Chaotic hypothesis

From Scholarpedia

Giovanni Gallavotti (2008), Scholarpedia, 3(1):5906. revision #29534 [link to/cite this article]

Curator: Dr. Giovanni Gallavotti, Physics, University di Roma, Italy

Chaotic Hypothesis: assumes that quite generally a system exhibiting chaotic motions does so in a maximal form so that it can be supposed to be a transitive hyperbolic system. This is analogous to the assumption that a system exhibiting only non chaotic motions is an integrable system, i.e. it is as ordered as possible and it can be, quite generally, either supposed to consist of harmonic oscillators or to be trivially related to such systems. The assumption implies that all observables have well defined time averages independent on the initial data, with the exception of initial data forming a zero volume set. The assumption also implies the ergodic hypothesis when the systems are Hamiltonian and can be viewed as its extension to non Hamiltonian ones.

Contents

Empirical chaoticity

In Nonequilibrium Statistical Mechanics and in Fluid Mechanics motions are often chaotic: this reflects the empirical property that initial data with representative points x, y, if extremely close (strictly speaking infinitesimally close) evolve in time into points $S_t x, S_t y$ separating from each other at an exponential rate on a time scale of the order of the one on which evolution is observable, i.e., on a scale in which the value of some among the observables of interest is perceived to change.

Mathematical representation: flows and maps

Mathematical models for time evolution can be differential equations whose solutions represent motions developing in continuous time t or, often, maps whose n-th iterate represents motions developing at discrete integer times n. The point representing the state of the system at time t is denoted $S_t x$ in the continuous time models (and the maps S_t are the flow generated by the equations of motion) or, at the n-th observation, $S^n \xi$ in the discrete time models. Here x, ξ will be points on a manifold X or Ξ respectively, called the phase space, or the space of the states, of the system.

Models of chaotic motions are evolutions which have at least one positive Lyapunov exponent. This is the physical meaning that is given to the statement that a motion is chaotic.

The Paradigm. Hyperbolicity and the statistics of motions

A paradigm for chaotic systems are the transitive hyperbolic systems, also called transitive Anosov systems. In the discrete time case these are smooth dynamical systems on a bounded manifold Ξ

whose dynamics $\xi \to S^n\xi$ can be proved to be chaotic and which at the same time are very well understood theoretically. For instance all initial data $\xi \in \Xi$ will admit a statistics in the sense that for all smooth observables $F(\xi)$ the time averages, i.e. the limits

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{j0}^{N-1} F(S^j\xi) = \int_\Xi F(\eta)\mu(d\eta),$$

exist, but for a zero volume set, and are independent of ξ and are equal to the average with respect to a probability distribution μ, which is called the SRB distribution. A continuous time version of the transitive Anosov maps are the mixing Anosov flows generated on a manifold X by a differential equation.

If a map $S = S_{\tau(\xi)}$ is obtained from a continuous time mixing Anosov flow via a timing event determined by a Poincare' map on a subset $\Xi \subset X$ then, in general, the map S generates an evolution which admits a statistics μ. The time averages in the continuous time evolution and the corresponding ones in the discrete timing are simply related by

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T F(S_t x)\,dt = \lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} \tilde F(S^j\xi) = \int_\Xi F(y)\mu(dy)$$

where $\tilde F(\xi) \stackrel{def}{=} \int_0^{\tau(\xi)} F(S_t\xi)\frac{dt}{\tau_+}$ and $\tau_+ = \int \tau(\xi)\mu(d\xi)$ is the average time interval between timing events.

Large deviations for time averages

If S is an Anosov map on Ξ and F is an observable the quantity

$$\varphi = \frac{1}{N}\sum_{j=0}^{N-1} \frac{F(S^j\xi)}{F_+}$$

can be regarded as a random variable with respect to the probability distribution μ. Its fluctuations are then controlled by a large deviation law in the sense that

probability$(\varphi \in [a,b]) = e^{N \max_{\varphi\in[a,b]} \zeta(\varphi) + O(1)}$

with $\zeta(\varphi)$ a function analytic and convex on an interval (φ_-, ϕ_+) and for $\phi \notin [\varphi_-, \varphi_+]$ it is usually extendeded by assigning it the value $-\infty$, meaning that intervals outside $[\varphi_-, \varphi_+]$ have a probability which tends to 0 faster than exponentially in N.

The hypothesis

In physical applications systems are often not isolated, \ie they interact with external objects (like thermostats) so that the complete equations of motion must involve also the evolution of the latter; furthermore the motions are dissipative, \ie their equations are not only not Hamiltonian but they imply that phase space volume contracts on the average. An important common feature is that the motions are usually chaotic, and develop approaching asymptotically an attacting set; then the following hypothesis has been proposed

Chaotic Hypothesis: The motions of a chaotic system develop asymptotically on an attracting set on which dynamics can be regarded as a transitive hyperbolic ("Anosov") evolution.

This means that it can be assumed that motions exhibit the same qualitative properties of those taking place in Anosov systems: a non trivial property akin to the well known hypothesis that generic isolated (Hamiltonian) systems are ergodic. Hence the hyperbolicity assumption concerns the evolution of the system towards the attractor and the stationary state as well as the stationary state itself. As in the case of the classical ergodic hypothesis the hyperbolicity has several consequences but only the ones that deal with the few observables of interest for the macroscopic properties of the system are to be considered relevant. This is illustrated by some of the important consequences discussed below.

It is interesting that the very first example of a non-trivial mechanical system satisfying the classical ergodic hypothesis has been, actually, an example (by E. Hopf) of a chaotic system which is rigorously an Anosov system (it is the geodesic flow on a surface of constant negative curvature).

Interest of the hypothesis

The interest of the Chaotic Hypothesis (CH) is that it implies the existence of the time averages with a probability distribution giving the statistics of the motion and satisfying a large deviation law; furthermore in the case of isolated systems the CH implies the ergodic hypothesis and identifies the SRB statistics with the usual microcanonical (equilibrium) distribution. The existence of the time averages has the physical interpretation of giving the statistical properties of stationary states.

Of great importance is that for Anosov systems an analytic expression for the SRB statistics can be found: this makes the CH suitable to derive relations between averages of observables without being necessarily able to compute such averages. In particular, this analytic expression can be employed to derive consequences of symmetry properties: a typical example is the Fluctuation theorem, [CG95], for Anosov time reversible systems and the associated Fluctuation relation that it implies for real systems as a consequence of the Time Reversal Symmetry.

Assuming the CH assigns a paradigmatic role to Anosov systems analogous to the paradigmatic role assigned to harmonic oscillators in the case of ordered motions. In this way one can say that a chaotic system can be physically understood if it can be modeled by a hyperbolic system, just as one says that a mechanical system with ordered motions is understood if it can be reduced to a system of harmonic oscillators.

Chaotic hypothesis and turbulence

The CH originates from the theory of turbulence where a similar hypothesis was formulated, [Ru80]. The CH received the above formulation for general mechanical systems in [GC95], and followed the intense research efforts and results made possible by the developments of electronic simulations in the 1980's, see [EM90], and in the early 1990's, see [GC95]; it was motivated by the need of a general systematization of their interpretation. The presentation discussed here can be found, extended, in [Ga06].

References

[Ru80] D. Ruelle, Measures describing a turbulent flow, Annals of the New York Academy of Sciences, **357**, 1--9, 1980.

[GC95] G. Gallavotti, E. G. D. Cohen, Dynamical ensembles in stationary states, Journal of Statistical Physics, **80**, 931--970, 1995.

[EM90] D. J. Evans, G. P. Morriss, Statistical Mechanics of Non-equilibrium Fluids, Academic Press, 1990, New-York.

[Ga06] G. Gallavotti, Entropy, Thermostats and Chaotic Hypothesis, Chaos, 16, 043114 (+6), 2006, DOI:10.1063/1.2372713.

See also

Attractor, Anosov Diffeomorphism, Chaos, Ergodic Theory, SRB Measure, Timing Event

Giovanni Gallavotti (2008) Chaotic hypothesis. Scholarpedia, 3(1):5906, (go to the first approved version)
Created: 14 December 2007, reviewed: 7 January 2008, accepted: 7 January 2008
Retrieved from "http://www.scholarpedia.org/article/Chaotic_hypothesis"

Categories: Dynamical Systems | Statistical Mechanics
