Nonperturbative Adler-Bardeen Theorem

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Abstract. *The Adler-Bardeen theorem has been proved only as a statement valid at all orders in perturbation theory, without any control on the convergence of the series. In this paper we prove a nonperturbative version of the Adler-Bardeen theorem in $d = 2$ by using recently developed technical tools in the theory of Grassmann integration.*

Keywords Nonperturbative merthods; Chiral anomaly; resummation of the perturbation expansion; Adler-Bardeen theorem; constructive QFT.

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1. Introduction and Main results

1.1 *Anomalies in QFT.* The chiral anomaly appears as a quantum correction to the conservation of the axial current for massless fermions. A crucial property is the *anomaly nonrenormalization*, which says that the chiral anomaly is given *exactly* by its lower order contribution. This property was proved first for QED_4 in [AB] in the well known *Adler-Bardeen Theorem*: it was shown that there is a dramatic cancellation, if a suitable regularization is assumed, among the infinite collection of Feynmann graphs contributing to the anomaly and at the end it turns out that the anomaly is given by a single graph (the famous "triangle graph"): the result can be condensed by the formula

$$\partial^\mu j^5_\mu = \frac{\alpha_0}{4\pi}\varepsilon_{\mu,\nu,\rho,\sigma} F^{\mu,\nu} F^{\rho,\sigma} \tag{1.1}$$

where α_0 is the unrenormalized coupling constant. Different proofs of (1.1) were given later in [Z] and [LS]: as the results in [AB], they were statements *valid at all orders in perturbation theory* and with no control on the convergence of the series itself. The property of the anomaly nonrenormalization holds also in the Electroweak model where it plays a crucial role even to prove the renormalizability; as the gauge fields couple to chiral currents, the chiral anomaly would break the renormalizability, but a remarkable cancellation between anomalies (not renormalized according the Adler-Bardeen theorem) of different fermion species saves the theory and gives a confirmation of the fermionic family structure as well.

Recent textbooks tend to present the anomaly nonrenormalization in a functional integral approach in which, following the elegant treatment of [F], one recovers it from the Jacobian associated to a chiral transformation. However, as explained for instance in [A1], such methods *cannot be considered* simpler proofs of the Adler-Bardeen theorem: the methods in [F] essentially treat the gauge fields as *classical fields* so that they produce essentially *one loop results* and eventual higher orders correction would be in any case not included. Hence it is the validity of such functional approach which is justified by [AB] rather than the contrary.

As the anomaly nonrenormalization is a quite delicate property, against which several objections has been raised along the years (see for instance [JJ], [AI] or [DMT]), it would be desirable to go beyond perturbation theory. This seems actually far from the present analytical possibilities in $d = 4$, for the difficulty of giving a real non-perturbative meaning to the functional integrals expressing the theory; it is worth then to consider $d = 2$ QFT models which have proven fruitful laboratories to test general properties.

In $d = 2$ the perturbative analysis in [AB] can be repeated with no essential modifications, see [GR], and still the anomaly nonrenormalization holds in the form (in the Euclidean case) $\partial_\mu j^5_\mu = -ie\alpha\varepsilon^{\mu,\nu}\partial_\mu A_\nu$, where e is the coupling and α is the value of the "bubble graph" (replacing the "triangle graph" in $d = 4$). It holds $\alpha = \frac{1}{4\pi}$ or $\alpha = \frac{1}{2\pi}$ depending if *dimensional* or *momentum* regularization is used. Again nonperturbative informations cannot be obtained by such a procedure, based on explicit cancellations between Feynmann graphs. It is also claimed that the anomaly nonrenormalization in $d = 2$ can be derived by an exact functional approach, see for instance [FSS]; indeed integrating out the fermions it turns out that the partition function for many $d = 2$ *QFT* models can be written as

$$\int P(dA) \frac{\det(\gamma_\mu[\partial_\mu + A_\mu])}{\det(\gamma_\mu \partial_\mu)} \tag{1.2}$$

where $A_{\mu,\mathbf{x}} = (A_{0,\mathbf{x}}, A_{1,\mathbf{x}})$ are fields with Gaussian measure $P(dA)$ with covariance $< A_{\mu,\mathbf{x}} A_{\nu,\mathbf{y}} >= e^2 \delta_{\mu,\nu} v(\mathbf{x} - \mathbf{y})$. A similar expression holds for the generating functional. It is well known [Se] that, *under suitably regularity conditions over* A_μ, $\log\det(\gamma_\mu\partial_\mu + \gamma_\mu A_\mu) - \log\det(\gamma_\mu \partial_\mu)$ is quadratic in A_μ; replacing the determinant with a quadratic exponential one gets easily,by an explicit integration of the Gaussian integrals, that the anomalies are *not renormalized by higher orders*. However in the above derivation *an approximation is implicit*; the fermionic determinant in (1.2) is given by a quadratic expression *only if* A_μ is sufficiently regular, but in (1.2) *the integral is over all possible fields* A, hence one is *neglecting* the contributions from the irregular fields. A peculiarity of $d = 2$ QFT is the existence of some exact solutions; indeed it has been claimed [GR] that the Adler-Bardeen theorem finds a nonperturbative verification from comparison with the operatorial exact solution of [J],[K] in the case of contact current-curent interaction. However the regularization in the functional integrals or in the operatorial exact solution are different, hence there is no guarantee [GL] that the Schwinger functions obtained from functional integrals converge, removing cutoffs and in the massless limit, to the exact ones (indeed this is not the case). In conclusion, even in $d = 2$ there are no rigorous verification of the Adler-Bardeen theorem in a functional integral approach to QFT beyond perturbation theory.

The rigorous construction of $d = 2$ QFT models from functional integrals is in general not trivial at all, as they appear to be related to the continuum limit of the correlations of coupled bidimensional Ising or vertex models [GM1], which are in general hard to compute [B]. Some $d = 2$ QFT models has been deeply investigated in the Eighties in the framework of Constructive QFT (see [GK],[Le]), and in recent times new powerful methods has been developed in [BM], overcoming the well known technical problem posed by the combination of a nonperturbative setting based on multiscale analysis [P],[G] with the necessity of exploiting cancellations due to local gauge symmetries. These new technical tools allow us to rigorously investigate, *for the first time*, the properties of anomalies of $d = 2$ QFT models constructed from functional integrals; in particular, we can prove a *non-perturbative* version of the Adler-Bardeen under suitable conditions on the bosonic propagator, avoiding completely Feynmann graphs expansions and with full rigor.

1.2 *Euclidean* QFT_2. We consider an Euclidean QFT in $d = 1 + 1$ whose Schwinger function can be obtained from the following functional integral

$$e^{\mathcal{W}_{N,L}(J,\phi)} = \int P_N(d\psi) P(dA) e^{\int d\mathbf{x} [e \bar\psi_{\mathbf{x}} (A_{\mu,\mathbf{x}}\gamma^\mu)\psi_{\mathbf{x}} + J_{\mu,\mathbf{x}} A_{\mu,\mathbf{x}} + \frac{\phi_{\mathbf{x}}\bar\psi_{\mathbf{x}}}{\sqrt{Z}} + \frac{\bar\phi_{\mathbf{x}}\psi_{\mathbf{x}}}{\sqrt{Z}}]} \tag{1.3}$$

where ϕ, J are external fields, Z is the *wave function renormalization* and:

-)in $\Lambda = [0, L] \times [0, L]$ a lattice Λ_a is introduced whose sites are given by the space-time points $\mathbf{x} = (x, x_0) = (na, n_0 a)$ with L/a integer and $n, n_0 = -L/2a, 1, \ldots, L/2a - 1$. We also consider the set $\mathcal{D}$ of space-time momenta $\mathbf{k} = (k, k_0)$ with $k = (m + \frac{1}{2})\frac{2\pi}{L}$ and $k_0 = (m_0 + \frac{1}{2})\frac{2\pi}{L}$ with $m, m_0 = 0, 1, \ldots, L/a - 1$. To simplify the notations we write $\int d\mathbf{x} = a^2 \sum_{\mathbf{x}\in\Lambda}$ and $\int d\mathbf{k} = \frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}}$.

-)$\psi_{\mathbf{x}}, \bar\psi_{\mathbf{x}}$, $\mathbf{x} \in \Lambda$ are a finite set of *Grassmann spinors* and $P_N(d\psi)$ is the fermionic integration with propagator

$$g(\mathbf{x} - \mathbf{y}) = \int d\mathbf{k} \frac{-i \not{\mathbf{p}} + m}{\mathbf{p}^2 + m^2} e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})} \chi_N(\mathbf{k}) \tag{1.4}$$

where $\chi_N(\mathbf{k})$ is a *smooth cutoff function* selecting momenta $|\mathbf{k}| \le \gamma^N$ with $\gamma > 1$ and N a positive integer. We assume $\gamma^N << a^{-1}$, that is the lattice cutoff is removed before the fermionic cutoff (we are essentially considering a continuum model with a momentum regularization).

-)The γ's matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \gamma^5 = -i\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1.5}$$

-)$A_{\mathbf{x}} = (A_{0,\mathbf{x}}, A_{1,\mathbf{x}})$ are Euclidean boson fields with periodic boundary conditions and Gaussian measure $P(dA)$ with covariance

$$< A_{\mu,\mathbf{x}} A_{\nu,\mathbf{y}} >= v(\mathbf{x}-\mathbf{y})\delta_{\mu,\nu} = \delta_{\mu,\nu} \int d\mathbf{p} e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})} v(\mathbf{p}) \tag{1.6}$$

Integrating the bosonic variables A one can rewrite (1.3) as

$$e^{\mathcal{W}_{N,L}(J,\phi)} = \int P_N(d\psi) e^{\frac{1}{4}\int d\mathbf{x}d\mathbf{y} v(\mathbf{x}-\mathbf{y})[e\bar\psi_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}} + J_{\mu,\mathbf{x}}][e\bar\psi_{\mathbf{y}}\gamma^\mu\psi_{\mathbf{y}} + J_{\mu,\mathbf{y}}] + \int d\mathbf{x}[\frac{\phi_{\mathbf{x}}\bar\psi_{\mathbf{x}}}{\sqrt{Z}} + \frac{\bar\phi_{\mathbf{x}}\psi_{\mathbf{x}}}{\sqrt{Z}}]} \tag{1.7}$$

The *Schwinger functions* are defined by

$$< \prod_{i=1}^n \psi_{\mathbf{x}_i} \prod_{i=1}^n \bar\psi_{\mathbf{y}_i} \prod_{i=1}^m j^\mu_{\mathbf{z}_i} >_{N,L} = \frac{\partial^{2n+m}\mathcal{W}_{N,L}(J,\phi,\bar\phi)}{\partial\phi_{\mathbf{x}_1}...\partial\phi_{\mathbf{x}_n}\partial\bar\phi_{\mathbf{y}_1}...\partial\phi_{\mathbf{y}_n}\partial J_{\mu_1,\mathbf{z}_1}...\partial J_{\mu_n,\mathbf{z}_m}}\Big|_{J=\phi=0} \tag{1.8}$$

where $j^\mu_{\mathbf{x}} = \bar\psi_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}}$. Of course the following trivial identities hold

$$ev(\mathbf{p}) < j^\mu_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} >=< A_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} > \tag{1.9}$$

and

$$ev(\mathbf{p}) < j^{5,\mu}_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} >= iev(\mathbf{p})\varepsilon^{\mu,\nu} < j^\nu_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} >= i\varepsilon_{\mu,\nu} < A_{\nu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} > \tag{1.10}$$

where we have used that $j^{5,\mu} = i\varepsilon^{\mu,\nu}j^\nu$, $\varepsilon_{\mu,\nu} = -\varepsilon_{\nu,\mu}$, $\varepsilon_{0,1} = -1$.

Depending on the explicit form of $v(\mathbf{p})$, to the functional integral (1.3) correspond several models: if $v(\mathbf{p}) = \mathbf{p}^{-2}$ and $m = 0$ it is a regularized version of the *Schwinger model*, if $m \neq 0$ is a version of QED_2 in the Feynmann gauge, if $v(\mathbf{p}) = (\mathbf{p}^2 + M^2)^{-1}$ it corresponds to the *Vector-gluon model* of [GR]; an ultraviolet cutoff can be eventually imposed if necessary. Particularly interesting is the case $v(\mathbf{p}) = 1$ (that is $v(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$) corresponding to the massive *Thirring model* (with a definite sign of the interaction).

The Schwinger functions (1.8) are well defined if the cutoffs (the volume L and the momentum cutoff N) are *finite*; the main problem is to show that, choosing properly the bare parameters Z, m (eventually depending from the cutoffs) $< \prod_{i=1}^n \psi_{\mathbf{x}_i} \prod_{i=1}^n \bar\psi_{\mathbf{y}_i} \prod_{i=1}^m j^\mu_{\mathbf{z}_i} >_{N,L}$ has a well defined non trivial limit as $N, L \to \infty$.

In this paper we will prove that, if the bosonic propagator decays fast enough in momentum space and for small coupling, the cutoffs L, N can be removed in the Schwinger functions for any finite m and Z. We will start from the fermionic representation (1.7) and the Grassmann functional integral is nonperturbatively evaluated by a multiscale analysis in which each step is proved to be well defined by tree expansion methods and determinant bounds (for a tutorial introduction to such techniques see [GM]); the massless limit is controlled using the methods introduced in [BM] allowing the implementation of WI (approximate,due to cutoffs) based on local Gauge invariance at each integration step.

By performing the local gauge transformation $\bar\psi_{\mathbf{x}} \to e^{\alpha_{\mathbf{x}}}\bar\psi_{\mathbf{x}}$, $\psi_{\mathbf{x}} \to e^{-\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}$ or $\bar\psi_{\mathbf{x}} \to e^{\gamma_5\alpha_{\mathbf{x}}}\bar\psi_{\mathbf{x}}$, $\psi_{\mathbf{x}} \to e^{-\gamma_5\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}$ in (1.3) we get, in the case $m = 0$

$$\begin{aligned} &-i\mathbf{p}_\mu\langle j^\mu_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle = \langle\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle - \langle\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}\rangle + \Delta^0_{N,L}(\mathbf{k},\mathbf{k}-\mathbf{p}) \\ &-i\mathbf{p}_\mu\langle j^{5,\mu}_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle = \gamma^5[\langle\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle - \langle\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}\rangle] + \Delta^5_{N,L}(\mathbf{k},\mathbf{k}-\mathbf{p}) , \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} \Delta^0_{N,L}(\mathbf{k},\mathbf{k}+\mathbf{p}) &= \int d\mathbf{k}' C^{\mu,N}_{\mathbf{k}',\mathbf{p}} < \bar\psi_{\mathbf{k}'}\gamma^\mu\psi_{\mathbf{k}'+\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} >_{N,L} \\ \Delta^5_{N,L}(\mathbf{k},\mathbf{k}+\mathbf{p}) &= \int d\mathbf{k}' C^{\mu,N}_{\mathbf{k}',\mathbf{p}} < \bar\psi_{\mathbf{k}'}\gamma^\mu\gamma^5\psi_{\mathbf{k}'+\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} >_{N,L} , \end{aligned} \tag{1.12}$$

with

$$C^{\mu,N}_{\mathbf{k},\mathbf{p}} = ([\chi_N(\mathbf{k})]^{-1} - 1)\mathbf{k}^\mu - ([\chi_N(\mathbf{k}-\mathbf{p})]^{-1} - 1)(\mathbf{k}^\mu - \mathbf{p}^\mu) \tag{1.13}$$

The last term in (1.11) is due to the presence of the cutoff function breaking the formal Gauge invariance of the action (it is formally vanishing if $\chi_N(\mathbf{k}) = 1$) and it is the average of the highly non local operator $\int d\mathbf{k}' C^{\mu,N}_{\mathbf{k}',\mathbf{p}} \bar\psi_{\mathbf{k}'}\gamma^\mu\psi_{\mathbf{k}'+\mathbf{p}}$. We prove the following result.

THEOREM 1 *Let us consider the generating functional (1.3) with* $Z = 1$, $|v(\mathbf{x})| \leq C$, $\int d\mathbf{x}[|v(\mathbf{x})| + |\partial v(\mathbf{x})| + |\mathbf{x}||v(\mathbf{x})|] \leq C$ *for a suitable constant* C *and and* e *small enough; then the Schwinger functions (1.8) are such that the limit*

$$\lim_{L,N\to\infty} \langle \prod_{i=1}^n \psi_{\mathbf{x}_i} \prod_{i=1}^n \bar\psi_{\mathbf{x}_i} \prod_{i=1}^m j^\mu_{\mathbf{z}_i} \rangle_{L,N} = \langle \prod_{i=1}^n \psi_{\mathbf{x}_i} \prod_{i=1}^n \bar\psi_{\mathbf{x}_i} \prod_{i=1}^m j^\mu_{\mathbf{z}_i} \rangle \tag{1.14}$$

exists at noncoinciding points uniformly in the fermionic mass and is non trivial.
In the massless case $m = 0$ *the WI (1.11) holds and*

$$\begin{aligned} \lim_{L,N\to\infty} \Delta^0_{N,L}(\mathbf{k},\mathbf{k}+\mathbf{p}) &= -\frac{e}{4\pi}(-i\mathbf{p}_\mu) < A_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} > \\ \lim_{L,N\to\infty} \Delta^5_{N,L}(\mathbf{k},\mathbf{k}+\mathbf{p}) &= \frac{e}{4\pi}(-i\mathbf{p}_\mu) i\varepsilon^{\mu,\nu} < A_{\nu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} > , \end{aligned} \tag{1.15}$$

The above result says that the correction terms $\Delta^0_{N,L}, \Delta^5_{N,L}$ to the Ward Identities (1.11), produced by the presence of the cutoff functions, generate the anomalies when the cutoffs are removed. Similar WI with any number of fermionic fields can be obtained and this can be read as

$$\partial_\mu j_\mu = -\frac{e}{4\pi}\partial_\mu A_\mu \quad \partial_\mu j^5_\mu = \frac{e}{4\pi} i\varepsilon^{\mu,\nu}\partial_\mu A_\nu \tag{1.16}$$

that is the anomaly is non-renormalized by higher orders, in agreement with the Adler-Bardeen theorem in $d = 2$ [GR] based on a cancellation between an infinite collection of Feynmann diagrams and with a momentum regularization for the fermionic loop. The main point is however that Theorem 1 is a *non perturbative* version of the Adler-Bardeen theorem, which is based neither on a Feynmann graphs expansion (for which convergence cannot be proved) nor on an exact evaluation of the functional integrals, which is not possible without approximations. The main technical tool is an expansion in terms of product of determinants, which allow us to implement the cancellations among Feynmann graphs due to the relative signs and it has good convergence properties. It turns out that all higher orders contributions to the anomaly vanish removing the cut-offs, and this is proved partly expanding the determinants in such a way that the good convergence properties are not lost.

From our construction an almost complete characterization of the Schwinger function is also obtained; for instance we can prove that the two point function $< \psi_{\mathbf{x}}\bar\psi_{\mathbf{y}} >$ decays for large $|\mathbf{x}-\mathbf{y}|$ as $e^{-m^{1+\hat\eta}|\mathbf{x}-\mathbf{y}|}|\mathbf{x}-\mathbf{y}|^{-1-\eta}$ where $\eta = ae^4 + O(e^6)$ and $\hat\eta = -be^2 + O(e^2)$ while for $\mathbf{x}\to\mathbf{y}$ it diverges as $|\mathbf{x}-\mathbf{y}|^{-1}$. The condition assumed in Theorem 1 for the bosonic propagator are verified for instance by

$$v(\mathbf{p}) = \int d\mathbf{p}\frac{e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{\mathbf{p}^2 + M^2}\chi_K(\mathbf{p}) \tag{1.17}$$

corresponding a massive boson propagator with an ultraviolet cut-off, which could be removed with some more technical effort.

1.3 *Local interaction.* The previous result says that the anomaly nonrenormalization holds if the bosonic propagator in momentum space decays fast enough and it is finite; the question then naturally rises if the anomaly nonrenormalization is valid also if the bosonic propagator does not decay at all, as in the case of the Thirring model in which $v(\mathbf{p})$ is a constant. In a companion paper [BFM] the case $v(\mathbf{p}) = 1$ has been studied, and it has been found that the functional integral (1.3) still defines a set of Schwinger functions removing cutoffs, in the limit $L,N\to\infty$, that is (1.14) still holds provided that we choose $Z=Z_N, m\equiv m_N$ depending on the ultraviolet cutoff, that is

$$Z_N=\gamma^{-N\eta}\big(1+\mathrm{O}(e^4)\big)\qquad m_M=m\gamma^{-\bar\eta N}\big(1+\mathrm{O}(e^2)\big) \tag{1.18}$$

with η and $\bar\eta$ independent of m and such that $\eta=ae^4+\mathrm{O}(e^6)$, $\bar\eta=be^2+\mathrm{O}(e^4)$, $a,b>0$. In the massless case $m=0$ the WI (1.11) holds but (1.15) has to be replaced by

$$\begin{aligned}&\lim_{L,N\to\infty}\Delta^0_{N,L}(\mathbf{k},\mathbf{k}+\mathbf{p})=[-\frac{e}{4\pi}+c_+e^3+F_\alpha]\mathbf{p}_\mu<A_{\mu,\mathbf{p}}\psi_\mathbf{k}\bar\psi_{\mathbf{k}+\mathbf{p}}>\\ &\lim_{L,N\to\infty}\Delta^0_{N,L}(\mathbf{k},\mathbf{k}+\mathbf{p})\Delta^5_{N,L}(\mathbf{k},\mathbf{k}+\mathbf{p})=i[\frac{e}{4\pi}+c_+e^3+F_\alpha]\mathbf{p}_\mu\varepsilon^{\mu,\nu}<A_{\nu,\mathbf{p}}\psi_\mathbf{k}\bar\psi_{\mathbf{k}+\mathbf{p}}>,\end{aligned} \tag{1.19}$$

with $c_+>0$ *non vanishing* and $|F_\alpha|\le Ce^5$. This means that if $v(\mathbf{p})=1$ *the anomaly has higher orders corrections*, that is (1.16) has to be replaced by

$$\partial_\mu j_\mu=i[-\frac{e}{4\pi}+c_+e^3+eF_+]\partial_\mu A_\mu\quad \partial_\mu j^5_\mu=[\frac{e}{4\pi}+c_+e^3+eF]i\varepsilon^{\mu,\nu}\partial_\mu A_\nu \tag{1.20}$$

This result of course implies that *one cannot replace the determinant in (1.2) by a quadratic exponential*; the contribution of the irregular fields is not negligible when $v(\mathbf{p})=1$. In Appendix 3 an explicit second order verification of (1.15) (1.20) has been included. (1.20) is apparently contrast with the Adler-Bardeen theorem [AB], but indeed this is not the case. In the [AB] analysis for QED_4, an ultraviolet cut-off K has been introduced for the boson propagator, and it is implicitly assumed that it is removed *after* the ultraviolet cut-off for the fermionic propagator; moreover the bare parameters are chosen as a function of K. In the model (1.3), if $v(\mathbf{p})$ decays for large momenta the theory is superrinormalizable, while if $v(\mathbf{p})=1$ is just renomalizable like QED_4. Proceeding analogously to [AB] (in a non perturbative framework) an ultraviolet cut-off can be introduced also in the bosonic propagator for instance by replacing $v(\mathbf{p})=1$ with $e^{-\mathbf{p}^2K^{-2}}$. There are then *two* ultraviolet cutoffs, corresponding to the fermionic or bosonic propagator, and *depending which cutoff is removed first different anomalies are found* as functions of the bare parameters. If K is removed before the fermionic cutoff N, that is $N\to\infty, K\to\infty$, we are essentially considering the case $v(\mathbf{p})=1$ discussed in [BFM]; it holds that the anomaly is given by (1.20), that is it is non linear in e but it is *renormalized* by higher orders. On the other hand if the fermionic cutoff is removed first a completely different result holds.

THEOREM 2 *Assume that $v(\mathbf{p})=e^{-\mathbf{p}^2K^{-2}}$; it is possible to find bare parameters $Z=Z_K, m=m_K$ such that the limit*

$$\lim_{L\to\infty}\lim_{K\to\infty}\lim_{N\to\infty}\langle\prod_{i=1}^n\psi_{\mathbf{x}_i}\prod_{i=1}^n\bar\psi_{\mathbf{x}_i}\prod_{i=1}^m j_\mu(\mathbf{z}_i)\rangle_{L,N,K} \tag{1.21}$$

exists and it is non trivial, and (1.11) holds togheter with (1.15), that is anomaly nonrenormalization holds.

This means that the the anomaly nonrenormalization holds if the fermionic cutoff is removed first, while is violated is the bosonic cutoff is removed first. The limit $N\to\infty, K\to\infty$ corresponds to a fermionic functional integral (1.7) with a local Thirring current-current interaction $j_\mu(\mathbf{x})j_\mu(\mathbf{x})$; the opposite limit $K\to\infty, N\to\infty$ is similar to the one used in the original [AB] paper.

In §2 we will describe our multiscale integration procedure, and in §3 theorem 1 and 2 are proved. A second order verification of our results is included for pedagogical reasons in Appendix 3.

2. Multiscale Integration

2.1 *Multiscale analysis*

It is convenient to adopt Weyl notation. Calling $\psi_\mathbf{x}=(\psi^-_{+,\mathbf{x}},\psi^-_{-,\mathbf{x}})$, and $\psi^\dagger_\mathbf{x}=(\psi^+_{+,\mathbf{x}},\psi^+_{-,\mathbf{x}})$, $\bar\psi=\psi^\dagger_\mathbf{x}\gamma_0$, the *Generating Functional* (1.3) can be written as

$$\begin{aligned}e^{\mathcal{W}_{N,L}(J,\phi)}=&\int P(d\psi)\exp\Big\{\lambda\frac{1}{2}\sum_{\omega=\pm}\int d\mathbf{x}\ \hat\psi^{(\le N)+}_{\mathbf{x},\omega}\hat\psi^{(\le N)-}_{\mathbf{x},\omega}\hat\psi^{(\le N)+}_{\mathbf{x},-\omega}\hat\psi^{(\le N)-}_{\mathbf{x},-\omega}+\\ &\int d\mathbf{x}J_{\mathbf{x},\omega}\psi^{(\le N)+}_{\mathbf{x},\omega}\psi^{(\le N)-}_{\mathbf{x},\omega}+\sum_\omega\int d\mathbf{x}\ \Big[\varphi^+_{\mathbf{x},\omega}\psi^{(\le N)-}_{\mathbf{x},\omega}+\psi^{(\le N)+}_{\mathbf{x},\omega}\varphi^-_{\mathbf{x},\omega}\Big]\Big\},\end{aligned} \tag{2.1}$$

where $\lambda=e^2$ and

$$P(d\psi^{(\leq N)})=\prod_{\mathbf{k}\in\mathcal{D}}\prod_{\omega=\pm}\frac{\mathrm{d}\hat\psi^{(\leq N)+}_{\mathbf{k},\omega}\mathrm{d}\hat\psi^{(\leq N)-}_{\mathbf{k},\omega}}{\mathcal{N}_N(\mathbf{k})}\exp\left\{-\frac{1}{L^2}\sum_{\omega,\omega'=\pm}\sum_{\mathbf{k}\in\mathcal{D}}\frac{T_{\omega,\omega'}(\mathbf{k})}{C_N^{-1}(\mathbf{k})}\hat\psi^{(\leq N)+}_{\mathbf{k},\omega}\hat\psi^{(\leq N)-}_{\mathbf{k},\omega'}\right\} \tag{2.2}$$

and

$$T_{\omega,\omega'}(k)=\begin{pmatrix} D_+(\mathbf{k}) & -m \\ -m & D_-(\mathbf{k})\end{pmatrix}_{\omega,\omega'} ; \qquad D_\omega(\mathbf{k})=-ik_0+\omega k_1 \ . \tag{2.3}$$

and $\{J_{\mathbf{x},\omega}\}_{\mathbf{x},\omega}$ are commuting variables, while $\{\varphi^\sigma_{\mathbf{x},\omega}\}_{\mathbf{x},\omega,\sigma}$ are anticommuting. Finally, the normalization of the fermionic measure is $\mathcal{N}_N(\mathbf{k})=-(1/L^4)|\mathbf{k}|^2C_N^2(\mathbf{k})$.

The function $C_N^{-1}(\mathbf{k})$ is defined in the following way; $\chi_0\in C^\infty(\mathbb{R}_+)$ is a non-negative, non-increasing function such that

$$\chi_0(t)=\begin{cases}1 & \text{if } 0\leq t\leq 1\\ 0 & \text{if } t\geq\gamma_0\ ,\end{cases}$$

for any choice of $\gamma_0: 1<\gamma_0\leq\gamma$; and we define, for any h,

$$f_j(\mathbf{k})=\chi_0\left(\gamma^{-j}|\mathbf{k}|\right)-\chi_0\left(\gamma^{-j+1}|\mathbf{k}|\right) \tag{2.4}$$

and $C_N^{-1}(\mathbf{k})=\sum_{j=-\infty}^N f_j(\mathbf{k})$; hence $C_N^{-1}(\mathbf{k})$ acts as a cutoff for momenta $|\mathbf{k}|\geq\gamma^{N+1}$ (ultraviolet region). By well known properties of Grassmann integrals (see for instance [GM]) we can write

$$P(d\psi^{(\leq N)})=\prod_{h=-\infty}^{N}P(d\psi^{(h)}) \tag{2.5}$$

where $P(d\psi^{(h)})$ is given by (2.2) with $f_h(\mathbf{k})$ replacing $C_N^{-1}(\mathbf{k})$. We integrate iteratively starting from the highests scales. We define

$$\int d\mathbf{x}|\mathbf{x}||v(\mathbf{x})|=\gamma^{-h_M} \tag{2.6}$$

and the integration procedure is different for scales greater or smaller than h_M.

2.2 *Ultraviolet integration*

We show inductively that, for any $h_M\leq k\leq N$

$$e^{-L^2F_k}\int P(d\psi^{(\leq k)})e^{-\mathcal{V}^{(k)}(\psi^{(\leq k)},\phi,\hat J)}\ , \tag{2.7}$$

where the Grassmann integration $P(d\psi^{(\leq k)})$ is equal to $P(d\psi^{(\leq N)})$ with the cutoff function $C_N(\mathbf{k})$ replaced by $C_k(\mathbf{k})$,

$$\mathcal{V}^{(k)}(\psi^{(\leq k)},\phi,J)=\bar{\mathcal{V}}^{(k)}(\psi^{(\leq k)},\hat J)+\mathcal{B}^{(k)}(\psi^{(\leq k)},\phi,J) \tag{2.8}$$

where

$$\bar{\mathcal{V}}^{(k)}=\sum_{l,\underline{\omega},\underline{\varepsilon}}\int d\mathbf{x}_1...d\mathbf{x}_{2l}W^{(k)}_{2l,m,\underline{\omega},\underline{\varepsilon}}\prod_{i=1}^{2l}\psi^{\varepsilon_i\leq k}_{\mathbf{x}_i,\omega_i}\prod_{i=1}^{m}J_{\omega_i}(\mathbf{x}_i) \tag{2.9}$$

and

$$\mathcal{B}^{(k)}=\sum_\omega\int d\mathbf{x}\ \left[\varphi^+_{\mathbf{x},\omega}\psi^{(\leq k)-}_{\mathbf{x},\omega}+\psi^{(\leq k)+}_{\mathbf{x},\omega}\varphi^-_{\mathbf{x},\omega}\right]+\sum_{\bar m}\sum_{\underline{\omega},\underline{\omega}',\underline{\sigma}}\int d\underline{\mathbf{x}}$$

$$\frac{\partial^{\bar m}\bar{\mathcal{V}}^{(k)}}{\partial\psi^{(\leq k)\sigma_1}_{\mathbf{x}_1,\omega_1}...\partial\psi^{(\leq k)\sigma_m}_{\mathbf{x}_m,\omega_m}}g^{[k,N]}_{\omega_1,\omega_1'}(\mathbf{x}_1-\mathbf{y}_1)\phi^{\sigma_1}_{\mathbf{y}_1,\omega_1'}...g^{[k,N]}_{\omega_m,\omega_m'}(\mathbf{x}_{\bar m}-\mathbf{y}_{\bar m})\phi^{\sigma_{\bar m}}_{\mathbf{y}_1,\omega'_{\bar m}}+\int d\mathbf{x}d\mathbf{y}g^{[k,N]}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})\phi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{y},\omega'}$$

where $g^{[k,N]}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) = \sum_{i=k}^{N} g^{(i)}(\mathbf{x}-\mathbf{y})$. In order to inductively prove (2.7) for $\gamma^{h_M} \le k < N$ we proceed as follows. We introduce the *localization operator* as a linear operator acting on the kernels $W^{(k)}_{2l,m,\underline{\omega},\underline{\varepsilon}}$ in the following way:

$$\mathcal{L}W^{(k)}_{2l,m} = \begin{cases} W^{(k)}_{2l,m} & \text{if} \quad l=1,2 \quad m=0 \\ W^{(k)}_{2l,m} & \text{if} \quad l=1 \quad m=1 \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

We also define $\mathcal{R}$ as $\mathcal{R} = 1-\mathcal{L}$ and rewrite the r.h.s. of (2.7) as

$$e^{-L^2F_k} \int P(d\psi^{(\le k)}) e^{-\mathcal{L}\mathcal{V}^{(k)}(\psi^{(\le k)},\phi,J)-\mathcal{R}\mathcal{V}^{(k)}(\psi^{(\le k)},\phi,J)} , \tag{2.11}$$

where by definition $\mathcal{L}\mathcal{V}^{(k)}$ can be written as

$$\mathcal{L}\bar{\mathcal{V}}^{(k)} = \sum_{\omega} \int d\mathbf{x}d\mathbf{y}\, n_{k,\omega}(\mathbf{x},\mathbf{y})\, \psi^{(\le k)+}_{\mathbf{x},\omega} \psi^{(\le k)-}_{\mathbf{y},\omega} + \sum_{\omega,\omega'} \int d\mathbf{x}d\mathbf{y}d\mathbf{z} (1+Z^{(2)}_{k,\omega',\omega}(\mathbf{z};\mathbf{x},\mathbf{y})) J_{\omega'}(\mathbf{z}) \psi^{(\le k)+}_{\mathbf{x},\omega} \psi^{(\le k)-}_{\mathbf{y},\omega} + \\ - \sum_{\omega,\omega'} \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4\, \lambda_{k,\omega,\omega'}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)\, \psi^{(\le k)+}_{\mathbf{x}_1,\omega} \psi^{(\le k)-}_{\mathbf{x}_2,\omega} \psi^{(\le k)+}_{\mathbf{x}_3,\omega'} \psi^{(\le k)-}_{\mathbf{x}_4,\omega'} . \tag{2.12}$$

We write

$$e^{-L^2\beta(F_k+t_k)} \int P(d\psi^{[\le k-1]}) \int P(d\psi^{(k)}) e^{-\mathcal{L}\mathcal{V}^{(k)}(\psi^{(\le k-1)}+\psi^{(k)},\phi,J)-\mathcal{R}\mathcal{V}^{(k)}(\psi^{(\le k-1)}+\psi^{(k)},\phi,J)} , \tag{2.13}$$

with $P(d\psi^{(k)})$ a Grassmann Gaussian integration with f_k replacing C_N^{-1}, and the corresponding propagator $g^{(k)}_{\omega,\omega'}(\mathbf{x},\mathbf{y})$ is bounded by,for any $N>1$

$$|g^{(k)}(\mathbf{x},\mathbf{y})| \le \gamma^k \frac{C_{\bar{N}}}{1+[\gamma^k|\mathbf{x}-\mathbf{y}|]^{\bar{N}}} \tag{2.14}$$

If we now define

$$e^{-\mathcal{V}^{(k-1)}(\psi^{(\le k-1)},\phi,J)-L^2\tilde{F}_k} = \int P(d\psi^{(k)}) e^{-\mathcal{L}\mathcal{V}^{(k)}(\psi^{(\le k-1)}+\psi^{(k)},\phi,J)-\mathcal{R}\mathcal{V}^{(k)}(\psi^{(\le k-1)}+\psi^{(k)},\phi,J)} , \tag{2.15}$$

it is easy to see that the procedure can be iterated. In this way we have written the kernels $W^{(k)}_{2l,m}$ as functions of *running coupling functions* $v_k(\underline{\mathbf{x}}) = \lambda_k, n_k, Z^2_k$ with $k \ge h$; the main advantage of this procedure is that the kernels $W^{(k)}_{2l,m}$ can be bounded if the running couplings are small enough. Denoting by $||f|| = \frac{1}{L^2}\int \prod_{i=1}^n d\mathbf{x}_i |f(\mathbf{x}_1,..,\mathbf{x}_n)|$ the kernels obey to the following dimensional bounds, see Appendix 1.

Lemma 1 *Assume that $||v_k|| \le C\lambda$ for $k \ge h$, for a suitable constant C; then it holds, if $\bar{C}$ is a constant*

$$||W^{(h)}_{2l,m}|| \le \bar{C}\lambda\gamma^{-h(l+m-2)} \tag{2.16}$$

In order to use the above result to prove that the kernels $W^{(h)}_{2l,m}$ are bounded, one has to show that the running coupling functions are small. By construction it holds that

$$\lambda_{k-1}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = \lambda v(\mathbf{x}_1-\mathbf{x}_3)\delta(\mathbf{x}_1-\mathbf{x}_2)\delta(\mathbf{x}_3-\mathbf{x}_4) + \sum_{h=k}^{N} W^{(h)}_{4,0} \quad n_{k-1}(\mathbf{x}_1,\mathbf{x}_2) = \sum_{h=k}^{N} W^{(h)}_{2,0}(\mathbf{x}_1,\mathbf{x}_2) \\ Z^{(2)}_k(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = 1 + \sum_{h=k}^{N} W^{(h)}_{2,1}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3); . \tag{2.17}$$

The bound (2.16) cannot be used in (2.17) to show that the running coupling constants are small, and we have to improve it. Defining

$$H^{(k)}_{n,m}=\sum_{h=k}^{N} W^{(h)}_{n,m} \tag{2.18}$$

we prove the following result.

Lemma 2 *Assume that,for a suitable constant* C

$$\sup_{j\ge k-1}||\lambda_j(\underline{\mathbf{x}})||\le C\lambda \quad \sup_{j\ge k-1}||n_j(\underline{\mathbf{x}})||\le C\lambda \quad \sup_{j\ge k-1}||Z^{(2)}_j(\mathbf{x})||\le C\lambda \tag{2.19}$$

Then, for a suitable C_1

$$||H^{(k)}_{2,0}(\mathbf{x},\mathbf{y})||\le C_1\gamma^{-k}\lambda \quad ||H^{(k)}_{4,0}(\mathbf{x}_1,..,\mathbf{x}_4)||\le C_1\gamma^{-k}\lambda^2 \quad ||H^{(k)}_{2,1}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)||\le C_1\gamma^{-k}\lambda \tag{2.20}$$

Proof. The proof is done by induction. First one proves (2.20) for $k=N$ ((2.19) is of course verified). Moreover if (2.20) is true for $j\ge k-1$, of course the running coupling constants are bounded; then it is sufficient to prove (2.20) for $j=k$. The proof then is reduced to the verification of the bounds (2.20) if (2.19) are verified, and this is done below distinguishing the different cases.

2.3 *Two fermionic lines*

We start considering the massless case $m=0$. We define the *truncated expectation*, if X_i are momomials in $\psi^{[k,N]}$, in the following way

$$\mathcal{E}^T_{k,N}(X_1;X_2;..;X_n)=\frac{\partial^n}{\partial\lambda_1...\partial\lambda_n}\log\int P_{k,N}(d\psi)e^{\lambda_1 X_1+..\lambda_n X_n}|_{\lambda_1=\lambda_2=..=\lambda_n=0} \tag{2.21}$$

where $P_{k,N}(d\psi)$ is given by (2.2) with $C_{k,N}$ replacing C_N. For semplicity of notations we also denotate

$$\mathcal{E}^T_{k,N}(X_1X_2...X_n)\equiv\mathcal{E}^T_{k,N}(X_1;X_2;...;X_n) \tag{2.22}$$

It holds that

$$H^{(k)}_{2,0}(\mathbf{x},\mathbf{y})=\frac{\partial^2}{\partial\phi^+_{\omega,\mathbf{x}}\partial\phi^-_{\omega,\mathbf{y}}}\sum_{n=1}^\infty\frac{1}{n!}\mathcal{E}^T_{k,N}(V(\psi+\phi)...V(\psi+\phi))\equiv\sum_{n=1}^\infty\frac{1}{n!}\;\frac{\partial^2}{\partial\psi^-_{\omega,\mathbf{x}}\partial\psi^+_{\omega,\mathbf{y}}}\mathcal{E}^T_{k,N}(V(\psi)...V(\psi)) \tag{2.23}$$

We define $\tilde\psi(\mathbf{x})=\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1}$, $\tilde\psi(\mathbf{y})=\psi^+_{\mathbf{y},-1}\psi^-_{\mathbf{y},-1}$ and $\tilde\psi(\mathbf{x}\cup\mathbf{y})=\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},-1}\psi^-_{\mathbf{y},-1}$. Hence V in (2.23) is given by $\int d\mathbf{x}d\mathbf{y}\lambda v(\mathbf{x}-\mathbf{y})\tilde\psi(\mathbf{x}\cup\mathbf{y})$ and we can write

$$\frac{\partial^2}{\partial\psi^+_{\mathbf{x}}\partial\psi^-_{\mathbf{y}}}\mathcal{E}^T_{k,N}(V(\psi)...V(\psi))= \tag{2.24}$$

$$n\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}\int d\tilde{\mathbf{y}}\lambda v(\mathbf{x}-\tilde{\mathbf{y}})\mathcal{E}^T_{k,N}(\psi^-_{\mathbf{x}}\tilde\psi(\tilde{\mathbf{y}});V;...;V)+n\delta(\mathbf{x}-\mathbf{y})\int d\tilde{\mathbf{y}}v(\mathbf{x}-\tilde{\mathbf{y}})\mathcal{E}^T_{k,N}(\tilde\psi(\tilde{\mathbf{y}});V;...;V(\psi)) \tag{2.25}$$

where $\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}$ means that the derivative cannot be applied over $\psi^-_{\mathbf{x}}$; in (2.25) we have separated the case in which the two external lines are connected to the same coordinate from the case in which are connected to different coordinates.

We use the following property of truncated expectations, see for instance [Le], if $\tilde\psi(P_1\cup P_2)=[\prod_{i\in P_1}\psi^{\varepsilon_i}_{\mathbf{x}_i}][\prod_{i\in P_2}\psi^{\varepsilon_i}_{\mathbf{x}_i}]$

$$\mathcal{E}^T(\tilde\psi(P_1\cup P_2)\tilde\psi(P_3)...\tilde\psi(P_n))= \tag{2.26}$$

$$\sum_{\substack{K_1,K_2,K_1\cap K_2=0\\ K_1\cup K_2=3,..,n}}(-1)^\pi\mathcal{E}^T(\tilde\psi(P_1)\prod_{j\in K_1}\tilde\psi(P_j))\mathcal{E}^T(\tilde\psi(P_2)\prod_{j\in K_2}\tilde\psi(P_j))+\mathcal{E}^T(\tilde\psi(P_1)\tilde\psi(P_2)...\tilde\psi(P_n)) \tag{2.27}$$

and $(-1)^\pi$ is the parity of the permutation necessary to bring the Grassmann variables on the r.h.s. of (2.26) to the original order. Note that the number of terms in the sum in the r.h.s. of (2.26) is bounded by C^n for a suitable constant C. Note that the same property holds if we replace ψ with $\psi+\phi$ where ϕ is an external line.

By using (2.26) for the first addend of (2.25) we get, $V(j)=\int d\mathbf{x}_j d\mathbf{y}_j\lambda v(\mathbf{x}_j-\mathbf{y}_j)\tilde\psi(\mathbf{x}_j\bigcup\mathbf{y}_j)$

$$\int d\tilde{\mathbf{y}}v(\mathbf{x}-\tilde{\mathbf{y}})\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}\mathcal{E}^T_{k,N}(\psi^-_{\mathbf{x}};\tilde\psi(\mathbf{y});V(1);...;V(n))+$$

$$\sum_{\substack{K_1,K_2,K_1\cap K_2=0\\ K_1\cup K_2=1,..,n}}(-1)^\pi\int d\tilde{\mathbf{y}}v(\mathbf{x}-\tilde{\mathbf{y}})[\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}\mathcal{E}^T_{k,N}(\psi^-_{\mathbf{x}}\prod_{j\in K_1}V(j))]\mathcal{E}^T_{k,N}(\tilde\psi(\mathbf{y})\prod_{j\in K_2}V(j))$$

where have used that the derivative applied on the second truncated expectation gives zero (it is the expectation of an odd number of fields).

If we define

$$<A_1;...;A_n>_T=\frac{\partial^n}{\partial\lambda_1...\partial\lambda_n}\log\int P_{k,N}(d\psi)e^{-V+\sum_{i=1}^n\lambda_iA_i}|_{\underline{\lambda}=0} \tag{2.28}$$

by summing over n we get

$$H^{(k)}_{2,0}(\mathbf{x},\mathbf{y})=\int d\tilde{\mathbf{y}}\lambda v(\mathbf{x}-\tilde{\mathbf{y}})<\tilde\psi(\tilde{\mathbf{y}})>\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi_{\mathbf{x}}>+$$

$$\int d\tilde{\mathbf{y}}\lambda v(\mathbf{x}-\tilde{\mathbf{y}})\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}})>_T+\delta(\mathbf{x}-\mathbf{y})\int d\tilde{\mathbf{y}}v(\mathbf{x}-\tilde{\mathbf{y}})<\tilde\psi(\tilde{\mathbf{y}})> \tag{2.29}$$

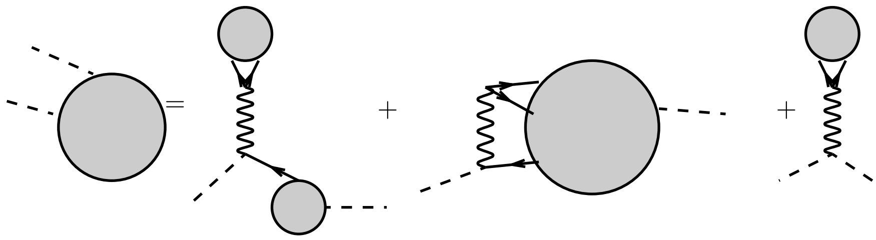
Fig. 1

Fig 1: Graphical representation of (2.29)

By a multiscale integration similar to the one in the previous section we get, see Appendix 1

$$||\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}})>_T||\le C\lambda\gamma^{-k} \tag{2.30}$$

Hence we get for the second addend in (2.29) the bound , using that $|v(\mathbf{x})|\le C$

$$\frac{1}{L^2}\int d\mathbf{x}d\mathbf{y}d\tilde{\mathbf{y}}|\lambda v(\mathbf{x}-\tilde{\mathbf{y}})\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}})>_T|\le C\lambda||\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}})>_T||\le C\lambda^2\gamma^{-k} \tag{2.31}$$

On the other hand the first and third term in (2.29) are vanishing in the massless case. In fact

$$<\tilde\psi(\mathbf{y})>=0 \tag{2.32}$$

as by translation invariance $<\tilde\psi(\mathbf{y})>=<\tilde\psi(0)>$. As there are only diagonal propagators we note that by it is given by the integral of $(4n+2)/2$ diagonal propagators and it is indipendent from $\tilde{\mathbf{y}}$, so it is vanishing by parity. Hence the integral of the first and last addend in (2.29) is vanishing.

2.4 *Two fermionic lines and one density line*

We have to bound

$$H_{2,1}^{(k)}(\mathbf{z};\mathbf{x},\mathbf{y})=\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\partial^2}{\partial\psi^+_{\mathbf{x}}\partial\psi^-_{\mathbf{y}}}\mathcal{E}^T_{k,N}(\tilde\psi(\mathbf{z})V...V) \tag{2.33}$$

and

$$\frac{\partial^2}{\partial\psi^+_{\mathbf{x}}\partial\psi^-_{\mathbf{y}}}\mathcal{E}^T_{k,N}(\tilde\psi(\mathbf{z})V...V)=n\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}\int d\tilde{\mathbf{y}}\lambda v(\mathbf{x}-\tilde{\mathbf{y}})\mathcal{E}^T_{k,N}(\psi^-_{\mathbf{x}}\tilde\psi(\tilde{\mathbf{y}});\tilde\psi(\mathbf{z});V;...;V)+ \tag{2.34}$$

$$n\lambda\delta(\mathbf{x}-\mathbf{y})\int d\tilde{\mathbf{y}}v(\mathbf{x}-\tilde{\mathbf{y}})\mathcal{E}^T_{k,N}(\tilde\psi(\tilde{\mathbf{y}})\tilde\psi(\mathbf{z})...V(\psi))$$

where again $\frac{\partial^*}{\partial\psi_{\mathbf{y}}}$ means that the derivative cannot be applied over $\psi^-_{\mathbf{x}}$; that is we have distinguished the case the two external lines comes put from different points or the same point. The first addend can be written,by (2.26) as

$$\int d\tilde{\mathbf{y}}v(\mathbf{x}-\tilde{\mathbf{y}})\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}\mathcal{E}^T_{k,N}(\psi^-_{\mathbf{x}}\tilde\psi(\tilde{\mathbf{y}});\tilde\psi(\mathbf{z});V;...;V)=\int d\tilde{\mathbf{y}}v(\mathbf{x}-\tilde{\mathbf{y}})\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}\mathcal{E}^T_{k,N}(\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}});\tilde\psi(\mathbf{z});V(j);...;V(n)])+$$

$$\sum_{\substack{K_1,K_2,K_1\cap K_2=0\\ K_1\cup K_2=1,..,n}}(-1)^\pi\int d\tilde{\mathbf{y}}v(\mathbf{x}-\tilde{\mathbf{y}})[\frac{\partial^*}{\partial\psi(\mathbf{y})}\mathcal{E}^T_{k,N}(\psi^-_{\mathbf{x}}\prod_{j\in K_1}V(j))]\mathcal{E}^T_{k,N}(\tilde\psi(\mathbf{z})\tilde\psi(\tilde{\mathbf{y}})\prod_{j\in K_2}V(j))$$

$$\sum_{\substack{K_1,K_2,K_1\cap K_2=0\\ K_1\cup K_2=3,..,n}}(-1)^\pi\int d\tilde{\mathbf{y}}v(\mathbf{x}-\tilde{\mathbf{y}})[\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}\mathcal{E}^T_{k,N}(\psi^-_{\mathbf{x}};\tilde\psi(\mathbf{z})\prod_{j\in K_1}V(j))]\mathcal{E}^T_{k,N}(\tilde\psi(\tilde{\mathbf{y}})\prod_{j\in K_2}V(j))$$

We finally obtain, by summing over n

$$H_{2,1}^{(k)}(\mathbf{z};\mathbf{x},\mathbf{y})=\int d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}\lambda v(\mathbf{x}-\tilde{\mathbf{y}})<\tilde\psi(\tilde{\mathbf{y}})>\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\mathbf{z})>_T+\int d\tilde{\mathbf{y}}\lambda v(\mathbf{x}-\tilde{\mathbf{y}})\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}});\tilde\psi(\mathbf{z})>_T$$

$$+\int d\tilde{\mathbf{y}}\lambda v(\mathbf{x}-\tilde{\mathbf{y}})<\tilde\psi(\tilde{\mathbf{y}});\tilde\psi(\mathbf{z})>_T\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^+_{\mathbf{x}}>+\delta(\mathbf{x}-\mathbf{y})\int d\tilde{\mathbf{y}}v(\mathbf{x}-\tilde{\mathbf{y}})<\tilde\psi(\mathbf{z});\tilde\psi(\tilde{\mathbf{y}})>_T \tag{2.35}$$

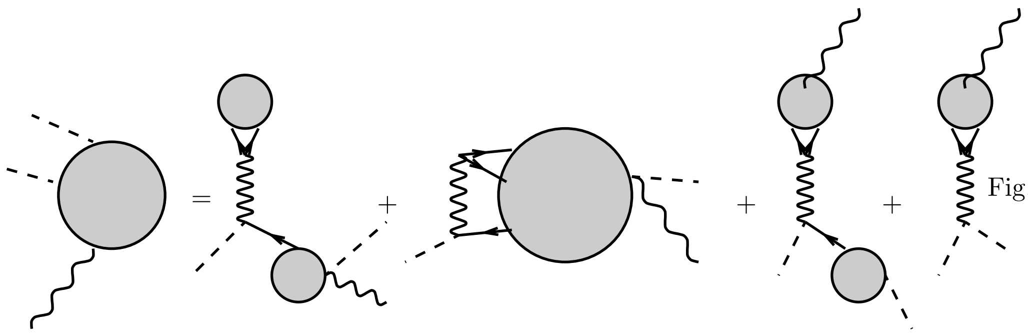

Fig. 2

Fig 2: Graphical representation of (2.35)

Again the first addend is vanishing in the massless case. The integral of the second addend is bounded by, as shown in the Appendix

$$\frac{1}{L^2}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}d\tilde{\mathbf{y}}|\lambda v(\mathbf{x}-\tilde{\mathbf{y}})\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}});\tilde\psi(\mathbf{z})>_T|\le C||\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}});\tilde\psi(\mathbf{z})>_T||\le C\lambda\gamma^{-2k} \tag{2.36}$$

In the third and fourth addend appear a density-density term which will be bounded in the following section by $C\gamma^{-k}$ so that they are also $O(\lambda\gamma^{-k})$.

2.5 *Two density lines*

We have to bound

$$H_{0,2}^{(k)}(\bar{\mathbf{z}},\mathbf{y})=\int d\mathbf{x}\lambda v(\bar{\mathbf{z}}-\mathbf{x})\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{E}_{k,N}^{T}(\tilde\psi(\mathbf{x})\tilde\psi(\mathbf{y})V...V)$$

We can distinguish the case in which the two fields $\tilde\psi(\mathbf{x})$ are contracted in the same point or not, so that

$$H_{0,2}^{(k)}(\bar{\mathbf{z}},\mathbf{y})=\int d\mathbf{x}d\mathbf{z}d\mathbf{z}'\lambda v(\bar{\mathbf{z}}-\mathbf{x})g_{\omega,\omega}^{[k,N]}(\mathbf{x}-\mathbf{z})g_{\omega,\omega}^{[k,N]}(\mathbf{x}-\mathbf{z}')\frac{\partial^*}{\partial\psi^-_{\mathbf{z}}}\frac{\partial}{\partial\psi^+_{\mathbf{z}'}}<\tilde\psi(\mathbf{y})>=$$

$$\int d\mathbf{x}d\mathbf{z}\lambda v(\bar{\mathbf{z}}-\mathbf{x})[g_{\omega,\omega}^{(k,N)}(\mathbf{x}-\mathbf{z})]^2[\delta(\mathbf{z}-\mathbf{y})+\int d\mathbf{z}'\lambda v(\mathbf{z}-\mathbf{z}')<\tilde\psi(\mathbf{z}');\tilde\psi(\mathbf{y})>_T]+$$

$$\int d\mathbf{x}d\mathbf{z}d\mathbf{z}'d\mathbf{z}''\lambda v(\bar{\mathbf{z}}-\mathbf{x})g_{\omega,\omega}^{(k,N)}(\mathbf{x}-\mathbf{z})g_{\omega,\omega}^{(k,N)}(\mathbf{x}-\mathbf{z}')v(\mathbf{z}'-\mathbf{z}'')\frac{\partial^*}{\partial\psi^-_{\mathbf{z}}}<\psi^-_{\mathbf{z}'}\tilde\psi(\mathbf{z}'');\tilde\psi(\mathbf{y})> \qquad (2.37)$$

where in the second line (2.34) has been used.

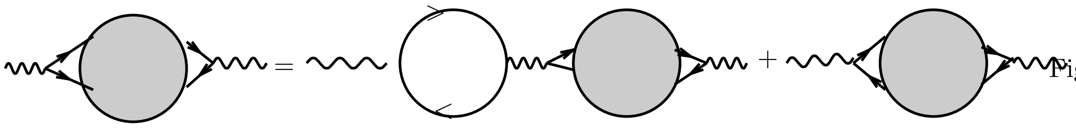

Fig 3: Graphical representation of (2.37)

The first addend of (2.37) can be rewritten as

$$\int d\mathbf{x}d\mathbf{z}v(\bar{\mathbf{z}}-\mathbf{z})][g_{\omega,\omega}^{(k,N)}(\mathbf{x}-\mathbf{z})]^2[\delta(\mathbf{z}-\mathbf{y})+\int d\mathbf{z}'\lambda v(\mathbf{z}-\mathbf{z}')<\tilde\psi(\mathbf{z}');\tilde\psi(\mathbf{y})>_T]+$$

$$\int d\mathbf{x}d\mathbf{z}[v(\bar{\mathbf{z}}-\mathbf{x})-v(\bar{\mathbf{z}}-\mathbf{z})][g_{\omega,\omega}^{(k,N)}(\mathbf{x}-\mathbf{z})]^2[\delta(\mathbf{z}-\mathbf{y})+\int d\mathbf{z}'\lambda v(\mathbf{z}-\mathbf{z}')<\tilde\psi(\mathbf{z}');\tilde\psi(\mathbf{y})>_T] \qquad (2.38)$$

The first line of (2.38) is vanishing in the massless case

$$\int d\mathbf{x}g_{\omega,\omega}^{(k,N)}(\mathbf{x}-\mathbf{z})=0 \qquad (2.39)$$

by the symmetry $g_{\omega,\omega}^{(k,N)}(r,r_0)=i\omega g_{\omega,\omega}^{(k,N)}(r_0,-r)$; on the other hand the second line can be written as

$$\int d\mathbf{z}d\mathbf{x}\lambda\int_0^1 dt[\partial_t v(\bar{\mathbf{z}}-\mathbf{z}+t(\mathbf{z}-\mathbf{x})](\mathbf{x}-\mathbf{z})[g_{\omega,\omega}^{[k,N]}(\mathbf{x}-\mathbf{z})]^2[\delta(\mathbf{z}-\mathbf{y})+\int d\mathbf{z}'\lambda v(\mathbf{z}-\mathbf{z}')<\tilde\psi(\mathbf{z}');\tilde\psi(\mathbf{y})>_T] \quad (2.40)$$

We perform the change of variables $\mathbf{r}_1=\mathbf{x}-\mathbf{z}\quad\mathbf{r}_2=\bar{\mathbf{z}}-\mathbf{z}+t(\mathbf{z}-\mathbf{x})$ with Jacobian -1, so that we can bound (2.40) as,using that $\int d\mathbf{x}|\partial v(\mathbf{x})|\le C$

$$||\int d\mathbf{r}_1d\mathbf{r}_2\lambda\partial v(\mathbf{r}_2)|\mathbf{r}_1|g_{\omega,\omega}^{[k,N]}(\mathbf{r}_1)]^2||\le C\lambda\sum_{k\le h_1\le h_2\le N}^{N}\gamma^{-3h_2}\gamma^{h_1}\gamma^{h_2}\le C\lambda\gamma^{-k}$$

$$||\int d\mathbf{r}_1d\mathbf{r}_2\lambda\partial v(\mathbf{r}_2)|\mathbf{r}_1|g_{\omega,\omega}^{[k,N]}(\mathbf{r}_1)]^2\int d\mathbf{z}'\lambda v(\mathbf{x}-\mathbf{r}_1-\mathbf{z}')<\tilde\psi(\mathbf{z}');\tilde\psi(\mathbf{y})>_T||\le C\lambda\gamma^{-k}||H_{0,2}^k|| \qquad (2.41)$$

where we have used that $g^{[k,N]}(\mathbf{r}) = \sum_{h=k}^N g^h(\mathbf{r})$. We consider now the second addend of (2.37); by (2.35) we get

$$\int d\mathbf{x}\lambda v(\bar{\mathbf{z}}-\mathbf{x})\int d\mathbf{z}d\mathbf{z}'d\mathbf{z}'' v(\mathbf{z}'-\mathbf{z}'')g^{[k,N]}(\mathbf{x}-\mathbf{z}')g^{[k,N]}(\mathbf{x}-\mathbf{z})\frac{\partial^*}{\partial\psi_{\mathbf{z}}} < \psi^-_{\mathbf{z}'};\tilde\psi(\mathbf{z}'');\tilde\psi(\mathbf{y}) >_T +$$
$$\int d\mathbf{x}\lambda v(\bar{\mathbf{z}}-\mathbf{x})\int d\mathbf{z}d\mathbf{z}'d\mathbf{z}''d\mathbf{z}''' g^{[k,N]}_{\omega,\omega}(\mathbf{x}-\mathbf{z})g^{[k,N]}_{\omega,\omega}(\mathbf{x}-\mathbf{z}')H_{2,0}(\mathbf{z}',\mathbf{z}'')g^{[k,N]}_{\omega,\omega}(\mathbf{z}''-\mathbf{z})]v(\mathbf{z}-\mathbf{z}''') < \tilde\psi(\mathbf{z}''');\tilde\psi(\mathbf{y}) >_T \tag{2.42}$$

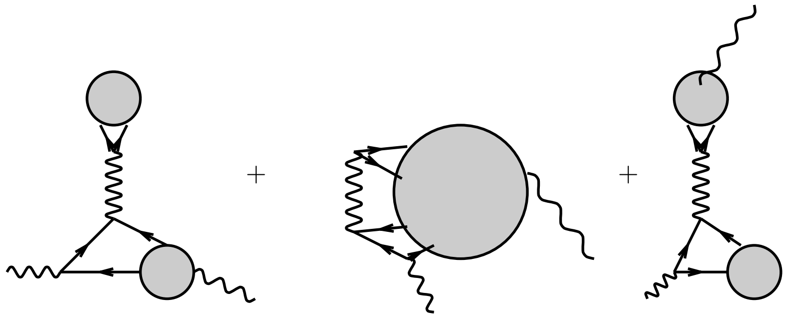

Fig. 4

Fig 4: Graphical representation of (2.42)

We prove in Appendix 1 that the first addend in (2.42) is bounded by

$$||g^{[k,N]}(\mathbf{x}-\mathbf{z}')g^{[k,N]}(\mathbf{x}-\mathbf{z})\frac{\partial^*}{\partial\psi_{\mathbf{z}}} < \psi^-_{\mathbf{z}'};\tilde\psi(\mathbf{z}'');\tilde\psi(\mathbf{y}) > || \le C\lambda\gamma^{-2k} \tag{2.43}$$

In order to bound the second addend

$$\int d\mathbf{z}d\mathbf{z}'d\mathbf{z}'' g^{[k,N]}_{\omega,\omega}(\mathbf{x}-\mathbf{z}')H^{(k)}_{2,0}(\mathbf{z}',\mathbf{z}'')g^{[k,N]}_{\omega,\omega}(\mathbf{z}''-\mathbf{z})g^{[k,N]}_{\omega,\omega}(\mathbf{x}-\mathbf{z})\int d\mathbf{z}'''\lambda v(\mathbf{z}-\mathbf{z}''') < \tilde\psi(\mathbf{z}''');\tilde\psi(\mathbf{y}) > \tag{2.44}$$

which we can rewrite as, using the compact support properties of the propagator

$$\sum_{k\le h_1,h_2\le N}\int d\mathbf{z}d\mathbf{z}'d\mathbf{z}'' g^{(h_1)}_{\omega,\omega}(\mathbf{x}-\mathbf{z}')H^{(k)}_{2,0}(\mathbf{z}',\mathbf{z}'')g^{(h_1)}_{\omega,\omega}(\mathbf{z}''-\mathbf{z})g^{(h_2)}_{\omega,\omega}(\mathbf{x}-\mathbf{z})\int d\mathbf{z}'''\lambda v(\mathbf{z}-\mathbf{z}''') < \tilde\psi(\mathbf{z}''');\tilde\psi(\mathbf{y}) > \tag{2.45}$$

We distinguish now the case $h_1 \le h_2$ or $h_1 \ge h_2$; if $h_1 \le h_2$ we integrate over $g^{(h_2)}_{\omega,\omega}$ and we use that $||H_{2,0}|| \le C\lambda\gamma^{-k}$ and $|g^{(h_1)}_{\omega,\omega}| \le C\gamma^{h_1}$ so that we get $\sum_{k\le h_1\le h_2\le N} C\lambda\gamma^{-k}\gamma^{h_1}\gamma^{-h_1}\gamma^{-h_2} \le \bar C\lambda\gamma^{-k}$. If $h_2 \le h_1$ we use that $|g^{(h_2)}_{\omega,\omega}| \le C\gamma^{h_2}$ so that we get $\sum_{k\le h_2\le h_1\le N} C\lambda\gamma^{-k}\gamma^{-2h_1}\gamma^{h_2} \le \bar C\lambda\gamma^{-k}$. In both case we can bound (2.44) by $C\gamma^{-k}||H^{(k)}_{0,2}||$.

By collecting all bounds we have found we have

$$||H^{(k)}_{0,2}|| \le C_1\lambda\gamma^{-k} - C_2\lambda\gamma^{-k}||H_{0,2}||$$

from which

$$||H^{(k)}_{0,2}|| \le \frac{C_1\lambda\gamma^{-k}}{1+C_2\lambda\gamma^{-k}} \le C_3\lambda\gamma^{-k} \tag{2.46}$$

2.6 *Four external lines*

In this case we can write

$$H^k_{4,0}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = \delta(\mathbf{x}_1-\mathbf{x}_2)\delta(\mathbf{x}_3-\mathbf{x}_4)\lambda v(\mathbf{x}_3-\mathbf{z}')H^k_{0,2}(\mathbf{x}_1,\mathbf{z}')+$$
$$\delta(\mathbf{x}_1-\mathbf{x}_2)\lambda v(\mathbf{x}_1-\mathbf{z})H^k_{1,2}(\mathbf{z};\mathbf{x}_3,\mathbf{x}_4) + \bar H^k_{4,0}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)$$

where the first two terms were evaluated before and the last term correspond to the four external lines attached at different points; proceeding as above it can be written as (in the massless case for semplicity)

$$\bar{H}^k_{4,0} = \int \lambda d\mathbf{z} v(\mathbf{x}_1 - \mathbf{z}) \frac{\partial^3}{\partial\psi_{\mathbf{x}_2}\partial\psi_{\mathbf{x}_3}\partial\psi_{\mathbf{x}_4}} < \psi^-_{\mathbf{x}_1}; \tilde{\psi}(\mathbf{z}) >_T + \int \lambda d\mathbf{z} v(\mathbf{x}_1 - \mathbf{z}) \frac{\partial}{\partial\psi_{\mathbf{x}_2}} < \psi^-_{\mathbf{x}_1} >_T \frac{\partial^2}{\partial\psi_{\mathbf{x}_3}\partial\psi_{\mathbf{x}_4}} < \tilde{\psi}(\mathbf{z}) >_T \tag{2.47}$$

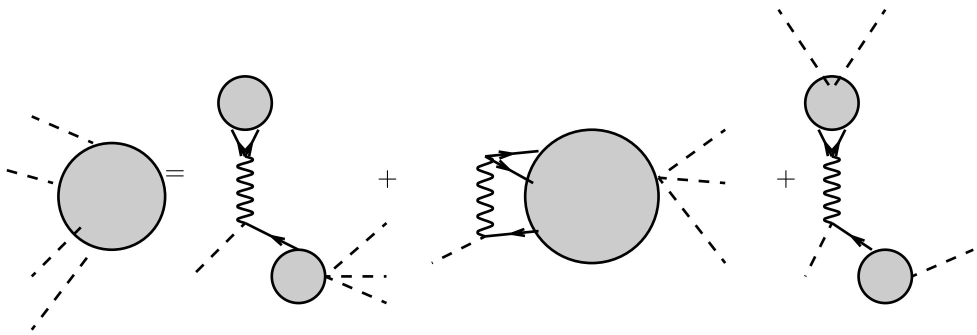

Fig. 5

Fig 5: Graphical representation of (2.47)

As it is proved in the Appendix 1

$$|| \frac{\partial^3}{\partial\psi_{\mathbf{x}_2}\partial\psi_{\mathbf{x}_3}\partial\psi_{\mathbf{x}_4}} < \psi^-_{\mathbf{x}_1}; \tilde{\psi}(\mathbf{z}) >_T || \leq C\lambda\gamma^{-k} \tag{2.48}$$

Finally the norm of last term in (2.47) is bounded by

$$|| \frac{\partial}{\partial\psi_{\mathbf{x}_2}} < \psi^-_{\mathbf{x}_1} >_T |||| \frac{\partial^2}{\partial\psi_{\mathbf{x}_3}\partial\psi_{\mathbf{x}_4}} < \tilde{\psi}(\mathbf{z}) >_T || \leq C\lambda\gamma^{-k} \tag{2.49}$$

as follows from the previous bounds.

We have then proved (2.20) in the massless case; in order to prove the bounds (2.20) in the massive case we note that we can write $g^k_{\omega,\omega}(\mathbf{x},\mathbf{y}) = \bar{g}^k_{\omega,\omega}(\mathbf{x},\mathbf{y}) + r^k_{\omega,\omega}(\mathbf{x},\mathbf{y})$ where $\bar{g}^k_{\omega,\omega}(\mathbf{x},\mathbf{y})$ is the propagator with $m = 0$ and $r^k_{\omega,\omega}(\mathbf{x},\mathbf{y})$ verifies the bound (2.14) with an extra $[\frac{m}{\gamma^k}]^2$; moreover $g^k_{\omega,-\omega}(\mathbf{x},\mathbf{y}) = \bar{g}^k_{\omega,-\omega}(\mathbf{x},\mathbf{y}) + r^k_{\omega,-\omega}(\mathbf{x},\mathbf{y})$ where $\bar{g}^k_{\omega,-\omega}(\mathbf{x},\mathbf{y})$ is the Fourier transform of $[\frac{m}{\gamma^k}]\bar{g}^k_{\omega,\omega}(\mathbf{k})\bar{g}^k_{-\omega,-\omega}(\mathbf{k})$ and $r^k_{\omega,-\omega}(\mathbf{x},\mathbf{y})$ verifies the bound (2.14) with an extra $[\frac{m}{\gamma^k}]^3$. Using the multilinearity of the determinants appearing in the fermionic expectations, we can separate the contribution in which all the propagators are massless (corresponding to the cases treated above) plus a rest, with an extra $[\frac{m}{\gamma^k}]$ with respect to the dimensional bound (2.16). The only case in which such improvment is not sufficient to get (2.20) is for $H^{(k)}_{2,0}$. However when the external line has the same ω index, there is surely at least or a propagator $r^k_{\omega,\omega}(\mathbf{x},\mathbf{y})$ or a nondiagonal propagator $g^k_{\omega,-\omega}(\mathbf{x},\mathbf{y})$, so that the improvment with respect to (2.16) is given by $[\frac{m}{\gamma^k}]^2$ and the bound (2.20) holds. The only remaining case is when the two external lines have different ω index and all the propagators are massless (if they are not there is an extra $[\frac{m}{\gamma^k}]$) and there is only a nondiagonal propagator $\bar{g}^k_{\omega,-\omega}(\mathbf{x},\mathbf{y})$; this case is then identical to the one trated in §2.4.

2.7 *The infrared integration*

After the integration of the scales $N, N-1, ..h_M$ we get a functional integral of the form

$$e^{-L^2 F_{h_M}} \int P(d\psi^{(\leq h_M)}) e^{-\mathcal{V}^{(h_M)}(\psi^{(\leq h_M)},\phi,J)} , \tag{2.50}$$

where $\mathcal{V}^{(h_M)}$ given by (2.9). The multiscale analysis of (2.9) has been done in [BM] in all details and we will not repeat it here; it turns out that after the integration of $h_M, ...h$ one gets

$$e^{-L^2 F_h} \int P_{Z_h,m_h}(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},\phi,J)} , \tag{2.51}$$

where $P_{Z_h,m_h}(d\psi^{(\leq h)})$ is given by (2.2) with C_N replaced by C_h, wave function renormalization Z_h and mass m_h; moreover the effective interaction $\mathcal{V}^{(h)}$ is $\lambda_h \int d\mathbf{x}\psi^{(\leq h)+}_{\omega,\mathbf{x}}\psi^{(\leq h)-}_{\omega,\mathbf{x}}\psi^{(\leq h)+}_{-\omega,\mathbf{x}}\psi^{(\leq h)-}_{\omega,\mathbf{x}}$ plus monomials in ψ of higher orders. As a consequence of remarkable cancellations due to the implementation of Ward identites based a local phase transformation at each iteration step, the effective counpling λ_h remains close to its initial value

$$\lambda_h = \lambda + O(\lambda^2) \tag{2.52}$$

and

$$Z_h = \gamma^{-\eta h}(1+O(\lambda)) \qquad \mu_h = \gamma^{-\tilde\eta h} m(1+O(\lambda)) \tag{2.53}$$

with $\eta = a\lambda^2 + O(\lambda^3)$ and $\tilde\eta = b\lambda + O(\lambda^2)$ with a, b positive constants. (2.51) is found by a procedure similar to the previous one in which the $\mathcal{L}$ operation consists in computing the kernel $W^h(\mathbf{k})$ at zero momentum or, in coordinate space, it consist of computing the external fields in the same coordinate point. Then, see [BM], to each kernel W^h is applied $1-\mathcal{L}$ which produces a derivative aplied on the external fields, giving an extra γ^h, and a factor $(\mathbf{x}-\mathbf{y})$, if $\mathbf{x}, \mathbf{y}$ are the coordinate of the external fields, wich can be bounded using that by $\int d\mathbf{z}|\mathbf{z}||g^i(\mathbf{z})| \leq C\gamma^{-2i}$ or $\int d\mathbf{z}|\mathbf{z}||v(\mathbf{z})| \leq \gamma^{-h_M} \leq \gamma^{-i}$. In this way an expansion for the Schwinger functions well defined in the limit $L, N \to \infty$ is obtained.

3. Ward Identities

3.1 *The anomaly* Performing the phase and chiral transformation $\psi^{\pm}_{\mathbf{x},\omega} \to e^{\pm\alpha_{\mathbf{x},\omega}}\psi^{\pm}_{\mathbf{x},\omega}$ in (2.1) and making derivatives with respect to the external fields we get,if $\rho_{\omega,\mathbf{x}} = \psi^+_{\omega,\mathbf{x}}\psi^-_{\omega,\mathbf{x}}$

$$D_{\omega'}(\mathbf{p})\langle\hat\rho_{\omega',\mathbf{p}}\hat\psi^-_{\omega,\mathbf{k}}\hat\psi^+_{\omega,\mathbf{k}-\mathbf{p}}\rangle = \delta_{\omega,\omega'}\left[\langle\hat\psi^-_{\omega,\mathbf{k}}\hat\psi^+_{\omega,\mathbf{k}}\rangle - \langle\hat\psi^-_{\omega,\mathbf{k}-\mathbf{p}}\hat\psi^+_{\omega,\mathbf{k}-\mathbf{p}}\rangle\right] + \Delta^{2,1,N}_{\omega',\omega}(\mathbf{p};\mathbf{k}) \tag{3.1}$$

where $\Delta^{2,1,N}_{\omega',\omega}(\mathbf{p};\mathbf{k})$ is the Fourier transform of

$$\Delta^{2,1,N}_{\omega',\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = \langle\psi^-_{\mathbf{y},\omega};\psi^+_{\mathbf{z},\omega};\delta T_{\mathbf{x},\omega'}\rangle_{L,N}\ , \tag{3.2}$$

where

$$\delta T_{\mathbf{x},\omega} = \frac{1}{L^4}\sum_{\substack{\mathbf{k}^+,\mathbf{k}^- \\ \mathbf{k}^+\neq\mathbf{k}^-}} e^{i(\mathbf{k}^+-\mathbf{k}^-)\mathbf{x}} C^{\varepsilon}_{N;\omega}(\mathbf{k}^+,\mathbf{k}^-)\hat\psi^+_{\mathbf{k}^+,\omega}\hat\psi^-_{\mathbf{k}^-,\omega}\ , \tag{3.3}$$

and

$$C_{N;\omega}(\mathbf{k}^+,\mathbf{k}^-) = [C_N(\mathbf{k}^-)-1]D_\omega(\mathbf{k}^-) - [C_N(\mathbf{k}^+)-1]D_\omega(\mathbf{k}^+)\ . \tag{3.4}$$

We write

$$\Delta^{2,1,N}_{\omega,\omega'}(\mathbf{p};\mathbf{k}) = v(\mathbf{p})\bar\nu_N(\mathbf{p})D_{-\omega}(\mathbf{p})G^{2,1,N}_{-\omega,\omega'}(\mathbf{p};\mathbf{k}) + \mathbf{p}_i R^{2,1,N}_{\omega,\omega',i}(\mathbf{p};\mathbf{k}) \tag{3.5}$$

where

$$\bar\nu(\mathbf{p}) = \lambda\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{C_{\omega,N}(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_{-\omega}(\mathbf{p})}g^{(\leq N)}_\omega(\mathbf{k})g^{(\leq N)}_\omega(\mathbf{k}-\mathbf{p}) \tag{3.6}$$

Note that if $\mathbf{p}$ is fixed, in the limit $N\to\infty$ we get, by changing variables $\mathbf{k}\to\gamma^N\mathbf{k}$ and expanding in $\gamma^{-N}\mathbf{p}$, $\bar\nu(\mathbf{p}) = \nu + O(\mathbf{p}\gamma^{-N})$, where $\nu = \bar\nu(\mathbf{p})|_{\mathbf{p}=0}$ and, using some cancellations due to the symmetry $g_\omega(\mathbf{k}) = -i\omega g_\omega(\mathbf{k}^*)$ with $\mathbf{k}^* = (k, k_0)$, it holds that

$$\bar\nu = \lambda\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{k_0}{|\mathbf{k}|}\chi_0'(|\mathbf{k}|)D^{-1}_\omega(\mathbf{k}) = \frac{\lambda}{4\pi}\int_0^\infty d\rho\chi_0'(\rho) = -\frac{\lambda}{4\pi} \tag{3.7}$$

We can obtain $\mathbf{p}_i R^{2,1,N}_{\omega,\omega',i}$ from the generating function

$$e^{\mathcal{W}_\Delta(J,\hat J,\phi)} = \int P(d\psi)e^{-V(\psi)+\sum_\omega\int d\mathbf{z}J(\mathbf{z})\psi^+_{\mathbf{z},\omega}\psi^-_{\mathbf{z},\omega}+\sum_\omega\int d\mathbf{z}[\psi^+_{\mathbf{z},\omega}\phi^-_{\mathbf{z},\omega}+\phi^+_{\mathbf{z},\omega}\psi^-_{\mathbf{z},\omega}]+T_0(\hat J,\psi)-\nu_{-,N}T_-(\hat J,\psi)} \tag{3.8}$$

with

$$T_0(\hat{J},\psi)=\frac{1}{L^4}\sum_{\mathbf{k},\mathbf{p}}\hat{J}(\mathbf{p})C_{N,\omega}(\mathbf{k},\mathbf{k}-\mathbf{p})\psi^+_{\mathbf{k}-\mathbf{p},\omega}\psi^-_{\mathbf{k},\omega}\equiv\frac{1}{L^2}\sum_{\mathbf{p}\neq 0}\hat{J}(\mathbf{p})\delta\rho_{\mathbf{p},\omega}$$
$$T_-(\hat{J},\psi)=\frac{1}{L^4}\sum_{\mathbf{k},\mathbf{p}}\hat{J}(\mathbf{p})D_{-\omega}(\mathbf{p})\psi^+_{\mathbf{k}-\mathbf{p},-\omega}\psi^-_{\mathbf{k},-\omega} \tag{3.9}$$

A crucial role in the analysis is played by the function

$$\Delta^{(i,j)}(\mathbf{k}^+,\mathbf{k}^-)=\frac{C_{N,\omega}(\mathbf{k}^+,\mathbf{k}^-)}{D_{-\omega}(\mathbf{p})}g^{(i)}_\omega(\mathbf{k}^+)g^{(j)}_\omega(\mathbf{k}^-)\ . \tag{3.10}$$

which is such that

$$\Delta^{(i,j)}_\omega(\mathbf{k}^+,\mathbf{k}^-)=0\qquad i,j<N \tag{3.11}$$

hence at least one between i or j must be equal to N. It holds that, for $i\leq N$

$$\Delta^{N,i}(\mathbf{k}^+,\mathbf{k}^-)=-\frac{f_j(\mathbf{k}^-)u_N(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+-\mathbf{k}^-)D_\omega(\mathbf{k}_-)}$$

where $u_N(\mathbf{k})=0$ for $|\mathbf{k}|\leq\gamma^N$ and $u_N(\mathbf{k})=1-f_N(\mathbf{k})$ for $|\mathbf{k}|\geq\gamma^N$. It is easy to verify that

$$\Delta^{N,i}(\mathbf{k}^+,\mathbf{k}^-)=\frac{\mathbf{p}}{D_{-\omega}(\mathbf{p})}S^{N,i}(\mathbf{k}^+,\mathbf{k}^-) \tag{3.12}$$

with

$$|\partial^m_{\mathbf{k}_+}\partial^l_{\mathbf{k}_-}S^{N,i}(\mathbf{k}^+,\mathbf{k}^-)|\leq C_{m+l}\gamma^{-i(1+l)}\gamma^{-l(1+N)} \tag{3.13}$$

from which

$$|S^{N,i}(\mathbf{z}-\mathbf{x},\mathbf{z}-\mathbf{y})|\leq C_{n+m}\frac{\gamma^N}{1+[\gamma^N|\mathbf{z}-\mathbf{x}|]^n}\frac{\gamma^i}{1+[\gamma^i|\mathbf{z}-\mathbf{y}|]^n} \tag{3.14}$$

3.2 *Multiscale analysis* The integration of $\mathcal{W}_\Delta(J,\hat{J},\phi)$ is done by a multiscale integration similar to the previous one. After the integration of $\psi^{(N)}$ the terms linear in $\hat{J}$ and quadratic in ψ in the exponent will be denoted by $K^{(N-1)}_J(\psi^{(\leq N-1)})$; we write $K^{(N-1)}_J=K^{(a,N-1)}_J+K^{(b,N-1)}_J$ where $K^{(a,N-1)}_J$ is obtained by the integration of T_0 and $K^{(b,N-1)}_J$ from the integration of T_-. We can write $K^{(a,N-1)}_J$ as

$$K^{(a,N-1)}_J(\psi^{(\leq N-1)})=\sum_{\tilde{\omega}}\int d\mathbf{y}d\mathbf{z}[F^{(N-1)}_{2,\omega,\tilde{\omega}}(\mathbf{x},\mathbf{y},\mathbf{z})+F^{(N-1)}_{1,\omega}(\mathbf{x},\mathbf{y},\mathbf{z})\delta_{\omega,\tilde{\omega}}]\psi^{+,\leq N-1}_{\mathbf{y},\tilde{\omega}}\psi^{-,\leq N-1}_{\mathbf{z},\tilde{\omega}} \tag{3.15}$$

where there is no $T_0(\hat{J},\psi^{\leq N-1})$ by (3.11), $F^{(N-1)}_{2,\omega,\tilde{\omega}}$ and $F^{(N-1)}_{1,\omega}$ represent the terms in which both or only one of the fields in $\delta\rho_{\mathbf{p},\omega}$, respectively, are contracted; we define

$$\mathcal{L}K^{(a,N-1)}_J=\sum_{\tilde{\omega}}\int d\mathbf{y}d\mathbf{z}F^{(N-1)}_{2,\omega,\tilde{\omega}}(\mathbf{x},\mathbf{y},\mathbf{z})\psi^{+,\leq N-1}_{\mathbf{y},\tilde{\omega}}\psi^{-,\leq N-1}_{\mathbf{z},\tilde{\omega}} \tag{3.16}$$

In the same way we decompose $K^{(b,N-1)}_J$ and we define $\mathcal{L}$ is a similar way; the above procedure leads to two new running coupling functions

$$\mathcal{L}K^{N-1}_J(\psi^{[\leq N-1]})=\frac{1}{L^4}\sum_{\mathbf{k},\mathbf{p}}\nu_{+,N-1}(\mathbf{k},\mathbf{p})\hat{J}(\mathbf{p})D_\omega(\mathbf{p})\psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k}+\mathbf{p},\omega}+\nu_{-,N-1}(\mathbf{k},\mathbf{p})\hat{J}(\mathbf{p})D_{-\omega}(\mathbf{p})\psi^+_{\mathbf{k},-\omega}\psi^-_{\mathbf{k}+\mathbf{p},-\omega} \tag{3.17}$$

The above integration procedure can be iterated with no important differences up to scale h_M; note that $F^{(k)}_{1,\omega}(\mathbf{k}^+,\mathbf{k}^-)$ is vanishing for $k \leq N-1$. If $W^{(h)}_{2l,m,\hat{m}}$ is the kernels in the effective potential moltiplying a monomial in $2l$ ψ-fields, m J fields and $\hat{m}$ $\hat{J}$ fields, the following result is proved in Appendix 2.

Lemma 3*Assume that for* $j \geq k$ $||\lambda_j||, ||n_j||, ||Z^{(2)}_j||$ *are small enough and*

$$||\nu_j|| \leq C\lambda\gamma^{\frac{1}{2}(k-N)} \tag{3.18}$$

then it holds the following bound,if $\hat{m} = 0, 1$

$$||W^{(h)}_{2l,m,\hat{m}}|| \leq C\lambda\gamma^{-h(l+m+\hat{m}-2)} \tag{3.19}$$

Moreover, if $\mathbf{k}, \mathbf{p}$ *are fixed to an* N *indipendent value*

$$\lim_{N\to\infty} R^{2,1,N}_{\omega,\omega',i}(\mathbf{k},\mathbf{p}) = 0 \tag{3.20}$$

3.3 *Proof of (2.20)*

We have now to prove (3.18). We can write

$$\nu_{k,-}(\mathbf{k},\mathbf{p}) = \nu^a_{k,-}(\mathbf{k},\mathbf{p}) + \nu^b_{k,-}(\mathbf{k},\mathbf{p}) \tag{3.21}$$

where in ν^b are the terms obtained contracting T_0 and in ν^a the terms obtained contracting T_-. It holds that

$$\nu^a_{k,-}(\mathbf{p},\mathbf{k}) = -v(\mathbf{p})\bar{\nu}(\mathbf{p}) - v(\mathbf{p})\bar{\nu}(\mathbf{p}) < \tilde{\psi}(\mathbf{p}); \psi^+_{\mathbf{k},-\omega}\psi^-_{\mathbf{k}+\mathbf{p},-\omega} > \tag{3.22}$$

Fig. 6

Fig 6: Graphical representation of (3.22)

On the other hand

$$\nu^b_k(\mathbf{p},\mathbf{k}) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial}{\partial\hat{J}(\mathbf{p})}\mathcal{E}^T_{k,N}(T_0; \frac{\partial^2}{\partial\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}+\mathbf{p}}}[V...V]) \tag{3.23}$$

By construction the two external fields cannot be attached to C (otherwise $\mathcal{L} = 0$). We distinguish now the case, as in §2.5, in which both the fermionic fields in T_0 are contracted with the same point with the case in which are contracted in different points.

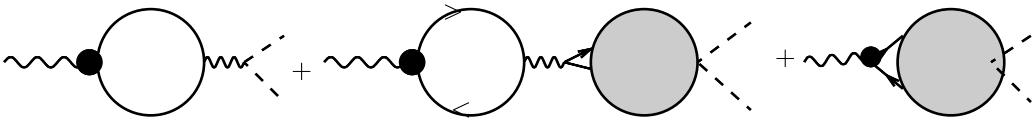

Fig. 7

Fig 7: Graphical representation of (3.23); the black dot represents C_N.

In momentum space the first case can be written

$$[\int \frac{d\mathbf{k}}{(2\pi)^2}\frac{C_{\omega,N}(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_{-\omega}(\mathbf{p})} g^{[k,N]}_\omega(\mathbf{k}) g^{[k,N]}_\omega(\mathbf{k}-\mathbf{p})]\lambda v(\mathbf{p})[1+ < \tilde\psi(\mathbf{p})\psi^+_{\mathbf{k}'}\psi^-_{\mathbf{k}'+\mathbf{p}} >] \tag{3.24}$$

so that summing (3.24) with (3.22) and using (3.6) we get a vanishing contribution for $k \le N-|\mathbf{p}|$; in fact it holds that

$$\int \frac{d\mathbf{k}}{(2\pi)^2}\frac{C_{\omega,N}(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_{-\omega}(\mathbf{p})} g^{[k,N]}_\omega(\mathbf{k}) g^{[k,N]}_\omega(\mathbf{k}-\mathbf{p})] = \int \frac{d\mathbf{k}}{(2\pi)^2}\frac{C_{\omega,N}(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_{-\omega}(\mathbf{p})} g^{\le N}_\omega(\mathbf{k}) g^{\le N}_\omega(\mathbf{k}-\mathbf{p})] \tag{3.25}$$

In fact $\Delta_{N,j}(\mathbf{k}^+,\mathbf{k}^-)$ is such that $|\mathbf{k}^+| \ge \gamma^N$ and $|\mathbf{k}^-| \ge \gamma^j$, then as $\mathbf{p}=\mathbf{k}^+-\mathbf{k}^-$ necessarily $j \ge N-|\mathbf{p}|$. On the other hand for $k \ge N-|\mathbf{p}|$ we have to prove that $||\nu_k|| \le \gamma^{-N+k} \le \gamma^{-|\mathbf{p}|}$ which is surely true as $\mathbf{p}$ is fixed.

It remain to consider the contribution to ν_b in which the two fermionic fields in T_0 are contracted with different points. We pass to coordinate space and we proceed exactly as in §2.5.

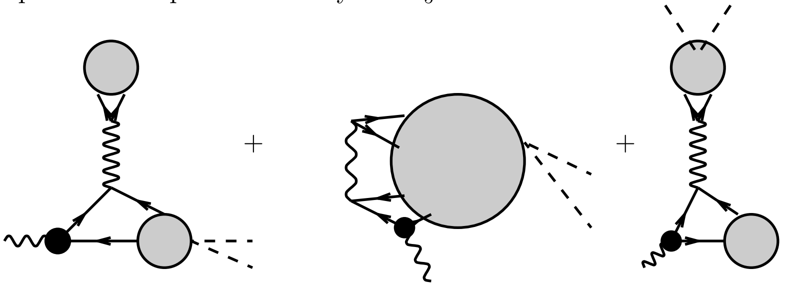

Fig. 8

Fig 8: Graphical representation of (3.27)

Such contribution can be written as

$$H(\mathbf{x},\mathbf{y}_1,\mathbf{y}_2) = \int d\mathbf{z}d\mathbf{z}'d\mathbf{z}'' \sum_{i=k}^N [S^{N,i}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}') + S^{i,N}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}')] v(\mathbf{z}'-\mathbf{z}'') \frac{\partial}{\partial\psi^-_{\mathbf{y}_1}}\frac{\partial}{\partial\psi^+_{\mathbf{y}_2}}\frac{\partial^*}{\partial\psi^-_{\mathbf{z}}} < \psi^-_{\mathbf{z}'}\tilde\psi(\mathbf{z}'') > \tag{3.26}$$

which can be rewritten as

$$\int d\mathbf{z}d\mathbf{z}'d\mathbf{z}''d\mathbf{z}''' v(\mathbf{z}'-\mathbf{z}'') \sum_{i=k}^N [S^{N,i}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}') + S^{i,N}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}')] \frac{\partial}{\partial\psi^-_{\mathbf{y}_1}}\frac{\partial}{\partial\psi^+_{\mathbf{y}_2}}\frac{\partial^*}{\partial\psi_{\mathbf{z}}} < \psi^-_{\mathbf{z}'};\tilde\psi(\mathbf{z}'') > +$$
$$\int d\mathbf{z}d\mathbf{z}'d\mathbf{z}'' \sum_{i=k}^N [S^{N,i}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}') + S^{i,N}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}')] H_{2,0}(\mathbf{z}',\mathbf{z}'') g^{(k,N)}_{\omega,\omega}(\mathbf{z}''-\mathbf{z})] v(\mathbf{z}-\mathbf{z}''') H_{1,2}(\mathbf{z}''',\mathbf{y}_1,\mathbf{y}_2) \tag{3.27}$$

We will prove in the Appendix that the first addend in (3.27) is bounded by

$$|| \sum_{i=k}^N [S^{N,i}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}') + S^{i,N}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}')] \frac{\partial^*}{\partial\psi_{\mathbf{z}}} < \psi^-_{\mathbf{z}'};\tilde\psi(\mathbf{z}'');\tilde\psi(\mathbf{y}) > || \le C\lambda\gamma^{-\frac{1}{2}(N+k)} \tag{3.28}$$

On the other hand the second addend in (3.27) is easily bounded by noting that, by momentum conservation,

$$||\sum_{i=k}^{N}[S^{N,i}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}')g^{(i)}_{\omega,\omega}(\mathbf{z}''-\mathbf{z})+S^{i,N}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}')g^{(N)}_{\omega,\omega}(\mathbf{z}''-\mathbf{z})]H_{2,0}(\mathbf{z}',\mathbf{z}'')v(\mathbf{z}-\mathbf{z}''')H_{1,2}(\mathbf{z}''',\mathbf{y}_1,\mathbf{y}_2)|| \quad (3.29)$$

We proceed as in §2.5, and in the first addend we integrate over the line $\mathbf{x}-\mathbf{z}$, using that $||H_{2,0}||\le C\lambda\gamma^{-k}$ and $|g^{(i)}_{\omega,\omega}|\le C\gamma^{-i}$ getting $\sum_{i=k}^N\gamma^{-N}\gamma^{-i}\gamma^{i}\gamma^{-k}\le |N-k|\gamma^{-N-k}\le C\gamma^{(-N-k)/2}$; in the second addend we integrate over the line $\mathbf{x}-\mathbf{z}'$, getting $\sum_{i=k}^N\gamma^{-2N}\gamma^{i}\gamma^{-k}\le C\gamma^{(-N-k)/2}$.

A similar analysis can be done for $\nu_{k,+}$ with the only difference that the first term in (3.22) and in Fig.7 is absent.

4. Appendix 1

4.1 *Proof of Lemma 1* For an introduction to the formalism used in this section we will refer to [GM]. We define a family of trees in the following way.

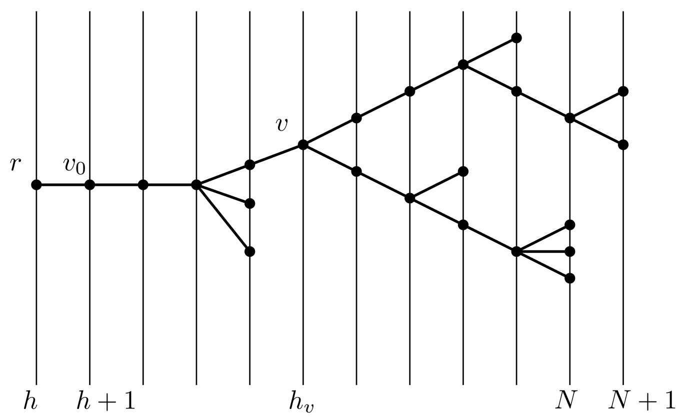

Fig. 9

Fig 9: an example of tree τ

1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of n points, the *endpoints* of the *unlabelled tree*, so that r is not a branching point. n will be called the *order* of the unlabelled tree and the branching points will be called the *non trivial vertices*. The unlabelled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order.

Two unlabelled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabelled trees with n end-points is bounded by 4^n.

We shall consider also the *labelled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabelled trees, as explained in the following items.

2) We associate a label $h\le N-1$ with the root and we denote by $\mathcal{T}_{h,n}$ the corresponding set of labelled trees with n endpoints. Moreover, we introduce a family of vertical lines, labelled by an integer taking values in $[h,N+1]$, and we represent any tree $\tau\in\mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained

in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root is on the line with index h. There is the constraint that, if v is an endpoint, $h_v > h+1$.
The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.
Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint (see above); its scale is $h+1$.
Finally, if there is only one endpoint, its scale must be equal to $h+2$.
3) With each endpoint v of scale $h_v = N+1$ we associate one of the monomials in the exponential of (2.1) and a set $\mathbf{x}_v$ of space-time points (the corresponding integration variables); with each endpoint of scale $h_v \le N$ we associate one of contributions in $\mathcal{L}\mathcal{V}^{(h_v)}$ (2.12). We impose the constraint that, if v is an endpoint, $h_v = h_{v'}+1$, if v' is the non trivial vertex immediately preceding v. Given a vertex v, which is not an endpoint, $\mathbf{x}_v$ will denote the family of all space-time points associated with one of the endpoints following v.
4) The trees containing only the root and an endpoint of scale $h+1$ (note that they do not belong to $\mathcal{T}_{h,N+1}$) will be called the *trivial trees*.
In terms of these trees, the effective potential $\mathcal{V}^{(h)}$, $h \le 1$, can be written as

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) + L^2 \tilde{F}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) \;, \tag{4.1}$$

where, if v_0 is the first vertex of τ and $\tau_1, .., \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, $\mathcal{V}^{(h)}(\tau, \psi^{(\le h)})$ is defined inductively by the relation

$$\mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1}[\bar{\mathcal{V}}^{(h+1)}(\tau_1, \psi^{(\le h+1)}); \ldots; \bar{\mathcal{V}}^{(h+1)}(\tau_s, \psi^{(\le h+1)})] \;, \tag{4.2}$$

and $\bar{\mathcal{V}}^{(h+1)}(\tau_i, \psi^{(\le h+1)})$,if $\mathcal{R} = 1 - \mathcal{L}$
a) is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i, \psi^{(\le h+1)})$ if the subtree τ_i is not trivial;
b) if τ_i is trivial and $h < N-1$, it is equal to $\mathcal{L}\mathcal{V}^{(h+1)}$ or, if $h = N-1$, to one of the monomials contributing to $\mathcal{V}^{(N)}(\psi^{\le N})$.
$\mathcal{E}^T_{h+1}$ denotes the truncated expectation with respect to the measure $P(d\psi^{(h+1)})$ We associate then to each vertex v is associated the set P_v, the set of labels of *external fields of* v, that is the field variables of type ψ which belong to one of the endpoints following v and either are not yet contracted in the vertex v (we call $P_v^{(n)}$ the set of these variables) or are contracted with the ψ variable of an endpoint of type φ through a propagator $g^{[h_v,N]}$. The sets $|P_v|$ must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \ldots, v_{s_v}$ are the vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$; if v is an endpoint, P_v is the set of field labels associated to it. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i} \backslash Q_{v_i}$, whose union $\mathcal{I}_v$ will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$. Given $\tau \in \mathcal{T}_{h,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints. We shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. Moreover, we associate with any $f \in \mathcal{I}_v$ a scale label $h(f) = h_v$. We call χ-vertices the vertices of τ such that their set $\mathcal{I}_v$ of internal lines is not empty; $V_\chi(\tau)$ will denote the set of all χ-vertices of τ.
With these definitions, we can rewrite $\mathcal{V}^{(h)}(\tau, \psi^{(\le h)})$ in the r.h.s. of (4.1) as:

$$\begin{aligned} \mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) &= \sum_{\mathbf{P} \in \mathcal{P}_\tau} \mathcal{V}^{(h)}(\tau, \mathbf{P}) \;, \\ \mathcal{V}^{(h)}(\tau, \mathbf{P}) &= \int d\mathbf{x}_{v_0} \tilde{\psi}^{(\le h)}(P_{v_0}) K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0}) \;, \end{aligned} \tag{4.3}$$

where

$$\tilde{\psi}^{(\le h)}(P_v) = \prod_{f \in P_v} \psi^{(\le h)\varepsilon(f)}_{\mathbf{x}(f),\omega(f)} \tag{4.4}$$

and $K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})$ is defined inductively by the equation, valid for any $v\in\tau$ which is not an endpoint,

$$K^{(h_v)}_{\tau,\mathbf{P},\Omega}(\mathbf{x}_v)=\frac{1}{s_v!}\prod_{i=1}^{s_v}[K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i})]\;\;\mathcal{E}^T_{h_v}[\tilde\psi^{(h_v)}(P_{v_1}\backslash Q_{v_1}),\ldots,\tilde\psi^{(h_v)}(P_{v_{s_v}}\backslash Q_{v_{s_v}})]\;, \tag{4.5}$$

where $\tilde\psi^{(h_v)}(P_{v_i}\backslash Q_{v_i})$ has a definition similar to (4.4). Moreover, if v is an endpoint and $h_v\le N$, $K^{(h_v)}_v(\mathbf{x}_v)=\lambda_{h_v},n_{h_v},1+Z^{(2)}_{h_v}$, while if $h_v=N+1$ $K^{(1)}_v$ is equal to one of the kernels of the monomials in (2.1).

(4.1)–(4.3) is not the final form of our expansion; we further decompose $\mathcal{V}^{(h)}(\tau,\mathbf{P})$, by using the following representation of the truncated expectation in the r.h.s. of (4.4). Let us put $s=s_v$, $P_i\equiv P_{v_i}\backslash Q_{v_i}$; moreover we order in an arbitrary way the sets $P_i^\pm\equiv\{f\in P_i,\varepsilon(f)=\pm\}$, we call $f^\pm_{ij}$ their elements and we define $\mathbf{x}^{(i)}=\cup_{f\in P_i^-}\mathbf{x}(f)$, $\mathbf{y}^{(i)}=\cup_{f\in P_i^+}\mathbf{x}(f)$, $\mathbf{x}_{ij}=\mathbf{x}(f^-_{i,j})$, $\mathbf{y}_{ij}=\mathbf{x}(f^+_{i,j})$. Note that $\sum_{i=1}^s|P_i^-|=\sum_{i=1}^s|P_i^+|\equiv n$, otherwise the truncated expectation vanishes. A couple $l\equiv(f^-_{ij},f^+_{i'j'})\equiv(f^-_l,f^+_l)$ will be called a line joining the fields with labels $f^-_{ij},f^+_{i'j'}$ connecting the points $\mathbf{x}_l\equiv\mathbf{x}_{i,j}$ and $\mathbf{y}_l\equiv\mathbf{y}_{i'j'}$, the *endpoints* of l. Then, it is well known [Le, GM] that, up to a sign, if $s>1$,

$$\begin{aligned}&\mathcal{E}^T_h(\tilde\psi^{(h)}(P_1)\tilde\psi^{(h)}(P_2)...\tilde\psi^{(h)}(P_s))=\\&=\sum_T\prod_{l\in T}\tilde g^{(h)}(\mathbf{x}_l-\mathbf{y}_l)\int dP_T(\mathbf{t})\det G^{h,T}(\mathbf{t})\;,\end{aligned} \tag{4.6}$$

where T is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}^{(i)}\cup\mathbf{y}^{(i)}$, that is T is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover $\mathbf{t}=\{t_{i,i'}\in[0,1],1\le i,i'\le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'}=\mathbf{u}_i\cdot\mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i\in\mathbb{R}^s$ of unit norm. Finally $G^{h,T}(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by

$$G^{h,T}_{ij,i'j'}=t_{i,i'}\tilde g^{(h)}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'}) \tag{4.7}$$

with $(f^-_{ij},f^+_{i'j'})$ not belonging to T.

In the following we shall use (4.6) even for $s=1$, when T is empty, by interpreting the r.h.s. as equal to 1, if $|P_1|=0$, otherwise as equal to $\det G^h=\mathcal{E}^T_h(\tilde\psi^{(h)}(P_1))$.

If we apply the expansion (4.6) in each non trivial vertex of τ, we get an expression of the form

$$\mathcal{V}^{(h)}(\tau,\mathbf{P})=\sum_{T\in\mathbf{T}}\int d\mathbf{x}_{v_0}\tilde\psi^{(\le h)}(P_{v_0})W^{(h)}_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0})\equiv\sum_{T\in\mathbf{T}}\mathcal{V}^{(h)}(\tau,\mathbf{P},T)\;, \tag{4.8}$$

where $\mathbf{T}=\bigcup_v T_v$. Given $\tau\in\mathcal{T}_{h,n}$ and the labels $\mathbf{P},T$, calling $v_i^*,\ldots,v_n^*$ the endpoints of τ and putting $h_i=h_{v_i^*}$, we get the bound

$$\begin{aligned}|W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0})|\le&\int\prod_{l\in T^*}d(\mathbf{x}_l-\mathbf{y}_l)\left[\prod_{i=1}^n|v_{h_i-1}(\mathbf{x}_{v_i^*})|\right]\cdot\\&\cdot\Big\{\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\max_{\mathbf{t}_v}|\det G^{h_v,T_v}(\mathbf{t}_v)|\prod_{l\in T_v}|g^{(h_v)}(\mathbf{x}_l-\mathbf{y}_l)|\Big\}\;,\end{aligned} \tag{4.9}$$

where T^* is a tree graph obtained from $T=\cup_vT_v$, by adding in a suitable (obvious) way, for each endpoint v_i^*, $i=1,\ldots,n$, the lines connecting the space-time points belonging to $\mathbf{x}_{v_i^*}$. A standard application of Gram–Hadamard inequality, combined with the dimensional bound on $g^{(h)}$ implies that

$$|\det G^{h_v,T_v}_\alpha(\mathbf{t}_v)|\le c^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)}\cdot\;\gamma^{h_v\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)\right)}\;. \tag{4.10}$$

Moreover

$$\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\int\prod_{l\in T^*}d(\mathbf{x}_l-\mathbf{y}_l)\prod_{i=1}^n|v_{h_i-1}(\mathbf{x}_{v_i^*})||g^{(h_v)}_{\omega_l}(\mathbf{x}_l-\mathbf{y}_l)|\le c^n\prod_{v\text{ note.p.}}\frac{1}{s_v!}\gamma^{-h_v(s_v-1)}\;. \tag{4.11}$$

so we can bound the r.h.s. of (4.9) by,if $\bar n+n_J+n_\phi=n$

$$(c\lambda)^{\bar n}\gamma^{h(2-\frac{|P_{v_0}|}{2}-n^J_{v_0}-n_{2,v_0})}\prod_{v\ \text{not e.p.}}\frac{1}{s_v!}\gamma^{-(\frac{|P_v|}{2}-2+n^J_v+n_{2,v})} \tag{4.12}$$

where n^J_v are the J fields associated to the endpoints following v, $n_{2,v}$ is the number of endpoints of type n_k following v and $\bar n+n_J+n_\phi=n$. We can bound (4.12) by

$$(c\lambda)^{\bar n}\gamma^{-hd_{v_0}}\prod_{\tilde v\in V_\chi(\tau)}\frac{1}{s_{\tilde v}!}\gamma^{-(h_{\tilde v}-h_{\tilde v'})d_{\tilde v}} \tag{4.13}$$

where d_v is the *dimension*, $d_v=\frac{|P_v|}{2}-2+n^J_v+n_{2,v}$ and and $\tilde v'$ is the χ-vertex immediately preceding $\tilde v$. By construction the vertices v with $|P_v|=2,4$ and $n^J_v=0$, or $|P_v|=2$ and $n^J_v=1$ are necessarily endpoints so they do not belong to V_χ, hence $d_v\ge 1$. In order to sum over τ and $\mathbf{P}$ we note that the number of unlabeled trees is $\le 4^n$; fixed an unlabeled tree, the number of terms in the sum over the various labels of the tree is bounded by C^n, except the sums over the scale labels and the sets $\mathbf{P}$. Regarding the sum over T, it is empty if $s_v=1$. If $s_v>1$ and $N_{v_i}\equiv|P_{v_i}|-|Q_{v_i}|$, the number of anchored trees with d_i lines branching from the vertex v_i can be bounded, by using Caley's formula, by

$$\frac{(s_v-2)!}{(d_1-1)!...(d_{s_v}-1)!}N_{v_1}^{d_1}...N_{v_{s_v}}^{d_{s_v}}\ ; \tag{4.14}$$

hence the number of addenda in $\sum_{T\in\mathbf{T}}$ is bounded by $\prod_{v\,\text{not e.p.}}s_v!\ C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}$.

In order to bound the sums over the scale labels and $\mathbf{P}$ we first use the inequality, following from (4.13)

$$\prod_{v\in V_\chi(\tau)}\gamma^{-(h_v-h_{v'})d_v}\le[\prod_{v\in V_\chi(\tau)}\gamma^{-\frac{1}{40}(h_v-h_{v'})}][\prod_{v\in V_\chi(\tau)}\gamma^{-\frac{|P_v|}{40}}] \tag{4.15}$$

The factors $\gamma^{-\frac{1}{40}(h_{\tilde v}-h_{\tilde v'})}$ in the r.h.s. of (4.15) allow to bound the sums over the scale labels by C^n. The sum over $\mathbf{P}$ can be bounded by using the following combinatorial inequality. Let $\{p_v,v\in\tau\}$ a set of integers such that $p_v\le\sum_{i=1}^{s_v}p_{v_i}$ for all $v\in\tau$ which are not endpoints; then (see for instance App. 6 of [GM])

$$\sum_{\mathbf{P}}\prod_{v\in V_\chi(\tau)}\gamma^{-\frac{|P_v|}{40}}\le\prod_{v\in V_\chi(\tau)}\sum_{p_v}\gamma^{-\frac{p_v}{40}}B(\sum_{i=1}^{s_v}p_v,p_v)\le C^n\ . \tag{4.16}$$

where $B(n,m)$ is the binomial coefficient. This concludes the proof of the Lemma.

Remark 1 If in τ there are two χ-vertices with scale h_1 and h_2, we can write the r.h.s. of (4.13) as

$$(c|\lambda|)^n\gamma^{-hd_{v_0}}\gamma^{-\frac12|h_1-h_2|}\prod_{\tilde v\in V_\chi(\tau)}\frac{1}{s_{\tilde v}!}\gamma^{-\frac12(h_{\tilde v}-h_{\tilde v'})d_{\tilde v}} \tag{4.17}$$

as $d_{\tilde v}\ge 1$; of course the sum over $\tau,\mathbf{P}$ can be performed as above and this implies that the dimensional bound (2.16) can be improved by $\gamma^{-\frac12|h_1-h_2|}$ for such trees; this property is called *short memory property.*

Remark 2 Let us consider a tree τ contributing to a kernel $W^{(h)}_{2l,m}$ for which $\mathcal{L}=1$ (see (2.10)); then $v_0\in V_\chi$; in fact if $v_0\notin V_\chi$ then v_0 is trivial and the external fields of v_0 and v_1 are the same,if v_1 is the vertex preceding v_0; then $\mathcal{RV}(\tau_1,\psi)=0$.

4.2 *Proof of (2.30)* It holds that

$$\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}})>_T=\frac{\partial^2}{\partial\phi(\mathbf{y})\phi(\mathbf{x})}\frac{\partial^2}{\partial J(\tilde{\mathbf{y}})\bar J(\mathbf{x})}\mathcal{H}(\phi,J,\bar J) \tag{4.18}$$

where

$$e^{\mathcal{H}(\phi,J,\bar J)}=\int P_{k,N}(d\psi)e^{-\mathcal{V}(\psi+\phi)+\sum_\omega\int d\mathbf{z}J(\mathbf{z})[\psi^+_{\mathbf{z},\omega}+\phi^+_{\mathbf{z},\omega}][\psi^-_{\mathbf{z},\omega}+\phi^-_{\mathbf{z},\omega}]+\int\bar J(\mathbf{z})\phi(\mathbf{z})[\psi_{\mathbf{z}}+\phi_{\mathbf{z}}]} \tag{4.19}$$

where the derivative over ϕ cannot be applied over $[\psi_{\mathbf{z}}+\phi_{\mathbf{z}}]$ otherwise a disconnected contribution is found. (4.19) can be integrated by a multiscale analysis as (1.3); after the integration of the scales $N,N-1,..,h$ we get $\int P_{k,h}(d\psi)e^{-\bar{\mathcal{V}}^{(h)}(\psi+\phi)}$ with

$$\bar{\mathcal{V}}^{(h)}(\psi)=\sum_{l,\underline{\omega},\underline{\varepsilon}}\int d\mathbf{x}_1...d\mathbf{x}_{2l}W^{(h)}_{2l,m,\bar m,\underline{\omega},\underline{\varepsilon}}\prod_{i=1}^{2l}\psi^{\varepsilon_i}_{\mathbf{x}_i,\omega_i}\prod_{i=1}^{m}J_{\omega_i}(\mathbf{x}_i)\prod_{i=1}^{\bar m}\bar J_{\omega_i}(\mathbf{x}_i) \tag{4.20}$$

and

$$\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}})>_T=\sum_{h=k}^{N}W^{(h)}_{2,1,1} \tag{4.21}$$

We integrate $\mathcal{W}(\phi,J,\bar J)$ by a multiscale procedure identical to the one for $\mathcal{W}(\phi,J)$ (in particular $\mathcal{L}W^{(h)}_{2l,m,\bar m}$ if $\bar m\neq 0$, so that no new running coupling functions are introduced) and we still get the bound (4.13) in which d_v is given by $\frac{|P_v|}{2}-2+n^J_v+n^{\bar J}_v+n_{2,v}$. There is an apparent problem due to the fact that $d_v=0$ for the vertices with one external $\bar J$ line and with two external fermionic lines. Let us consider the terms $\int d\mathbf{k}d\mathbf{p}W^{(h_v)}_{2,0,1}(\mathbf{k},\mathbf{p})\bar J(\mathbf{p})\phi^+_{\mathbf{k}}\psi^-_{\mathbf{k+p}}$ associated with such vertices; of course $W^{(h_v)}_{2,0,1}=g^{[h_v,N]}(\mathbf{k}+\mathbf{p})G_2(\mathbf{k}+\mathbf{p})$; the momentum $\mathbf{k}+\mathbf{p}$ of the external ψ fields has scale $\gamma^{h_{v'}}$, and $g^{h_{v'}}(\mathbf{k}+\mathbf{p})g^{h_v}(\mathbf{k}+\mathbf{p})$ is nonvanishing only if $|h_{v'}-h_v|\le 2$; hence, as $\gamma^{h_v-h_{v'}}\le\gamma^2$, we can replace d_v with $d_v+\varepsilon_v$,with $\varepsilon_v=1$ when $|P_v|=2,n^{\bar J}_v=1$ so that we get $||W^{(h)}_{2l,m,\bar m}||\le C\lambda\gamma^{-h(l+m+\bar m-2)}$ implying $||W^{(h)}_{2,1,1}||\le C\lambda\gamma^{-h}$.

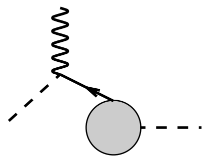

Fig. 10

Fig 10: Graphical representation of marginal terms $\bar J\phi\psi$

In the same way

$$\frac{\partial^*}{\partial\psi^-_{\mathbf{y}}}<\psi^-_{\mathbf{x}};\tilde\psi(\tilde{\mathbf{y}});\tilde\psi(\mathbf{z})>_T=\frac{\partial^2}{\partial\phi(\mathbf{y})\phi(\mathbf{x})}\frac{\partial^2}{\partial J(\tilde{\mathbf{y}})J(\mathbf{z})\bar J(\mathbf{x})}\mathcal{H}(\phi,J,\bar J) \tag{4.22}$$

and $||W^{(h)}_{2,2,1}||\le C\lambda\gamma^{-2h}$ from which (2.36) follows. Finally (2.48) follows fromthe fact that $||W^{(h)}_{3,2,1}||\le C\lambda\gamma^{-2h}$.

4.3 *Proof of (2.43)*

We can write

$$\int d\mathbf{z}g^{[k,N]}(\mathbf{x}-\mathbf{z}')g^{[k,N]}(\mathbf{x}-\mathbf{z})\frac{\partial^*}{\partial\psi_{\mathbf{z}}}<\psi^-_{\mathbf{z}'};\tilde\psi(\mathbf{z}'');\tilde\psi(\mathbf{y})>=$$

$$=\frac{\partial}{\partial\bar J(\mathbf{z}')\bar J(\mathbf{x})}\frac{\partial}{\partial J(\mathbf{z}'')\partial J(\mathbf{y})}\log\int P^{[k,N]}_{\sigma}(d\psi)P^{[k,N]}_{-\sigma}(d\psi)e^{-\mathcal{V}(\psi_\sigma)}e^{\int d\mathbf{z}[\sum_\varepsilon\bar J(\mathbf{z})\psi^\varepsilon_{\sigma,\mathbf{z}}\psi^{-\varepsilon}_{\sigma,\mathbf{z}}+\bar J(\mathbf{z})\psi^\varepsilon_{\sigma,\mathbf{z}}\psi^{-\varepsilon}_{-\sigma,\mathbf{z}}]} \tag{4.23}$$

where $\mathcal{V}(\psi)=-\lambda\frac{1}{2}\sum_{\omega=\pm}\int d\mathbf{x}\;\hat\psi^{(\le N)+}_{\mathbf{x},\omega}\hat\psi^{(\le N)-}_{\mathbf{x},\omega}\hat\psi^{(\le N)+}_{\mathbf{x},-\omega}\hat\psi^{(\le N)-}_{\mathbf{x},-\omega}$ and $\psi_\sigma,\psi_{-\sigma}$ are indipendent fields. In order to check (4.23) we note that the r.h.s. of (4.23) gives

$$<\psi^+_{\sigma,\mathbf{z}'}\psi^-_{-\sigma,\mathbf{z}'};\psi^-_{\sigma,\mathbf{x}}\psi^+_{-\sigma,\mathbf{x}};\tilde\psi(\mathbf{z}'');\tilde\psi(\mathbf{y})>=g^{[k,N]}(\mathbf{x}-\mathbf{z}')<\psi^+_{\mathbf{z}'};\psi^-_{\mathbf{x}};\tilde\psi(\mathbf{z}'');\tilde\psi(\mathbf{y})> \tag{4.24}$$

and we use the identity $\psi^-_{\mathbf{x}}=\int d\mathbf{z} g^{[k,N]}(\mathbf{x}-\mathbf{z})\frac{\partial^*}{\partial\psi_{\mathbf{z}}}$, where we have used that $\frac{\partial}{\partial\psi_{\mathbf{z}}}$ cannot be applied over $\psi^+_{\mathbf{z}'}$ otherwise a disconnected contribution is found. The r.h.s.of (4.23) can be integrated by a multiscale analysys as above, and after the integration of the scales $N, N-1, ..k$ and the effective potential have the form (2.9) and kernels $W^k_{l_1,l_2,m_1,m_2}$ multiplying l_1 fields ψ_σ, l_1 fields $\psi_{-\sigma}$, m_1 fields J and m_2 $\bar{J}$. There are new terms with vanishing dimension: there are no vertices with external lines $\bar{J}\psi^+_\sigma\psi^-_\sigma$, as the contraction of the fields $\psi_{-\sigma}$ means that there are at least two external lines $\bar{J}$; the vertices with external lines $\bar{J}\psi^+_\sigma\psi^-_{-\sigma}$ have surely a propagator with the same momentum as the external line, then $\gamma^{h_v-h'_v}\le\gamma^2$ and the sum over trees can be done without introducing anynew coupling. Then the norm of the l.h.s. of (4.23) is bounded by $\sum_{h=k}^N ||W^k_{0,0,2,2}||\le C\lambda\gamma^{-2h}$ so that we get (2.43)

5. Appendix 2

5.1 *Proof of Lemma 3* As the case $\hat{m}=0$ is identical to the previous one, we consider only the case $\hat{m}=1$. The trees are essentially identical with the ones in Appendix 1, with the only difference that there is or an end-point associated to $\nu_{j,\pm}$ at scale j, or an endpoint associated to T_0. In the first case $d_v\ge 1$ for any v by construction. In the second case there is surely a χ-vertex at scale $N-1$, by (3.11); moreover the only vertex v with $d_v=0$ has one external line $\hat{J}$ and two external ψ-lines; it has necessarily scale $N-1$ and the form

$$F^{(N-1)}_{1,\omega}(\mathbf{k}^+,\mathbf{k}^-)=[\frac{[C_N(\mathbf{k}^-)-1]D_\omega(\mathbf{k}^-)\hat{g}^{(N)}_+(\mathbf{k}^+)-u_N(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+-\mathbf{k}^-)}G^{(2)}(\mathbf{k}^+) \tag{5.1}$$

for a suitable function $G^{(2)}(\mathbf{k})$; by the support properties of the functions $\hat{g}^{(N)}_+(\mathbf{k}^+)$, $u_N(\mathbf{k}^+)$, there is a nonvanishing contribution only if the external line with momentum $\mathbf{k}^+$ is contracted at scale $N-2$, so that $\gamma^{h_v-h_{v'}}\le\gamma^2$. Hence we get

$$||\mathcal{W}_\tau||\le C^n\lambda^n\gamma^{-h(2-\frac{n_\phi}{2}-n_{\hat{J}}-n_J)}\prod_v\gamma^{-\frac{1}{2}(h_v-h_{v'})(-2+\frac{|P_v|}{2}+n^{\hat{J}}_v+n^J_v+\varepsilon_v)} \tag{5.2}$$

where $\varepsilon_v=1$ if $|P_v|=2, n^{\hat{J}}_v=1$. The sum over τ can be done as in Appendix 1 and (3.19) is found. Moreover (3.20) follows noting that the trees contributing to $R^{2,1,N}_{\omega,\omega',i}(\mathbf{k},\mathbf{p})$ have an endpoint associated to T_0 or an $\nu_{j,\pm}$ at scale j. There is then a gain, with respect to the dimensional bound, of a factor $\gamma^{\frac{1}{2}(\bar{h}-N)}$, if $\bar{h}$ is the scale of $\mathbf{k}$; in fact the trees contributing to $R^{2,1,N}_{\omega,\omega',i}(\mathbf{k},\mathbf{p})$ have surely a χ-vertex at scale $\bar{h}$, and an end-point associated to $\nu_{j,\pm}$ at scale j, or an endpoint associated to T_0; by the short memory property and (3.18) it follows (3.20).

5.2 *Proof of (3.28)* We can write

$$\begin{aligned}&H^{(k)}_{0,0,2,2}(\mathbf{z}',\mathbf{x},\mathbf{z}'',\mathbf{y}_1,\mathbf{y}_2)=[S^{N,i}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}')+S^{i,N}(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}')]\frac{\partial}{\partial\psi^-_{\mathbf{y}_1}}\frac{\partial}{\partial\psi^+_{\mathbf{y}_2}}\frac{\partial^*}{\partial\psi_{\mathbf{z}}}<\psi^-_{\mathbf{z}'};\tilde{\psi}(\mathbf{z}'')>=\\&\frac{\partial}{\partial\hat{J}(\mathbf{z}')\bar{J}(\mathbf{x})}\frac{\partial}{\partial J(\mathbf{z}'')}\frac{\partial^2}{\partial\phi(\mathbf{y}_1)\partial\phi(\mathbf{y}_2)}\log\int P^{[k,N]}_\sigma(d\psi)P^{[k,N]}_{-\sigma}(d\psi)e^{-\mathcal{V}(\psi_\sigma)}e^{\int\sum_\varepsilon\hat{J}(\mathbf{z})\delta\rho(\mathbf{z})+\bar{J}(\mathbf{z})\psi^\varepsilon_{\sigma,\mathbf{z}}\psi^{-\varepsilon}_{-\sigma,\mathbf{z}}}\end{aligned} \tag{5.3}$$

where $\mathcal{V}(\psi)=-\lambda\frac{1}{2}\sum_{\omega=\pm}\int d\mathbf{x}\ \hat{\psi}^{(\le N)+}_{\mathbf{x},\omega}\hat{\psi}^{(\le N)-}_{\mathbf{x},\omega}\hat{\psi}^{(\le N)+}_{\mathbf{x},-\omega}\hat{\psi}^{(\le N)-}_{\mathbf{x},-\omega}$ and

$$\delta\rho(\mathbf{z})=\int d\mathbf{p}e^{i\mathbf{p}\mathbf{z}}\int d\mathbf{k}C_N(\mathbf{k},\mathbf{k}+\mathbf{p})\psi^+_{\mathbf{k},\sigma}\psi^-_{\mathbf{k}+\mathbf{p},-\sigma}$$

The r.h.s.of (5.3) can be integrated by a multiscale analysys as above, and after the integration of the scales $N, N-1, ..k$ and the effective potential have the form (2.9) and kernels $W^k_{l_1,l_2,m_1,m_2}$ multiplying l_1 fields ψ_σ, l_1 fields $\psi_{-\sigma}$, m_1 fields J and m_2 $\bar{J}$. There are new terms with vanishing dimension: there are no vertices with external lines $\bar{J}\psi^+_\sigma\psi^-_\sigma$, as the contraction of the fields $\psi_{-\sigma}$ means that there are at least two external lines $\bar{J}$;

the vertices with external lines $\bar{J}\psi^+_\sigma\psi^-_{-\sigma}$ have surely a propagator with the same momentum as the external line, then $\gamma^{h_v-h_{v'}}\le\gamma^2$ and the sum over trees can be done without introducing any new coupling.

We can write (analogously to (2.18))

$$H^{(k)}_{2,0,1,2}=\sum_{i=k}^N\sum_{\tau\in\mathcal{T}^*_i}W^{(k)}_{\tau;2,0,1,2} \tag{5.4}$$

where $\mathcal{T}^*_i$ is the set of trees with root at scale i and such that v_0 is a χ-vertex. Note that, by construction, there is surely a χ vertex at scale N, hence the dimensional bound is improved by a factor $\gamma^{\frac{1}{2}(i-N)}$,see §5.1, so that

$$||H^{(k)}_{2,0,1,2}||=\sum_{i=k}^N C\lambda^2\gamma^{-2i}\gamma^{\frac{1}{2}(i-N)}\le\tilde{C}\lambda^2\gamma^{-k}\gamma^{-\frac{1}{2}N} \tag{5.5}$$

5.3 *The limit of local interaction.* In order to prove Theorem 2, we consider $v(\mathbf{p})=e^{\frac{-\mathbf{p}^2}{K^2}}$ with suitable Z_K,m_K and $\gamma^{-h_M}=K$. We fix K and then we proceed as above by taking the limit $N\to\infty$, and from the previous analysis it follows that the WI verifyes (1.15). We consider then the limit $K\to\infty$; the contribution of a tree with a vertex with scale $h_v\ge h_M$ to a Schwinger function with fixed coordinate is vanishing as $K\to\infty$, by the short memory property, and by (2.53) the bare parameters have to be chosen as in (5.6) .

6. Appendix 3:Perturbative Computations

We can check the WI (1.11), (1.15), (1.19) by a naive perturbative computation at lowest orders. Note that $j^0_{\mathbf{z}}=\sum_{\omega'=\pm}\psi^+_{\omega',\mathbf{x}}\psi^-_{\omega',\mathbf{x}}$ and $j^1_{\mathbf{z}}=i\sum_{\omega'=\pm}\omega'\psi^+_{\omega',\mathbf{x}}\psi^-_{\omega',\mathbf{x}}$ and therefore $\mathbf{p}_\mu\hat{j}^\mu_{\mathbf{p}}=i\sum_{\omega'=\pm}D_{\omega'}(\mathbf{p})\hat\rho_{\omega',\mathbf{p}}$, $\mathbf{p}_\mu\hat{j}^{5,\mu}_{\mathbf{p}}=i\sum_{\omega'=\pm}\omega'D_{\omega'}(\mathbf{p})\hat\rho_{\omega',\mathbf{p}}$ so that

$$\begin{aligned}D_{\omega'}(\mathbf{p})\langle\hat\rho_{\omega',\mathbf{p}}\hat\psi^-_{\omega,\mathbf{k}}\hat\psi^+_{\omega,\mathbf{k}+\mathbf{p}}\rangle=&\delta_{\omega,\omega'}\left[\langle\hat\psi^-_{\omega,\mathbf{k}}\hat\psi^+_{\omega,\mathbf{k}}\rangle-\langle\hat\psi^-_{\omega,\mathbf{k}+\mathbf{p}}\hat\psi^+_{\omega,\mathbf{k}+\mathbf{p}}\rangle\right]\\&+\nu_+(\mathbf{p})D_{\omega'}(\mathbf{p})\langle\hat\rho_{\omega',\mathbf{p}}\hat\psi^-_{\omega,\mathbf{k}}\hat\psi^+_{\omega,\mathbf{k}+\mathbf{p}}\rangle+\nu_-(\mathbf{p})D_{-\omega'}(\mathbf{p})\langle\hat\rho_{-\omega',\mathbf{p}}\hat\psi^-_{\omega,\mathbf{k}}\hat\psi^+_{\omega,\mathbf{k}+\mathbf{p}}\rangle\end{aligned} \tag{6.1}$$

can be also written as

$$-i\mathbf{p}_\mu\langle j^\mu_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}}\rangle=[\langle\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}\rangle-\langle\psi_{\mathbf{k}+\mathbf{p}}\bar\psi_{\mathbf{k}+\mathbf{p}}\rangle]+(\nu_+(\mathbf{p})+\nu_-(\mathbf{p}))(-i\mathbf{p}_\mu\langle j^\mu_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}}\rangle) \tag{6.2}$$

$$-i\mathbf{p}_\mu\langle j^{5,\mu}_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}}\rangle=\gamma^5[\langle\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}\rangle-\langle\psi_{\mathbf{k}+\mathbf{p}}\bar\psi_{\mathbf{k}+\mathbf{p}}\rangle]+(\nu_+(\mathbf{p})-\nu_-(\mathbf{p}))(-i\mathbf{p}_\mu\langle j^{5,\mu}_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}}\rangle) \tag{6.3}$$

We can write $\langle\hat\rho_{\omega',\mathbf{p}}\hat\psi^-_{\omega,\mathbf{k}}\hat\psi^+_{\omega,\mathbf{k}+\mathbf{p}}\rangle\equiv\hat{G}^{2,1}_{\omega';\omega}(\mathbf{p};\mathbf{k})$ and $\langle\psi_{\omega,\mathbf{k}}\psi^+_{\omega,\mathbf{k}}\rangle\equiv\hat{G}^2_\omega(\mathbf{k})$ as a (non convergent) power series in λ

$$\hat{G}^{2,1}_{\omega';\omega}(\mathbf{p};\mathbf{k})=\sum_{n=0}^\infty\hat{G}^{2,1(n)}_{\omega';\omega}(\mathbf{p};\mathbf{k})\lambda^n\qquad\hat{G}^2_\omega(\mathbf{k})=\sum_{n=0}^\infty\hat{G}^{2(n)}_\omega(\mathbf{k})\lambda^n \tag{6.4}$$

The perturbative contributions to $\hat{G}^{2,1}_{\omega';\omega}(\mathbf{p};\mathbf{k})$, $\hat{G}^2_\omega(\mathbf{k})$ can be obtained by a standard Feynman graph expansion with propagator $g^{(\le N)}_\omega(\mathbf{k})$. A crucial role will be played by the following identity

$$g^{(\le N)}_\omega(\mathbf{k})g^{(\le N)}_\omega(\mathbf{k}+\mathbf{p})=\frac{g^{(\le N)}_\omega(\mathbf{k})-g^{(\le N)}_\omega(\mathbf{k}+\mathbf{p})}{D_\omega(\mathbf{p})}-g^{(\le N)}_\omega(\mathbf{k})g^{(\le N)}_\omega(\mathbf{k}+\mathbf{p})\frac{C_N(\mathbf{k},\mathbf{k}+\mathbf{p})}{D_\omega(\mathbf{p})} \tag{6.5}$$

where $C_N(\mathbf{k},\mathbf{k}+\mathbf{p})$ is given by (3.4). Of course if $|\mathbf{k}|\le\gamma^{N-2}$ the second addend in (6.5) is vanishing. We get

$$\hat{G}^{2(0)}_\omega(\mathbf{k})=g^{(\le N)}_\omega(\mathbf{k})\qquad\hat{G}^{2,1(0)}_{\omega;\omega}(\mathbf{p};\mathbf{k})=g^{(\le N)}_\omega(\mathbf{k})g^{[h,N]}_\omega(\mathbf{k}+\mathbf{p}) \tag{6.6}$$

If $\mathbf{k}, \mathbf{p}$ are "far" from the cutoffs, that is $|\mathbf{k}|, |\mathbf{k}+\mathbf{p}| \le \gamma^{N-2}$ we get from (6.5)

$$\hat{G}^{2,1(0)}_{\omega';\omega}(\mathbf{p};\mathbf{k}) = g^{(\le N)}_{\omega}(\mathbf{k}) g^{(\le N)}_{\omega}(\mathbf{k}+\mathbf{p}) = \frac{g^{(\le N)}_{\omega}(\mathbf{k}) - g^{(\le N)}_{\omega}(\mathbf{k}+\mathbf{p})}{D_\omega(\mathbf{p})} = \frac{G^{2(0)}_{\omega}(\mathbf{k}) - G^{2(0)}_{\omega}(\mathbf{k}+\mathbf{p})}{D_\omega(\mathbf{p})}$$

and we see that (6.1) holds with $\nu^{(0)}_{+} = \nu^{(0)}_{-} = 0$. At first order in λ

$$\hat{G}^{2(1)}_{\omega}(\mathbf{k}) = g^{(\le N)}_{\omega}(\mathbf{k}) g^{(\le N)}_{\omega}(\mathbf{k}) \int d\mathbf{k}' g^{(\le N)}_{\omega}(\mathbf{k}') = 0$$

by parity; that is the tadpole contribution is vanishing. Moreover $\hat{G}^{2,1(0)}_{\omega;\omega}(\mathbf{p};\mathbf{k}) = 0$ as there is no graph contributing to it.

At the second order in λ we find, if $B^{(\le N)}_{-\omega}(\mathbf{k}_1, \mathbf{k}_2) = g^{(\le N)}_{-\omega}(\mathbf{k}_1) g^{(\le N)}_{-\omega}(\mathbf{k}_2) v^2(\mathbf{k}_1 - \mathbf{k}_2)$

$$\hat{G}^{2(2)}_{\omega}(\mathbf{k}) = g^{(\le N)}_{\omega}(\mathbf{k}) [\int d\mathbf{k}_1 \int d\mathbf{k}_2 B^{(\le N)}_{-\omega}(\mathbf{k}_1, \mathbf{k}_2) g^{(\le N)}_{\omega}(\mathbf{k} - \mathbf{k}_2 + \mathbf{k}_1)] g^{(\le N)}_{\omega}(\mathbf{k}) \quad (6.7)$$

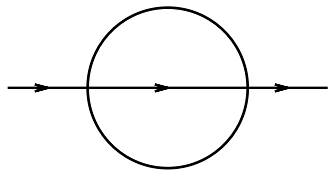

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Fig 11: Feynmann graph of $\hat{G}^{2(2)}_{\omega}(\mathbf{k})$

On the other hand $\hat{G}^{2,1(2)}_{\omega,\omega}$ is given by three graphs

$$\hat{G}^{2,1(2)}_{\omega,\omega}(\mathbf{k},\mathbf{p}) = \hat{G}^{2,1(2)}_{a,\omega,\omega}(\mathbf{k},\mathbf{p}) + \hat{G}^{2,1(2)}_{b,\omega,\omega}(\mathbf{k},\mathbf{p}) + \hat{G}^{2,1(2)}_{c,\omega,\omega}(\mathbf{k},\mathbf{p}) \quad (6.8)$$

where $\hat{G}^{2,1(2)}_{a,\omega}(\mathbf{k},\mathbf{p})$ is given by

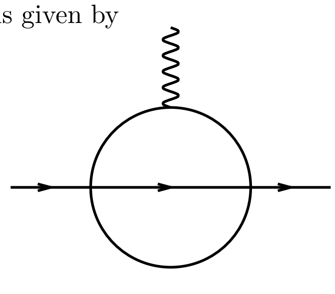

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Fig 12: Feynmann graph of $\hat{G}^{2,1(2)}_{a,\omega}$

while $\hat{G}^{2,1(2)}_{b,\omega}(\mathbf{k},\mathbf{p})$ is given by

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Fig 13: Feynmann graph of $\hat{G}^{2,1(2)}_{b,\omega}$

and $\hat{G}^{2,1(2)}_{c,\omega}(\mathbf{k},\mathbf{p})$ is given by

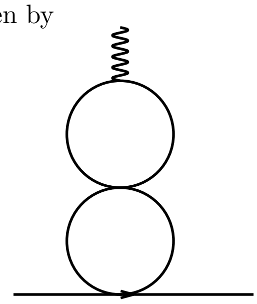

Fig 14: Feynmann graph of $\hat{G}^{2,1(2)}_{c,\omega}$

We get, using (6.5) and if $|\mathbf{k}|, |\mathbf{k}+\mathbf{p}| \leq 2^{N-2}$

$$D_\omega(\mathbf{p})\hat{G}^{2,1(2)}_{b,\omega,\omega}(\mathbf{k},\mathbf{p}) - \hat{G}^{2(2)}_\omega(\mathbf{k}) + \hat{G}^{2(2)}_\omega(\mathbf{k}+\mathbf{p}) =$$

$$g^{(\leq N)}_\omega(\mathbf{k})\{\int d\mathbf{k}_1 d\mathbf{k}_2 B^{(\leq N)}_{-\omega}(\mathbf{k}_1,\mathbf{k}_2)[g^{(\leq N)}_\omega(\mathbf{k}+\mathbf{p}-\mathbf{k}_2+\mathbf{k}_1) - g^{(\leq N)}_\omega(\mathbf{k}-\mathbf{k}_2+\mathbf{k}_1)]\}g^{(\leq N)}_\omega(\mathbf{k}+\mathbf{p}) \qquad (6.9)$$

and using again (6.5) the r.h.s. of (6.9) can be written as

$$-D_\omega(\mathbf{p})\hat{G}^{2,1(2)}_{a,\omega,\omega}(\mathbf{k},\mathbf{p}) + g^{(\leq N)}_\omega(\mathbf{k})g^{(\leq N)}_\omega(\mathbf{k}+\mathbf{p})[\int d\mathbf{k}_1 d\mathbf{k}_2 B^{(\leq N)}_{-\omega} g^{(\leq N)}_\omega(\mathbf{k}+\mathbf{p}-\mathbf{k}_2+\mathbf{k}_1)g^{(\leq N)}_\omega(\mathbf{k}-\mathbf{k}_2+\mathbf{k}_1)C_{N,\omega}]$$

Finally, by using again (6.5) and the fact that, by parity $\int d\mathbf{k} g^{(\leq N)}_\omega(\mathbf{k}) = 0$, we get

$$\hat{G}^{2,1(2)}_{c,\omega}(\mathbf{k},\mathbf{p}) = [\int d\mathbf{k}_1 g^{(\leq N)}_\omega(\mathbf{k}_1)g^{(\leq N)}_\omega(\mathbf{k}_1+\mathbf{p})C_{h,N}(\mathbf{k}_1,\mathbf{k}_1+\mathbf{p})][\int d\mathbf{k}_2 g^{(\leq N)}_{-\omega}(\mathbf{k}_2)g^{(\leq N)}_{-\omega}(\mathbf{k}_2+\mathbf{p})]g^{(\leq N)}_\omega(\mathbf{k})g^{(\leq N)}_\omega(\mathbf{k}+\mathbf{p})]$$

and,using that

$$\hat{G}^{2,1(1)}_{\omega,-\omega}(\mathbf{k},\mathbf{p}) = \int d\mathbf{k}_2 g^{(\leq N)}_{-\omega}(\mathbf{k}_2)g^{(\leq N)}_{-\omega}(\mathbf{k}_2+\mathbf{p})$$

we find at the end,putting togheter all terms

$$D_\omega(\mathbf{p})\hat{G}^{2,1(2)}_{\omega,\omega}(\mathbf{k},\mathbf{p}) = \hat{G}^{2(2)}_\omega(\mathbf{k}) - \hat{G}^{2(2)}_\omega(\mathbf{k}+\mathbf{p}) + \tilde{\nu}^{(1)}_-(\mathbf{k},\mathbf{p})D_{-\omega}(\mathbf{p})\hat{G}^{2,1(1)}_{-\omega,\omega}(\mathbf{k},\mathbf{p}) + \tilde{\nu}^{(2)}_+(\mathbf{k},\mathbf{p})D_\omega(\mathbf{p})\hat{G}^{2,1(0)}_{\omega,\omega}(\mathbf{k},\mathbf{p}) \qquad (6.10)$$

where

$$\tilde{\nu}^{(1)}_-(\mathbf{k},\mathbf{p}) = \int d\mathbf{k}_1 g^{(\leq N)}_\omega(\mathbf{k}_1)g^{(\leq N)}_\omega(\mathbf{k}_1+\mathbf{p})\frac{C_{\omega,N}(\mathbf{k}_1,\mathbf{k}_1+\mathbf{p})}{D_{-\omega}(\mathbf{p})} \qquad (6.11)$$

$$\tilde{\nu}^{(2)}_+(\mathbf{k},\mathbf{p}) = \int d\mathbf{k}_1 \int d\mathbf{k}_2 v^2(\mathbf{k}_1-\mathbf{k}_2)g^{(\leq N)}_{-\omega}(\mathbf{k}_1)g^{(\leq N)}_{-\omega}(\mathbf{k}_2)g^{(\leq N)}_\omega(\mathbf{k}+\mathbf{p}-\mathbf{k}_2+\mathbf{k}_1)g^{(\leq N)}_\omega(\mathbf{k}-\mathbf{k}_2+\mathbf{k}_1)\frac{C_{\omega,N}}{D_\omega(\mathbf{p})}$$

Note that $\tilde{\nu}^{(1)}_-(\mathbf{p})$ coincides with (3.6). On the other hand the value of $\tilde{\nu}^{(2)}_+(\mathbf{p})$ depends crucially on $v(\mathbf{p})$.
If $v(\mathbf{p})$ decays for large $\mathbf{p}$, using (3.10) we can write

$$\tilde{\nu}^{(2)}_+(\mathbf{k},\mathbf{p}) = \int d\mathbf{k}' \sum_{h=-\infty}^{N} v^2(\mathbf{k}-\mathbf{k}')[\Delta^{h,N}(\mathbf{k}',\mathbf{k}'+\mathbf{p}) + \Delta^{N,h}(\mathbf{k}',\mathbf{k}'+\mathbf{p})]A(\mathbf{k}-\mathbf{k}'')$$

where

$$A(\mathbf{p}) = \int d\mathbf{k}'' g^{(\leq N)}_{-\omega}(\mathbf{k}'') g^{(\leq N)}_{-\omega}(\mathbf{p}-\mathbf{k}'')$$

As $\sup_{\mathbf{p}} |A(\mathbf{p})| \leq C$ (it is the same as ν_- using (6.5) and noting that $\int d\mathbf{k} g$ is vanishing) we get, remembering that $\mathbf{k}, \mathbf{p}$ are fixed and $|v(\mathbf{p})| \leq C(\mathbf{p}^2+1)^{-1}$ and if $C_{\mathbf{k},\mathbf{p}}$ is a $\mathbf{k},\mathbf{p}$-dependent constant

$$|\tilde\nu^{(2)}_+(\mathbf{p})| \leq C_{\mathbf{k},\mathbf{p}} \sum_{h=-\infty}^{N} \lambda \frac{1}{\gamma^{2h}+1} \gamma^h \gamma^{-N} \leq \lambda \bar C_{\mathbf{k},\mathbf{p}} \gamma^{-N}$$

which is vanishing for $N \to \infty$.

On the other hand if $v(\mathbf{p}) = 1$ we get,for $\mathbf{k}, \mathbf{p} = 0$

$$\nu^{(2)}_+(0) = \int \frac{d\mathbf{k}}{(2\pi)^2} \left[\frac{u_0(|\mathbf{k}|)\chi_0(|\mathbf{k}|)}{|\mathbf{k}|^4} - \frac{\chi_0'(|\mathbf{k}|)}{2|\mathbf{k}|^3} \right] A(\mathbf{k}) D^2_{-\omega}(\mathbf{k})$$

which is *nonvanishing*; on the other hand the rest is vanishing as $N \to \infty$ as $\mathbf{p}, \mathbf{k}$ are fixed,by dimensional reasons.

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