

# Absolute Stability Limit for Relativistic Charged Spheres

Alessandro Giuliani\* and Tony Rothman†

*Princeton University, Princeton NJ 08540*

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## Abstract

We find an exact solution for the stability limit of relativistic charged spheres for the case of constant gravitational mass density and constant charge density. We argue that this provides an absolute stability limit for any relativistic charged sphere in which the gravitational mass density decreases with radius and the charge density increases with radius. We then provide a cruder absolute stability limit that applies to any charged sphere with a spherically symmetric mass and charge distribution. We give numerical results for all cases. In addition, we discuss the example of a neutral sphere surrounded by a thin, charged shell.

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## 1 Introduction

Over the past decade, extremal black holes—black holes for which the charge equals the mass in geometric units—have been the subject of considerable interest, largely because such objects were the ones originally employed to derive the Bekenstein-Hawking entropy directly from string theory[1]. Aside from developments in string

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\*E-mail: giuliani@princeton.edu. On leave from Dipartimento di Matematica, Università di RomaTre, Largo San Leonardo Murialdo 1, 00146, Roma, Italy.

†E-mail: trothman@princeton.edu.

theory, however, there has long been ample motivation to study extremal black holes because from the classical and semi-classical point of view they provide the “zero-temperature” limit in black hole thermodynamics. At the same time, substantial evidence suggests that one should not view extremal black holes as any sort of continuous limit of their sub-extremal counterparts, black holes for which the charge is less than the mass. For example, the horizon structure of classical, charged black holes changes completely at extremality[2]. Some studies have also concluded that entropy is not well-defined for extremal black holes[3, 4]. More definitely, one knows from Israel’s proof of the third law of black hole dynamics[5] that extremality cannot be attained in a finite time, and that the conclusion holds even under Hawking radiation and superradiance, which violate the assumptions of Israel’s proof[3, 6]. Thus, it appears impossible to create an extremal black hole from a subextremal one, and the only remaining possibility is to produce one from the collapse of an already extremal object.

For this reason it is of interest to investigate the stability of relativistic charged spheres. Previous studies along these lines have been mainly numerical[7, 6] and have concluded that while for  $Q < M$  collapse always takes place at a critical radius  $R_c$  outside the horizon, as  $Q$  approaches  $M$ , this critical radius approaches the horizon itself,  $R_+$ . The present paper is intended as an analytic companion to the numerical investigations. Our point of departure is the classic proof of Buchdahl[8], who showed that for uncharged stars gravitational collapse into a black hole will always take place when  $R < (9/4)M$ , regardless of equation of state.<sup>1</sup> (See Weinberg’s *Gravitation and Cosmology*[10] for a clear presentation of Buchdahl’s argument, or §3 of this paper.) Because in a charged sphere Coulomb repulsion tends to oppose the gravitational force,  $R_c$  should be less than  $(9/4)M$ . However, relativistically, the charge increases gravitational energy as well, and so at some point gravity always wins out and col-

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<sup>1</sup>Throughout we use units in which  $G = c = 1$ .

lapse into a black hole takes place. At  $Q = M$ , the Coulomb repulsion equals the gravitational force and one finds numerically that  $R_c = R_+$ . Thus one should have  $M \leq R_c < (9/4)M$ , always.

Although for a given charge distribution one can indeed find  $R_c$  numerically, one does suspect that there must be an analytic proof, analogous to Buchdahl's, that applies to relativistic charged spheres.<sup>2</sup> In other words, given a value of  $Q/M$  we should be able to find an absolute bound on  $R/M$ , independent of other physical parameters, below which the object collapses into a black hole. Anninos and Rothman[6] (henceforth AR) intended to include such a proof as a supplement to their numerical investigation but as that project neared completion they learned that Yunqiang and Siming (henceforth YS) had already claimed to have given such a proof[9]. The YS demonstration, however, is far from transparent and does not appear to have ever been published. Moreover it does not provide a sharp value for the collapse radius, as in Buchdahl's proof, but rather gives a general lower bound on it. The important feature of this lower bound is that for  $Q < M$  it is *always larger than*  $R_+$ , as expected from numerics. Not long ago we decided to take the opportunity to present a simplified version of this interesting result. In the process we have found an exact solution for the case of constant mass and charge densities, and this allows us, in a manner complete analogous to Buchdahl's, to put an absolute (sharp) stability limit on a very large class of objects, all those with charge density increasing radially, and gravitational mass density decreasing radially. This stability limit, which should cover essentially all cases of interest, is the main result of our paper. For the remaining cases, we present a proof similar to that of YS, but we hope with greater clarity, and give explicit numerical results for a lower bound on  $R_c$ . We also give an exact solution for the stability limit of a neutral sphere surrounded by a charged shell.

The paper is organized as follows. In Section 2 we introduce the relevant Einstein

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<sup>2</sup>In this paper we tend to speak of charged spheres rather than charged stars, as there is no good reason to think that charged stars, in the usual sense of the word, exist.

equations and introduce notation. In Section 3 we review the  $Q = 0$  case and derive an exact solution for the case of a neutral sphere surrounded by a charged shell. Section 4 is devoted to the main result of our paper: we solve exactly the case of constant charge density and constant gravitational mass density, derive its critical stability radius and show that it gives an absolute bound on the critical stability radii for all spherically symmetric distributions in which gravitational mass density gradient is negative and the charge density gradient is positive. In Section 5 we calculate a general lower bound on the critical stability radius. Finally in Section 6 we summarize the results and draw conclusions.

## 2 Einstein Equations

As mentioned above, the plan is to find an absolute stability limit on  $R/M$  for relativistic charged spheres that is independent of the equation of state and depends only on  $Q/M$ . We will assume throughout that the pressure  $p$  and density  $\rho$  are both positive, that the charge density is positive and that  $Q \leq M$  in order to avoid naked singularities; this last assumption ensures that spacetime is asymptotically predictable[2].

We also restrict attention to spherically symmetric mass and charge distributions, for which the metric can be written in the form

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (2.1)$$

where the metric components  $e^{2\Phi(r)}$  and  $e^{2\Lambda(r)}$  are positive.

As is well-known, the classic Reissner-Nordström (RN) solution for the charged spherically symmetric case gives

$$e^{-2\Lambda(r)} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = e^{+2\Phi(r)}, \quad (2.2)$$

where  $r \geq R$  and  $R$  is the outer radius of the sphere. The Schwarzschild-Droste (SD) solution is of course recovered by setting  $Q = 0$ . Both the RN and SD, however,

are vacuum solutions, concerned solely with the metric outside  $R$ . For the collapse problem we need to study the behavior of the metric functions  $\Lambda(r)$  and  $\Phi(r)$  for  $r < R$ , where the pressure, the charge and mass densities are nonzero. The procedure for solving the “interior Reissner-Nordström equations” is nevertheless much the same as for the exterior case. One assumes (see AR or de Felice[7] for more details) a perfect-fluid stress-energy tensor for the hydrodynamic part

$$(T_{\mu\nu})_{hydro} = (p + \rho)u_\mu u_\nu + pg_{\mu\nu}, \quad (2.3)$$

while for the electromagnetic part

$$4\pi(T_{\mu\nu})_{EM} = F_\mu^\alpha F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = \frac{q^2(r)}{8\pi r^4} \text{diag}[e^{2\Phi}, -e^{2\Lambda}, r^2, r^2\sin^2\theta]. \quad (2.4)$$

In these expressions  $\rho = \rho_{rm} + e$  is the total mass density,  $\rho_{rm}$  is the rest mass density,  $e$  is the internal energy density,  $p$  is the fluid pressure,  $u_\mu$  is the four-velocity and  $F_{\mu\nu}$  is the electromagnetic field strength tensor.  $(T_{\mu\nu})_{EM}$  here is of the same form as for the exterior RN solution, as it must be by Gauss’s law, except that instead of the total charge  $Q$ , we now have a  $q(r)$ , the charge within any given radius  $r$ . Indeed, by definition

$$q(r) = 4\pi \int_0^r e^{\Phi(r')+\Lambda(r')} r'^2 j^0(r') dr', \quad (2.5)$$

where  $j^0$  is the charge density. (This is the usual definition of charge, modified only for metric curvature.) The boundary condition requires  $Q = q(R)$ .

With the above stress-energy tensor, the (00) Einstein equation is found to be

$$\Phi'' + \Phi'^2 - \Phi'\Lambda' + \frac{2\Phi'}{r} = 4\pi e^{2\Lambda} \left[ \rho + 3p + \frac{q^2(r)}{4\pi r^4} \right], \quad (2.6)$$

where  $\prime$  denotes derivatives with respect to  $r$ . Similarly, the (11) equation is

$$-\Phi'' - \Phi'^2 + \Phi'\Lambda' + \frac{2\Lambda'}{r} = 4\pi e^{2\Lambda} \left[ \rho - p - \frac{q^2(r)}{4\pi r^4} \right], \quad (2.7)$$

and the (22) equation is

$$e^{2\Lambda} - 1 + \Lambda'r - \Phi'r = 4\pi r^2 e^{2\Lambda} \left[ \rho - p + \frac{q^2(r)}{4\pi r^4} \right]. \quad (2.8)$$

The left-hand-side of these equations is necessarily the same as for the exterior SD or RN solutions; only the right-hand-side differs because of the nonzero stress-energy tensor. Following the standard procedure for deriving the exterior solutions, we can take linear combinations of Eqs. (2.6), (2.7) and (2.8) to eliminate the terms in  $\Phi$ . One easily finds that for any  $r \leq R$

$$e^{-2\Lambda(r)} = 1 - \frac{2m_i(r)}{r} - \frac{\mathcal{F}(r)}{r}, \quad (2.9)$$

where

$$m_i(r) \equiv 4\pi \int_0^r \rho r'^2 dr' \quad \text{and} \quad \mathcal{F}(r) \equiv \int_0^r \frac{q^2(r')}{r'^2} dr'. \quad (2.10)$$

Here,  $m_i(r)$  is the usual definition of the mass within a radius  $r$ . The subscript  $i$  denotes “internal” to emphasize that  $m_i$  contains both rest and internal energy. We use the designation because it will become necessary to distinguish  $m_i(r)$  from the gravitational mass  $m_g(r)$ , defined momentarily. Requiring that (2.9) matches the exterior solution (2.2) at  $r = R$ , gives

$$1 - \frac{2M}{R} + \frac{Q^2}{R^2} = 1 - \frac{1}{R} \int_0^R (8\pi\rho r^2 + \frac{q^2}{r^2}) dr, \quad (2.11)$$

or

$$M = \frac{1}{2} \int_0^R (8\pi\rho r^2 + \frac{q^2}{r^2}) dr + \frac{Q^2}{2R}, \quad (2.12)$$

which defines the gravitational mass at  $R$  (the mass measured by a satellite in orbit around the object). By Gauss’s law, however, the same must be true at any radius, and so using the definition of  $\mathcal{F}$  from Eq. (2.10),

$$m_g(r) = m_i(r) + \frac{\mathcal{F}(r)}{2} + \frac{q^2(r)}{2r}, \quad (2.13)$$

which clarifies the distinction between  $m_i$  and  $m_g$ . In terms of the gravitational mass, the metric function  $e^{\Lambda(r)}$  is

$$e^{\Lambda(r)} = \left(1 - \frac{2m_i(r)}{r} - \frac{\mathcal{F}(r)}{r}\right)^{-1/2} = \left(1 - \frac{2m_g(r)}{r} + \frac{q^2(r)}{r^2}\right)^{-1/2}. \quad (2.14)$$

One can write these functions either in terms of  $m_i$  and  $\mathcal{F}$ , or  $m_g$  and  $q$ , but because we do not in general know the charge distribution  $q(r)$  and hence  $\mathcal{F}(r)$ , when thinking about boundary conditions it is much more convenient to use  $m_g$ , since in that case  $e^{\Lambda(r)}$  matches onto  $e^{\Lambda(R)}$  in the expected way. We will therefore generally use the second form.

The pressure in these equations can be eliminated by taking three times Eq. (2.7) and adding it to Eq. (2.6), which yields

$$\Phi'' + \Phi'^2 - \Phi'\Lambda' - \frac{\Phi'}{r} = \frac{3\Lambda'}{r} - \left(8\pi\rho - \frac{q^2}{r^4}\right) e^{2\Lambda}.$$

Upon multiplication by  $e^{-\Lambda+\Phi}/r$ , the left-hand-side turns out to be an exact differential, and so, letting  $\zeta(r) \equiv e^{\Phi(r)}$  as in Weinberg's notation [10], one has

$$\left(\frac{1}{r}e^{-\Lambda}\zeta'\right)' = \left[\frac{3\Lambda'e^{-2\Lambda}}{r^2} - \frac{8\pi\rho}{r} + \frac{q^2}{r^5}\right] e^{\Lambda}\zeta. \quad (2.15)$$

Eq. (2.15) will prove to be the fundamental equation of our analysis. It can be brought into perhaps more familiar form by noting by that  $8\pi\rho/r = 2m'_i(r)/r^3$  and  $\mathcal{F}'(r) = q^2/r^2$ . Then

$$\left(\frac{1}{r}e^{-\Lambda}\zeta'\right)' = e^{\Lambda(r)} \left[ \left(\frac{m_i(r)}{r^3}\right)' + \frac{1}{2r^3} \left(\frac{5q^2(r)}{r^2} - \frac{3\mathcal{F}(r)}{r}\right) \right] \zeta. \quad (2.16)$$

This is the equivalent of Weinberg's Eq.(11.6.14), employed in Buchdahl's proof of the absolute limit of stability for ordinary stars. Our equation, however, contains two more terms within the square brackets than the usual one, as well as an extra term in the expression (2.14) defining  $e^{\Lambda}$ . We point out that although  $e^{\Lambda(r)}$  has the simple form given in Eq. (2.14), no such closed form exists for  $\zeta(r)$  for  $r \leq R$ . Indeed the differential equation (2.15) should be regarded as the equation defining  $\zeta$  in the interior of the sphere *for given input distributions*  $\rho$  and  $q$ .

Requiring that  $\zeta$  and  $\zeta'$  match on to the exterior RN solution at  $r = R$  gives the following important boundary conditions:

$$\zeta(R) = \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{1/2}$$

$$\zeta'(R) = \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{-1/2} \left(\frac{M}{R^2} - \frac{Q^2}{R^3}\right), \quad (2.17)$$

with  $M$  the gravitational mass in (2.12).

Once  $\zeta$  and  $\Phi$  have been computed using (2.15), one can easily show from (2.8), (2.9) and (2.13) that the pressure is given in terms of  $\Phi'$ ,  $m_g$  and  $q$  by

$$p = \frac{1}{4\pi r^2} \left( \Phi' r e^{-2\Lambda} - \frac{m_g}{r} + \frac{q^2}{r^2} \right). \quad (2.18)$$

Clearly, in order for the solution of (2.16) to be physically acceptable, we must require that  $\zeta > 0$  and that  $p$ , as computed from (2.18), is nonnegative and satisfies a proper equation of state, usually assumed of the form  $p = p[\rho_{rm}]$ . For a given class of distributions  $\rho, q$  we define the critical stability radius  $R_c$  as the smallest possible radius for which a physically acceptable solution to (2.15) can be found.

Then, in order to find a bound on the stability of the charged star, it is sufficient to show that if the radius is smaller than  $R_c$ , then no physically acceptable solution exists, whatever the choice of  $\rho, q$ .

### 3 Neutral Sphere Surrounded by Charged Shell

In this section we first review Buchdahl's  $Q = 0$  case, which illustrates the general strategy for finding  $R_c$ . The question then naturally arises as to whether the result changes for a neutral sphere surrounded by a charged shell. We show that the basic  $Q = 0$  scenario is fairly easily adapted to cover this case.

For the moment, then, let us set  $Q = 0$  and drop the subscript  $i$  on  $m$  (there is now no distinction between between  $m_g$  and  $m_i$ ). Eq. (2.16) then becomes

$$\left(\frac{1}{r} e^{-\Lambda} \zeta'\right)' = \left(1 - \frac{2m(r)}{r}\right)^{-1/2} \left[\left(\frac{m(r)}{r^3}\right)'\right] \zeta, \quad (3.1)$$

where we have used  $e^{-2\Lambda(r)} = 1 - 2m(r)/r$ .



Let us now assume that for any physically reasonable star  $\rho' \leq 0$ . Then, because  $(m/r^3)' = 4\pi r^{-4} \int_0^r \rho'(x)x^3 dx \leq 0$ , it follows that

$$\left(\frac{1}{r}\left(1 - \frac{2m}{r}\right)^{1/2}\zeta'\right)' \leq 0 \quad (3.2)$$

Integrating this expression from  $r$  to  $R$  gives

$$\frac{1}{R}\left(1 - \frac{2M}{R}\right)^{1/2}\zeta'(R) - \frac{1}{r}\left(1 - \frac{2m}{r}\right)^{1/2}\zeta'(r) \leq 0. \quad (3.3)$$

We now make use of the boundary conditions Eq. (2.17) with  $Q = 0$ . Inserting the expression for  $\zeta'(R)$  into Eq. (3.3) gives

$$\zeta'(r) \geq \frac{Mr}{R^3}\left(1 - \frac{2m(r)}{r}\right)^{-1/2}. \quad (3.4)$$

Integrating again from 0 to  $R$  yields

$$\zeta(R) - \zeta(0) \geq \frac{M}{R^3} \int_0^R \frac{r dr}{\left(1 - \frac{2m(r)}{r}\right)^{1/2}}. \quad (3.5)$$

with  $\zeta(R) = \left(1 - \frac{2M}{R}\right)^{1/2}$ . Now, in order to have  $\zeta(0) > 0$ , we require that

$$0 < \zeta(0) \leq \left(1 - \frac{2M}{R}\right)^{1/2} - \frac{M}{R^3} \int_0^R \frac{r dr}{\left(1 - \frac{2m(r)}{r}\right)^{1/2}}. \quad (3.6)$$

Note that, as remarked above,  $(m/r^3)' \leq 0$ , and so  $m/r^3 \geq M/R^3$  for all  $r \leq R$ .

Plugging this into (3.6) we find:

$$0 < \left(1 - \frac{2M}{R}\right)^{1/2} - \frac{M}{R^3} \int_0^R \frac{r dr}{\left(1 - \frac{2Mr^2}{R^3}\right)^{1/2}}. \quad (3.7)$$

The integral is now trivially performed to get

$$0 < \left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2} \left[1 - \left(1 - \frac{2M}{R}\right)^{1/2}\right], \quad (3.8)$$

which immediately implies Buchdahl's result  $R > (9/4)M$ .

Note that for stars with constant density all the above inequalities become equalities at the critical radius, and so the value  $(9/4)M$  is precisely their critical stability

radius.<sup>3</sup> The value  $R_c = (9/4)M$  gives an absolute (sharp!) stability limit for all stars with distributions satisfying  $\rho' \leq 0$ . If any such star is compressed to the point that  $R < R_c$  gravitational collapse necessarily takes place.

Let us now modify the above computation to handle the case of a neutral sphere of constant density surrounded by a thin shell of internal (“inertial”) mass  $K$  that carries a uniformly distributed charge  $Q \leq M$ . In such a situation, Eq. (2.14) shows that  $\Lambda$  suffers a discontinuity at  $r = R$ . Let  $R^-$  and  $R^+$  represent the inner and outer radii of the shell,  $M_{int}$  be the mass interior to the shell and  $M_s = K + Q^2/2R$  be the gravitational mass of the shell (cf. Eq. (2.12)). Then  $M = M_{int} + M_s$  is the total mass and  $e^{-2\Lambda(R^-)} = 1 - 2M_{int}/R$ , while  $e^{-2\Lambda(R^+)} = 1 - 2M/R + Q^2/R^2$ . Since the jump in  $\Lambda$  is finite, however, Eq. (2.18) implies that any discontinuity in  $\Phi'$  and hence in  $\zeta'$  is finite as well. Thus  $\zeta$  itself is continuous at the boundary with precisely the value given by the first of Eqs. (2.17).

One can greatly simplify the calculations by assuming that  $K = 0$ , in which case  $\Lambda$  and  $\Phi$  are both *continuous* at  $r = R$  and only their derivatives suffer a discontinuity at the surface. (For a more detailed discussion of these issues we refer the reader to Cohen and Cohen [11], who derive the solution for a thin charged shell of radius  $R$ , with  $M_{int} = 0$ .)

So, let us take  $K = 0$  and assume that interior to the shell  $\rho = 3M_{int}/(4\pi R^3)$ , with  $M_{int} = M - Q^2/(2R)$ . In order for the shell to be stable against gravitational collapse, it is necessary to have a nonzero elastic stress tensor concentrated on the surface, as assumed in [11]. This means that the stress energy tensor must be modified by the addition of a term  $(T_{\mu\nu})_{el}$  whose only nonzero elements are  $T_{\theta\theta} = r^2 S\delta(r - R)$  and  $T_{\phi\phi} = r^2 \sin^2\theta S\delta(r - R)$ , where  $S$  is the elastic energy and the delta function

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<sup>3</sup>It is straightforward to check that in the uncharged case, if  $\zeta > 0$ , then  $p > 0$  and finite,  $\forall r < R$ . This means that the only condition to be imposed for the solution to be physically acceptable is  $\zeta > 0$ , in other words, precisely the condition we imposed above.

is normalized such that  $\int dr 4\pi r^2 \delta(r - R) = 1$ . The presence of  $(T_{\mu\nu})_{el}$  modifies the Einstein equations as follows: Eq. (2.6) contains an extra term  $8\pi e^{2\Lambda} S \delta(r - R)$  on the right hand side; Eq. (2.7) contains an extra term  $-8\pi e^{2\Lambda} S \delta(r - R)$  on the right hand side; Eq. (2.8) is unchanged. With these additions Eq. (2.16) becomes

$$\left(\frac{1}{r}e^{-\Lambda}\zeta'\right)' = e^{\Lambda(r)} \left[ \left(\frac{m_i(r)}{r^3}\right)' + \frac{1}{2r^3} \left(\frac{5q^2(r)}{r^2} - \frac{3\mathcal{F}(r)}{r}\right) + \frac{8\pi}{r} S \delta(r - R) \right] \zeta, \quad (3.9)$$

where

$$e^{-2\Lambda(r)} = \begin{cases} 1 - 2m_i(r)/r & r < R \\ 1 - 2M/r + Q^2/r^2 & r > R \end{cases} \quad (3.10)$$

and  $\zeta = e^{-\Lambda}$  for  $r \geq R$ . Integrating both sides of Eq. (3.9) between  $R^-$  and  $R^+$ , and using the fact that  $\Lambda$  and  $\Phi$  are both continuous at  $r = R$ , we see that  $e^{-\Lambda(R)}(\zeta'(R^+) - \zeta'(R^-)) = 2\zeta e^\Lambda S/R^2$ , or

$$\zeta'(R^-) = \frac{1}{\sqrt{1 - 2M_{int}/R}} \left[ \frac{M_{int}}{R^2} - \frac{Q^2}{2R^3} - \frac{2S}{R^2} \right]. \quad (3.11)$$

Computing  $\zeta'(R^-)$  from Eq. (2.18) yields

$$\zeta'(R^-) = \frac{1}{\sqrt{1 - 2M_{int}/R}} \left[ \frac{M_{int}}{R^2} + 4\pi R p_- \right], \quad (3.12)$$

and comparing the two expressions shows that

$$4\pi R p_- = -\frac{Q^2}{2R^3} - \frac{2S}{R^2}. \quad (3.13)$$

The parameter  $S$  should be chosen such that  $p_- \geq 0$ .

We can now proceed as in the  $Q = 0$  case, integrating  $\left(\frac{1}{r}e^{-\Lambda}\zeta'\right)' = 0$  from 0 to  $R^-$  with the new boundary conditions  $\zeta(R^-) = \sqrt{1 - 2M_{int}/R}$  and  $\zeta'(R^-) = \frac{1}{\sqrt{1 - 2M_{int}/R}} \left[ \frac{M_{int}}{R^2} + 4\pi R p_- \right]$ . The result is that the new critical radius is smallest when  $p_- = 0$ , in which case  $S = -Q^2/4R$ , consistent with the result in [11] for  $K = 0$ . We then find that  $R_c$  is precisely  $(9/4)M_{int}$ , as one might expect from Gauss's law. With  $M_{int} = M - Q^2/(2R_c)$ , solving for  $R_c$  in terms of  $M$  and  $Q$  gives

$$R_c = \frac{9}{8} \left( M + \sqrt{M^2 - \frac{8}{9}Q^2} \right). \quad (3.14)$$

For  $Q = 0$ ,  $R_c = 9/4M$ , as expected, and a nonzero  $Q$  indeed lowers  $R_c$ . The extremal case,  $Q = M$ , gives  $R_c = 3/2M$ , which is plausible, as below we will find that for the full extremal charged sphere  $R_c = M$ .

## 4 Constant Density Case

The special case of perhaps greatest interest (and the easiest one to handle), is that of constant density,  $m_g \propto r^3$  and  $q \propto r^3$ . Remarkably, we are able to find an exact solution in this situation. Moreover, because a neutral test particle senses the gravitational mass  $m_g$  within a radius  $r$ , it is evidently  $m_g$  that plays the role  $m_i$  did in the  $Q = 0$  case. In other words, it is  $m_g$  that determines the weight of material in the sphere, and a physically reasonable requirement for stability is that  $\rho'_g \leq 0$ , where  $\rho_g$  is the gravitational mass density. If we additionally impose the requirement that the charge density is positive and increases outwards (that is,  $q' \geq 0$  and  $(q/r^3)' \geq 0$ ), which also seems reasonable if like charges repel, then we will also be able to find, in complete analogy with  $Q = 0$  case, an absolute lower bound on the critical radius of any charged sphere meeting the two conditions.

We begin by rewriting the fundamental equation (2.16) in terms of  $m_g$ :

$$\left(\frac{1}{r}e^{-\Lambda}\zeta'\right)' = \left(1 - \frac{2m_g(r)}{r} + \frac{q^2(r)}{r^2}\right)^{-1/2} \left[\left(\frac{m_g}{r^3}\right)' - q\left(\frac{q}{r^4}\right)'\right]\zeta \quad (4.1)$$

With the *ansatz* that  $m_g = M(r/R)^3$  and  $q = Q(r/R)^3$ , the first term in the square brackets vanishes and Eq. (4.1) becomes

$$\left(\frac{1}{r}e^{-\Lambda_c}\zeta'\right)' = e^{\Lambda_c}\frac{Q^2r}{R^6}\zeta, \quad (4.2)$$

where now

$$e^{-\Lambda_c} = \sqrt{1 - 2\frac{M}{R}\left(\frac{r}{R}\right)^2 + \frac{Q^2}{R^2}\left(\frac{r}{R}\right)^4}. \quad (4.3)$$

Here and in what follows the subscript  $c$  refers to “constant-density case.”

Let us define a new variable  $\tilde{\zeta}$  such that

$$\tilde{\zeta}(f_c(r)) = \zeta(r) \quad ; \quad f_c(r) \equiv R^{-2} \int_0^r dx x e^{\Lambda_c(x)}. \quad (4.4)$$

Substituting (4.4) into Eq. (4.2) gives at once

$$\frac{d^2 \tilde{\zeta}}{df_c^2} = \frac{Q^2}{R^2} \tilde{\zeta}, \quad (4.5)$$

which has the obvious solution

$$\tilde{\zeta}_c(f_c) = c_1 e^{Qf_c/R} + c_2 e^{-Qf_c/R}. \quad (4.6)$$

Moreover,  $f_c(r)$  is a standard integral:

$$\begin{aligned} f_c(r) &= \int_0^r dx \frac{x/R^2}{\sqrt{1 - 2\frac{M}{R} \left(\frac{x}{R}\right)^2 + \frac{Q^2}{R^2} \left(\frac{x}{R}\right)^4}} = \frac{R}{2M} \int_0^{Mr^2/R^3} \frac{dy}{\sqrt{1 - 2y + \left(\frac{Q^2}{M^2}\right)y^2}} \\ &= -\frac{R}{2Q} \log \left( \frac{M}{Q} - \frac{Q}{M}y + \sqrt{1 - 2y + \left(\frac{Q^2}{M^2}\right)y^2} \right) \Big|_0^{Mr^2/R^3}, \end{aligned} \quad (4.7)$$

or

$$f_c(r) = \frac{R}{2Q} \log \frac{M/Q + 1}{M/Q - Qr^2/R^3 + e^{-\Lambda_c(r)}}. \quad (4.8)$$

Imposing the boundary conditions  $\tilde{\zeta}_c(f_c(R)) = e^{-\Lambda_c(R)}$  and  $d\tilde{\zeta}_c(f_c(R))/df_c = M/R - Q^2/R^2$  we find after some algebra

$$\begin{aligned} c_1 &= \frac{1}{2} \frac{(M/Q - Q/R + e^{-\Lambda_c(R)})^{3/2}}{\sqrt{M/Q + 1}} \\ c_2 &= -\frac{1}{2} \left( M/Q - Q/R - e^{-\Lambda_c(R)} \right) \sqrt{\frac{M/Q + 1}{M/Q - Q/R + e^{-\Lambda_c(R)}}}, \end{aligned} \quad (4.9)$$

and so, finally, the exact solution for  $\zeta$  is

$$\zeta_c(r) = \frac{1}{2} \left\{ \frac{(M/Q - Q/R + e^{-\Lambda_c(R)})^{3/2}}{\sqrt{M/Q - Qr^2/R^3 + e^{-\Lambda_c(r)}}} - \right.$$

$$-\left(M/Q - Q/R - e^{-\Lambda_c(R)}\right) \sqrt{\frac{M/Q - Qr^2/R^3 + e^{-\Lambda_c(r)}}{M/Q - Q/R + e^{-\Lambda_c(R)}}} \quad (4.10)$$

The condition for this solution to be physical is  $\zeta_c(0) > 0$ .<sup>4</sup> As in the  $Q = 0$  case we get the equation for the critical radius by setting  $\zeta_c(0) = 0$ , which yields:

$$\left(M/Q - Q/R + e^{-\Lambda_c(R)}\right)^2 = (M/Q + 1) \left(M/Q - Q/R - e^{-\Lambda_c(R)}\right) \quad (4.11)$$

One easily sees from Eq. (4.11) that  $Q = M$  implies  $R_c = M$ , as claimed in the Introduction. For other values of  $Q/M$  we solve this equation for  $R/M$ . Letting  $\mu = M/R$  and  $\sigma = Q/M$  in Eq. (4.11), we find after some further algebra:

$$4\sigma^4\mu^3 - 12\sigma^2\mu^2 + (9 + 3\sigma^2)\mu - 4 = 0. \quad (4.12)$$

Thus the exact solution for the critical radius boils down to finding the roots of this cubic equation for  $\mu$  in terms of  $\sigma$ . One immediately sees that  $Q = 0$  implies that  $R_c/M = 9/4$ . Numerical results for various values of  $Q/M$  are given in Table 1 and plotted in Figure 1.

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<sup>4</sup>A straightforward computation shows that if  $\zeta_c > 0$  then  $p$ , as computed from (2.18), is automatically positive and finite, as expected.

$Q/M$	$R_c/M$
0	2.250
.1	2.244
.2	2.226
.3	2.196
.4	2.152
.5	2.093
.6	2.016
.7	1.915
.8	1.781
.9	1.586
.99	1.224
.999	1.091
.9999	1.039

Table 1. The stability limit  $R_c/M$  for the constant-density sphere, tabulated for various values of  $Q/M$ .

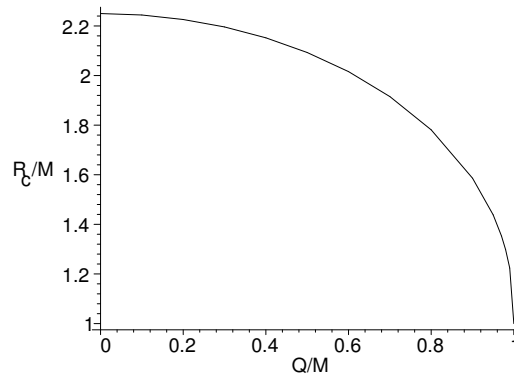


Figure 1: The stability limit  $R_c/M$  plotted against  $Q/M$  for the constant density case.

Thus, we have found an exact solution for the constant density case. We now argue, as indicated earlier, that any physically reasonable solution should have  $\rho'_g \leq 0$  and, for a positive charge density,  $(q/r^3)' \geq 0$ . If so, the constant density case maximizes the expression in square brackets in the right-hand-side of Eq. (4.1) and then  $\left((1/r)e^{-\Lambda}\zeta'\right)' \leq e^\Lambda(Q^2r/R^6)\zeta$ , as long as  $\zeta \geq 0$ . In terms of  $\tilde{\zeta}$  and

$f = R^{-2} \int_0^r dx x e^{\Lambda(x)}$ , Eq. (4.5) is now replaced by the inequality

$$\frac{d^2 \tilde{\zeta}}{df^2} \leq \frac{Q^2}{R^2} \tilde{\zeta}. \quad (4.13)$$

Note that now  $f \neq f_c$ , because the  $e^\Lambda$  appearing in the definition of  $f$  is no longer that of the constant-density solution (4.3) but instead the general  $(1 - 2m_g/r + q^2/r^2)^{-1/2}$ .

As discussed in Appendix 1, the differential inequality (4.13) implies that  $\tilde{\zeta} \leq \tilde{\zeta}_0$ , for all  $f \leq f(R)$ , where  $\tilde{\zeta}_0(f)$  is the solution to the differential equation  $d^2 \tilde{\zeta}_0/df^2 = (Q^2/R^2) \tilde{\zeta}_0$  satisfying the same boundary conditions as  $\tilde{\zeta}$  does. One finds that  $\tilde{\zeta}_0(f) = c_1^0 e^{Qf/R} + c_2^0 e^{-Qf/R}$ , with

$$\begin{aligned} c_1^0 &\equiv \frac{1}{2} e^{-\frac{Q}{R}f(R)} \left( \frac{M}{Q} - \frac{Q}{R} + e^{-\Lambda(R)} \right) \\ c_2^0 &\equiv - \frac{1}{2} e^{\frac{Q}{R}f(R)} \left( \frac{M}{Q} - \frac{Q}{R} - e^{-\Lambda(R)} \right). \end{aligned} \quad (4.14)$$

Note that the conditions  $\rho'_g \sim (m_g/r^3)' \leq 0$  and  $(q/r^3)' \geq 0$  imply that  $e^{-2\Lambda(r)}$  is always smaller than  $e^{-2\Lambda_c} = (1 - 2Mr^2/R^3 + Q^2r^4/R^6)$ . Therefore  $\tilde{\zeta}_0(f(r)) \leq \tilde{\zeta}_c(f_c(r))$ .

The conclusion is that  $\tilde{\zeta}(f(r)) \leq \tilde{\zeta}_0(f(r)) \leq \tilde{\zeta}_c(f_c(r))$  or, equivalently  $\zeta(r) \leq \zeta_c(r)$ , where  $\zeta(r)$  is the general solution to (4.1). Consequently, the critical radius for any distribution with  $\rho'_g \leq 0$ ,  $q' \geq 0$  and  $(q/r^3)' \geq 0$  is always larger than the critical stability radius plotted in Fig.1. The constant density case therefore provides us with an absolute stability limit for any relativistic charged sphere satisfying these conditions.

Of course, we do not know exactly what a “realistic” charge distribution is for such objects, and so below we provide a somewhat cruder bound that is independent of any assumptions whatsoever about the charge distribution. The remainder of the paper consists of a formal proof along the lines of YS, but we have streamlined the presentation, filled in a number of gaps and present explicit numerical results.



## 5 General Case

The plan is now to bound the behavior of the solution to Eq.(4.1) for  $0 \leq Q < M$  under the most general conditions possible. To reiterate, we assume that any physically acceptable solution meets only the following conditions:

$$\begin{aligned} p(r) &\geq 0, & \zeta(r) &> 0, \\ 0 &\leq Q < M, & R &> R_+. \end{aligned} \quad (5.1)$$

Here  $R_+ \equiv M + \sqrt{M^2 - Q^2}$  is the outer horizon of a RN black hole; if  $R = R_+$  then gravitational collapse of the charged sphere has already taken place. The quantities  $m_i$ ,  $m_g$  and  $q$  will be considered inputs that are related by Eq.(2.13) and they satisfy the conditions

$$\begin{aligned} m_g(R) &= M, & q(R) &= Q, \\ m_g &\geq q, & m_g + \sqrt{m_g^2 - q^2} &< r. \end{aligned} \quad (5.2)$$

The last two conditions are required to avoid naked singularities, as discussed in Section 2.

Once again, we define the *critical instability radius*  $R_c(M, Q)$  as the smallest radius  $R > R_+$  for which a physically acceptable solution can be found in  $[0, R]$  for any input functions  $m_g, q$  satisfying (5.2). We also parameterize the difference between  $Q$  and  $M$  by

$$\Delta \equiv \sqrt{1 - Q^2/M^2} < 1 \quad (5.3)$$

and assume  $R = R_+(1 + \varepsilon)$ , where  $\varepsilon$  is some number (not necessarily small). With this notation,  $R_+ = (1 + \Delta)M$  and

$$R = (1 + (1 + \varepsilon)\Delta)M. \quad (5.4)$$

We shall prove below that, under the conditions (5.1)-(5.2), the critical stability radius

admits a general lower bound of the form

$$R_c \geq (1 + (1 + \varepsilon_0)\Delta) M \quad (5.5)$$

with  $\varepsilon_0 \simeq 1/264$ . We do not believe that this specific value of  $\varepsilon_0$  has any physical relevance: it is merely a byproduct of our estimates and it can certainly be improved.

Let us now turn to the proof of (5.5). The strategy will consist in demonstrating that, whenever  $R < (1 + (1 + \varepsilon_0)\Delta) M$ , for some suitable  $\varepsilon_0$  to be constructed below, then every possible solution  $\zeta$  of Eq. (4.1) becomes negative somewhere in  $[0, R)$ , for any  $m_g, q$  satisfying conditions (5.2). Thus the solution becomes physically unacceptable. In the proof we shall need the following preliminary estimate:

**Lemma 1** *If  $R < (1 + (1 + \varepsilon_0)\Delta) M$  and  $\delta = (\varepsilon_0^2 + 2\varepsilon_0)^{1/2}$ , then*

$$\sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} \leq \delta \left( \frac{M}{R} - \frac{Q^2}{R^2} \right) \quad (5.6)$$

For a proof of Lemma 1, see Appendix 2.

We now begin to study Eq.(4.1). Integrating both sides between  $r$  and  $R$  and using the boundary conditions (2.17) we find, in analogy to Eq. (3.3):

$$\frac{1}{R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right) - \frac{1}{r} e^{-\Lambda(r)} \zeta'(r) = \int_r^R dx e^{\Lambda(x)} \left[ \left( \frac{m_g(x)}{x^3} \right)' - q(x) \left( \frac{q(x)}{x^4} \right)' \right] \zeta(x) \quad (5.7)$$

A crucial point in our proof consists in finding a uniform bound on the right-hand-side, independent of  $\zeta$ ,  $m_g$  and  $q$ , at least for  $r$  close enough to  $R$ . This will allow us to dispense with the details of  $\zeta$ . In Appendix 3 we prove the following key estimate.

**Lemma 2** *Let  $\beta < 1$ . If  $R < (1 + (1 + \varepsilon_0)\Delta) M$  and  $\varepsilon_0 = \beta^4 \alpha_0$ , with*

$$\alpha_0 = \frac{(\sqrt{2} - 1)^2/4}{1 + \sqrt{1 + (\sqrt{2} - 1)^2/4}},$$

then either the r.h.s. of (5.7) is uniformly bounded from above by  $\frac{1}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right)$  for all  $r \in [\beta R, R]$  or  $\zeta(r) = 0$  for some  $r \in [\beta R, R]$ .

**Remark.** Note that if  $\varepsilon_0$  is chosen as in Lemma 2 then the constant  $\delta$  in Lemma 1 satisfies  $\delta < \beta^2(\sqrt{2}-1)/2$  (see the proof of Lemma 2). This fact will be used below.

Now assume that  $\zeta > 0$  for any  $r \leq R$ . By Lemma 2, for any  $r \in [\beta R, R]$  with  $\beta < 1$ , if  $R < (1 + (1 + \varepsilon_0)\Delta) M$  where now  $\varepsilon_0 = \beta^4 \alpha_0$ , then the right hand side of (5.7) is uniformly bounded above by  $\frac{1}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right)$ . As a consequence, from (5.7) we find

$$\zeta'(r) \geq \frac{r e^{\Lambda(r)}}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right).$$

Integrating this inequality between  $\beta R$  and  $R$  and using  $e^\Lambda \geq 1$ , we get:

$$\sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} - \zeta(\beta R) \geq \frac{1}{2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right) \int_{\beta R}^R \frac{dr}{R^2}, \quad (5.8)$$

or

$$0 < \zeta(\beta R) \leq \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} - \frac{1-\beta}{4} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right) \quad (5.9)$$

On the other hand, under the assumption that  $R < (1 + (1 + \varepsilon_0)\Delta) M$  with  $\varepsilon_0 = \beta^4 \alpha_0$ , we have from Lemma 1 and the remark after Lemma 2,

$$0 > \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} - \beta^2 \frac{\sqrt{2}-1}{2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right).$$

If we merely set  $\beta^2 \frac{\sqrt{2}-1}{2} = \frac{1-\beta}{4}$ , which gives

$$\beta = \beta_0 \equiv \frac{\sqrt{1 + 8(\sqrt{2}-1)} - 1}{4(\sqrt{2}-1)},$$

we then have a contradiction, and this implies that no physical solution to the Einstein equations can be found. This proves (5.5), with an explicit bound on  $\varepsilon_0$  given by  $\beta_0^4 \alpha_0$ , or  $\varepsilon_0 \simeq 1/264$ .

We can now tabulate  $R_c/M > 1 + (1 + \varepsilon_0)\Delta$  for various values of  $Q/M$ . The results are shown in Table 2.

$Q/M$	$R_c/M$
0	2.250
.1	1.999
.2	1.983
.3	1.956
.4	1.920
.5	1.869
.6	1.830
.7	1.717
.8	1.602
.9	1.437
.99	1.142
.999	1.045
.9999	1.014

Table 2. The lower bound on  $R_c/M$  is tabulated for and various values of  $Q/M$ .

Because these figures represent a lower bound on  $R_c/M$ , they should all lie beneath the corresponding numbers of Table 1, and indeed they do. *Any* relativistic charged sphere, regardless of equation of state, must have a critical stability value of  $R_c/M$  greater than the values presented here. The discussion above fails for  $Q = M$ , in which case the stability bound is simply  $R > R_+$ .

## 6 Conclusions

The main result of this paper is an exact solution for the stability limit of constant-density relativistic charged spheres for all  $Q \leq M$ . We also argued that in any “physically reasonable” case where the gravitational mass density decreases with the radius and the charge density increases with radius, the constant-density case provides an absolute stability limit for all charged spheres. If calculation of a stability limit for rotating objects proves tractable we would expect a quantitatively similar behavior for  $R_c/M$ , given a value of  $a/M$ , the angular-momentum-parameter-to-mass ratio.

For the most general charged case we found a cruder bound that is independent of any assumption about the mass and charge distribution, except for the basic conditions (5.2). Both “physically reasonable” and general bounds approach the horizon  $R_+$  in the limit  $Q \rightarrow M$ . That in this limit the critical “collapse” radius is precisely  $R_+$  is intuitively reasonable because  $Q = M$  is the point at which the Coulomb repulsion equals the gravitational force. Such a state is evidently unstable, but our results apparently do not exclude what seems to be the only route to producing an extremal black hole: to first create an extremal charged sphere, then compress it to the horizon. Nevertheless, given the other issues surrounding extremal bodies mentioned in the Introduction, one should continue to hesitate before regarding such objects as a smooth limit of the sub-extremal state.

**Note Added:** Just prior to submission of this paper we learned that a similar proof of our main result has been independently given in [12].

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## Appendix 1: On the differential inequality (4.13)

In this Appendix we prove that if  $\tilde{\zeta}(f)$  satisfies inequality (4.13) with boundary conditions  $\tilde{\zeta}(f(R)) = e^{-\Lambda(R)}$  and  $d\tilde{\zeta}(f(R))/df = M/R - Q^2/R^2$ , then  $\tilde{\zeta}(f) \leq \tilde{\zeta}_0(f)$ , for all  $0 \leq f \leq f(R)$ . Again,  $\tilde{\zeta}_0(f)$  is the solution to the differential equation  $d^2\tilde{\zeta}_0(f)/df^2 = (Q^2/R^2)\tilde{\zeta}_0(f)$  satisfying the same boundary conditions as  $\tilde{\zeta}(f)$  does. It follows that the constant density case  $m_g \propto r^3, q \propto r^3$  provides an absolute stability limit for any relativistic charged sphere in which  $\rho'_g \leq 0, q' \geq 0$  and  $(q/r^3)' \geq 0$ .

In order to prove that  $\tilde{\zeta} \leq \tilde{\zeta}_0$ , we define  $g(x) = \tilde{\zeta}_0(f(R) - x) - \tilde{\zeta}(f(R) - x)$  and

show that  $g(x) \geq 0$ , for all  $0 \leq x \leq f(R)$ . Note that  $g(0) = dg(0)/dx = 0$  and from (4.13) that  $d^2g/dx^2 \geq (Q^2/R^2)g$ , for all  $0 \leq x \leq f(R)$ . Without loss of generality, we can assume that  $g(x)$  is not identically zero in a right-neighborhood of the origin. Let us pick some  $x_0 < R/Q$  (the reason for this choice will become clearer below) and let  $0 \leq x_1 \leq x_0$  satisfy  $g(x_1) = \min_{x \in [0, x_0]} g(x)$ . There are two cases:

(1)  $g(x_1) = 0$ . This means that  $g(x)$  must be nonnegative in  $[0, x_0]$ , which in turn implies that  $g(x)$  is nonnegative for all  $0 \leq x \leq f(R)$ . If this were not the case, then there would be some  $x^*$ , for  $0 < x^* < f(R)$ , such that  $g(x^*) > 0$  would be a local maximum. But this would mean that  $d^2g(x^*)/dx^2 \leq 0$ , which contradicts the condition  $d^2g(x^*)/dx^2 \geq (Q^2/R^2)g(x^*) > 0$ .

(2)  $g(x_1) < 0$ . By the mean-value theorem  $g(x_1) = x_1 dg(x_2)/dx$ , for some  $0 < x_2 < x_1$ . Again by the mean-value theorem,  $dg(x_2)/dx = x_2 d^2g(x_3)/dx^2$ , for some  $0 < x_3 < x_2$ . On the other hand  $d^2g(x_3)/dx^2 \geq (Q^2/R^2)g(x_3)$ , and so we have  $0 > g(x_1) \geq x_1 x_2 (Q^2/R^2) g(x_3)$ , implying in particular that  $g(x_3) < 0$ . Since  $g(x_1)$  is the minimum of  $g(x)$  in  $[0, x_0]$  we also have  $|g(x_3)| \leq |g(x_1)|$ . Finally:

$$0 < |g(x_1)| \leq x_1 x_2 (Q^2/R^2) |g(x_1)| \quad (\text{A1.1})$$

Because  $0 < x_2 < x_1 \leq x_0 < R/Q$ , we see that the r.h.s. of this inequality is strictly smaller than  $|g(x_1)|$ , but this is a contradiction. Thus  $g(x_1) < 0$  is an impossibility and the proof is concluded.

## Appendix 2: Proof of Lemma 1

Let  $R = (1 + (1 + \varepsilon)\Delta)M$  as in Eq. (5.4) for  $0 < \varepsilon < \varepsilon_0$ . Then in terms of  $\Delta$  and  $\varepsilon$ :

$$\frac{M}{R} = \frac{1}{1 + (1 + \varepsilon)\Delta}, \quad \frac{Q^2}{R^2} = \frac{1 - \Delta^2}{[1 + (1 + \varepsilon)\Delta]^2} \quad (\text{A2.1})$$

Inequality (5.6), which we want to prove, now takes the form:

$$1 - \frac{2}{1 + (1 + \varepsilon)\Delta} + \frac{1 - \Delta^2}{[1 + (1 + \varepsilon)\Delta]^2} \leq \delta^2 \left( \frac{1}{1 + (1 + \varepsilon)\Delta} - \frac{1 - \Delta^2}{[1 + (1 + \varepsilon)\Delta]^2} \right)^2. \quad (\text{A2.2})$$

Multiplying both sides by  $[1 + (1 + \varepsilon)\Delta]^2$  gives

$$\varepsilon^2 + 2\varepsilon \leq \delta^2 \left( \frac{1 + \varepsilon + \Delta}{1 + (1 + \varepsilon)\Delta} \right)^2. \quad (\text{A2.3})$$

Notice that for  $0 \leq Q < M$  the right-hand-side is always  $\geq \delta^2$ . So if we choose  $\delta^2 = \varepsilon_0^2 + 2\varepsilon_0$ , Eq. (A2.3) is certainly satisfied and the lemma is proven.  $\blacksquare$

## Appendix 3: Proof of Lemma 2

Let us denote the right-hand-side of (5.7) by  $G(r)$ . The integral vanishes if the lower limit is  $R$ ; hence  $G(R) = 0$ . So, by continuity,  $G(r)$  will be less than  $\frac{1}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right)$  in a small enough interval of the form  $[r_0, R]$ . Let us pick some  $\beta < 1$  and let  $\varepsilon_0 = \beta^4 \alpha_0$ , with  $\alpha_0 = \frac{(\sqrt{2}-1)^2/4}{1 + \sqrt{1+(\sqrt{2}-1)^2/4}}$  (the relevance of this specific number will be made clear below). We want to show that either  $G(r) \leq \frac{1}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right)$  for all  $r \in [\beta R, R]$ , or  $\zeta(r) = 0$  for some  $r \in [\beta R, R]$ , in which case the solution becomes unphysical.

We proceed by contradiction. Assume that  $\zeta(r) > 0$  in  $[\beta R, R]$  and that  $G(r) - \frac{1}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right)$  changes sign in the same interval. This means that there is some  $\beta' > \beta$  such that both  $G(\beta' R) = \frac{1}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right)$  and  $G(r) < \frac{1}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right)$ ,  $\forall r \in (\beta' R, R]$ . Now, from the definition of  $e^{-\Lambda}$  (Eq. (2.9)), one has

$$\frac{d(e^{-\Lambda(x)})}{dx} = -x^2 e^\Lambda \left[ \left( \frac{m_g(x)}{x^3} \right)' - q(x) \left( \frac{q(x)}{x^4} \right)' + 2 \frac{m_g}{x^4} - 3 \frac{q^2}{x^5} \right], \quad (\text{A3.1})$$

so that  $G(\beta' R)$  can be immediately rewritten as:

$$G(\beta' R) = \int_{\beta' R}^R dx \left[ -\frac{1}{x^2} \frac{d(e^{-\Lambda})}{dx} \right] \zeta(x) + \int_{\beta' R}^R dx e^{\Lambda(x)} \left( -2 \frac{m_g}{x^4} + 3 \frac{q^2}{x^5} \right) \zeta(x) \quad (\text{A3.2})$$

By (5.2), using in particular  $q(x) \leq m_g(x) \leq x$ , we find that  $-2m_g/x^4 + 3q^2/x^5 \leq 1/x^3$ . Thus, after integrating the first term by parts,

$$G(\beta' R) \leq -\frac{\zeta(R)}{R^2} e^{-\Lambda(R)} + \frac{\zeta(\beta' R)}{(\beta' R)^2} e^{-\Lambda(\beta' R)} + \int_{\beta' R}^R dx e^{-\Lambda} \left( \frac{\zeta'}{x^2} - 2 \frac{\zeta}{x^3} \right) + \int_{\beta' R}^R dx \frac{\zeta e^\Lambda}{x^3} \quad (\text{A3.3})$$

Note that from (2.18), under the condition that  $p \geq 0$  and  $r \geq m_g \geq q$ , we must have  $\zeta' \geq 0$  for  $\zeta \geq 0$ . So, neglecting the negative terms in (A3.3), and using the fact that  $e^{-\Lambda} \leq 1$ ,  $\zeta' \geq 0$  and that  $x \geq \beta'R$ , the inequality becomes:

$$G(\beta'R) \leq 2 \frac{\zeta(R)}{(\beta'R)^2} + \frac{\zeta(R)}{(\beta'R)^3} \int_{\beta'R}^R dx e^{\Lambda(x)}. \quad (\text{A3.4})$$

Now, with the supposition that  $G(\beta'R) = \frac{1}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right)$  as well as the boundary condition (2.17) for  $\zeta$ ,

$$\int_{\beta'R}^R dx e^{\Lambda(x)} \geq \frac{(\beta')^3 R}{2} \frac{\left( \frac{M}{R} - \frac{Q^2}{R^2} \right)}{\sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}}} - 2\beta'R \quad (\text{A3.5})$$

On the other hand, in view of (5.7) and of the condition that  $G(r) < \frac{1}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right)$ ,  $\forall r \in (\beta'R, R]$ , we have

$$\zeta'(r) > \frac{re^{\Lambda(r)}}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right), \quad \forall r \in (\beta'R, R]. \quad (\text{A3.6})$$

Integrating between  $\beta'R$  and  $R$  yields

$$\zeta(R) - \zeta(\beta'R) > \frac{1}{2R^2} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right) \int_{\beta'R}^R dr r e^{\Lambda(r)}, \quad (\text{A3.7})$$

and employing the boundary conditions on  $\zeta(R)$  once again gives

$$\zeta(\beta'R) < \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} - \frac{\beta'}{2R} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right) \int_{\beta'R}^R dr e^{\Lambda(r)}. \quad (\text{A3.8})$$

Since inequality (A3.5) gives a minimum for the integral in this expression we can insert (A3.5) into the right-hand-side here to get, finally,

$$0 < \zeta(\beta'R) < \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} - \frac{\beta'}{2R} \left( \frac{M}{R} - \frac{Q^2}{R^2} \right) \left[ \frac{(\beta')^3 R}{2} \frac{\left( \frac{M}{R} - \frac{Q^2}{R^2} \right)}{\sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}}} - 2\beta'R \right]. \quad (\text{A3.9})$$

If we let  $y = \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} / \left( \frac{M}{R} - \frac{Q^2}{R^2} \right)$ , then the previous expression becomes

$$y^2 + (\beta')^2 y - \frac{1}{4}(\beta')^4 > 0, \quad (\text{A3.10})$$



which implies that

$$y > \beta'^2 \frac{\sqrt{2}-1}{2} > \beta^2 \frac{\sqrt{2}-1}{2}. \quad (\text{A3.11})$$

On the other hand, we see that Lemma 1 states that  $y \leq \delta$ , where  $\delta = (\varepsilon_0^2 + 2\varepsilon_0)^{1/2}$ . Take  $\varepsilon_0$  smaller than the positive root of the equation  $x^2 + 2x - \beta^4(\sqrt{2}-1)^2/4 = 0$ , that is, smaller than  $x_+ = -1 + \sqrt{1 + \beta^4(\sqrt{2}-1)^2/4}$ . For example,  $\varepsilon_0 = \beta^4/47.13 < \beta^4(\sqrt{2}-1)^2/[4 + 2\sqrt{4 + (\sqrt{2}-1)^2}]$  does the job. Then  $y < \beta^2 \frac{\sqrt{2}-1}{2}$ . We have reached a contradiction, and the Lemma is proved. ■

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