ are interpreted as expectations with respect to the measure $\hat{P}_0^{(N)}(dz)$.

The expectations involved in (7.17) can be easily evaluated, with some patience, since they are gaussian integrals. To express the results it turns out that it is very useful to introduce a generalization of the symbol $:x^n:$ where $x$ is a gaussian random variable, defined in sec. 2. We shall consider $k$ gaussian random variables (not necessarily independent) $x_1, x_2, \cdots, x_k$ and denote, given $k$ non negative integers $n_1, \cdots, n_k$:

$$:x_1^{n_1} \cdots x_k^{n_k}: \tag{7.18}$$

a polynomial in $x_1, \cdots, x_k$ which can be recursively defined so that

$$:\left(\sum_{i=1}^{k} \lambda_i x_i\right)^n: = \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1! \cdots n_k!} \lambda_1^{n_1} \cdots \lambda_k^{n_k} :x_1^{n_1} \cdots x_k^{n_k}: \tag{7.19}$$

for all the $\lambda_1, \cdots, \lambda_k$ real and the $n$'s (integer), or, better, as the polynomial such that in the evaluation with the gaussian integral contraction rule (Wick's theorem [22]) of quantities like:

$$\mathrm{E}\,(:x_1^{n_1}\cdots x_k^{n_k}: z_1 \cdots z_s) \tag{7.20}$$

(where the $z_i$'s are other gaussian random variables) no graphs with contractions among the lines representing $x_1, \cdots, x_k$ occur and, furthermore, the polynomial that we are defining is constrained to have the coefficient of the monomial of highest degree equal to 1.

Define, for $\Delta, \Delta' \in Q_{N-1}$:

$$\delta^{(i)}_{\Delta\Delta'}(N) = \sum_{\substack{\bar\Delta \subset \Delta, \bar\Delta \in Q_N \\ \bar\Delta' \subset \Delta', \bar\Delta' \in Q_N}} 2^{-i}\,[(\overline{C}^{[N]}_{\bar\Delta\bar\Delta'} + \overline{C}^{[\leq N-1]}_{\Delta\Delta'})^i - \overline{C}^{[\leq N-1]i}_{\Delta\Delta'}] \tag{7.21}$$

where $\overline{C}^{[N]}_{\bar\Delta\bar\Delta'}$ and $\overline{C}^{[\leq N-1]}_{\Delta\Delta'}$ denote the covariances:

$$\begin{aligned} &\int z_{\bar\Delta}\, z_{\bar\Delta'}\, \hat{P}^{(N)}_0\,(dz) && \bar\Delta, \bar\Delta' \in Q_N \\ &\int X_\Delta\, X_{\Delta'}\, P^{(N)}_0\,(dz) && \Delta, \Delta' \in Q_{N-1} \end{aligned} \tag{7.22}$$

respectively. We define also:

$$W^{(N-1)} = \lambda^2 \sigma_N^2 \sum_{\Delta,\Delta' \in Q_{N-1}} \frac{4^2\, 6^2\, 2}{2^3}\, \delta^{(2)}_{\Delta\Delta'}(N) : X^2_\Delta\, X^2_{\Delta'} : \tag{7.23}$$

then the expression appearing under square bracket in (7.17) can be written:

$$\begin{aligned} &V^{(N-1)} + W^{(N-1)} - \int V^{(N-1)}\, W^{(N-1)}\, P^{(N)}_0\,(dz) + \\ &+ \lambda^2 \sigma_N^2 \sum_{\Delta,\Delta' \in Q_{N-1}} \sum_{i=1,3} A_{4-i}\, \delta^{(i)}_{\Delta\Delta'}\,(: X^{4-i}_\Delta\, X^{4-i}_{\Delta'} : - \delta_{i,3} : X^2_\Delta :) + \\ &+ \lambda^3 \sigma_N^3 \sum_{\Delta,\Delta' \in Q_{N-1}} \sum_{i=1,2} \delta^{(i)}_{\Delta\Delta'}\, A^1_{4-i}\, \mu_N(\Delta') : X^{4-i}_\Delta\, X^{2-i}_{\Delta'} : + \\ &- \lambda^3 \sigma_N^3 \sum_{\Delta,\Delta',\Delta'' \in Q_{N-1}} \sum_{\substack{0<i_1+i_2+i_3<12 \\ 0\leq i_\cdot \leq 4}} \delta^{(i_1 i_2 i_3)}_{\Delta\Delta'\Delta''} : X^{4-i_1}_\Delta\, X^{4-i_2}_{\Delta'}\, X^{4-i_3}_{\Delta''} \end{aligned} \tag{7.24}$$

where $\delta^{(i_1 i_2 i_3)}_{\Delta\Delta'\Delta''}$ are suitable coefficients, described later, and the A's are some numbers (e.g. $a_1 = 4^3\, 6$, $A_2 = 4\, 6^2$ etc.).

The relevant properties of the coefficients $\delta^{\{\cdots\}}_{\{\cdots\}}$ are their decay properties when two of the tesserae which appear in their arguments separate.

The relevant property of the A's is their N-independence: their actual values are totally irrelevant.

Before stating the decay properties of the $\delta$ coefficients it is convenient to collect the terms of (7.24) so that the sums over the pairs and triples of tesserae run only over pairs and triples of distinct tesserae.

The result of this rearrangement of the terms of (7.24) is that it can be written as:

$$V^{(N-1)} + W^{(N-1)} - \langle V^{(N-1)} W^{(N-1)} \rangle_{(0)} + E^{(N-1)} \tag{7.25}$$

where the symbol $\langle\ \rangle_{(0)}$ denotes the integration with respect to $P_0^{(N)}$, i.e. with respect to all the $z$'s of any order, and:

$$\begin{aligned} E^{(N-1)} = {} & \lambda^2 \sigma_N^2 \sum_{\Delta \in Q_{N-1} \cap I} \overline{\delta}_\Delta (N) : X_\Delta^6 : + \\ & + \lambda^2 \sigma_N^2 \sum_{\substack{\Delta, \Delta' \in Q_{N-1} \cap I \\ \Delta \neq \Delta}} \sum_{i=1,3} A_{4-i}\, \delta_{\Delta\Delta'}^{(i)} (: X_\Delta^{4-i} X_{\Delta'}^{4-i} : - \delta_{i,3} : X_\Delta^2 :) + \\ & + \lambda^3 \sigma_N^3 \sum_{\Delta \in Q_{N-1} \cap I} \sum_{i=1}^{5} \overline{\delta}_\Delta^{(j)'} (N) : X_\Delta^{2j} : + \\ & + \lambda^3 \sigma_N^3 \sum_{\substack{\Delta, \Delta' \in Q_{N-1} \cap I \\ \Delta \neq \Delta'}} \sum_{\substack{0 < i+j=2,6 \\ j, i > 0}} \overline{\delta}_{\Delta\Delta'}^{(ij)} (: X_\Delta^j X_{\Delta'}^j : - \delta_{i,j}\, \delta_{i,1} : X_\Delta^2 :) \\ & + \lambda^3 \sigma_N^3 \sum_{\substack{\Delta, \Delta', \Delta'' \in Q_{N-1} \cap I \\ \Delta \neq \Delta' \neq \Delta'' \neq \Delta}} \sum_{\substack{2 \le i+j+k < 12 \\ 0 < i, k, j}} \overline{\delta}_{(\Delta\Delta'\Delta'')}^{(ijk)} : X_\Delta^i X_{\Delta'}^j X_{\Delta''}^k : . \end{aligned} \tag{7.26}$$

The properties of the coefficients $\overline{\delta}$ are the following: they may depend on $\lambda$ and I but there exist constant $C_{27}$, $C_{28}$ such that:

$$|\overline{\delta}_\Delta^{(j)}|\ ,\ |\overline{\delta}_\Delta^{(j)'}| \le C_{27} (1 + N) \qquad \forall j \tag{7.27}$$

$$|\overline{\delta}_{\Delta\Delta'}^{(ij)}|\ ,\ |\overline{\delta}_{\Delta\Delta'}^{(i)}| \le (N+1)\, C_{27} \exp - C_{28}\, l_N^{-1}\, d(\xi_\Delta, \xi_{\Delta'}) \quad \forall i,j, \forall \Delta, \Delta' \in Q_{N-1}$$

(7.28)

$$|\overline{\delta}_{\Delta'\Delta\Delta''}^{(ijk)}| \le (N+1)\, C_{27} \exp - C_{28}\, l_N^{-1} d(\xi_\Delta, \xi_{\Delta'}, \xi_{\Delta''}) \quad \forall i,j,k, \forall \Delta, \Delta', \Delta'' \in Q_{N-1}$$

where $d(\xi_1, \cdots, \xi_p)$ denotes, in general, the length of the shortest graph containing $\xi_1, \cdots, \xi_p$ and connecting these points ("graph distance of $(\xi_1, \cdots, \xi_p)$").

To continue we would like to integrate the function:

$$\exp [V^{(N-1)} + W^{(N-1)} - \langle W^{(N-1)} V^{(N-1)} \rangle_{(0)} + E^{(N-1)}] \tag{7.29}$$

with respct to $\hat{P}_0^{(N-1)}$ by using the "cumulant rule", as we just did. However we have to prove that this is possible since we cannot use any more the theory of the hard polymers of sec. 5 because the functions $W^{(N-1)}$ and $E^{(N-1)}$ are no longer "local", i.e. they are not of the form $\sum_\Delta g_\Delta (z_\Delta)$.

We shall therefore generalize the content and results of sec. 5 to deal with expectations of more general quantities (like those in which we are interested.) This will be done in the next section.

If we believe, for the moment, that the cumulant formula can be used also in the present case we can easily perform the successive integrations with respect to $\hat{P}_0^{(N)}, \hat{P}_0^{(N-1)}, \dots, \hat{P}_0^{(N-k+1)}$ and the result of this formal calculation can be expressed, after some simple algebra, as:

$$(7.30)\quad \exp\left[V^{(N-k)} + \sum_{h=1}^{k} \left(W^{(N-h,k-h)} - \langle W^{(N-h,k-h)} V^{(N-k)}\rangle_{(0)}\right) + \sum_{h=1}^{k} E^{(N-h,k-h)}\right] \cdot$$
$$\cdot \exp \sum_{h=0}^{k-1} \text{“}O_{29}\text{”}\, (\lambda^4 \sigma_{N-k+h}^4 (N-k+h)^{52} (\log \lambda)^{32} l_{N-k+h}^{-3}) \mid I \mid$$

where “$O_{29}$” recalls that the calculation is formal (i.e. we do not control the error which, however, we hope to be of the indicated size), and

$$(7.31)\quad W^{(N-h,k-h)} = \int W^{(N-h)} \hat{P}_0^{(N-h)}(dz) \cdots \hat{P}_0^{(N-k+1)}(dz) \qquad h < k$$
$$W^{(N-h,0)} \equiv W^{(N-h)}$$

and $E^{(N-h,k-h)}$ is given by an expression similar to that of $E^{(N-h)}$ with the coefficients $\bar{\delta}(N-h)$ replaced by new coefficients $\bar{\delta}(N-h, k-h)$ which verify the relations: $\exists C'_{29}$ such that:

$$(7.32)\quad \mid \bar{\delta}^{\{\dots\}}(N-h, k-h) \mid \leq C'_{29}\, 2^{-\frac{n}{2}(k-h)}\, \bar{\delta}^{\{\dots\}}(N-h)$$

where $n$ is the degree of the Wick product to which $\bar{\delta}^{\{\dots\}}$ is the coefficient. Furthermore to obtain $E^{(N-h,k-h)}$ from $E^{(N-h)}$ the variables $X_\Delta, \Delta \in Q_{N-k}$ should be replaced by $X_{\bar{\Delta}}, \bar{\Delta} \in Q_{N-k}$, where $\bar{\Delta} \supset V$.

If the above cumulant formula could be really applied $k$ times with a rigorous estimate of error, like the one guessed in the above formula, we could iterate N times and find exactly as in the Bernoulli case a lower bound of the form:

$$(7.33)\quad E_- = -C_{29}\, \lambda^4 (\log \lambda)^{32} \sum_{N=0}^{\infty} \sigma_N^4 N^{52} l_N^{-3}$$

for $\lambda$ small enough.

Before explaining in the next section the simple extension of the polymer technique which allows to use the cumulant formula with the wanted error estimate thereby allowing to verify recursively, as we did in the Bernoulli case, the formula (7.30) and to explicitely evaluate the numerical constant $c_{29}$, let us informally discuss the only new problem which appear with respect to the Bernoulli case. It comes from the term in each $E^{(\bar{N}-k)}$ of the form:

$$(7.34)\quad \sigma_{\bar{N}}^2 \sum_{\Delta \neq \Delta' \in Q_{\bar{N}}} \bar{\delta}^{(2)}_{\Delta,\Delta'}(\bar{N}, k)\, (: X_{\bar{\Delta}} X_{\bar{\Delta}'} : - : X_{\bar{\Delta}}^2 :)$$

where $\bar{\Delta}, \bar{\Delta}' \in Q_{\bar{N}-k}$ and $\bar{\Delta} \supset \Delta, \bar{\Delta}' \supset \Delta'; \Delta, \Delta' \in Q_{\bar{N}}$. This term was missing in the Bernoulli case since $\bar{\delta}^{(2)}_{\Delta,\Delta'}$ was identically zero when $\Delta \neq \Delta'$.

In the present situation, if one does not take explicitely into account that $: X_{\bar{\Delta}} X_{\bar{\Delta}'} : \equiv : X_{\bar{\Delta}}^2 :$ if $\bar{\Delta} = \bar{\Delta}'$ even though $\Delta \neq \Delta'$ one could only estimate

the value of $E^{(\bar{N},k)}$ as

$$\sigma_{\bar{N}}^2 \, 2^{-k} \, 2^{3\bar{N}} \, B_{\bar{N}-k}^3 \, |I| \tag{7.35}$$

which at the end of the iterations ($k = \bar{N}$) would be $o(1)\,|I|$ which is not enough for our purposes because each frequency index would contribute the same $o(1)$ to $E_-$ thus making it diverge with N as $\sum_{\bar{N}=0}^{N} o(1) = o(1)\,N$.

However the cancellation just mentioned allows to bound $E^{(N,k)}$ with the much better quantity:

$$\sigma_{\bar{N}}^2 \, 2^{-k} \, 2^{3(\bar{N}-k)} \, |I| \, 2^{2k} \, B_{\bar{N}-k}^2 \tag{7.36}$$

because only the tesserae ($2^{2k}$ in number) which are close to the boundaries of the $2^{3(\bar{N}-k)}\,|I|$ cubes of $Q_{N-k}$ really count as a consequence of the decay properties of the $\bar{\delta}$ functions.

The reader might also be warried by the high power polynominals which appear in the exponential to be integrated (via the W's and E's); this is not a problem because the integrations run on domains where the arguments of the polynomials are bounded (because of the characteristic functions): even in the Bernoulli case we could have written the recursive hypothesis replacing the products (cfr. (4.15)) with the exponentials of suitable sums, (for all values of the iteration parameter $k$ such that $N - k \geq k_0(\lambda)$, cfr. (4.16), if we applied the techniques that we are using to the Bernoulli case we would realize that the restrictions to be put on $\lambda$ to be able to use the cumulant formula discussed above imply that $k_0(\lambda)$ is zero).

Let us formulate the problem of checking the cumulant formula: let $\mathscr{F}$ be the set of all the pairs of nearest neighbour tesserae of $Q_{N-1}$ as well as the set of all the tesserae, pairs of tesserae, triples of tesserae appearing in the sums expressing $E^{(N-1)}$, $W^{(N-1)}$ subjected to the further restriction that we consider only pairs $\Delta, \Delta'$ or triples $\Delta, \Delta', \Delta''$ such that

$$d(\xi_\Delta, \xi_{\Delta'}) \quad , \quad d(\xi_\Delta, \xi_{\Delta'}, \xi_{\Delta''}) \leq \tau_N = C_{28}^{-1} N (C_{31} + (\log \lambda)^2) \tag{7.37}$$

where the constant $C_{31}$ will be chosen shortly.

Then write:

$$V^{(N-1)} + W^{(N-1)} + E^{(N-1)} = \sum_{f \in \mathscr{F}} H_f(z_f) + \tag{7.38}$$

$$+ \text{(contributions coming from the missing terms)}$$

and $H_f(z_f)$ represents the contribution of the addend with index $f$ in the sum over sets of 1, 2, 3 tesserae which expresses $V^{(N-1)}+W^{(N-1)}+E^{(N-1)}$; the "contributions by the missing terms" are given by the sum over the pairs and triples which appear in the sum expressing $V^{(N-1)}+W^{(N-1)}+E^{(N-1)}$ but which are "too spread" to be in $\mathscr{F}$. The variables $X_\Delta$ with $\Delta$ in $Q_{N-2}$ appear as

parameters in $H_f$; their values are restricted, in the cases of interest to us, to be smaller than $B_{N-2}$.

The decay properties of the $\bar{\delta}$ coefficients tell us that if the constant $C_{31}$ is suitably chosen the contribution of the " missing terms " can be estimated by:

$$(7.39) \qquad |I| O_{32} (\lambda \sigma_N B_N^4 I_N^{-3} \exp - \tfrac{1}{2} (\log \lambda)^2 N) \exp - C_{31} N$$

and we shall take $C_{31}$ so large that this term is dominated by $\lambda^4 \sigma_N^4 I_N^{-3} |I|$.

We shall also put:

$$(7.40) \qquad \bar{H}_f (z_f) = \frac{\beta}{2} (z_\Delta - z_{\Delta'})^2 \qquad f = (\Delta , \Delta') \in \mathcal{F}_0$$

and $\bar{H}_f (z_f) = 0$ if $f \in \mathcal{F}$ but $f$ is not a couple of nearest neighbours.

The problem we wish to study is that of the analysis of the cumulant expansion of the integral:

$$(7.41) \qquad \frac{\int \exp - \sum_{f \in \mathcal{F}} H_f (z_f) \exp - \sum_{f \in \mathcal{F}} \bar{H}_f (z_f) \prod_{\Delta \in Q_{N-1}} \chi (|X_\Delta| < B_{N-1}) \, dz_\Delta}{\int \exp - \sum_{f \in \mathcal{F}} \bar{H}_f (z_f) \prod_{\Delta \in Q_{N-1}} \chi (|X_\Delta| < B_{N-1}) \, dz_\Delta} .$$

In the next section we concentrate our attention to develop the algorithm necessary for this purpose and we shall not come back to the technical details related to the verification of the validity of the recursive hypothesis described above. Neither we shall attack the problem of the upper bounds which can be reduced to that relative to the lower bound via the same ideas of the corresponding Bernoulli case once the corollaries of sec. 5 (in the $d = 1$ case) or of sec. 8 (in the $d = 2$ case) are taken into account.

## 8. More polymer theory

The content of this section is also extracted from [12].

Let $Z_1$ be the unit square lattice in $R^{d+1}$ which we think as the set of the centres of the pavement Q of $R^{d+1}$.

Let $\mathcal{F}$ be a family of subsets of Q consisting of single cubes, pairs of cubes, triples of cubes and let $\mathcal{F}_0$ denote the set of the nearest neighbour pairs of Q. Suppose $\mathcal{F}_0 \subset \mathcal{F}$.

Given a cube M, centred at the origin and paved by Q, consider the measure:

$$(8.1) \qquad \exp - \sum_{\xi \in M \cap Z_1} \alpha^2 S_\xi^2 - \sum_{f \in \mathcal{F}} (\bar{H}_f (S_f) + \tau_f H_f (z_f)) \prod_{\xi \in M \cap Z_1} \psi_\xi (S_\xi) \, dS_\xi$$

where $(\psi_\xi)_{\xi \in M}$ is a family of functions with values in $[0, 1]$, (measurable),

$$(8.2) \qquad \begin{aligned} &\bar{H}_f = \varepsilon (S_{\xi_\Delta} - S_{\xi_{\Delta'}})^2 \quad \text{if } f = (\Delta , \Delta') \in \mathcal{F}_0 \\ &\bar{H}_f = 0 \qquad\qquad\qquad\qquad \text{otherwise} \end{aligned}$$

and the functions $H_f(S_f)$ are, for the moment, arbitrary (measurable) functions of the variables $(S_\xi)_{\xi \in f}$; the parameters $(\tau_f)_{f \in \mathscr{F}}$ are assumed to be in $[0,1]$; $\alpha$ and $\varepsilon$ are positive parameters.

Define:

$$Z_\tau = \int \left( \prod_{\xi \in M \cap Z_1} \psi_\xi(S_\xi)\, e^{-\alpha^2 S_\xi^2} dS_\xi \right) \exp - \sum_{f \in \mathscr{F}} (\overline{H}_f(S_f) + \tau_f H_f(S_f)) . \tag{8.3}$$

We wish to study the $\tau$-dependence of $Z_\tau$. Clearly this generalizes the problem treated in sec. 5.

It is now natural to define a "polymer" R as a connected subset of a set $F \subset \mathscr{F}$ where the definition of connectedness is the same as that used in sec. 5 provided the $\mathscr{F}$ considered there is replaced by the one we are considering now. Similarly we can define, thogheter whith the set $\mathscr{R}$ of the polymers, the set $\mathscr{N}$ of the overlapping polymers: two polymers $R, R'$ overlap $(R \cap R' \neq \varnothing)$ if they have at least one cube, among those in the sets which form them, in common; a set $\Gamma = (R_1, R_2, \cdots, R_n)$ of polymers belongs to $\mathscr{N}$ if, given any two polymers $R, R' \in \Gamma$, there is a chain of polymers in $\Gamma$ with end elements R and R' and with successive non empty intersections.

The same formal discussion and the same formulae of sec 5 hold: in fact the functions $\varphi^T(\Gamma)$ have a purely combinatorial nature and their expression can be obtained from the expression of the analogous functions considered in sec 5 by just changing the meaning of the symbol $\mathscr{F}$ from that considered in sec 5 to the one we are considering now. The functions $\varphi^T$ will have the same formal properties of those of sec. 5.

If the series in the following expression is absolutely convergent one can then write,

$$\frac{Z_\tau}{Z_0} = \exp \sum_{\Gamma \in \mathscr{N} \cap M} (\zeta_\tau(\Gamma) - \zeta_0(\Gamma))\, \varphi^T(\Gamma) \tag{8.4}$$

where, if $\Gamma = (R_1, \ldots, R_n)$:

$$\zeta_\tau(\Gamma) = \prod_{i=1}^{n} \zeta_\tau(R_i) \tag{8.5}$$

and

$$\zeta_\tau(R) = \frac{\displaystyle\int \prod_{\xi \in R \cap Z_1} (\psi_\xi(S_\xi)\, e^{-\alpha^2 S_\xi^2} dS_\xi) \prod_{f \in R} (e^{-H_f(S_f)\tau_f - \overline{H}_f(S_f)} - 1)}{\displaystyle\int \prod_{\xi \in R \cap Z_1} \psi_\xi(S_\xi)\, e^{-\alpha^2 S_\xi^2} dS_\xi} . \tag{8.6}$$

Proceeding as in sec 5 we define, to formulate sufficient conditions for the absolute convergence of the above series, the following quantities:

$$b(f) = \sup_{(S_\xi)} \sup_{\xi \in f} \sup_{\tau \in [0,1]} | e^{-\tau_f H_f(S_f)} - 1 |\, e^{-\frac{\alpha^2}{4} \frac{1}{\nu} \sum_{\xi \in f} S_\xi^2} \qquad f \in \mathscr{F} \setminus \mathscr{F}_0 \tag{8.7}$$

$$b(f) = \sup_{(S_\xi)} \sup_{\xi \in f} \sup_{\tau', \tau \in [0,1]} | e^{-\tau \overline{H}_f(S_f) - \tau' H_f(S_f)} - 1 |\, e^{-\frac{\alpha^2}{4} \frac{1}{\nu_0} \sum_{\xi \in f} S_\xi^2} \qquad f \in \mathscr{F}_0 \tag{8.8}$$

where $\bar{\nu}$ denotes the maximum number of $f$'s in $\mathscr{F}$ which can contain a given cube and $\bar{\nu}_0 = 2(d+1)+1$. Then:

$$S_\xi(R) \leq \prod_{f \in R} (\hat{b} b(f)) \tag{8.9}$$

if

$$\hat{b} = \sup_{\xi \in M} \left( \int \left( \exp - \frac{\alpha^2}{4} S^2 \right) \psi_\xi(S)\, ds \right) \Big/ \left( \int (\exp - \alpha^2 S^2)\, \psi_\xi(S)\, ds \right),$$

as before.

To fix the ideas we remark that, eventually, we shall be interested in a case where the sets $f \in \mathscr{F}$ will only be sets of 1, 2, 3 cubes with diameter smaller than $r_N$ (cfr. (7.37)):

$$r_N = C_{28}^{-1} N (C_{31} + (\log \lambda)^2) \tag{8.10}$$

hence if $d = 2$:

$$\bar{\nu} \leq (\text{const})\, r_N^9 . \tag{8.11}$$

The conditions of convergence are given in the following Lemma:

LEMMA [12]. *Given $z_1 > 0$ there exists $\tilde{z}_0 > 0$ such that:*

$$\sum_{\substack{\Gamma \ni 0 \\ \text{diam}\Gamma \geq l}} \left( \prod \tilde{z}_0\, e^{-z_1 d(f)} \right) |\varphi^T(\Gamma)| < 2^{-l} \qquad l \leq 0 \tag{8.12}$$

*where $d(f)$ = graph distance of the centres of the cubes in $f$.*

We shall suppose $\tilde{z}_0 < z_0$ where $z_0$ is the constant appearing in the Lemma of sec. 5.

To analyze the $\tau$-dependence we proceed exactly as in sec. 5.

Let us define $z_1 = C_{27}$ and:

$$B_0 = \sup_f e^{z_1 d(f)}\, b_f \tag{8.13}$$

$$B_k = \sup_{f \in \mathscr{F}/\mathscr{F}_0} \sup_{\substack{(S_\xi)_{\xi \in f} \\ \tau_f \in [0,1]}} e^{-\frac{\alpha^2}{4} \frac{1}{\bar{\nu}} \sum_{\xi \in f} S_\xi^2} |H_f(S_f)|^k\, e^{-H_f(S_f)\tau_f} \qquad k > 0 \tag{8.14}$$

$$E_k = \sum_{h=0}^{k+1} (h!)^{-1} B_h . \tag{8.15}$$

Then the Taylor's formula gives, if $p$ = number of $f$ in R not belonging to $\mathscr{F}_0$:

$$X_\tau(R) = \zeta_\tau(R) - \zeta_0(R) = \prod_{f \in R} (\hat{b} e^{-z_1 d(f)}) \cdot \tag{8.16}$$

$$\cdot \sum_{h=1}^{k+1} \sum_{(h_f)_{f \in R},\, \Sigma_f h_f = h} \left( \prod_{f \in R} \frac{\tau_f^{h_f}}{h_f!} B_{h_f} \right) \pi_h((h_f)_{f \in R})\, (1 - E_k)^{-1}$$

where $|\pi_1|, \cdots, |\pi_{k+1}|$ are $\geq 1$ and $\pi_1, \cdots, \pi_k$ are $\tau$-independent functions of the integers $(h_f)_{f \in R}$.

Hence:

$$(8.17)\quad \prod_{R\in\Gamma}\zeta_\tau(R)-\prod_{R\in\Gamma}\zeta_0(R)=(1-4E_k(1-E_k)^{-1})^{-1}\prod_{f\in\Gamma}\left(\frac{4E_k}{1-E_k}e^{-z_1 d(f)}\right)\cdot \cdot\sum_{h=1}^{k+1}\sum_{(h_f)_{f\in\Gamma},\,\Sigma h_f=h}\frac{\tau_f^{h_f}}{h_f!}\pi'_h(h_f)_{f\in R}$$

where $|\pi'_1|,\cdots,|\pi'_{k+1}|$ are $\leq 1$ and $\pi'_1,\cdots,\pi'_k$ are $\tau$-independent.

Therefore, as in sec. 5:

COROLLARY. *Let* $\tau=(\tau_f)_{f\in\mathscr{F}}$ *be a family of parameters in* $[0,1]$ *and let* $\theta\in[0,1]$ *and put* $\theta\tau=(\theta\tau_f)_{f\in\mathscr{F}}$, *then consider the ratio* $Z_{\theta\tau}/Z_0$ *and suppose that for some* $\eta\in[0,1]$:

$$(8.18)\qquad 4\,\theta E_k(1-\theta E_k)^{-1}<z_0\eta$$

*then:*

$$(8.19)\qquad Z_{\theta\tau}/Z_0=\exp\sum_{h=0}^{k}\theta^h\frac{\delta(h)}{h!}+\theta^{k+1}\frac{\delta(k+1,\theta)}{(k+1)!}$$

*and* $\delta(1),\cdots,\delta(h)$ *are* $\theta$-*independent and if* $\Lambda=$ *support of the set of the* $f\in\mathscr{F}$ *such that* $H_f\not\equiv 0$:

$$(8.20)\qquad \begin{aligned}&|\delta(h)|\leq h!\,|\Lambda|\eta && h=0,1,\cdots,k\\ &|\delta(k+1,\theta)|\leq(k+1)!\,|\Lambda|\eta\,.\end{aligned}$$

The obvious analogues of the other corollaries and remarks of the previous section 5 hold in the natural sense.

The application of these results is as follows: it is a matter of simple algebraic computations to check that the condition for the cumulant formula integration of

$$(8.21)\qquad \int\hat{P}^{(N-1)}(dz)\exp(V^{(N-1)}+W^{(N-1)}+E^{(N-1)})\,\chi_B^{(N-1)}$$

as expressed by the above lemma are met. Furthermore, by induction as we saw in the Bernoulli case, one can show that the cumulant formula can be iterated $k$ times producing the result guessed formally in (7.30) as long as $N-k\geq k_0(\lambda)$ where $k_0(\lambda)$ is suitably chosen (and $k_0(\lambda)\equiv 0$ if $\lambda$ is small enough) and if $C_{29}$ is chosen large enough.

## 9. CONCLUSIONS

It seems interesting to make an analysis of the $d=3$ case along the above lines.

As the reader has certainly remarked, no diagrams have appeared in the above discussions. We interpret this as a manifestation of the fact that

in the above approach only the primitively divergent diagrams really matter. In the $d = 3$ case a treatment of the above type (as it is widely believed), does not seem to be free from the preliminary need to classify or somehow understand the structure of the primitively divergent diagrams. The amount of information presently available does not seem to be enough to try the study of the $d = 3$ case.

It seems tempting, before looking at the $d = 3$ problem, to study an apparently simpler case: the model obtained by taking $d = 3$ and setting $\gamma_N = (l_N^{-2+\varepsilon} - l_{N-1}^{-2+\varepsilon})$. In this case in fact the $\sigma_N \propto l_N^{\varepsilon} \xrightarrow[N\to\infty]{} 0$ and it might be possible to analyze this "$3 - \varepsilon$"-dimensional case with the methods of this work. It is not however immediately clear if the recursive scheme would still work, even in the Bernoulli case.

The models introduced in sec. 2 and 4 allow some flexibility to create models with intermediate degrees of divergence: for instance in the $d = 2$ case choosing $\gamma_N = (l_N^{-1+\varepsilon} - l_{N-1}^{-1+\varepsilon})$ one would eliminate the "mass divergence" as well as the third order vacuum divergence. Such cases provide some interesting models from a pedagogical point of view, though they are not completely trivial.

Finally it would be interesting to attack the problem of the infrared and ultraviolet regularity with other standard techniques of Statistical Mechanics and Probability Theory, like the superstability methods [18, 19] and the conditional probability techniques [20, 21] which are so fruitful in the theory of continuous spin systems. The work [21] gives evidence that this might be possible.

Another completely different direction of research would be to use the hierarchical fields to get some insights on the non-standard field theories like the non renormalized ones. In these cases one expects that the $z$'s are no longer good random variables but it would be interesting to realize the precise sense in which this happens.

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