Time evolution problems in classical statistical mechanics and the wind-tree model

GIOVANNI GALLAVOTTI

Rockefeller University, New York 10021[1]

1 Introduction

The main problem of the statistical dynamics is the explanation of the approach to macroscopic equilibrium for large systems of interacting particles.

The discovery of the Boltzmann equation [1] and of its approximate solutions [2] leading to the various hydrodynamic equations have provided a first answer and a first insight into the microscopic mechanisms underlying the macroscopic evolution. However it is now well realized that the Boltzmann equation can hold rigorously only in very special limiting cases [3] and, furthermore, because of its phenomenological character, this equation leaves open the famous question of how a microscopically reversible dynamic can lead to an irreversible macroscopic evolution.

Bogoliubov has been the first to successfully relate the Liouville equations to the Boltzmann equation and to the hydrodynamic behaviour [4]. Although he did have a considerable success and found generalizations of the Boltzmann equation expected to hold for "dense" gases (i.e. for gases so dense that the many particle collisions are not neglegible), one cannot consider Bogoliubov's theory as completely satisfactory especially from the point of view of the mathematical physics.

[1] Physique Théorique, C.N.R.S., chemin Joseph-Aiguier, Marseille.

In this lecture we shall review a number of problems which arise when one tries to rigorously investigate the evolution of macroscopic system from the microscopic dynamics. After having considered the problems in the general case of continuous classical systems of interacting particles and given, without proof, the few existing results we shall reconsider the same problems in the simpler case of the wind-tree model which will allow us to pursue a deeper investigation. In the case of the wind-tree model the results will be presented and proved in some detail.

2 Formal derivation of the time evolution equations

In what follows we shall consider our macroscopic system as infinite. We shall make this assumption not for aesthetic reasons but because this is apparently the simplest way to get rid of the boundary conditions at the boundaries of finite systems which, if one wants to be realistic, have to be very complicated and eventually lead any way to consider infinite systems (i.e. one should specify the state of the walls surrounding the system and the mechanism of interaction of the walls with their own surroundings).

We shall see that, considering the system as infinite, it will be possible to realize and, to some extent, study the physical situation arising when one considers the evolution of a finite system in contact with a heat reservoir.

In other words one should look, as we shall see in detail later, at the consideration of an infinite system as a tool for describing in convenient and mathematically handable form the boundary conditions.

We remark that a mathematically handable situation would seem to arise by considering a finite (but large) system with ideal boundary conditions, e.g. smooth hard walls with reflection conditions; in this case one would not encounter the difficulties coming from the presence of an infinite number of particles but the whole problem will by no means be simpler than the one connected with the infinite system because now new difficulties will be met in the study of the long time behaviour, e.g. in the approach to equilibrium because of the quasi-periodic nature of the evolution of finite systems related to the well-known Poincaré cycles [6].

The above "apology" for using infinite systems has been done because their use is sometimes regarded as a useless luxury and not merely as a possible and sometimes even fruitful way of attack of a difficult problem.

Suppose, now, that the system is constituted of identical particles whose phase coordinates (p, q) will be denoted by x and suppose that the hamiltonian of these particles is

$$H(x_1, x_2, \ldots) = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} \varphi(q_i - q_j) \tag{2.1}$$

where m is the mass of the particles and φ is a (differentiable) interaction potential which should be nice enough in order that the bulk thermodynamics exists: i.e. it should be stable and such that

$$|\varphi(q)| \underset{q\to\infty}{<} \frac{C}{|q|^{\nu+\varepsilon}} \quad \text{for some } \varepsilon \tag{2.2}$$

where v is the dimension of the space.

We denote $S_t(x_1 \ldots x_n)$ the configuration into which $(x_1, \ldots, x_n)$ n finite evolves in time under the Hamilton equations:

$$\begin{aligned} p_i &= -\frac{\partial H}{\partial q_i} \\ q_i &= \frac{\partial H}{\partial p_i} \end{aligned} \tag{2.3}$$

Now the problem of the time evolution can be put as follows: given an initial state described by a set of correlation functions

$$\rho(x_1, \ldots, x_n; 0) \quad n = 1, 2, \ldots \tag{2.4}$$

how can one describe the state of the system at time $t \neq 0$? (We are giving to the words "correlation functions $\rho(x_1, \ldots x_n; 0)$" the usual meaning of probability densities for finding n particles with phase coordinates in a volume $\frac{dx_1 \ldots dx_n}{n!}$ around $x_1 \ldots x_n$ (where $dx_i = dp_i dq_i$) irrespective of the location of the other particles.)

Of course one expects the answer to be that the state at time t is described by a set of correlation functions $\{\rho(x_1, x_2, \dots x_n; t)\}$ uniquely determined by the initial ones.

We shall devote this and the following sections to the investigation of this problem which, although only preliminary, is already very difficult and can be solved only in special cases. In the remaining part of this section we shall give a formal derivation of the equations which should be verified by $\rho(x_1 \dots x_n t)$.

Consider a finite system enclosed in a box Λ and suppose that the initial distribution is $D_N(x_1, \dots x_N; 0)$: this function gives the probability density for finding inside Λ exactly N particles and to find them in a volume $\frac{dx_1 \dots dx_N}{N!}$ around $x_1, \dots, x_n$. At time t the distribution will be $D_N(x_1 \dots x_N; t) = D_N(S_{-t}(x_1 \dots x_N) j 0)$ where S_t is the evolution operator introduced in connection with the equations (2.3) above. It is immediate to verify that:

$$\frac{\partial D_N(x_1 \dots x_N; t)}{\partial t} = [D_N(x_1 \dots x_N; t), H(x_1 \dots x_N)] \tag{2.5}$$

where $[\cdot, \cdot]$ denotes the Poisson bracket.

The expressions of the correlation functions associated with a distribution $D_N(x_1 \dots x_N; t)$ are, according to their probabilistic meaning:

$$\rho(x_1 \dots x_n; t) = \sum_{m=0}^{\infty} \int_{\Lambda^m} D_{n+m}(x_1 \dots x_n y_1 \dots y_m; t) \frac{dy_1 \dots dy_m}{m!} \tag{2.6}$$

and now, differentiating with respect to time both sides of (2.6) and using (2.5) and (2.1) one finds

$$\begin{aligned} \frac{\partial_\rho(x_1 \dots x_n; t)}{\partial t} &= [\rho(x_1 \dots x_n; t), H(x_1 \dots x_n)] + \\ &+ \sum_{i=1}^{n} \int \frac{\partial \rho(x_1 \dots x_n y; t)}{\partial p_i} \frac{\partial \varphi(q_i - q)}{\partial q_i} dy \end{aligned} \tag{2.7}$$

where $y = (p, q)$, $dy = dpdq$ and we have neglected the presence of the walls of Λ and assumed that $D_N(x_1 \dots x_N; t)$ assumes equal values on

apposite sites on the boundaries of Λ and vanishes for large momenta.

Because of the hidden assumptions described above equation (2.7) can only hold for infinite boxes Λ, but of course the above arguments don't really prove anything.

Equations (2.7) are called the BBGKY [5] hierarchy of integro-differential equations; they relate the n-particles correlation functions to the $(n+1)$-particles ones. These equations are the starting point of the Bogoliubov theory of the Boltzmann equation. There are no existence theorems for the system (2.7) for arbitrarily given initial conditions $\rho(x_1 \dots x_n; 0)$ and in the next section we shall discuss a set of initial conditions which, because of their physical meaning, can be expected to be initial values of a solution of (2.7).

3 The initial states and their evolution

The first difficulty met in a rigorous theory of the time evolution is found in the definition of the initial state. As mentioned in the preceding section, if one wants to avoid the delicate problems of the boundary conditions, one has to consider infinite systems. Once one considers an infinite system it becomes a problem to find "reasonable" initial conditions.

By reasonable we mean conditions which are first, physically realizable, at least approximately, and broad enough to allow a discussion of the hydrodynamic phenomena and, secondly, well defined mathematically. A guide to the definition of convenient initial conditions is given by the following experiment: one has a system which is enclosed in a (finite) container in thermal equilibrium with a heat reservoir; the system has been put in a non-equilibrium situation by some external means (for instance there is a density gradient obtained through an external field etc.), then at a certain instant one suppresses the external cause of non-equilibrium and looks how the systems tends to reach equilibrium with itself and the thermostat. There is a simple way of idealizing the above situation in a mathematically treatable fashion: we can simulate the thermostat and the non-equilibrium cause by considering an infinite system containing identical particles in an initial state which is an equi-

librium state at temperature β^{-1} and density n under the action of the mutual pair potential φ and of an external field h localized in a finite region Λ_h: the fact that the system is infinite and the external perturbation h is localized in a finite region guarantees that our system is in equilibrium, at temperature β^{-1} and density n, far enough from Λ_h i.e. the system is its own container and thermostat and also that the state of our system is a non-equilibrium state in the senise that if at $t=0$ we switch off the external field h (i.e. if we remove the local non-equilibrium cause) the system will no longer be in equilibrium and will evolve towards, as we intuitively expect, the equilibrium state with temperature β^{-1} and density n i.e. the local disturbance will spread out and disappear at infinity (or, by suggestive words, will be absorbed by the heat reservoir). The correlation functions describing the state considered as initial in the above discussion are given by limit the [19]:

$$\rho(x_1 \dots x_n; 0) = \lim_{\Lambda\to\infty} \frac{\sum_{m=0}^{\infty} Z^{n+m} \int_{\Lambda^m} e^{-\beta H(x_1 \dots x_n y_1 \dots y_m) - \beta H \sum_{i}^{n} h(x_i) - \beta \sum_{j}^{m} h(y_j)} \frac{dy_1 \dots dy_m}{m!}}{\sum_{m=0}^{\infty} Z^m \int_{\Lambda^m} e^{-\beta H(y_1 \dots y_m) - \beta \sum j h(y_j)} \frac{dy_1 \dots dy_m}{m!}} \tag{3.1}$$

Where we recognize the usual definition of the correlation functions of an infinite system in equilibrium at temperature β^{-1} and activity Z (we use the grand canonical ensemble and therefore we use the activity Z instead of the density n to define the correlation functions) under the action of the external field h and the Hamiltonian H introduced in (2.1).

At this point we have to pay a first tribute to the elimination of the boundary problems through the consideration of an infinite system: in fact we have to guarantee the existence of the limit (3.1) defining the initial conditions and such limit can be shown (with the present knowledges) to exist only if Z is small enough by using the Kirkwood-Saltzburg equations which are usually discussed in the absence of an external field [7] but which hold also in presence of an external field.

Therefore if we want to discuss the time evolution of the states described above we must restrict ourselves to the case of low activity (i.e. to low density gases).

If Z is small enough the initial state is well defined and one can ask if its time evolution is well defined when, at $t=0$, we suppress the external field h. The time evolving correlation functions are defined by the limit.

$$\rho(x_1 \dots x_n; t) = \lim_{\Lambda\to\infty} \frac{\sum_{m=0}^{\infty} Z^{m+n} \int_{\Lambda^m} e^{-\beta H(x_1 \dots x_n y_1 \dots y_m) - \beta h(S_{-t}(x_1 \dots y_m))} \frac{dy_1 \dots dy_m}{m!}}{\text{(same factor as in (3.1))}} \tag{3.2}$$

where $h(S_{-t}(x_1 \dots y_m))$ denotes the external potential evaluated on the configuration $S_{-t}(x_1 \dots y_m)$ and we have suppressed the evolution operator from $H(\dots)$ because of the energy conservation. To avoid confusion we remember that the evolution operator S_t corresponds to the equations of motion (2.3) (i.e. to the hamiltonian (2.1)) and therefore S_t has nothing to do with the external field h (suppressed at $t=0$) [20].

The expression (3.2) resembles to the (3.1) and one is tempted to try to prove the existence of the limit (3.2) at low Z by regarding $h(S_{-t}(x_1 \dots x_n))$ as an external (time dependent) potential energy and using the Kirkwood-Saltzburg equations. However this is not possible because the theory of the Kirkwood-Saltzburg equations which allows to prove the existence of the infinite volume correlation functions at low Z uses the fact that the potentials acting on the particle are short range potentials (e.g. the mutual potential should verify condition (2.2) and the external potentials should be concentrated in finite regions): these short range conditions are realized at $t=0$ by hypothesis but cannot be guaranteed for $t\neq 0$ because $h(S_{-t}(x_1 \dots x_n))$ a priori depends on all the particles $x_1 \dots x_n$ even if they are very far from the region Λ_h when h is non-vanishing and our knowledge of the properties of the operator S_t (i.e. of the solutions of the equations of motion) is not good enough to give evaluations on how fast and how much a particle which is very far from Λ_h can influence the motion of a particle inside Λ_h. These evaluations would be needed to estimate the range of the potentials giving rise to the potential energy $h(S_{-t}(x_1 \dots x_n))$ [21].

The point is that if there are n particles it will, a priori, take a time increasingly shorter as n increases in order that a far particle influences a particle originally in Λ_h and, in (3.2), the number of particles in $h(S_{-t}(x_1 \dots y_m))$ can be arbitrarily large.

From the physical point of view one expects that for essentially all the configurations the particles which are far do not influence, in a given time t, a particle which is originally in a fixed region, say Λ_h, and therefore apart from "zero probability" sets of "pathological" configurations the potentials giving rise to $h(S_{-t}(x_1 \dots x_n))$ are short range potentials and it should be possible to prove the existence of the limit (3.2) either using the Kirkwood-Saltzburg equations or by some other method based on this finite range property. That the situation is really of this kind is illustrated by the case of one dimensional systems with finite range twice differentiable mutual interaction φ: for such systems it has been shown that the set of configurations not showing the above pathologies in their time evolution is large enough to allow the proof of very general results on time evolution of infinite systems [8] and to allow a proof of the existence of the limit (3.2) at low Z and also of the fact that the functions defined by (3.2) obey the BBGKY hierarchy [9] and therefore provide the first example of an existence theorem of a solution of the system (2.7) with initial condition of the type discussed above [22].

4 Other problems on the time evolution

As we see the results on the preliminary problem of the existence of the time evolution for infinite systems are very partial and therefore unsatisfactory.

Furthermore there are a number of problems to be solved after one has a solution to the existence problem and that are waiting for a solution even in the one dimensional case where the existence problem can be solved.

A list of some of these problems is the following:

1) Are the functions $\rho(x_1 \dots x_n; t)$ analytic in Z at small Z?
2) If (1) is true, does the radius of convergence of the power series of $\rho(x_1 \dots; t)$ depend on t?
3) Is it true that $\lim_{t\to\infty} \rho(x_1 \dots x_n; t) = \hat{p}(x_1 \dots x_n)$ where $\hat{p}(\dots)$ are the equilibrium correlation functions at temperature β^{-1} and activity Z (i.e. the ρ corresponding to $h = 0$)?
4) Do the hydrodynamic coefficients exist?

5) How the approximate validity of the Boltzmann equation can be rigorously established?

Let us illustrate the importance of the above questions.

Question (1), (2) are interesting because one can easily derive the coefficients of a formal power series in Z of the form

$$\rho(x_1 \dots x_n; t) = \sum_{N=0}^{\infty} Z^N c_N(x_1 \dots x_n; t) \tag{4.1}$$

but the derivation leaves open the question of the convergence [10]. Assuming the convergence of the series (4.1) it makes sense to ask question (2); the expressions of the coefficients $c_N(x_1 \dots x_N; t)$ have a behaviour in t which is asymptotically for large t, proportional to t^N [11] and, therefore, the radius of convergence a priori shrinks to zero as t^{-1} (assuming that it is different from zero).

However in spite of the t^N behaviour of the coefficients c_N it could happen that the radius of convergence does not shrink to zero at all and this would imply the existence of nice relations between all the coefficients c_N which could be exploited to make a resummation of the series (4.1) to obtain a new series (no longer a power series in Z) which is less singular than (4.1) as $t\to\infty$ and therefore is more suitable to investigate the asymptotic behaviour as $t\to\infty$.

The resummation of the series (4.1) would be particularly welcome because the evaluation of the hydrodynamic coefficients based on (4.1) in power series in Z gives rise to divergent coefficients to all orders of the series (except the lowest one) [12]. This fact gives a new importance to question (2) and to the conjecture that the radius of convergence does not shrink to zero.

The importance of questions (3), (5) above is clearly evident. We should stress that no one of the above questions has, so far, been answered even partially and one of the reasons for this failure is the lack of good techniques to deal with the problem discussed in section 3 and concerning the existence of the functions $\rho(x_1 \dots x_n; t)$.

It is therefore of great interest to investigate the questions (1), (5) in less realistic models so simplified that at least some of the above questions can be answered.

There are a number of models for which the problem of the existence of the time depending correlation functions is either trivial or solved. An example is given by the quantum spin systems on a lattice [13], but for this model nothing can, by now, be said about the problems raised in (1)÷(5). Other examples are given by soluble models like the (trivial) free gas or the system of hard rods in one dimension [14]: this last model has been solved with a different kind of initial state but the results are compatible with the conjectures of a constant radius of convergence for the $\rho(x_1 \dots x_n; t)$ and with the existence of the resummation discussed above; it would be of some interest a solution of this hard rods model with the initial conditions we are considering.

Here we want to examine another class of models for which the problem of existence is easily solved because one can control the phenomenon of the influence of far particles on the motion of particles in a given region but the question (1)÷(5) are non trivial and can be partially answered; the models are the so-called wind-tree models one of which will be described and discussed in detail in the following few sections.

5 The wind-tree model

This model consists in a system with two kinds of particles: a light particle (the wind-particle) and particle with a large mass (the tree-particle). The light particles do not interact between themselves but they interact with the trees which are supposed so heavy that they do not recoil in the shocks and, therefore, the wind particle is scattered from the surface of the trees (which we suppose to be, as far as the wind particles are concerned, hard spheres), according to the law of equal angles of incidence and reflection, without changing the absolute value of velocity.

The trees are supposed to be distributed randomly and one obtains different models by assigning different tree-distributions.

For instance one could suppose that the trees are in a distribution which corresponds to the equilibrium distribution of a gas of particles interacting through a certain given pair potential Φ at a temperature β^{-1} and density n.

In order to give an illustration of the underlying ideas we shall consider the easiest case in which the trees are a perfect gas i.e. their distribution is the one which corresponds to a pair potential $\Phi \equiv 0$. More concretely the distribution of the trees is given by the set of correlation functions:

$$\rho(c_1, \dots c_N) = n^N \tag{5.1}$$

where c_i denote the spatial coordinates of the trees (we never consider the distribution of the momenta of the trees since they are supposed to be infinitely heavy and therefore have zero momentum) and n is the density of the trees. A more useful way of describing the tree distribution is obtained by giving the probability density $\mathscr{P}_\Lambda(c_1 \dots c_N)$ for finding exactly N-trees inside a volume and for finding them within $dc_1 \dots dc_N$ around $c_1 \dots c_N$; from (5.1) it follows that this probability density is given by the Poisson distribution [15]:

$$\mathscr{P}_\Lambda(c_1 \dots c_N) = e^{-V(\Lambda)n} \frac{n^N}{N!} \tag{5.2}$$

Let us now suppose that at time $t = 0$ the wind particles are described by a set of correlation functions which correspond to the equilibrium correlation functions of the wind at temperature β^{-1} and activity Z in interaction with the trees and with an external bounded potential h localized in a finite region Λ_h. We suppose, as said above, that the trees are distributed as a perfect gas at temperature β^{-1} and density n. A precise mathematical definition of the initial state can be given as follows: suppose the system finite and enclosed in a cubic region K and suppose also that the trees are fixed in position $c_1 \dots c_N$, then one would find for the wind the following correlation functions:

$$\rho_{c_1 \dots c_N}(x_1, \dots x_m; 0) = Z^m e^{-\beta \sum_i \frac{p_i^2}{2m}} \chi_{c_1 \dots c_N}(x_1 \dots x_m) e^{-\beta \sum_i h(q_i)} \tag{5.3}$$

where $x_i = (p_i, q_i)$ are the coordinates of a wind particle and $\chi_{c_1 \dots c_N}(x_1 \dots x_m)$ denotes 1 if the positions $q_i\, i = 1, \dots m$ are outside the spheres associated with the trees $c_1 \dots c_N$ and zero otherwise—Formula (5.3) gives simply the correlation functions of a free wind gas which is not allowed to occupy the volume already occupied by the spheres associated with the trees and is subject to the action of an external field. Now to

take into account that the trees are not fixed but distributed randomly we have to average over all the coordinates $c_1 \dots c_N$ and we obtain using (5.2):

$$\rho(x_1 \dots x_m; 0) = \langle p_{\dots}(x_1 \dots x_m; 0) \rangle = \tag{5.4}$$

$$= \lim_{K\to\infty} e^{-nV(K)} \sum_{M=0}^{\infty} n^M \int_K \frac{dc_1 \dots dc_M}{M!} \rho_{c_1 \dots c_M}(x_1 \dots x_n; 0)$$

Let us now look at the time evolution of the initial state described by (5.4). Define $S_t^{c_1 \dots c_M}(x_1 \dots x_m)$ to be the configuration into which $x_1 \dots x_m$ evolves in presence of the trees $c_1 \dots c_M$ in the time t: then the correlation functions at time t are defined by

$$\rho(x_1 \dots x_m; t) = \lim_{K\to\infty} e^{-nV(K)} \sum_{M=0}^{\infty} n^M \int_K \frac{dc_1 \dots dc_M}{M!}$$

$$\times \rho_{c_1 \dots c_M}(S_{-t}^{c_1 \dots c_M}(x_1 \dots x_m); 0) \tag{5.5}$$

We now are faced, as in the realistic case, with the problems of proving the existence of the limits (5.4), (5.5). But it is clear that in the present case we do not have at all the problems related with the bad knowledge of the properties of the solutions of the equations of motion and we can control very easily how much a particle which is far from a given region can influence what happens in this region: in fact since the wind particles originally at $x_1 \dots x_m$ can travel; in time t, at most a distance $\max |p_i| t$ (because of the velocity conservation) we see that the trees that influence the motion of $x_1 \dots x_m$ are the trees located in a strictly finite region $R(x_1 \dots x_m; t)$ around the coordinates $q_1 \dots q_m$ of $x_1 \dots x_m$ (the dimensions of this region being of the order of $\max_i |p_i| t$) [16].

The existence of the limits (5.4), (5.5) is immediately proved after the above remarks: in fact if K is larger than $R(x_1 \dots x_m; t)$ the quantities under the limit sign become constant and we get: [17]

$$\rho(x_1 \dots x_m; t) = e^{-nV(R(x_1 \dots; t))} \sum_{M=0}^{\infty} n^M \int_{R(x_1 \dots x_m; t)} \frac{dc_1 \dots dc_M}{M!}$$

$$\times \rho_{c_1 \dots c_m}(S_{-t}^{c_1 \dots c_M}(x_1 \dots x_m); 0) \tag{5.6}$$

This shows that the problem of the existence of the time evolution is solved in this case; however from (5.6) it follows rather easily [18] that if one developes $\rho(x_1 \dots x_n; t)$ in a power series in n as

$$\rho(x_1 \dots x_m; t) = \sum_{M=0}^{\infty} n^M c_M(x_1 \dots x_m; t) \tag{5.7}$$

the coefficients $c_M(x_1 \dots x_m; t)$ behave as $t\to\infty$ as t^N and it also follows that the power series in n of the hydrodynamic coefficients (say the diffusion coefficient) contains divergent terms in all orders.

However the wind-tree model is interesting because one can go farther than the above observations and one can give some answers to the questions (1) ÷ (5) posed in the preceding section.

For instance it is clear from (5.6) and (5.3) that $\rho(x_1 \dots x_m; t)$ is, at fixed t, an entire analytic function of the tree-density n and therefore the radius of convergence of the series (5.7) does not shrink to zero as $t\to\infty$ but stays constant and in fact infinite. This answers questions (1) and (2): this could seem due to the fact that the radius is infinite and therefore the argument for its shrinking seems not valid; however this is due to the fact that the trees are a perfect gas and the non-shrinking property is preserved in other less simple wind-tree models (obtained by assuming that the tree gas is not a free gas) where the radius of convergence of the series (5.7) is finite [18].

Taking into account also these results on more general wind-tree models we shall regard questions (1), (2) satisfactorily answered and we therefore look for the possibility of a resummation of the series (5.7) which leads to an expression for $\rho(x_1 \dots x_n; t)$ less singular as $t\to\infty$.

This resummation can really be performed explicitely through a method that will be discussed in detail later. Let us first describe the results of this resummation and their relevance on questions (4), (5).

The resumed expression for the functions $\rho(x_1 \dots x_m; t)$ decomposes naturally into two terms the first being the equilibrium correlations functions (i.e. the ρ's defined by (5.1) when $h \equiv 0$) and the second is a certain series:

$$\rho(t, x_1 \dots x_m) = \bar{\rho}(x_1 \dots x_m) + \sum_{N=0}^{\infty} \gamma_N(x_1 \dots x_m; t, n) n^N \tag{5.8}$$

the coefficients $\gamma_N(x_1 \dots x_m; t, n)$ are expressed through certain integrals of the form

$$\gamma_N(x_1 \dots x_m; t, n) = \int_0^\infty e^{-nt\gamma} \mathcal{f}_N(\gamma, x_1 \dots x_m) d\gamma \tag{5.9}$$

where $\mathcal{f}_N(\gamma, x_1 \dots x_m)$ is a certain distribution in γ. The resumed series (5.8) is absolutely convergent and the sum of the moduli of its terms is uniformly bounded in time. Furthermore each of the coefficients γ_N stays bounded in time. The above expression is interesting first because it gives rise naturally to an expression for the departure from equilibrium $\rho - \bar{\rho}$ and, furthermore, this departure from equilibrium is expressed in (5.9), (5.8) by a superposition of exponentials decreasing in time and therefore reflects the tendence of the system to attain equilibrium.

However in our particular case one can show that $\rho - \bar{\rho}$ does not tend to zero as $t \to \infty$: the physical reason is related to the fact that the trees are hard spheres with respect to the wind but they can overlap freely because they are distributed as a free gas: this implies that one can construct tree configurations which have non vanishing probability and which forbid a wind particle from leaving a certain region (i.e. we can form "boxes" of trees) and this implies that a particle originally in this region will never leave it i.e. there are particles which always "remember" their initial situation and this establishes a long time correlation which according to well-known arguments, should prevent approach to equilibrium [18]. Mathematically this phenomenon appears as a singularity in γ around $\gamma = 0$ of the distributions $\mathcal{f}_N(\gamma, \dots)$. We want to stress that, as the infinite radius of convergence of (5.7), this is another unpleasant feature of our simplistic assumption that the trees are a free gas and, therefore, can overlap: if we suppose that the trees interact in such a way that they cannot overlap (i.e. if the mutual pair potential Φ contains a hard core which forbids overlapping) one can show that all the $\gamma_N \underset{t\to\infty}{\to} 0$ i.e. that the $\mathcal{f}_N(\gamma, \dots)$ are regular for $\gamma = 0$, but, even in this case, one is not able to prove that the sum of the series (5.8) tends to zero (approach to equilibrium) because the series (5.8), although term by term free of divergences as $t \to \infty$, is not uniformely convergent [18]. The above discussion shows that the problem of the approach to equilibrium even in this simple model is far from being settled. However it is possible from the explicit expressions found after the resummation (see below) to obtain some results on the approach to equilibrium in a limiting case: this limiting case is the so-called Boltzmann limit and corresponds to the limit in which the radius σ of the trees tends to zero while the mean free path $\lambda \propto (n\sigma^{v-1})^{-1}$ stays constant (v is the dimension of the space). In this limit the expressions for the coefficients γ_N simplify considerably and one can show explicitely a number of interesting properties: for instance one can show that the time evolution is governed by the Boltzmann equation and, therefore, the system approaches equilibrium and that the approach to equilibrium is realized through a diffusion process [18].

The wind-tree model is, so far, the only case in which one can rigorously show that the evolution in the Boltzmann limit is described by the Boltzmann equation (which in this case is rather simple being linear). In the above limit one can also compute the diffusion coefficient and the other hydrodynamic coefficients which turn out to be given, of course, by the Boltzmann "first order" approximation.

6 Resummation in the wind-tree model

The resummation is suggested by the idea that everything should depend on the probability $g(p, r; t)$ that a particle leaving the origin with velocity p reaches the position r after time t. To evaluate this probability we observe that there are many ways for this wind particle to reach r after a time t: the wind particle can travel without suffering any collision, or can collide with just one tree in c_1 or just with two trees in c_1, c_2 and so on. We can label these different possibilities with the coordinates $c_1, \dots, c_N$ of the N trees hit by the wind particle. In general if $N > 1$ each of the trees can be hit more that once. Now the probability of a path is given by the probability of finding the position $c_1 \dots c_N$ occupied by the trees and simoultaneously, the region $\mathbf{R}(t, c_1 \dots c_N)$ swept by an ideal tree when its center is moved along the path, empty of other trees. According to (5.2) this probability $P(t; c_1 \dots c_N)$ is proportional to:

$$n^N (\exp - nV(\mathbf{R}(t; c_1 \dots c_N))) \frac{dc_1 \dots dc_N}{N!} \tag{6.1}$$

the proportionate factor being $\exp nV_0$ where V_0 is the volume of a tree

(this factor is the reciprocal of the probability for finding a region of volume V_0 empty of trees and is needed to take into account the fact that the wind particle leaves the origin at time $t=0$ and, therefore, there should be no tree containing the origin at $t=0$).

Now from the discussion leading to (6.1) it follows that $g(p, r, t)$ can be written as

$$g(p, r; t) = e^{nV_0} \sum_{N\geq 0} n^N \int_{A_{x,t,N}} e^{-nV(\mathbf{R}(t_1 \dots c_N, t))} \frac{dc_1 \dots dc_N}{N!} \times \delta((S_{-t}^{c_1 \cdots c_N} x)_2 - 2) \quad \chi_{c_1 \dots c_N}(x) \tag{6.2}$$

where $x=(p,0)$; $(S_{-t}^{c_1 \cdots c_N} x)_2$ denotes the spatial coordinate of $S_{-t}^{c_1 \cdots c_N} x$; the region $A_{x,t,N}$ is the set of tree configurations $c_1 \dots c_N$ such that a wind particle with initial phase coordinate x hits, in the time t, all the trees $c_1 \dots c_N$ at least once.

We remark that the series giving $g(p; r; t)$ is a series of positive terms that sums up to a probability distribution and so one should be able to show that, in some sense, each of the terms of the series stays bounded as $t \to \infty$. Furthermore the functions $n^N e^{-nV(\mathbf{R}(c_1 \dots c_N; t))} e^{nV_0}$ should be regarded as cut-offs obtained "resuming" the power series in n for $g(p, r, t)$ which can be obtained from (6.2) by developing all the exponentials in powers. The name cut-off given to $n^N e^{-nV(\mathbf{R}(c_1 \dots c_N; t))}$ is due to the fact that this term tends to zero as $V(\mathbf{R}(\varepsilon_1 \dots \varepsilon_N; t)) \to \infty$ i.e. it is very small for very long trajectories, which, on the other hand, are responsible, for the long time divergences of the coefficients of the power series of $g(p, r; t)$ in powers on n (because these coefficients are expressed through integrals of the type of the ones in (6.2) but without the "cut-offs"). Since we expect that everything depends on $g(p, r; t)$, we can expect that the "resumed" form (6.2) for $g(p, r, t)$ leads to a resumed expression for the correlation functions and we now show that things really go this way. We consider only the case of the one-particle correlation function $p(x; t)$ (the general case being essentially identical). From formula (5.6) and (5.3) we can write, if $x=(p, q)=(p, 0)$ (we can suppose that $q=0$ without loss of generality):

$$\rho(x; t) = z e^{-\beta \frac{p^2}{2m}} (e^{-nV(R(x,t))} \sum_{M=0}^{\infty} n^M \int \frac{dc_1 \dots dc_M}{M!} \chi_{c_1 \dots c_M}(x) \exp -\beta h(S_{-t}^{c_1 \cdots c_M} x)) \tag{6.3}$$

therefore if $\rho(x)$ is the equilibrium correlation function (i.e. the one given by (6.3) with $h=0$):

$$\rho(x; t) - \rho(x) = z e^{-\beta \frac{p^2}{2m}} (e^{-nV(R(x,t))} \sum_{M=0}^{\infty} n^N \int_{R(x,t)M} \frac{dc_1 \dots dc_M}{M!} \chi_{c_1 \dots c_M}(x) \cdot \cdot (\exp -\beta h(S_{-t}^{c_1 \cdots c_M} x) - 1)) \tag{6.4}$$

Let

$$I(x, t) = e^{-nV(R(x,t))} \sum_{M=0}^{\infty} n^N \int_{R(x,t)} \frac{d_{c_1} \dots d_{c_M}}{M!} \chi_{c_1 \dots c_M}(x) \cdot \cdot (\exp -\beta h(S_{-t}^{c_1 \cdots c_M} x) - 1) \tag{6.5}$$

then it is possible to show after the straightforward calculations exposed below [18] that:

$$I(x, t) = e^{-nV_0} \int (e^{-\beta h(r)} - 1) g(p, r, t) dr \tag{6.6}$$

and this expression together with (6.4), (6.5), (6.2) proves (5.8).

In fact we can decompose the integration region

$$R(x, t)^M \text{ as}$$

$$R(x, t)^M = \bigcup_{N=0}^{M} \bigcup_{j_1, \dots j_N} R_{j_1 \dots j_N} \tag{6.7}$$

where $R_{j_1 \dots j_N}$ is the set of all the $c_1, \dots, c_N$ such that

$$S_{-\tau}^{c_1 \cdots c_M} x \equiv S_{-\tau}^{c_{j1} \cdots j_N} x, \text{ for all } 0 \leq \tau \leq t \tag{6.8}$$

i.e. is the set of configurations $c_1, \dots c_M$ such that the trajectory $S_{-\tau}^{c_1 \cdots c_M} x$ hits only $c_{j_1} \dots c_{j_N}$ in the time t; the union $\bigcup_{j_1 \dots j_N}$ runs over all the choises of N indices between M.

Now using (6.7), (6.8) and using the symmetry of the integration over $c_1 \dots c_M$ to relabel the indices $j_1 \dots j_N$ as 1, 2, ..., N, the integral (6.5) becomes:

$$I(x, t) = \sum_{M \geq 0} \sum_{N=0}^{M} \sum_{j_1 \dots j_N} \int_{Rj_1 \dots j_N} \frac{dc_1 \dots dc_M}{M!} n^M e^{-V(R(x,t))n} \chi_{c_1 \dots c_M} (\exp -\beta h(S_{-t}^{c_{j1} \cdots c_{jN}} x) - 1) =$$

$$= \sum_{N=0}^{\infty} \int_{A_{x,t,N}} n^N \frac{dc_1 \dots dc_N}{N!} (\exp -\beta h(S_{-t}^{c_1 \dots c_N} x) - 1) \cdot \chi_{c_1 \dots c_N} x$$

$$\left[\sum_{T=0}^{\infty} \int_{\substack{c' \in R(x,t) \\ c' \notin \mathbf{R}(t, c_1 \dots c_N)}} \frac{dc'_1 \dots dc'_T}{T!} n^T e^{-V(R(x,t))n} \right] \tag{6.9}$$

where the region $A_{x,t,N}$ coincides with the one defined after (6.2) and $\mathbf{R}(t, c_1 \dots c_N)$ is the region introduced before (6.1) i.e. the region swept by an ideal tree when its center is moved along the path $S_{-\tau}^{c_1 \dots c_N} x$, $0 \leq \tau \leq t$ and $T = M - N$.

The integral in square brackets in (6.8) is simply $\exp -nV(\mathbf{R}(t, c_1 \dots c_N))$ and therefore:

$$I(x, t) = \sum_{N=0}^{\infty} \int_{A_{x,t,N}} n^N \frac{dc_1 \dots dc_N}{N!} e^{-nV(\mathbf{R}(t, c_1 \dots c_N))} \chi_{c_1 \dots c_N}(x)$$

$$(\exp -\beta h(S_{-t}^{c_1 \dots c_N} x) - 1)$$

$$= \int dr (e^{-\beta h(r)} - 1) \left[\sum_{N=0}^{\infty} \int_{A_{x,t,N}} n^N \frac{dc_1 \dots dc_N}{N!} e^{-nV(\mathbf{R}(t, c_1 \dots c_N))} \chi_{c_1 \dots c_N}(x) \right.$$

$$\left. S((S_{-t}^{c_1 \dots c_N} x)_2 - r) \right] \tag{6.10}$$

and this together with definition (6.2) proves (6.6).

Conclusion

The problem of the existence of time evolution for v-dimensional systems ($v \geq 2$) is completely open together with the questions (1)÷(5) in section 3. In one dimension there are some results but most of the problems are still open.

The situation is by far better in the wind-tree models. We have discussed in detail a very special case of the wind-tree models. However the results obtained hold for a much larger class of models but need more technicalities to be obtained.

Although it has been proved the existence of a resummation in the wind-tree models the problems of proving the approach to equilibrium and of evaluating the diffusion coefficient or of finding corrections to the Boltzmann equation near the Boltzmann limit remain essentially unanswered.

Although the problem of even proving the existence of the diffusion coefficient remains open it may be of some consolation the fact that using the resumed form of the Green functions (6.2) one gets a formal series (no longer a power series in the density n) for the diffusion coefficient which is divergence free at all orders.

We conclude with some few words about the relation between macroscopic irreversibility and the microscopic reversibility looked at when the initial states are of the type considered in this lecture. Since our systems are infinite it makes sense to talk only of mean energy $u(t)$ and of mean entropy $S(t)$. These quantities are global quantities and since our system is only locally perturbed from equilibrium they should stay constant in time and equal to their equilibrium values.

This result has been proved rigorously in the case of one dimensional systems with short range weak potentials and holds without any doubt for the wind-tree models. In particular there is no privileged direction of the time since the mean entropy is constant and all the problems connected with the idea that entropy increases lose their signification in this contest and the system is expected to attain equilibrium either for $t \to +\infty$ or for $t \to -\infty$.

References

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4. E. G. D. Cohen, in *Fundamental problems in statistical mechanics*, Ed. E. G. D. Cohen (North Holland, Amsterdam, 1962).
5. BBGKY are the initials of Bogoliubov, Born, Green, Kirkwood, Yvon.

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10. at our knowledge there are no published accounts of this calculation but a similar one can be found in: E. G. D. Cohen: in *Fundamental problems in statistical mechanics*, Ed. E. G. D. Cohen, North Holland, Amsterdam, 1968, p. 235–244.
11. This statement has not been proved in general but there is little doubt on its truth: see ref. [10].
12. again these calculation are not done in the literature where one usually considers density expansions instead of activity expansions: see for instance ref. [10].
13. D. Robinson, *Comm. Math. Phys.*, **7**, 337 (1968).
14. D. W. Jepsen, *J. Math. Phys.*, **6**, 405 (1965) and J. L. Lebowitz, J. K. Percus.
15. D. Ruelle: see ref. [7].
16. In order that the developments given below hold the region $R(x_1, \ldots, x_m; t)$ has to be large enough to contain the spheres of the same size of the trees and centered at $q_1, \ldots, q_m$ (this is due to the factors $\chi_{c_1 \ldots c_n}(x_1, \ldots, x_m)$ in the definition of the ρ's).
17. The arbitrary of the choice of the region $R(x_1, \ldots, x_m; t)$ does not infirm formula (5.6) because it is immediately verified that if (5.6) holds for a given $R(x_1, \ldots, x_m; t)$ it also holds for any $R(x_1, \ldots, x_m) \supset R(x_1, \ldots, x_m; t)$.
18. G. Gallavotti, *Phys. Rev.*, (1969).
19. Here as well as everywhere in this paper we give to the symbol $\Lambda \to \infty$ the simplest meaning: i.e. Λ is a cube centered at the origin with sides tending to infinity.
20. The time evolution operator could transform a configuration initially inside Λ into a configuration outside Λ. One could avoid this phenomenon by introducing periodic boundary conditions on the box Λ (and periodizing the hamiltonian): the two approaches should give the same results (at least at low density).
21. In general these potential have many-particle components and not only the two-particle (pair potential) component: see ref. [18].
22. However the results in [8] are not strong enough to allow the use of the Kirkwood-Saltzburg equations but only to prove general properties of the time evolution and the existence, at low z, of a solution of the BBGKY hierarchy.