Luttinger liquid fixed point for a 2D flat Fermi surface

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Abstract

We consider a system of 2D interacting fermions with a flat Fermi surface. The apparent conflict between Luttinger and non Luttinger liquid behavior found through different approximations is resolved by showing the existence of a line of non trivial fixed points, for the RG flow, corresponding to Luttinger liquid behavior; the presence of marginally relevant operators can cause flow away from the fixed point. The analysis is non-perturbative and based on the implementation, at each RG iteration, of Ward Identities obtained from local phase transformations depending on the Fermi surface side, implying the partial vanishing of the Beta function.

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1 Introduction

The properties of the $2D$ interacting fermions are still largely unknown, despite the tremendous effort devoted to their understanding in the last years. One of the most debated questions is on the possible existence of a *Luttinger liquid phase*, first suggested by Anderson [1] as an explanation of some properties of high T_c superconductors, as observed also in recent experiments, see e.g. [2] .

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It has been proved, in the case of symmetric, smooth and convex Fermi surfaces (like in the Jellium model [3] or in the Hubbard model in the non half filled case [4]), that the wave function renormalization Z is essentially temperature independent up to exponentially small temperatures. As in a Luttinger liquid one expects instead a logarithmic behavior in this regime, *i.e.* $Z \simeq 1 + O(U^2 \log \beta)$, such results rule out for sure the possibility of Luttinger liquid behavior.

On the contrary the presence in the Fermi surface of flat regions can produce non Fermi liquid behavior. The simplest model with a flat Fermi surface is the 2D Hubbard model at half filling, in which the Fermi surface is a square. It was proved in [5] that the wave function renormalization is $Z = 1 + O(U^2 \log^2 \beta)$ up to exponentially small temperatures; the presence of the $\log^2 \beta$ is a consequence of the Van Hove singularities, related to the fact that the Fermi velocity is vanishing at the corners of the squared Fermi surface, and implies that also such a model does *not* show Luttinger liquid behavior.

It is important to stress that the results in [3],[4],[5] are rigorous as they are based on expansions which are *convergent* provided that the temperature is not too low, the finite temperature acting as an infrared cut-offs; however such expansions cannot give any information on the zero temperature properties.

A lot of attention has been devoted in recent years to the zero temperature properties of Fermi surfaces with flat regions and no corners, which share some features with the Fermi surfaces of some cuprates as seen in photoemission experiments. Parquet methods results [6] and perturbative Renormalization Group (RG) analysis [7] truncated at one loop indicate that, for repulsive interactions, there is no indication of a Luttinger liquid phase at zero temperature; the effective couplings flow toward a strong coupling regime related to the onset of d-wave superconductivity. In a more recent RG analysis truncated at 2 loops [8] one still gets a flow to strong coupling, but in some intermediate region some indication of Luttinger liquid behavior is found.

Apparently conflicting results are found by applying bosonization: in [9], [10] a model of electrons on a square Fermi surface was mapped in a collection of fermions on coupled chains, and it is found that the correlations at zero temperature in momentum space are similar to the one of the Luttinger model. A related but somewhat different strategy consists in proposing an exactly solvable 2D analogue of the Luttinger model; this approach was pursued

in [11] and [12] and again Luttinger liquid behavior up to zero temperature was found.

A possible explanation of such conflicting results was suggested in [13], postulating the existence for the RG flow, in addition to the trivial fixed point associated to non interacting fermions, of a *non trivial* fixed point associated to Luttinger behavior, which could be made instable by the presence of marginally relevant operators. In this paper we provide a quantitative verification of such hypothesis showing explicitly the existence of a line non trivial Luttinger fixed points for the RG flow of a system of 2D interacting fermions with a flat Fermi surface. It is would be not possible to derive such result directly from the perturbative expansions, as it is related to cancellations between graphs to all orders of the expansion which are too complex to be seen explicitly; it is indeed well known that even in 1D Ward Identities (WI) are necessary to prove the existence of a Luttinger liquid fixed point [14]. Our analysis is based on the implementation, in an exact RG approach, of WI with corrections due the the cut-offs introduced in the multiscale analysis, extending a technique already used to establish Luttinger liquid behavior in a large class of 1D fermionic systems [15],[16] or 2D spin systems [17]. Such methods are the only ones which can be applied to *non exactly solvable* models, like the model analyzed in this paper.

2 The model

We consider a model with a square Fermi surface similar to the one considered in [6], [8] or [9]; the Schwinger functions are given by functional derivatives of the generating functional

$$e^{\mathcal{W}(\phi)} = \int P(d\psi) e^{\mathcal{V}(\psi) + \int d\mathbf{x} [\psi_\mathbf{x}^+ \phi_\mathbf{x}^- + \psi_\mathbf{x}^- \phi_\mathbf{x}^+]} \tag{1}$$

with $\psi_\mathbf{k}^\pm$ are Grassmann variables, $\mathbf{k} = (k_-, k_+, k_0)$, $k_\pm = \frac{2\pi}{L_\pm} n_\pm$, $k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2})$, $n_\pm, n_0 = 0, \pm 1, \pm 2, \dots$ and $P(d\psi)$ is the fermionic integration with propagator

$$g_\mathbf{k} = \sum_{\sigma=\pm} \sum_{\omega=\pm} \frac{H(k_{-\sigma}) C_0^{-1}(\sqrt{a_0^{-2}(k_0^2 + v_F^2(|k_\sigma| - p_F)^2)})}{-ik_0 + v_F(k_\sigma - \omega p_F)} \equiv \sum_{\sigma,\omega=\pm} g_{\sigma,\omega,\mathbf{k}} \tag{2}$$

$H(k_{-\sigma}) = \chi(a^{-2} k_{-\sigma}^2)$, $\chi(t) = 1$ if $t < 1$ and 0 otherwise, $C_0^{-1}(t)$ is a smooth compact support function $= 1$ for $t < 1$ and $= 0$ for $t \geq \gamma$, $\gamma > 1$. We

assume, for definiteness, $a \le \frac{p_F}{4}, a_0 \le \frac{p_F}{20}$ so that the support of $g_{\mathbf{k}}$ is over 4 disconnected regions; the Fermi surface is defined as the set of in which $g_{\mathbf{k}}$ for $k_0 = 0$ is singular, in the limit $\beta \to \infty$.

By using well known properties of Grassmann integrals, see [18], (6) allows to write the Grassmann field as a sum of independent fields

$$\psi_{\mathbf{k}}^{\pm} = \sum_{\sigma=\pm} \sum_{\omega=\pm} \psi_{\omega,\sigma,\mathbf{k}}^{\pm} \tag{3}$$

with $\psi_{\omega,\sigma,\mathbf{x}}^{\pm}$ independent Grassmann variables with propagator $g_{\omega,\sigma,\mathbf{k}}$. As in [6],[8] or [9] we can consider only interactions between parallel patches $(V = L_+ L_-)$

$$\mathcal{V} = \sum_{\sigma} \sum_{\underline{\omega}} \frac{1}{(V\beta)^4} \sum_{\mathbf{k}_1,\ldots\mathbf{k}_4} U\hat{v}(\mathbf{k}_1-\mathbf{k}_2)\psi_{\omega_1,\sigma,\mathbf{k}_1}^{+}\psi_{\omega_2,\sigma,\mathbf{k}_2}^{-}\psi_{\omega_3,\sigma,\mathbf{k}_3}^{+}\psi_{\omega_4,\sigma,\mathbf{k}_4}^{-}\delta(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4) \tag{4}$$

with $v(\mathbf{x})$ a short range potential. The 2-point Schwinger function is given by

$$S_2(\mathbf{x}, \mathbf{y}) = \frac{\partial^2 \mathcal{W}(\phi)}{\partial \phi_{\mathbf{x}} \partial \phi_{\mathbf{y}}}|_{\phi=0} \tag{5}$$

Figure 1: The Fermi surface corresponding to the singularities of $g_{\mathbf{k}}$; the four sides are labelled by $(\sigma, \omega) = (\pm, \pm)$.

3 Renormalization Group analysis

As the interaction does not couple different σ we can from now on fix $\sigma = +$ for definiteness and forget the index σ. We analyze the functional integral

(1) by performing a multiscale analysis, using the methods of constructive Quantum Field Theory (for a general introduction to such methods, see [18]). The propagator (2) can be written as sum of "single slice" propagators in the following way

$$g_\omega(\mathbf{x} - \mathbf{y}) = \sum_{h=-\infty}^{0} e^{i\omega p_F(x_+ - y_+)} g_\omega^{(h)}(\mathbf{x} - \mathbf{y}) \tag{6}$$

where

$$g_\omega^{(h)}(\mathbf{x} - \mathbf{y}) = \frac{1}{V\beta} \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{H(k_-)f_h(\mathbf{k})}{-ik_0 + \omega v_F k_+} \tag{7}$$

and $f_h(\mathbf{k})$ has support in a region $O(\gamma^h)$ around each flat side of the Fermi surface, at a distance $O(\gamma^h)$ from it, that is $a_0 \gamma^{h-1} \leq k_0^2 + v_F^2 k_+^2 \leq a_0 \gamma^{h+1}$; note that in each term in (6) the change of variables $k_+ \to k_+ + \omega p_F$ has been performed. The single scale propagator verify the following bound, for any integer M

$$|g_\omega^{(h)}(\mathbf{x})| \leq C_M |\frac{\sin ax_-}{x_-}| \frac{\gamma^h}{1 + [\gamma^h(|x_+| + |x_0|)]^M} \tag{8}$$

The integration is done iteratively integrating out the fields with momenta closer and closer to the Fermi surface, renormalizing at each step the wave function. After the integration of the fields $\psi^{(0)}, ..., \psi^{(h+1)}$ we obtain

$$\int P_{Z_h}(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})} \tag{9}$$

where $P_{Z_h}(d\psi^{(\leq h)})$ is the fermionic integration with propagator $Z_h^{-1}(\mathbf{k})g_{\omega,\mathbf{k}}^{(\leq h)}$, with $g_{\omega,\mathbf{k}}^{(\leq h)} = \sum_{k=-\infty}^{h} g_{\omega,\mathbf{k}}^{(k)}$ and Z_h is defined iteratively starting from $Z_0 = 1$; moreover, if $\vec{p}_F = (0, p_F, 0)$

$$\mathcal{V}^{(h)}(\psi^{\leq h}) = \sum_{n=1}^{\infty} \sum_{\underline{\omega}} \frac{1}{(\beta V)^{2n}} \sum_{\mathbf{k}_1,...,\mathbf{k}_{2n}} \delta(\sum_i \varepsilon_i(\mathbf{k}_i + \omega_i \vec{p}_F))$$

$$\left[\prod_{i=1}^{2n} \hat{\psi}_{\omega_i,\mathbf{k}_i}^{(\leq h)\varepsilon_i} \right] \widehat{W}_{2n}^{(h)}(\mathbf{k}_1, ..., \mathbf{k}_{2n-1}) \tag{10}$$

By using that $\int d\mathbf{k} |g_{\omega,\sigma,\mathbf{k}}^{(k)}| \leq C\gamma^k$ and $|g_{\omega,\sigma,\mathbf{k}}^{(k)}| \leq C\gamma^{-k}$ we see that the kernels $\widehat{W}_{2n}^{(k)}$ are $O(\gamma^{-k(n-2)})$; this means that the terms quadratic in the fields have

5

positive scaling dimension and the quartic terms have vanishing scaling dimension, and all the other terms have negative dimension; we have then to properly renormalize the terms with non-negative dimension.

Calling $\bar{\mathbf{k}} = (k_-, 0, 0)$ we define an $\mathcal{L}$ operator acting linearly on the kernels of the effective potential:

1)$\mathcal{L}\widehat{W}_{2n}^{(h)} = 0$ if $n \geq 2$

2)If $n = 1$

$$\mathcal{L}\widehat{W}_2^h(\mathbf{k}) = \widehat{W}_2^h(\bar{\mathbf{k}}) + k_0 \partial_{k_0} \widehat{W}_2^h(\bar{\mathbf{k}}) + k_+ \partial_+ \widehat{W}_2^h(\bar{\mathbf{k}}) \tag{11}$$

.

3) If $n = 2$

$$\mathcal{L}\widehat{W}_4^h(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \delta_{\sum_i \varepsilon_i \omega_i, 0} \widehat{W}_4^h(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3) . \tag{12}$$

Calling $\partial_0 \widehat{W}_2^h(\bar{\mathbf{k}}) = -iz_h(k_-)$, $\partial_+ \widehat{W}_2^h(\bar{\mathbf{k}}) = \omega z_h(k_-)$ (symmetry considerations are used) and
$l_h(k_{-,1}, k_{-,2}, k_{-,3}) = \widehat{W}_4^h(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)$ we obtain

$$\mathcal{L}\mathcal{V}^h = \frac{1}{\beta V} \sum_{\mathbf{k}} [z_h(k_-)\omega k_+ - ik_0 z_h(k_-)] \widehat{\psi}_{\mathbf{k},\omega}^{+(\leq h)} \widehat{\psi}_{\mathbf{k},\omega}^{-(\leq h)} + \tag{13}$$

$$\sum_{\underline{\omega},\underline{\sigma}}^* \frac{1}{(\beta V)^4} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4} l_h(k_{-,1}, k_{-,2}, k_{-,3}) \widehat{\psi}_{\mathbf{k}_1,\omega_1}^{+(\leq h)} \widehat{\psi}_{\mathbf{k}_2,\omega_2}^{-(\leq h)} \widehat{\psi}_{\mathbf{k}_3,\omega_3}^{+(\leq h)} \widehat{\psi}_{\mathbf{k}_4,\omega_4}^{-(\leq h)} \delta(\sum_i \varepsilon_i \mathbf{k}_i))$$

where $\sum_{\underline{\omega}}^*$ is constrained to the condition $\sum_i \varepsilon_i \omega_i \vec{p}_F = 0$ and we have used that, by symmetry, $W_2^h(\bar{k}) = 0$.

We write (9) as

$$\int P_{Z_h}(d\psi^{(\leq h)}) e^{-\mathcal{L}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) - \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})} \tag{14}$$

with $\mathcal{R} = 1 - \mathcal{L}$. The non trivial action of $\mathcal{R}$ on the kernel with $n = 2$ can be written as

$$\mathcal{R}\widehat{W}_4^h(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = [\widehat{W}_4^h(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) - \widehat{W}_4^h(\bar{\mathbf{k}}_1, \mathbf{k}_2, \mathbf{k}_3)] \tag{15}$$
$$+ [\widehat{W}_4^h(\bar{\mathbf{k}}_1, \mathbf{k}_2, \mathbf{k}_3) - \widehat{W}_4^h(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \mathbf{k}_3)] + [\widehat{W}_4^h(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \mathbf{k}_3) - \widehat{W}_4^h(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)]$$

6

The first addend can be written as

$$k_{0,1} \int_0^1 dt \partial_{k_{0,1}} \widehat{W}_4^h(k_{-,1}, k_{+,1}, tk_{0,1}; \mathbf{k}_2, \mathbf{k}_3) +$$

$$k_{+,1} \int_0^1 dt \partial_{k_{+,1}} \widehat{W}_4^h(k_{-,1}, tk_{+,1}, 0; \mathbf{k}_2, \mathbf{k}_3) \tag{16}$$

The factors $k_{0,1}$ and $k_{+,1}$ are $O(\gamma^{h'})$, for the compact support properties of the propagator associated to $\widehat{\psi}_{\omega_1, \mathbf{k}_1}^{+(\leq h)}$, with $h' \leq h$, while the derivatives are dimensionally $O(\gamma^{-h-1})$; hence the effect of $\mathcal{R}$ is to produce a factor $\gamma^{h'-h-1} < 1$ making its scaling dimension negative. Similar considerations can be done for the action of $\mathcal{R}$ on the $n = 1$ terms. The effect of the $\mathcal{L}$ operation is to replace in $W_2^h(\mathbf{k})$ the momentum $\vec{k}$ with its projection on the closest flat side of the Fermi surface. Hence the fact that the propagator is singular over an extended region (the Fermi surface) and not simply in a point has the effect that the renormalization point cannot be fixed but it must be left moving on the Fermi surface.

In order to integrate the field $\psi^{(h)}$ we can write (14) as

$$\int P_{Z_{h-1}}(d\psi^{(\leq h)}) e^{-\mathcal{L}\widetilde{\mathcal{V}}^h(\sqrt{Z_h}\psi^{(\leq h)}) - \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})} \tag{17}$$

where $P_{Z_{h-1}}(d\psi^{(\leq h)})$ is the fermionic integration with propagator

$$\frac{1}{Z_{h-1}(\mathbf{k})} \frac{H(k_-)C_h^{-1}(\mathbf{k})}{-ik_0 + \omega v_F k_+} \tag{18}$$

with $C_h^{-1}(\mathbf{k}) = \sum_{k=-\infty}^h f_k$ and

$$Z_{h-1}(\mathbf{k}) = Z_h(k_-)[1 + H(k_-)C_h^{-1}(\mathbf{k})z_h(k_-)] \tag{19}$$

Moreover $\mathcal{L}\widetilde{\mathcal{V}}^h$ is the second term in (13).

We rescale the fields by rewriting the r.h.s. of (14) as

$$\int P_{Z_{h-1}}(d\psi^{(\leq h)}) e^{-\mathcal{L}\widehat{\mathcal{V}}^h(\sqrt{Z_{h-1}}\psi^{(\leq h)}) - \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})} \tag{20}$$

where

$$\mathcal{L}\widehat{\mathcal{V}}^h(\psi) = \sum_{\underline{\omega}}^* \frac{1}{(\beta V)^4} \sum_{\mathbf{k}_1,..,\mathbf{k}_4} g_h(k_{-,1}, k_{-,2}, k_{-,3}) \widehat{\psi}_{\mathbf{k}_1,\omega_1}^+ \widehat{\psi}_{\mathbf{k}_2,\omega_2}^- \widehat{\psi}_{\mathbf{k}_3,\omega_3}^+ \widehat{\psi}_{\mathbf{k}_4,\omega_4}^- \delta(\sum_i \varepsilon_i \mathbf{k}_i) \tag{21}$$

and the *effective couplings*

$$g_h(k_{-,1}, k_{-,2}, k_{-,3}) = [\prod_{i=1}^{4} \sqrt{\frac{Z_h(k_{-,i})}{Z_{h-1}(k_{-,i})}}] l_h(k_{-,1}, k_{-,2}, k_{-,3}) \qquad (22)$$

After the integrations of the fields $\psi^{(0)}, \psi^{(-1)}, ..., \psi^{(h)}$ we get an effective theory describing fermions with wave function renormalization Z_h and effective interaction (21). Note that Z_h and g_h are non trivial functions of the momentum parallel to the Fermi surface.

We write

$$\int P_{Z_{h-1}}(d\psi^{(\leq h-1)}) \int P_{Z_{h-1}}(d\psi^{(h)}) e^{-\mathcal{L}\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) - \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})} \quad (23)$$

and the propagator of $P_{Z_{h-1}}(d\psi)$ is

$$\hat{g}_{\omega,\sigma}^h(\mathbf{k}) = H(k_-)\frac{1}{Z_{h-1}(k_-)}\frac{\widetilde{f}_h(\mathbf{k})}{-ik_0 + \omega v_F k_+}$$

and

$$\widetilde{f}_h(\mathbf{k}) = Z_{h-1}(k_-)[\frac{C_h^{-1}(\mathbf{k})}{Z_{h-1}(\mathbf{k})} - \frac{C_{h-1}^{-1}(\mathbf{k})}{Z_{h-1}(k_-)}] \qquad (24)$$

with $H(k_-)\widetilde{f}_h(\mathbf{k})$ having the same support that $H(k_-)f_h(\mathbf{k})$. We integrate then the field $\psi^{(h)}$ and we get

$$\int P_{Z_{h-1}}(d\psi^{(\leq h-1)}) e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})} \qquad (25)$$

and the procedure can be iterated.

The above procedure allows us to write $W_{2n}^{(h)}$ as a series in the effective couplings g_k, $k \geq h$, which is *convergent*, see [18], provided that L_- is finite and $\varepsilon_h = \sup_{k\geq h} ||g_k||$ small enough; moreover $||W_{2n}|| = O(\gamma^{-h(n-2)})$. A similar analysis can be repeated for the 2-point function.

However even if the couplings g_k starts with small values, they can possibly increase iterating the RG and at the end reach the boundary of the (estimated) convergence domain; if this happen, all the above procedure looses its consistency. A finite temperature acts as an infrared cut-off saying that the RG has to be iterated up to a maximum scale $h_\beta = O(\log \beta)$ and, up to exponentially small temperatures *i.e.* $\beta \leq O(e^{\kappa|U|^{-1}})$, then surely the effective couplings are in the convergence domain; however, in order to get lower temperatures, more information on the effective couplings are necessary.

8

4 RG flow and Luttinger liquid fixed point

The RG analysis seen in the previous section implies that the effective coupling g_h verify a flow equation of the form

$$g_{h-1} = g_h + \beta_g^{(h)}(g_h; ...; g_0) \qquad (26)$$

where the r.h.s. of the above equation is called *Beta function*, which is expressed by a convergent expansion in the couplings if ε_h is small enough. The first non trivial contribution to $\beta_g^{(h)}$, called $\beta_g^{(2)(h)}$, is quadratic in the couplings and it is given by

$$\beta_g^{(2)(h)} = \beta_h^{(a)} + \beta_h^{(b)} \qquad (27)$$

where

$$\beta_h^{(a)} = \int d\mathbf{p}\, H(k_{1,-} - p_-) H(k_{3,-} + p_-) g_h(k_{1,-}, k_{1,-} - p_-, k_{3,-}) \qquad (28)$$

$$g_h(k_{1,-} - p_-, k_{2,-}, k_{3,-} + p_-) \frac{f_h(\mathbf{p}) C_h(\mathbf{p})}{p_0^2 + v_F^2 p_+^2}$$

$$\beta_h^{(b)} = - \int d\mathbf{p}\, H(k_{2,-} - p_-) H(k_{3,-} - p_-) g_h(k_{1,-}, k_{2,-} - p_-, k_{3,-} - p_-)$$

$$g_h(k_{2,-} - p_-, k_{2,-}, k_{3,-}) \frac{f_h(\mathbf{p}) C_h(\mathbf{p})}{p_0^2 + v_F^2 p_+^2} \qquad (29)$$

The above expression essentially coincides with the one found in [6] or [8]; it is indeed well known that the lowest order contributions to the Beta function are essentially independent by RG procedure one follows.

The flow equation (26) encodes most of the physical properties of the model, but its analysis is extremely complex. Some insights can be obtained by truncating the beta function at second order, and by the numerical analysis of the resulting flow by discretization of the Fermi surface; it is found, see [6] or [8], that $g_h(k_{-,1}, k_{-,2}, k_{-,3})$ has a flow which, for certain values of $k_{-,1}, k_{-,2}, k_{-,3}$ increases and reach the estimated domain of convergence of the series for $W_n^{(k)}$. While this increasing can be interpreted as a sign of instability, mathematically speaking this means that the truncation procedure becomes inconsistent.

A basic question is about the fixed points of the flow equation (26); in particular if there is, in addition to the trivial fixed point $g_h = 0$, a non trivial

fixed point corresponding to Luttinger liquid behavior. Note first that the set

$$g_h(k_{-,1}, k_{-,2}, k_{-,3}) = \frac{1}{L_-}\delta(k_{1,-} - k_{2,-})\delta_{\omega_1,\omega_2}\delta_{\omega_3,\omega_4}\lambda_h \tag{30}$$

with λ_h constant in $\mathbf{k}$, is invariant under the RG flow, in the sense that if $\mathcal{L}\mathcal{V}^{(k)}$ has the form,for $k \geq h$

$$\frac{1}{(\beta V)^3}\sum_{\substack{\mathbf{k},\mathbf{k}',\mathbf{p} \\ \omega,\omega'}}\lambda_h\delta_{p_-,0}\widehat{\psi}^+_{\omega,\mathbf{k}}\widehat{\psi}^-_{\omega,\mathbf{k}+\mathbf{p}}\widehat{\psi}^+_{\omega',\mathbf{k}'}\widehat{\psi}^-_{\omega',\mathbf{k}'-\mathbf{p}} \tag{31}$$

the same is true for $\mathcal{L}\mathcal{V}^{(h-1)}$. This can be checked by the graph expansion. In the graphs contributing to $W_4^{(h)}$, the external lines of the graphs contributing to $W_4^{(h)}$ either comes out from a single point, or are connected by a chain of propagators with the same ω, k_-. Moreover in each Feynman graph the only dependence from the momenta of the external lines is through the function $H(k_-)$ which are 1 in the support of the external fields $\int d\mathbf{k}H(k_-)\psi^{\pm}_{\mathbf{k}} = \int d\mathbf{k}\psi^{\pm}_{\mathbf{k}}$. For the same reasons also Z_h is independent from k_-.

The crucial point is that, in the invariant set (30), some dramatic cancellation are present implying the following *asymptotic vanishing of the beta function* (which will be proved in the subsequent sections)

$$\beta_g^{(h)} = O(\gamma^h\varepsilon_h^2) \tag{32}$$

saying that there is a *cancellations* between the graphs with four external lines and the graphs with two lines contributing to the square of Z_h , see (22); such graphs are $O(1)$ but there are cancellations making the size of the sum of them $O(\gamma^h)$. At the second order (32) can be verified from (29) and (30); at third order, it is compatible with (A16),(A15) and (4.8) of [8].

The validity of (32) immediately implies the existence of a line of *non-trivial fixed points* for (26) of the form

$$g_{-\infty}(k_{-,1}, k_{-,2}, k_{-,3}) = \frac{1}{L_-}\delta(k_{1,-} - k_{2,-})\delta_{\omega_1,\omega_2}\delta_{\omega_3,\omega_4}\lambda_{-\infty} \tag{33}$$

with $\lambda_{-\infty}$ $\mathbf{k}$-independent and continuous function of U, $\lambda_{-\infty} = \lambda_0 + O(U^2)$, $\lambda_0 = cU$ for a suitable constant c.

Note also that to such fixed point is associated Luttinger liquid behavior, as, from (19), $Z_h \simeq \gamma^{2\eta h}$, with $\eta = a\lambda_0^2 + O(U^3)$ and $\bar{\mathbf{p}} = (0, \bar{p}_+, \bar{p}_0)$

$$a = \lim_{h\to-\infty}\sum_{\omega'}\frac{\lambda_0^2}{L_-}\frac{1}{h}\int d\mathbf{k}'d\bar{p}_+d\bar{p}_0\frac{H(k'_-)C_h(\mathbf{k}')}{-ik'_0 + \omega'v_Fk'_+} \tag{34}$$

$$\frac{H(k'_-)C_h(\mathbf{k}'+\bar{\mathbf{p}})}{-i(k'_0+\bar{p}_0)+\omega'v_F(k'_++\bar{p}_+)}\frac{\partial}{\partial k_+}\frac{H(k_-)C_h^{-1}(\mathbf{k}-\bar{\mathbf{p}})}{-i(k_0-\bar{p}_0)+\omega v_F(k_+-\bar{p}_+)}\Big|_{k_0=k_+=0}$$

Indeed the 2-point Schwinger function can be written as

$$S_2(\mathbf{x},\mathbf{y}) = \sum_{\omega=\pm} e^{i\omega p_F(x_+-y_+)}\frac{1}{V\beta}\sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{g_\omega^{(h)}(\mathbf{k})}{Z_h}[1+A^{(h)}(\mathbf{k})] \qquad (35)$$

with $A^{(h)}(\mathbf{k}) = O(\varepsilon_h)$, so that

$$S_2(\mathbf{x},\mathbf{y}) = \sum_{\omega=\pm} e^{i\omega p_F(x_+-y_+)} \qquad (36)$$

$$\frac{1}{V\beta}\sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{H(k_-)C_0^{-1}(\mathbf{k})}{-ik_0+\omega v_F k_+}\frac{1+A(\mathbf{k})}{|k_0^2+v_F^2 k_+^2|^\eta}$$

with $|A(\mathbf{k})| \le C|U|$. This means that to the fixed point is associated Luttinger liquid behavior, as the wave function renormalization vanishes at the Fermi surface as a power like with a non-universal critical index;the Luttinger liquid behavior is found only if L_- is finite, as if $L_- \to \infty$ the critical index is vanishing. Note also that the cancellation in (32) reduce to the one in 1D if $k_- = k'_-$ in (31).

5 The auxiliary model

There is essentially no hope of proving a property like (32) directly from the graph expansion, as the algebra of the graphs is too cumbersome (except than at one loop in which it is easy to check). We will follow instead the same strategy for proving the asymptotic vanishing of the Beta function in 1D followed in [15],[16], considering an *auxiliary model* with the same beta function, up to irrelevant terms, but verifying extra symmetries, from which a set of Ward Identities can be derived. In the present case, such identities are related to the invariance under local phase transformations depending on the Fermi surface side, which in the model (1) is broken by the cut-off function $C_0^{-1}(\mathbf{k})$ and by the lattice.

We consider an *auxiliary model* with generating function is

$$\int \mathcal{D}\psi \prod_\omega e^{\int d\mathbf{k}H^{-1}(k_-)C_{h,N}(\mathbf{k})(-ik_0+\omega k_+ v_F)\widehat{\psi}^+_{\omega,\mathbf{k}}\widehat{\psi}^-_{\omega,\mathbf{k}}+\bar{\mathcal{V}}(\psi)+\sum_{\varepsilon=\pm}\int d\mathbf{x}\psi^\varepsilon_{\omega,\mathbf{x}}\phi^{-\varepsilon}_{\omega,\mathbf{x}}+\int d\mathbf{x}J_{\mathbf{x}}\rho_{\omega,\mathbf{x}}}$$

$$(37)$$

with $C_{h,N}^{-1}(\mathbf{k}) = \sum_{k=h}^{N} f_k(k_0^2 + v_F^2 k_+^2)$, $h \leq 0$ $\rho_{\omega,\mathbf{x}} = \psi_{\mathbf{x},\omega}^+ \psi_{\mathbf{x},\omega}$

$$\bar{\mathcal{V}} = \frac{U}{L_-} \sum_{\omega,\omega'} \int d\mathbf{x} dy_- v(x_0 - y_0, x_+ - y_+) \psi_{\omega,x_-,x_+,x_0}^+ \psi_{\omega,x_-,x_+,x_0}^- \psi_{\omega',y_-,x_+,x_0}^+ \psi_{\omega',y_-,x_+,x_0}^-$$

$$= \frac{U}{L_-} \frac{1}{(\beta V)^2} \sum_{\mathbf{k},\mathbf{k}'} \frac{1}{\beta L_+} \sum_{p_0,p_+} \widehat{v}(p_0,p_+) \widehat{\psi}_{\omega,\mathbf{k}}^+ \widehat{\psi}_{\omega,\mathbf{k}+(0,p_+,p_0)}^- \widehat{\psi}_{\omega',\mathbf{k}'}^+ \widehat{\psi}_{\omega',\mathbf{k}'-(0,p_+,p_0)}^- \tag{38}$$

with $v(x_0, x_+)$ a short range interaction. The above functional integral is very similar to the previous one, with the difference that there is an ultraviolet cut–off γ^N on the $+$ variables , which will be removed at the end, and an ultraviolet cut-off $O(1)$ on the $-$ variables; such features are present also in the models introduced in [9] or [12].

Again (37) can be analyzed by a multiscale integration based on a decomposition similar to (6), with the difference that the scale are from h to N.

In the integration of the scales between N and 0, the *ultraviolet scales*, there is no need of renormalization; apparently the terms with two or four external lines have positive or vanishing dimension but one can use the non locality of the interaction to improve their scaling dimension. We integrate (with $\mathcal{L} = 0$) the fields $\psi^{(N)}, \psi^{(N-1)}, .., \psi^{(k)}$ and we call $W_{2n,m}^{(k)}$ the kernels in the effective potential multiplying $2n$ fermionic fields and m J fields. Again the dimension is $\gamma^{-k(n+m-2)}$, $k \geq 0$ and we have to improve the bounds using the non-locality of the interaction. We can write

$$W_{2,0}^{(k)}(\mathbf{x},\mathbf{y}) = \int d\mathbf{y}_1 U \frac{v(x_0 - y_{1,0}, x_+ - y_{1,+})}{L_-} W_{0,1}^{(k)}(\mathbf{y}_1) g^{(k,N)}(\mathbf{x} - \mathbf{y}_2) W_{2,0}^{(k)}(\mathbf{y}_2;\mathbf{y})$$

$$+U \int d\mathbf{y}_2 \frac{v(x_0 - y_{0,1}, x_+ - y_{+,1})}{L_-} g^{(k,N)}(\mathbf{x} - \mathbf{y}_2) W_{2,1}^{(k)}(\mathbf{y}, \mathbf{y}_2; \mathbf{y}_1) + \tag{39}$$

$$U\delta(\mathbf{x} - \mathbf{y}) \int d\mathbf{y}_1 \frac{v(x_0 - y_{1,0}, x_+ - y_{1,+})}{L_-} W_{0,1}^{(k)}(\mathbf{y}_1)$$

The first and the third addend of Fig.2 are vanishing, by the symmetry $g(k_0, k_+, k_-) = -g(-k_0, -k_+, k_-)$; hence, using that $\|g^{(j)}\|_1 \leq \tilde{C}\gamma^{-j}$ and that $W_{2,1}^{(k)}$ is $O(U)$ (by induction), we obtain the following bound

$$\|W_{2,0}^{(k)}\| \leq C \frac{|U|}{L_-} \|W_{2,1}^{(k)}\| \cdot \sum_{j=k}^{N} \|g^{(j)}\|_1 \leq C|U|\gamma^k L_-^{-1}\gamma^{-2k} \tag{40}$$
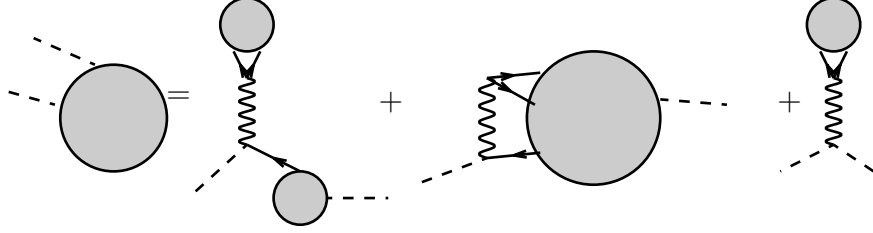
12

Figure 2: Graphical representation of (39); the blobs represent $W_{n,m}^{(k)}$, the wiggly lines represent v, the lines $g^{(k,N)}$

Note that we have a gain $O(L_-^{-1}\gamma^{-2k})$, due to the fact that we are integrating over a fermionic instead than over a bosonic line.

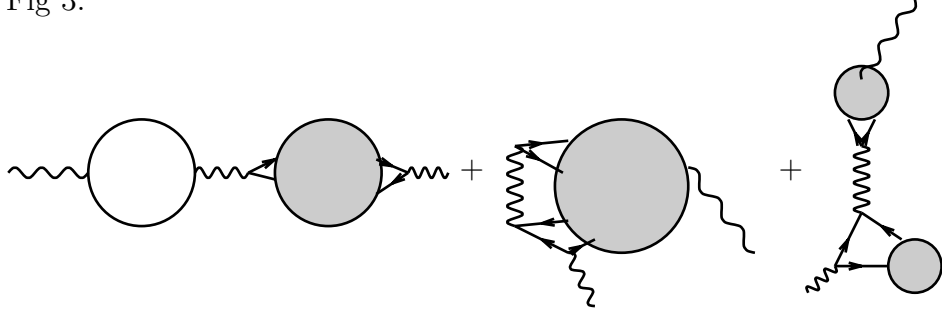Similar arguments can be repeated for $W_{0,2}^{(k)}$, which can be decomposed as in Fig 3.

Figure 3: Decomposition of $W_{2,0}^{(k)}$

The second term in the figure is bounded by $O(|U|L_-^{-1}\gamma^{-2k})$. A similar bound is found for the third term in Fig.3; regarding the first term, we can rewrite it as

$$\int d\mathbf{x} d\bar{\mathbf{z}} [g^{(k,N)}(\mathbf{z} - \mathbf{x})]^2 \frac{U}{L_-} v(x_0 - \bar{z}_0, x_+ - \bar{z}_+) W_{0,2}^{(k)}(\bar{\mathbf{z}}, \mathbf{y}) =$$

$$\int d\mathbf{x} d\bar{\mathbf{z}} U \frac{v(\bar{z}_0 - z_0, \bar{z}_+ - z_+)}{L_-} [g^{(k,N)}(\mathbf{x} - \mathbf{z})]^2 W_{0,2}^{(k)}(\bar{\mathbf{z}}, \mathbf{y}) \qquad (41)$$

$$+ \int d\mathbf{x} d\bar{\mathbf{z}} \frac{U}{L_-} [v(\bar{z}_0 - x_0, \bar{z}_+ - x_+) - v(\bar{z}_0 - z_0, \bar{z}_+ - z_+)][g^{(k,N)}(\mathbf{x} - \mathbf{z})]^2 W_{0,2}^{(k)}(\bar{\mathbf{z}}, y_0)$$

13

and using that

$$\int d\mathbf{x}[g^{(k,N)}(\mathbf{x} - \mathbf{z})]^2 = \int dk_- H(k_-) \int dk_0 dk_+ \frac{C_{k,N}^{-2}(\mathbf{k})}{(-ik_0 + k_+)^2} = 0 \qquad (42)$$

the first addend is vanishing; the second addend, by using the interpolation formula for $v(\bar{z}_0 - x_0, \bar{z}_+ - x_+) - v(\bar{z}_0 - z_0, \bar{z}_+ - z_+)$, can be bounded by $C|U|\gamma^{-k}$, as by induction $||W_{0,2}^{(k)}|| \leq C|U|$. A similar analysis proves the bound for $W_{4,0}^{k)}$.

After the integration of the fields $\psi^{(N)}, \psi^{(N-1)}, ..., \psi^{(-1)}$ we get a Grassmann integral very similar to (1); the integration of the remaining fields $\psi^{(0)}, \psi^{(-1)}, ..$ is done following the same procedure as in section 3, with the effective coupling of the form (30) and $\mathcal{L}\mathcal{V}^{(k)}$ of the form (31). The crucial point is that the beta function *coincides* with the beta function for the model (1) up to $O(\gamma^h)$ terms; hence it is enough to prove the validity of (32) in the auxiliary model.

6 Ward Identities

We derive now a set of Ward Identities relating the Schwinger functions of the auxiliary model (37); by performing the change of variables

$$\psi_{\omega,\mathbf{x}}^{\pm} \to e^{\pm i\alpha_{\omega,\mathbf{x}}}\psi_{\omega,\mathbf{x}}^{\pm} \qquad (43)$$

and making a derivative with respect to $\alpha_{\mathbf{x},\omega}$ and to the external fields we obtain

$$\int d\mathbf{k}'[H^{-1}(k'_- + p_-)C_{h,N}(\mathbf{k}' + \mathbf{p})(-i(k'_0 + p_0) + \omega v_F(k'_+ + p_+)) -$$
$$H^{-1}(k'_-)C_{h,N}(\mathbf{k}')(-ik'_0 + \omega v_F k'_+)]\langle \widehat{\psi}_{\omega,\mathbf{k}'+\mathbf{p}}^+ \widehat{\psi}_{\omega,\mathbf{k}'}^- \widehat{\psi}_{\omega',\mathbf{k}-\mathbf{p}}^+ \widehat{\psi}_{\omega',\mathbf{k}}^- \rangle =$$
$$\delta_{\omega,\omega'}[\langle \widehat{\psi}_{\omega',\mathbf{k}-\mathbf{p}}^+ \widehat{\psi}_{\omega',\mathbf{k}-\mathbf{p}}^- \rangle - \langle \widehat{\psi}_{\omega',\mathbf{k}}^+ \widehat{\psi}_{\omega',\mathbf{k}}^- \rangle] \qquad (44)$$

where $\langle \widehat{\psi}_{\omega,\mathbf{k}'+\mathbf{p}}^+ \widehat{\psi}_{\omega,\mathbf{k}'}^- \widehat{\psi}_{\omega',\mathbf{k}-\mathbf{p}}^+ \widehat{\psi}_{\omega',\mathbf{k}}^- \rangle$ is the derivative with respect to $J_{\mathbf{p}}, \phi_{\omega',\mathbf{k}-\mathbf{p}}^+, \phi_{\omega',\mathbf{k}}^-$ of (37). Computing (44) for $p_- = 0$ we get,if $\bar{\mathbf{p}} = (0, \bar{p}_+, \bar{p}_0)$

$$(-i\bar{p}_0 + \omega v_F \bar{p}_+)\langle \rho_{\bar{\mathbf{p}},\omega} \widehat{\psi}_{\mathbf{k},\omega'}^+ \widehat{\psi}_{\mathbf{k}-\bar{\mathbf{p}},\omega'}^- \rangle + \Delta(\mathbf{k}, \bar{\mathbf{p}}) =$$
$$\delta_{\omega,\omega'}[\langle \widehat{\psi}_{\omega',\mathbf{k}-\bar{\mathbf{p}}}^+ \widehat{\psi}_{\omega',\mathbf{k}-\bar{\mathbf{p}}}^- \rangle - \langle \widehat{\psi}_{\omega',\mathbf{k}}^+ \widehat{\psi}_{\omega',\mathbf{k}}^- \rangle] \qquad (45)$$

14

and

$$\Delta(\mathbf{k}, \bar{\mathbf{p}}) = \int d\mathbf{k}' C(\mathbf{k}', \bar{\mathbf{p}}) \langle \widehat{\psi}^+_{\omega, \mathbf{k}'+\bar{\mathbf{p}}} \widehat{\psi}^-_{\omega, \mathbf{k}'} \widehat{\psi}^+_{\omega', \mathbf{k}-\bar{\mathbf{p}}} \widehat{\psi}^-_{\omega', \mathbf{k}} \rangle \tag{46}$$

with

$$C(\mathbf{k}, \bar{\mathbf{p}}) = (-ik_0 + \omega v_F k_+) \tag{47}$$
$$[C_{h,N}(\mathbf{k} + \bar{\mathbf{p}}) - C_{h,N}(\mathbf{k})] + (-i\bar{p}_0 + \omega v_F \bar{p}_+)[C_{h,N}(\mathbf{k} + \bar{\mathbf{p}}) - 1]$$

In deriving the above equation we have used that

$$\int d\mathbf{k}' H^{-1}(k'_-) \langle \widehat{\psi}^+_{\omega, \mathbf{k}'+\bar{\mathbf{p}}} \widehat{\psi}^-_{\omega, \mathbf{k}'} \widehat{\psi}^+_{\omega', \mathbf{k}-\bar{\mathbf{p}}} \widehat{\psi}^-_{\omega, \mathbf{k}} \rangle =$$
$$\int d\mathbf{k}' \langle \widehat{\psi}^+_{\omega, \mathbf{k}'+\bar{\mathbf{p}}} \widehat{\psi}^-_{\omega, \mathbf{k}'} \widehat{\psi}^+_{\omega', \mathbf{k}-\bar{\mathbf{p}}} \widehat{\psi}^-_{\omega', \mathbf{k}} \rangle \tag{48}$$

for the compact support properties of the fields $\psi_\mathbf{k}$ and $H^2 = H$; note also the crucial role of the condition $p_- = 0$ in the above derivation.

The presence of the term $\Delta(\mathbf{k}, \bar{\mathbf{p}})$ in the Ward Identity (45) is related to the presence of the ultraviolet cut-off; as in 1D, such a term is not vanishing even in the limit $N \to \infty$ and it is responsible of the anomalies, see [15],[16]. The following correction identity holds, similar to the one in the 1D case

$$\Delta(\mathbf{k}, \bar{\mathbf{p}}) = \nu(-i\bar{p}_0 - \omega v_F \bar{p}_+) \sum_{\omega''=\pm} \langle \rho_{\bar{\mathbf{p}}, \omega''} \widehat{\psi}^+_{\omega', \mathbf{k}-\bar{\mathbf{p}}} \widehat{\psi}^-_{\omega', \mathbf{k}} \rangle + R^{2;1}_\omega(\mathbf{k}, \bar{\mathbf{p}}) \tag{49}$$

with $R^{2;1}_\omega$ a small correction. Indeed $R^{2;1}_\omega$ can be written as functional derivative, with respect to ϕ^+, ϕ^-, J, of

$$e^{\mathcal{W}_\Delta(J,\phi)} = \int P(d\psi) e^{-V(\psi) + \sum_\omega \int d\mathbf{z} [\psi^+_{\omega, \mathbf{z}} \phi^-_{\omega, \mathbf{z}} + \phi^+_{\omega, \mathbf{z}} \psi^-_{\omega, \mathbf{z}}] + T_0(J,\psi) - T_-(J,\psi)} \tag{50}$$

with

$$T_0(\psi) = \int \frac{d\bar{p}_+}{(2\pi)} \frac{d\bar{p}_0}{(2\pi)} \frac{d\mathbf{k}}{(2\pi)^3} C(\mathbf{k}, \bar{\mathbf{p}}) J_{\bar{\mathbf{p}}} \widehat{\psi}^+_{\mathbf{k}+\bar{\mathbf{p}}, \omega} \widehat{\psi}^-_{\mathbf{k}, \omega} \tag{51}$$

$$T_-(\widehat{\psi}) = \sum_{\omega'} \int \frac{d\bar{p}_+}{(2\pi)} \frac{d\bar{p}_0}{(2\pi)} \frac{d\mathbf{k}}{(2\pi)^3} \nu J_{\bar{\mathbf{p}}} (-i\bar{p}_0 - \omega v_F \bar{p}_+) \widehat{\psi}^+_{\mathbf{k}+\bar{\mathbf{p}}, \omega'} \widehat{\psi}^-_{\mathbf{k}, \omega'} \tag{52}$$

(50) can be evaluated by a multiscale integration similar to the previous one, the only difference being that $\int J\rho$ is replaced by $T_0 - T_-$. The terms with vanishing scaling dimension of the form $J\psi^+\psi^-$ can be obtained from the contraction of T_0 and T_-; in the first case we can perform a decomposition similar to the one in Fig.3, see Fig. 4. Regarding the second and third term,
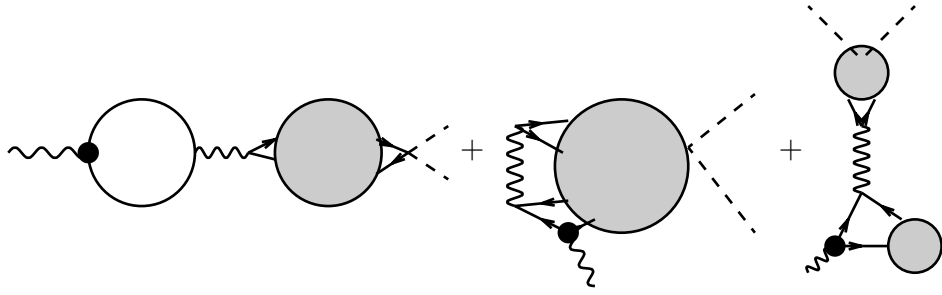
15

Figure 4: Terms obtained from the contraction of T_0; the black dot represents $C(\mathbf{k}, \bar{\mathbf{p}})$

we can proceed exactly as in the previous section, the main difference being that at least one of the two fields have scale N so that they obey to the bound $O(\gamma^{-2k}\gamma^{-(1/2)(N-k)})$. This follows from the fact that when C is multiplied by two propagators we get

$$C(\mathbf{k}, \bar{\mathbf{p}})g^{(i)}(\mathbf{k})g^{(j)}(\mathbf{k}+\bar{\mathbf{p}}) = \frac{f_i(\mathbf{k})}{-ik_0 + \omega v_F k_+}[\frac{f_j(\mathbf{k}+\bar{\mathbf{p}})}{C_{h,N}^{-1}(\mathbf{k}+\bar{\mathbf{p}})} - f_j(\mathbf{k}+\bar{\mathbf{p}})]$$

$$-\frac{f_j(\mathbf{k}+\bar{\mathbf{p}})}{-i(k_0+\bar{p}_0) + \omega v_F(k_+ + \bar{p}_+)}[\frac{f_i(\mathbf{k})}{C_{h,N}^{-1}(\mathbf{k})} - f_i(\mathbf{k})] \qquad (53)$$

which is non vanishing only if one among i or j are equal to h or N.

The main difference with the analysis in the previous section is that in the first term of Fig 4; the "bubble" in Fig. 3 was vanishing, while here it is not. We choose ν in (50) equal to the value of this bubble, in order to cancel it. The value of the bubble is given by

$$\nu = Uv(\bar{p}_0, \bar{p}_+) \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{C(\mathbf{k}, \bar{\mathbf{p}})}{-i\bar{p}_0 - \omega v_F \bar{p}_+} g_\omega^{(\le N)}(\mathbf{k})g_\omega^{(\le N)}(\mathbf{k}+\bar{\mathbf{p}}) \qquad (54)$$

and,in the limit $N \to \infty$, $\nu = Uv(\bar{p}_0, \bar{p}_+)\frac{a}{4\pi^2}$. Hence for $k_0, k_+ = O(\gamma^h)$

$$|R_\omega^{2,1}(\mathbf{k}, \bar{\mathbf{p}})| \le C\varepsilon_h^2 \gamma^{-2h} \qquad (55)$$

7   Schwinger-Dyson equation

An immediate consequence of the analysis in the previous section is that, for momenta computed at the infrared scale $|\mathbf{k}| = |\mathbf{k}'| = |\mathbf{k}+\widehat{\mathbf{p}}| = |\mathbf{k}'-\widehat{\mathbf{p}}| =$

16

$$|\widehat{\mathbf{p}}| = \gamma^h$$

$$\langle \widehat{\psi}^-_{\omega,\mathbf{k}} \widehat{\psi}^+_{\omega,\mathbf{k}+\widehat{\mathbf{p}}} \widehat{\psi}^-_{\omega',\mathbf{k}'} \widehat{\psi}^+_{\omega',\mathbf{k}'-\widehat{\mathbf{p}}} \rangle =$$

$$\frac{1}{(Z_h)^2} \frac{\lambda_h}{L_-} g^{(h)}_{\omega,\mathbf{k}} g^{(h)}_{\omega',\mathbf{k}'} g^{(h)}_{\omega,\mathbf{k}+\widehat{\mathbf{p}}} g^{(h)}_{\omega',\mathbf{k}'-\widehat{\mathbf{p}}} (1 + O(\varepsilon_h))$$

$$\langle \widehat{\psi}^-_{\omega,\mathbf{k}} \widehat{\psi}^+_{\omega,\mathbf{k}} \rangle = \frac{g^{(h)}_{\omega,\mathbf{k}}}{Z_h} (1 + O(\varepsilon_h)) \tag{56}$$

This says that relations between the effective couplings at a certain scale $h \leq 0$ can be obtained from relations between the Schwinger functions of the auxiliary model (37) computed at the infrared cut-off scale. The starting point for deriving such relations is the Schwinger-Dyson equation for the 4-point function, given by, if the external momenta are computed at the infrared scale and $\widehat{\mathbf{p}} = (0, \widehat{p}_+, \widehat{p}_0)$ and $\bar{\mathbf{p}} = (0, \bar{p}_+, \bar{p}_0)$

$$\langle \widehat{\psi}^-_{\omega,\mathbf{k}} \widehat{\psi}^+_{\omega,\mathbf{k}-\widehat{\mathbf{p}}} \widehat{\psi}^-_{\omega',\mathbf{k}'} \widehat{\psi}^+_{\omega',\mathbf{k}'+\widehat{\mathbf{p}}} \rangle =$$

$$\sum_{\omega''} \{ \frac{U}{L_-} v(\widehat{p}_0, \widehat{p}_+) g_{\omega',\mathbf{k}'+\widehat{\mathbf{p}}} \langle \widehat{\psi}^-_{\omega',\mathbf{k}'} \widehat{\psi}^+_{\omega',\mathbf{k}'} \rangle \langle \rho_{\widehat{\mathbf{p}},\omega''} \widehat{\psi}^-_{\omega,\mathbf{k}} \widehat{\psi}^+_{\omega,\mathbf{k}-\widehat{\mathbf{p}}} \rangle + \tag{57}$$

$$\frac{U}{L_-} g_{\omega',\mathbf{k}'+\widehat{\mathbf{p}}} \int \frac{d\bar{p}_0}{(2\pi)} \frac{d\bar{p}_+}{(2\pi)} v(\bar{p}_0, \bar{p}_+) \langle \rho_{\bar{\mathbf{p}},\omega''} \widehat{\psi}^-_{\omega,\mathbf{k}} \widehat{\psi}^+_{\omega,\mathbf{k}-\widehat{\mathbf{p}}} \widehat{\psi}^-_{\omega',\mathbf{k}'} \widehat{\psi}^+_{\omega',\mathbf{k}'+\widehat{\mathbf{p}}-\bar{\mathbf{p}}} \rangle \}$$

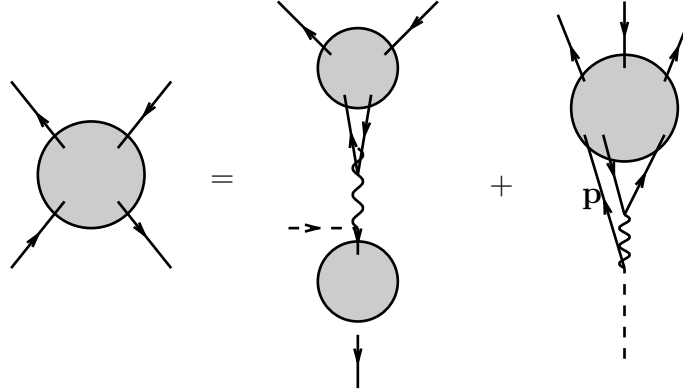

Figure 5: Graphical representation of (57); the dotted line represent the free propagator

By the WI (45),(49)

$$(-i\widehat{p}_0 + \omega' v_F \widehat{p}_+) \bar{\langle} \rho_{\widehat{\mathbf{p}},\omega'} \widehat{\psi}^-_{\mathbf{k},\omega} \widehat{\psi}^+_{\mathbf{k}-\widehat{\mathbf{p}},\omega} \rangle =$$

$$A_{\omega,\omega'}(\mathbf{p}) [\langle \widehat{\psi}^-_{\omega,\mathbf{k}-\widehat{\mathbf{p}}} \widehat{\psi}^+_{\omega,\mathbf{k}-\widehat{\mathbf{p}}} \rangle - \langle \widehat{\psi}^-_{\omega,\mathbf{k}} \widehat{\psi}^+_{\omega,\mathbf{k}} \rangle] + H^{(2,1)}_{\omega,\omega'}(\mathbf{k}, \mathbf{p}) \tag{58}$$

17

with $A_{\omega,\omega}(\mathbf{p}) = 1 + O(\varepsilon_h)$, $A_{\omega,-\omega}(\mathbf{p}) = O(\varepsilon_h)$ and even in $\mathbf{p}$; moreover $H_{\omega,\omega'}(\mathbf{k},\mathbf{p})$ is a linear combination of the $R^{2,1}$ functions in (49) with bounded coefficients. The WI for the 4 point function is given by

$$(-i\bar{p}_0 + \omega v_F \bar{p}_+)\langle \rho_{\bar{\mathbf{p}},\omega}\widehat{\psi}^-_{\omega,\mathbf{k}}\widehat{\psi}^+_{\omega,\mathbf{k}-\widehat{\mathbf{p}}}\widehat{\psi}^-_{-\omega,\mathbf{k}'}\widehat{\psi}^+_{-\omega,\mathbf{k}'+\widehat{\mathbf{p}}-\bar{\mathbf{p}}}\rangle = \tag{59}$$
$$\langle \widehat{\psi}^-_{\mathbf{k}-\bar{\mathbf{p}},\omega}\widehat{\psi}^+_{\mathbf{k}-\widehat{\mathbf{p}},-\omega}\widehat{\psi}^-_{\mathbf{k}',-\omega}\widehat{\psi}^+_{\mathbf{k}'+\widehat{\mathbf{p}}-\bar{\mathbf{p}},-\omega}\rangle - \langle \widehat{\psi}^-_{\mathbf{k},\omega}\widehat{\psi}^+_{\mathbf{k}-\widehat{\mathbf{p}}+\bar{\mathbf{p}},\omega}\widehat{\psi}^-_{\mathbf{k}',-\omega}\widehat{\psi}^+_{\mathbf{k}'+\widehat{\mathbf{p}}-\bar{\mathbf{p}},-\omega}\rangle$$
$$+\nu\sum_{\omega'}(-i\bar{p}_0 - \omega v_F \bar{p}_+)\langle \rho_{\bar{\mathbf{p}},\omega}\widehat{\psi}^-_{\mathbf{k},\omega}\widehat{\psi}^+_{\mathbf{k}-\widehat{\mathbf{p}},\omega}\widehat{\psi}^-_{\mathbf{k}',-\omega}\widehat{\psi}^+_{\mathbf{k}'+\widehat{\mathbf{p}}-\bar{\mathbf{p}},-\omega}\rangle + R^{4,1}_{\omega}(\mathbf{k},\mathbf{k}',p_0)$$

and similar ones, so that the second addend of the l.h.s. is given by

$$\int d\bar{p}_0 d\bar{p}_+ \chi_\varepsilon(\bar{\mathbf{p}})\frac{\widehat{v}(\bar{p}_0,\bar{p}_+)}{-i\bar{p}_0 + \omega'' v_F \bar{p}_+}[A_1(\bar{\mathbf{p}})\langle \widehat{\psi}^-_{\mathbf{k}-\bar{\mathbf{p}},\omega}\widehat{\psi}^+_{\mathbf{k}-\widehat{\mathbf{p}},\omega}\widehat{\psi}^-_{\mathbf{k}',\omega'}\widehat{\psi}^+_{\mathbf{k}'+\widehat{\mathbf{p}}-\bar{\mathbf{p}},\omega'}\rangle$$
$$+A_2(\bar{\mathbf{p}})\langle \widehat{\psi}^-_{\mathbf{k},\omega}\widehat{\psi}^+_{\mathbf{k}-\widehat{\mathbf{p}}+\bar{\mathbf{p}},\omega}\widehat{\psi}^-_{\mathbf{k}',\omega'}\widehat{\psi}^+_{\mathbf{k}'+\widehat{\mathbf{p}}-\bar{\mathbf{p}},\omega'}\rangle + A_3(\bar{\mathbf{p}})\langle \widehat{\psi}^-_{\mathbf{k},\omega}\widehat{\psi}^+_{\mathbf{k}-\widehat{\mathbf{p}},\omega}\widehat{\psi}^-_{\mathbf{k}'-\bar{\mathbf{p}},\omega'}\widehat{\psi}^+_{\mathbf{k}'+\widehat{\mathbf{p}}-\bar{\mathbf{p}},\omega'}\rangle$$
$$+A_4(\bar{\mathbf{p}})\langle \widehat{\psi}^-_{\mathbf{k},\omega}\widehat{\psi}^+_{\mathbf{k}-\widehat{\mathbf{p}},\omega}\widehat{\psi}^-_{\mathbf{k}',\omega'}\widehat{\psi}^+_{\mathbf{k}'+\widehat{\mathbf{p}},\omega'}\rangle + H^{4,1}(\mathbf{k},\mathbf{k}',\bar{\mathbf{p}})] \tag{60}$$

where $\chi_\varepsilon(\bar{\mathbf{p}})$ is a compact support function vanishing for $\bar{\mathbf{p}} = 0$, and such that it becomes the identity in the limit $\varepsilon \to 0$; moreover the functions $A_i(\mathbf{p})$ are bounded and even in $\mathbf{p}$, and $H^{4,1}$ is a linear combination of the $R^{4,1}$ functions in (59) with bounded coefficients.

We have now to bound all the sums in the r.h.s. of (60). Note first that, by parity

$$\int d\bar{p}_0 d\bar{p}_+ \chi_\varepsilon(\bar{\mathbf{p}})A_4(\bar{\mathbf{p}})\frac{\widehat{v}(\bar{p}_0,\bar{p}_+)}{-i\bar{p}_0 + \omega'' v_F \bar{p}_+}\langle \widehat{\psi}^-_{\mathbf{k},\omega}\widehat{\psi}^+_{\mathbf{k}-\widehat{\mathbf{p}},\omega}\widehat{\psi}^-_{\mathbf{k}',\omega'}\widehat{\psi}^+_{\mathbf{k}'+\widehat{\mathbf{p}},\omega'}\rangle = 0 \tag{61}$$

Moreover the first term in the r.h.s. of (60) verifies

$$|\int d\bar{p}_0 d\bar{p}_+ \frac{\widehat{v}(\bar{p}_0,\bar{p}_+)}{-i\bar{p}_0 + v_F \omega'' \bar{p}_+}A_1(\bar{\mathbf{p}})\chi_\varepsilon(\bar{\mathbf{p}})\langle \widehat{\psi}^-_{\mathbf{k}-\bar{\mathbf{p}},\omega}\widehat{\psi}^+_{\mathbf{k}-\widehat{\mathbf{p}},\omega}\widehat{\psi}^-_{\mathbf{k}',\omega'}\widehat{\psi}^+_{\mathbf{k}'+\widehat{\mathbf{p}}-\bar{\mathbf{p}},\omega'}\rangle| \leq C\varepsilon_h \frac{\gamma^{-3h}}{Z_h^2} \tag{62}$$

and a similar bound is true for the second and third term. Finally, as in [16]

$$|\int d\bar{p}_0 d\bar{p}_+ \frac{\widehat{v}(\bar{p}_0,\bar{p}_+)}{-i\bar{p}_0 + v_F \omega'' \bar{p}_+}H^{4,1}(\mathbf{k},\mathbf{k}',\bar{\mathbf{p}})| \leq C\varepsilon_h \frac{\gamma^{-3h}}{Z_h^2} \tag{63}$$

By inserting (56),(58),(60),(61),(62) in (57) we get $\lambda_h = \lambda_0 + O(U^2)$, which means that the effective interaction remain close to initial value for any RG iteration; a contradiction argument shows that this can be true only if the beta function is asymptotically vanishing: as this beta function is the same of the 2D model (1) with effective couplings (30), up to $O(\gamma^h)$ terms, then (32) follows.

18

8 Conclusions

We have shown that the RG flow for a system of spinless fermions with flat Fermi surface has, in addition to the trivial fixed point, a line of Luttinger liquid fixed points, corresponding to vanishing wave function renormalization and anomalous exponents in the 2-point function; such fixed point is in the invariant set (30). This makes quantitative the analysis in [10], in which the existence of a Luttinger fixed point in 2D was postulated on the basis of bosonization. With respect to previous perturbative RG analysis,the key novelty is the implementation of WI at each RG iteration, in analogy to what is done in 1D.

Of course the other effective interactions should cause flows away from this fixed point; a similar phenomenon happens in the 1D (spinning) Hubbard model, in which there is a Luttinger liquid fixed point in the invariant set obtained setting all but the backscattering and umklapp scattering terms equal to zero (that is the set $g_{1,h} = g_{3,h} = 0$ in the g-ology notation, see [19]), but *attractive* backscattering interaction produces a flow to a strong coupling regime. We can in any case expect, as in 1D, that even if the Luttinger fixed point in 2D is not stable its presence has an important role in the physical properties of the system.

Fermi surfaces with flat or almost flat pieces and no van Hove singularities are found in the Hubbard model with next to nearest neighbor interactions or in the Hubbard model close to half filling, and it is likely that our results can be extended, at least partially, to such models. Note however that in such models the sides of the Fermi surface are not perfectly flat, so that one expects a renormalization of the shape of the Fermi surface, as in [3], which is absent in the case of flat sides by symmetry. Another simplifying property of the model considered here is that the modulation of the Fermi velocity is taken constant along the Fermi surface, contrary to what happens in more realistic models; a momentum dependent Fermi velocity produces extra terms in the WI, as it is evident from (43) (v_F should be replaced by $v_F(k'_-)$ in the first line and $v_F(k'_- + p_-)$ in the second line), and their effect deserves further analysis.

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