Renormalization Group and Ward Identities in regularized QED4 with large photon mass

Vieri Mastropietro

Università di Roma "Tor Vergata"
via della Ricerca Scientifica, I-00133, Roma

Abstract. *We analyze, by Renormalization Group methods, a regularized version of Euclidean QED4 with any value, including zero, of the fermion mass and a large photon mass. We will prove that the Schwinger functions are expressed by convergent series and verify the Ward Identities up to corrections which are small for momentum scales far from the ultraviolet cut-off.*

Keywords QED; Ward Identities; Renormalization Group; Constructive QFT.

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1. Introduction

1.1 *Ward Identities and Renormalization Group*

The gauge invariance of classical Electrodynamics leads to well known difficulties in its quantization, and the standard method to overcome them consists to add to the action a gauge-fixing term $\xi(\partial_\mu A_\mu)^2$; particularly convenient is the choice $\xi = 1/2$ so that the photon propagator aquires the simple form $\delta_{\mu,\nu} v(\mathbf{p})$ with $v(\mathbf{p}) = (\mathbf{p}^2 + M^2)^{-1}$, where M is a photon mass which can be added to make the infrared problems less severe. The formal Euclidean generating function of QED$_4$ then becomes

$$e^{W(J^A_\mu,\phi,\bar\phi)} = \int d\psi d\bar\psi dA e^{-\int d\mathbf{x}[\frac{1}{4}F_{\mu,\nu,\mathbf{x}}F_{\mu,\nu,\mathbf{x}}+\frac{1}{2}(\partial_\mu A_{\mu,\mathbf{x}})^2+\frac{1}{2}M^2 A_{\mu,\mathbf{x}}A_{\mu,\mathbf{x}}+\bar\psi_\mathbf{x}(-\gamma_\mu\partial_\mu+\gamma_\mu A_{\mu,\mathbf{x}})\psi_\mathbf{x}+J^A_{\mu,\mathbf{x}}A_{\mu,\mathbf{x}}+\phi_\mathbf{x}\bar\psi_\mathbf{x}+\bar\phi_\mathbf{x}\psi_\mathbf{x}]} \tag{1.1}$$

and the gauge invariance of the classical theory is replaced by a weaker invariance charapterized by the validity of an infinite set of non trivial identities between the correlations, called *Ward Identities* (WI) [W],[T], like

$$-i\mathbf{p}_\mu\langle A_{\mu,\mathbf{p}}\psi_\mathbf{k}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle = ev(\mathbf{p})[\langle\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle - \langle\psi_\mathbf{k}\bar\psi_\mathbf{k}\rangle] \tag{1.2}$$

and similar ones with any number of fields, obtained from (1.1) with the change of variables $\psi \to e^{i\chi_\mathbf{x}}\psi_\mathbf{x}, \bar\psi_\mathbf{x} \to \bar\psi_\mathbf{x} e^{-i\chi_\mathbf{x}}$. The WI can be also obtained by Feynman graph expansions, using that the fermionic propagator verify

$$\frac{1}{\not\mathbf{k}} - \frac{1}{\not\mathbf{k}+\not\mathbf{p}} = \frac{1}{\not\mathbf{k}}\not\mathbf{p}\frac{1}{(\not\mathbf{k}+\not\mathbf{p})} \tag{1.3}$$

or using an operatorial approach, from the relation

$$\partial_\mu\langle j^\mu_\mathbf{z}\psi_\mathbf{x}\bar\psi_\mathbf{y}\rangle = \langle\partial_\mu j^\mu_\mathbf{z}\psi_\mathbf{x}\bar\psi_\mathbf{y}\rangle + \delta(z_0-x_0)\langle[j^0_\mathbf{z},\psi_\mathbf{x}]\bar\psi_\mathbf{y}\rangle - \delta(z_0-y_0)\langle\psi_\mathbf{x}[j^0_\mathbf{z},\bar\psi_\mathbf{x}]\rangle = \delta(\mathbf{z}-\mathbf{x})\langle\psi_\mathbf{x}\bar\psi_\mathbf{y}\rangle - \delta(\mathbf{z}-\mathbf{y})\langle\psi_\mathbf{x}\bar\psi_\mathbf{y}\rangle \tag{1.4}$$

where the delta functions arise from the derivative of the θ-functions of the time order product and in the last step one uses the conservation of the current $\partial_\mu j^\mu_\mathbf{z} = 0$ and the commutation relation $\delta(z_0-x_0)[j^0_\mathbf{z},\psi_\mathbf{x}] = \delta(\mathbf{z}-\mathbf{x})\psi_\mathbf{x}$.

Of course all such derivations of the WI are purely *formal*, and indeed the correlations appearing in the r.h.s. or the l.h.s. of (1.2) are plagued by ultraviolet divergences. In order to construct a quantum gauge theory like

QED one has to regularize the theory introducing cut-offs, and then prove that, with a proper choice of the bare parameters, the cut-offs can be be removed and the correlations not only verify the Osterwalder-Schroeder axioms, but also the infinite set of WI associate to the gauge symmetry.

By using regularizations respecting (1.3), like the dimensional one, a classical inductive computation shows that the Ward Identities, understood as identities valid order by order in the perturbative expansion, are preserved through the renormalization procedure, see for instance [IZ]. More complex is to prove a similar statement with the *momentum* regularizations used in the Wilsonian Renormalization Group approach to QFT, introduced in [P] and [G], which has on its side the merit to be suitable in principle for the *non-perturbative* construction of QFT models. For finite values of the cut-offs, the momentum regularizations violate (1.3) and produce additional terms in the WI, so that one has to show that WI are finally restored removing the cut-offs. This was proved in [FHRW], as an order by order statement, by using the Gallavotti-Nicolo' tree formalism with two different regularizations, namely a momentum and a Pauli-Villar loop regularization; later on in [H] a similar statement was proved by using only momentum regularizations, assuming a non-vanishing photon mass. The renormalizability of QED_4 and the restoration of the WI removing the cut-offs was also obtained in [KK1] and [KK], respectively with massive or massive photons, using the Polchinski method. The interplay between WI and Wilsonian Renormalization Group in gauge theory has been also extensively analyzed in the physical literature, see for instance [B], [BAM].

In all the above analysis, QED is studied at a perturbative level, writing the n-point functions as formal power series whose convergence cannot be proved and which are probably not convergent at all; the WI are proved as order by order identities among formal series expansions. It is generally believed that a non-perturbative construction of QED_4, with no cut-offs, is possible only considering it as part of the Electroweak theory, which is asymptotically free. A non-perturbative construction of QED_4 with an ultraviolet cut-off was achieved in [DH]: the fermionic fields can be integrated out obtaining an effective boson theory which can be studied by cluster expansion techniques. Under the assumption of *massless* photons and that the fermionic mass is much *larger* than the electric charge, it was shown that the theory is asymptotically free in the infrared, that is that the behaviour of the photon n-point functions is the same as in the free case. The properties of the WI were however not analyzed; indeed they appear quite involved to study once that the fermionic degree of freedom are integrated out.

In this paper a regularized version of QED_4 is analyzed at a non-perturbative level following the opposite route, that is by integrating out the photon fields; in this way a purely fermionic theory is obtained which can be considered a non-local version of the Jona-Lasinio model. We assume a *large* photon mass, but our results are valid also for a small or even *vanishing* fermionic mass; note that the electron mass-charge ratio in adimensional units is $O(10^{-20})$ so that the requirement of large mass with respect to charge is quite unrealistic. The use of a fermionic formalism makes possible the study of the WI. We will show that the WI are verified up to corrections which are small for momentum scales far from the ultraviolet cut-off, provided that the charge renormalization is taken into account. Such results will be proved by an extension of the methods previously adopted in $d = 2$ QFT [M],[BFM] (and based on [BM],[BM1]), where they allow the removal of the ultraviolet cut-off and the complete construction of a non-trivial theory.

1.2 *Main results.*

We consider the following *generating function*

$$e^{\mathcal{W}_{N,L}(J^A,\phi)} = \int P(d\psi)P(dA)e^{\int d\mathbf{x}[e\bar\psi_\mathbf{x}(A_{\mu,\mathbf{x}}\gamma_\mu)\psi_\mathbf{x}+J^A_{\mu,\mathbf{x}}A_{\mu,\mathbf{x}}+\phi_\mathbf{x}\bar\psi_\mathbf{x}+\bar\phi_\mathbf{x}\psi_\mathbf{x}]} \tag{1.5}$$

where

-)in $\Lambda = [0,L]\times[0,L]\times[0,L]\times[0,L]$ a lattice Λ_a is introduced whose sites are given by the space-time points $\mathbf{x}_\mu = n_\mu a$, $\mu = 0,1,2,3$ with L/a integer and $n_\mu = -L/2a, 1, \ldots, L/2a-1$. We also consider the set $\mathcal{D}$ of space-time momenta $\mathbf{k}$ with $k_\mu = (m_\mu + \frac{1}{2})\frac{2\pi}{L}$ and with $m_\mu = 0,1,\ldots,L/a-1$.

-)The γ-matrices verify $\{\gamma_\mu,\gamma_\nu\} = -2\delta_{\mu,\nu}$ and are chosen as

$$\gamma^0 = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}, \qquad \gamma^\alpha = \begin{pmatrix} 0 & -\sigma_\alpha \\ \sigma_\alpha & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{1.6}$$

and σ^α, $\alpha=1,2,3$, are the σ-matrices

$$\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\;,\qquad \sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\;,\qquad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\;. \tag{1.7}$$

-)With each $\mathbf{k}\in\mathcal{D}$ we associate two *Grassmann spinors* $\psi_\mathbf{k},\bar\psi_\mathbf{k}$, and we define the functional integration $\mathcal{D}\psi$ as the linear functional on the Grassmann algebra generated be the variables $\psi_\mathbf{k},\bar\psi_\mathbf{k}$ such that, given a monomial $Q(\psi)$ in the variables $\psi_\mathbf{k},\bar\psi_\mathbf{k}$, its value its zero except in the case $Q(\psi)=\prod_{\mathbf{k}\in\mathcal{D}}\psi_\mathbf{k}\bar\psi_\mathbf{k}$, up to a permutation of the variables; in such a case $\int\prod_{\mathbf{k}\in\mathcal{D}}\psi_\mathbf{k}\bar\psi_\mathbf{k}=1$. We define $\psi_\mathbf{x}=\frac{1}{L^4}\sum_{\mathbf{k}\in\mathcal{D}}e^{-i\mathbf{k}\mathbf{x}}\psi_\mathbf{k}$ and $\bar\psi_\mathbf{x}=\frac{1}{L^4}\sum_{\mathbf{k}\in\mathcal{D}}e^{i\mathbf{k}\mathbf{x}}\bar\psi_\mathbf{k}$ and the *fermionic integration* is

$$P(d\psi^{(\le N)})=\mathcal{N}^{-1}\mathcal{D}\psi\exp[-\frac{Z_N}{L^4}\sum_{\mathbf{k}\in\mathcal{D}}\bar\psi_\mathbf{k}\chi_N^{-1}(\mathbf{k})(i\not\mathbf{k}+mI)\psi_\mathbf{k}] \tag{1.8}$$

where $\not\mathbf{k}=\gamma_\mu k_\mu$ and $\chi_N(\mathbf{k})$ is a non-vanishing smooth cut-off function, never vanishing and selecting momenta $|\mathbf{k}|\le\gamma^N$; a specific choice will be done in the following section. The propagator corresponding to (1.8) is

$$g(\mathbf{x}-\mathbf{y})=\frac{1}{L^4}\sum_{\mathbf{k}\in\mathcal{D}}\frac{-i\not\mathbf{k}+mI}{\mathbf{k}^2+m^2}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\chi_N(\mathbf{k}) \tag{1.9}$$

We will call $j_\mu=\bar\psi\gamma_\mu\psi$ and $j^5_\mu=\bar\psi\gamma_\mu\gamma_5\psi$.

-)For any $\mathbf{x}\in\Lambda_a$, $A_{\mu,\mathbf{x}}$ is an euclidean boson field defined by the Gaussian measure $P(dA)$ with covariance $v_{\mu,\nu}(\mathbf{x}-\mathbf{y})=\delta_{\mu,\nu}v(\mathbf{x}-\mathbf{y})$. We will consider the case

$$v(\mathbf{x}-\mathbf{y})=\frac{1}{L^4}\sum_{\mathbf{p}\in\mathcal{D}}e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}\frac{\chi_N(\mathbf{p})}{\mathbf{p}^2+M_N^2} \tag{1.10}$$

with $M_N=\gamma^N M$.

The *Schwinger functions* are defined by

$$\langle\psi_{\mathbf{x}_1};...;\psi_{\mathbf{x}_n};\bar\psi_{\mathbf{y}_1};..;\bar\psi_{\mathbf{y}_n};A_{\mu_1,\mathbf{z}_1};...;A_{\mu_m,\mathbf{z}_m}\rangle=\frac{\partial^{2n+m}\mathcal{W}_{N,L}(J^A,\phi)}{\partial\phi_{\mathbf{x}_1}...\partial\phi_{\mathbf{x}_n}\partial\bar\phi_{\mathbf{y}_1}...\partial\phi_{\mathbf{y}_n}\partial J^A_{\mu_1,\mathbf{z}_1}...\partial J^A_{\mu_m,\mathbf{z}_m}}\Big|_{J=\phi=0} \tag{1.11}$$

We will prove the following result.

Theorem. *There exists ε_0, independent from m,N, such that the Schwinger functions (1.11) are analytic in $|e|\le\varepsilon_0$ and verify the Ward Identity, for $|\mathbf{p}|,|\mathbf{k}|,|\mathbf{k}-\mathbf{p}|\le\kappa$*

$$-i\mathbf{p}_\mu\langle A_{\mu,\mathbf{p}};\psi_\mathbf{k}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle=e_0v(\mathbf{p})[\langle\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle-\langle\psi_\mathbf{k}\bar\psi_\mathbf{k}\rangle](1+O(e\kappa\gamma^{-N})) \tag{1.12}$$

where

$$e_0=e(1-c_+e^2+O(e^4))$$

where c_+ is a constant given by (4.4),(4.6) below. In the massless case $m=0$ the following chiral Ward Identity holds

$$-i\mathbf{p}_\mu\langle j^5_{\mu,\mathbf{p}};\psi_\mathbf{k}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle=a(\langle\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle-\langle\psi_\mathbf{k}\bar\psi_\mathbf{k}\rangle)[1+O(e(\kappa\gamma^{-N})^\theta)] \tag{1.13}$$

with $a=1+O(e^2)$ and $\theta>0$ is a constant.

Remarks

1)(1.12) says that the WI (1.2) is verified up to corrections $O(e(\kappa\gamma^{-N})^\theta)$, *provided that the bare charge e is replaced by the dressed charge e_0*; a similar statement holds for all the other WI and it could be proved by a straightforward extension of the analysis in this paper.

2)As a consequence of (2.4), (1.12) can be written as a conservation equation

$$-i\mathbf{p}_\mu\langle j_{\mu,\mathbf{p}};\psi_\mathbf{k}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle=\bar a(\langle\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle-\langle\psi_\mathbf{k}\bar\psi_\mathbf{k}\rangle)[1+O(e(\kappa\gamma^{-N})^\theta)] \tag{1.14}$$

with $\bar{a} = 1 - c_+ e^2 + O(e^4)$; if we define the current as $e_0 \bar{\psi}\gamma_\mu \psi$ and the axial current as $e_0 \bar{\psi}\gamma_\mu \gamma^5 \psi$, we see that (1.14) and (1.12) are compatible with the current conservation (up to small corrections) but not with the axial current conservation; this is the well known *anomaly* in QED$_4$.

3)Our results hold also for the Jona-Lasinio model, whose Schwinger functions are given by (1.11) with $v(\mathbf{x}-\mathbf{y}) = \lambda\delta(\mathbf{x}-\mathbf{y})$, assuming $\lambda = O(\gamma^{-2N})$.

4)The value of the bare photon mass M_N should be chosen to make vanishing the photon mass at physical scales; a second order perturbative computations says that the renormalization of the photon mass is $O(\gamma^N e)$. While it has the N-dependence we assumed in (1.10), the choice $M = O(e)$ is outside the range of validity of our theorem.

5)In the $N \to \infty$ a non-interacting (trivial) theory is found, with fermionic wave function renormalization $Z_N = 1 + O(e)$ and current renormalization $Z_N^{(1)} = 1 + O(e)$.

2. Renormalization Group analysis

2.1 *Multiscale Integration*

We will consider for notational simplicity the $m = 0$ case; the massive case poses no additional difficulty and it can be analyzed in a similar way up to some trivial modifications (see for instance [BFM] for the $d = 2$ case). Integrating the bosonic variables A one can rewrite (1.5) as a purely fermionic theory

$$e^{\bar{\mathcal{W}}_{N,L}(J^A,\phi)} = \int P(d\psi) e^{\frac{1}{2}\int d\mathbf{x}d\mathbf{y}v(\mathbf{x}-\mathbf{y})[e\bar{\psi}_\mathbf{x}\gamma_\mu\psi_\mathbf{x}+J^A_{\mu,\mathbf{x}}][e\bar{\psi}_\mathbf{y}\gamma_\mu\psi_\mathbf{y}+J^A_{\mu,\mathbf{y}}]+\int d\mathbf{x}[\phi_\mathbf{x}\bar{\psi}_\mathbf{x}+\bar{\phi}_\mathbf{x}\psi_\mathbf{x}]} \tag{2.1}$$

where $\int d\mathbf{x}$ is a shorthand for $a^4 \sum_{\mathbf{x}\in\Lambda_a}$.

We will find more convenient to consider the following generating function, writing $\lambda = e^2/2$

$$e^{\mathcal{W}_{N,L}(J,\phi)} = \int P(d\psi^{(\leq N)}) e^{-\mathcal{V}^{(N)}(\psi)+\int d\mathbf{x} J_{\mu,\mathbf{x}}\bar{\psi}_\mathbf{x}\gamma_\mu\psi_\mathbf{x}+\int d\mathbf{x}[\phi_\mathbf{x}\bar{\psi}_\mathbf{x}+\bar{\phi}_\mathbf{x}\psi_\mathbf{x}]} \tag{2.2}$$

where

$$\mathcal{V}^{(N)} = -\lambda \int d\mathbf{x}d\mathbf{y} v(\mathbf{x}-\mathbf{y})(\bar{\psi}_\mathbf{x}\gamma_\mu\psi_\mathbf{x})(\bar{\psi}_\mathbf{y}\gamma_\mu\psi_\mathbf{y}) \tag{2.3}$$

Of course the correlations obtained from (2.2) are trivially related to the correlations from (2.1) (see for instance (2.4)); for instance

$$ev(\mathbf{p})\langle j_{\mu,\mathbf{p}}; \psi_\mathbf{k}\bar{\psi}_{\mathbf{k}+\mathbf{p}}\rangle = \langle A_{\mu,\mathbf{p}}; \psi_\mathbf{k}\bar{\psi}_{\mathbf{k}+\mathbf{p}}\rangle \tag{2.4}$$

$\mathcal{W}_{N,L}$ is invariant under the Euclidean transformation

$$\psi'(\mathbf{x}') = S(\Lambda)\psi(\mathbf{x}) \qquad \bar{\psi}'(\mathbf{x}') = \bar{\psi}(\mathbf{x})S(\Lambda)^{-1} \qquad \mathbf{x}'_\nu = \Lambda_{\mu,\nu}\mathbf{x}_\nu$$

with $S(\Lambda)\gamma^\nu S(\Lambda)^{-1} = (\Lambda^{-1})_{\mu,\nu}\gamma_\nu$. The local quadratic invariant terms are then only $\bar{\psi}_\mathbf{x}\psi_\mathbf{x}$ and $\bar{\psi}_\mathbf{x} \not\partial\psi_\mathbf{x}$.

We describe in more detail our choice of the cut-off fuction $\chi_N(\mathbf{k})$. Let $\chi_0 \in C^\infty(\mathbb{R}_+)$ be a non-negative, non-increasing smooth function such that

$$\chi_0(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t \geq \gamma_0\ , \end{cases}$$

for any choice of $\gamma_0 : 1 < \gamma_0 \leq \gamma$; and we define, for any integer k

$$f_k(\mathbf{k}) = \chi_0\left(\gamma^{-k}|\mathbf{k}|\right) - \chi_0\left(\gamma^{-k+1}|\mathbf{k}|\right) \tag{2.5}$$

and $f_k(\mathbf{k})$ are functions with support $\gamma^{k-1} \le |\mathbf{k}| \le \gamma^{k+1}$, so that $C_N^{-1}(\mathbf{k}) = \sum_{k=-\infty}^{N} f_k(\mathbf{k})$ is a compact support cut-off function vanishing for $|\mathbf{k}| \ge \gamma^{N+1}$. We define

$$\chi_N(\mathbf{k}) = C_N^{\varepsilon}(\mathbf{k})^{-1} = \sum_{k=-\infty}^{N} f_k^{\varepsilon}(\mathbf{k})$$

where $f_k^{\varepsilon}(\mathbf{k}) = f_k(\mathbf{k})$ for $-\infty \le k \le N-1$, while $f_N^{\varepsilon}(\mathbf{k})$ is a C^{∞} function of $|\mathbf{k}|$, such that $f_N^{\varepsilon}(\mathbf{k}) = f_N(\mathbf{k})$ for $\gamma^{N-1} \le |\mathbf{k}| \le \gamma^N$, $f_N(\mathbf{k}) > 0$ for $|\mathbf{k}| \ge \gamma^N$ and, if $|\mathbf{k}| \ge \gamma^{N+1}$, $0 < |\mathbf{k}| \le e^{-|\mathbf{k}|}$.

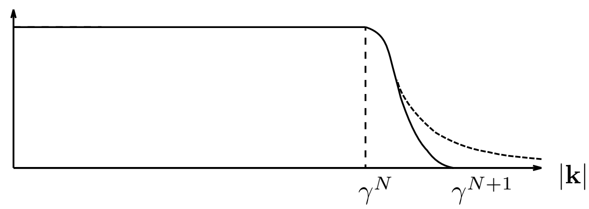

Fig. 1

Fig 1: Graphical representation of $\chi_N(\mathbf{k})$

The functional integral is analyzed through a multiscale integration procedure; the starting point is to write the fermionic propagator as

$$g(\mathbf{x}-\mathbf{y}) = \sum_{h=-\infty}^{N} g^{(h)}(\mathbf{x}-\mathbf{y}) \tag{2.6}$$

with, by integrating by parts, for any positive integer K

$$|g^{(h)}(\mathbf{x}-\mathbf{y})| \le \gamma^{3h} \frac{C_K}{1+(\gamma^h|\mathbf{x}-\mathbf{y}|)^K} \tag{2.7}$$

In a similar way the following bound for the boson propagator is obtained:

$$|v(\mathbf{x}-\mathbf{y})| \le \gamma^{2N} \frac{C_K}{1+(\gamma^N|\mathbf{x}-\mathbf{y}|)^K} \tag{2.8}$$

By well known properties of Grassmann integrals (see for instance [GM]) we can write

$$e^{\mathcal{W}_{N,L}(J,\phi)} = \int P(d\psi^{(\le N-1)}) \int P(d\psi^{(N)}) e^{-\mathcal{V}^{(N)}(\psi^{(\le N)}) + \int d\mathbf{x} J_{\mu,\mathbf{x}} \bar\psi^{(\le N)}_{\mathbf{x}} \gamma_\mu \psi^{(\le N)}_{\mathbf{x}} + \int d\mathbf{x} [\phi_{\mathbf{x}} \bar\psi^{(\le N)}_{\mathbf{x}} + \bar\phi_{\mathbf{x}} \psi^{(\le N)}_{\mathbf{x}}]} \tag{2.9}$$

where $P(d\psi^{(N)})$ and $P(d\psi^{(\le N-1)})$ are Grassmann integrations given by (1.9) with $C_N^{-1}(\mathbf{k})$ replaced by $f_N^{\varepsilon}(\mathbf{k})$ and $C_{N-1}^{-1}(\mathbf{k}) = \sum_{k=-\infty}^{N-1} f_j(\mathbf{k})$ respectively (and Z_N replaced by 1). We can integrate the field $\psi^{(N)}$ obtaining

$$e^{\mathcal{W}_{N,L}(J,\phi)} = e^{-L^4 E_N + S_N(\phi,J)} \int P(d\psi^{(\le N-1)}) e^{-\mathcal{V}^{(N-1)}(\psi^{(\le N-1)}) + \mathcal{B}^{(N-1)}(\psi,J,\phi)} \tag{2.10}$$

where $\mathcal{V}^{(N-1)}$ is the *effective potential* which can be written , if ψ^{ε} is ψ or $\bar\psi$, as

$$\mathcal{V}^{(N-1)}(\psi^{(\le N-1)}) = \sum_{l,m} \int d\mathbf{x}_1 ... d\mathbf{x}_{2l} W_{2l}^{(N-1)}(\mathbf{x}_1, .., \mathbf{x}_{2l}) \prod_{i=1}^{2l} \psi_{\mathbf{x}_i}^{(\varepsilon_i \le N-1)} \tag{2.11}$$

and $\mathcal{B}^{(N-1)}$ is the *effective source*, which can be written as

$$\mathcal{B}^{(N-1)}(\psi,J,\phi) = \mathcal{B}_1^{(N-1)}(\psi^{(\le N-1)},J) + \mathcal{B}_2^{(N-1)}(\psi^{(\le N-1)},\phi) + \mathcal{B}_3^{(N-1)}(\psi^{(\le N-1)},J,\phi) \tag{2.12}$$

where $\mathcal{B}_1^{(N-1)}$ contains all the terms linear in J, $\mathcal{B}_2^{(N-1)}$ all the terms linear in ϕ or $\bar\phi$ and $\mathcal{B}_3^{(N-1)}$ is the rest.
We split the effective potential $\mathcal{V}^{(N-1)}$ as $\mathcal{L}\mathcal{V}^{(N-1)}+\mathcal{R}\mathcal{V}^{(N-1)}$, where $\mathcal{R}=1-\mathcal{L}$ and $\mathcal{L}$ is a linear operator acting on functions like (2.11). Its action on the kernels $W_{2l}^{(N-1)}$ is $\mathcal{L}W_{2l}^{(N-1)}=0$ if $l>1$,while if $l=1$

$$\mathcal{L}\int d\mathbf{x}d\mathbf{y}W_2^{(N-1)}(\mathbf{x},\mathbf{y})\bar\psi_{\mathbf{x}}\psi_{\mathbf{y}}=\int d\mathbf{x}d\mathbf{y}W_2^{(N-1)}(\mathbf{x},\mathbf{y})[\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}}+(\mathbf{x}_\mu-\mathbf{y}_\mu)\bar\psi_{\mathbf{x}}\partial_\mu\psi_{\mathbf{x}}] \tag{2.13}$$

Note that by parity $\int d\mathbf{y}W_2^{(N)}(\mathbf{x},\mathbf{y})=0$ and by Euclidean invariance $\int d\mathbf{y}(\mathbf{x}_\mu-\mathbf{y}_\mu)W_2^{(N)}(\mathbf{x},\mathbf{y})=\gamma_\mu S$ with S a scalar so that

$$\mathcal{L}\mathcal{V}^{(N-1)}=z_{N-1}\int d\mathbf{x}\psi_{\mathbf{x}}\ \partial\!\!\!/\bar\psi_{\mathbf{x}} \tag{2.14}$$

Analogously $\mathcal{B}_1^{(N-1)}(\psi^{(\le N-1)},J)$ is given by an expression similar to (2.11), sum of momonials with l Grassmann fields and a single $\gamma_\mu J_\mu$ fields, integrated over kernels $W_{2l,1}^{(N-1)}$. We define the $\mathcal{L}$ operation as $\mathcal{L}W_{2l,1}^{(N-1)}=0$ if $l>1$ and if $l=1$

$$\mathcal{L}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}W_{2,1}^{(N-1)}(\mathbf{z},\mathbf{x},\mathbf{y})\gamma_\mu J_{\mu,\mathbf{z}}\bar\psi_{\mathbf{x}}\psi_{\mathbf{y}}=\int d\mathbf{x}d\mathbf{y}d\mathbf{z}W_{2,1}^{(N-1)}(\mathbf{z},\mathbf{x},\mathbf{y})\gamma_\mu J_{\mu,\mathbf{z}}\bar\psi_{\mathbf{z}}\psi_{\mathbf{z}} \tag{2.15}$$

and again by Euclidean invariance

$$\mathcal{L}\mathcal{B}_1^{(N-1)}(\psi,J)=Z_{N-1}^{(1)}\int d\mathbf{z}J_{\mu,\mathbf{z}}\bar\psi_{\mathbf{z}}\gamma_\mu\psi_{\mathbf{z}} \tag{2.16}$$

It is convenient to write

$$\mathcal{R}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}W_{2,1}^{(N-1)}(\mathbf{z},\mathbf{x},\mathbf{y})\gamma_\mu J_{\mu,\mathbf{z}}\bar\psi_{\mathbf{x}}\psi_{\mathbf{y}}=$$

$$\int d\mathbf{x}d\mathbf{y}d\mathbf{z}W_{2,1}^{(N-1)}(\mathbf{z},\mathbf{x},\mathbf{y})\gamma_\mu J_{\mu,\mathbf{z}}[(\mathbf{x}_\mu-\mathbf{z}_\mu)\int_0^1 dt\partial_\mu\bar\psi_{\mathbf{x}(t)}\psi_{\mathbf{z}}+(\mathbf{y}_\mu-\mathbf{z}_\mu)\int_0^1 dt\bar\psi_{\mathbf{x}}\partial_\mu\psi_{\mathbf{y}(t)}] \tag{2.17}$$

where $\mathbf{x}(t)=\mathbf{x}+t(\mathbf{z}-\mathbf{x})$ and $\mathbf{y}(t)=\mathbf{y}+t(\mathbf{z}-\mathbf{y})$ are called *interpolated points*; a similar expression holds for (2.13).
$\mathcal{B}_2^{(N-1)}(\psi^{(\le N-1)},\phi)$ has the form

$$\mathcal{B}_2^{(N-1)}(\psi,\phi)=\int d\mathbf{x}\left[\bar\phi_{\mathbf{x}}\psi_{\mathbf{x}}^{(\le N-1)}+\bar\psi_{\mathbf{x}}^{(\le N-1)}\phi_{\mathbf{x}}\right]+\int d\mathbf{x}d\mathbf{y}\left[\bar\phi_{\mathbf{x}}g^{(N)}(\mathbf{x}-\mathbf{y})\frac{\partial}{\partial\psi_{\mathbf{y}}}\mathcal{V}^{(N-1)}+\frac{\partial}{\partial\bar\psi_{\mathbf{y}}}\mathcal{V}^{(N-1)}g^{(N)}(\mathbf{x}-\mathbf{y})\phi_{\mathbf{x}}\right] \tag{2.18}$$

and the $\mathcal{L}$ operation is defined decomposing in (2.18) $\mathcal{V}^{(N-1)}$ as $\mathcal{L}\mathcal{V}^{(N-1)}+\mathcal{R}\mathcal{V}^{(N-1)}$ using the definition in (2.13). Finally we define $\mathcal{L}\mathcal{B}_3^{(N-1)}=0$.
We write then

$$e^{\mathcal{W}_{N,L}(J,\phi)}=e^{-L^4E_N+S_N(\phi,J)}\int P(d\psi^{(\le N-1)})e^{-\mathcal{V}^{(N-1)}}=e^{-L^4E_N+S_N(\phi,J)}$$

$$\int P(d\psi^{(\le N-2)})\int P(d\psi^{(N-1)})e^{-\mathcal{L}\mathcal{V}^{(N-1)}-\mathcal{R}\mathcal{V}^{(N-1)}}=e^{-L^4E_{N-1}+S_{N-1}(\phi,J)}\int P(d\psi^{(\le N-2)}))e^{-\mathcal{V}^{(N-2)}} \tag{2.19}$$

and the procedure can be iterated; after the fields $\psi^{(N-2)},...,\psi^{(k+1)}$ we arrive to expressions similar to (2.11) with $N-1$ replaced by k, and in the analogous of (2.18) $g^{(N)}(\mathbf{y}-\mathbf{y})$ is replaced by $\sum_{i=k+1}^N g^{(i)}(\mathbf{x}-\mathbf{y})$.

2.2 *The tree expansion*

At the end of the iterative procedure we obtain

$$\mathcal{W}_{N,L}(J,\phi)=-L^4E_N+\sum_{m^\phi+n^J\ge 1}S_{2m^\phi,n^J}(\phi,J) \tag{2.20}$$

where E_N and $S_{2m^\phi,n^J}(\phi,J)$ can be written as sum of trees defined in the following way.

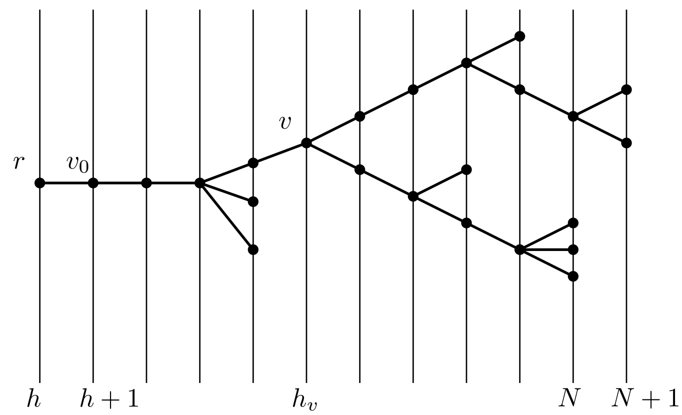

Fig. 2

Fig 2: an example of tree τ

1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n. We shall consider also the *labeled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label $h \leq N-1$ with the root and we denote $\mathcal{T}_{h,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an an integer taking values in $[h, N+1]$, and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root is on the line with index h. The tree will intersect the vertical lines in set of points different from the root and the end-points; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$. Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $h+1$. There is the constraint, for the end-points of scale h_v, that $h_v = h_{v'}+1$, if v' is the first non trivial vertex immediately preceding v.

3) There are two kind of end-points, *normal* and *special*. With each normal endpoint of scale h_v we associate $z_{h_v-1}\int d\mathbf{x}\bar\psi_\mathbf{x}\ \not\partial\psi_\mathbf{x}$ or $\lambda\int d\mathbf{x}d\mathbf{y}v(\mathbf{x}-\mathbf{y})(\bar\psi_\mathbf{x}\gamma_\mu\psi_\mathbf{x})(\bar\psi_\mathbf{y}\gamma_\mu\psi_\mathbf{y})$ and we call it respectively z or λ end-point. We will call $\bar m_{4,v}$ the number of λ end-points with scale h_v+1, and $m_{4,v} = \sum_{\bar v \geq v}\bar m_{4,\bar v}$. There are two types of special endpoints, called of type ϕ and J and to which is associated $Z^{(1)}_{h_v-1}\int d\mathbf{x}J_{\mu,\mathbf{x}}\bar\psi_\mathbf{x}\gamma_\mu\psi_\mathbf{x}$ and $\int d\mathbf{x}[\bar\phi_\mathbf{x}\psi_\mathbf{x}+\phi_\mathbf{x}\bar\psi_\mathbf{x}]$ respectively. Given $v\in\tau$, we shall call n^ϕ_v and n^J_v the number of endpoints of type ϕ and J following v in the tree.

4) We introduce a *field label* f to distinguish the field variables appearing in the terms $\mathcal{V}$ associated with the endpoints. The set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$ will denote the space-time point of the field variable with label f. We call *trivial tree* a tree containing only the root

and an endpoint.

Defining $V^{(k)}(\psi^{(\le k)},J,\phi)=\mathcal{V}^{(k)}(\psi^{(\le k)})+\mathcal{B}^{(k)}(\psi^{(\le k)},J,\phi)$, we can write

$$V^{(k)}(\psi^{(\le k)},J,\phi)+L^4E^2_{k+1}=\sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{k,n}}V^{(k)}(\tau) \tag{2.21}$$

where, if v_0 is the first vertex of τ and $\tau_1,..,\tau_s$ ($s=s_{v_0}$) are the subtrees of τ with root v_0, $\bar V^{(k)}$ is defined inductively by the relation, $k\le N-1$

$$V^{(k)}(\tau)=\frac{(-1)^{s+1}}{s!}\mathcal{E}^T_{k+1}[\bar V^{(k+1)}(\tau_1);..;\bar V^{(k+1)}(\tau_s)] \tag{2.22}$$

where $\mathcal{E}^T_{k+1}$ is the *truncated expectation* and $\bar V^{(k+1)}(\tau)=\mathcal{R}V^{(k+1)}(\tau)$ if the subtree τ_i contains more then one end-point, while if τ_i contains only one end-point $\bar V^{(k+1)}(\tau)$ is equal to ($z_N=0$)

$$z_{k+1}\int d\mathbf{x}\bar\psi^{(\le k+1)}_{\mathbf{x}}\ \partial\!\!\!/\psi^{(\le k+1)}_{\mathbf{x}};\qquad \lambda\int d\mathbf{x}d\mathbf{y}v(\mathbf{x}-\mathbf{y})(\bar\psi^{(\le k+1))}_{\mathbf{x}}\gamma_\mu\psi^{(\le k+1)}_{\mathbf{x}})(\bar\psi^{(\le k+1)}_{\mathbf{y}}\gamma_\mu\psi^{(\le k+1)}_{\mathbf{y}})$$

if it is a normal end-point, or ($Z^{(1)}_N=1$)

$$Z^{(1)}_{k+1}\int d\mathbf{x}J_{\mu,\mathbf{x}}\bar\psi^{(\le k+1))}_{\mathbf{x}}\gamma_\mu\psi^{(\le k+1))}_{\mathbf{x}};\qquad \int d\mathbf{x}[\bar\phi_{\mathbf{x}}\psi^{(\le k+1))}_{\mathbf{x}}+\phi_{\mathbf{x}}\bar\psi^{(\le k+1))}_{\mathbf{x}}]$$

if it is a special end-point.

We will use the following well known representation of the fermionic truncated expectation, see for instance [GM], if P is a set of indices and $\tilde\psi^{(h)}(P)=\prod_{f\in P}\psi^{(h)\varepsilon(f)}_{\mathbf{x}(f)}$

$$\mathcal{E}^T_h(\tilde\psi^{(h)}(P_1);\tilde\psi^{(h)}(P_2);...;\tilde\psi^{(h)}(P_s))=\sum_T\prod_{l\in T}g^{(h)}(\mathbf{x}_l-\mathbf{y}_l)]\int dP_T(\mathbf{t})\det G^{h,T}(\mathbf{t}) \tag{2.23}$$

where T is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}(f)_{f\in P_i}$, that is T is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover $\mathbf{t}=\{t_{i,i'}\in[0,1],1\le i,i'\le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'}=\mathbf{u}_i\cdot\mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i\in\mathbb{R}^s$ of unit norm. Finally $G^{h,T}(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by

$$G^{h,T}_{ij,i'j'}=t_{i,i'}g^{(h)}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'}) \tag{2.24}$$

and $\mathbf{x}_{ij}$ is the one of the points in P_j. In the following we shall use (2.23) even for $s=1$, when T is empty, by interpreting the r.h.s. as equal to 1, if $|P_1|=0$, otherwise as equal to $\det G^h=\mathcal{E}^T_h(\tilde\psi^{(h)}(P_1))$.

By using (2.21), (2.23) and the definition of the $\mathcal{R}$ operation, we get a simple expression for $V^{(h)}(\tau)$, for any $\tau\in\mathcal{T}_{h,n}$. We associate with any vertex v of the tree a set P_v, the *external fields* of v. The set P_v includes both the field variables of type ψ which belong to one of the λ,z,J endpoints following v and are not yet contracted at scale h_v (in the iterative integration procedure), to be called *normal external fields*, and those which belong to an endpoint of type λ,z,J and are contracted with a field variable belonging to an endpoint $\tilde v$ of type ϕ through a propagator $g^{(h_{\tilde v}-1)}$, to be called *special external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1,\ldots,v_{s_v}$ are the s_v vertices immediately following it, then $P_v\subset\cup_iP_{v_i}$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v=\cup_iQ_{v_i}$. The subsets $P_{v_i}\backslash Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v>1$, that is if v is a non trivial vertex. Moreover, if the set P_{v_0} contains only special external fields, that is if $|P_{v_0}|=n^\phi$, and $\tilde v_0$ is the vertex immediately following v_0, then $|P_{v_0}|<|P_{\tilde v_0}|$.

Given $\tau\in\mathcal{T}_{h,n}$, there are many possible choices of the subsets P_v, $v\in\tau$, compatible with all the constraints; we shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. Then we can write, calling

$P_{v_0} = P^a_{v_0} \bigcup P^b_{v_0}$ where $P^a_{v_0}$ correspond to the normal external fields and $P^b_{v_0}$ to the special external fields (spinor indices are omitted)

$$\bar V^{(k)}(\tau) = \sum_{\mathbf{P}\in\mathcal{P}_\tau} \sum_{T\in\mathbf{T}} \sum_{\alpha\in A_T} \int d\mathbf{x}_{v_0} W_{\tau,\mathbf{P},T,\alpha}(\mathbf{x}_{v_0}) [\prod_{f\in P^a_{v_0}} \psi^{\varepsilon(f)(\le k)}_{\mathbf{x}(f)}][\prod_{f\in P^b_{v_0}} g^{(h_{\bar v}-1)}(\mathbf{x}(f)-\mathbf{y}(f))\phi^{\varepsilon(f)(\le k)}_{\mathbf{y}(f)}][\prod_f J(\mathbf{x}_f)] \tag{2.25}$$

where

$$W_{\tau,\mathbf{P},T,\alpha}(\mathbf{x}_{v_0}) = \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \cdot \det G^{h_v,T_v}_\alpha(\mathbf{t}_v)$$

$$\Big[\prod_{l\in T_v} \int_0^1 dt_l \int_0^1 ds_l \tilde\partial^{q_\alpha(f_l^-)} \tilde\partial^{q_\alpha(f_l^+)} [(\mathbf{x}_l(t_l)-\mathbf{y}_l(s_l))^{b_\alpha(l)} \partial^{m_l} g^{(h_v)}(\mathbf{x}_l(t_l)-\mathbf{y}_l(s_l))]\Big][\prod_{v\in v^*_\lambda} (\mathbf{r})^{b_\alpha}\lambda v(\mathbf{r})][\prod_{v\in v^*_z} z_{h_v}][\prod_{v\in v^*_Z} Z^{(1)}_{h_v}] \tag{2.26}$$

and $\mathbf{x}'(t), \mathbf{y}'(t)$ are the interpolated points (see (2.17)) produced in the renormalization, A_T is a set of indices necessary to label the terms produced by the $\mathcal{R}$ operation, $\mathbf{T}$ is the set of the tree graphs on $\mathbf{x}_{v_0}$, obtained by putting together an anchored tree graph T_v for each non trivial vertex v and adding lines connecting the space-time points belonging to the set $\mathbf{x}_v$ for each endpoint v; $v^*_\lambda, v^*_z, v^*_Z$ are the λ, z and $Z^{(1)}$ endpoints of τ, f_l^- and f_l^+ are the labels of the two fields forming the line l and $G^{h_v,T_v}_\alpha(\mathbf{t}_v)$ is the matrix with elements

$$G^{h_v,T_v}_{\alpha,ij,i'j'} = t_{v,i,i'} \partial^{q_\alpha(f_l^-)} \partial^{q_\alpha(f_l^+)} \partial^{m(f^-_{i,j})+m(f^+_{i',j'})} g^{(h_v)}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'}) \; . \tag{2.27}$$

with $(f^-_{ij}, f^+_{i'j'})$ not belonging to T. By construction, there is a constant C such that, $\forall T\in\mathbf{T}_\tau$, $|A_T|\le C^n$; moreover, for any $\alpha\in A_T$, $q_\alpha(l)=0,1,2$,$b_\alpha(l)=0,1,2$ and by the definition of $\mathcal{R}$ the following inequality is satisfied

$$\Big[\prod_{f\in I_{v_0}} \gamma^{h_\alpha(f) q_\alpha(f)}\Big]\Big[\prod_{l\in T} \gamma^{-h_\alpha(l) b_\alpha(l)}\Big] \le \prod_{v \text{ not e.p.}} \gamma^{-z(P_v)} \; , \tag{2.28}$$

where $h_\alpha(f) = h_{v_0}-1$ if $f\in P_{v_0}$, otherwise it is the scale of the vertex where the field with label f is contracted; $h_\alpha(l) = h_v$, if $l\in T_v$ and

$$z(P_v) = \begin{cases} 2 & \text{if } |P_v|=2,\ n^\phi_v = 0,1 \quad n^J_v = 0\ , \\ 1 & \text{if } |P_v|=2 \text{ and } n^J_v = 1, \quad n^\phi_v = 0\ , \\ 0 & \text{otherwise.} \end{cases} \tag{2.29}$$

In order to derive (2.25) one has to follow an iterative procedure writing the $\mathcal{R}$ operation as in (2.17) and decomposing the "zero" factors $(\mathbf{x}_\mu - \mathbf{z}_\mu)$ along the propagators in T and the interactions $v(\mathbf{r})$. Note that, from (2.6), $\int d\mathbf{r}|\mathbf{r}||g^{(h)}(\mathbf{r})| \le C\gamma^{-2h}$, and from (2.7), $\int d\mathbf{r}|\mathbf{r}||v(\mathbf{r})| \le C\gamma^{-3N}$; hence each zero factor produces an extra γ^{-h} or $\gamma^{-N}\le\gamma^{-h}$ in the bounds. An important property that one has to check is that the zero factors do not accumulate, that is $b_\alpha \le 3$; such procedure has been extensively described in §3 of [BM] to which we refer for more details.

We will use now the following bound

$$|\det G^{h_v,T_v}_\alpha(\mathbf{t}_v)| \le C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)}$$

$$\gamma^{3\frac{h_v}{2}\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)\right)} \gamma^{h_v \sum_{i=1}^{s_v}\left[q_\alpha(P_{v_i}\backslash Q_{v_i})+m(P_{v_i}\backslash Q_{v_i})\right]} \gamma^{-h_v \sum_{l\in T_v}\left[q_\alpha(f_l^+)+q_\alpha(f_l^-)+m(f_l^+)+m(f_l^-)\right]} \tag{2.30}$$

The proof is based on the well known *Gram-Hadamard inequality*, stating that, if M is a square matrix with elements M_{ij} of the form $M_{ij} = < A_i, B_j >$, where A_i, B_j are vectors in a Hilbert space with scalar product $<\cdot,\cdot>$, then

$$|\det M| \le \prod_i ||A_i||\cdot||B_i|| \; . \tag{2.31}$$

where $||\cdot||$ is the norm induced by the scalar product.

Let $\mathcal{H}=\mathbb{R}^s\otimes\mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of complex four dimensional vectors $F(\mathbf{k})=(F_1(\mathbf{k}),\ldots,F_4(\mathbf{k}))$, $F_i(\mathbf{k})$ being a function on the set $\mathcal{D}$, with scalar product

$$< F,G >=\sum_{i=1}^4\frac{1}{L^4}\sum_{\mathbf{k}}F_i^*(\mathbf{k})G_i(\mathbf{k})\ . \tag{2.32}$$

If $h_v\le 0$, it is easy to verify that

$$G^{h_v,T_v}_{ij,i'j'}=t_{i,i'}g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'})=<\mathbf{u}_i\otimes A^{(h_v)}_{\mathbf{x}(f^-_{ij}),\omega(f^-_{ij})},\mathbf{u}_{i'}\otimes B^{(h_v)}_{\mathbf{x}(f^+_{i'j'}),\omega(f^+_{i'j'})}>\ , \tag{2.33}$$

where $\mathbf{u}_i\in\mathbb{R}^s$, $i=1,\ldots,s$, are the vectors such that $t_{i,i'}=\mathbf{u}_i\cdot\mathbf{u}_{i'}$, and

$$\begin{aligned}
A^{(h)}_{\mathbf{x},\omega}(\mathbf{k})&=e^{i\mathbf{kx}}\frac{\sqrt{f_h(\mathbf{k})}}{|\mathbf{k}|}\cdot\begin{cases}(ik_0+k_3,0,k_1-ik_2,0,0,0,0,0), & \text{if }\omega=1,\\ (0,k_1+ik_2,0,ik_0-k_3,0,0,0,0), & \text{if }\omega=2,\\ (0,0,0,0,ik_0+k_3,0,-k_1+ik_2,0), & \text{if }\omega=3,\\ (0,0,0,0,0,-k_1-ik_2,0,ik_0+k_3), & \text{if }\omega=4,\end{cases}\\
B^{(h)}_{\mathbf{x},\omega}(\mathbf{k})&=e^{i\mathbf{ky}}\frac{\sqrt{f_h(\mathbf{k})}}{|\mathbf{k}|}\cdot\begin{cases}(1,1,0,0,0,0,0,0), & \text{if }\omega=1,\\ (0,0,1,1,0,0,0,0), & \text{if }\omega=2.\\ (0,0,0,0,1,1,0,0), & \text{if }\omega=3,\\ (0,0,0,0,0,0,1,1), & \text{if }\omega=4.\end{cases}
\end{aligned} \tag{2.34}$$

By using (2.26) and (2.30) and $|\int d\mathbf{r}v(\mathbf{r})|\le C\gamma^{-2N}$ and assuming $|z_k|\le C|\lambda|$,$|Z^{(1)}_k-1|\le C|\lambda|$ we find

$$\int d\mathbf{x}_{v_0}|W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0})|\le L^4\gamma^{-h[-4+\frac{3|P_{v_0}|}{2}-2m_{4,v_0}+n^J_{v_0}]}$$
$$\prod_{v\text{ not e.p.}}\left\{\frac{1}{s_v!}C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}\gamma^{-[-4+\frac{3|P_v|}{2}-2m_{4,v}+n^J_v+z_v]}\right\}[\prod_v\gamma^{-2N\bar m_{4,v}}] \tag{2.35}$$

where for the integration over the interpolated points we proceed as discussed in §3.15 of [BM].
Finally by using the identity

$$\gamma^{h2m_{4,v_0}}\prod_v\gamma^{(h_v-h_{v'})2m_{4,v}}=\prod_v\gamma^{h_v2\bar m_{4,v}} \tag{2.36}$$

we finally obtain

$$\int d\mathbf{x}_{v_0}|W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0})|\le L^4\gamma^{-h[-4+\frac{3|P_{v_0}|}{2}+n^J_{v_0}]}C^n|\lambda|^{\bar n}\prod_{v\text{ not e.p.}}\left\{\frac{1}{s_v!}\gamma^{-[-4+\frac{3|P_v|}{2}+n^J_v+z_v]}\right\}[\prod_v\gamma^{-2(N-h_v)\bar m_{4,v}}] \tag{2.37}$$

2.3 *Bound for the effective potential*
We bound the contribution to the effective potential, which is given by

$$V^{(k)}(\psi^{(\le k)},J,\phi)+L^4\tilde E^2_{k+1}=\sum_{n=1}^\infty\sum_{\tau\in\mathcal{T}^*_{k,n}}V^{(k)}(\tau) \tag{2.38}$$,

where $\mathcal{T}^*_{k,n}$ is the set of trees with $n_\phi=n_J=0$; note that in such a case $4-\frac{3|P_v|}{2}+z_v>0$.
Regarding the sum over T, it is empty if $s_v=1$. If $s_v>1$ and $N_{v_i}\equiv|P_{v_i}|-|Q_{v_i}|$, the number of anchored trees with d_i lines branching from the vertex v_i can be bounded, by using Caley's formula, by (see for instance [GM])

$$\frac{(s_v-2)!}{(d_1-1)!...(d_{s_v}-1)!}N^{d_1}_{v_1}...N^{d_{s_v}}_{v_{s_v}}\ ;$$

hence the number of addenda in $\sum_{T\in\mathbf{T}}$ is bounded by $\prod_{v \text{ not e.p.}} s_v!\ C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}$.
In order to bound the sums over the scale labels and $\mathbf{P}$ we first use the inequality

$$O(e(\kappa\gamma^{-N})^\theta)\prod_{v \text{ not e.p.}}\gamma^{-[-4+\frac{3|P_v|}{2}+z_v]}\le[\prod_{\tilde v}\gamma^{-\frac{1}{40}(h_{\tilde v}-h_{\tilde v'})}][\prod_{v \text{ not e.p.}}\gamma^{-\frac{|P_v|}{40}}] \tag{2.39}$$

where $\tilde v$ are the non trivial vertices, and $\tilde v'$ is the non trivial vertex immediately preceding $\tilde v$ or the root. The factors $\gamma^{-\frac{1}{40}(h_{\tilde v}-h_{\tilde v'})}$ in the r.h.s. of (2.39) allow to bound the sums over the scale labels by C^n.
Finally the sum over $\mathbf{P}$ can be bounded by using the following combinatorial inequality. Let $\{p_v, v\in\tau\}$ a set of integers such that $p_v\le\sum_{i=1}^{s_v}p_{v_i}$ for all $v\in\tau$ which are not endpoints; then

$$\prod_{v \text{ not e.p.}}\sum_{p_v}\gamma^{-\frac{p_v}{40}}\le C^n\ . \tag{2.40}$$

We have finally to show that z_h is bounded for any h. By construction z_h verifies an equation of the form

$$z_{h-1}=z_h+\beta_z^{(h)}(z_{h+1},..,z_N,\lambda) \tag{2.41}$$

and $\beta_z^{(h)}(z_{h+1},..,z_N,\lambda)$ is sum of trees with at least one end-point of type λ, as the trees with only z end-points are vanishing at zero momenta, for the compact support properties of the propagators. Calling $\bar h$ the scale of a λ end-point, the factor $\gamma^{-2(N-\bar h)}$ in (2.37) and the fact that $4-\frac{3|P_v|}{2}+z_v>0$ implies that, for a suitable constant $\theta>0$

$$|\beta_z^{(h)}(z_{h+1},..,z_N,\lambda)|\le C|\lambda|\gamma^{2\theta(h-N)} \tag{2.42}$$

and by iteration

$$|z_h|\le\sum_{k=h}^N|\beta_z^{(h)}|\le C|\lambda| \tag{2.43}$$

This implies that the kernels $W_{2l}^{(h)}$ of the effective potential verify the bound

$$\frac{1}{L^4}\int d\underline{\mathbf{x}}|W_{2l}^{(h)}(\underline{\mathbf{x}})|\le C^l\gamma^{-h(-4+3l)}|\lambda|^{l-1} \tag{2.44}$$

2.4 *Bounds for the correlations*
We have obtained an expression for $\langle j^5_{\mu,\mathbf{p}};\psi_\mathbf{k}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle$ of the form

$$\langle j_{\mu,\mathbf{p}};\psi_\mathbf{k}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle=\sum_{n=0}^\infty\sum_{j_0=-\infty}^{N-1}\sum_{\tau\in\mathcal{T}_{j_0,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}\\|P_{v_0}|=2}}G_\tau(\mathbf{p},\mathbf{k})\ ,$$

Note first that $Z_h^{(1)}=1+O(\lambda))$, as also the beta function for $Z_h^{(1)}$ verifies the bound (2.42).
Let us define, for any $\mathbf{k}\neq 0$, $h_\mathbf{k}=\min\{j: f_j(\mathbf{k})\neq 0\}$ and suppose that $\mathbf{p}$, $\mathbf{k}$, $\mathbf{p}-\mathbf{k}$ are all different from 0. It follows that, given τ, if h_- and h_+ are the scale indices of the ψ fields belonging to the endpoints associated with ϕ^+ and ϕ^-, while h_J denotes the scale of the endpoint of type J, $G^{2,1}_\tau(\mathbf{p},\mathbf{k})$ can be different from 0 only if $h_-=h_\mathbf{k},h_\mathbf{k}+1$, $h_+=h_{\mathbf{k}-\mathbf{p}},h_{\mathbf{k}-\mathbf{p}}+1$ and $h_J\ge h_\mathbf{p}-\log_\gamma 2$. Moreover, if $\mathcal{T}_{j_0,n,\mathbf{p},\mathbf{k}}$ denotes the set of trees satisfying the previous conditions and $\tau\in\mathcal{T}_{j_0,n,\mathbf{p},\mathbf{k}}$, $|G_\tau(\mathbf{p},\mathbf{k})|$ can be bounded by $\int d\mathbf{z}d\mathbf{x}|G_\tau(\mathbf{z};\mathbf{x},\mathbf{y})|$. Hence we get, if $d_v=-4+\frac{3|P_v|}{2}+n^J_v+z_v$

$$|\langle j_{\mu,\mathbf{p}};\psi_\mathbf{k}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle|\le C\gamma^{-h_\mathbf{k}}\gamma^{-h_{\mathbf{k}-\mathbf{p}}}\sum_{n=0}^\infty\sum_{j_0=h-1}^{N-1}\sum_{\tau\in\mathcal{T}_{j_0,n,\mathbf{p},\mathbf{k}}}\sum_{\substack{\mathbf{P}\in\mathcal{P}\\|P_{v_0}|=2}}(C|\lambda|)^n\prod_{\text{v not e.p.}}\gamma^{-d_v}\ . \tag{2.45}$$

Given $\tau \in \mathcal{T}_{j_0,n,\mathbf{p},\mathbf{k}}$, let v_0^* the higher vertex preceding all three special endpoints and $v_1^* \geq v_0^*$ the higher vertex preceding either the two endpoints of type ϕ (to be called $v_{\phi,+}$ and $v_{\phi,-}$) or one endpoint of type ϕ and the endpoint of type J (to be called v_J). It turns out that $d_v > 0$, except for the vertices belonging to the path $\mathcal{C}^*$ connecting v_1^* with v_0^*, where, if $|P_v| = 2$ and $n_v^J = 1$, $d_v = 0$. Hence, we can sum over the scale and P_v labels of τ, only if we fix the scale indices h_0^* and h_1^* of v_0^* and v_1^*, after multiplying by $\gamma^{-\delta(h_1^*-h_0^*)}$, δ being any positive number. Of course, we have also to perform the sum over h_0^*, h_1^* of $\gamma^{\delta(h_1^*-h_0^*)}$, which is divergent, if we proceed exactly in this way.

In order to solve this problem, we note that, if $v \notin \mathcal{C}^*$, $d_v - 1/4 > 0$. Hence, before performing the sums over the scale and P_v labels, we can extract from each γ^{-d_v} factor associated with the vertices belonging to the paths connecting the three special endpoints with v_0^* or v_1^*, a $\gamma^{-1/4}$ piece, to be used to perform safely the sums over h_0^*, h_1^* in the following way.

Let us consider first the family $\mathcal{T}^{(1)}_{j_0,n,\mathbf{p},\mathbf{k}}$ of trees such that the two special endpoints following v_1^* are $v_{\phi,+}$ and $v_{\phi,-}$ and let us suppose that $|\mathbf{k}| \geq |\mathbf{k}-\mathbf{p}|$. In this case, before doing the sums over the the scale and P_v labels, we fix also the scale h_J of v_J. We get, if $h_J^* \equiv \max\{h_\mathbf{p}+2, h_0^*+1\}$, if $d_v = -4 + \frac{3|P_v|}{2} + n_v^J + z_v$:

$$\sum_{n=0}^{\infty} \sum_{j_0=h-1}^{N-1} \sum_{\tau \in \mathcal{T}^{(1)}_{j_0,n,\mathbf{p},\mathbf{k}}} \sum_{\substack{\mathbf{P}\in\mathcal{P} \\ |P_{v_0}|=2}} (C|\lambda|)^n \prod_{\text{v not e.p.}} \gamma^{-d_v} \leq$$

$$\leq C \sum_{h_1^*=-\infty}^{h_{\mathbf{k}-\mathbf{p}}} \sum_{h_0^*=-\infty}^{h_1^*} \sum_{h_J=h_J^*}^{+\infty} \gamma^{\delta(h_1^*-h_0^*)} \gamma^{-\frac{1}{4}[(h_\mathbf{k}-h_1^*)+(h_{\mathbf{k}-\mathbf{p}}-h_1^*)+(h_J-h_0^*)]} \tag{2.46}$$

and the r.h.s. can be bounded by, multiplying and dividing by $\gamma^{\delta(h_\mathbf{k}-h_\mathbf{p})}$

$$C' \gamma^{\delta(h_\mathbf{k}-h_\mathbf{p})} \sum_{h_1^*=-\infty}^{h_{\mathbf{k}-\mathbf{p}}} \gamma^{\delta(h_1^*-h_\mathbf{k})} \gamma^{-\frac{1}{4}[(h_\mathbf{k}-h_1^*)+(h_{\mathbf{k}-\mathbf{p}}-h_1^*)} \sum_{h_0^*=-\infty}^{h_1^*} \gamma^{\delta(h_\mathbf{p}-h_0^*)} \gamma^{-\frac{1}{4}(h_J^*-h_0^*)} \tag{2.47}$$

so that the r.h.s. of (2.47) is bounded by $C\gamma^{\delta(h_\mathbf{k}-h_\mathbf{p})}$, if $\delta \leq 1/8$. If $|\mathbf{k}-\mathbf{p}| \geq |\mathbf{k}|$, we get a similar result, with $h_{\mathbf{k}-\mathbf{p}}$ in place of $h_\mathbf{k}$.

Let us consider now the family $\mathcal{T}^{(2,+)}_{j_0,n,\mathbf{p},\mathbf{k}}$ of trees such that the two special endpoints following v_1^* are v_J and $v_{\phi,+}$. We get, if $h_J^* \equiv \max\{h_\mathbf{p}+2, h_1^*+1\}$ and $\bar{h}_0 = \min\{h_{\mathbf{k}-\mathbf{p}}, h_1^*\}$:

$$\sum_{n=0}^{\infty} \sum_{j_0=h-1}^{N-1} \sum_{\tau \in \mathcal{T}^{(1,+)}_{j_0,n,\mathbf{p},\mathbf{k}}} \sum_{\substack{\mathbf{P}\in\mathcal{P} \\ |P_{v_0}|=2}} (C|\lambda|)^n \prod_{\text{v not e.p.}} \gamma^{-d_v} \leq$$

$$\leq C \sum_{h_1^*=-\infty}^{h_\mathbf{k}} \sum_{h_0^*=-\infty}^{\bar{h}_0} \sum_{h_J=h_J^*}^{+\infty} \gamma^{\delta(h_1^*-h_0^*)} \gamma^{-\frac{1}{4}[(h_\mathbf{k}-h_1^*)+(h_{\mathbf{k}-\mathbf{p}}-h_0^*)+(h_J-h_1^*)]} \tag{2.48}$$

and it is easy to prove that, if $\delta \leq 1/8$, the r.h.s. of (2.48) is bounded by $C\gamma^{\delta(h_\mathbf{k}-h_{\mathbf{k}-\mathbf{p}})}$, if $|\mathbf{k}| \geq |\mathbf{k}-\mathbf{p}|$, by a constant, otherwise. The family $\mathcal{T}^{(2,-)}_{j_0,n,\mathbf{p},\mathbf{k}}$ of trees such that the two special endpoints following v_1^* are v_J and $v_{\phi,-}$ can be treated in a similar way and one obtains a bound $C\gamma^{\delta(h_{\mathbf{k}-\mathbf{p}}-h_\mathbf{k})}$, if $|\mathbf{k}-\mathbf{p}| \geq |\mathbf{k}|$, or a constant, otherwise.

By putting together all these bounds, we get, for any positive $\delta \leq 1/8$:

$$|\langle j_{\mu,\mathbf{p}}; \psi_\mathbf{k} \bar{\psi}_{\mathbf{k}-\mathbf{p}} \rangle| \leq \frac{C_\delta}{|\mathbf{k}||\mathbf{k}-\mathbf{p}|} [\left(\frac{|\mathbf{k}|}{|\mathbf{p}|}\right)^\delta + \left(\frac{|\mathbf{k}-\mathbf{p}|}{|\mathbf{p}|}\right)^\delta + \left(\frac{|\mathbf{k}|}{|\mathbf{k}-\mathbf{p}|}\right)^\delta + \left(\frac{|\mathbf{k}-\mathbf{p}|}{|\mathbf{k}|}\right)^\delta] \tag{2.49}$$

with $C_\delta \to \infty$ as $\delta \to 0$.

Note that the contributions from all trees with at least a λ end-point admit a similar bound with an extra $\gamma^{-\theta(N-h_\mathbf{k})}$, for the presence of the factor $\prod_v \gamma^{-2(N-h_v)\bar{m}_{4,v}}$ in (2.37). This means that the only non vanishing trees, at fixed $\mathbf{k},\mathbf{p}$ and in the limit $N\to\infty$, are the ones with one $Z^{(1)}$ end-point and any number of Z end-points; the corresponding Feynman graphs are chain graphs. For similar reasons, only trees with no λ end-points contribute in the limit $N\to\infty$ at finite $\mathbf{k}$ to $\langle\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}\rangle$.

3. Ward Identities

3.1 *Ward Identities with cut-offs*
With the transformations

$$\psi_{\mathbf{x}}\to e^{i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}\qquad\bar\psi_{\mathbf{x}}\to\bar\psi_{\mathbf{x}}e^{-i\alpha_{\mathbf{x}}}\tag{3.1}$$

the generating function becomes

$$e^{\mathcal{W}_{N,L}(J,\phi)}=\int P(d\psi^{(\le N)})e^{-\int d\mathbf{x}\bar\psi_{\mathbf{x}}(e^{i\alpha_{\mathbf{x}}}D^{(\le N)}e^{-i\alpha_{\mathbf{x}}}-D^{(\le N)})\psi^-_{\mathbf{x}}}e^{-V+(\int d\mathbf{x}[\phi_{\mathbf{x}}\bar\psi_{\mathbf{x}}e^{-i\alpha_{\mathbf{x}}}+e^{-i\alpha_{\mathbf{x}}}\bar\phi_{\mathbf{x}}\psi_{\mathbf{x}}]}\tag{3.2}$$

where $D^{(\le N)}\psi_{\mathbf{x}}=\frac{1}{L^4}\sum_{\mathbf{k}}e^{-i\mathbf{k}\mathbf{x}}C^{\varepsilon}_N(\mathbf{k})i\not{\mathbf{k}}\psi_{\mathbf{k}}$. In deriving the above expression we have used that that the Jabobian corresponding to the transformation (3.1) is 1; in fact

$$\int\mathcal{D}\psi[\prod_{\mathbf{x}\in A^+}\bar\psi_{\mathbf{x}}][\prod_{\mathbf{y}\in A^-}\psi_{\mathbf{y}}]=\frac{1}{L^{4|A^+|}}\sum_{\mathbf{k}_1\ldots\mathbf{k}_{|A^+|}}e^{i\sum_{i=1}^{|A^+|}\mathbf{k}_i\mathbf{x}_i}\frac{1}{L^{4|A^-|}}\sum_{\mathbf{p}_1\ldots\mathbf{p}_{|A^-|}}e^{-i\sum_{i=1}^{|A^-|}\mathbf{p}_i\mathbf{y}_i}\int\mathcal{D}\psi[\prod_{i=1}^{|A^+|}\bar\psi_{\mathbf{k}_i}][\prod_{i=1}^{|A^-|}\psi_{\mathbf{p}_i}]\tag{3.3}$$

By definition of Grassman variables the only nonvanishing term in the sum in the r.h.s. correspond to an addend with all the $\mathbf{k},\mathbf{p}$ distinct; moreover one can have a nonvanishing term only if $\{\mathbf{k}_i\}_{i=1}^{A^+}=\{\mathbf{p}_i\}_{i=1}^{A^-}=\mathcal{D}$, and this is possible only if the $\mathbf{x}_i,\mathbf{y}_i$ are all different and such that $\{\mathbf{x}_i\}_{i=1}^{A^+}=\{\mathbf{y}_i\}_{i=1}^{A^-}=\Lambda_a$, so that

$$\int\mathcal{D}\psi[\prod_{\mathbf{x}\in A^+}\bar\psi_{\mathbf{x}}][\prod_{\mathbf{y}\in A^-}\psi_{\mathbf{y}}]=\int\mathcal{D}\psi\prod_{\omega}[\prod_{\mathbf{x}\in A^+}e^{i\alpha_{\mathbf{x}}}\bar\psi_{\omega,\mathbf{x}}][\prod_{\mathbf{x}\in A^-}e^{-i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}]\tag{3.4}$$

We use write the exponent of (3.2) as

$$\int d\mathbf{x}[\bar\psi_{\mathbf{x}}D^{(\le N)}(\alpha_{\mathbf{x}}\psi_{\mathbf{x}})-\alpha_{\mathbf{x}}\bar\psi_{\mathbf{x}}D^{(\le N)}\psi_{\mathbf{x}}]=\int d\mathbf{x}\alpha_{\mathbf{x}}[(D^{(\le N)}\bar\psi_{\mathbf{x}})\psi_{\mathbf{x}}+\bar\psi_{\mathbf{x}}(D^{(\le N)}\psi_{\mathbf{x}})]=\int d\mathbf{x}\alpha_{\mathbf{x}}[\not\partial(\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}})+\alpha_{\mathbf{x}}\delta j_{\mathbf{x}}]\tag{3.5}$$

where

$$\delta j_{\mathbf{x}}=\frac{1}{L^8}\sum_{\mathbf{k}^+\ne\mathbf{k}^-}e^{i(\mathbf{k}^+-\mathbf{k}^-)\mathbf{x}}C(\mathbf{k}^+,\mathbf{k}^-)\bar\psi_{\mathbf{k}^+}\psi_{\mathbf{k}^-}\tag{3.6}$$

and

$$C(\mathbf{k}^+,\mathbf{k}^-)=(\chi_N^{-1}(\mathbf{k}^-)-1)i\not{\mathbf{k}}_--(\chi_N^{-1}(\mathbf{k}^+)-1)i\not{\mathbf{k}}_+\tag{3.7}$$

Differentiating with respect to $\alpha_{\mathbf{x}}$ both sides of (3.2), and setting $\alpha_{\mathbf{x}}=0$ we find and finally making a derivative with respect to $\phi_{\mathbf{x}},\phi_{\mathbf{y}}$ and passing to Fourier transform

$$-i\mathbf{p}_\mu\langle j^\mu_{\mathbf{p}};\bar\psi_{\mathbf{k+p}}\bar\psi^-_{L,\mathbf{k}}\rangle_{N,L}=\langle\bar\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}}\rangle_{N,L}-\langle\bar\psi^+_{\mathbf{k+p}}\psi^-_{\mathbf{k+p}}\rangle_{N,L}-\langle\delta j_{\mathbf{p}};\bar\psi_{\mathbf{k+p}}\psi_{\mathbf{k}}\rangle_{N,L}\tag{3.8}$$

By comparing (3.8) with (1.4), we see that they differ form the presence of the last term in (3.8), which take into account of the presence of the ultraviolet cut-off and it is indeed of the same order of magnitude of the other correlations in (3.8). We will prove in the following section the following identity

$$\langle\delta j_{\mathbf{p}};\bar\psi_{\mathbf{k+p}}\psi_{\mathbf{k}}\rangle_{N,L}=-\nu_N\langle j_{\mathbf{p}};\bar\psi_{\mathbf{k+p}}\psi_{\mathbf{k}}\rangle_{N,L}+R_{N,L}(\mathbf{p};\mathbf{k})\tag{3.9}$$

where $\nu_N=-i\not{\mathbf{p}}(c_+e^2+O(e^4))$, for a suitable constant c_+ and $R_N(\mathbf{k},\mathbf{p})$ is a small correction, as it verifies the same bound as $\mathbf{p}_\mu\langle j^\mu_{\mathbf{p}};\bar\psi_{\mathbf{k+p}}\bar\psi^-_{L,\mathbf{k}}\rangle$ up to an extra factor $\gamma^{-\theta(N-h_{\mathbf{k}})}$, if $h_{\mathbf{k}}$ is the scale of $\mathbf{k}$. In order to do this we write $R_{N,L}(\mathbf{p},\mathbf{k})$ as

$$R_{N,L}(\mathbf{p};\mathbf{k})=\frac{\partial^3}{\partial J_{\mathbf{p}}\partial\phi_{\mathbf{k}}\partial\bar\phi_{\mathbf{k-p}}}\hat W_{N,L}|_{J=\phi=\bar\phi=0}$$

$$e^{\hat{\mathcal{W}}_{N,L}(J,\phi)}=\int P(d\psi^{(\leq N)})e^{-V^{(N)}(\psi)+\frac{1}{L^4}\sum_{\mathbf{p}\in\mathcal{D}}J_{\mathbf{p}}\bar\chi_N(\mathbf{p})\delta j_{\mathbf{p}}+\nu_N\frac{1}{L^4}\sum_{\mathbf{p}\in\mathcal{D}}J_{\mathbf{p}}\mathbf{p}_\mu j^\mu_{\mathbf{p}}+\frac{1}{L^4}\sum_{\mathbf{k}\in\mathcal{D}}[\psi_{\mathbf{k}}\bar\phi_{\mathbf{k}}+\bar\psi_{\mathbf{k}}\phi_{\mathbf{k}}]} \tag{3.10}$$

and $\hat W_{N,L}$ is analyzed by a multiscale integration similar to the one on §2: note that we can freely multiply $\delta j_{\mathbf{p}}$ by a smooth compact support function $\bar\chi(\mathbf{p})$, equal to 1 for $|\mathbf{p}|\leq 3\gamma^N$ and 0 for $|\mathbf{p}|\geq 4\gamma^N$, as by the compact support properties of the propagators the values of $\mathbf{p}$ are such that $\bar\chi(\mathbf{p})=1$.

In the massless case, by the change of variables $\psi_{\mathbf{x}}\to e^{i\gamma^5\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}\qquad \bar\psi_{\mathbf{x}}\to\bar\psi_{\mathbf{x}}e^{-i\gamma^5\alpha_{\mathbf{x}}}$ and proceeding in the same way we get

$$-i\mathbf{p}_\mu\langle j^5_{\mu,\mathbf{p}};\bar\psi_{\mathbf{k}+\mathbf{p}}\bar\psi^-_{L,\mathbf{k}}\rangle_{N,L}=\langle\bar\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}}\rangle_{N,L}-\langle\bar\psi^+_{\mathbf{k}+\mathbf{p}}\psi^-_{\mathbf{k}+\mathbf{p}}\rangle_{N,L}-\langle\delta j^5_{\mathbf{p}};\bar\psi_{\mathbf{k}+\mathbf{p}}\psi_{\mathbf{k}}\rangle_{N,L} \tag{3.11}$$

and also in such case a decomposition like (3.9) holds; in the following we will prove (3.9), but all the computations can be repeated in the case (3.10) up to some obvious modifications.

3.2 *Analysis of the corrections*

We integrate $\hat{\mathcal{W}}_{N,L}(J,\phi)$ following a multiscale procedure writing

$$e^{\hat{\mathcal{W}}_{N,L}(J,\phi)}=\int P(d\psi^{(\leq N-1)})\int P(d\psi^{(N)})e^{-V^{(N)}(\psi)+\frac{1}{L^4}\sum_{\mathbf{p}\in\mathcal{D}}J_{\mathbf{p}}\delta j_{\mathbf{p}}+\nu_N\frac{1}{L^4}\sum_{\mathbf{p}\in\mathcal{D}}J_{\mathbf{p}}\mathbf{p}_\mu j^\mu_{\mathbf{p}}+\frac{1}{L^4}\sum_{\mathbf{k}\in\mathcal{D}}[\psi_{\mathbf{k}}\bar\phi_{\mathbf{k}}+\bar\psi_{\mathbf{k}}\phi_{\mathbf{k}}]} \tag{3.12}$$

and integrating iteratively the fields $\psi^{(N)},\psi^{(N-1)}...,\psi^{(h)}$.

In the expansion for $R_{N,L}(\mathbf{p},\mathbf{k})$ the following quantity will appear

$$\Delta^{h,k}(\mathbf{k}^+,\mathbf{k}^-)=g^{(h)}(\mathbf{k}^+)C^\mu(\mathbf{k}^+,\mathbf{k}^-)g^{(k)}(\mathbf{k}^-)=\bar\chi(\mathbf{p})\frac{f_h(\mathbf{k}^+)}{i\not{\mathbf{k}}^+}[\frac{f_k(\mathbf{k}^-)}{\chi_N(\mathbf{k}^-)}-f_k(\mathbf{k}^-)]-\frac{f_k(\mathbf{k}^-)}{i\not{\mathbf{k}}^-}[\frac{f_h(\mathbf{k}^+)}{\chi_N(\mathbf{k}^+)}-f_h(\mathbf{k}^+)] \tag{3.13}$$

such that

$$\Delta^{h,k}(\mathbf{k}^+,\mathbf{k}^-)=0\qquad h,k<N \tag{3.14}$$

as $\chi_N(\mathbf{k})=1$ in the support of $g^{(h)}(\mathbf{k})$ when $h\leq N-1$.

Moreover for $j\leq N-1$ we can write

$$\Delta^{N,j}(\mathbf{k}^+,\mathbf{k}^-)=-\chi(\mathbf{p})\frac{f_j(\mathbf{k}^-)u_N(\mathbf{k}^+)}{i\not{\mathbf{k}}_-}=-\chi(\mathbf{p})\frac{f_j(\mathbf{k}^-)}{i\not{\mathbf{k}}_-}\mathbf{p}_\mu\int_0^1 dt\partial_\mu u_N(\mathbf{k}_++t\mathbf{p})\equiv\mathbf{p}_\mu S^{N,j}_\mu(\mathbf{k}_+,\mathbf{k}_-)$$

where $u_N(\mathbf{k})=0$ for $|\mathbf{k}|\leq\gamma^N$ and $u_N(\mathbf{k})=1-f_N(\mathbf{k})$ for $|\mathbf{k}|\geq\gamma^N$; in fact $\frac{f_j(\mathbf{k})}{\chi_N(\mathbf{k})}-f_j(\mathbf{k})=0$ for $j\leq N-1$ and $\frac{f_N(\mathbf{k})}{\chi_N(\mathbf{k})}-f_N(\mathbf{k})\equiv u_N(\mathbf{k})$ is vanishing for $|\mathbf{k}|\leq\gamma^N$ and it is equal to $1-f_N(\mathbf{k})$ for $|\mathbf{k}|\geq\gamma^N$. A similar expression can be written for $\Delta^{j,N}$, using that $\Delta^{h,k}$ is symmetric. The Fourier transform

$$S^{(N,j)}_\mu(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{z})=\int d\mathbf{k}_+d\mathbf{k}_-e^{-i(\mathbf{k}_+-\mathbf{k}_-)\mathbf{x}}e^{i\mathbf{k}_+\mathbf{y}}e^{-i\mathbf{k}_-\mathbf{z}}S^{(N,j)}(\mathbf{k}_+,\mathbf{k}_-) \tag{3.15}$$

verifies the bound

$$|S^{N,j}_\mu(\mathbf{z}-\mathbf{x},\mathbf{z}-\mathbf{y})|\leq C_n\frac{\gamma^{3N}}{1+[\gamma^N|\mathbf{z}-\mathbf{x}|]^n}\frac{\gamma^{3j}}{1+[\gamma^j|\mathbf{z}-\mathbf{y}|]^n} \tag{3.16}$$

The bound follows integrating by parts, noting that volume factors are γ^{4j} (for the support of f_j) and γ^{4N} (for the support of $\bar\chi(\mathbf{p})u_N(\mathbf{k}^+)$), and each derivative with respect to $\mathbf{k}_-$ gives and extra γ^{-j} while each derivative with respect to $\mathbf{k}_+$ gives an extra γ^{-N}. In the same way

$$\Delta^{N,N}(\mathbf{k}^+,\mathbf{k}^-)=\chi(\mathbf{p})\mathbf{p}_\mu[\frac{f_N(\mathbf{k}^-)}{\not{\mathbf{k}}_-}\int_0^1 dt\partial_\mu u_N(\mathbf{k}_++t\mathbf{p})-\frac{f_j(\mathbf{k}^+)}{\not{\mathbf{k}}_+}\int_0^1 dt\partial_\mu u_N(\mathbf{k}_-+t\mathbf{p})]$$

After the integration of ψ^N in (3.12) we get

$$e^{\hat{\mathcal{W}}_{N,L}(J,\phi)}=e^{-L^4E_N+\hat S_N(\phi,J)}\int P(d\psi^{(\leq N-1)})e^{-\mathcal{V}^{(N-1)}(\psi^{(\leq N-1)})+\mathcal{B}^{(N-1)}(\psi,J,\phi)} \tag{3.17}$$

where $\mathcal{V}^{(N-1)}$ is the same as in (2.9) and $\mathcal{B}^{(N-1)}$ can be again decomposed as in (2.12); we can write

$$\mathcal{B}_1^{(N-1)}(\psi^{(\leq N-1)},J)=\mathcal{B}_1^{a(N-1)}(\psi^{(\leq N-1)},J)+\mathcal{B}_1^{b(N-1)}(\psi^{(\leq N-1)},J) \tag{3.18}$$

where $\mathcal{B}_1^{b(N-1)}(\psi^{(\leq N-1)},J)$ is obtained from the contraction of $\nu_N\frac{1}{L^4}\sum_{\mathbf{p}\in\mathcal{D}}J_{\mathbf{p}}\mathbf{p}_\mu j_{\mu,\mathbf{p}}$ and $\mathcal{B}_1^{a(N-1)}(\psi^{(\leq N-1)},J)$ is obtained from the contraction of $\frac{1}{L^4}\sum_{\mathbf{p}\in\mathcal{D}}J_{\mathbf{p}}\delta j_{\mathbf{p}}$. The $\mathcal{L}$ operation is defined as in §2 for $\mathcal{V}^{(N-1)}$, $\mathcal{B}_2^{(N-1)}$ and $\mathcal{B}_3^{(N-1)}$. We can write

$$\mathcal{B}_1^{b(N-1)}=\sum_{l=2}^\infty\frac{1}{L^4}\sum_{\mathbf{p}}\frac{1}{L^4}\sum_{\mathbf{k}_1}\cdots\frac{1}{L^4}\sum_{\mathbf{k}_{2l}}\mathbf{p}_\mu W_{\mu,2l}^{(N-1)}(\mathbf{p},\underline{\mathbf{k}})J(\mathbf{p})\delta(\mathbf{p}+\sum_i\varepsilon_i\mathbf{k}_i)\prod_{i=2}^{2l}\psi_{\mathbf{k}_i}^{\varepsilon_i}$$

and we define $\mathcal{L}=0$ for $l>1$ and

$$\mathcal{L}W_{\mu,2}^{(N-1)}(\mathbf{p},\underline{\mathbf{k}})=W_{\mu,2}^{(N-1)}(\mathbf{0},\underline{\mathbf{0}}) \tag{3.19}$$

and again by symmetry $W_{\mu,2}^{(N-1)}(\mathbf{0},\underline{\mathbf{0}})=\gamma_\mu Z$ where Z is a constant.

Finally we define $\mathcal{L}=0$ for all the terms in $\mathcal{B}_1^{a(N-1)}$ with more than two fermionic fields; the terms with two fields can be written as

$$\frac{1}{L^4}\sum_{\mathbf{p}}J_{\mathbf{p}}\mathbf{p}_\mu\frac{1}{L^4}\sum_{\mathbf{k}}\bar\psi_{\mathbf{k}}\psi_{\mathbf{k}+\mathbf{p}}\{\frac{1}{L^4}\sum_{\tilde{\mathbf{k}}}S_\mu^{N,N}(\tilde{\mathbf{k}},\tilde{\mathbf{k}}-\mathbf{p})G_4(\tilde{\mathbf{k}},\mathbf{k},\mathbf{k}-\mathbf{p})+ \tag{3.20}$$

$$[(\chi_N(\mathbf{k})^{-1}-1)i\not{\mathbf{k}}g^{(N)}(\mathbf{k}+\mathbf{p})-u_N(\mathbf{k}+\mathbf{p})]G^2(\mathbf{k}+\mathbf{p})+[(\chi_N(\mathbf{k}+\mathbf{p})^{-1}-1)(i\not{\mathbf{k}}+i\not{\mathbf{p}})g^{(N)}(\mathbf{k})-u_N(\mathbf{k})]G_2(\mathbf{k})\}$$

where we distinguish the terms obtained from the contraction of δj are such that or the two fields in δj or one field are contracted; the case corresponding the two fields non contracted is absent thanks to (3.14).

We can replace $G_2(\mathbf{k})$ in (3.20) with $G_2(\mathbf{k})-G_2(0)$, using that, by parity, $G_2(0)=0$; this improve the bound of a factor $\gamma^{-N+h'}$, where h' is the scale at which one of the external ψ fields are contracted. Finally we define the $\mathcal{L}$ operation over the first addend of (3.20) as

$$\mathcal{L}\frac{1}{L^4}\sum_{\mathbf{p}}J_{\mathbf{p}}\mathbf{p}_\mu\frac{1}{L^4}\sum_{\mathbf{k}}\bar\psi_{\mathbf{k}}\psi_{\mathbf{k}+\mathbf{p}}[\frac{1}{L^4}\sum_{\tilde{\mathbf{k}}}S_\mu^{N,N}(\tilde{\mathbf{k}},\tilde{\mathbf{k}}-\mathbf{p})G_4^N(\tilde{\mathbf{k}},\mathbf{k},\mathbf{k}-\mathbf{p})]=$$

$$\frac{1}{L^4}\sum_{\mathbf{p}}J_{\mathbf{p}}\mathbf{p}_\mu\frac{1}{L^4}\sum_{\mathbf{k}}\bar\psi_{\mathbf{k}}\psi_{\mathbf{k}+\mathbf{p}}[\frac{1}{L^4}\sum_{\tilde{\mathbf{k}}}S_\mu^{N,N}(\tilde{\mathbf{k}},\tilde{\mathbf{k}})G_4(\tilde{\mathbf{k}},0,0)] \tag{3.21}$$

By Euclidean invariance we find

$$\mathcal{L}\mathcal{B}_1^{(N-1)}(\psi^{(\leq N-1)},J)=\nu_{N-1}\int d\mathbf{x}\bar\psi_{\mathbf{x}}\not{\partial}\psi_{\mathbf{x}} \tag{3.22}$$

The procedure can be iterated integrating the fields $\psi^{N-1},\psi^{N-2},..\psi^h$; we can still decompose $\mathcal{B}_1^{(h)}$ as $\mathcal{B}_1^{a(h)}+\mathcal{B}_2^{b(h)}$ where $\mathcal{B}_2^{b(h)}$ is obtained from the contraction of $\nu_k\int d\mathbf{x}\bar\psi_{\mathbf{x}}\not{\partial}\psi_{\mathbf{x}}$ for some $k\geq h$ and

$$\mathcal{B}_1^{a(h)}=\frac{1}{L^4}\sum_{\mathbf{p}}J_{\mathbf{p}}\mathbf{p}_\mu\frac{1}{L^4}\sum_{\mathbf{k}}\bar\psi_{\mathbf{k}}\psi_{\mathbf{k}+\mathbf{p}}[\frac{1}{L^4}\sum_{\tilde{\mathbf{k}}}[S_\mu^{N,h}(\tilde{\mathbf{k}},\tilde{\mathbf{k}}-\mathbf{p})+S_\mu^{h,N}(\tilde{\mathbf{k}},\tilde{\mathbf{k}}-\mathbf{p})]G_4^h(\tilde{\mathbf{k}},\mathbf{k},\mathbf{k}-\mathbf{p})] \tag{3.23}$$

and the $\mathcal{L}$ operation is defined as before.

Assume now that

$$|\nu_h|\leq c_0|\lambda|\gamma^{\theta(h-N)}\ , \tag{3.24}$$

We have obtained an expression for $R_{N,L}$ (3.10) of the form

$$R_{N,L}(\mathbf{p};\mathbf{k})=\sum_{n=0}^\infty\sum_{j_0=-\infty}^{N-1}\sum_{\tau\in\mathcal{T}_{j_0,n,2,1}}\sum_{\substack{\mathbf{P}\in\mathcal{P}\\|P_{v_0}|=2}}R_\tau(\mathbf{p},\mathbf{k})\ ,$$

and the trees are defined as before, with the only important difference that there are endpoints v to which is associated $\nu_{h_v} \frac{1}{L^4} \sum_{\mathbf{p}\in\mathcal{D}} J_{\mathbf{p}} \mathbf{p}_\mu j_{\mu,\mathbf{p}}$ or, if $h_v = N$, $\frac{1}{L^4} \sum_{\mathbf{p}\in\mathcal{D}} J_{\mathbf{p}} \delta j_{\mu,\mathbf{p}}$: in both case we have the constraint that $h_v - h_{v'} = 1$, if v' is the first non trivial vertex following v. We call such end-points of type T_ν and T_0 respectively.

The sum over the trees such that the endpoint is of type T_ν can be bounded as in (2.45), the only difference being that, thanks to the bound (3.24), one has to multiply the r.h.s. by a factor $|\lambda|\gamma^{-\theta(N-h_J)}$, which has to be inserted also in the r.h.s. of the bounds (2.46) and (2.48). Hence the analogous of the r.h.s. of (2.46) becomes

$$\sum_{h_1^*=-\infty}^{h_{\mathbf{k}-\mathbf{p}}} \sum_{h_0^*=-\infty}^{h_1^*} \sum_{h_J=h_J^*}^{+\infty} \gamma^{\delta(h_1^*-h_0^*)} \gamma^{-\frac{1}{4}[(h_{\mathbf{k}}-h_1^*)+(h_{\mathbf{k}-\mathbf{p}}-h_1^*)+(h_J-h_0^*)]} \gamma^{-\theta(N-h_J)} \tag{3.25}$$

and the r.h.s. can be bounded by, multiplying and dividing by $\gamma^{\delta(h_{\mathbf{k}}-h_{\mathbf{p}})}$

$$C' \gamma^{\delta(h_{\mathbf{k}}-h_{\mathbf{p}})} \sum_{h_1^*=-\infty}^{h_{\mathbf{k}-\mathbf{p}}} \gamma^{\delta(h_1^*-h_{\mathbf{k}})} \gamma^{-\frac{1}{4}[(h_{\mathbf{k}}-h_1^*)+(h_{\mathbf{k}-\mathbf{p}}-h_1^*)]} \sum_{h_0^*=-\infty}^{h_1^*} \gamma^{\delta(h_{\mathbf{p}}-h_0^*)} \gamma^{-\frac{1}{4}(h_J^*-h_0^*)} \gamma^{-\theta(N-h_J^*)} \tag{3.26}$$

so that the r.h.s. of (2.47) is bounded by $C\gamma^{\delta(h_{\mathbf{k}}-h_{\mathbf{p}})}(\gamma^{-N}\kappa)^\theta$, if $\delta \le 1/8$ and $|\mathbf{p}|, |\mathbf{k}-\mathbf{p}|, |\mathbf{k}| \le \kappa$; in fact $\gamma^{h_J^*} = \gamma^{h_{\mathbf{p}}+2}$ if $h_{\mathbf{p}} \ge h_0^* - 1$ and $\gamma^{h_J^*} = \gamma^{h_0^*+1} \le \gamma^{h_1^*+1} \le \gamma^{h_{\mathbf{k}-\mathbf{p}}+1}$ if $h_{\mathbf{p}} < h_0^* - 1$. In a similar way also the other cases have an extra factor $(\gamma^{-N}\kappa)^\theta$.

Let us now consider the trees with an endpoint of type T_0; in such case the fields of the T_0 endpoint are contracted at scale j, N; this implies that the sum over h_J is missing in the r.h.s. of the bounds (2.46) and (2.48) and $h_J = N$; the analogous of (2.46) becomes then

$$\sum_{h_1^*=-\infty}^{h_{\mathbf{k}-\mathbf{p}}} \sum_{h_0^*=-\infty}^{h_1^*} \gamma^{\delta(h_1^*-h_0^*)} \gamma^{-\frac{1}{4}[(h_{\mathbf{k}}-h_1^*)+(h_{\mathbf{k}-\mathbf{p}}-h_1^*)+(N-h_0^*)]} \tag{3.27}$$

and $\gamma^{-(N-h_0^*)} \le \gamma^{-(N-h_{\mathbf{k}-\mathbf{p}})}$ so that again there is an improvement in the bounds of a factor $(\gamma^{-N}\kappa)^\theta$. Hence $R_{N,L}(\mathbf{p};\mathbf{k})$ verifies a bound similar to (2.49), with an extra factor $(\gamma^{-N}\kappa)^\theta$.

3.3 *The flow of the counterterms*

We have finally to prove (3.24). The flow equation has the form

$$\nu_{j-1} = \nu_j + \beta_\nu^{(j)}(\nu_j; \ldots; \nu_j; \lambda) , \tag{3.28}$$

and

$$\beta_\nu^{(j)}(\nu_j; \ldots; \nu_N; \lambda) = \beta_\nu^{(j,1)}(\lambda) + \sum_{j'=j}^{N} \nu_{j'} \tilde\beta_\nu^{(j,j')}(\lambda) . \tag{3.29}$$

Moreover given a positive $\theta < 1/4$, there are constants c_1 and c_2 such that

$$|\beta_\nu^{(j,1)}(\lambda)| \le c_1|\lambda|\gamma^{2\theta(j-N)} \quad , \quad |\tilde\beta_\nu^{(j,j')}(\lambda)| \le c_2\lambda^2\gamma^{2\theta(j-j')} . \tag{3.30}$$

This follows from the fact that $\beta_\nu^{(j,1)}$ and $\tilde\beta_\nu^{(j,j')}$ are given by a sum of trees verifying the bound (3.31) , with at least an end-point respectively at scale N and at scale j', hence one can improve the bound respectively by a factor $\gamma^{2\theta j}$ and $\gamma^{2\theta(j-j')}$.

By a simple iteration, (3.28) can also be written in the form

$$\nu_{j-1} = \nu_N + \sum_{j'=j}^{N} \beta_\nu^{(j')}(\nu_{j'}; \ldots; \nu_N; \lambda) . \tag{3.32}$$

By choosing

$$\nu_N = - \sum_{j=-\infty}^{N} \beta_\nu^{(j)}(\nu_{j,\omega};\ldots;\nu_N;\lambda) \ , \tag{3.33}$$

we see, by inserting (3.33) in the r.h.s. of (3.32), that we have to show that there is a sequence $\underline{\nu} = \{\nu_j,\, j \le N\}$, such that ν_N is of order λ and

$$\nu_j = - \sum_{j'=-\infty}^{j} \beta_\nu^{(j')}(\nu_{j'},..,\nu_N;\lambda) \ . \tag{3.34}$$

In order to prove that, we introduce the space $\mathfrak{M}_\theta$ of the sequences $\underline{\nu} = \{\nu_j,\, j \le N\}$ such that $|\nu_j| \le c\bar{\lambda}_h \gamma^{\theta j}$, for some c; we shall think $\mathfrak{M}_\theta$ as a Banach space with norm $||\underline{\nu}||_\theta = \sup_{\le j \le N} |\nu_j|\gamma^{-\theta(j-N)}|\lambda|^{-1}$. We then look for a fixed point of the operator $\mathbf{T} : \mathfrak{M}_\theta \to \mathfrak{M}_\theta$ defined as:

$$(\mathbf{T}\underline{\nu})_j = - \sum_{j'=-\infty}^{j} \beta_\nu^{(j')}(\nu_{j'},..,\nu_N;\lambda) \ . \tag{3.35}$$

Note that, if λ is sufficiently small, then $\mathbf{T}$ leaves invariant the ball $\mathfrak{B}_\theta$ of radius $c_0 = 2c_1 \sum_{n=0}^\infty \gamma^{-n}$ of $\mathfrak{M}_\theta$, c_1 being the constant in (3.30). In fact, by (3.29) and (3.30), if $||\underline{\nu}||_\theta \le c_0$, then

$$|(\mathbf{T}\underline{\nu})_j| \le \sum_{j'=-\infty}^{j} c_1|\lambda|\gamma^{2\theta j'} + \sum_{j'=-\infty}^{j}\sum_{i=j'}^{0} c_0|\lambda|\gamma^{\theta(i-N)}c_2\lambda^2\gamma^{2\theta(j'-i)} \le c_0|\lambda|\gamma^{\theta(j-N)} \ , \tag{3.36}$$

if $2c_2|\lambda|(\sum_{n=0}^\infty \gamma^{-n})^2 \le 1$.

$\mathbf{T}$ is a also a contraction on $\mathfrak{B}_\theta$, if λ is sufficiently small; in fact, if $\underline{\nu}, \underline{\nu}' \in \mathfrak{M}_\theta$,

$$\begin{aligned} |(\mathbf{T}\nu)_j - (\mathbf{T}\underline{\nu}')_j| &\le \sum_{j'=-\infty}^{j} |\beta_\nu^{(j')}(\nu_{j'},..,\nu_N;\lambda) - \beta_\nu^{(j')}(n'_{j'},..,\nu'_N;\lambda)| \\ &\le \sum_{j'=-\infty}^{j}\sum_{i=j'}^{0} ||\underline{\nu}-\underline{\nu}'||_\theta|\lambda|\gamma^{\theta(i-N)}c_2\lambda^2\gamma^{2\theta(j'-i)} \le \frac{1}{2}||\underline{\nu}-\underline{\nu}'||_\theta|\lambda|\gamma^{\theta(j-N)} \ , \end{aligned} \tag{3.37}$$

if $c_2\lambda^2(\sum_{n=0}^\infty \gamma^{-n})^2 \le 1/2$. Hence, by the contraction principle, there is a unique fixed point $\underline{\nu}^*$ of $\mathbf{T}$ on $\mathfrak{B}_\theta$.

4. Appendix 1: Computation of c_+

The value of ν at lowest order is obtained from the localization of the following terms

$$\begin{aligned} I_1 + I_2 = v(\mathbf{p}) \int \frac{d\mathbf{k}'}{(2\pi)^4} \mathrm{Tr}[\gamma_\nu \frac{\chi_N(\mathbf{k}')}{i\,\not{\mathbf{k}}'} C_N(\mathbf{k}',\mathbf{k}'+\mathbf{p}) \frac{\chi_N(\mathbf{k}'+\mathbf{p})}{i(\not{\mathbf{k}}'+\not{\mathbf{p}})}]\gamma_\nu + \\ \int \frac{d\mathbf{k}'}{(2\pi)^4} v(\mathbf{k}')[\gamma_\nu \frac{\chi_N(\mathbf{k}')}{i\,\not{\mathbf{k}}'} C_N(\mathbf{k}',\mathbf{k}'+\mathbf{p}) \frac{\chi_N(\mathbf{k}'+\mathbf{p})}{i(\not{\mathbf{k}}'+\not{\mathbf{p}})}\gamma_\nu] \end{aligned} \tag{4.1}$$

Using that

$$g^{(\le N)}(\mathbf{k})C_N(\mathbf{k},\mathbf{k}+\mathbf{p})g^{(\le N)}(\mathbf{k}+\mathbf{p}) = (1-\chi_N(\mathbf{k}))\chi_N(\mathbf{k}+\mathbf{p})[\frac{1}{i(\not{\mathbf{k}}+\not{\mathbf{p}})} - \frac{1}{i\,\not{\mathbf{k}}}] + (\chi_N(\mathbf{k})-\chi_N(\mathbf{k}+\mathbf{p}))\frac{1}{i\,\not{\mathbf{k}}} \tag{4.2}$$

we find that

$$\mathcal{L}I_1 = v(\mathbf{p})\mathbf{p}_\alpha \int \frac{d\mathbf{k}}{(2\pi)^4}(1-\chi_N(\mathbf{k}))\chi_N(\mathbf{k})\frac{\partial}{\partial \mathbf{k}_\alpha}\frac{\mathbf{k}_\mu}{i\mathbf{k}^2}\mathbf{Tr}(\gamma_\nu\gamma_\mu)\gamma_\nu - v(\mathbf{p})\mathbf{p}_\alpha \int \frac{d\mathbf{k}}{(2\pi)^4}[\frac{\partial}{\partial \mathbf{k}_\alpha}\chi_N(\mathbf{k})]\frac{\mathbf{k}_\mu}{i\mathbf{k}^2}\mathbf{Tr}(\gamma_\nu\gamma_\mu)\gamma_\nu \tag{4.3}$$

and using that $\mathbf{Tr}(\gamma_\nu\gamma_\mu)=\delta_{\nu,\mu}$ and $\int d\mathbf{k}\mathbf{k}_\alpha\mathbf{k}_\beta F(|\mathbf{k}|)$ is vanishing unless $\nu=\alpha$ we find

$$\mathcal{L}I_1=-iv(\mathbf{p})\mathbf{p}_\mu\int\frac{d\mathbf{k}}{(2\pi)^4}(1-\chi_N(\mathbf{k}))\chi_N(\mathbf{k})\frac{\partial}{\partial\mathbf{k}_\mu}\frac{\mathbf{k}_\mu}{\mathbf{k}^2}\gamma_\mu+iv(\mathbf{p})\mathbf{p}_\mu\int\frac{d\mathbf{k}}{(2\pi)^4}[\frac{\partial}{\partial\mathbf{k}_\mu}\chi_N(\mathbf{k})]\frac{\mathbf{k}_\mu}{\mathbf{k}^2}\gamma_\mu\equiv -i\not{\mathbf{p}}c_1 \quad (4.4)$$

In the same way

$$\mathcal{L}I_2=\mathbf{p}_\alpha\int\frac{d\mathbf{k}}{(2\pi)^4}v(\mathbf{k})(1-\chi(\mathbf{k}))\chi(\mathbf{k})\frac{\partial}{\partial\mathbf{k}_\alpha}\frac{\mathbf{k}_\mu}{i\mathbf{k}^2}\gamma_\nu\gamma_\mu\gamma_\nu-\mathbf{p}_\alpha\int\frac{d\mathbf{k}}{(2\pi)^4}v(\mathbf{k})[\frac{\partial}{\partial\mathbf{k}_\alpha}\chi(\mathbf{k})]\frac{\mathbf{k}_\mu}{i\mathbf{k}^2}\gamma_\nu\gamma_\mu\gamma_\nu \quad (4.5)$$

which can be rewritten as,using that $\gamma_\nu\gamma_\mu\gamma_\nu=2\gamma_\mu$

$$\mathcal{L}I_2=-i2\mathbf{p}_\mu\int\frac{d\mathbf{k}}{(2\pi)^4}v(\mathbf{k})(1-\chi_N(\mathbf{k}))\chi_N(\mathbf{k})\frac{\partial}{\partial\mathbf{k}_\mu}\frac{\mathbf{k}_\mu}{\mathbf{k}^2}\gamma^\mu+2i\int\frac{d\mathbf{k}}{(2\pi)^4}v(\mathbf{k})\frac{\chi'(\mathbf{k})}{|\mathbf{k}|}\not{\mathbf{p}}\equiv -i\not{\mathbf{p}}c_2 \quad (4.6)$$

We will call $c_+=c_1+c_2$.

5. Appendix 2: The WI at lowest orders

It is possible to check at lowest order the WI (1.12) by perturbation theory, using the identity

$$\frac{\chi_N(\mathbf{k})}{i\not{\mathbf{k}}}-\frac{\chi_N(\mathbf{k}+\mathbf{p})}{i\not{\mathbf{k}}+\mathbf{p}}=\frac{\chi_N(\mathbf{k})}{i\not{\mathbf{k}}}i\not{\mathbf{p}}\frac{\chi_N(\mathbf{k}+\mathbf{p})}{i\not{\mathbf{k}}+i\not{\mathbf{p}}}+\frac{\chi_N(\mathbf{k})}{i\not{\mathbf{k}}}C_N(\mathbf{k},\mathbf{k}+\mathbf{p})\frac{\chi_N(\mathbf{k}+\mathbf{p})}{i(\not{\mathbf{k}}+\not{\mathbf{p}})} \quad (5.1)$$

replacing (1.4) when a momentum cut-off is present. Calling $\langle\psi_\mathbf{k}\bar\psi_\mathbf{k}\rangle^{(1)}=g(\mathbf{k})\Sigma^{(1)}g(\mathbf{k})$ the first order term of the perturbative expansion of the 2-point function, we obtain

$$[(\Sigma^{(1)}(\mathbf{k})-\Sigma^{(1)}(\mathbf{k}+\mathbf{p})]|_{\mathbf{k}=0}=-\int\frac{d\mathbf{k}'}{(2\pi)^4}v(\mathbf{k}')[\gamma_\nu\frac{\chi_N(\mathbf{k}')}{i\not{\mathbf{k}}'}\gamma_\nu]+\int\frac{d\mathbf{k}'}{(2\pi)^4}v(\mathbf{k}')[\gamma_\nu\frac{\chi_N(\mathbf{k}'+\mathbf{p})}{i(\not{\mathbf{k}}'+\not{\mathbf{p}})}\gamma_\nu]$$

$$=-\int\frac{d\mathbf{k}'}{(2\pi)^4}v(\mathbf{k}')[\gamma_\nu\frac{\chi_N(\mathbf{k}')}{i\not{\mathbf{k}}'}i\not{\mathbf{p}}\frac{\chi_N(\mathbf{k}'+\mathbf{p})}{i(\not{\mathbf{k}}'+\not{\mathbf{p}})}\gamma_\nu]+\int\frac{d\mathbf{k}'}{(2\pi)^4}v(\mathbf{k}')[\gamma_\nu\frac{\chi_N(\mathbf{k}')}{i\not{\mathbf{k}}'}C_N(\mathbf{k}',\mathbf{k}'+\mathbf{p})\frac{\chi_N(\mathbf{k}'+\mathbf{p})}{i(\not{\mathbf{k}}'+\not{\mathbf{p}})}\gamma_\nu] \quad (5.2)$$

and calling $\langle j^\mu_\mathbf{p};\bar\psi_{\mathbf{k}+\mathbf{p}}\bar\psi^-_\mathbf{k}\rangle^{(1)}=g(\mathbf{k})g(\mathbf{k}+\mathbf{p})\Gamma^{(1)}_\mu(\mathbf{k},\mathbf{p})$ the first order contribution to $\langle j^\mu_\mathbf{p};\bar\psi_{\mathbf{k}+\mathbf{p}}\bar\psi^-_\mathbf{k}\rangle$ we obtain

$$\mathbf{p}_\mu\Gamma^{(1)}_\mu(\mathbf{k},\mathbf{p})=\int\frac{d\mathbf{k}'}{(2\pi)^4}v(\mathbf{k}')[\gamma_\nu\frac{\chi_N(\mathbf{k}')}{i\not{\mathbf{k}}'}\not{\mathbf{p}}\frac{\chi_N(\mathbf{k}'+\mathbf{p})}{i(\not{\mathbf{k}}'+\not{\mathbf{p}})}\gamma_\nu]+v(\mathbf{p})\int\frac{d\mathbf{k}'}{(2\pi)^4}\mathrm{Tr}[\gamma_\nu\frac{\chi_N(\mathbf{k}')}{i\not{\mathbf{k}}'}\not{\mathbf{p}}\frac{\chi_N(\mathbf{k}'+\mathbf{p})}{i(\not{\mathbf{k}}'+\not{\mathbf{p}})}]\gamma_\nu \quad (5.3)$$

and the second term, the renormalization of the photon mass, is equal , using (5.1), to $i\mathrm{Tr}[\gamma_\nu\frac{\chi_N(\mathbf{k}')}{i\not{\mathbf{k}}'}C_N(\mathbf{k}',\mathbf{k}'+\mathbf{p})\frac{\chi_N(\mathbf{k}'+\mathbf{p})}{i(\not{\mathbf{k}}'+\not{\mathbf{p}})}]\gamma_\nu$. Hence we obtain

$$-i\mathbf{p}_\mu\langle j^\mu_\mathbf{p};\bar\psi_{\mathbf{k}+\mathbf{p}}\bar\psi^-_\mathbf{k}\rangle^{(1)}-\langle\psi_\mathbf{k}\bar\psi_\mathbf{k}\rangle^{(1)}+\langle\psi_{\mathbf{k}+\mathbf{p}}\bar\psi_{\mathbf{k}+\mathbf{p}}\rangle^{(1)}= \quad (5.4)$$

$$\int\frac{d\mathbf{k}'}{(2\pi)^4}v(\mathbf{k}')[\gamma_\nu\frac{\chi_N(\mathbf{k}')}{i\not{\mathbf{k}}'}C_N(\mathbf{k}',\mathbf{k}'+\mathbf{p})\frac{\chi_N(\mathbf{k}'+\mathbf{p})}{i(\not{\mathbf{k}}'+\not{\mathbf{p}})}\gamma_\nu]+v(\mathbf{p})\int\frac{d\mathbf{k}'}{(2\pi)^4}\mathrm{Tr}[\gamma_\nu\frac{\chi_N(\mathbf{k}')}{i\not{\mathbf{k}}'}C_N(\mathbf{k}',\mathbf{k}'+\mathbf{p})\frac{\chi_N(\mathbf{k}'+\mathbf{p})}{i(\not{\mathbf{k}}'+\not{\mathbf{p}})}]\gamma_\nu]$$

and the last addend is equal to, up to terms vanishing as $N\to\infty$, to $-c_+\not{\mathbf{p}}_\mu$, in agreement with (1.12).

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