Although Spin-Charge (SC) separation has been well established in one dimensional fermionic systems, the theoretical situation for two dimensions, the relevant case for describing high $T_c$ cuprates, is rather controversial. In this paper we consider a system of two-dimensional electrons with a flat Fermi Surface, and we prove that a weak interaction between electrons on the same sector produces SC separation. Our treatment is rigorous and based on novel methods introduced in the framework of Constructive Renormalization Group: SC separation emerges from the anomalies in the Ward Identities obtained from local phase transformations depending on the Fermi surface side.

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I. INTRODUCTION AND MAIN RESULTS

The proposal that Spin-Charge (SC) separation is the cause of the peculiar properties of the high $T_c$ cuprates has been debated since several years [1], and it is apparently far from being settled. Some experimental evidence of SC separation seems to be present in all the four relevant phases of high $T_c$ cuprates [2]. However such materials are described in terms of two-dimensional (2D) interacting fermionic systems and, while in $d = 1$ SC separation is theoretically well established, its occurrence in $d = 2$ is rather controversial. SC separation in $d = 2$ indeed emerges in certain analysis based on bosonization [3], but it has been pointed out, see e.g. [4] and [5], that 2D bosonization is based on questionable and uncontrolled approximations. Moreover such results are apparently contradicted by a number of papers based on parquet methods results [4] and perturbative one and two loops Renormalization Group (RG) analysis [6],[7]; such analysis, which are quite effective in indicating the onset of $d$-wave superconductivity, do not show any indication of SC separation. However such results cannot really exclude SC separation, as it is a non-perturbative phenomenon which escapes to a purely perturbative investigation, see e.g. [8] or [9].

Spanning interacting fermions in 1D are described by the $g$-ology model [10], in which the dispersion relation is linearized around each of the two Fermi points (labelled by $\omega = \pm$) and the interaction is decomposed in three different channels (in the not half filled case), with $g_1, g_2$ and $g_4$ couplings. The model has a rather complex behavior, but SC separation is produced even by the $g_4$ channel alone. In such a case the Hamiltonian can be written as the sum of two independent hamiltonians, corresponding to charge and spin excitation, and the full Hilbert space can be represented as a product of spin and charge excitations. The two point Schwinger the has the form (in the case of local, spin symmetric interactions)

$$S(x) = \sum_{\omega = \pm} \frac{1}{2\pi} \frac{1}{(i\omega x + v_\sigma x_0)^{1/2}} \frac{1}{(i\omega x + v_\sigma x_0)^{1/2}} e^{i\omega(x-y)}$$

(1)

with $v_\sigma = 1$, $v_\rho = 1 + g_4/2\pi$; the Schwinger function is splitted in the product of two Schwinger functions with different Fermi velocities, indicating that the fermion is fractionalized and that a description in terms of individual particles, as in Fermi liquids, is impossible. Note that any perturbative analysis would fail to detect such a factorization; if we write eq. (1) as a power series its second order has the form $g_4 A(x)/(i\omega x + x_0)$ with $A(x)$ bounded, which is perfectly compatible also with a Schwinger function not exhibiting SC separation. When we switch on also the other interaction channels other interesting phenomena appear ($g_2$ produces anomalous exponents while $g_1$ negative produces a spin gap), but the SC separation phenomenon is left unaffected.

It is widely believed that a basic model for Cuprates is the 2D Hubbard model. It has been proved in [11] that Fermi liquid behavior (which excludes SC separation) is present at $T \neq 0$ far enough from half filling. The most interesting region is however close to half-filling (corresponding to cuprates in the low-doping regime), where the Fermi surface is almost a square. If the small rounded portions at the corners of the Fermi surface are neglected and the Fermi velocity is assumed constant one gets the 2D flat Fermi surface model, extensively analyzed in literature, see e.g. [3], [4],[7],[12]. This model has several simplifying features, but it has still a rather complex behavior which is quite difficult to analyze. It is not exactly solvable and there is no hope to get an exact factorization in the hamiltonian; SC separation can be only detected by observing a factorization in the Schwinger functions like in (1), at least asymptotically.

The Schwinger functions of the 2D flat Fermi surface model are given by functional derivatives of the generating functional

$$e^{i\mathcal{W}(\phi)} = \int P(d\psi) e^{\mathcal{V}(\psi)+\sum_\sigma \int dx [\psi^+_{\sigma,x} \phi^-_{\sigma,x} + \psi^-_{\sigma,x} \phi^+_{\sigma,x}]}$$

(2)

with $x = (x_0, x_-, x_+)$, $\psi^\pm_{\sigma,x}$ are Grassman variables, $\sigma = \pm$ is the spin,

$$\mathcal{V} = U \sum_{\sigma, \sigma'} \int dx \psi^+_{\sigma,x} \psi^-_{\sigma',x} + \psi^+_{\sigma',x} \psi^-_{\sigma,x}$$

(3)
\[
\hat{g}(k) = \sum_{\alpha=\pm} \sum_{\omega=\pm} \frac{H(k_{\alpha})C_0^{-1}(k_\alpha)}{-ik_0 + \omega v_F (k_\alpha - \omega p_F)} 
\]

where \( k_\alpha = (k_0, k_\alpha) \), \( C_0^{-1}(k_\alpha) \) is a compact support function selecting momenta \( k_0^2 + \vec{p}_F^2(k_\alpha - \vec{p}_F)^2 \leq p_F^2/10 \) and \( H(k_{-\alpha}) = 1 \) if \( |k_{-\alpha}| < p_F/4 \) and 0 otherwise. The 2-point Schwinger function is

\[
S(x-y) = \frac{\partial^2 W(\phi)}{\partial \phi_{\sigma,x} \partial \phi_{\sigma,y}} |_{\phi=0} 
\]

The free Fermi surface is defined as the set of momenta such that \( \hat{g}(k) \) is singular as \( k_0 = 0 \), see Fig.1. Due to special symmetries of the model, the free and interacting Fermi surface are equal; this would not be true in the 2D Hubbard model, except at half-filling.

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**FIG. 1:** The Fermi surface, defined as the singularity of (4) when \( k_0 = 0 \); the four sides are labelled by \( (\alpha, \omega) = (\pm, \pm) \)

By well-known properties of Grassmann integrals we can write the Grassmann field as a sum of independent fields

\[
\psi_{\sigma,x}^{\pm} = \sum_{\alpha=\pm} \sum_{\omega=\pm} e^{\pm i \vec{p}_F \cdot \vec{x}} \psi_{\omega,\alpha,\sigma,x}^{\pm} \tag{6}
\]

with \( \vec{p}_F = (0, p_F) \) and \( \vec{p}_F = (p_F, 0) \) and \( \psi_{\omega,\sigma,x}^{\pm} \) independent Grassmann variables with propagator

\[
g_{\omega,\alpha}(x-y) = \frac{1}{\beta V} \sum_k e^{-ik(x-y)} \frac{H(k_{\alpha})C_0^{-1}(k_\alpha)}{-ik_0 + \omega v_F k_\alpha} \tag{7}
\]

By inserting (6) in the interaction (3) we get a sum over a certain number of terms. Among such interaction channels we select the interactions between electrons on the same sector of the Fermi surface with no transfer momentum orthogonal to the flat sides same side. The fields with different \( \alpha \) are decoupled so that we only consider \( \alpha = + \) for definiteness. The interaction channel we select is

\[
\mathcal{V} = U \sum_{\omega} \sum_{\alpha=\pm} \delta_{\omega,0} \psi_{\omega,\sigma,k}^+ \psi_{\omega,\sigma,k}^- + \sum_{k,k',p} \delta_{p,0} \psi_{\omega,\sigma,k}^+ \psi_{\omega,\sigma,k'}^+ \psi_{\omega,\sigma',k'}^- \psi_{\omega,\sigma',k'+p}^- \tag{8}
\]

where \( V = L_x L_y \). Our main result is that the 2-point Schwinger function, if the interaction is (8) and \( U \) is small enough, is given by

\[
S(x) = G(x_{-}) \sum_{\omega} e^{i\omega p_F x_{-}} \frac{1}{2\pi} \frac{1}{v_\sigma v_p} \frac{1}{v_F} \int \frac{dx_{+}}{\sqrt{1 - \frac{v_F^2}{v_\sigma^2}}} \tag{9}
\]

with \( A \) given by (34) below. The 2-point Schwinger function is asymptotically factorized in the product of two Schwinger functions, with different Fermi velocities, which is a clear signal of electron fractionalization and SC separation.

**II. RENORMALIZATION GROUP ANALYSIS**

The analysis is performed computing the Grassmann integral (2) iteratively integrating out the fields with momenta closer and closer to the Fermi surface. The propagator can be written as sum of "single slice" propagators in the following way

\[
g_{\omega}(x-y) = \sum_{h=-\infty}^{0} e^{i\omega p_F (x_{+} - y_{+})} g_{\omega}^{(h)}(x-y) \tag{11}
\]

where \( g_{\omega}^{(h)}(x-y) \) is identical to \( g_{\omega}(x-y) \) but \( C_0^{-1}(k) \), with \( \hat{k} = (k_0, k_+) \), is replaced by \( f_h(k) \), with support in a region \( O(\gamma^h) \), \( \gamma \gg 1 \), around each flat side of the Fermi surface, at a distance \( O(\gamma^h) \) from it. After the integration of the fields \( \psi^{(0)}, ..., \psi^{(h+1)} \) we obtain (in the case \( \phi = 0 \) for definiteness)

\[
\int P(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\psi^{(\leq h)})} \tag{12}
\]

where \( P(d\psi^{(\leq h)}) \) is the fermionic integration with propagator \( g^{(\leq h)}(k) \) and \( \mathcal{V}^{(h)} \) is sum of monomials in \( \psi \) with kernels \( W^{(h)}_{2n}(k_1, ..., k_{2n-1}) \).

By using that \( \int dk |g^{(k)}(k)| \leq C_\gamma^k \) and \( |g^{(k)}(k)| \leq C_\gamma^{-k} \) we see that the kernels \( W^{(h)}_{2n} \) are \( O(\gamma^{-h(n-2)}) \); this means that the terms quadratic in the fields have positive scaling dimension and the quartic terms have vanishing scaling dimension, and all the other terms have negative dimension; we have then to properly renormalize the terms with non-negative dimension. Calling \( \hat{k} = (k_- , 0, 0) \) we define an \( \mathcal{L} \) operator acting linearly on the kernels of the effective potential: \( \mathcal{L} \hat{W}_{2n}^{(h)} = 0 \) if \( n \geq 2 \), if \( n = 2 \)

\[
\mathcal{L} W^{(h)}_{4}(k_1, k_2, k_3) = W^{(h)}_{4}(\hat{k}_1, \hat{k}_2, \hat{k}_3) \tag{13}
\]
while if \( n = 1 \)
\[
\mathcal{L} W_{2}^{(h)}(\mathbf{k}) = W_{2}^{(h)}(\mathbf{k}) + k_0 \partial_{k_0} W_{2}^{(h)}(\mathbf{k}) + \partial_{k_+} W_{2}^{(h)}(\mathbf{k})
\]  
(14)

We get
\[
\mathcal{L} \psi = \frac{1}{\beta V} \sum_{k} z_h(k_-(i k_0 + \omega k_+)) \tilde{\psi}^{+,(h)}(k_0,\omega) \psi^{-(h)}(k_0,\omega) + \sum_{\sigma=\sigma'} \frac{1}{(\beta V)^4} \sum_{k_1,\ldots,k_4} g_h(k_{-1},k_{-2},k_{-3}) \tilde{\psi}_1^{+,(h)}(k_{1,\sigma}) \psi_2^{-(h)}(k_{1,\sigma}) \times \tilde{\psi}_3^{+(h)}(k_{2,\sigma'}) \psi_4^{-(h)}(k_{2,\sigma'}) \delta(\sum_i \varepsilon_i k_i)
\]  
(15)

where we have used that, by symmetry, \( W_{2}^{\dagger}(\mathbf{k}) = 0 \).

We write (12) as
\[
\int P(d\psi^{(h)}) e^{-\mathcal{L} \psi^{(h)}(\sqrt{z_h} \psi^{(h)}) - \mathcal{R} \psi^{(h)}(\sqrt{z_h} \psi^{(h)})} \mathcal{R}^{(h)}(\psi^{(h)})
\]  
(16)

with \( \mathcal{R} = 1 - \mathcal{L} \). The non trivial action of \( \mathcal{R} \) on the kernel with \( n = 2 \) can be written as

\[
\mathcal{R} W_{4}^{(h)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = W_{4}^{(h)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) - W_{4}^{(h)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + [W_{4}^{(h)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) - W_{4}^{(h)}(\mathbf{\bar{k}}_1,\mathbf{\bar{k}}_2,\mathbf{\bar{k}}_3)] + [W_{4}^{(h)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) - W_{4}^{(h)}(\mathbf{\bar{k}}_1,\mathbf{\bar{k}}_2,\mathbf{\bar{k}}_3)]
\]  

(17)

The first term can be written as
\[
k_{0,1} \int_{0}^{1} dt \partial_{k_0,1} \tilde{W}_{4}^{h}(k_{-1,1},k_{+1,1},t k_{0,1};\mathbf{k}_2,\mathbf{k}_3) + k_{+,1} \int_{0}^{1} dt \partial_{k_+,1} \tilde{W}_{4}^{h}(k_{-1,1},k_{+1,1},0;\mathbf{k}_2,\mathbf{k}_3)
\]  

(18)

The factors \( k_{0,1} \) and \( k_{+,1} \) are \( O(\gamma^h) \), for the compact support properties of the propagator associated to \( \tilde{\psi}^{+(h)}(k_0,\omega) \), with \( h' \leq h \), while the derivatives are dimensionally \( O(\gamma^{h-1}) \); hence the effect of \( \mathcal{R} \) is to produce a factor \( \gamma^{h-h-1} \) making its scaling dimension negative. Similar considerations can be done for the action of \( \mathcal{R} \) on the \( n = 1 \) terms. The effect of the \( \mathcal{L} \) operation is to replace in \( W_{2}^{(h)}(\mathbf{k}) \) the momentum \( \mathbf{k} \) with its projection on the closest flat side of the Fermi surface. Hence the fact that the propagator is singular over an extended region (the Fermi surface) and not simply in a point has the effect that the renormalization point cannot be fixed but it must be left moving on the Fermi surface.

With a generic interaction the running coupling constants \( g_h(k_{-1,1},k_{-2,1},k_{-3,1}) \) have a non trivial dependence from the momenta and the scale index \( h \). However in the case of the interaction (8), involving fermions in the same side of the Fermi surface, some dramatic cancellations are present which have the effect that the running coupling constants are independent from the tangential momentum and with a bounded flow. Indeed, if \( W_{m}^{(h),p} \) is the \( p \)-order contribution to \( W_{m}^{(h)} \), one has
\[
W_{4}^{(h),p}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = 0 \quad p \geq 2
\]  
(19)
\[
W_{2}^{(h),p}(\mathbf{k}) = \partial_{\mathbf{\bar{k}}} \tilde{W}_{2}^{(h),p}(\mathbf{\bar{k}}) = \partial_{\mathbf{\bar{k}}} \tilde{W}_{2}^{(h),p}(\mathbf{\bar{k}}) = 0 \quad p \geq 2
\]  
(20)

with \( \partial_{\mathbf{\bar{k}}} = i v_F \partial_0 + \omega \partial_+ \). Moreover for any integer \( p \geq 1 \), we define \( \mathbf{k}^* \) such that \( k_-^* = k_- \) and
\[
\begin{pmatrix}
 k_0^* \\
 k_+^*
\end{pmatrix} = \begin{pmatrix}
 \cos(\frac{\omega}{v_F}) & -v_F \sin(\frac{\omega}{v_F}) \\
 v_F^{-1} \sin(\frac{\omega}{v_F}) & \cos(\frac{\omega}{v_F})
\end{pmatrix} \begin{pmatrix}
 k_0 \\
 k_+
\end{pmatrix}
\]  
(21)

implying \( g_h(\mathbf{k}) = e^{i\omega \mathbf{\bar{x}}} g_h(\mathbf{k}^*) \). Then
\[
W_{4}^{(h),p}(\mathbf{k}^*) = e^{-i\omega \pi(1-\frac{1}{3})} W_{4}^{(h),p}(\mathbf{k})
\]  

(22)

which implies \( W_{4}^{(h),p}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = 0 \) for any \( p \geq 2 \); in the same way
\[
W_{2}^{(h),p}(\mathbf{k}^*) = e^{-i\omega \pi(1-\frac{1}{3})} W_{2}^{(h),p}(\mathbf{k})
\]  

(23)

\[
\mathcal{R} W_{4}^{(h),p}(\mathbf{k}^*) = e^{-i\omega \pi(1-\frac{1}{3})} \mathcal{R} W_{4}^{(h),p}(\mathbf{k})
\]  

(24)

\[
(\partial_{\mathbf{\bar{k}}} W_{2}^{(h),p})(\mathbf{k}^*) = e^{-i\omega \pi(1-\frac{1}{3})} (\partial_{\mathbf{\bar{k}}} W_{2}^{(h),p})(\mathbf{k})
\]  

(25)

so that (19) is proved for any \( p \geq 2 \); this implies that \( g_h(k_{-1,1},k_{-2,1},k_{-3,1}) = U \delta_{k_{-1,1},k_{-2,1},k_{-3,1}} \). Moreover the first and second of (22) implies \( W_{2}^{(h),1}(\mathbf{k}) = \partial_{\mathbf{\bar{k}}} W_{2}^{(h),1}(\mathbf{k}) = 0 \), and an explicit computation shows that \( \partial_{\mathbf{\bar{k}}} W_{2}^{(h),1}(\mathbf{k}) = 0(\gamma^h) \).

In the exponent of (12), \( \mathcal{L} \psi^{(h)} \) is essentially equal to (8) with \( \psi^{(h)} \) replacing \( \psi \), that is the effective coupling is not renormalized. The \( \mathcal{R} \) operator makes the dimension of all kernels non-negative, so that \( W_{2}^{(h)} \) is written as a series in \( \mathcal{U} \) which is convergent provided that \( \mathcal{U} \) is small; moreover \( ||W_{2}^{(h)}|| = O(\gamma^{-h(n-2)}) \). Convergence follows from constructive RG methods, like tree and cluster expansions and determinant bounds for fermionic expectations (for an introduction to such methods, see [13]).

A similar analysis can be repeated for the 2-point function, and it is found that the difference between the free and the interaction Schrödinger function is bounded by \( C \mathcal{U} ||\mathbf{k}| \). The same conclusions are true even if we replace in (8) \( \delta_{\mathbf{\bar{k}},0} \) with \( v(\mathbf{p}) \) such that \( v(\mathbf{p}) = v(\mathbf{p}^*) \).

III. MODIFIED WARD IDENTITIES

We introduce an auxiliary model whose Schrödinger functions have the same asymptotic properties than (2), but verifying extra symmetries, from which a set of Ward Identities can be derived. Such identities are related to the invariance under local phase transformations depending on the Fermi surface side, which in the model (2) is broken by the cut-off function \( C_0^{-1}(\mathbf{k}) \) and by the lattice. The generating function of the auxiliary model is given by
\[
e^{\mathcal{M}(\phi,J)} = \int P(d\psi^{(h)}) e^{\mathcal{M}(\psi^{(h)}) + \sum \int d\mathbf{\bar{k}} \phi^+_{\mathbf{\bar{k}}} \psi_{\mathbf{\bar{k}}} + \phi^+_{\mathbf{\bar{k}}} \psi_{\mathbf{\bar{k}}}}
\]  
(26)
with \( \rho_{\sigma,x} = \psi_{\sigma,x}^+ \psi_{\sigma,x}^- \), \( P(d\psi^{(\leq N)}) \) is the fermionic integration with propagator (7), with \( C^{-1}_N(k) = \sum_{h=-\infty}^{N} f_h(k) \) replacing \( C^{-1}(k) = \sum_{h=-\infty}^{0} f_h(k) \) and \( V \) identical to (8) but with a non local interaction, namely
\[
V = U \sum_{\sigma,\sigma'} \frac{1}{L^{-1}} \int dx dy (\bar{x} - \bar{y}) \psi_{\sigma,x}^+ \psi_{\sigma,x}^- \psi_{\sigma',y}^+ \psi_{\sigma',y}^- \quad (24)
\]
with \( \bar{x} = (x_0, x_+) \), \( v(p) = v(p^*) \), \( v(0) = 1 \) and \( |v(p)| \leq C e^{-|k|} \). The Schwinger functions are obtained by performing derivatives with respect to the external fields \( J, \phi \).

Again (23) can be analyzed by a multiscale integration based on a decomposition similar to (11), with the difference that the scale are from \( h \) to \( N \). In the integration of the scales between \( N \) and 0, the *ultraviolet scales*, there is no need of renormalization; apparently the terms with two or four external lines have positive or vanishing dimension but one can use the non locality of the interaction to improve their scaling dimension. We integrate (with \( \mathcal{L} = 0 \)) the fields \( \psi^{(N)} \), \( \psi^{(N-1)} \), ..., \( \psi^{(k)} \) and we call \( W^{(k)}_{2n,m} \), the kernels in the effective potential multiplying \( 2n \) fermionic fields and \( m J \) fields. Again the dimension is \( \gamma^{-k(n+m-2)} \), \( k \geq 0 \) and we have to improve the bounds using the non-locality of the interaction. The kernels \( W^{(k)}_{2,0} \) can be written as (see Fig.2)
\[
W^{(k)}_{2,0}(x,y) = \frac{U}{L^{-1}} \int dw dw' v(\bar{x} - \bar{w}) g^{(k+1,N)}_w(x - w') W^{(k)}_{1,2}(w;w',y)
\]
By integrating over the fermionic line rather than over the interaction line, and using that \( \int dw |g^{(k,N)}_w(x)| \leq C \gamma^{-k} \) and that the interaction is bounded, we get an improvement, namely
\[
\int dw W^{(k)}_{2,0}(x,0) \leq C |U| \frac{\gamma^{-k}}{L^{-1}} \quad (26)
\]
Regarding \( W^{(k)}_{1,2} \) it can be written as sum of three kinds of terms, represented in Fig.3,4,5. The terms in Fig.3 are \( O(\frac{1}{L^{-1}}) \), using (26). Regarding the contribution in Fig.4, it is convenient to decompose the three propagators \( g_{\omega}, g_{\omega}, g_{\omega} \) into scales, \( \sum_{j, i, i' = k}^{N} g_{\omega}^{(j)} g_{\omega}^{(i)} g_{\omega}^{(i')} \) and then, for any realization of \( j, i, i' \), to take the \( | \cdot |_1 \) norm on the two propagator on the higher scales, and the \( | \cdot |_\infty \) norm on the propagator with the lowest one. In this way we obtain:
\[
\frac{|U|}{L^{-1}} \sum_{j=k}^{N} \sum_{i=k}^{j} \sum_{i'=k}^{j} |g^{(j)}_{\omega}||g^{(i)}_{\omega}||g^{(i')}_{\omega}| \leq \frac{C_2}{L^{-1}} |U| \gamma^{-2k} \quad (27)
\]
Finally the last kind of contributions are \( O(\gamma^{-k}) \), be-

\[
\int dx g_{\omega}(x)^2 = 0 \quad (28)
\]
After the integration of the fields \( \psi^{(N)}, \psi^{(N-1)}, ..., \psi^{(-1)} \)

\[
\int dx W^{(k)}_{1,2}(x,0) \leq C |U| \frac{\gamma^{-k}}{L^{-1}} \quad (29)
\]
we get a Grassmann integral very similar to (2), the only difference being that the exponent differ only for irrelevant terms. The integration of the remaining fields \( \psi^{(0)}, \psi^{(-1)} \) is then done following the same procedure as in the previous section and the Schwinger functions of the two models have the same asymptotic behavior, in the sense that their difference is \( O(|\bar{x} - \bar{y}|^{-\theta}) \), \( \theta > 1 \) (for a detailed proof in a similar case, see [14]); hence it is sufficient to prove (9) for the auxiliary model. We derive a set of set of Ward Identities for the auxiliary model; by performing the change of variables, with
\[ \psi^\pm_k \rightarrow e^{\pm i \alpha x} \psi^\pm_k \]  
with \( \alpha \) a local phase independent from \( x \), and making a derivative with respect to \( \alpha \) and to the external fields we obtain, if \( \mathbf{p} = (0, \bar{p}_z, \bar{p}_0), \mathbf{k} = (0, \bar{k}_z, \bar{k}_0) \), \( D_\omega(\mathbf{p}) = -i \bar{p}_0 + \omega v_F p \), \( \rho_\omega = \sum_\sigma \rho_\sigma p_\sigma \),

\[ D_\omega(\mathbf{p}) (\hat{\rho}_\sigma \hat{\psi}_k^{+\sigma} \hat{\psi}_k^{-\sigma} + \hat{\psi}_k^{+\sigma} \hat{\psi}_k^{-\sigma}) = \\
(\hat{\psi}_k^{-\sigma})^* (\hat{\rho}_\sigma \hat{\psi}_k^{+\sigma} \hat{\psi}_k^{-\sigma}) + \Delta(\mathbf{k}, \mathbf{p}) \]  
(30)

and

\[ \Delta(\mathbf{k}, \mathbf{p}) = \sum_{\sigma'} \frac{1}{\beta V} \sum_{k'} C(\mathbf{k}', \mathbf{p}) \langle \hat{\psi}_{k'+\sigma'}^+ \hat{\psi}_{k'-\sigma'}^0 \hat{\psi}_{k,\sigma}^+ \hat{\psi}_{k,\sigma}^- \rangle \]  
(31)

with, \( C(\mathbf{k}, \mathbf{p}) = D_\omega(\mathbf{k})[C_N(\mathbf{k} + \mathbf{p}) - C_N(\mathbf{k})] + D_\omega(\mathbf{p})[C_N(\mathbf{k} + \mathbf{p}) - 1] \)

(32)

The presence of the term \( \Delta(\mathbf{k}, \mathbf{p}) \) in the Ward Identity (30) is related to the presence of the ultraviolet cutoff function \( C_N(\mathbf{k}) \), which breaks the formal gauge invariance under the phase transformation (29). Even if \( C(\mathbf{k}, \mathbf{p}) \rightarrow 0 \) in the limit \( N \rightarrow \infty \), \( \Delta(\mathbf{k}, \mathbf{p}) \) is not vanishing in the limit \( N \rightarrow \infty \) but it produces an anomaly which is at the end responsible of the SC separation phenomenon.

IV. ANOMALIES AND SC SEPARATION

The analysis of \( \Delta(\mathbf{k}, \mathbf{p}) \) can be performed by an extension of the methods developed in [14],[15] for the 1D case. The main point is the following identity

\[ \Delta(\mathbf{k}, \mathbf{p}) = \nu v(\mathbf{p}) D_{-\omega}(\mathbf{p}) \langle \hat{\rho}_\sigma \hat{\psi}_k^{+\sigma} \hat{\psi}_k^{-\sigma} \rangle + R(\mathbf{k}, \mathbf{p}) \]  
(33)

where \( R(\mathbf{k}, \mathbf{p}) \) is a small correction vanishing as \( N \rightarrow \infty \) and \( \nu \equiv AU \)

\[ A = 2 \frac{1}{\beta V} \sum_k C(\mathbf{k}, \mathbf{p}) \left[ -i \bar{p}_0 - \omega v_F p_+ \right] g^{(-N)}(k) g^{(-N)}(\mathbf{k} + \mathbf{p}) \big|_{\mathbf{p}=0} \]

\[ = 2 \frac{1}{\beta L} \sum_{k_z} H(k_z) \frac{1}{\beta L} \sum_k C_N'(\mathbf{k}) \]  
(34)

The identity (33) can be derived by writing \( R \) as a Grassman integral similar to the one for \( \langle \hat{\rho}_\sigma \hat{\psi}_k^{+\sigma} \hat{\psi}_k^{-\sigma} \rangle \), with only difference that, in the exponent, the term \( \frac{1}{\beta V} \sum_{k',\sigma} J_{\mathbf{k}',\mathbf{p}} \psi_{k',\mathbf{k}+\mathbf{p}}^{+\sigma} \psi_{k',\mathbf{k}+\mathbf{p}}^{-\sigma} - \frac{1}{\beta V} \sum_{k',\sigma} \nu J_{\mathbf{k}',\mathbf{k}+\mathbf{p}}^{+\sigma} \psi_{k',\mathbf{k}+\mathbf{p}}^{-\sigma} \)

(35)

We perform a multiscale integration and the main difference with respect to the previous case is that there

\[ \langle \psi_{\sigma,\mathbf{k}+\mathbf{p}}^{+\sigma} \rangle = g_N(k) + \frac{1}{\beta L} \sum_\mathbf{p} v(\mathbf{p}) \langle \hat{\rho}_\sigma \hat{\psi}_{\sigma,\mathbf{k}+\mathbf{p}}^{+\sigma} \hat{\psi}_{\sigma,\mathbf{k}+\mathbf{p}}^{-\sigma} \rangle \]  
(38)

are in the effective potential marginal terms of the form \( J \psi^+ \psi^- \), whose kernel is obtained from the contraction of the first or the second term in (35), see Fig. 6. One decomposes the terms coming from the first term in (35) and gets the analogue of the decomposition in Fig.3,4,5. An important difference comes from the fact

\[ C(\mathbf{k}, \mathbf{p}) g^{(i)}(k) g^{(j)}(\mathbf{k} + \mathbf{p}) \]

(36)

is different from 0 only if \( i \) or \( j \) are equal to \( N \).
The term in the second line is vanishing in the limit whose solution has the form

\[ \langle \hat{\psi}_{\sigma,k}^+ \hat{\psi}_{\sigma,k} \rangle = g(k)[1 + \frac{U}{L_- \beta L_+} \sum_p \frac{v(p)}{D_+(p)} + \nu v(p) D_-^{-1}(p) R(k,p) + \frac{U}{L_- \beta L_+} \sum_p \frac{v(p)}{D_+(p)} + \nu v(p) D_-^{-1}(p) \langle \hat{\psi}_{\sigma,k-p}^+ \hat{\psi}_{\sigma,k-p} \rangle] \]

The term in the second line is vanishing in the limit \( N \to \infty \), so that, in the limit \( N \to \infty \), \( S(k) = H(k_-) \hat{S}(k_0, k_+) \) with \( \hat{S}(\hat{x}) \) verifying

\[ (\partial_0 + i \omega_v \partial_1) \hat{S}(\hat{x}) = \delta(\hat{x}) + \frac{U}{L_-} A(\hat{x}) S(\hat{x}) \tag{39} \]

with

\[ A(\hat{x}) = \int dp_0 dp_+ \frac{v(p)}{D_+(p)} + \nu_v(p) D_-^{-1}(p) e^{i(p_0 x_0 + p_+ x_+)} \tag{40} \]

whose solution has the form

\[ S(\hat{x}) = g(\hat{x}) e^{\int d\hat{x} [g(\hat{x} - \hat{z}) - g(\hat{x})] A(\hat{x})} \tag{41} \]

Using that

\[ \int d\hat{x} [g(\hat{x} - \hat{z}) - g(\hat{x})] A(\hat{x}) = \frac{1}{2\pi \nu} \log \frac{x_0 + i \omega x_+}{x_0 (1 + \frac{\pi \nu}{1-\nu}) + H(\hat{x})} \tag{42} \]

with \( H(\hat{x}) \) vanishing as \( |\hat{x}| \to \infty \), (9) finally follows.

V. CONCLUSIONS

In conclusion we have proved by non-perturbative RG that the interactions between electrons on the same sector (the analogue of the \( g_4 \) interaction in 1D) produces SC separation in 2D fermions with a flat Fermi surface. The deep origin of such phenomenon is identified in the anomalies in the Ward Identities related to the approximate local phase transformations depending from the Fermi surface side. Note that the two velocities \( v_\sigma \) and \( v_R \) are different uniformly in the volume, despite the fact that we have selected only the \( p_- \) term in the sums.

The presence of the other interaction channels makes the resulting behavior of the flat Fermi surface model much more complex (for instance the interaction between opposite sides of the Fermi surface produces Luttinger liquid behavior, see [12]) but, in analogy with the 1D case, we can expect the SC separation phenomenon is left unaffected.