Non-perturbative aspects of Chiral Anomalies

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Abstract. *We investigate the properties of chiral anomalies in $d = 2$ in the framework of Constructive Quantum Field Theory. The condition that the gauge propagator is sufficiently soft in the ultraviolet is essential for the anomaly non-renormalization; when it is violated, as for contact current-current interactions, the anomaly is renormalized by higher order corrections. The same conditions are also essential for the validity, in the massless case, of the closed equation obtained combining Ward Identities and Schwinger-Dyson equations; this solves the apparent contradiction between perturbative computations and exact analysis.*

Keywords. Nonperturbative renormalization; Chiral anomaly; renormalization or non-renormalization of the anomalies

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1. Introduction

Anomalies are the breaking of certain classical symmetries happening in Quantum Field Theory (QFT), see [A], [BJ]. A well known example is in QED_4, in which the axial Ward Identity (WI) is

$$\mathbf{p}_\mu \hat\Gamma^\mu_5(\mathbf{p},\mathbf{k}) = \gamma^5(\hat S(\mathbf{k}-\mathbf{p}))^{-1} - \gamma^5(\hat S(\mathbf{k}))^{-1} + 2im\hat\Gamma_5(\mathbf{p},\mathbf{k}) + i\alpha \hat F(\mathbf{p},\mathbf{k}) \tag{1.1}$$

where $\Gamma^\mu_5(\mathbf{p},\mathbf{k})$ and $\hat\Gamma_5(\mathbf{p},\mathbf{k})$ are the Fourier transform of $\langle T(j^5_{\mu,\mathbf{z}}\bar\psi_\mathbf{x}\psi_\mathbf{y})\rangle_A$ and $\langle T(j^5_\mathbf{z}\bar\psi_\mathbf{x}\psi_\mathbf{y})\rangle_A$ (A means truncation with respect to the fermion interacting propagator $\hat S(\mathbf{k})$), and $\hat F(\mathbf{p},\mathbf{k})$ is the Fourier transform of $\varepsilon_{\mu,\nu,\rho,\sigma}\langle T(F_{\mu,\nu,\mathbf{z}}F_{\rho,\sigma,\mathbf{z}}\,\bar\psi_\mathbf{x}\psi_\mathbf{y}\rangle_A$. The last term, which is unexpected from formal applications of the classical Noether theorem, is the anomaly; it was shown by Adler [A] that α is exactly quadratic in the charge $\alpha = \frac{e^2}{16\pi^2}$, that is the anomaly is *non-renormalized* by higher order corrections. An accurate derivation of this property, known as *Adler-Bardeen Theorem*, was given in [AB], in which (1.1) with $\alpha = \frac{e^2}{16\pi^2}$ was proved as a *perturbative order by order* identity among formal expansions in Feynman graphs. In most textbooks, the properties of the anomalies are actually derived following the functional integral approach in [F], in which however the gauge fields are treated as external classical fields, so that higher order corrections to the anomaly would be in any case neglected. On the other hand several objections have been rised against the validity of the anomaly non-renormalization along the years, starting from [JJ] (see [A1] for a recent review), so that a non-perturbative derivation of it would be higly desirable, in view also of the role of such a property in the proof of renormalizability of the Electroweak model. This is actually far from the present possibilities in $d = 4$, so that anomalies have been investigated in $d = 2$, with the hope of getting results beyond a purely perturbative level and to have insights for the $d = 4$ case.

In [GR] it was shown by a formal expansion in Feynman graphs that the anomaly non-renormalization holds also for $d = 2$ QFT models *either for massive gauge and for Thirring interactions*. The advantage of the $d = 2$ case is that one can use one the operatorial exact solutions to get an "explicit verification of the perturbation-theory calculations". Indeed, following the analysis in [J], the anomaly non-renormalization appears as a consequence of the validity of the Ward-Identities for the total and axial current and of the Schwinger-Dyson equation; such equations can be combined in a *closed equation* for the two and four point function and from a self-consistency argument the explicit value of the anomalies is obtained, showing the absence of higher orders corrections.

Note that the validity of the anomaly non-renormalization for Thirring interactions does not follow from the Adler-Bardeen theorem in $d = 2$, as there the fast decay of the bosonic propagator plays an essential role. Indeed other perturbative computations [AF] have showed that in the $d = 2$ massless Thirring model there are higher orders contribution to the anomaly. Note also that the question on wether or not one can use exact results to infer properties about the correlations computed in a functional integral approach is not trivial at all and it was the subject of extensive debates, see for instance [GL].

The recent developments in the mathematical analysis of quantum models at low dimension make finally possible to investigate the properties of the Chiral anomalies at a non-perturbative level in the framework of Constructive QFT. In such an approach, the Euclidean n-point functions are obtained as the limit of functional integrals suitably regularized through lattice or momentum cut-offs; Feynman graph expansions are avoided for their bad combinatorial properties, and cluster expansions are instead used, which allow to prove the convergence of the series involved. While a well known problem in this approach is posed by the basic conflict between the scale decompositions used in a non-perturbative setting [P], [G] and the local symmetries, the methods recently developed in [BM] overcome such a problem, at least in $d = 2$, and allow the rigorous construction of QFT models in $d = 2$ showing that the momentum cut-offs can be removed and that the resulting Schwinger functions verify the axioms. By using such methods it has been rigorously proved in [M], [BFM] that the condition that the gauge propagator is sufficiently soft in the ultraviolet is *essential* for the anomaly non-renormalization in a functional integral approach; when it is violated, as for Thirring current-current interactions, the anomaly is *renormalized* by higher order corrections. Such results confirm, at a non-perturbative level, the perturbative analysis in [AB] in which the decay of the boson propagator plays an essential role; they are however in apparent contrast with the results based on the exact solutions in which the anomaly non-renormalization seems not to require such conditions.

In this paper we will explain how to resolve such apparent contradiction, and we finally clarify the relation betwen exact analysis and functional approach in $d = 2$ models. By combining the WI with the Schwinger-Dyson equation, at *finite* cut-offs, one does not obtain a closed equation as there are additional corrections depending in a complicate way from the cut-offs. We will prove that such corrections are indeed vanishing when the cut-offs are removed *provided that* the same conditions ensuring the anomaly non-renormalization are verified. Such conditions require that the boson propagator decay fast enough for large momenta, or at least that the boson cut-off is removed *after* the fermionic one; in purely fermionic models, it is necessary to start with *non local* current-current interactions taking the local limit *after* the removal of the fermionic cut-off. If such conditions are not verified, as for Thirring contact interactions, the corrections are not vanishing so that the closed equation postulated by the exact solutions is *not verified* by the correlations computed from functional integrals; this solves the apparent contradictions between functional integral approach and exact analysis.

The paper is organized in the following way. In §2 the main results are presented; in §3 and §4 we construct the model and study the anomalies, referring for the complete proofs, which are quite long and technical, to [M],[BMF]. Finally in §5 we analyze the corrections to the closed equations obtained combining the WIs with Schwinger-Dyson equations.

2. Main results

2.1. *The model*

We consider the (Euclidean) $d=2$ QFT model whose correlations can be obtained from the *generating function*

$$\mathcal{W}_{K,N}(J,\phi)=\log\int P_{Z_2}(d\psi^{(\le N)})P_{Z_3}(dA^{\le K})e^{\int d\mathbf{x}[eZ_1\bar\psi_{\mathbf{x}}(A_{\mu,\mathbf{x}}\gamma_\mu)\psi_{\mathbf{x}}+J_{\mu,\mathbf{x}}A_{\mu,\mathbf{x}}+\phi_{\mathbf{x}}\bar\psi_{\mathbf{x}}+\bar\phi_{\mathbf{x}}\psi_{\mathbf{x}}]} \tag{2.1}$$

$\phi_{\mathbf{x}},\bar\phi_{\mathbf{x}},J_{\mu,\mathbf{x}}$ are external fields, Z_2 and Z_3 are the fermionic and bosonic wave function renormalizations, Z_1 is the charge renormalization, $\psi,\bar\psi$ are Grassmann variables and $P_{Z_2}(d\psi^{(\le N)})$ is the Grassmannian integration with propagator

$$g_N(\mathbf{x}-\mathbf{y})=\frac{1}{Z_2}\int d\mathbf{p}\frac{-i\not\mathbf{p}+Z_4m}{\mathbf{p}^2+Z_4^2m^2}e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}\chi_N(\mathbf{p}) \tag{2.2}$$

where $\chi_N(\mathbf{k})$ is a *smooth cutoff function* non vanishing for $|\mathbf{k}|\le 2^N$, N a positive integer. Finally $A_{\mu,\mathbf{x}}=(A_{0,\mathbf{x}},A_{1,\mathbf{x}})$ are Euclidean boson field with Gaussian measure $P_{Z_3}(dA^{\le K})$ and propagator $<A_{\mu,\mathbf{x}}A_{\nu,\mathbf{y}}>=\delta_{\mu,\nu}v_K(\mathbf{x}-\mathbf{y})$; if A_μ is a massive vector field its covariance is

$$v_K(\mathbf{x}-\mathbf{y})=\frac{1}{Z_3}\int\frac{d\mathbf{p}}{(2\pi)^2}e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}\frac{\chi_K(\mathbf{p})}{\mathbf{p}^2+M^2} \tag{2.3}$$

but we will mostly consider the case

$$v_K(\mathbf{x}-\mathbf{y})=\frac{1}{Z_3}\int\frac{d\mathbf{p}}{(2\pi)^2}e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}\chi_K(\mathbf{p}) \tag{2.4}$$

The reason is that the theory with propagator (2.4) has a perturbative structure much more similar to $d=4$ gauge models, as it is renormalizable with divergence index is $2-\frac{f}{2}-b$, if b,f are the external bosonic and fermionic lines (to be compared with $4-\frac{3}{2}f-b$ for QED_4), while with the choice (2.3) the the theory is superrinormalizable and the index is $2-n-f/2$, if n is the perturbative order. Different properties will be found, in the case (2.4), depending if the fermionic or the bosonic cut-off is removed first.

The truncated Euclidean Schwinger functions are defined as

$$<\psi_{\mathbf{x}_1}...\psi_{\mathbf{x}_n}\bar\psi_{\mathbf{y}_1}...\bar\psi_{\mathbf{y}_n}A_{\mu_1,\mathbf{z}_1}...A_{\mu_m,\mathbf{z}_m}>_{N,K}=\frac{\partial^{2n+m}\mathcal{W}_{N,L}(J^A,\phi)}{\partial\phi_{\mathbf{x}_1}...\partial\phi_{\mathbf{x}_n}\partial\bar\phi_{\mathbf{y}_1}...\partial\phi_{\mathbf{y}_n}\partial J^A_{\mu_1,\mathbf{z}_1}...\partial J^A_{\mu_m,\mathbf{z}_m}}\Big|_0 \tag{2.5}$$

We remark that the Schwinger functions (2.5) *cannot be* explicitly computed, even when $m=0$, unless some approximation is done, as in [FGS], which is equivalent to treat the gauge field as a classical field; indeed the model (2.1) is strictly related to certain non-solvable statistical mechanics models describing $d=2$ Ising models coupled by a quartic spin interaction, see [GM1].

By integrating the boson field a purely fermionic theory is obtained, equivalent in the limit $K\to\infty$ to a regularized massive Thirring model

$$e^{\mathcal{W}_{N,K}(J,\phi)}=\int P_{Z_2}(d\psi^{(\le N)})e^{\frac{e^2}{2}\int d\mathbf{x}d\mathbf{y}v_K(\mathbf{x}-\mathbf{y})[Z_1e\bar\psi_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}}+J_{\mu,\mathbf{x}}][Z_1e\bar\psi_{\mathbf{y}}\gamma^\mu\psi_{\mathbf{y}}+J_{\mu,\mathbf{y}}]+\int d\mathbf{x}[\phi_{\mathbf{x}}\bar\psi_{\mathbf{x}}+\bar\phi_{\mathbf{x}}\psi_{\mathbf{x}}]} \tag{2.6}$$

2.2 . *Removal of cut-offs and construction of the theory.*

If the boson propagator is (2.4), we have to choose different bare parameters depending if the fermionic or bosonic cut-off is removed first. It has been proved in [M] that, if e is small enough, by choosing the bare parameters as

$$Z_3 = 1, \quad Z_1 = Z_2 \equiv Z = \gamma^{-\eta K}, \quad Z_4 = \gamma^{-\eta_1 K} \tag{2.7}$$

with η, η_1 analytic functions of e and $\eta = ae^4 + O(e^6)$, $\eta_1 = be^2 + O(e^4)$, $a, b > 0$ suitable constants, the limit

$$\lim_{K\to\infty} \lim_{N\to\infty} < \psi_{\mathbf{x}_1} ... \psi_{\mathbf{x}_n} \bar\psi_{\mathbf{y}_1} ... \bar\psi_{\mathbf{y}_n} A_{\mu_1,\mathbf{z}_1} ... A_{\mu_m,\mathbf{z}_m} >_{N,K} \tag{2.8}$$

exists at non coinciding points and verifies the Osterwalder-Schroeder axioms [OS]. On the other hand it has be proven in [BFM] that, by choosing

$$Z_3 = 1, \quad Z_1 = Z_2 \equiv Z = \gamma^{-\eta N}, \quad Z_4 = \gamma^{-\eta_1 N} \tag{2.9}$$

the limit $\lim_{N\to\infty} \lim_{K\to\infty} < \psi_{\mathbf{x}_1} ... \psi_{\mathbf{x}_n} \bar\psi_{\mathbf{y}_1} ... \bar\psi_{\mathbf{y}_n} A_{\mu_1,\mathbf{z}_1} ... A_{\mu_m,\mathbf{z}_m} >_{N,K}$ exists at non coinciding points and verifies the axioms.

Finally if the gauge propagator is (2.3), the limit $N \to \infty$ of (2.5) can be taken assuming that $Z_3 = Z_1 = Z_2 = Z_4 = 1$.

2.3 .*Renormalization or non renormalization of the anomalies.*

The Ward Identity for the axial current is

$$\begin{aligned} &\mathbf{p}^\mu \langle j^5_{\mu,\mathbf{p}} \psi_{\mathbf{k}} \bar\psi_{\mathbf{k}-\mathbf{p}} \rangle_{K,N} = \gamma^5 \langle \psi_{\mathbf{k}-\mathbf{p}} \bar\psi_{\mathbf{k}-\mathbf{p}} \rangle_{K,N} - \\ &\gamma^5 \langle \psi_{\mathbf{k}} \bar\psi_{\mathbf{k}} \rangle_{K,N} + m \langle j^5_{\mathbf{p}} \psi_{\mathbf{k}} \bar\psi_{\mathbf{k}-\mathbf{p}} \rangle_{K,N} + \alpha_{K,N}(\mathbf{p},\mathbf{k}) \varepsilon_{\mu,v} i \mathbf{p}_\mu < A_{\nu,\mathbf{p}} \psi_{\mathbf{k}} \bar\psi_{\mathbf{k}-\mathbf{p}} >_{K,N} \end{aligned} \tag{2.10}$$

with $j^5_\mu = Z_1 \bar\psi \gamma_\mu \gamma^5 \psi$, $j^{\mu,5} = Z_1 \bar\psi \gamma^5 \psi$. In the case (2.4) if the fermionic cut-off is removed first, at fixed momenta

$$\lim_{K\to\infty} \lim_{N\to\infty} \alpha_{K,N}(\mathbf{p},\mathbf{k}) \equiv \alpha_{\infty,1} = \frac{e}{2\pi} \tag{2.11}$$

while if the bosonic cut-off is removed first, at fixed momenta

$$\lim_{N\to\infty} \lim_{K\to\infty} \alpha_{K,N}(\mathbf{p},\mathbf{k}) \equiv \alpha_{\infty,2} = \frac{e}{2\pi} + Ae^3 + O(e^4) \tag{2.12}$$

with $A > 0$ a non vanishing constant (see (4.8) below). Different WI are then verified depending if the bosonic or fermionic cut-off is removed first. The same result (2.11) holds with the choice (2.3).

2.4 .*Anomalies and Schwinger-Dyson equation.*

The *Schwinger-Dyson* equation in the massless case for the regularized theory is, in the case (2.4)

$$< \psi_{\mathbf{k}} \bar\psi_{\mathbf{k}} >_{K,N} = g_N(\mathbf{k})[Z^{-1} - e^2 \int \frac{d\mathbf{p}}{(2\pi)^2} \hat v_K(\mathbf{p}) \gamma_\mu < j_{\mu,\mathbf{p}} \psi_{\mathbf{k}} \bar\psi_{\mathbf{k}-\mathbf{p}} >_{K,N}] \tag{2.13}$$

which can be combined with the WI (2.10) and its analogous for the current at finite cut-off.

We will prove that when the fermionic cut-off is removed first it holds

$$< \psi_{\mathbf{k}} \bar\psi^+_{\mathbf{k}} >_{K,N} = \mathcal{B}_{1,K,N}(\mathbf{k}) + \frac{g_N(\mathbf{k})}{Z} - e^2 \frac{[\bar a_1 - a_1]}{2} g_N(\mathbf{k}) \int \frac{d\mathbf{p}}{(2\pi)^2} i \hat v_K(\mathbf{p}) \frac{\gamma_\mu \mathbf{p}_\mu}{\mathbf{p}^2} < \psi_{\mathbf{k}-\mathbf{p}} \bar\psi_{\mathbf{k}-\mathbf{p}} >_{K,N} \tag{2.14}$$

where $a_1^{-1}=1-\alpha_{\infty,1}$ and $\bar a_1^{-1}=1-\bar\alpha_{\infty,1}$, with $\bar\alpha_{\infty,1}=-\frac{e}{2\pi}$, and $\mathcal{B}_{1,K,N}(\mathbf{k})$ is a term depending on the integrated difference between the WI with or without cut-off, and it is such that

$$\lim_{K\to\infty}\lim_{N\to\infty}\mathcal{B}_{1,K,N}(\mathbf{k})=0 \tag{2.15}$$

An opposite situation is encountered, if the bosonic cut-off is removed first; in such a case the equation for the propagator is

$$<\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}>_{K,N}=\mathcal{B}_{2,K,N}(\mathbf{k})+\frac{g_N(\mathbf{k})}{Z}-e^2\frac{[a_2-\bar a_2]}{2}g_N(\mathbf{k})\int\frac{d\mathbf{p}}{(2\pi)^2}i\hat v_K(\mathbf{p})\frac{\gamma_\mu\mathbf{p}_\mu}{\mathbf{p}^2}<\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}>_{K,N} \tag{2.16}$$

and $a_2^{-1}=1-\alpha_{\infty,2}$ and $\bar a_2^{-1}=1-\bar\alpha_{\infty,2}$, with $\bar\alpha_{\infty,2}=-\frac{e}{2\pi}+Ae^3+O(e^4)$; again $\mathcal{B}_{2,K,N}(\mathbf{k})$ is a term depending on the integrated difference between the WI with or without cut-off, but in this case it is not vanishing at all, but it holds

$$\mathcal{B}_{2,K,N}(\mathbf{k})=\sigma\frac{g_N(\mathbf{k})}{Z}+\rho<\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}>+H_{K,N}(\mathbf{k}) \tag{2.17}$$

with $\rho=\bar Ae^4+O(e^6)$, with $\bar A>0$, $\sigma=O(e^6)$ and $\lim_{N\to\infty}\lim_{K\to\infty}H_{K,N}(\mathbf{k})=0$.

Finally with the choice (2.3), (2.15) holds.

2.5. *Remarks.*

1)The presence of the fermionic cut-off produces corrections to the Ward Identities (see (2.10) below), but the bare parameters can be still chosen to verify the relation $Z_1=Z_2$, essential for preserving Gauge invariance.

2)The results in §2.3 say that the anomaly non-renormalization requires in an essential way that the gauge propagator is sufficiently soft in the ultraviolet; when this is not true, as with the choice (2.4) with the bosonic cut-off removed first, the anomaly has indeed higher orders contributions.

3)The results in §2.4 says that, even if the WI and SD equations are verified removing cut-offs, their combination is true *only if* the gauge propagator is soft enough in the ultraviolet. When this is not the case there is a non vanishing correction and the n-point functions verify a different equation.

4)The properties of the exact solution of the massless Thirring model are recovered only starting from a non local current-current interaction, and performing the local limit *after* the removal of the fermionic cut-off. On the contrary, if one starts from a local interaction, different properties are found. This solves the apparent contradiction between functional integral approach and the exact analysis.

3. Removal of cut-offs and construction of the theory

We consider first the case (2.4) and $N\ge K$, and we call $\lambda=e^2$ and $\mathcal{V}^{(N)}(\psi,J,\phi)$ the exponent in the r.h.s. of (2.6). Using the properties of the Grassmanian integrations (see [GM] for a tutorial introduction to multiscale techniques for Grassman integrals)

$$\begin{aligned}&e^{\mathcal{W}_{N,K}(J,\phi)}=\int P_Z(d\psi^{(\le N)})e^{\mathcal{V}^{(N)}(\psi,J,\phi)}=\\&\int P_Z(d\psi^{(\le N-1)})\int P_Z(d\psi^{(N)})e^{\mathcal{V}^{(N)}(\psi,J,\phi)}=\int P_Z(d\psi^{(\le N-1)})e^{\mathcal{V}^{(N-1)}(\psi,J,\phi)}\end{aligned} \tag{3.1}$$

where $\chi_N(\mathbf{k})=\chi_{N-1}(\mathbf{k})+f^{(N)}(\mathbf{k})$, with $f^{(N)}(\mathbf{k})$ with support in $2^{N-1}\le|\mathbf{k}|\le 2^{N+1}$ and $P_Z(d\psi^{(N)})$ has propagator $g^{(N)}(\mathbf{k})$ coinciding with (2.2) with $\chi_N(\mathbf{k})$ replaced by $f^{(N)}(\mathbf{k})$; moreover $\mathcal{V}^{(N-1)}=\log\sum_{n=1}^\infty\frac{1}{n!}\mathcal{E}_N^T(\mathcal{V}^{(N)}...\mathcal{V}^{(N)})$ where $\mathcal{E}_N^T$ is the truncated fermionic expectation.

We can integrate in the same way the fields $\psi^{N-1},...\psi^{h+1}$ obtaining a sequence of effective potentials $\mathcal{V}^{(N-2)}...\mathcal{V}^{(h)}$, where $\mathcal{V}^{(h)}$ is a series of monomials in the $\psi^{(\leq h)}$, ϕ and J fields integrated over kernels $H^{(h)}_{n,m}$, if n is the number of ψ or ϕ fields and m the number of J fields. A crucial technical ingredient for bounding such kernels is the following classical formula

$$\mathcal{E}^T_h(\tilde\psi(P_1)...\tilde\psi(P_n))=\sum_T(\prod_{l\in T}g^{(h)}(\mathbf{x}_l-\mathbf{y}_l))\int dP_T(t)detG^T \tag{3.2}$$

where $\mathcal{E}^T_h$ is the truncated fermionic expectation with propagator $g^{(h)}$, $\tilde\psi(P)=\prod_{i\in P}\psi_{\mathbf{x}_i}$, T is a set of lines form a tree between the clusters of points $P_1,..,P_n$ and dP_T is a suitable normalized probability measure. By the *Gram-Hadamard* inequality for determinants it follows that $|detG^T|\leq 2^{(\sum_i |P_i|-n)h}$; this determinant bound allows to exploit the cancellations due to the anticommutativity. By the above formula, and using that $\int d\mathbf{r}|g^{(h)}(\mathbf{r})|\leq C2^{-h}$ and $\int d\mathbf{r}|v_K(\mathbf{r})|\leq C$, we find, if $||f||=\frac{1}{L^2}\int d\underline{\mathbf{r}}|f(\underline{\mathbf{r}})|$ and $|\lambda|\leq\varepsilon$

$$||H^{(h)}_{n,m}||\leq C\varepsilon 2^{h(2-\frac{n}{2}-m)} \tag{3.3}$$

Of course $H^{(h)}_{n,m}$ could be written as sum of Feynman graphs; at order k one would get a bound similar to (3.3) with an extra $k!$, which would spoil convergence. However the bound (3.3) is still not suitable to take the limit $N\to\infty$ as there are terms with non-negative *scaling dimension*, and we have to improve the bounds in such cases. By the properties of the truncated expectation, we can write, if $g^{(h,N)}(\mathbf{x}-\mathbf{y})=\sum_{k=h}^N g^{(k)}(\mathbf{x}-\mathbf{y})$

$$\begin{aligned}&H^{(h)}_{2,0}(\mathbf{x},\mathbf{y})=\int d\mathbf{y}_1\lambda v_K(\mathbf{x}-\mathbf{y}_1)H^{(h)}_{0,1}(\mathbf{y}_1)g^{(h,N)}(\mathbf{x}-\mathbf{y}_2)H^{(h)}_{2,0}(\mathbf{y}_2;\mathbf{y})+\\&\lambda\int d\mathbf{y}_2 v_K(\mathbf{x}-\mathbf{y}_1)g^{(h,N)}(\mathbf{x}-\mathbf{y}_2)H^{(h)}_{2,1}(\mathbf{y},\mathbf{y}_2;\mathbf{y}_1)+\lambda\delta(\mathbf{x}-\mathbf{y})\int d\mathbf{y}_1 v_K(\mathbf{x}-\mathbf{y}_1)H^{(h)}_{0,1}(\mathbf{y}_1)\end{aligned} \tag{3.4}$$

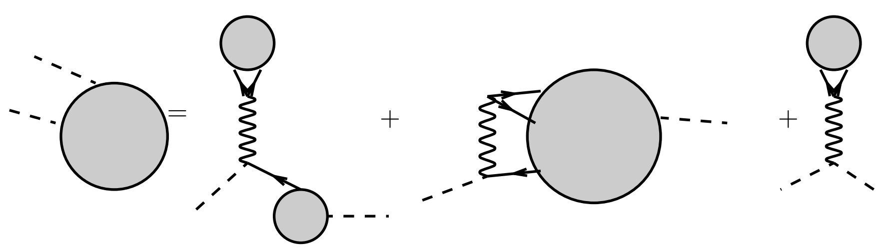

Fig 1: Decomposition of $H^{(h)}_{2,0}$

In the massless case $m=0$ the first and the third addend are vanishing; hence,using that $||g^{(h)}(\mathbf{x})||\leq C2^{-h}$, for $h\geq K$

$$||H^{(h)}_{2,0}||\leq C\varepsilon(\sup|v_K|)2^{-h}||H^{(h)}_{2,1}||\leq C\lambda 2^h 2^{-2h+2K} \tag{3.5}$$

where we have used that $\sup|v_K|\leq C2^{2K}$; we have then a gain $O(2^{-2(h-K)})$ with respect to the bound (3.3), due to the fact that we are integrating over a fermionic instead than over a bosonic line. In the massive case, the first and third addend are not vanishing but a gain $O(m^2Z_4^2 2^{-2h})$ is obtained from the non-diagonal propagator.

Similar arguments can be repeated for all the kernels with $(2-\frac{n}{2}-m)\geq 0$. For instance for $n=0, m=2$ we can decompose the truncated expectation in the terms in which the field attached to J are contracted with the same or with different points.

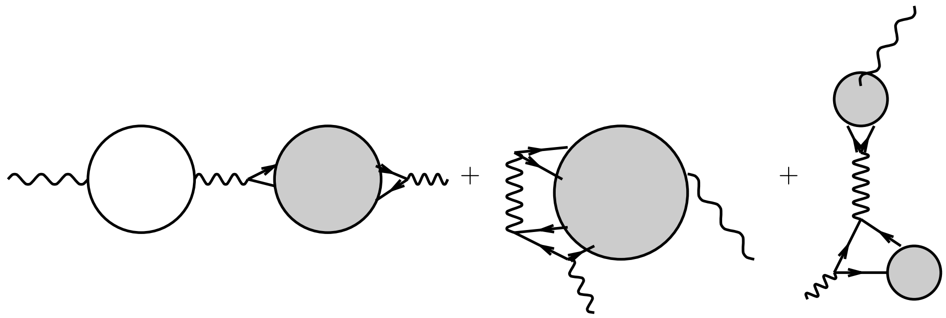

Fig 2: Decomposition of $H^{(h)}_{2,0}$ in the massless case

The terms in the second added in the above picture can be bounded as above, integrating over a fermionic instead that over a fermionic line; they are bounded by $C\varepsilon 2^{2(h-K)}$; a similar bound is obtained also for the third addend for dimensional reason. Regarding the first addend, we can rewrite it as

$$\int d\mathbf{x}d\mathbf{z}\lambda v_K(\bar{\mathbf{z}}-\mathbf{z})[g^{(h,N)}_{\omega,\omega}(\mathbf{x}-\mathbf{z})]^2 H^{(h)}_{0,2}+\int d\mathbf{x}d\mathbf{z}[v_K(\bar{\mathbf{z}}-\mathbf{x})-v_K(\bar{\mathbf{z}}-\mathbf{z})][g^{(h,N)}_{\omega,\omega}(\mathbf{x}-\mathbf{z})]^2\lambda v_K(\mathbf{z}-\mathbf{z}')H^{(h)}_{0,2} \tag{3.6}$$

where, in the massless case, $\int d\mathbf{x}[g^{(h,N)}_{\omega,\omega}(\mathbf{x}-\mathbf{z})]^2=0$ and the second terms has an extra 2^{K-h}.

The above procedure allow us to integrate the fields $\psi^{(N)},\psi^{(N-1)},..,\psi^{(K)}$, the *ultraviolet regime*. The integration of the fields with scale $K-1, K-2, ...$ (the *infrared regime*) is done in a different way, by *localizing* the terms quadratic and quartic in the fields, and adding at each step the quadratic terms to the free fermionic integration, so obtaining

$$e^{-F_h(\phi,J)}\int P_{Z_h}(d\psi^{(\leq h)})e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},\phi,J)}\;, \tag{3.7}$$

where $P_{Z_h}(d\psi^{(\leq h)})$ has propagator similat to (2.2) with χ_N replaced by χ_h, and Z_2, Z_4 replaced by Z_h, Z^m_h. The result of this procedure is that the n-point functions are expressed by expansions in the effective couplings $\lambda_h.., \lambda_N$ which are *convergent* if $\sup_{h\leq j\leq N}|\lambda_h|$ is small enough. The extraction of the local part produces an improvement in the size of the kernels, producing derivatives applied on the external fields, giving an extra 2^h, and factors $(\mathbf{x}-\mathbf{y})$, if $\mathbf{x},\mathbf{y}$ are the coordinate of the external fields; this last factor can be bounded using that $2^h\int d\mathbf{z}|\mathbf{z}||g^{(h)}(\mathbf{z})|\leq C2^{-h}$ or $\int d\mathbf{z}|\mathbf{z}||v(\mathbf{z})|\leq C2^{-K}\leq C2^{-h}$. The major problem is to show that that λ_h and $\frac{Z^{(1)}_h}{Z_h}$ remain close to their initial value; this can be proved with the techniques in [BM], based on Ward Identities combined with Schwinger-Dyson equations at each integration step; the presence of the cut-offs break the local phase invariance and produce corrections in the identities which must be taken into account. The consequence are the following relations

$$\lambda_h=\lambda+O(\lambda^2),\quad Z_h=2^{-\eta h}(1+O(\lambda)),\quad \frac{Z^{(1)}_h}{Z_h}=(1+O(\lambda)),\quad Z^m_h=2^{-\eta_1 h}(1+O(\lambda)) \tag{3.8}$$

As $Z_K=Z_N(1+O(\lambda)), Z^{(1)}_K=Z^{(1)}_N(1+O(\lambda))$, with the choice (2.7) we can remove first the fermionic cut-off $N\to\infty$ and then the bosonic one $K\to\infty$.

In the case of propagator (2.4) we proceed in a similar way except that the cut-off K remain fixed. In the case (2.3) with $K \geq N$ we can perform the limit $K \to \infty$ first and the integration of the scale $N, N-1, ..$ is done using the multiscale integration procedure used in the infrared regime of the previous case, with bare parameters chosen as in (3.8) with $h = N$.

4. Renormalization or non-renormalization of the anomalies

Let us consider for simplicity the case of massless fermions $m = 0$ and propagator (2.3); performing in (2.6) the local phase transformation $\psi_{\mathbf{x}} \to e^{i\alpha_{\mathbf{x}} + i\alpha^5_{\mathbf{x}}\gamma^5}\psi_{\mathbf{x}}$ and $\bar\psi_{\mathbf{x}} \to \bar\psi_{\mathbf{x}} e^{-i\alpha_{\mathbf{x}} + i\alpha^5_{\mathbf{x}}\gamma^5}$ we find

$$\mathbf{p}_\mu \langle j^5_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle_{K,N} = \gamma^5\langle\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle_{K,N} - \gamma^5\langle\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}\rangle_{K,N} + \langle\delta j^5_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle_{K,N} \tag{4.1}$$

where $j^{\mu,5}_{\mathbf{p}} = Z\int\frac{d\mathbf{k}'}{(2\pi)^2}\bar\psi_{\mathbf{k}'}\gamma^\mu\gamma^5\psi_{\mathbf{k}'-\mathbf{p}}$ and $\delta j^5_{\mathbf{p}} = Z\int\frac{d\mathbf{k}'}{(2\pi)^2}C_\mu(\mathbf{k}',\mathbf{k}'-\mathbf{p})\bar\psi_{\mathbf{k}'}\gamma_\mu\gamma^5\psi_{\mathbf{k}'-\mathbf{p}}$ with

$$C_\mu(\mathbf{k}_-,\mathbf{k}_+) = (\chi_N^{-1}(\mathbf{k}_-) - 1)\mathbf{k}_{-,\mu} - (\chi_N^{-1}(\mathbf{k}_+) - 1)\mathbf{k}_{+,\mu} \tag{4.2}$$

The presence of the last term in (2.10) is due to the presence of fermionic cut-off which breaks local invariance. By using that $e\hat v_K(\mathbf{p}) < j^{5,\mu}_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} >= i\varepsilon_{\mu,\nu} < A_{\nu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} >$, and using Weyl notations $\psi_{\mathbf{x}} = (\psi^-_{\mathbf{x},+},\psi^-_{\mathbf{x},-})$, $j^0_{\mathbf{x}} = \sum_{\omega'=\pm}\rho_{\omega',\mathbf{x}}$, $j^1_{\mathbf{x}} = i\sum_{\omega'=\pm}\omega'\rho_{\omega',\mathbf{x}}$, with $\rho_{\omega',\mathbf{x}} = \psi^+_{\omega,\mathbf{x}}\psi^-_{\omega',\mathbf{x}}$ we can write

$$\langle\delta j^5_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle_{K,N} = \alpha_\infty\varepsilon_{\mu,v}i\mathbf{p}_\mu < A_{\nu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}} >_{K,N} + \sum_\omega \omega R_{\omega,K,N}(\mathbf{k},\mathbf{p}) \tag{4.3}$$

where $D_\omega(\mathbf{p}) = -ip_0 + \omega p$ and

$$R_{\omega,K,N}(\mathbf{k},\mathbf{p}) = \frac{\partial^3}{\partial J_{\mathbf{p},\omega}\partial\phi^-_{\mathbf{k}}\partial\phi^+_{\mathbf{k}-\mathbf{p}}}\mathcal{W}_R(J,\phi)|_{0,0} \tag{4.4}$$

with

$$e^{\mathcal{W}_R(J,\phi)} = \int P_Z(d\psi^{(\leq N)})e^{\mathcal{V}^{(N)}(\psi,0,\phi)+\int d\mathbf{p}J_{\mathbf{p},\omega}Z[\delta\rho_{\mathbf{p},\omega}-\nu_+\hat v_K(\mathbf{p})D_+(\mathbf{p})\rho_{\mathbf{p},\omega}-\nu_- D_-\hat v_K(\mathbf{p})(\mathbf{p})\rho_{\mathbf{p},-\omega}]} \tag{4.5}$$

and $\delta\rho_{\mathbf{p},\omega} = \int\frac{d\mathbf{k}'}{(2\pi)^2}C_\omega(\mathbf{k}',\mathbf{k}'-\mathbf{p})\psi^+_{\omega,\mathbf{k}'}\psi^-_{\omega,\mathbf{k}'-\mathbf{p}}$, $C_\omega(\mathbf{k},\mathbf{k}-\mathbf{p}) = D_\omega(\mathbf{k}-\mathbf{p})(\chi_N^{-1}(\mathbf{k}-\mathbf{p}) - 1) - (\chi_N^{-1}(\mathbf{k}) - 1)D_\omega(\mathbf{k})$, $\alpha_\infty = \nu_+ - \nu_-$; ν_+,ν_- are found requiring that $R_{\omega,K,N}(\mathbf{k},\mathbf{p})$ is vanishing removing the cut-offs.

The advantage of writing $R_{\omega,K,N}$ as a functional integral (4.5) is that is can be non-perturbatively evaluated using a multiscale analysis similar to the one of the previous section. Again we start from the case (2.4) and $N \geq K$. The integration of the ultraviolet scale is done as above, and after the integration of $N, N-1, ..h$ the exponent is a series of monomials in the $\psi^{(\leq h)}$, ϕ and J fields integrated over kernels $G^{(h)}_{n,m}$. If $\nu_+ = \nu_- = 0$, it is an immediate consequence of the property $C^\mu(\mathbf{k}_-,\mathbf{k}_+)g^{(i)}(\mathbf{k}_-)g^{(j)}(\mathbf{k}_+) = 0$ for $i,j < N$ the bound

$$||G^{(h)}_{n,m}|| \leq C\varepsilon 2^{h(2-\frac{n}{2}-m)}2^{\frac{1}{2}(h-N)} \tag{4.6}$$

We will choose $\nu_\pm$ to improve the above bound in the case $n = 2, m = 1$. We can write $G^{(h)}_{2,1} = G^{(h)}_{a,2,1} + G^{(h)}_{b,2,1}$ where $G^{(h)}_{a,2,1}$ contans all the terms obtained from the contraction of $\delta\rho_{\mathbf{p},\omega}$. Proceeding as above $G^{(h)}_{a,2,1}$ can be decomposed as in the following picture

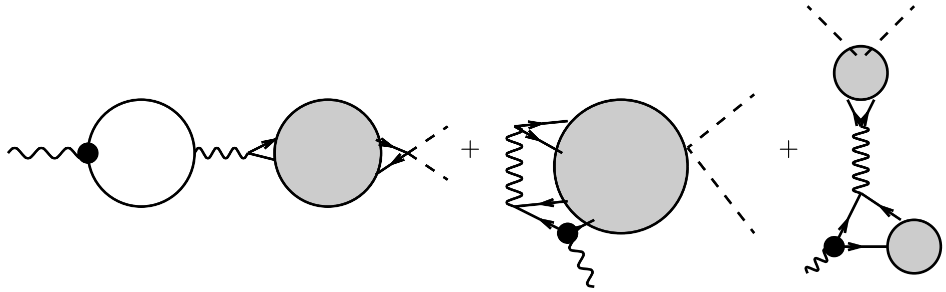

Fig 3: the black dot represents the operator (4.2)

In order to bound the second and the third addend we integrate, as in the previous section,over a fermionic instead that over a fermionic line, so obtaining the bound $\varepsilon 2^{-2(h-K)}2^{\frac{1}{2}(h-N)}$ for $h \ge K$.

Regarding the first term we note that the fermionic loop in the first addend is not vanishing, for the presence of the operator (4.2); its value is given by $-\frac{D_\omega(\mathbf{p})}{4\pi}+r_N$ with r_N vanishing as $N\to\infty$. On the other hand,if we choose $\nu_- = -\frac{1}{4\pi}$ and $\nu_+ = 0$ we see that the terms $G^{(h)}_{b,2,1}$ cancels out with the first addend in fig. 3; the conclusion is that, for $h \ge K$

$$||G^{(h)}_{2,1}|| \le C\varepsilon 2^{h(2-\frac{n}{2}-m)}2^{-2(h-K)}2^{\frac{1}{2}(h-N)} \tag{4.7}$$

From the above bounds it follows that the bound for $R_{\omega,K,N}$ is similar to the one for $\langle\rho_{\omega,\mathbf{p}}\psi^+_{\mathbf{k}-\mathbf{p},\omega'}\psi^-_{\mathbf{k},\omega'}\rangle$ up to an extra factor $2^{\frac{1}{2}(h_{\mathbf{k}}-N)}$, if $h_{\mathbf{k}}$ is the scale of $\mathbf{k}$; hence it follows that

$$\lim_{N\to\infty} R^{(5)}_{K,N}(\mathbf{k},\mathbf{p}) = 0 \tag{4.8}$$

On the other hand in the case $K\to\infty$ at N fixed there is no improvement in the bounds due to the bosonic line. Hence after the integration of $N, N-1, ...h$ we arrive to

$$\int P_{Z_h}(d\psi^{(\le h)})e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi,0)+\int d\mathbf{p}J_{\mathbf{p},\omega}Z_h(\nu_{+,h}D_+(\mathbf{p})\rho_{\mathbf{p},\omega}+\nu_{-,h}D_-(\mathbf{p})\rho_{\mathbf{p},-\omega})+\mathcal{B}^h(J,\psi)}\ , \tag{4.9}$$

where $\mathcal{B}^h(J,\psi)$ is sum of terms with negative scalig dimension. It is possible to control the flow of the coupling $\nu_{\pm h}$ by choosing $\nu_\pm$; indeed by a fixed point argument $\nu_{\pm,h} = O(2^{\frac{1}{2}(h-N)}\varepsilon)$ provided that

$$\nu_- = \frac{\lambda}{4\pi} + O(\lambda^2) \qquad \nu_+ = A\lambda^2 + O(\lambda^3) \tag{4.10}$$

with

$$A = \int \frac{d\mathbf{k}}{(2\pi)^2}\left[\frac{u_0(|\mathbf{k}|)\chi_0(|\mathbf{k}|)}{|\mathbf{k}|^4} - \frac{\chi_0'(|\mathbf{k}|)}{2|\mathbf{k}|^3}\right]\int \frac{d\mathbf{k}''}{(2\pi)^2}g^{(\le N)}_{-\omega}(\mathbf{k}'')g^{(\le N)}_{-\omega}(\mathbf{k}-\mathbf{k}'')D^2_{-\omega}(\mathbf{k}) > 0 \tag{4.11}$$

5. Anomalies and Schwinger-Dyson equation

After inserting the WI in the SD equation (2.13) at finite cut-off, we get (2.14) or (2.16) with

$$\mathcal{B}_{i,K,N}(\mathbf{k}) = g_{\omega,N}(\mathbf{k})\sum_{\varepsilon=\pm}\frac{a_i-\varepsilon\bar{a}_i}{2}\int\frac{d\mathbf{p}}{(2\pi)^2}\frac{\hat{v}_K(\mathbf{p})}{D_{-\omega}(\mathbf{p})}R_{\varepsilon\omega,K,N}(\mathbf{k},\mathbf{p}) \tag{5.1}$$

where $a_i^{-1} = 1 - \nu_- - \nu_+$, $\bar{a}_i^{-1} = 1 + \nu_- + \nu_+$ and $R_{\varepsilon\omega,K,N}(\mathbf{k},\mathbf{p})$ are the corrections (4.4) to the WI appearing in (4.3). We have seen that such corrections vanish removing cut-off and *at fixed momenta*; however in (5.1) the corrections are integrated up to the cut-off scale, precisely where such corrections are not small, so that one is not legitimate to exchange the limit with the integrals. In order to bound (5.1) it is convenient to write it as

$$\mathcal{B}_{i,K,N}(\mathbf{k}) = \sum_{\varepsilon=\pm} \frac{a_i - \varepsilon\bar{a}_i}{2} \frac{\partial^2}{\partial h_{\mathbf{k}} \partial \phi^-_{\mathbf{k}}} \mathcal{W}_{h,\varepsilon,K,N}|_{h=\phi=0} \tag{5.2}$$

where $e^{\mathcal{W}_{h,\varepsilon,K,N}(h,\phi)} =$

$$\int P_Z(d\psi^{(\le N)}) e^{\mathcal{V}^{(N)}(\psi,0,\phi) + Z_K \int \frac{d\mathbf{p}}{(2\pi)^2} g_\omega(\mathbf{k}) h_{\mathbf{k},\omega} \psi_{\mathbf{k}-\mathbf{p},\omega} v_K(\mathbf{p}) [\frac{\delta\rho_{\mathbf{p},\varepsilon\omega}}{D_{-\omega}(\mathbf{p})} - \nu_+ \frac{D_{\varepsilon\omega}(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \rho_{\mathbf{p},\varepsilon\omega} - \nu_- \frac{D_{-\varepsilon\omega}(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \rho_{\mathbf{p},-\varepsilon\omega}]} \tag{5.3}$$

The key point is that the generating function for the correction $\mathcal{W}_{h,\varepsilon,K,N}(h,\phi)$ is very similar to the generating function $\mathcal{W}_{K,N}$ (2.1) at $J = 0$, the main difference being that one external fermionic field comes out from the complex interaction expressed from the exponent in (5.3).

Again we start from the case (2.4) and $N \ge K$. The integration of the ultraviolet scales is done as in §3, and after the integration of $N, N-1, ..h$ the exponent is a series of monomials in the $\psi^{(\le h)}$, ϕ and h fields integrated over kernels $D^{(h)}_{n,m}$, where m is the number of h fields; if $m = 0$ such kernels coincide with the one previously analyzed in §3 and we can concentrate on the case $m = 1$. By the properties of (4.2), if $\nu_\pm = 0$

$$||D^{(h)}_{n,1}|| \le C\varepsilon 2^{h(2-\frac{n+1}{2})} 2^{\frac{1}{2}(h-N)} \tag{5.4}$$

so that, as in §3, we have to improve the bound in the case $n = 1, 3$.

On the other hand, we can decompose the kernels $D^{(h)}_{n,1}$ in the following way

$$D^{(h)}_{n,1} = D^{a(h)}_{n,1} + D^{b(h)}_{n,1} \tag{5.5}$$

where in $D^{a(h)}_{n,1}$ are the terms such that the field $\psi^-_{\mathbf{k}-\mathbf{p},\varepsilon\omega}$ appearing in (5.3) is not contracted, while $D^{b(h)}_{n,1}$ are the terms obtained contracting $\psi^-_{\mathbf{k}-\mathbf{p},\varepsilon\omega}$. The terms in $D^{b(h)}_{n,1}$ obtained from the contraction of $\delta\rho_{\mathbf{p},\varepsilon\omega}$ can be decomposed as in the following picture.

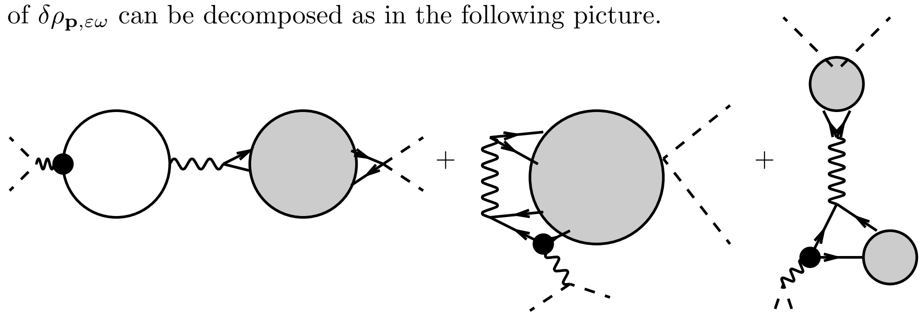

Fig 4: the black dot represents the operator (4.2)

One immediately recognizes that

$$D^{a(h)}_{n-1,1} = G^{(h)}_{n,1} \tag{5.6}$$

so that one can use the analysis in §4 to infer that $G^{(h)}_{n,m}$ verifys (4.7).

On the other hand $D^{b(h)}_{n,1}$ has the structure shown in the following picture.

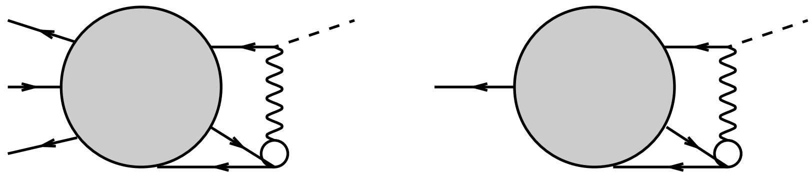

Fig 5: the white box represent $\delta\rho_{\mathbf{p},\omega} - \nu_+ D_+(\mathbf{p})\rho_{\mathbf{p},\omega} - \nu_- D_-(\mathbf{p})\rho_{\mathbf{p},-\omega}$

The terms with $n = 3$ can be written as

$$D^{b(h)}_{3,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \int \lambda v_K(\mathbf{x}_1 - \mathbf{z}_1) g(\mathbf{x}_1 - \mathbf{z}_2) G^{(h)}_{4,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{z}_2; \mathbf{z}_1) \tag{5.7}$$

and by (4.6)

$$||D^{b(h)}_{3,1}|| \leq C\varepsilon(\sup|v_K|) 2^{-2h} 2^{\frac{1}{2}(h-N)} \leq C\varepsilon 2^{2(K-h)} 2^{\frac{1}{2}(h-N)} \tag{5.8}$$

We can proceed in the same way for $D^{b(h)}_{1,1}$ obtaining

$$||D^{b(h)}_{1,1}|| \leq C\varepsilon(\sup|v_K|) 2^{-h} 2^{\frac{1}{2}(h-N)} \leq C\varepsilon 2^{h} 2^{2(K-h)} 2^{\frac{1}{2}(h-N)} \tag{5.9}$$

From the above bounds it follows that the bound for $\mathcal{B}_{1,K,N}(\mathbf{k})$ is similar to the one for $\langle \psi^+_{\mathbf{k},\omega} \psi^-_{\mathbf{k},\omega} \rangle$ up to an extra factor $2^{\frac{1}{2}(h_{\mathbf{k}}-N)}$, if $h_{\mathbf{k}}$ is the scale of $\mathbf{k}$; hence it follows that $\mathcal{B}_{1,K,N}(\mathbf{k}) \to 0$ at $\mathbf{k}$ fixed, as $N \to \infty$.

On the other hand in the case (2.4) and $K \geq N$ there is no the gain in the bound obtained integrating over a bosonic instead that over a fermionic line. Hence after the integration of $N, N-1, ...h$ we arrive to, see [BFM]

$$\int P_{Z_h}(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},\phi,0) + \int \frac{d\mathbf{p}}{(2\pi)^2} g_\omega(\mathbf{k}) h_{\mathbf{k},\omega} \psi_{\mathbf{k}-\mathbf{p},\varepsilon\omega} Z_h [\nu_{+,h} \frac{D_{\varepsilon\omega}(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \rho_{\mathbf{p},\varepsilon\omega} + \nu_{-,h} \frac{D_{-\varepsilon\omega}(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \rho_{\mathbf{p},-\varepsilon\omega}]}$$
$$e^{\tilde{\lambda}_h \frac{Z_h^2}{Z} \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{d\mathbf{p}}{(2\pi)^2} g_\omega(\mathbf{k}) h_{\mathbf{k},\omega} \psi^-_{\mathbf{k}-\mathbf{p},\omega} \rho_{\mathbf{p},\varepsilon\omega} + \int \frac{d\mathbf{k}}{(2\pi)^2} \sum_{i=h}^{N-1} \tilde{z}_i \frac{Z_i}{Z_N} D_\omega(\mathbf{k}) g_\omega(\mathbf{k}) h_{\mathbf{k},\omega} \psi^-_{\mathbf{k},\omega} + \mathcal{B}^h(J,\psi)} \tag{5.10}$$

where $\hat{\mathcal{B}}^h(J,\psi)$ is sum of terms with negative scaling dimension, $\nu_{\pm,h}$ are the effective coupling appearing in (5.3) and $\tilde{\lambda}_k, \tilde{z}_k$ are effective coupling coming from the localization of the terms in Fig.5. While the presence of the counterterms $\nu_\pm$ allow to prove that $\nu_{\pm,h} = O(2^{\frac{1}{2}(h-N)}\varepsilon)$, there are no counterterms for $\tilde{\lambda}_k, \tilde{z}_k$; they have a nontrivial flow verifying $|\tilde{\lambda}_h - \alpha_\varepsilon \lambda_h| \leq C\lambda^2$ and $|\tilde{z}_h - \alpha_\varepsilon z_h| \leq C\lambda^2$, with $\alpha_- = O(\lambda)$ and $\alpha_+ = O(1)$, so that, by (3.8), they are non-vanishing and consequently the limit of $B_{K,N}$ is also non-vanishing in this case.

6. Conclusions

In a non-perturbative functional integral setting, even if both the WI and the Schwinger-Dyson equation are true when cut-off are removed, the closed equation formally obtained by combining such equations is true *only if* the gauge propagator is sufficiently soft in the ultraviolet; the same conditions ensure also the validity of the anomaly non-renormalization. This explains the apparent contradiction between exact results and functional integral analysis.

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