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BOUNDARY CONDITIONS FOR THE HEAT EQUATION

by

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Abstract: we interpret in terms of the Brownian motion the boundary condition $\partial u = \varkappa u + \frac{1}{2}\sigma(\Delta u)$ for the heat equation in a polygonal region.

1.- Introduction

The purpose of this note is to explain from the point of view of the Brownian motion the significance of the "elastic" boundary condition $\partial u = \varkappa u$ and the "sticky" boundary condition $\partial u = \frac{1}{2}(\Delta u)$ and, more generally, of the boundary condition $\partial u = \varkappa u + \frac{1}{2}\sigma(\Delta u)$ for solutions of the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}(\Delta u)$ in a ν-dimensional open polygonal region D; here and below the symbol ∂ stands for differentiation in the inward pointing direction perpendicular to the boundary B of D and $\varkappa, \sigma$ are non negative continous functions on B.

The "reflecting" Brownian motion ($\varkappa = \sigma = 0$) is constructed from scratch in section 3 and its sample paths are used to build the Brownian motion corresponding to $\partial u = \varkappa u + \frac{1}{2}\sigma(\Delta u)$.

The correspondence is meant in the following sense: the transition density of the Brownian motion is an elementary solution (i e a Green's function) of the problem:

$$\text{a)}\quad \frac{\partial u}{\partial t} = \frac{1}{2}(\Delta u) \qquad \text{in } D \qquad \forall t>0$$

$$\text{b)}\quad \partial u = \varkappa u + \frac{1}{2}\sigma(\Delta u) \qquad \text{on } B' \qquad \forall t>0$$

$$\text{c)}\quad \|u(\cdot,t)\|_\infty \le C < \infty \qquad \forall t \ge 0 \quad \text{and}$$

$$(1.1)\qquad u \in C^\infty(D)\,,\quad u \in C^1(D\cup B') \qquad \forall t \ge 0$$

$$\text{d)}\quad |\operatorname{grad} u(x,t)| \le C'\,(\text{distance of } x \text{ from the "corners"})^{-1} + C \quad \text{on } D\cup B',\ \forall t \ge 0$$

$$\text{e)}\quad u(x,0+) = f(x) \qquad f \in C_0^\infty(\bar{D}),\ \|f\|_\infty < \infty$$

where B' is the part of B obtained considering only the $(\nu-1)$-dimensional faces of B (i.e., more precisely, excluding from B the corners in 2-dimensions, the corners and the edges in 3-dimensions etc.). We shall refer to the set $B-B'$ as the set of "corners" of B

We shall prove a uniqueness theorem for problem (1.1) for $\sigma = 0$ and D bounded however such a result holds also if $\sigma \neq 0$ and D is bounded but the proof will be given elsewhere.

We have tried to write this paper keeping it reasonably self-contained and understandable to readers not familiar with the Brownian motion.

In section 2 the basic definitions and properties of the free Brownian motion are reviewed this section also contains some results which we need later. Most of them are proved in appendix

In section 3 we construct the reflecting Brownian motion i e we solve problem (1.1) with $\varkappa = \sigma = 0$

In section 4 we use the results of section 3 to deal with the "elastic" barrier ($\varkappa \neq 0$, $\sigma \equiv 0$)

In section 5 we prove the uniqueness theorem for the elastic barrier problem in a bounded D We also compare the solution obtained through the Brownian motion technique with the one obtained using the usual variational method

In section 6 we construct the general Brownian motion, i e we find a solution of problem (1.1).

The paper is written so that after reading section 2 (which can be skipped by the reader familiar with the Brownian motion), one can go through the rest of the paper without reading the proof of the theorems and still get a clear idea of the phylosophy of the article. The philosophy being to describe the considered boundary conditions in terms of simple geometric operations on the sample paths of the free Brownian motion

We have not attempted at obtaining the best possible results: the chief point being to explain the idea without getting bogged down in technical details

Our interest on the subject has been spurred by a

number of recent applications of the Brownian motion to problems in Statistical mechanics. [1]

2.- The free Brownian motion

We list here the definitions and a number of properties of the free Brownian motion which will be the key for the investigation of our problem.

First we remember the definition of the free Brownian motion.

Let $\dot{R}^\nu$ be the one point compactification of the ν-dimensional space. Consider the product space:

$$\Omega = \prod_{0 \le t < \infty} \dot{R}^\nu \tag{2.1}$$

which is a compact in the product topology. The elements of Ω are paths $t \to \omega(t), 0 \le t < \infty$. The space $C(\Omega)$ of the continous functions on Ω is densely spanned by the functions $F(\omega)$ of the form $F(\omega) = f(\omega(t_1), \ldots, \omega(t_n))$ where $0 \le t_1 < t_2 < \ldots < t_n < \infty$ are n "times" and $f(\xi_1, \ldots, \xi_m)$ is a continous function on $(\dot{R}^\nu)^m$.

We can define a regular measure $P_x(d\omega)$ on the Borel field of Ω (through the Riesz representation theorem) by means of the formula

$$\int_\Omega P_x(d\omega) F(\omega) = \int g_{t_1}(x, \xi_1)\, g_{t_2 - t_1}(\xi_1, \xi_2) \ldots g_{t_n - t_{n-1}}(\xi_{n-1}, \xi_n) \cdot f(\xi_1, \ldots, \xi_m)\, d\xi_1 \ldots d\xi_n \tag{2.2}$$

where $g_t(x,y) = (2\pi t)^{-\frac{\nu}{2}} \exp - \frac{(x-y)^2}{2t}$ is the Gauss kernel.

It is possible to prove that $P_x(d\omega)$ is a probability measure which is concentrated on the (measurable) set of the continuous paths $\omega \in \Omega$ such that $\omega(0) = x$, (Wiener's theorem). [2]

It is also possible to prove that $P_x(d\omega)$ describes "paths without memory" in the precise sense that if $F(\omega)$ is a P_x-measurable function which depends only on the values of $\omega(t)$ for $0 \leq t \leq \tau$ and if $G(\omega)$ is a P_x-measurable function which depends only on the values of $\omega(t)$ for $t \geq \tau$ then:

$$(2.3) \qquad \int_\Omega P_x(d\omega) F(\omega) G(\omega) = \int_\Omega P_x(d\omega) F(\omega) \cdot \left[\int_\Omega P_{\omega(\tau)}(d\omega') G(\omega \underset{\tau}{\cup} \omega') \right]$$

where the symbol $\omega \underset{\tau}{\cup} \omega'$ means the trajectory which coincides with $\omega(t)$ for $0 \leq t \leq \tau$ and with $\omega'(t-\tau)$ for $t \geq \tau$ (observe that for the G we are considering, $G(\omega \underset{\tau}{\cup} \omega')$ depends only on ω').

In fact the free Brownian motion has a much stronger "lack of memory": let $0 \leq m(\omega) \leq \infty$ be a "hitting time" for the path ω : i.e. let $m(\omega)$ be the first time ω hits a given closed set $\Gamma \subset \dot{R}^\nu$: $m(\omega) = \inf \{ t : \omega(t) \in \Gamma \}$. Then if $F(\omega)$ is a measurable function which depends only on the values of $\omega(t)$ for $t \leq m(\omega)$ (for all ω) and if $G(\omega)$ is another measurable function which depends only on the values of $\omega(t)$ for $t \geq m(\omega)$ then [3]:

$$\int_{\Omega_\infty} P_x(d\omega)F(\omega)G(\omega) = \int_{\Omega_\infty} P_x(d\omega)F(\omega)\left[\int_\Omega P_{\omega(m(\omega))}(d\omega')\cdot G(\omega \underset{m}{\cup} \omega')\right] \tag{2.4}$$

where $\Omega_\infty = \{\omega : m(\omega) < +\infty\}$.

Property (2.4) called the "strong Markov property" holds for a more general class of "Markov times" $m(\omega)$ but we do not need here the more general formulation. [3]

We now state in form of theorems other properties of the free Brownian motion:

Theorem 1: *let* $f(\vec{x})$ *be a* C^∞*-function on* $\dot{R}^\nu$ *and* Γ *be a closed set in* $\dot{R}^\nu$. *Consider the set* $A \subset \Omega$:

$$A = \{\omega \mid \omega(t) \notin \Gamma, \forall\, t \geq 0\}$$

and the function:

$$u(x,t) = \int_A P_x(d\omega) f(\omega(t)) \tag{2.5}$$

then $u(x,t)$ *is* C^∞ *around any point* (x,t) *with* x *at a positive distance from* Γ *and* $t \geq 0$. *Furthemore* $u(x,t)$ *verifies in this region of* (x,t) *the heat equation with* $f(\vec{x})$ *as initial condition:*

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}(\Delta u)(x,t) \quad , \quad u(x,0+) = f(x) \tag{2.6}$$

finally the gradient of $u(x,t)$ *can be bounded by:*

$$|\text{grad } u| \leq C\,(\text{dist. of } x \text{ from } \Gamma)^{-1} \qquad C < \infty,\ \forall t \geq 0 \tag{2.7}$$

the proof of this theorem which shall appear more or less

obvious to the reader familiar with the Brownian motion, is in Appendix 1.

Theorem 2: *(P. Levy) Consider a 1-dimensional free Brownian motion and let* $x_0 \in \mathbb{R}^1$ *be a point and* $\omega \in \Omega$ *be a path. Let*

$$(2.8) \qquad \underset{\sim}{t}_{x_0,\varepsilon}(\omega, t) = \frac{1}{2\varepsilon} \text{ measure } \{\theta : 0 \le \theta \le t, |\omega(\theta) - x_0| \le \varepsilon\}$$

then, for P_x*-almost all paths* $\omega \in \Omega$ *the following limit exists:*

$$(2.9) \qquad \underset{\sim}{t}_{x_0}(\omega, t) = \lim_{\varepsilon \to 0} \underset{\sim}{t}_{x_0,\varepsilon}(\omega, t)$$

and defines what is called the "local time" spent by ω *in* x_0.

For the proof the reader is referred to the literature [4] however in Appendix 2 we give some arguments in the hope to make (2.9) intuitive. These arguments could in fact be turned into a rigorous proof with some extra-work and at the same time they provide an intuitive idea for why the theorem 3 below holds:

Theorem 3: *Let* $f(\xi)$ *be a* C^∞*-function on the compactified positive semiaxis. Let* $\varkappa \ge 0$ *and define*

$$(2.10) \qquad u(x,t) = \int_\Omega P_x(d\omega) f(|\omega(t)|)\, e^{-\varkappa \underset{\sim}{t}_0(\omega,t)}$$

then $u(x,t)$ *regarded as a function of* $x \ge 0$ *verifies the*

heat equation on the positive semiaxis (with initial value $f(\xi)$) *together with the boundary condition:*

$$\left.\frac{\partial u}{\partial x}\right|_{x=0^+} = \varkappa \left. u\right|_{x=0^+} \qquad \forall\, t>0 \tag{2.11}$$

finally the following theorem holds:

Theorem 4: *let* $f(\xi)$ *be a* C^∞*-function on the compactified positive semiaxis. Define, for* $\sigma \geq 0$, *the mapping* $T_\sigma : \Omega \to \Omega$:

$$(T_\sigma \omega)(t) = \omega(\Theta(t)) \tag{2.12}$$

where $\Theta(t)$ *is the inverse function of*

$$\Sigma(t) = t + \sigma \int_0^t d\underline{t}_0(\omega, \tau) \tag{2.13}$$

then the function:

$$u(x,t) = \int P_x(d\omega) f(|(T_\sigma \omega)(t)|) \tag{2.14}$$

verifies on the positive semiaxis the heat equation (with initial value $u(x, 0+) = f(x)$ *)together with the boundary condition:*

$$\left.\frac{\partial u}{\partial x}\right|_{x=0^+} = \frac{1}{2}\sigma \left.(\Delta u)\right|_{x=0^+} \qquad \forall\, t>0 \tag{2.15}$$

and this explains why (2.15) is called a sticky boundary condition.

The proof of this theorem is sketched in Appendix 2.

for a complete proof see [6].

Theorems 3 and 4 are well known from the theory of the Brownian motion [5,6].

Remark Theorem 1 holds under more general assumptions which look, a priori, quite technical but that will be needed later. Therefore in Appendix 1 theorem 1 is in fact proved in the form explained below.

Denote ω_s^+ the path $\omega_s^+(t) = \omega(t+s)$, $\quad 0 \leq t < \infty$

Given a function $f(\omega,t)$ suppose that there is an open square $Q_{x_0,L}$ centered at a point $x_0 \in R^\nu$ and with side $L > 0$ such that

i) if $\omega(\tau) \in Q_{x_0,L}$, $\quad 0 \leq \tau \leq t$ (i.e. if ω stays in $Q_{x_0,L}$ up to t) then:

(2.16) $$f(\omega,t) = \varphi(\omega(t))$$

where $\varphi(\xi)$ is a continuous function on $Q_{x_0,L}$ almost everywhere differentiable with an essentially bounded gradient; φ is also supposed to be the integral of its own gradient.

ii) if $\omega(\tau) \notin Q_{x_0,L}$ for some τ such that $0 \leq \tau \leq t$ let m be the hitting time associated with $\partial Q_{x_0,L}$ then

(2.17) $$f(\omega,t) = f(\omega_m^+, t-m)$$

iii) f is uniformly bounded: $\sup_{\omega,t} |f(\omega,t)| = \|f\|_\infty < \infty$

then:

Theorem 1': *the function defined* $\mathcal{Q}_{x_0,L}$ *by:*

(2.18) $$u(x,t) = \int f(\omega,t)\, P_x(d\omega)$$

is ∞ *- differentiable in* (x,t) *for* $t>0$ *; furthemore:*

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}(\Delta u)(x,t) \qquad x \in \mathcal{Q}_{x_0,L} \quad t>0$$

$$u(x,0+) = \varphi(x) \qquad x \in \mathcal{Q}_{x_0,L}$$

(2.19)

$$\|u(\cdot,t)\|_\infty \leq C < \infty \qquad C \text{ depends only on } \nu \text{ and } \|f\|_\infty$$

$$|\operatorname{grad} u(x_0,t)| \leq \frac{C'}{L} \qquad C' \text{ depends only on } \nu, \|f\|_\infty, \sup|\operatorname{grad}\varphi|, L \text{ and is bounded as } L \to 0.$$

Clearly theorem 1' contains theorem 1 as particular case. The additional differentiability for $t = 0$ follows from the additional assumptions on f .

3. - Reflecting Brownian motion

We treat for simplicity and ease of illustration only the case $\nu = 2$: the results and methods are valid for $\nu \geq 2$.

Let D be a polygon with sides labelled $1, 2, \ldots, n$. The idea is to define a mapping $T^+ : \Omega \to \Omega^+$ where $\Omega^+ = \prod_{0 \leq t < \infty} \dot{\bar{D}}$ and $\dot{\bar{D}}$ is the one-point compactification of $\bar{D}$ and to use this mapping to carry the measure $P_x(d\omega)$ on Ω into a measure $P_x^+(d\omega)$ on Ω^+ . If the mapping T^+ is conveniently definied then the solution to problem (1.1) with $\varkappa = \sigma = 0$ will be expressed as:

(3.1) $$u(x,t) = \int f(\omega(t))\, P_x^+(d\omega) \equiv \int f(T^+\omega(t))\, P_x(d\omega)$$

To define τ^+ let us first construct a covering K of $\dot{\bar{D}}$: reflect repeatedly $\dot{\bar{D}}$ across its sides, using every time a side different from the one last used; each new reflection produces a copy (image) of the fundamental domain which is labelled by a string of integers listing from left to right, the successive sides employed for reflection. These images are declared to be all different, so that one gets an infinite sheeted covering K of $\dot{\bar{D}}$ with one sheet associated to each possible finite string of reflections. Each image of $\dot{\bar{D}}$ has n corners (resp. sides) which, when regarded as points of K will be called corners of K (resp. sides of K).

Let x be a point on K-(corners). Let ω be a path in Ω that starts from x regarded as a point of the plane. Suppose that ω is a continuous path which never passes through the denumerable set of the images of the corners (the numerability of the set of images follows from the fact that they are in correspondence with rationals in a base-n representation).

Let Ω_D be this subset of paths: since a Brownian motion never passes through a set of Newtonian capacity zero [2], it follows that $P_x(\Omega_D) = 1$.

Since $\omega \in \Omega_D$ is continuous we can call τ_1 the first time that ω, thought as walking on K, hits an image of a side of D; the point where this side is hit cannot be a corner because $\omega \in \Omega_D$ and therefore we can define $\tau_2 > \tau_1$ as the first instant after τ_1 when ω hits a new side (we still

think of ω as walking on K); again the point of collision cannot be a corner of K: so we can construct a sequence $0 \leq T_1 < T_2 < T_3 < \dots$ of successive instants. We can see that $\lim_{i \to \infty} T_i = +\infty$. Otherwise let $\bar{T} = \lim_{i \to \infty} T_i < \infty$ and observe that the sequence of sides successively hit at the times T_i must consist of images of only two out of n sides: for, if this does not happen, the path must make jumps of finite lenght in intervals of time that are arbitrarily small; and for the same reason, these images must all have a common corner Q to which $\omega(T_i)$ must tend as $i \to \infty$: since this $Q = \omega(\bar{T})$ would be the image of a corner under a finite number of reflections this contradicts the assumption $\omega \in \Omega_D$.

As a consequence of the above argument it is possible to think every path $\omega \in \Omega_D$ that starts at $x \in$ K-(corners) (in the sense explained above) as a path on $\Omega^K = \prod_{0 \leq t < \infty} K$ Hence it is possible to define for each $x \in$ K-(corners) a mapping $T^K : \Omega \to \Omega^K$ which is P_x-almost everywhere defined. This mapping can be used to define a measure $P_x^K(d\omega)$ over Ω^K for $x \in$ K-(corners) as follows: if $F \in C(\Omega^K)$ then by definition:

$$(3.2) \qquad \int F(\omega)\, P_x^K(d\omega) = \int F(T^K\omega)\, P_x(d\omega) \qquad \forall x \in \text{K-(corners)}$$

We can now define the mapping $T^+ : \Omega \to \Omega^+$ using the natural mapping of K on D: T^+ sends $\omega \in \Omega$ into the natural image $(T^K\omega)^+$ on $\dot{D}$ of $T^K\omega$ (the "natural" mapping denoted by $(\cdot)^+$ is of course the mapping obtained by folding back

the images of $\dot{\bar{D}}$ onto $\dot{\bar{D}}$): the mapping T^+ is P_x-almost everywhere defined for $x \in D \cup$(B-corners). In terms of measures the mapping T^+ carries P_x into a measure P_x^+ such that for $F \in C(\Omega^+)$:

$$(3.3)\quad \int P_x^+(d\omega)\, F(\omega) = \int P_x^K(d\omega)\, F((\omega)^+) = \int P_x(d\omega)\, F((T^K\omega)^+).$$

If $f \in C^\infty(\dot{\bar{D}})$ then the integral:

$$(3.4)\quad u(x,t) = \int P_x^+(d\omega)\, f(\omega(t)) = \int P_x^K(d\omega)\, f((\omega)^+(t))$$

is defined for $x \in \bar{D}$-(corners) and the second integral in (3.4) makes sense also for $x \in K$-(corners) and coincides with the integral:

$$(3.5)\quad \tilde{u}(x,t) = \int P_x^K(d\omega)\, \tilde{f}(\omega(t)) = \int P_x(d\omega)\, \tilde{f}(T^K\omega)(t))$$

where $\tilde{f}(x)$ is defined as the natural extension of f to K (i.e. as the extension through successive reflections). It is clear that the function $\tilde{u}(x,t)$ is defined on K-(corners) and is even in a perpendicular direction across any side of K (because such is $\tilde{f}$) and, therefore, has a vanishing normal derivative on the sides of K provided this derivative exists and is continuous.

An application of theorem 1' to the function $f(\omega,t) = \tilde{f}((T^K\omega)(t))$ proves the infinite differentiability on K-(corners) of $\tilde{u}(x,t)$ and the fact that $u(x,t)$ (i.e. the restriction of $\tilde{u}(x,t)$ to D $\cup$ (B-corners)) is a solution of problem (1.1) with $\varkappa = \sigma = 0$.

4. - Elastic barrier

We shall prove a generalization of theorem 4 for $\nu = 2$. The case $\nu > 2$ is as easy.

Let us first define what we mean by local time distribution on a line Λ : $\underline{\underline{t}}_{\Lambda}(\omega, t; d\underline{\xi})$.

Let ω be a two-dimensional path of a free Brownian motion. Given a line Λ we choose it as the 2-axis. Let $\omega = (\omega_1, \omega_2)$ where ω_1 and ω_2 are the components of ω along the 1 and 2-axes: they are independent free 1-dimensional Brownian motions.

Let $\varkappa(\underline{\xi})$ be a continuous function of $\underline{\xi} \in \dot{\Lambda}$ (here as above and below a point on a set will denote its one point compactification).

Let also $\underline{t}_0(\omega_1, t)$ be the local time function at $x_2 = 0$ of the Brownian motion ω_1 (see section 2). We can define the measure $\underline{\underline{t}}_{\Lambda}(\omega, t; d\underline{\xi})$ on $\dot{\Lambda}$ through the Riesz theorem and the formula:

$$(4.1) \qquad \int_{\dot{\Lambda}} \varkappa(\underline{\xi})\, \underline{\underline{t}}_{\Lambda}(\omega, t; d\underline{\xi}) = \int_0^t \varkappa(\omega_2(\tau))\, d\underline{t}_0(\omega_1, t)$$

this formula makes sense for P_x-almost all ω's ($x \in R^2$) since for P_x-almost all ω's, $\underline{t}_0(\omega_1, t)$ exists and is a finite non-decreasing function of t so that

$$(4.2) \qquad \left| \int_0^t \varkappa(\omega_2(\tau))\, d\underline{t}_0(\omega_1, \tau) \right| \leq \|\varkappa\|_\infty\, \underline{t}_0(\omega_1, t)$$

The quantity $\int_{\Lambda} \underline{\underline{t}}_{\Lambda}(\omega, t; d\underline{\xi})$ will be called the local time

spent on $\lambda \subset \dot{\Lambda}$.

To define the local time measure $\underline{\underline{t}}_B(\omega, t; d\underline{\xi})$ over a polygon B, boundary of a polygonal region D, we proceed as follows: let $\omega \in \Omega$ be a path which does not go through the image of any of the corners of D. Then we can define the local time spent on B by following the path ω and by adding the local times spent on the images of the sides of D that are crossed as ω walks on the covering K of $\bar{D}$. It is important to note that this local time function is not the integral over the images of B of the local time measure defined in (4.1) (observe also that these images form, in general, a dense set).

It is easy to see that up to a time $t < \infty$ the trajectory ω can cross only a finite number of different sides and on each of them it can spend only a finite local time: so we see that $\underline{\underline{t}}_B(\omega, t) < \infty$ almost always (See sect. 3).

It is now possible to define $\underline{\underline{t}}_B(\omega, t; d\underline{\xi})$ on B through:

$$(4.2') \qquad \int_{\dot{B}} \underline{\underline{t}}_B(\omega, t; d\underline{\xi})\, \varkappa(\underline{\xi}) = \int_0^t \varkappa(\omega(\tau))\, d\underline{\underline{t}}_B(\omega, \tau) \qquad \varkappa \in C(\dot{B})$$

where $\varkappa(\underline{\xi})$ is arbitrarily put equal to zero when $\underline{\xi} \notin B$.

Let us define for $\varkappa \in C(\dot{B})$, $\varkappa \geq 0$:

$$(4.3) \qquad I(\varkappa, \omega; t) = \int_B \varkappa(\underline{\xi})\, \underline{\underline{t}}_B(\omega, t; d\underline{\xi})$$

It is natural to hope that if $f \in C^\infty(\dot{\bar{D}})$ then:

$$(4.4)\quad u(x,t)=\int_{\Omega} e^{-I(x,\omega;t)} f((T^{+}\omega)(t))\,P_x(d\omega) \qquad x\in D\cup B'$$

is a solution to (1.1) with $\sigma=0$.

We remark at this point that $u(x,t)$ above is defined in an obvious way for $x\in$ (K - the corners), and is even perpendicularly to any side of B.

Let λ be a side (without the corners) of B and let

$$(4.5)\quad \psi(\omega,t)= \exp\left[-\int_B \kappa(\underline{\xi})\,\underline{t}_{=B}(\omega,t;d\underline{\xi})+\int_{\lambda}\underline{t}_{=\lambda}(\omega,t;d\underline{\xi})\,\kappa(\underline{\xi})\right]\cdot f((T^{+}\omega)(t))$$

and therefore

$$(4.6)\quad u(x,t)=\int_{\Omega} e^{-\int_{\lambda}\underline{t}_{=\lambda}(\omega,t;d\underline{\xi})\kappa(\underline{\xi})}\,\psi(\omega,t)\,P_x(d\omega)$$

From now on the proof is much on the same lines of the proof of theorems 1 and 1' but we describe it here in some detail. However we suggest the reader to read first the proof of theorem 1' in appendix 1.

Let first $\kappa(\underline{\xi})=\kappa=$ constant. Let $Q_{\xi_0,L}$ be an open square centered at $\xi_0\in\lambda$ and with side L small enough so that $Q_{\xi_0,L}$ does not contain any of the corners of $\bar{D}$. We suppose that $Q_{\xi_0,L}$ has sides parallel and perpendicular to the 2-axis which is assumed to contain λ; we also suppose that $\xi_0=(0,0)$.

Let $\eta(\omega)$ be the hitting time associated with the path

ω and $\partial \Omega_{\varphi_{0,L}}$, $(\omega(0) = x \in \Omega_{\varphi_{0,L}})$.

From the form of (4.5) we realize that if $\tau(\omega) < t$:

$$(4.7)\quad e^{-\kappa \int_\lambda \xi_\lambda(\omega,\tau;d\varphi)} \psi(\omega,t) = F_\kappa(\omega_1,\tau)\,\psi(\omega^+_\tau, t-\tau)$$

with

$$(4.7')\quad F_\kappa(\omega_1,\tau) = \exp -\kappa \underline{t}_0(\omega_1;\tau) \qquad \tau \geq 0$$

therefore using the strong Markov property and denoting μ_1 and μ_2 the exit times of the components ω_1, ω_2 of ω from $\left(-\frac{L}{2}, \frac{L}{2}\right)$ we find:

$$(4.8)\quad \begin{aligned} u(x,t) = &\int_0^t \int_{-\frac{L}{2}}^{+\frac{L}{2}} d\tau \quad f^0_{x_2,+}(\tau)\, G^{L,\kappa}_\tau(x_1,\varphi)\,d\varphi\; u((\varphi,\tfrac{L}{2}), t-\tau) + \\ &+ \int_0^t \int_{-\frac{L}{2}}^{+\frac{L}{2}} d\tau\, f^\kappa_{x_1,+}(\tau)\; G^{L,0}_\tau(x_2,\varphi)\,d\varphi\; u((\tfrac{L}{2},\varphi), t-\tau) + \end{aligned}$$

+similar terms coming from the remaining sides +

$$+ \int_{-\frac{L}{2}}^{+\frac{L}{2}} \int_{-\frac{L}{2}}^{+\frac{L}{2}} f(|\varphi|,\eta)\, d\varphi\, d\eta\, G^{L,\kappa}_t(x_1,\varphi) G^{L,0}_t(x_2,\eta)$$

where $x = (x_1,x_2) \in \Omega_{\varphi_{0,L}}$ and we have put, for $i = 1,2$:

$$(4.9)\quad G^{L,\kappa}_\tau(x_i,\varphi)\,d\varphi = \int_{\{\omega_i \mid \omega_i(\tau) \in d\varphi,\ \tau < \mu_i\}} e^{-\kappa \underline{t}_0(\omega_i,\tau)}\, P_{x_i}(d\omega_i)$$

$$(4.10)\quad f^\kappa_{x_i,\pm}(\tau)\,d\tau = \int_{\{\omega_i \mid \omega_i(\vartheta) = \pm\frac{L}{2},\ \tau < \vartheta < \tau + d\tau\}} e^{-\kappa \underline{t}_0(\omega_i;\tau)}\, P_{x_i}(d\omega_i)$$

we also observe that $u((\varphi, \pm\tfrac{L}{2}), \tau)$ is even in $\varphi \in [-\frac{L}{2}, \frac{L}{2}]$

as a consequence of the remark after (4.4).

Expressions (4.9), (4.10) are quantities associated with one-dimensional Brownian motions and can be computed explicitely (see appendix 3): they are differentiable in $x_i \geq 0$ up to $x_i = 0$ and verify at $x_1 = 0$ the elastic boundary condition with parameter $\varkappa$:

$$\frac{\partial u}{\partial x_1}\Big|_{x_1=0^+} = \varkappa\, u\Big|_{x_1=0^+} \tag{4.11}$$

furthemore they verify in $\left[-\frac{L}{2}, \frac{L}{2}\right]$ the heat equation for $t > 0$; their gradients also admit the same type of bounds found in appendix 1 for the analogous quantities. Using these facts we realize that $u(x,t)$ is indeed a solution of problem (1.1) with $\sigma \equiv 0$, $\varkappa(\underline{\varphi}) = \varkappa$.

If $\varkappa(\underline{\varphi})$ is not constant the above proof requires some minor changes. In this case one has to take into account the formula below, valid for $t <$ exit time from the box $Q_{\underline{\varphi}_0, L}$:

$$\left| \exp - \int_{-\frac{L}{2}}^{+\frac{L}{2}} \varkappa(\underline{\varphi})\, \underline{\underline{t}}_{\lambda}(\omega, t; d\underline{\varphi}) - \exp - \varkappa(\underline{\varphi}_0)\, \underline{t}_0(\omega, t) \right| \equiv$$

$$\equiv \left| \exp - \int_{-\frac{L}{2}}^{\frac{L}{2}} \varkappa(\underline{\varphi})\, \underline{\underline{t}}_{\lambda}(\omega, t; d\underline{\varphi}) - \exp - \int_{-\frac{L}{2}}^{\frac{L}{2}} \varkappa(\underline{\varphi}_0)\, \underline{\underline{t}}_{\lambda}(\omega, t; d\underline{\varphi}) \right| \leq$$

$$\leq \left(\max_{|\underline{\varphi} - \underline{\varphi}_0| \leq \frac{L}{2}} \left| \varkappa(\underline{\varphi}) - \varkappa(\underline{\varphi}_0) \right| \right) \underline{t}_0(\omega, t) = o(L)\, \underline{t}_0(\omega, t)$$

and the probability distribution of the local time [7] and the kind of techniques used in appendix 1 and in this sec-

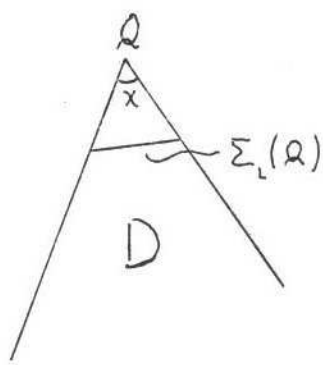
tion.

The calculation is tedious but straigth forward and we skip it.

5. - Uniqueness problems

We investigate the uniqueness properties of the solution of (1.1) only for the case D is finite and $\sigma = 0$. We also suppose (without a real loss of generality) that $\nu = 2$.

In this section we shall also show that the solution of (1.1), with $\sigma = 0$ coincides with the usual variational solution in the case D is finite. The non trivial part of this statement being the fact that the variational solution and (1.1) belong to the same functional space. The "usual" solutions are for instance, in [12] [13].



Consider a domain D obtained from D by excluding little triangular regions near the corners (see figure). For a given corner Q the triangular region has two sides of equal lenght lying on the two sides of B converging in Q, the third side of the triangle will be called $\Sigma_L(Q)$ and we suppose that its length is L. We call $B_L = B \cap (D_L) = \partial D \cap \partial D_L$.

Integrating the solution of (1.1) over D_L and $[0,t]$ we

find:

$$\int_{D_L} u(x,\tau)^2\,dx - \int_{D_L} u(x,0)^2\,dx = -\frac{1}{2}\int_0^t d\tau \int_{D_L} |\operatorname{grad} u(x,\tau)|^2 d\tau -$$

$$(5.1)\qquad -\frac{1}{2}\int_0^t d\tau \int_{B_L} \varkappa(\underline{\xi})\,|u(\underline{\xi},\tau)|^2 d\tau - \frac{1}{2}\int_0^t d\tau \int_{\cup_\alpha \Sigma_L(\alpha)} u(\underline{\xi},\tau)\,\frac{\partial u(\underline{\xi},\tau)}{\partial n}\,d\underline{\xi}$$

Using c) and d) of (1.1) it follows that the contribution of the last integral to (5.1) is uniformly bounded as $L \to 0$. Hence, using again c), for almost all $t \geq 0$:

$$(5.2)\qquad \int_D |\operatorname{grad} u(x,t)|^2\,dx < +\infty$$

and therefore the following limit exists for all $t \geq 0$:

$$(5.3)\qquad \psi(t) = \lim_{L\to 0} \int_0^t d\tau \int_{\Sigma_L} u(\underline{\xi},\tau)\,\frac{\partial u(\underline{\xi},\tau)}{\partial n}\,d\underline{\xi}$$

and, from c) and d), is absolutely continuous in t.

From c) we deduce that the functions $\int_{\Sigma_L} u(\underline{\xi},\tau)\,\frac{\partial u(\underline{\xi},\tau)}{\partial n}\,d\underline{\xi}$ are uniformly bounded in L, hence they form a conditionally weakly compact set in $L_1([0,T])$ for any $T < \infty$; this fact together with (5.3) implies:

$$(5.4)\qquad \lim_{L\to 0} \int_{\Sigma_L} u(\underline{\xi},\tau)\,\frac{\partial u(\underline{\xi},\tau)}{\partial n}\,d\underline{\xi} = \frac{d\psi}{dt}(t) \qquad \text{weakly in } L_1([0,T])$$

We want to prove that $\psi(t) \equiv 0, \frac{d\psi}{dt} \equiv 0$. Suppose the contrary; then there is a $T > 0$ and an $\varepsilon > 0$ such that:

(5.5) $$\mu^{+}_{\varepsilon,T} = mes\{t : 0 \le t \le T , \frac{d\psi}{dt} > \varepsilon\} > 0$$

(5.5') or $$\mu^{-}_{\varepsilon,T} = mes\{t : 0 \le t \le T , \frac{d\psi}{dt} < -\varepsilon\} > 0$$

Suppose (5.5) is true (if not, then (5.5') is true and the argument below is essentially the same). From the weak L_1-convergence we deduce

(5.6) $$mes\left\{t : \int_{\Sigma_L} u(\xi,t)\,\frac{\partial u(\xi,t)}{\partial n}\,d\xi > \frac{\varepsilon}{2}\right\} > \frac{\varepsilon\mu^{+}_{\varepsilon,T}}{2CC'}\,\frac{1}{2tg\frac{\chi}{2}}$$

Hence for small enough L: (if χ is the angle of B in Q):

(5.7) $$mes\left\{\underline{\xi} \mid \underline{\xi} \in \Sigma_L , \left|\frac{\partial u}{\partial n}(\underline{\xi},t)\right| \ge \frac{\varepsilon/2}{2CL}\right\} \ge \frac{\varepsilon}{4CC'}\,\frac{L}{2tg\frac{\chi}{2}}$$

for all the t's in the set with positive measure considered in (5.6), therefore:

(5.8) $$\int_0^T d\tau \int_{D-D_L} \left|\frac{du}{dn}\right|^2 dx \ge \frac{\varepsilon\mu^{+}_{\varepsilon,T}}{2CC'} \int_0^L \left(\frac{\frac{\xi}{2}}{2C}\right)^2 \frac{1}{l^2}\,\frac{\varepsilon l}{8CC'tg\frac{\chi}{2}\left(2tg\frac{\chi}{2}\right)^2}\,dl = +\infty$$

which is in contradiction with (5.1) which implies that $|grad\, u(x,t)|^2$ has a uniformly bounded integral over x,t if $u(x,t)$ verifies (1.1). It is now easy to prove uniqueness: let u, V be two solutions of (1.1) verifying properties a) ÷ e) with $\sigma \equiv 0$ and suppose D is finite. Then, from what has been proved in this section, it follows that the function $W = u - V$ is such that $W(x,0^+) = 0$ and, from (5.6), (5.9):

$$(5.9)\qquad \int_D w(x,t)^2\,dt\,dx = -\int_0^t d\tau \int_D |\operatorname{grad} w|^2\,dx - \int_0^t d\tau \int_B \kappa(\underline{\xi})\,|w(\xi,\tau)|^2\,d\underline{\xi}$$

since $\kappa(\cdot) \geq 0$ it follows that $\int w(x,t)^2\,dx\,d\tau = 0 \quad \forall\, t \geq 0$; i.e. $w(x,t) = 0$ almost everywhere in (x,t).

An alterative proof is the following: by proving (5.2), (5.9) we have implicitely proved that the solution of (1.1) is in the Sobolev space $V_2(D\times(0,T))$ and is a generalized solution of the heat equation. Since we know that there can be only one such solution of the heat equation we deduce that any solution of (1.1) is the same as the one obtained from the variational approach [13] [which, in turn, must have the properties listed in (1.1)].

6. - Sticky barrier and general barrier

Suppose $P_x^+(d\omega)$, $x \in D \cup B^-$ is the reflecting Brownian motion in the polygonal region D.

Given a continuous non negative function $\sigma \in C(\dot{B})$ we can define P_x^+-almost everywhere, the function:

$$(6.1)\qquad \Sigma(t) = t + \int_B \sigma(\underline{\xi})\, \underline{t}_{=B}(\omega, t; d\underline{\xi})$$

where $\underline{t}_{=B}$ is the local time on B introduced in section 4. $\Sigma(t)$ is strictly increasing in t and one can consider

its inverse $\Theta(t)$: i.e. let $\Theta(t)$ be such that $\Theta(\Sigma(t)) \equiv t$. The functions Θ , Σ are defined for P_x^+-almost-all ω's $\in \Omega^+$ and can be used to define a mapping $T_\sigma : \Omega^+ \to \Omega^+$:

$$(T_\sigma \omega)(t) = \omega(\Theta(t)) \tag{6.2}$$

this is P_x^+-almost everywhere defined. Clearly $T_\sigma \omega$ has the same geometric form as ω but is run through with a different law of motion: $T_\sigma \omega$ spends more time on B.

Since T_σ is defined almost-everywhere on Ω^+ and is Borel measurable, it can be used to carry the measure P_x^+ into $P_x^{(0,\sigma)}(d\omega)$ on Ω^+ in such a way that if $f(\omega) \in C(\Omega^+)$:

$$\int f(\omega)\, P_x^{(0,\sigma)}(d\omega) = \int f(T_\sigma \omega)\, P_x^+(d\omega) \tag{6.3}$$

in view of theorem 4 it is hopeful that if $f \in C^\infty(\bar{D})$ then:

$$u(x,t) = \int P_x^{(0,\sigma)}(d\omega)\, f(\omega(t)) \tag{6.4}$$

solves the problem (1.1) with $\varkappa = 0$. The way one proceeds to prove this theorem is clear enough from the proofs of theorem 1' and of the validity of the elastic boundary condition in section 4: i.e. one reduces the problem to the one dimensional case using the strong Markov property of the free Brownian motion (See sect. 2 theorem 4).

The measure $P_x^{(\varkappa,\sigma)}(d\omega)$ associated with the general case of problem (1.1) can also be easily constructed using the local time on B: the solution of problem (1.1) is in fact given by:

$$(6.5)\qquad u(x,t)=\int P_x^{(\kappa,\sigma)}(d\omega)\,f(\omega(t))=\int P_x^{+}(d\omega)\,e^{-I(\kappa,\omega;\vartheta(t))}\,f(\omega(\vartheta(t)))$$

where:

$$(6.6)\qquad \bar{I}(\omega;\vartheta(t))=\int_{\mathcal{B}}\kappa(\underline{\xi})\,\underline{t}_{=\mathcal{B}}(\omega,\vartheta(t);d\underline{\xi})$$

The proof that (6.5) is indeed the solution of (1.1) again proceeds through reduction to the one dimensional case on the same lines as before and will also be skipped

Appendix 1

We prove here the statements of theorem 1' (hence of theorem 1).

Consider the function $u(x,t)$ introduced in (2.18). Since $P_x(d\omega)$ is a probability measure it is clear that for $C \geq \|f\|_\infty$:

$$|u(x,t)| \leq C < \infty \qquad \forall\, t \geq 0 \tag{A.1}$$

Using (2.17) and the strong Markov property (see section 2) we can write:

$$u(x,t) = \int_{\partial Q_{\xi_0,L}} d\underline{\xi} \int_0^t d\eta\, u(\underline{\xi}, t-\eta)\, g_x(\underline{\xi},\eta) + \int_{\{\omega:\, \omega(\tau) \in Q_{\xi_0,L},\ 0\leq\tau\leq t\}} P_x(d\omega)\, \varphi(\omega(t)) \tag{A.2}$$

where $g_x(\underline{\xi},\eta)\, d\underline{\xi}\, d\eta$ is the probability (P_x-measure) on that a path ω leaves x at $t=0$ and reaches $\partial Q_{x_0,L}$ between $\underline{\xi}$ and $\underline{\xi}+d\underline{\xi}$ at a time between η and $\eta+d\eta$.

We can find an explicit expression for $g_x(\underline{\xi},\eta)\, d\underline{\xi}\, d\eta$ using the properties of the one-dimensional Brownian motion.

Choose the origin so that $x_0 = (\frac{L}{2}, \frac{L}{2})$ suppose $x = (x_1, x_2) \in Q_{x_0,L}$ and that $\underline{\xi}$ belongs to the upper side of $\partial Q_{x_0,L}$ i.e $\underline{\xi} = (\xi, L)$ $0 \leq \xi \leq L$. Then if $\omega = (\omega_1, \omega_2)$ with ω_1, ω_2 being the two projection of ω on the coordinate axes (ω_1, ω_2 are two one-dimensional independent Brownian motions) we have:

(A.3) $$\varrho_x(\xi,\tau)\,d\xi\,d\tau = P_{x_1}\left(\{\omega_1 \mid \mu_1 > \tau,\ \omega_1(\tau)\in d\xi\}\right) f_{x_2}(\tau)\,d\tau$$

where μ_1 is the exit time from $(0,L)$ of ω_1 and $f_{x_2}(\tau)$ is the probability distribution that the component ω_2 of ω gets out of $(0,L)$ through L at a time between τ and $\tau + d\tau$.

The two functions appearing in (A.2) are well known from the theory of the one-dimensional Brownian motion [8,9]:

(A.4) $$P_{x_1}\left(\{\omega_1 \mid \mu_1 > \tau,\ \omega_1(\tau)\in d\xi\}\right) = G^{0}_{\tau}(x_1,\xi) = \left(\frac{2}{L}\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2}{2L^2}\tau} \sin\frac{n\pi}{L}x_1 \sin\frac{n\pi}{L}\xi\right) d\xi \qquad 0\le x,\xi\le L$$

where we see that $G^{0}_{\tau}(x,\xi)$ defined by (A.4) is the Green's function for the heat equation in $Q_{x_0,L}$ with vanishing boundary conditions

Furthermore:

(A.5) $$f_{x_2}(\tau) = \frac{\pi}{L^2}\sum_{n=1}^{\infty}(-1)^{n+1} e^{-\frac{n^2\pi^2}{2L^2}\tau} \sin\frac{n\pi}{L}x_2$$

or also:

(A.6) $$\hat{f}_{x_2}(\alpha) = \int_0^{\infty} e^{-\alpha t} f_{x_2}(t)\,dt = \frac{\operatorname{sh}\sqrt{2\alpha}\,x_2}{\operatorname{sh}\sqrt{2\alpha}\,L}$$

Similar formulae hold for the other three sides of $\partial Q_{x_0,L}$.

It is now clear that the contribution to $u(x,t)$ in (A.2) is a sum of convolutions of a bounded function $u(\xi,\tau)$ with a function $g_x(\xi,\tau)$ which is ∞-differentiable in x and τ for $x \in \Omega_{x_0,L}$ and $\tau \geq 0$.

The bulk contribution can be written, using (A.4), (2.16), as

$$\text{(A.7)} \qquad \int_0^L \int_0^L G_t^0(x_1,\xi)\, G_t^0(x_2,\eta)\, \varphi(\xi,\eta)\, d\xi\, d\eta$$

i.e. this contribution is the solution of the heat equation on the square $\Omega_{x_0,L}$ with vanishing boundary conditions and with $\varphi(\xi,\eta)$ as initial data. it is well known that such a solution is infinitely differentiable in x and t for $t > 0$ and is differentiable in the generalized sense at $t = 0$.

The fact that $u(x,t)$ verifies the heat equation in D follows, for instance, from (A.2) and the fact that, as is obvious from (A.3) (A.4), both $f_x(t)$ and $G_t^0(x,\xi)$ verify the one-dimensional heat equation in (x,t) and $f_x(0)=0$.

The property $u(x,0+) = \varphi(x)$, $x \in \Omega_{x_0,L}$, also follows from the explicit representation (A.2): the contribution from $\partial\Omega_{x_0,L}$ tends to zero as $t \to 0$ uniformly for x in a closed subset of $\Omega_{x_0,L}$, while (A.7) tends to $\varphi(x)$ uniformly in the same region. We leave the easy check to the reader.

To find a bound for the gradient consider, for $\underline{\xi} = (\xi, L) \in \partial Q_{x_0, L}$, the function $\frac{\partial}{\partial x_1} g_x(\xi, \tau)$ we have:

$$\left.\frac{\partial g_x(\xi, \tau)}{\partial x_1}\right|_{x = x_0} =$$

(A.8)

$$= \left(\frac{2\pi}{L^2} \sum_{n=1}^{\infty} e^{-\frac{n^2 \pi^2 \tau}{2L^2}} n \cos\frac{n\pi}{2} \sin\frac{n\pi}{L}\xi \right) f_{\frac{L}{2}}(\tau)$$

therefore, for some $\gamma < \infty$:

$$\left| \frac{\partial g_x(\xi, \tau)}{\partial x_1} \right| \le \frac{2\pi}{L^2} \gamma \frac{L^2}{\tau} f_{\frac{L}{2}}(\tau) \tag{A.9}$$

from (A.5), (A.6) it follows that $f_{\frac{L}{2}}(\tau) = \frac{1}{L^2} \psi\left(\frac{\tau}{L^2}\right)$ where $\psi(\vartheta)$ tends to zero exponentially in ϑ as $\vartheta \to \infty$ and in $\frac{1}{\vartheta}$ as $\vartheta \to 0$. Therefore the contribution to $\frac{\partial u(x, t)}{\partial x_1}$ of the upper boundary is not larger than :

$$L \frac{2\pi\gamma}{L^2} C \int_0^t \frac{L^2}{\tau} \frac{1}{L^2} \psi\left(\frac{\tau}{L^2}\right) d\tau \le \frac{2\pi\gamma C}{L} \int_0^{\infty} \frac{\psi(\tau)\, d\tau}{\tau} \tag{A.10}$$

Analogously, for some $\gamma' < \infty$:

$$\left| \frac{\partial g_x(\xi, \tau)}{\partial x_2} \right|_{x_0} =$$

(A.11)

$$\left| G^{\circ}_{\tau}\left(\frac{L}{2}, \xi\right) \left.\frac{\partial f_{x_2}(\tau)}{\partial x_2}\right|_{x_2 = \frac{L}{2}} \right| \le \frac{\gamma'}{L} \sqrt{\frac{L^2}{\tau}} \left| \left.\frac{\partial f_{x_2}(\tau)}{\partial x_2}\right|_{x_2 = \frac{L}{2}} \right|$$

but from (A.6) it follows that $\frac{\partial f_{x_2}(m_t)}{\partial x_2}\Big|_{x_2=\frac{L}{2}} = \frac{1}{L^3}\chi\left(\frac{m_t}{L^2}\right)$ with $\chi(\vartheta)$ tending to zero exponentially in ϑ as $\vartheta \to \infty$ and in $\frac{1}{\vartheta}$ as $\vartheta \to 0$. Therefore the contribution from the upper boundary to $\frac{\partial u}{\partial x_2}(x,t)\Big|_{x=x_0}$ is not larger than

$$(A.12)\quad L\frac{\gamma' c}{L}\int_0^t \sqrt{\frac{L^2}{m_t}}\ \frac{1}{L^3}\chi\left(\frac{m_t}{L^2}\right)dm_t \le \frac{\gamma' c}{L}\int_0^{t\infty}\frac{\chi(\tau)}{\tau}\,d\tau$$

Similar bounds can be obtained for the contributions from the other sides of $\partial\Omega_{x_0,L}$.

Finally we have to consider the contribution from the bulk term (A.7). We have alredy remarked that (A.7) is the solution of the heat equation on a square and with vanishing boundary conditions; on such a domain the equation is elementary and therefore, as is well known, it can be bounded, independently of L, from a constant (dependent only on the dimension of the space) times $\sup_{\Omega_{x_0,L}}|\operatorname{grad}\varphi|$ which in our case is supposed to be finite. This bound can also be obtained directly from (A.7) and (A.4) by tranferring the derivative from χ to ξ and then integrating by parts (to transfer the derivative from χ to ξ one has to write the sines in terms of exponentials and then treat separately each term).

Hence we conclude by adding the contributions from

the boundary $\partial\Omega_{x_0,L}$ and from the bulk that $|\mathrm{grad}\, u(x_0,t)|$ can be bounded at all t by a C' times $\max\left(\|f\|_\infty, \sup_{\Omega_{x_0,L}} |\mathrm{grad}\,\varphi|\right)$ divided by L(with C' bounded as $L \to 0$).

Appendix 2.

Consider the function $V_\varepsilon(x) = \frac{\kappa}{2\varepsilon}$ if $|x| \leq \varepsilon$, $V_\varepsilon(x) \equiv 0$ if $|x| > \varepsilon$.

Consider the eigenvalue equation in $0 < x < \infty$:

$$\frac{1}{2}\frac{d^2 u_\varepsilon}{dx^2} - V_\varepsilon(x)\, u_\varepsilon = -\alpha\, u_\varepsilon$$

$$\frac{du_\varepsilon}{dx}\Big|_{x=0} = 0 \qquad \text{(B.1)}$$

whose solution is, up to a multiplicative constant:

$$u_\varepsilon(x,\alpha) = \cos\sqrt{2\alpha - \frac{\kappa}{\varepsilon}}\, x \qquad 0 < x < \varepsilon$$

$$u_\varepsilon(x,\alpha) = B_\varepsilon \cos\sqrt{2\alpha}\, x + C_\varepsilon \sin\sqrt{2\alpha}\, x \qquad \varepsilon < x < \infty \qquad \text{(B.2)}$$

The constants B_ε and C_ε are determined by the condition of continuity at $x = \varepsilon$ of u and its derivative. We have:

$$B_\varepsilon = \cos\sqrt{2\alpha}\,\varepsilon \cos\sqrt{2\alpha - \frac{\kappa}{\varepsilon}}\,\varepsilon + \sqrt{\frac{2\alpha - \frac{\kappa}{\varepsilon}}{2\alpha}} \sin\sqrt{2\alpha - \frac{\kappa}{\varepsilon}} \sin\sqrt{2\alpha}\,\varepsilon \xrightarrow[\varepsilon \to 0]{} 1 \qquad \text{(B.3)}$$

$$C_\varepsilon = \sin\sqrt{2\alpha}\,\varepsilon\cos\sqrt{2\alpha-\frac{\kappa}{\varepsilon}}\,\varepsilon - \sqrt{\frac{2\alpha-\frac{\kappa}{\varepsilon}}{2\alpha}}\cos\sqrt{2\alpha-\frac{\kappa}{\varepsilon}}\,\varepsilon\cos\sqrt{2\alpha}\,\varepsilon \xrightarrow[\varepsilon\to 0]{} \frac{\kappa}{\sqrt{2\alpha}}$$

Hence we see that $\lim_{\varepsilon\to 0} u_\varepsilon(x,\alpha) = u_0(x,\alpha)$ where $u_0(x,\alpha)$ is the solution of the eigenvalue problem:

$$\frac{1}{2}\frac{d^2 u_0(x)}{dx^2} = -\alpha u_0(x) \qquad x>0$$

$$\frac{du_0(x)}{dx}\bigg|_{x=0} = \kappa\, u_0(x)\bigg|_{x=0} \tag{B.4}$$

From the above observation and with some technical work one can show that the problem

$$\frac{\partial u_\varepsilon(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u_\varepsilon(x,t)}{\partial x^2} - V_\varepsilon(x)\,u_\varepsilon(x,t) \qquad x,t>0$$

$$\frac{\partial u_\varepsilon(0,t)}{\partial x} = 0 \qquad t>0 \;;\; u_\varepsilon(x,0+) = f(x) \tag{B.5}$$

has a solution $u_\varepsilon(x,t)$ which tends, as $\varepsilon\to 0$, pointwise for $x>0$ to $u_0(x,t)$ solution of the problem:

$$
\begin{aligned}
&\frac{\partial u_0(x,t)}{\partial t} = \frac{1}{2}\,\frac{\partial^2 u_0(x,t)}{\partial x^2} && x,t>0\\
&u_0(x,0+) = f(x)\\
&\frac{\partial u_0(0,t)}{\partial x} = \kappa\, u_0(0,t) && t>0
\end{aligned}
\tag{B.6}
$$

On the other hand the solution to (B.5) can easily be expressed in terms of the Feynman-Kac formula [10].

$$
u_\varepsilon(x,t) = \int P_x(d\omega)\left(\exp -\int_0^t V_\varepsilon(\omega(\tau)d\tau\right) f(|\omega(\tau)|) \qquad x>0 \tag{B.7}
$$

and we realize that

$$
\int_0^t V_\varepsilon(\omega(\tau))\,d\tau = \kappa\, \underline{t}_{0,\varepsilon}(\omega,t) \tag{B.8}
$$

From (B.8) and the fact that $u_\varepsilon(x,t) \xrightarrow[\varepsilon\to 0]{} u_0(x,t)$ it is possible to conclude (again after some measure theoretic work) that in some average sense the limit as $\varepsilon \to 0$ of $\underline{t}_{0,\varepsilon}(\omega,t)$ exists.

The fact that this limit exists pointwise and almost-everywhere is more delicate and comes from the evaluation of integrals of the type:

$$\int P_x(d\omega)\int_t^{t'} V_\varepsilon(\omega(\tau))\,d\tau$$

which allow to find continuity properties in t of $t_{o,\varepsilon}(\omega,t)$ (for almost all ω) which hold uniformly in ε as $\varepsilon \to 0$.

We observe that the above remarks give almost a proof of theorem 3. the question that is left open is the existence of the derivatives at $x = o$. This point could be proved with a little work but we will not do that because the spirit of this appendix is to only provide some intuition to the reader. For a formal proof see [5].

The intuitive background to theorem 4 is explained below: again what is discussed here is very near to a proof (which would need only a little more work). A formal proof is in [6].

Let $\delta_\varepsilon(x) = 2\left(\frac{\sigma}{2\varepsilon} + 1\right)$ for $|x| < \varepsilon$, $\delta_\varepsilon(x)$ for $|x| > \varepsilon$.

Consider the eigenvalue problem:

$$\frac{1}{\delta_\varepsilon(x)}\frac{d^2}{dx^2}u_\varepsilon = -\alpha v_\varepsilon \qquad x > 0$$
$$\frac{du_\varepsilon}{dx}\Big|_{x=o} = 0 \tag{B.9}$$

proceding as before we see that the pointwise limit

$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u_0(x)$ where $u_0(x)$ is the solution to the problem:

$$\text{(B.10)} \qquad \begin{aligned} \frac{1}{2}\,\frac{d^2 u_0}{dx^2} &= -\alpha u_0 \\ \left.\frac{du_0}{dx}\right|_{x=0} &= \frac{\sigma}{2}\,\frac{d^2 u_0}{dx^2} \end{aligned}$$

Again after some measure theoretic work this fact implies that the problem

$$\text{(B.11)} \qquad \begin{aligned} &\frac{\partial u_\varepsilon(x,t)}{\partial t} = \frac{1}{s_\varepsilon(x)}\,\frac{\partial^2 u(x,t)}{\partial x^2} && x>0 \\ &u_\varepsilon(x,0+) = f(x) && \text{with } f(x)\equiv 0 \text{ for large } x \\ &\frac{\partial u_\varepsilon(0,t)}{\partial x} = 0 && t>0 \end{aligned}$$

has a solution $u_\varepsilon(x,t)$ which as $\varepsilon \to 0$ tends to $u_0(x,t)$ solution of:

$$\text{(B.12)} \qquad \begin{aligned} &\frac{\partial u_0(x,t)}{\partial t} = \frac{1}{2}\,\frac{\partial^2 u_0(x,t)}{\partial x^2} && x>0 \\ &u_0(x,0+) = f(x) \\ &\frac{\partial u_0(0,t)}{\partial x} = \frac{\sigma}{2}\,\frac{\partial^2 u_0(0,t)}{\partial x^2} \end{aligned}$$

On the other hand it is possible to write the solution to (B.11) in terms of the Brownian motion: consider the

function:

$$\text{(B.13)} \qquad \overline{\Sigma}_\varepsilon(t) = t + \sigma \int_0^t d\underline{t}_{-o,\varepsilon}(\omega,\tau)$$

and let Θ_ε be its inverse $\Theta_\varepsilon(\Sigma_\varepsilon(t)) \equiv t$, $t \geq 0$. $\overline{\Sigma}_\varepsilon$, Θ_ε are defined for almost all paths (indeed they are defined on the continous paths).

Suppose $|x| < \varepsilon$ and consider a little interval $[-L,L]$ containing x and entirely contained in $(-\varepsilon,\varepsilon)$. Let η be the exit time of x from $(-L,L)$. The function:

$$\text{(B.14)} \qquad u_\varepsilon(x,t) = \int P_x(d\omega)\, f(|\omega(\Theta_\varepsilon(t))|)$$

can be written for the x under consideration as:

$$\text{(B.15)} \qquad \begin{aligned} u_\varepsilon(x,t) = & \int_{\{\omega \mid \omega(\Theta(\tau)) \in (-L,L),\ o \leq \tau \leq t\}} P_x(d\omega)\, f\left(\left|\omega\left(\frac{t}{1+\frac{\sigma}{2\varepsilon}}\right)\right|\right) + \\ & + \int_0^t P_L(x,\eta)\, d\eta\, u_\varepsilon(L, t-\eta) + \\ & + \int_0^t P_{-L}(x,\eta)\, d\eta\, u_\varepsilon(-L, t-\eta) \end{aligned}$$

where $P_{\pm L}(\tau)d\tau$ is the probability that the path $\omega(\Theta(t)) = \omega\left(\frac{t}{1+\frac{\sigma}{2\varepsilon}}\right)$ reaches $\pm L$ starting from x for $t \in (\tau, \tau+d\tau)$, without hitting any of the two sides for $\tau < \tau$.

Since, up to τ $\Theta(t) = \frac{t}{(1+\frac{\sigma}{2\varepsilon})}$ it is easy to write down explicit expressions for P_{tL} and to check on these expressions that $u_\varepsilon(x,t)$ is such that

$$\frac{1}{2\left(1+\frac{\sigma}{2\varepsilon}\right)} \frac{d^2 u_\varepsilon}{dx^2} = \frac{\partial u_\varepsilon}{\partial t} \qquad |x| < \varepsilon$$

(B.16)
$$\frac{\partial u_\varepsilon}{\partial x}\Big|_{x=0} = 0$$

$$u_2(x, 0+) = f(x) \qquad |x| < \varepsilon$$

An analogous calculation shows that for $x > \varepsilon$, $u_\varepsilon(x,t)$ verifies the equations $u_\varepsilon(x,0) = f(x)$, $\frac{1}{2}\frac{d^2 u_\varepsilon}{dx^2} = \frac{\partial u_\varepsilon}{\partial t}$

Finally it is also possible to check that both u_ε and $\frac{\partial u_\varepsilon}{\partial x}$ are continous through ε. Therefore (B. 14) is the solution of the problem (B.9).

Using the fact that, as $\varepsilon \to 0$, $u_\varepsilon(x,t) \to u_o(x,t)$, where u_o is the solution to the problem (B.12), and using theorem 2, we see that

$$u_0(x,t) = \int P_x(d\omega)\, f(\omega(\Theta(t))) \tag{B.17}$$

where $\Theta(t)$ is P_x-almost- everywhere defined and:

$$\Theta(t) = \lim_{\varepsilon \to 0} \Theta_\varepsilon(t) = \Sigma^{(-1)}(t) \tag{B.18}$$

$$\Sigma(t) = t + \sigma \int_0^t d\underline{t}_{-0}(\omega,\tau)$$

and this ends our intuitive interpretation of theorem 4 (for a complete prove we still lack of the existence of the derivatives on the boundary, which, however, at this stage is merely a technicality). For a formal and complete proof see [6].

Appendix 3

We shall here compute explicit expressions for (4.9), (4.10).

In appendix 2 we have used the fact that the Green's functions for the one dimensional elastic barrier is the even part (i.e. the part obtained considering only the even eigenfunctions) of the $\varepsilon \to 0$ limit of the problem:

$$(C.1)\quad \frac{1}{2}\frac{d^2u}{dx^2} - V_\varepsilon(x)\,u = -\alpha u \qquad V_\varepsilon(x) \begin{cases} 0 & |x|>\varepsilon \\ \frac{\kappa}{2\varepsilon} & |x|<\varepsilon \end{cases}$$

It is easy to derive from this remark explicit expressions for (4.9) (4.10). Since what is relevant in (4 8) is only the even part of (4.9) (even part means a measure on $[-\frac{L}{2},\frac{L}{2}]$ which gives the same value as (4.9) to the integral of any even function) and the function $\int_{x,+}^{\kappa}(\eta)+\int_{x,-}^{\kappa}(\eta)$ we shall confine our calculation to the evaluation of these objects.

Formula (4.9) defines the Green's function for the $\varepsilon\to 0$ limit of the problem (C.1) with zero boundary condition ($u(\frac{L}{2})=0=u(-\frac{L}{2})$). the normalized even eigenvalues for such a problem are, in the limit $\varepsilon\to 0$,

$$(C.2)\quad \psi_\alpha(x) = \frac{\sin\left(\sqrt{2\alpha}\,|x|+\varphi(\sqrt{2\alpha})\right)}{L\left(1-\frac{\sin\sqrt{2\alpha}\,L}{\sqrt{2\alpha}\,L}\right)}$$

where

$$(C.3)\quad \varphi(\lambda) = \operatorname{arctg}\frac{\lambda}{\kappa}$$

$$(C.4)\quad \sqrt{2\alpha}\,\frac{L}{2}+\varphi(\sqrt{2\alpha}) = n\pi \qquad n=0,1,\ldots$$

therefore the even part of the Green's function is (if $\lambda=\sqrt{2\alpha}$):

$$(C.5)\quad G_t^{even,L,\kappa}(x,\varphi)=\sum_{n=0}^{\infty} e^{-\lambda_n^2\frac{t}{2}}\ \frac{\sin(\lambda_n|x|+\varphi(\lambda_n))\sin\lambda_n|y|+\varphi(\lambda_n))}{L\left(1-\frac{\sin\lambda_n L}{\lambda_n L}\right)}$$

It is important that the minimum of the denominator in (C 5) is not o but $\geq L\left(1-\frac{1}{\pi}\right)$.

The function $f_x^{\kappa}(m)=f_{x,+}^{\kappa}(m)+f_{x,-}^{\kappa}(m)$ is related to (C 5) by

$$(C.6)\quad \int_0^t f_x^{\kappa}(m)\,dm+\int_{-\frac{L}{2}}^{+\frac{L}{2}} d\xi\, G_t^{L,\kappa}(x,\xi)=$$
$$=\int_{-\infty}^{+\infty} d\xi\ G_t^{\infty,\kappa}(x,\xi)=\lim_{L'\to\infty}\int_{-\frac{L'}{2}}^{+\frac{L'}{2}} G_t^{L',\kappa}(x,\xi)\,d\xi$$

which, if written in terms of the paths using (4 9), (4.10), has an obvious meaning.

We have $\int_{-\frac{L}{2}}^{+\frac{L}{2}} G_t^{L,\kappa}(x,\xi)\,d\xi=\int_{-\frac{L}{2}}^{\frac{L}{2}} G_t^{even,L,\kappa}(x,\xi)\,d\xi$ and

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} G_t^{even,L,\kappa}(x,\xi)\,d\xi=2\sum_n \frac{e^{-\lambda_n^2\frac{t}{2}}}{\lambda_n}\sin\left(\lambda_n|x|+\varphi(\lambda_n)\right)\cdot$$

$$\cdot\ \frac{\cos\varphi(\lambda_n)-(-1)^n}{L\left(1-\frac{\sin\lambda_n L}{\lambda_n L}\right)}=\left(\text{using } \cos\left(\lambda_n L+\varphi(\lambda_n)\right)=(-1)^n\right)=$$

$$= 2\sum_n \frac{e^{-\lambda_n^2 \frac{t}{2}}}{\lambda_n} \sin\left(\lambda_n |x| + \varphi(\lambda_n)\right) \cos\varphi(\lambda_n) \cos\left(\lambda_n L + \varphi(\lambda_n)\right) \frac{(-1)^n}{\left(1 - \frac{\sin\lambda_n L}{\lambda_n L}\right)} +$$

(C.7)

$$- 2\sum_n \frac{e^{-\lambda_n^2 \frac{t}{2}}}{\lambda_n} \sin\left(\lambda_n |x| + \varphi(\lambda_n)\right) \frac{(-1)^n}{\left(1 - \frac{\sin\lambda_n L}{\lambda_n L}\right)}$$

we now remark that $(-1)^n \left(1 - \frac{\sin\lambda_n L}{\lambda_n L}\right)^{-1}$ is the residuum in λ_n of the function $\sin\left(\lambda \frac{L}{2} + \varphi(\lambda)\right)$; therefore (C.7) can be written as a complex integral over a contour in the λ-plane which goes clockwise around the positive axis; computing explicitely the right hand side of (C.7) (which is the limit as $L \to \infty$ of (C.7)) and then differentiating (C.6) with respect to t we find:

$$f_x^x(t) = -\frac{1}{2\pi i} \oint \varkappa\lambda d\lambda \, e^{-\lambda^2 \frac{t}{2}} \frac{\varkappa \sin\lambda|x| + \lambda\cos\lambda|x|}{\varkappa^2 + \lambda^2} \frac{\varkappa\cos\lambda L - \lambda\sin\lambda L}{\lambda\cos\lambda L + \varkappa\sin\lambda L} +$$

$$(C.8) + \frac{1}{2\pi i} \oint \lambda d\lambda \, e^{-\lambda^2 \frac{t}{2}} \frac{\varkappa\sin\lambda|x| + \lambda\cos\lambda|x|}{\varkappa\sin\lambda L + \lambda\cos\lambda L} - \frac{1}{2\pi} \int_0^{+\infty} \varkappa\lambda d\lambda \, e^{-\lambda^2 \frac{t}{2}} \cdot$$

$$\cdot \frac{\varkappa\sin\lambda|x| + \lambda\cos\lambda|x|}{\varkappa^2 + \lambda^2} = -\frac{1}{2\pi i} \oint \varkappa d\vartheta \, e^{-\vartheta \frac{t}{2}} \frac{\varkappa \sin\sqrt{\vartheta}|x| + \sqrt{\vartheta}\cos\sqrt{\vartheta}|x|}{\varkappa^2 + \vartheta} \cdot$$

$$\cdot \frac{\cos\sqrt{\vartheta}L - \sqrt{\vartheta}\sin\sqrt{\vartheta}L}{\sqrt{\vartheta}\cos\sqrt{\vartheta}L + \varkappa\sin\sqrt{\vartheta}L} + \frac{1}{2\pi i}\oint d\vartheta\, e^{-\vartheta\frac{\tau}{2}} \frac{\varkappa\sin\sqrt{\vartheta}|x| + \sqrt{\vartheta}\cos\sqrt{\vartheta}|x|}{\varkappa\sin\sqrt{\vartheta}L + \sqrt{\vartheta}\cos\sqrt{\vartheta}L} +$$

(C.8)

$$-\frac{1}{2\pi}\oint d\vartheta\, e^{-\vartheta\frac{\tau}{2}} \frac{\varkappa\sin\sqrt{\vartheta}|x| + \sqrt{\vartheta}\cos\sqrt{\vartheta}|x|}{\varkappa^2 + \vartheta}$$

changing $\vartheta \to -p$ we find

$$f_x^{\varkappa}(t) = -\frac{1}{2\pi i}\oint \varkappa dp\, e^{p\frac{\tau}{2}} \frac{\varkappa\, sh\sqrt{p}|x| + \sqrt{p}\, ch\sqrt{p}|x|}{\varkappa^2 - p}\left[\frac{\varkappa ch\sqrt{p}L + \sqrt{p}sh\sqrt{p}L}{\sqrt{p}ch\sqrt{p}L + \varkappa sh\sqrt{p}L} - \right.$$

(C 9)

$$\left. - 1\right] + \frac{1}{2\pi i}\oint dp\, e^{p\frac{\tau}{2}} \frac{\varkappa\, sh\sqrt{p}|x| + \sqrt{p}\, ch\sqrt{p}|x|}{\varkappa\, sh\sqrt{p}L + \sqrt{p}\, ch\sqrt{p}L}$$

the contour being around the negative real axis in the λ-plane.

We easily recognize from the above formula that:

$$\text{(C.10)}\quad \hat{f}_x^{\varkappa}(p) = \int_0^{\infty} e^{-pt} f_x^{\varkappa}(t)\, dt = \frac{\varkappa\, sh\sqrt{2p}|x| + \sqrt{2p}\, ch\sqrt{2p}|x|}{\varkappa\, sh\sqrt{2p}L + \sqrt{2p}\, ch\sqrt{2p}L} +$$

$$(C.10)\quad +\varkappa\,\frac{\varkappa \operatorname{sh}\sqrt{2p}\,|x| + \sqrt{2p}\operatorname{ch}\sqrt{2p}\,|x|}{\varkappa^2 - 2p}\qquad\left[\frac{\varkappa \operatorname{ch}\sqrt{2p}\,L + \sqrt{2p}\operatorname{sh}\sqrt{2p}\,L}{\sqrt{2p}\operatorname{ch}\sqrt{2p}\,L + \varkappa \operatorname{sh}\sqrt{2p}\,L} - 1\right]$$

For the particular case $\varkappa = 0$ see [9].

Formula (C.10) implies that $\hat{f}^{\varkappa}_{\varkappa}(p) \to 0$ exponentially in $\sqrt{p}$ if $p \to \infty$ and also that $\hat{f}^{\varkappa}_{\varkappa}(p)$ is analytic around $p = 0$; so $f^{\varkappa}_{\varkappa}(t) \to 0$ exponentially in t as $t \to \infty$ and $f^{\varkappa}_{\varkappa}(t) \to 0$ exponentially in $\frac{1}{t}$ as $t \to 0$.

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4 - -----: pag. 40 - 45

5 - -----: pag. 45

6 - -----: pag. 186

7 - -----: pag. 45. Problem 3

8 - -----: pag. 30. Problem 8

9 - -----: pag. 29. Problem 6

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ERRATA CORRIGE

Pag. 10: Inserire, dopo la formula (2.4), la frase: "Formula(2.4) holds true also if $\sigma(\omega)$ is replaced by a function $\sigma(\omega, m(\omega))$ where $\sigma(\omega, t)$ is measurable for each t and t-continous for almost all ω's ".

Pag. 10: Inserire alla terz'ultima riga "near Γ " fra bounded e by.

Pag. 13: Inserire alla riga 9, prima di s. suppose: "measurable for each t and t-con-inou tinous for ω continous.".

Pag. 15: Inserire alla riga 14, dopo passes, la frase: ",for t>o,".

Pag. 19: Inserire alla riga 3, dopo go, la frase: ",for t>o,".

Pag. 20: Inserire alla penultima riga: "and that the 1-axis points inwards to D".

Pag. 25: Inserire alla riga 5: "(for small L)".

Pag. 26: Alla penultima riga aggiungere 4 dopo section. All'ultima riga cambiare il 4 in[14] .

Pag. 48: Aggiungere la nota [14] il cui testo è il seguente:"Strictly speaking,the local time measure $\underline{l}_\beta$ has been defined only for $\omega \in \Omega^\kappa$ or $\omega \in \Omega$ .However it is clear that if $(\omega)^t = (\omega')^t$ for $\omega, \omega' \in \Omega^\kappa$ then $\underline{l}_\beta(\omega, t, d\underline{\xi}) \equiv \underline{l}_\beta(\omega', t, d\underline{\xi})$. Therefore $\underline{l}_\beta(\omega, t, d\underline{\xi})$ is defined in a natural way if $\omega \in \Omega^t$".