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Divergences and the Approach to Equilibrium in the Lorentz and the Wind-Tree Models

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We prove that for wide class of physically interesting initial states the time evolution of the wind particles's correlation functions can be described at any finite time by a convergent power series in the density of the tree particles, provided this density is small enough. We show that all the coefficients (except the lowest ones) of this power series contain terms diverging as $t \to +\infty$. Nevertheless, we prove that the radius of convergence does not shrink to zero as $t \to +\infty$ and that the divergent terms can be resummed into a cutoff, thereby constructing a new series for the correlation functions having each term bounded as $t \to +\infty$. Although divergence free, this series does not converge uniformly in time; however, it can be used to show that equilibrium cannot be reached if the tree-tree interaction allows overlapping and to study the limiting case of vanishing tree size but nonvanishing free path; in this last case we find an exact expression for the Green's functions showing that the approach to equilibrium is described by a diffusion process which rigorously verifies the Boltzmann equation.

1. INTRODUCTION

A class of models was introduced by Lorentz and Ehrenfest, to all of which we will briefly refer as "wind-tree" models,[1-3] in order to understand some of the basic difficulties of the kinetic theory of gases.

In these models there are two types of particles: The "wind" particles move through the space interacting only with the particles of the second kind (the "tree" particles) which, however, are not supposed to be affected in their motion by the "light" wind particles and so are supposed to be in internal equilibrium under the action of their mutual forces. Each model is completely described by the wind-tree potential and by the tree-tree potential.

In this paper we shall examine only the case in which the trees are, with respect to the wind, either hard spheres or equally oriented hard cubes which reflect the wind on their surface.

We show that, if at time $t = 0$ the wind particles are in a nonequilibrium state which can be visualized as a state in which the wind is in equilibrium at temperature β^{-1} and chemical potential μ outside a certain finite region (where it is not in equilibrium), then the time evolution of the wind is described by a set of correlation functions (Sec. 3) that are analytic in the density n of the trees around $n = 0$ at any finite t (Sec. 4). We give explicit expressions for the coefficients of the expansion of the wind correlation functions in powers of the tree density n and we show that these coefficients (except the lowest ones) are sums of terms diverging in the limit $t \to \infty$ (Sec. 4). We study how these divergences can be eliminated by a resummation and we prove that the difference between the equilibrium correlation functions and the time-dependent correlation functions is given by a series (which is no longer a power series in n) in which the moduli of each term are uniformly bounded in time as is their sum (Sec. 5). Then we discuss the approach to equilibrium and show how to use the resummed series to exclude the possibility of approach to equilibrium in case the mutual interaction between the trees allows overlapping (Sec. 6). In Sec. 7 we study, for a free-tree gas, the limit-

ing situation (the so-called Boltzmann limit) in which the size of the trees is vanishingly small but the mean free path is finite, and show that the approach to equilibrium is described by a diffusion process which is studied in some detail. Finally, we show in Sec. 8 that this diffusion process verifies exactly the Boltzmann equation, while in Sec. 9 we give a summary of the results and compare them with those already known.

This paper is inspired by some recent results on the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy and on the equilibrium properties of the wind-tree model.[4-8]

2. DESCRIPTION OF THE INITIAL STATE

The initial states we are going to consider are nonequilibrium states only in a finite region; outside this region they are equilibrium states at chemical potential μ and inverse temperature β. The precise description is given below.

Let us fix an inverse temperature β, a chemical potential μ, an external field $h(q)$ acting on the wind particles, and let us choose a configuration $C=(c_1,\dots,c_m)$ of the tree particles. Then, let us consider the equilibrium correlation functions of the wind at inverse temperature β, chemical potential μ, and under the action of the forces due to the trees and to the external field h, which we suppose to be bounded below and which acts on a finite region Λ_h (the fact that h vanishes outside a bounded region will be used essentially only in the discussion of the approach to equilibrium in Sec. 6). Clearly, since the wind is free in the region outside the trees, these correlation functions are simply given by

$$\rho_C(x_1,\dots,x_n)=\exp[\beta\mu n-\beta T(x_1,\dots,x_n)-\beta H(x_1,\dots,x_n)]\,\chi_C(x_1)\cdots\chi_C(x_n), \tag{1}$$

where we have denoted $x=(p,q)$ the phase-space coordinates of a wind particle,

$$T(x_1,\dots,x_n)=\frac{1}{2}\sum_{i=1}^{n}p_i^2,\qquad H(x_1,\dots,x_n)=\sum_{i=1}^{n}h(q_i),$$

and $\chi_C(x)=0$ if the position coordinate q of the wind particle x is inside some of the trees in C, $\chi_C(x)=1$ otherwise.

Now, suppose that the probability distribution of the trees is the equilibrium distribution of a system of particles mutually interacting through a pair potential $\varphi(c-c')$ at inverse temperature $\tilde{\beta}$ and activity z (we need not suppose $\tilde{\beta}=\beta$ since there is no exchange of energy between the trees and the wind). So we are led to consider the state of the wind particles average of (1) over the distribution of the tree configurations C defined by the correlation functions

$$\rho(0,x_1,\dots,x_n)=\lim_{\Lambda\to\infty}Z_\Lambda^{-1}\sum_{m=0}^{\infty}z^m\int_{\Lambda^m}\exp[-\tilde{\beta}V(c_1,\dots,c_m)]\,\frac{dc_1\cdots dc_m}{m!}\,\rho_C(x_1,\dots,x_n), \tag{2}$$

where $C=(c_1,\dots,c_m)$, $V(c_1,\dots,c_m)=\sum_{i<j}\varphi(c_i-c_j)$,

and Z_Λ is the grand canonical partition function for a tree system confined in a cube Λ centered around the origin and with temperature and activity parameters given by $\tilde{\beta}^{-1}$ and z, respectively.

In order to deal with definition (2) we need to suppose that the tree-tree potential φ is stable and sufficiently decreasing at infinity in order that the thermodynamic limit exists[9,10] and that the low z Mayer expansions of the tree-correlation functions converge.[11-14] We shall prove the existence of the limit (2) at low z in Sec. 3.

Now, if at time $t=0$ we switch off the external field h, the wind particles will no longer be in equilibrium and their correlation functions will change in time.

3. TIME EVOLUTION AT SMALL-TREE DENSITY

Let $x=(p,q)$ be the phase-space coordinates of a wind particle and let $S_t{}^C x$ be the coordinates into which x evolves in the time t when the tree configuration is $C=(c_1,\dots,c_m)$.

Taking into account that the kinetic energy of the wind is conserved, we can write the correlation functions at time $t\neq 0$ as [(cf. (1), (2)]

$$\rho(t;x_1,\ldots,x_n)=\lim_{\Lambda\to\infty} Z_\Lambda^{-1}\sum_{m\geq 0} z^m\int_{\Lambda^m}\frac{dc_1\cdots dc_m}{m!}\,e^{-\tilde\beta V(C)}\,e^{\beta\mu n-\beta T(x_1,\ldots,x_n)}$$

$$\times\exp[-\beta H(S_{-t}{}^{C}(x_1),\ldots,S_{-t}{}^{C}(x_n))]\,\chi_C(x_1)\cdots\chi_C(x_n)\,, \tag{3}$$

where we have abbreviated $C=(c_1,\ldots,c_m)$ and the functions $H(\cdot)$, $\chi_C(\cdot)$ have been introduced in (1). We now show that the limit (3) exists for $t\gtrless 0$ and for z small. In fact, if we introduce the tree-particle correlation functions

$$n(c_1,\ldots,c_m)=\lim_{\Lambda\to\infty} Z_\Lambda^{-1}\sum_{h\geq 0} z^{h+m}\int_{\Lambda^h}\frac{dc'_1\cdots dc'_h}{h!}\exp[-\tilde\beta V(c_1,\ldots,c_m;c'_1,\ldots,c'_h)] \tag{4}$$

then[11-14] $n(c_1,\ldots,c_m)$ exist and, as functions of z, are simultaneously analytic for z in a circle K around $z=0$ and there exists a $B>0$ such that for $z\in K$

$$|n(c_1,\ldots,c_m)|\leqq B^m\,. \tag{5}$$

So, we can define for any finite set R the functions

$$f_R(c_1,\ldots,c_m)=\sum_{h\geq 0}\int_{R^h}(-1)^h\frac{dc'_1\cdots dc'_h}{h!}\,n(c_1,\ldots,c_m;c'_1,\ldots,c'_h) \tag{6}$$

(for $c_1,\ldots,c_m\in R$); these functions have the physical meaning of being the probability densities for finding inside R, exactly m tree particles, and for finding them exactly with coordinates $c_1,\ldots,c_m$.[15]

Now we remark that (3) is the grand canonical average of the function $\chi_C(x_1)\cdots\chi_C(x_n)\exp[-\beta H(S_{-t}{}^{C}(x_1),\ldots,S_{-t}{}^{C}(x_n))]$ over the tree-configurations C at fixed $t,x_1,\ldots,x_n$; but at fixed t, $x_1,\ldots,x_n$ this function depends only on the trees in C that are located in a finite region $R(t;x_1,\ldots,x_n)$ around the positions $q_1,\ldots,q_n$ of the wind particles $(x_1,\ldots,x_n)$ (the linear dimensions of this region being of the order

$$\max_{1\leq i\leq n}|p_i|t)\,.$$

Using this remark we can immediately write the value of the limit (3) in terms of the functions in (6)

$$\rho(t;x_1,\ldots,x_n)=\{\exp[\beta\mu n-\beta T(x_1,\ldots,x_n)]\}\,I(x_1,\ldots,x_n;t;H)\,,$$

where $I(x_1,\ldots,x_n;t;H)$ is given [using the probabilistic meaning of the functions in (6)] by

$$I(x_1,\ldots,x_n;t;H)=\sum_{k\geq 0}\int_{R(t;x_1,\ldots,x_n)}\frac{dc_1\cdots dc_k}{k!}\,f_{R(t;x_1,\ldots,x_n)}(c_1,\ldots,c_k)$$

$$\times\exp[-\beta H(S_{-t}{}^{C}(x_1),\ldots,S_{-t}{}^{C}(x_n))]\,\chi_C(x_1)\cdots\chi_C(x_n)\,, \tag{7}$$

the integral I being convergent because of (5) and the boundedness of $R(t;x_1,\ldots)$.

4. ANALYTICITY IN THE TREE DENSITY

From formulas (5) and (6) it follows that for z inside the analyticity circle K the functions $f_R(c_1,\ldots,c_n)$ are analytic in z and denoting the volume of R by $V(R)$ we have

$$f_R(c_1,\ldots,c_n)\leqq B^m e^{BV(R)} \tag{8}$$

This bound, together with formula (7), implies that the functions $\rho(t;x_1,\ldots,x_n)$ are analytic functions of

$z \in K$. This analyticity property can be transformed in the analyticity of $\rho(t;x_1,\dots)$ in terms of the density n of the trees in a circle around $n=0$, because of the analyticity of the activity as a function of n for n in a neighborhood of the origin (see, for instance, Lebowitz and Penrose[16]).

So far we have established the existence of our activity (density) expansion for the wind correlation functions valid in the region of activities (densities) where the tree correlation functions can be proven to be analytic.

Now the problem of finding explicit expressions for the coefficients of the power series naturally arises. It is indeed possible to derive formally these coefficients and to make the formal derivation rigorous, at least in the case when the mutual tree-tree interaction contains a hard core. We discuss these points in the rest of the section; the following discussion is, however, rather technical and can be skipped (and we suggest the reader do so) by accepting one of the results, i.e., the fact that the coefficients of the activity series (or at least some of them) diverge as $t \to +\infty$ and that, therefore, it would be desirable to show that there is a way of resumming the series which eliminates such divergences. This resummation problem will be our main concern in the next sections.

Let us consider the numerator of Eq. (3): Forgetting the factor $\exp[\beta\mu n - \beta T(p_1,\dots,p_n)]$, it can be written

$$Z_\Lambda(x_1,\dots,x_n;t) = \sum_{m\geq 0} z^m \int_{\Lambda^m} \frac{dc_1\cdots dc_m}{m!} \exp[-\tilde\beta V(C) - \tilde\beta V_{x_1\cdots x_m t}(C)], \tag{9}$$

$$\text{where}\quad \exp[-\tilde\beta V_{x_1\cdots x_n t}(C)] = \chi_C(x_1)\cdots\chi_C(x_n)\exp[-\beta H(S_{-t}{}^C(x_1),\dots,S_{-t}{}^C(x_n))]. \tag{10}$$

The potential energy $V_{x_1\cdots x_n t}(C)$ can be considered as due to the action of two potentials acting on the trees. The first potential is an external 1-particle potential defined as: $\Phi_{x_1\cdots x_n}(c) = +\infty$, if at least one of the wind particles x_i is inside the tree located at c and $\Phi_{x_1\cdots x_n}(c) = 0$ otherwise. Clearly if

$$V_1(C) = \sum_{i=1}^{m} \Phi_{x_1\cdots x_n}(c_i), \quad \text{we have } \chi_C(x_1)\cdots\chi_C(x_n) = \exp[-\tilde\beta V_1(C)]. \tag{11}$$

The second potential contributing to $V_{x_1\cdots x_n t}(C)$ is a potential which produces a potential energy $V_2(c_1\cdots c_m)$ such that

$$\tilde\beta V_2(c_1,\dots,c_m) = \beta H(S_{-t}{}^C(x_1),\dots,S_{-t}{}^C(x_n)). \tag{12}$$

This potential energy does not, in general, come from a simple pair potential but involves many-body interactions between the trees $c_1,\dots,c_n$ in C, i.e.,

$$V_2(C) = \sum_{D\subset C} \Phi(D/x_1,\dots,x_n;t), \tag{13}$$

where the sum over D runs not only over the pairs of trees in C but over all the subconfigurations D of C. The explicit expression for Φ is

$$\Phi(C/x_1,\dots,x_n;t) = (\beta/\tilde\beta)\sum_{D\subset C}(-1)^{N(C)-N(D)} H(S_{-t}{}^D(x_1),\dots,S_{-t}{}^D(x_n)), \tag{14}$$

where the sum runs over all the subconfigurations D of C and $N(D)$ denotes the number of trees in D. We have, for instance,

$$\Phi(\emptyset/x_1,\dots,x_n;t) = (\beta/\tilde\beta)H(S_{-t}{}^{\emptyset}(x_1),\dots,S_{-t}{}^{\emptyset}(x_n)),$$

$$\Phi(c_1/x_1,\dots,x_n;t) = (\beta/\tilde\beta)[H(S_{-t}{}^{c_1}(x_1),\dots,S_{-t}{}^{c_1}(x_n)) - H(S_{-t}{}^{\emptyset}(x_1),\dots,S_{-t}{}^{\emptyset}(x_n))],$$

$$\Phi(c_1c_2/x_1,\dots,x_n;t) = (\beta/\tilde\beta)[H(S_{-t}{}^{c_1c_2}(x_1),\dots,S_{-t}{}^{c_1c_2}(x)) - H(S_{-t}{}^{c_1}(x_1),\dots,S_{-t}{}^{c_1}(x_n))]$$

$$- H[(S_{-t}{}^{c_2}(x_1),\dots,S_{-t}{}^{c_2}(x_n)) + H(S_{-t}{}^{\emptyset}(x_1),\dots,S_{-t}{}^{\emptyset}(x_n))].$$

After these definitions we see that (9) can be interpreted as the grand-partition function of the tree sys-

tem corresponding to the activity z, temperature $\tilde{\beta}^{-1}$ and to an interaction which is the sum of the mutual-pair interaction and an external many-body interaction dependent on the wind particles $x_1, \ldots, x_n$ and on the time t.

We remark that the potential $\Phi(C/x_1, \ldots, x_n; t)$, in spite of its complicated expression, is, in fact, a nice potential: By fixing $(x_1, \ldots, x_n; t)$ and taking advantage of the crucial fact that the particles $x_1 \cdots x_n$ can travel in time t on a distance at most $\max|p_i|t$, it is easy to verify that the potential $\Phi(C/x_1, \ldots, x_n; t)$ is localized in a finite region around $x_1, \ldots, x_n$. In fact suppose that the tree configuration $C = (c_1, \ldots, c_m)$ contains at least one tree, say c_1, which is so far from the positions of $x_1, \ldots, x_n$ that $S_{-t}{}^{C}(x_1)$ does not depend on c_1 ("far" is of the order of $\max|p_i|t$), then

$$\Phi(c_1, \ldots, c_m/x_1, \ldots, x_n t) = (\beta/\tilde{\beta}) \sum_{D \subset C, D \not\ni c_1} (-1)^{N(C)-N(D)} H(S_{-t}{}^{D}(x_1), \ldots, S_{-t}{}^{D}(x_n))$$

$$+ (\beta/\tilde{\beta}) \sum_{D \subset C, D \not\ni c_1} (-1)^{N(C)-N(D)-1} H(S_{-t}{}^{D \cup c_1}(x_1), \ldots, S_{-t}{}^{D \cup c_1}(x_n)) = 0,$$

because $$S_{-t}{}^{D \cup c_1}(x_i) \equiv S_{-t}{}^{D}(x_i).$$

Therefore, the potentials which originate the potential energy $V(c_1, \ldots, c_n)$ in (9) are finite-range localized potentials.

Having given (9) the interpretation of grand-canonical partition function of the tree system under suitable interactions we observe that formula (3) can be written using definition (9) as

$$\rho(t; x_1, \ldots, x_n) = \exp[\beta\mu n - \beta T(p_1, \ldots, p_n)] Z_\Lambda(x_1, \ldots, x_n; t)/Z_\Lambda \,. \tag{15}$$

Now it is well known from the theory of the Mayer expansions that the partition function $\tilde{Z}_\Lambda$ corresponding to a potential energy $E(c_1, \ldots, c_m)$ of the form

$$E(c_1, \ldots, c_m) = E(C) = \sum_{D \subset C} \Phi(D)$$

can be written

$$\tilde{Z}_\Lambda = \exp\left(\sum_{m=\infty}^{\infty} z^m \int_{\Lambda^m} \frac{dc_1 \cdots dc_m}{m!} U(c_1, \ldots, c_m) \right), \tag{16}$$

where the Ursell functions $U(C)$ are defined as

$$U(C) = \sum_{\Gamma} \prod_{T \in \Gamma} [e^{-\beta\Phi(T)} - 1] \,. \tag{17}$$

Here $\sum_\Lambda$ has the familiar [familiar at least if the potential $\Phi(T)$ is a pair potential, i.e., if $\Phi(T) = 0$ unless $N(T) = 2$] meaning of sum over the "connected" diagrams and the notion of connectedness of a diagram has to be generalized in an obvious way by allowing not only pair bonds [i.e., $N(T) = 2$] but also many-body bonds [$N(T) \geq 2$] if $\Phi(T)$ is not a pure pair potential.

Using (15)–(17) we can now easily produce the desired expression for the coefficients of the activity expansion of $\rho(t, x_1, \ldots, x_n)$.

Let us regard the mutual-pair potential $\varphi(c - c')$ as a many-body potential $\varphi(C)$ [i.e., define $\varphi(C) = \varphi(c_1, \ldots, c_m) \equiv 0$ if $m \neq 2$ and $\varphi(c, c') = \varphi(c - c')$] and let us introduce the following Ursell functions:

$$U(C) = \sum_{\Gamma} \prod_{T \in \Gamma} [e^{-\tilde{\beta}\varphi(T)} - 1] \,, \tag{18}$$

$$U(C/X) = \chi_C(x_1) \cdots \chi_C(x_n) U(C) \,, \tag{19}$$

$$U(C/X;t) = \chi_C(x_1)\cdots\chi_C(x_n)\exp\Big(-\tilde{\beta}\sum_{j=1}^{m}\Phi(c_j/X;t)\Big)\sum_{\Gamma}\prod_{T\in\Gamma}\{\exp[-\beta\varphi(T)-\tilde{\beta}\Phi(T/X;t)]-1\}, \tag{20}$$

where $C=(c_1,\ldots,c_m)$, $X=(x_1,\ldots,x_n)$, and $\sum$ has the above meaning of sum over the "connected diagrams."

In terms of (18)–(20) we can define the following coefficients:

$$\gamma_m(t;x_1,\ldots,x_n) = \int[U(c_1,\ldots,c_m/X;t)-U(c_1,\ldots,c_m)]\frac{dc_1\cdots dc_m}{m!}, \tag{21}$$

$$\gamma_m(x_1,\ldots,x_n) = \int[U(c_1,\ldots,c_m/X)-U(c_1,\ldots,c_m)]\frac{dc_1\cdots dc_m}{m!}. \tag{22}$$

These integrals are well defined because $\chi_C(x_1)\cdots\chi_C(x_n)$ and $\Phi(C/X;t)$ differ, respectively, from 1 and 0 only if some tree in C is contained in a bounded region (depending on t, $x_1,\ldots,x_n$) and because all the potentials vanish at infinity (see, for instance, Ref. 13).

Now, applying formulas (15)–(17) it follows [denoting $S_{-t}(x)$ the free evolution, i.e., $S_{-t}\phi(x)$]

$$\rho(t;x_1,\ldots,x_n) = \exp[-\beta\mu n-\beta T(x_1,\ldots,x_n)-\tilde{\beta}H(S_{-t}(x_1),\ldots,S_{-t}(x_n))]\exp\Big(\sum_{m\geq 1}z^m\gamma_m(t;x_1,\ldots,x_n)\Big), \tag{23}$$

where the unusual term $\exp[\tilde{\beta}H(S_{-t}(x_1),\ldots)]$ comes from the ("unusual") fact that the potential $\Phi(C/X;t)$ gives to the empty configuration the energy $(\beta/\tilde{\beta})H[(S_{-t}(x_1),\ldots,S_{-t}(x_n))]$ instead of zero. This term describes the evolution of the free wind.

From (23) one immediately derives the desired expansion of ρ in terms of z (and n).

In order to prove that (23) gives us the correct expansion of $\rho(t;X_1,\ldots,x_n)$ in terms of z (which we have shown to exist), we need to overcome the convergence problems that arise in a rigorous derivation of (23). These problems are essentially the same as the problems encountered in proving the convergence of the Mayer expansions and never seem to have been investigated for the case in which many-body interactions are involved – except in the lattice-gas case[17–19] – but it is clear that the proofs given for the lattice-gas case carry without essential change in the case where the mutual interaction between the trees contains a hard core.[20] If this is not the case, although we know that a z expansion exists, we cannot be sure that it is given by (23), but this seems to be only a technical problem which for our purposes is not very essential; we will not deal with it since we shall be able to give an expansion for the ρ's which is surely divergence free as $t\to\infty$ [contrary to the series (23), see below] and which holds for an arbitrary mutual tree-tree interaction. We remark that in the case $h=0$, formula (23) becomes

$$\rho(x_1,\ldots,x_n)=e^{\beta\mu n-\beta T(x_1,\ldots,x_n)}\exp\sum_{m\geq 1}z^m\gamma_m(x_1,\ldots,x_n), \tag{24}$$

which gives the power series of the equilibrium correlation functions and for which there are no doubts about the convergence of the series at low z because $\gamma_m(x_1,\ldots,x_n)$ involves only external and pair potentials and we can use the theory of the Mayer expansions and the known properties of the cluster integrals.[11–13]

We conclude this section by asserting that if one writes the integrand in (21) as a sum of the differences of the corresponding terms in (18) and (20), and considers only the contribution to the integrand (21) of one fixed diagram, say Γ_0, then one can verify in simple cases that this contribution will diverge as $t\to\infty$ except for the lowest values of m [take, for instance, $m=3$, h constant in Λ and zero outside, consider only one wind particle, i.e., $n=1$ in (19), (20), ..., and assume the trees to be hard cubes, or take $m=2$, $n=1$, h as before, and assume the trees are hard spheres; in both cases consider only the simplest possible diagram].

Instead of discussing the possibility or the impossibility of compensations between the various diagrams and the various orders we note that the existence of these divergences is a well-known phenomenon,[1–3,21,22] that there are not enough cancellations, i.e., the divergences are intrinsically inherent in the nature of the density expansions, and an expansion of the ρ's with coefficients not diverging as $t\to\infty$ does not exist. We avoid this complicated problem by giving nonsingular expressions for the correlation functions.

5. RESUMMATION OF THE ACTIVITY SERIES

In order not to hide behind apparently complicated deductions the physical meaning and the main ideas

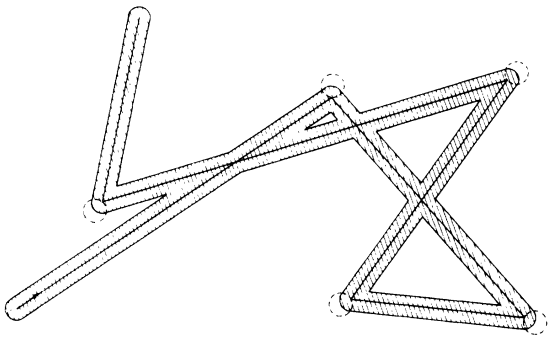

FIG. 1. Set $R(x,t/c_1,\ldots,c_N)$ is the hatched region. The dashed circles represent the trees $c_1,\ldots,c_N$ $(N=5)$ and the length of the trajectory in the hatched region is $|p|t$.

underlying the resummation procedure, we sketch briefly the arguments suggesting the formal proofs in the case of the one-particle correlation function.

We abandon the power-series approach of the preceding sections and try to give a more intrinsic "non-perturbative" characterization of the evolving state. Clearly, everything should depend on the probability $g(p,r;t)$ that a particle leaving the origin with velocity p reaches the position r after a time t has elapsed.

There are many ways for this wind particle to reach r after a time t: The wind particle can travel without suffering any collision or can collide with just one tree in c, or with just two trees in c_1, c_2, and so on. We can label these different possibilities with the coordinates $c_1,\ldots,c_N$ of the N trees hit by the wind particle. In general if $N>1$, each of the trees can be hit more than once (Fig. 1 refers, for simplicity, to the particular case in which the wind particle hits each tree only once). Now the probability of a path is given by the probability of finding the positions $c_1,\ldots,c_N$ occupied by the trees and, simultaneously, the region $R(x,t/c_1,\ldots,c_N)$, swept by an ideal tree when its center is moved along the trajectory (see Fig. 1), empty of other trees. According to the physical meaning of the function $f_R(\cdot)$ introduced in (6), the above-mentioned probability is given by the following formula, which also defines implicitly the function F which will be needed later:

$$P(x,t;c_1,\ldots,c_N) = F(x,t;c_1,\ldots,c_N) f_{R(x,0/\phi)}(\phi)^{-1} = f_{R(x,t;c_1,\ldots,c_N)}(c_1,\ldots,c_N) f_{R(x,0/\phi)}(\phi)^{-1} \chi_C(x), \tag{25}$$

where $x=(0,p)$ denotes the phase coordinate of the wind particle at time $t=0$ and the factor $f_{R(x,0/\phi)}(\phi)^{-1} \times \chi_{c_1\cdots c_N}(x)$ is introduced to take into account the fact that we must suppose that at time $t=0$ there is no tree which "contains" the wind particle, i.e., there is no tree in the tree-shaped region $R(x,0/\phi)$. From the probabilistic meaning of (25) it follows:

$$g(p,r;t) = \sum_{N\geq 0} \int_{A_{x,t,N}} \frac{dc_1\cdots dc_N}{N!} P(x,t/c_1,\ldots,c_N)\delta([S_{-t}{}^{c_1\cdots c_N}(x)]_2 - r). \tag{26}$$

Here the integration over $c_1,\ldots,c_N$ must be restricted to the region $A_{x,t,N}$ of the $c_1,\ldots,c_N$ such that the wind particle leaving the origin at time $t=0$ with velocity p hits all of them at least once as the time evolves from 0 to t; the symbol $[S_{-t}{}^{c_1\cdots c_N}(x)]_2$ denotes the spatial coordinate of $[S_{-t}{}^{c_1\cdots c_N}(x)]$.

Now, we remark that the series giving $g(p,r;t)$ is a series of positive terms that sums up to a probability distribution and so, using the fact that it is a normalized distribution, one should be able to show that in some sense each term of the series stays bounded as $t\to\infty$. Furthermore, the functions $f_{R(x,t/c_1\cdots c_N)}(c_1,\ldots,c_N)$ in (25), which in the lowest order in the tree density are simply n^N, should be regarded as cutoffs [they deserve this name because they are expected to vanish as the volume $V(R(x,t/c_1,\ldots,c_N))\to\infty$, see (34) below, i.e., they give a small weight to the long trajectories] and it should be possible to interpret these cutoffs as obtained by resumming the divergent terms in the activity series.

In the rest of this section we make the ideas above precise. We shall deal only with the one-particle correlation function $\rho(t,x)$, since the consideration of the higher-order correlation functions needs only trivial changes. The main result will be formulas (31)–(33) below, which give a rather simple expression for the evolution in time of the "departure from equilibrium" $\rho(t,x)-\rho(x)$ and for some of the time-depen-

dent Green's functions.

Given $x=(0,p)$, t and a tree configuration $c_1,\dots,c_N$, such that all its trees are hit during the time t by the wind particle which at $t=0$ has phase coordinate x, we can consider the probability introduced in (25). We want to prove first that the integral $I(x;t;H)$ appearing in (7) is given by

$$I(x;t;H)=\sum_{N\geq 0}\int_{A_{x,t,N}}\frac{dc_1\cdots dc_N}{N!}F(x,t;c_1,\dots,c_N)\exp[-\beta H(S_{-t}^{c_1\cdots c_N}(x))]\,. \tag{27}$$

Furthermore, $I(x;t;0)\equiv I(x;0;0)\leq 1\,.$ (28)

To prove the statements above we proceed as follows: $I(x;t;H)$ [defined in (7)] can be transformed as

$$\sum_{k\geq 0}^{\infty}\int_{R(t;x)^k}\frac{dc_1\cdots dc_k}{k!}f_{R(t;x)}(c_1,\dots,c_k)\chi_{c_1\cdots c_k}(x)\exp[-\beta H(S_{-t}^{c_1\cdots c_k}(x))]=\sum_{k\geq 0}\sum_{N=0}^{k}\sum_{j_1,\dots,j_N}$$

$$\times\int_{A(j_1,\dots,j_N)}\frac{dc_1\cdots dc_k}{k!}f_{R(t;x)}(c_1,\dots,c_k)\;\chi_{c_1\cdots c_k}(x)\exp[-\beta H(S_{-t}^{c_{j_1}\cdots c_{j_N}}(x))]\,, \tag{29}$$

where $A(j_1,\dots,j_N)$ is the set of all the $c_1,\dots,c_k$ contained in $R(x;t)$ such that

$$S_{-\tau}^{c_1\cdots c_k}(x)=S_{-\tau}^{c_{j_1}\cdots c_{j_N}}(x),\quad \text{for all } 0\leq\tau\leq t\,,$$

i.e., such that the trajectory $S_{-\tau}^{c_1\cdots c_k}(x)$ hits only $c_{j_1},\dots,c_{j_N}$ in the time t; the $\Sigma_{j_1,\dots,j_N}$ runs over all the possible choices of N indices between k indices.

Now, using the symmetry of the f's as functions of the c's we can write the second integral in (29) as

$$=\sum_{N=0}^{\infty}\int_{A_{x,t,N}}\frac{dc_1\cdots dc_N}{N!}\exp[-\beta H(S_{-t}^{c_1\cdots c_N}(x))]\chi_{c_1\cdots c_N}(x)$$

$$\times\left[\sum_{h\geq 0}\int_{\substack{c'\in R(x;t)\\ c'\notin R(x;t,c_1,\dots,c_N)}}\frac{dc'_1\cdots dc'_h}{h!}f_{R(x;t)}(c_1,\dots,c_N;c'_1,\dots,c'_h)\right]\,, \tag{30}$$

where the integration region $A_{x,t,N}$ is the same as the one appearing in (26); now it is immediate to see that (30) coincides with (27) because the term in the square bracket in (30) coincides with $f_{R(x,t/c_1,\dots,c_N)}(c_1,\dots,c_N)$ as a consequence of the probabilistic meaning of the functions f_R [see discussion after (6)].

Formulas (28) follow from the alternative definition of $I(x;t;H)$ through the limit [compare (3) and (7)]

$$I(x;t;H)=\lim_{\Lambda\to\infty}Z_\Lambda^{-1}\sum_{m\geq 0}\int_{\Lambda^m}\frac{dc_1\cdots dc_m}{m!}z^m\exp[-\beta H(S_{-t}^{c_1\cdots c_m}(x))]\chi_{c_1\cdots c_m}(x)\,e^{-\tilde{\beta}V(c_1,\dots,c_M)}\,.$$

Having proven (27) and (28) we can write the difference between the $\rho(x;t)$ and the equilibrium correla-function $\rho(x)$ (which corresponds to $H=0$ and is z analytic around $z=0$): Using (27) and (7) we get for $\rho(x;t)-\rho(x)=\exp[\beta\mu-\beta T(x)]\,[I(x;t;H)-I(x;0;0)]$ and

$$I(x;t;H)-I(x;0;0)=\sum_{N\geq 0}\int_{A_{xtNH}}\frac{dc_1\cdots dc_N}{N!}F(x,t;c_1,\dots,c_N)\{\exp[-\beta H(S_{-t}^{c_1\cdots c_N}(x))]-1\}\,, \tag{31}$$

where the region A_{xtNH} is defined as $A_{x,t,N}$ in (26) but with the further condition that the trajectory

$S_{-t}{}^{c_1\cdots c_N}(x)$ must end inside the region Λ_h where the initial external potential h is different from zero.

We see from (31) and (28) that, since h is supposed to be bounded below, all the terms of the series (31) are uniformly bounded in t.

Remembering that $\rho(x)$ is z analytic for small z, we can say that (31) should be regarded as a resummation of the activity series because developing $F(x;t;c_1,\dots,c_N)$ in powers of z [such a development is possible as it can be seen from the definitions (6)] and grouping the coefficients of the powers of z coming from the various terms in (31) one gets back the activity series. Inequality (28) ensures that no term in the new series for the ρ's blows up as $t\to\infty$.

Formula (31) would be very useful if one could prove that the involved series is uniformly convergent in time; however, one can easily convince himself that this is not the case (cf. Sec. 7). So it is a legitimate question to ask which is the practical use of the resummed series. The answer is that the above resummed series can be used to prove some interesting theorems such as the impossibility of the approach to equilibrium in the case where the tree-tree interaction allows overlapping or the rigorous validity of the Boltzmann equation in the Boltzmann limit (see Secs. 6 and 7). But one of the main reasons for which we think that the series (31) deserves some attention is that, using it as a starting point for the usual formal manipulations which lead to expressions for the diffusion coefficient and other transport coefficients, one automatically gets divergence-free expressions without having to deal with any further resummation [this is a consequence of the damping function $F(x,t;c_1,\dots,c_N)$]; however, we shall not deal with this point since it involves developments which, although usual, are not rigorous and in this paper we are concerned mainly with rigorous results and we shall discuss the formal (and perhaps more interesting) implications of the rigorous resummation achieved in this paper elsewhere.

We conclude this section by observing that, using the functions defined in (25) and their probabilistic interpretation [cf. discussion preceding (25)], one can easily find expressions for the most relevant Green's functions. For instance, the Green's function $G(\hat n,r;t)$, defined as the probability distribution that a particle leaving the origin at time $t=0$ with a velocity $p=|p|\hat n$ arrives in r after a time t has elapsed, is [if $x_0=(p,0)$]

$$G(\hat n,r;t)=\Omega_\nu^{-1}\sum_{N\geq 0}\int_{A_{x_0,t,N}}\frac{dc_1\cdots dc_N}{N!}\,P(x,t;c_1,\dots,c_N)\,\delta[(S_{-t}{}^{c_1\cdots c_N}(x_0))_2-r]\,, \tag{32}$$

where Ω_ν is the surface of a ν-dimensional sphere and $(S_{-t}{}^{c_1\cdots c_N}(x_0))_2$ is the space coordinate of $S_{-t}{}^{c_1\cdots c_N}(x_0)$ and $A_{x_0,t,N}$ has been defined in (20).

A similar expression holds for the Green's function $G(\hat n,r;t/\hat n_0,0)$ expressing the probability of finding, at time t, a wind particle with phase coordinates $(p=|p|\hat n,r)$, if its phase coordinate at $t=0$ was $(p_0=|p|\hat n_0,0)$: It is given by [denoting $x=(p,r)$ and $x_0=(p_0,0)$]

$$G(\hat n,r;t/\hat n_0,0)=\sum_{N\geq 0}\int_{A_{x_0,t,N}}\frac{dc_1\cdots dc_N}{N!}\,P(x_0,t;c_1,\dots,c_N)\,\delta[S_{-t}{}^{c_1\cdots c_N}(x_0)-x]\,. \tag{33}$$

In order to be able to use formulas (31)–(33) one should know some expression for the $F(x,t;\dots)$ or for the functions $f_R(\cdot)$ introduced in (25) and (6) and here we are faced with the difficulty that the equilibrium statistical mechanics of a continuous system is not well understood, at least not at the point of giving even approximate formulas for the $f_R(\cdot)$, when R has the shape of interest here (a very thin tube). However, there is one important case in which the f's are known and it is obviously the case of a free-tree gas: We have, in fact,

$$f_R(c_1,\dots,c_N)=z^N e^{-zV(R)}\,, \tag{34}$$

where $V(R)$ is the volume of R.

6. APPROACH TO EQUILIBRIUM

All the results that have been derived so far did not depend on the fact that the field h is supposed to be concentrated in a finite region Λ_h. The situation is radically different when we turn to the study of the approach to equilibrium. In fact, if h is concentrated on a finite region we expect on physical grounds that the wind system approaches the equilibrium state corresponding to a temperature β^{-1} and a chemical potential μ (if it does approach any equilibrium) because the initial state differs from this equilibrium state only in a finite region Λ_h and the rest of the system acts as a reservoir (with infinite inertia).

If h is not concentrated on a finite region it is

difficult to see which will be the equilibrium state reached by the system, since the removal of the external field involves an infinite energy change and so we expect the new equilibrium to have different values of the temperature and the chemical potential.

The rate of approach to equilibrium is given, according to formula (31), by the rate of approach to zero of the function

$$J(t) = \sum_{N \geq 0} \int_{A_{x,t,N,H}} \frac{dc_1, \ldots, dc_N}{N!} \times F(x,t;c_1,\ldots,c_N) \leq 1 . \qquad (35)$$

This function has [cf. (31), (26), (28)] the very simple physical meaning of being proportional to the probability that the wind particle $x = (p, 0)$ reaches the region Λ_h in the time t in the assigned random distribution of the trees [the proportionality coefficient is $f_{R(x;0/\phi)}(\phi)^{-1}$; see (25)].

We now show that a necessary condition for the return probability $J(t) \to 0$ as $t \to \infty$ (i.e., for the approach to equilibrium) is that the mutual interaction between the trees contains a hard core with radius (or edge) larger than the radius of the wind-tree core. This condition is called for obvious reasons "nonoverlapping condition."

This result is intuitively a consequence of the fact that, if the trees can overlap, the probability that a wind particle is trapped in a finite region is nonzero even for very small density. Clearly, if the trees were allowed to recoil in the shocks with the wind this argument would no longer apply but it seems to suggest that the approach to equilibrium would have a slow tail when large overlapping is allowed and the mass of the trees is much larger than that of the wind.

The formal proof is based on the expression (35). In fact, if the tree-tree hard core is smaller than the wind-tree core we can find N so large that we can arrange N trees in such a way to surround completely (as far as the wind is concerned) a given region; for simplicity we suppose that this region if contained in Λ_h and contains the origin. Clearly, the set of arrangements of N trees which "surround" a given region has a nonzero measure, under our overlapping hypothesis, and so we see that the contribution to $J(t)$ from the Nth term $J_N(t)$ in (35) cannot tend to zero, provided the cutoff function [introduced in (25)] does not vanish for the paths trapped in the chosen finite region [the cutoff functions cannot vanish if z is small enough, because for small z they are proportional to z^N with an approximation uniform in z if the volume of the region $R(x,t;\ldots)$ varies in a bounded set].

In the case that the tree-tree core is equal or larger than the wind-tree hard core, the set of tree configurations which trap a wind particle is of zero measure and the wind particle will eventually escape from any finite region as $t \to \infty$ except for a zero-measure set of values $c_1, \ldots, c_N$ and of wind initial coordinates x (this reasonable statement has been proved only in the case of spherically shaped trees and, in this case, is a by-product of the proof of the ergodicity of a system of hard spheres[23-25]). Since we can expect on physical grounds that the trapping or quasi-trapping phenomena are negligible in the limit $t \to \infty$, the system will approach equilibrium. However, a proof of this expected result is lacking, but it seems possible that one could at least prove that each term in (35) tends to zero if there is no overlapping: This "almost" follows from the fact that each term in (35) is uniformly bounded and from the Sinai result[25] mentioned above, and it should be possible to fill the gaps of this argument without any new idea.

We conclude this section with a remark. We observe that we have proved that, if the trees can overlap, the system does not approach the Gibbs distribution as $t \to \infty$; however, we can expect that if one subtracts from $J(t)$ in (35) the contribution $J_T(t)$ from the tree configurations which divide the space (as far as the wind particle originally at x is concerned) into two separate regions, then $J(t) - J_T(t)$ should tend to zero as $t \to \infty$ and so we can say that this function gives the rate of approach to the asymptotic "regime" which differs from the Gibbs distribution by $J_T(t) \exp[\beta\mu - \beta T(x)]$ [see (31)]. We can even expect, as in the case of nonoverlapping trees, that the approach to the asymptotic regime is described by a diffusion process.

7. BOLTZMANN LIMIT AND THE RELATED DIFFUSION PROCESS

In this section we analyze in some detail the limiting case in which the radius σ of the trees (supposed, to be definite, spherically shaped) tends to zero while the density n tends to infinity in such a way that the mean free path $\lambda = (2n\sigma)^{-1}$ stays constant; we suppose also that the trees are a perfect gas (so $n = z$) and, for simplicity, that the dimension of the space is 2.

We shall study only the Green's function $G(p, \vec{r}; t)$ obtained averaging (32) over the possible directions of the initial velocity $p = |\vec{p}|\hat{n}$ and we shall prove that the asymptotic form of $G(p, \vec{r}; t)$ is given by

$$G(p, \vec{r}; t) \underset{t \to +\infty}{\sim} \exp(-r^2/D'pt)/\pi D'pt \quad \text{for some } D' > 0 , \qquad (36)$$

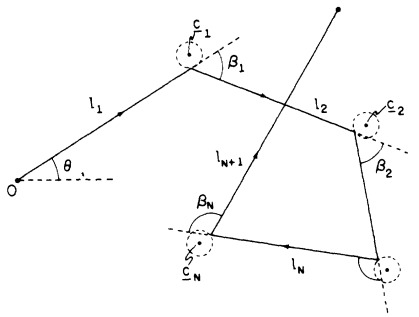

FIG. 2. Change of variables from $c_1,\dots,c_N$ to $l_1,\dots,l_{N+1}$; $\beta_1,\dots,\beta_N$.

i.e., the Green's function $G(p,\vec{r};t)$, through which one can express the time evolution of a distribution of wind particles with a given velocity $|\vec{p}|$ but with the directions of the vector $\vec{p}$ equally distributed, is asymptotically a diffusion kernel verifying an ordinary diffusion equation.

Instead of proving (36) we shall prove the equivalent statement[26]:

$$\lim_{t\to\infty} \int e^{\vec{y}\cdot\vec{r}/(pt)^{1/2}} G(p,r;t)\,dr = e^{D'y^2/4} \quad \text{for all vectors } \vec{y}\ . \tag{37}$$

Consider, in fact, the Nth contribution to $G(p,\vec{r};t)$ in (24). This term is expressed as an integral over the positions $c_1,\dots,c_N$ of N trees disposed in such a way to be hit by the wind particle leaving from the origin with velocity p during the time interval t.

Divide this region into the union $\bigcup_{j_1\cdots j_N} R_{j_1\dots j_N}(t)$, where $R_{j_1\ \dots\ j_N}(t)$ is defined as the set of configurations $c_1\cdots c_N$ such that our wind particle hits, in the time t, c_1 j_1-times, ..., c_N j_N-times.

Let us now consider the contribution from the integration region $R_{1,\dots,1}(t)$. It is very convenient to change the variables $c_1,\dots,c_N$ into the variables $l_1,\dots,l_{N+1}$; $\beta_1,\dots,\beta_N$ defined in Fig. 2. The Jacobian of this transformation is simply given by:

$$\frac{dc_1\cdots dc_N}{N!} = \sigma^N \delta\Big(\sum_{i=1}^{N+1} l_i - pt\Big)\,dl_1\cdots dl_{N+1}(\sin\tfrac{1}{2}\beta_1\, d\tfrac{1}{2}\beta_1)\cdots(\sin\tfrac{1}{2}\beta_N\, d\tfrac{1}{2}\beta_N)\,, \tag{38}$$

and if we write $G(\vec{p},\vec{r};t)$ as the average over the directions of the initial velocity p of the function defined in (32), the integral appearing in (37) becomes a series whose Nth term is given by:

$$(2z\sigma)^N \int_0^\infty \prod_{i=1}^{N+1} dl_i \int_0^{2\pi} \prod_{i=1}^{N} (\sin\tfrac{1}{2}\beta_i\, d\tfrac{1}{4}\beta_i) \int_0^{2\pi} \frac{d\theta}{2\pi}\, \delta\Big(\sum_{i=1}^{N+1} l_i - pt\Big)\, \exp\Big(\sum_{i}^{N+1} \vec{l}_i\cdot\vec{y}/(pt)^{1/2}\Big)$$

$$\times e^{-zV(R(x,t;c_1,\dots,c_N))}\, e^{z\pi\sigma^2}\ , \tag{39}$$

where the extra integration $d\theta/2\pi$ has been introduced in view of the fact that we have to consider the average of (32) over the initial velocity directions and the vectors $\vec{l}_i$ are the vectors represented by arrows in Fig. 2. The integration region over $l_1,\dots,l_{N+1}$; $\beta_1,\dots,\beta_N$ is subject to some further restrictions besides the ones indicated in (39), i.e., the values of $l_1,\dots,l_{N+1}$; $\beta_1,\dots,\beta_N$ which are allowed must have the property that no tree can be found either intercepting the straight segments of the broken line representing the trajectory in Fig. 2 or containing the starting point.

Now in the limit of vanishing size ($\sigma\to 0, n\sigma = \text{const}$) of the trees these further constraints disappear and, furthermore,

$$zV(R(x,t;c_1,\dots,c_N) \to \lambda^{-1} \sum_{j=1}^{N+1} l_j = \lambda^{-1} pt\,, \quad \text{as } \sigma\to 0\,, \tag{40}$$

where $\lambda = (2z\sigma)^{-1} = (2n\sigma)^{-1}$ is the mean free path (we have $n = z$ because the gas of trees is supposed to be perfect). So we find that the regions $R_{1,\dots,1}(t)$ contribute to $G(p,\vec{r};t)$ as

$$\sum_{N=0}^{\infty}(2n\sigma)^N e^{-\lambda^{-1}pt}\int_0^{\infty}\prod_{i=1}^{N+1}dl_i\int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^{2\pi}\prod_{i=1}^{N}(\sin\tfrac{1}{2}\beta_i\, d\tfrac{1}{4}\beta_i)\exp\sum_{i=1}^{N+1}(\vec{l}_i\cdot\vec{y})/(pt)^{\frac{1}{2}}\cdot\delta\Big(\sum_{i=1}^{N+1}l_i-pt\Big). \tag{41}$$

A priori this formula gives only a part of (37) since it comes from the consideration of $R_1,\ldots,_1(t)$ only; however, we realize that (41) gives the correct value of $G(p,\vec{r};t)$ because we are considering the limiting case $\sigma\to 0, n\sigma=\text{const}$ and the integrals coming from the regions

$$R_{j_1,\cdots,j_N}(t)$$

with some of the j's different from unity vanish in this limit.

Now to calculate (41) we first note that it can be interpreted as the average of $\exp[\vec{r}\cdot\vec{y}/(pt)^{1/2}]$ over $\vec{r}$ when the process describing the random variable $\vec{r}$ is the following "random flight" process: A particle leaves the origin with a velocity p in a direction θ with probability $d\theta/2\pi$ and flies in the time t a path of length pt suffering N shocks with probability $e^{-\lambda^{-1}pt}(\lambda^{-1}pt)^N/N!$ and at each shock it is deviated by an angle β with probability $\sin\frac{1}{2}\beta\, d\frac{1}{4}\beta$. The random flight problem has been investigated by several people [see Ref. 26 for flights in which each step has a given length; for flights in which the length of each step is distributed according to a given distribution[27]; flights of the type in which we are interested have been studied by Rayleigh[28] and van Leeuwen and Weyland[29] (but with $\sin\frac{1}{2}\beta\, d\frac{1}{4}\beta$ replaced by $d\beta/2\pi$)], and applying their methods, in particular see Ref. 26, we find that (36) holds with D' given by

$$D'=\tfrac{3}{4}n\sigma=\tfrac{3}{2}\lambda \tag{42}$$

and the calculation leading to this result is reported in the Appendix.

We remark that the value $D'p$ of the diffusion coefficient, as given by (42), is the lowest-order expression for the diffusion coefficient already known by other methods[2,29] and it is the exact expression for the diffusion coefficient in our limit.

8. BOLTZMANN EQUATION

We conclude this paper by showing that in the limiting case considered in Sec. 7 the time evolution of our system is rigorously described by the Boltzmann equation.[30]

This property has been conjectured by several people and has a very clear physical meaning (see, for instance, Grad[31]) but it seems to have never been proved rigorously.

Let us consider only the two-dimensional case; in this case let us call θ the angle between the unit vector $\hat{n}$ in the direction of the velocity p and the x axis; in what follows we shall sometimes identify $\hat{n}$ and θ.

Consider the Green's functions $G(\theta,\vec{r};t)$ and $G(\theta,\vec{r};t/\theta_0,0)$ introduced in (32) and (33), respectively. According to their physical meaning one expects that, in the Boltzmann limit, these functions verify the Boltzmann equation which, for $G(\theta,\vec{r};t)$, is easily seen to be [$\lambda=(2n\sigma)^{-1}$]:

$$\begin{aligned}\frac{\partial G(\theta,\vec{r};t)}{\partial t}&=-\hat{n}\cdot\frac{\partial G(\theta,\vec{r};t)}{\partial\vec{r}}+\lambda^{-1}\int_0^{2\pi}[G(\theta',\vec{r};t)-G(\theta,\vec{r};t)]\,\Big|\sin\frac{\theta'-\theta}{2}\Big|\frac{d\theta'}{4}\\&=-\hat{n}\cdot\frac{\partial G(\theta,\vec{r};t)}{\partial\vec{r}}+\lambda^{-1}\int_0^{2\pi}G(\theta',\vec{r};t)\Big|\sin\frac{\theta'-\theta}{2}\Big|\frac{d\theta}{4}-\lambda^{-1}G(\theta,\vec{r};t),\end{aligned} \tag{43}$$

and an analogous equation is expected to hold for $G(\theta,\vec{r};t/\theta_0,0)$.

We give a proof of (43); the proof of the analogous equation for $G(\theta,\vec{r};t/\theta_0,0)$ is essentially the same.

Using the results and the notations of Sec. 7, the expression of (24) in our limit is

$$G(\theta,\vec{r};t)=\frac{e^{-\lambda^{-1}pt}}{2\pi}\sum_{N\geq 0}\lambda^{-N}\int_0^{\infty}\Big(\prod_{i=1}^{N+1}dl_i\Big)\Big(\prod_{j=1}^{N}\sin\frac{\beta_j}{2}\frac{d\beta_j}{4}\Big)\delta\Big(\sum_{j=1}^{N+1}\vec{l}_j-\vec{r}\Big)\delta\Big(\sum_{i=1}^{N+1}l_i-pt\Big). \tag{44}$$

So we recognize that the last term in (43) comes from the time derivative of $\exp(-\lambda^{-1}pt)$ whereas the two other terms are found as follows: Consider the time derivative of the Nth term $G_N(\theta,\vec{r};t)$ in (44) and denote $\hat{n}_i$ a unit vector in the direction of l_i (therefore, $\hat{n}_1=\hat{n}=\hat{\theta}$) then it is straightforward to go through the following chain of equations:

$$\frac{\partial G_N(\theta,\vec{r};t)}{\partial t} = \frac{\partial}{\partial t}\int_0^{pt}\left(\prod_{i=1}^{N+1} dl_i\right)\int_0^{2\pi}\left(\prod_{i=1}^{N}\sin\frac{\beta_i}{2}\frac{d\beta_i}{4}\right)\delta\left(\sum_{i=1}^{N+1}\vec{l}_i-\vec{r}\right)\delta\left(\sum_{i=1}^{N+1} l_i - pt\right) = \frac{\partial}{\partial t}\int_0^{pt} dl_{N+1}$$

$$\times\int_0^{pt-l_{N+1}} dl_N\cdots\int_0^{pt-l_{N+1}\cdots-l_3} dl_2\int_0^{2\pi}\left(\prod_{i=1}^{N}\sin\frac{\beta}{2}\frac{d\beta_i}{4}\right)\delta\left(\sum_{i=2}^{N+1} l_i(\hat{n}_i-\hat{n}_1)+pt\hat{n}_1-\vec{r}\right)$$

$$= \int_0^{pt} dl_{N+1}\int_0^{pt-l_{N+1}} dl_N\cdots\frac{\partial}{\partial t}\int_0^{pt-l_{N+1}\cdots-l_3} dl_2\int_0^{2\pi}\left(\prod_{i=1}^{N}\sin\frac{\beta_i}{2}\frac{d\beta_i}{4}\right)$$

$$\times\,\delta\left(\sum_{i=2}^{N+1} l_i(\hat{n}_i-\hat{n}_1)+pt\hat{n}_1-\vec{r}\right) = -p\hat{n}\cdot\frac{\partial G_N}{\partial\vec{r}}(\theta,\vec{r};t) + \int_0^{2\pi}\sin\frac{\beta_1}{2}\cdot\frac{d\beta_1}{4}\cdot G_{N-1}(\theta+\beta_1,\vec{r};t)$$

and now using this recurrence relation (43) follows immediately.

9. SUMMARY AND COMPARATIVE DISCUSSION OF THE RESULTS

(i) The time-dependent correlation functions describing the evolution of a certain class of initial states have been proved to exist and to be analytic in the density of the trees (at small enough density), and in spite of the fact that the coefficients of their density expansions diverge as $t\to\infty$ the radius of convergence of these expansions does not shrink to zero at $t\to\infty$.

(ii) The above-mentioned fact of the constance of the radius of convergence leads us to believe that there is the possibility of resumming the divergent terms in a cutoff; in fact, we have given a rigorous proof of the existence of this resummation which has been explicitly exhibited.

We believe that the rigorous resummation obtained in this paper is closely related to the formal resummations demonstrated[3,29,32,33] by using diagrammatic techniques based on the binary collision expansions.[34] However, our results seem to include a deeper resummation process than the "resummation of the uncorrelated virtual collisions" used in the above-mentioned references because our expressions for the Green's functions contain only real collisions and consequently the cutoff function is more complicated.

It seems possible that, using our expressions for the Green's functions as a starting point for the investigation of the properties of the diffusion coefficient, the divergence found in Ref. 3 in the case of overlapping square trees (and only four possibilities for the directions of the wind velocity) can be renormalized.

(iii) We have given two applications of the resummed series for the Green's functions and the correlation functions, i.e., we have proved that, in the Boltzmann limit (i.e., vanishing size of the trees but nonvanishing free path), the approach to equilibrium is described by a diffusion process of which we find the asymptotic form and which we prove to verify exactly the Boltzmann equation. As a second application we have shown that equilibrium cannot be attained in the case of overlapping trees.

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APPENDIX

Formula (41) can be written

$$\sum_{N=0}^{\infty}(2n\sigma)^N\exp(-\lambda^{-1}pt)I_N(\vec{y},pt)\ , \tag{A1}$$

and the main problem is to find an expression for $I_N(\vec{y},pt)$ as $t\to\infty$; we have

$$I_N(\vec{y},pt)=\int_0^\infty \prod_{j=1}^{N+1} dl_j \int_0^{2\pi}\frac{d\theta}{2\pi}\left(\prod_{j=1}^{N+1}\sin\frac{\beta_j}{2}\frac{d\beta_j}{4}\right)\prod_{j=1}^{N+1}\exp(l_j y\cos\alpha_j)\delta\left(\sum_{j=1}^{N+1} l_j - pt\right)\,, \tag{A2}$$

where $\alpha_j=\theta+\beta_1+\cdots+\beta_{j-1}$. So, developing the exponential in powers, (A2) becomes

$$\sum_{n_0\cdots n_N}^{0,\infty}\int_0^{2\pi}\frac{d\theta}{2\pi}\int^{2\pi}\prod_{j=1}^{N+1}\left(\sin\frac{\beta_j}{2}\frac{d\beta_j}{4}\right)\left(\prod_{j=1}^{N+1}dl_j\right)\prod_{j=1}^{N+1}\left(\frac{l_j y}{(pt)^{1/2}}\right)^{n_{j-1}}\frac{(\cos\alpha_j)^{n_{j-1}}}{(n_{j-1})!}\,, \tag{A3}$$

and using the formula

$$\int_0^\infty\prod_{j=1}^{N+1}dl_j\prod_{j=1}^{N+1}l_j^{n_j-1}\delta\left(\sum_{j=1}^{N+1}l_j-pt\right)=\frac{(pt)^{N+\sum_j n_{j-1}}}{(N+\sum_j n_{j-1})!}\prod_j(n_{j-1}!)\,, \tag{A4}$$

we see that (A3) becomes

$$\sum_{k=0}^{\infty}\frac{(pt)^{N+k/2}}{(N+k)!}y^k\sum_{\sum_i n_i=k}\int_0^{2\pi}\frac{d\theta}{2\pi}\left(\prod_{j=1}^{N}\sin\frac{\beta_j}{2}\frac{d\beta_j}{4}\right)\prod_{j=1}^{N+1}(\cos\alpha_j)^{n_j-1}\,. \tag{A5}$$

We now sketch the way to find $I_k(N)$ defined as

$$\begin{aligned}I_k(N)&=\sum_{\sum_i n_i=k}\int_0^{2\pi}\frac{d\theta}{2\pi}\left(\prod_{j-1}^{N}\sin\frac{\beta_j}{2}\frac{d\beta_j}{4}\right)\prod_{j=1}^{N+1}(\cos\alpha_j)^{n_j-1}\\&=\sum_{\sum_i n_i=k}\int_0^{2\pi}\frac{d\theta}{2\pi}(\cos\theta)^{n_0}\frac{d\beta}{4}\sin\frac{\beta}{2}(\cos\theta+\beta)^{n_1}\cdots\frac{d\beta_N}{4}\sin\frac{\beta_N}{2}(\cos\theta+\beta_j+\cdots\beta_N)^{n_N}\,,\end{aligned} \tag{A6}$$

and after the change, $\eta_0=\theta,\eta_1=\theta+\beta_1,\ldots,\eta_N=\theta+\cdots+\beta_N$, we find that

$$\begin{aligned}\sum_{k=0}^{\infty}z^k I_k(N)&=\int_0^{2\pi}\frac{d\eta_0}{2\pi}\frac{1}{1-z\cos\eta_0}\frac{d\eta}{4}\left|\sin\frac{\eta_1-\eta_0}{2}\right|\frac{1}{1-z\cos\eta_1}\cdots\frac{d\eta_N}{4}\left|\sin\frac{\eta_N-\eta_{N-1}}{2}\right|\frac{1}{1-z\cos\eta_N}\\&=\langle\Phi,K^N\Phi\rangle\,,\end{aligned} \tag{A7}$$

where the kernel K is defined as

$$K(\eta^1,\eta)=\frac{1}{4}\frac{1}{(1-z\cos\eta^1)^{1/2}}\left|\sin\frac{(\eta^1-\eta)}{2}\right|\frac{1}{(1-z\cos\eta)^{1/2}} \tag{A8}$$

and Φ is the function

$$\Phi(\eta)=[2\pi(1-z\cos\eta)]^{1/2}\,. \tag{A9}$$

Now we remark that if $\lambda_0(z)$ is the highest eigenvalue of K and Φ_0 is the corresponding eigenfunction, we have [using (A7)]

$$\sum_{1=0}^{\infty}z^k I_k(N)\simeq\lambda_0(z)^N|\langle\Phi,\Phi_0\rangle|^2,\quad\text{as } N\to\infty\,; \tag{A10}$$

so, writing $\lambda_0(z)=1+z^2\gamma+z^4(\cdots)\,,$ (A11)

note that only even powers of z appear in this expansion), it follows that

$$I_k(N) = 0, \qquad \text{if } k \text{ is odd},$$
$$\simeq N^{k/2}\gamma^{k/2}/(k/2)!, \qquad \text{as } N\to\infty, \quad \text{if } k \text{ is even} \tag{A12}$$

[the fact that $I_k(N)$ is 0 when k is odd is directly verified from its definition (A6) by changing the coordinates η_i into $\eta_i+\pi$].

From Formula (A12) we see that

$$\sum_{N=0}^{\infty}(2n\sigma)^N e^{-\lambda^{-1}pt} I_N(y,tp) = e^{-\lambda^{-1}tp}\sum_{N=0}^{\infty}\sum_{k=0}\lambda^{-N}\frac{(pt)^{N+k}}{(N+2k)!}y^{2k}I_N(2k)$$

$$\underset{t\to\infty}{\simeq} e^{-\lambda^{-1}pt}\sum_{k,N>0}\frac{(\lambda^{-1}pt)^{N+k}}{(N+2k)!}\frac{N^k}{k!}\left(\frac{\gamma y^2}{\lambda^{-1}}\right)^k \underset{t\to\infty}{\simeq}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\gamma y^2}{\lambda^{-1}}\right)^k = \exp\lambda\gamma y^2, \tag{A13}$$

so we have found [cf. (36)] $D' = 4\lambda\gamma$. (A14)

An elementary application of the perturbation theory to K gives the coefficient γ which turns out to be

$$\gamma = \tfrac{3}{8} . \tag{A15}$$

The details of this calculation can be found in Ref. 26 where an essentially identical perturbation calculation is done.

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