Resonances and summation of divergent series

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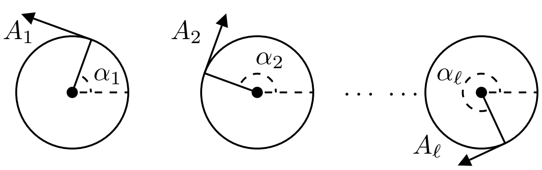
Abstract: *Resonances in classical mechanics lead to divergent formal series requiring multiscale analysis, i.e. renormalization group methods, to be summed.*

1. Resonances

The simplest situations arise perturbing a system of ℓ points $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)$ on a circle by a potential $\varepsilon f(\boldsymbol{\alpha})=\varepsilon\sum_{\boldsymbol{\mu}\in\mathbb{Z}^\ell} f_{\boldsymbol{\mu}}\, e^{i\boldsymbol{\mu}\cdot\boldsymbol{\alpha}}$, with $f_{\boldsymbol{\mu}}\equiv 0$ if $|\boldsymbol{\mu}|>N$, see Fig.1. The unperturbed ordered motions give way, in general, to chaotic motions which develop amidst a rather dense net of motions which are still ordered.



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Representation of phase space in terms of ℓ rotators: $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{T}^\ell$

Chaotic motions "take the place" near the trajectories of unperturbed resonant motions: an unperturbed resonant motion is a motion $\boldsymbol{\alpha}\to\boldsymbol{\alpha}+\boldsymbol{\omega}t$ with $\boldsymbol{\omega}_0\cdot\boldsymbol{\mu}=0$ for some non zero $\boldsymbol{\mu}\in\mathbb{Z}^\ell$. Without real loss of generality suppose that $\boldsymbol{\omega}_0=(\boldsymbol{\omega},\mathbf{0})$ with $\boldsymbol{\omega}\in\mathbb{Z}^r$, $r<\ell$ and suppose that $\boldsymbol{\omega}\in\mathbb{R}^r$ are independent in the sense

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|>\frac{1}{C|\boldsymbol{\nu}|^\tau},\qquad\forall\quad \mathbf{0}\neq\boldsymbol{\nu}\in\mathbb{Z}^r \tag{1.1}$$

for some $C,\tau>0$ ($\tau\ge r-1$, necessarily). Therefore this is a motion in which $\boldsymbol{\alpha}=(\boldsymbol{\gamma},\boldsymbol{\beta})$ moves as $\boldsymbol{\alpha}\to(\boldsymbol{\gamma}+\boldsymbol{\omega}\,t,\boldsymbol{\beta})$. There are many such motions because $\boldsymbol{\beta}\in\mathbb{T}^s$ ($s=\ell-r$) is arbitrary: they form a s–dimensional family of invariant tori of dimension r. The angles $\boldsymbol{\gamma}\in\mathbb{T}^r$ are called *fast angles* while the angles $\boldsymbol{\beta}\in\mathbb{T}^s$ are called *slow angles*. However the chaotic motions do not completely destroy the resonant motions and some of them remain and one should think that the chaotic motions really develop "around" the few resonant motions that remain.

This explains why the theory of the surviving resonant motions is interesting: a deeper reason is that in presence of (small) friction the chaotic motions dissipate more energy and motions concentrate around the resonant motions: therefore the resonant motions are the places where to look to find the eventual behavior of random initial data (e.g. in the planetary system one finds many resonances even though the resonances fill a very small part of phase space, actually of 0 volume).

A resonant motion *in presence of a perturbation* with potential εf, "of strength ε", is defined as a motion that can be represented in the form

$$\boldsymbol{\alpha}(t)\equiv(\boldsymbol{\gamma}(t),\boldsymbol{\beta}(t))=(\boldsymbol{\psi}+\boldsymbol{\omega}t+\mathbf{g}(\boldsymbol{\psi}+\boldsymbol{\omega}t),\boldsymbol{\beta}_0+\mathbf{k}(\boldsymbol{\psi}+\boldsymbol{\omega}\,t)) \tag{1.2}$$

where $\boldsymbol{\beta}_0$ is some point in $\mathbb{R}^s$ and $\mathbf{h}(\boldsymbol{\psi})\equiv(\mathbf{g}(\boldsymbol{\psi}),\mathbf{k}(\boldsymbol{\psi}))$ is a function of ε which should tend to 0 as $\varepsilon\to 0$ for some $\boldsymbol{\beta}_0$ (hence $\boldsymbol{\beta}_0$ identifies the surviving torus). Thus to look for resonant motions with frequencies $(\boldsymbol{\omega},\mathbf{0})$ means looking for functions $\mathbf{h}(\boldsymbol{\psi})=(\mathbf{g}(\boldsymbol{\psi}),\mathbf{k}(\boldsymbol{\psi}))$ periodic and smooth on $\mathbb{T}^r$ and for $\boldsymbol{\beta}_0\in\mathbb{R}^s$ such that $t\to\boldsymbol{\alpha}(t)\equiv(\boldsymbol{\psi}+\boldsymbol{\omega}\,t,\boldsymbol{\beta}_0)+\mathbf{h}(\boldsymbol{\psi}+\boldsymbol{\omega}\,t)$ solves the equations of motion $\ddot{\boldsymbol{\alpha}}=-\varepsilon\boldsymbol{\partial}_{\boldsymbol{\alpha}}f(\boldsymbol{\alpha})$, or

$$(\boldsymbol{\omega}\cdot\boldsymbol{\partial}_{\boldsymbol{\psi}})^2\mathbf{h}(\boldsymbol{\psi})=-\varepsilon\,\boldsymbol{\partial}_{\boldsymbol{\alpha}}f(\boldsymbol{\psi}+\mathbf{g}(\boldsymbol{\psi}),\boldsymbol{\beta}_0+\mathbf{k}(\boldsymbol{\psi})) \tag{1.3}$$

It should be clear that the value of $\boldsymbol{\beta}_0$ has to be very special: namely such that the s points standing in $\boldsymbol{\beta}_0$ feel a $\mathbf{0}$ average force at least in a first approximation (i.e. to lowest order in ε). This means that $\boldsymbol{\beta}_0$ has to be a stationarity point for the average of the force $\boldsymbol{\partial}_{\boldsymbol{\alpha}}f(\boldsymbol{\gamma},\boldsymbol{\beta})$ over the fast angles $\boldsymbol{\gamma}$: i.e. $\boldsymbol{\beta}_0$ has to satisfy

$$\boldsymbol{\partial}_{\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0)=\mathbf{0},\quad\text{if}\quad \overline{f}(\boldsymbol{\beta})\stackrel{def}{=}\int f(\boldsymbol{\gamma},\boldsymbol{\beta})\,\frac{d^r\boldsymbol{\gamma}}{(2\pi)^r} \tag{1.4}$$

The function $f(\boldsymbol{\gamma},\boldsymbol{\beta})$ will be written

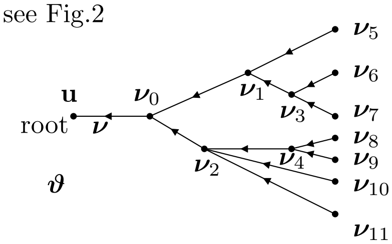
$$f(\boldsymbol{\gamma},\boldsymbol{\beta})=\sum_{|\boldsymbol{\nu}|<N} f_{\boldsymbol{\nu}}(\boldsymbol{\beta})\,e^{i\boldsymbol{\gamma}\cdot\boldsymbol{\nu}} \tag{1.5}$$

so that $\overline{f}(\boldsymbol{\beta})\equiv f_{\mathbf{0}}(\boldsymbol{\beta})$. The potential is supposed to be a trigonometric polynomial in $\boldsymbol{\gamma}$ just for simplicity of exposition: the main result holds, however, even under the weaker assumption that the potential is analytic.

The problem has a formal solution that described in the next section: the results and properties are discussed in Sect.3.

2. Diagrams for the resonant series

An algorithm to solve the (1.3) is easy to devise and is described below. Consider a (rooted) tree graph $\boldsymbol{\vartheta}$ formed by k (pairwise distinct) lines oriented towards the root, see Fig.2



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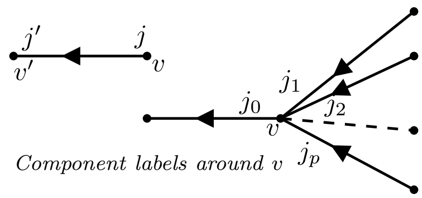
The extremes of each line will be called *nodes* but it is convenient that the root will not be considered a node. The notation $v<w$ will denote that a node v of a tree precedes the node w of the (same) tree in the partial

order induced on the tree by the orientation of its lines. A line wil be identified with its two extremes $v'v$, with $v' < v$, and if v is the last node before the root (which by our convention would not be a node) we set nevertheless $v' = r$. Various labels will be attached to the tree, see Fig.3,

(1) To each node $v < r$ of $\boldsymbol{\vartheta}$ attach a *harmonic* $\boldsymbol{\nu}_v \in \mathbb{Z}^r$.
(2) To line $\lambda \equiv v'v$ attach a *current* $\boldsymbol{\nu}(\lambda) = \sum_{w \le v} \boldsymbol{\nu}_w$.
(3) At the two extremes of each line $\lambda = v'v$ attach two *component labels* j', j,



Component labels around v 3

In this way it is posible to define the *coupling constant* associated with the node $v < r$: if $v \to J_v = (j_0, \ldots, j_p)$ are the labels at the extremes of the lines entering or exiting $v < r$ (see Fig.3) the coupling is defined as $\partial_{J_v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0)$ where $\partial_j f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0)$ is the derivative with respect to β_j if $j > r$ and it is, naturally, multiplication by $i\nu_j$ if $j \le r$.

The labels allow us to define the *value* of the labeled tree graph: it is defined as proportional to the product of the couplings of each node times the product of the *line propagators* $g_{j'_\ell, j_\ell}$, it the line $\ell = v'v$ has labels j'_ℓ, j_ℓ associated with v', v; precisely

$$\mathrm{Val}(\boldsymbol{\vartheta}) = \frac{\varepsilon^k}{k!}\Big(\prod_v \varepsilon \boldsymbol{\partial}_{J_v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0)\Big)\Big(\prod_{lines\,\ell} g_{j_\ell j'_\ell}\Big) \tag{2.1}$$

where, if $M_0 \stackrel{def}{=} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\varepsilon \boldsymbol{\partial}^2_{\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0))^{-1} \end{pmatrix}_{ij}$ denotes the $\ell \times \ell$ matrix with all elements 0 except the principal $s \times s$ matrix in the lowest corner (equal to $\partial^2_{\boldsymbol{\beta}\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0)^{-1}$), the propagators are defined by

$$\begin{aligned} g_{ij} &\stackrel{def}{=} \frac{\delta_{ij}}{(\boldsymbol{\omega} \cdot \boldsymbol{\nu}(\ell))^2}, \qquad &&\text{if } \boldsymbol{\nu}(\ell) \ne \mathbf{0} \qquad \text{or} \\ g_{ij} &\stackrel{def}{=} (M_0)_{ij}, &&\text{if } \boldsymbol{\nu}(\ell) = \mathbf{0} \end{aligned} \tag{2.2}$$

Trees can contain nodes with just one entering line and one exiting line (the example in Fig.2 does not contain any such node): calling *trivial* such nodes consider *only* tree graphs without trivial nodes v with $\nu_v = \mathbf{0}$ *and, at the same time,* with the entering line having $\mathbf{0}$ current. Then

$$h_{j,\boldsymbol{\nu}} = \sum_{\boldsymbol{\vartheta}}{}^{*} \mathrm{Val}(\boldsymbol{\vartheta}) \tag{2.3}$$

where the sum is over all labeled trees with a fixed current $\boldsymbol{\nu}$ on the root line and with a fixed label j, "root label", at the highest extreme of the root line; the $*$ reminds us that only tree graphs without trivial nodes with $\mathbf{0}$ incoming current are considered. Note that the root label j is fixed so tht the function (2.3) is a ℓ components vector.

This is a formal power series in ε (the $*$ in (2.3) implies that if k is the number of nodes $v < r$ then the number of lines with $\mathbf{0}$ current is not too large, so that the value is always a monomial in ε of order $\ge \frac{1}{2}k$, so the $\mathbf{0}$ current lines cannot lower too much the power of ε in $\mathrm{Val}(\boldsymbol{\vartheta})$).

The series is formal and convergence is doubtful because the best estimate that we are able to find for the coefficient $\mathbf{h}^{(k)}$, sum of all contributions of degree k in ε, is

$$|h^{(k)}_{\boldsymbol{\nu}}| \le b B^k \varepsilon^k k!^{2\tau} \tag{2.4}$$

which is established by taking into account (1.1).

3. Summation results

The main result can be summarized by the following theorem, taken from [GG05] where it was not noted that the proof implied that the domain in Fig.4 below instead of having a vertical tangent at the origin could be enlarged to a domain with a horizontal cusp at the origin (unpublished).

Theorem: *If* $\partial^2_{\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0) < 0$ *the tree series can be rearranged to yield a convergent series representation of* $\mathbf{h} = (\mathbf{g}(\boldsymbol{\psi}), \mathbf{k}(\boldsymbol{\psi}))$*, hence its existence, in* $\mathcal{D}(\mathcal{E})$ *of the form in Fig.4*

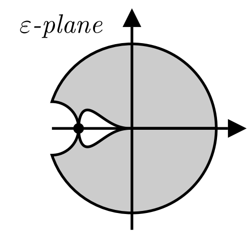
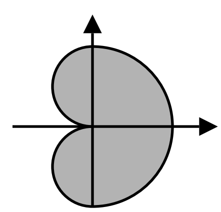


Fig.4: $\varepsilon \in \mathcal{E}$; $\mathcal{E}$ dense at 0. The figure illustrates the analyticity domain $\mathcal{D}(\varepsilon)$ associated with a *single* $\varepsilon \in \mathcal{E}$, represented by the dot, ("elliptic case"). The cusp at the origin is quadratic. 4

where $\mathcal{E} \subset (-\varepsilon_0, 0]$ *is a set with open dense complement in* $(-\varepsilon_0, 0]$ 0 *is a density point. The set* $\cap_{\varepsilon\in\mathcal{E}} \mathcal{D}(\varepsilon)$ *contains a domain of the form in Fig.5:*



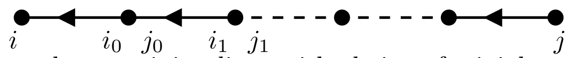
The domains of analyticity for $\varepsilon < 0, \varepsilon \in \mathcal{E}$, contain a set of this form with quadratic cusp at the origin. 5

Furthermore the functions $\mathbf{h}$ *defined in each* $\mathcal{D}(\varepsilon)$ *for* $\varepsilon \in \mathcal{E}$ *are real on the positive axis and define a solution to (1.3), i.e. also a quasi periodic motion. All functions defined in the above domains have the formal powers series as an asymptotic series at the origin and have derivatives everywhere bounded by* $O(k!^{3\tau})$*. At the origin the bound is better:* $O(k!^{2\tau})$

Remark: a heuristic linear stability analysis shows that the Lyapunov coefficients of the motions are either 0 ($2r$

of them) or real if $\varepsilon > 0$ and imaginary if $\varepsilon < 0$ ($2s$ of them) . This is the reason why the invariant tori, if really existent, are called *hyperbolic* if $\varepsilon > 0$ and *elliptic* if $\varepsilon < 0$. A direct proof that hyperbolic tori with analytic equations in a domain of the form in Fig.5 exist would be easier and it can be found in [GG02].

The proof is based on the remark that the extra $k!$ arise, *for instance*, because of the existence of graphs which contain chains of trivial nodes, like those in Fig.6,



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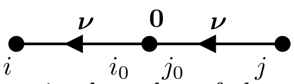
and graphs containing lines with chains of trivial nodes can be *eliminated* at the price of changing the propagators. In fact insertion of a trivial node v on a line as in Fig.7



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generates a change in the value of the graph that can be obtained, if M_0 is the matrix defined after (2.1), by replacing the propagator matrix as

$$\frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \to \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big(M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big)^k_{ij} \tag{3.1}$$

after summing over the labels i_0, j_0 relative to the node v and properly taking into account the combinatorial factors.

Hence summing on the arbitrary number of insertions one can simply imagine that in (2.3) the sum extends only over graphs containing *no trivial node* provided the propagator matrix of *each line* (with $\boldsymbol{\nu}\ne\mathbf{0}$) is replaced by

$$\begin{aligned}\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \Rightarrow &\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\sum_{k=0}^\infty\big(M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big)^k \equiv\\ &\equiv \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2-M_0}\end{aligned} \tag{3.2}$$

Of course this means that we are using the rule

$$\sum_{k=0}^\infty z^k = \frac{1}{1-z} \text{ for } z = M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \tag{3.3}$$

However it is easy to realize that *necessarily* there will be $\boldsymbol{\nu}$'s with $|z|>1$. In other words using (3.2) as propagator implies acepting a rule like $\sum_{k=0}^\infty 2^k = 1+2+4+\ldots = -1$, infinitely many times since there are infinitely many $\boldsymbol{\nu}$'s for which $|z|>1$.

On the other hand we see that if $M_0\le 0$, i.e. if $\boldsymbol{\beta}_0$ is a maximum point for $\overline{f}(\boldsymbol{\beta})$ as supposed here, the theory is likely to be easier for $\varepsilon > 0$. For $\varepsilon < 0$ it becomes even possible that the denominators in the *r.h.s.* of (3.2) vanish, so that to continue the discussion we will be forced to discard the values of ε too close to values such that $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| = \pm\sqrt{-\varepsilon\mu_j}$ with μ_j a positive eigenvalue of the matrix $\partial^2_{\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0)$.

The latter problem is dwarfed by the realization that there are a lot of other "chains" that one can form by inserting on any line a subtree with node harmonics adding up to $\mathbf{0}$: it is easy to check that also such more elaborated insertions produce values which can be bounded by a power of $k!$ too. The idea is then to eliminate all such chains by *resummations* similar to the ones just discussed. The reason one embarks into such a program is that the following theorem by Siegel holds:

Theorem: *Given a tree θ let $\mathcal{N}_n$ be the number of lines of "scale n": i.e. s.t.*

$$2^{-n} < C|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| \le 2^{-n+1} \tag{3.4}$$

$n=0,1,\ldots$. *If a graph θ contains no pair of lines $\ell' > \ell$ with only lower intermediate scales and with $\boldsymbol{\nu}(\ell') = \boldsymbol{\nu}(\ell)$, then*

$$\mathcal{N}_n \le 4N2^{-n/\tau}\,k \tag{3.5}$$

Remark: by "intermediate" between $\ell' > \ell$ and ℓ it is meant a line $\overline{\ell} < \ell'$ and *either* $> \ell$ *or* not comparable to ℓ.

It is immediate to see that the sum of the values of the graphs subject to the constraint that no lime ℓ' which follows a line ℓ has the same current is a convergent power series: because the product of the *couplings* is bounded by

$$\prod_v |\partial_{J^v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0)| \le \prod_v N^{|J^v|}F^k \le N^{2k}F^k; \tag{3.6}$$

the number of node harmonics is bounded by $(2N+1)^k$ (if N is the bound on the harmonics in the potential, see (1.5)); the number of trees is bounded by k^{k-1}; and the product of the small divisors is, by (3.5),

$$\begin{aligned}\prod_v \frac{1}{(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu})^2} \le &C^{2k}\prod_{n=0}^\infty 2^{2n\mathcal{N}_n} \le\\ \le &C^{2k}\big(\prod_{n=0}^\infty 2^{2n(4N2^{-n/\tau})}\big)^k.\end{aligned} \tag{3.7}$$

If $|\varepsilon| < \big(N^2C^2(2N+1)^\ell 3F2^{8N\sum_n n2^{-n/\tau}}\big)^{-1}$ convergence follows.

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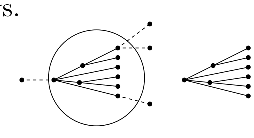
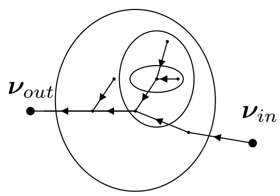


An example of a cluster: surrounded by a circle are the lines of a tree graph with scale $\le n$. The dashed lines have scale $> n$ and one at least among those encircled has scale n. The second figure shows the lines of the cluster as a set detached from the surrounding tree.

To perform the resummation aiming at the elimination of the problem just pointed out we proceed as in renormalization theory and introduce the notion of *clusters* to organize the lines of θ according to their "scale" (see (3.1)).

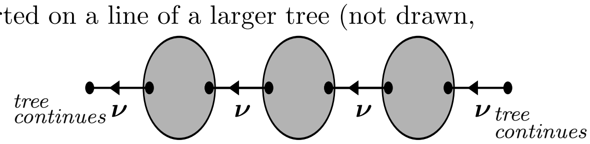
A cluster of lines of scale n *in a tree graph is a maximal connected set of lines of* θ *with scales* $p \le n$ *and with one line at least of scale* n.

Among the clusters particularly important are the *self–energy clusters*: they are clusters with only one entering line and only one exiting line, see Fig.9



The circles surround various clusters that can exist inside a sels-energy cluster 9

Such graphs are "dangerous" as they can be inserted into any line of a tree graph leaving unchanged the current of the line as in Fig.10 where self-energy clusters are inserted on a line of a larger tree (not drawn,



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To proceed we imagine to consider trees with no self-energy cluster of scale 0: then we construct all other trees by inserting an arbitrary number of self-energy clusters of scale 0 on each line. Inserting a single self-energy cluster Γ of scale 0 modifies the value in a way equivalent to replacing the propagator $\frac{1}{\boldsymbol{\omega}\cdot\boldsymbol{\nu}^2-M_0}$ by the new propagator $\frac{1}{\boldsymbol{\omega}\cdot\boldsymbol{\nu}^2-M_0} m_\Gamma \frac{1}{\boldsymbol{\omega}\cdot\boldsymbol{\nu}^2-M_0}$ where m_Γ ia a matrix depending on the cluster shape and obtained by multiplying coupling constants and propagators of the nodes and lines in the cluster. Since *no self energy clusters* can be found inside the self-energy clusters of scale 0 the sum $\sum_\Gamma m_\Gamma = M_1$ over all possible self energy clusters Γ that can be inserted is convergent (by Siegel's lemma) if the propagators of the lines in the cluster with current $\boldsymbol{\nu}$ can be bounded above proportionally to $\boldsymbol{\omega}\cdot\boldsymbol{\nu}^{-2}$: this certainly happens (as remarked above) if $M_0 \ge 0$. Suppose first that this is the case.

An insertion of k self-energy clusters can, then, be taken into account by changing the propagator of a line into $g^{[\ge 1]}$ defined as follows

$$
\begin{aligned}
g^{[\ge 1]}(x) =&\frac{1}{x^2-M_0}\sum_{k=0}^{\infty}(M_1\frac{1}{x^2-M_0})^k \;\; \to \\
&\to \frac{1}{x^2-M_0-M_1}
\end{aligned}
\tag{3.8}
$$

and assuming that $\mathbf{h}$ is defined as the sum of the values of all tree graphs *without* any self-energy cluster of scale 0.

At this point the idea is to iterate and eliminate, successively, the self energy cluster of scale $1, 2, 3, \ldots$ by recursively modifying the propagators.

The difficulty is that the matrix M_1 *might no longer have a structure allowing us to bound the denominator in the propagator from below by* $\boldsymbol{\omega}\cdot\boldsymbol{\nu}^2$ *even if* $M_0 \le 0$. Because the 0 elements in the matrix M_0 (defined after (2.1)) become in general $\ne 0$ when M_0 is changed into $M_0 + M_1$ and the matrix may become of indefinite sign even though $M_0 \le 0$.

The next key remark is that this *does not happen*: the reason is that the self energy cluster values m_Γ have special properties which imply that *cancellations* occurr when the sum $\sum_\Gamma$ is performed. The cancellations have the same origin of the well known cancellations that occurr in the proof of the convergence ot the KAM theorem.

After remarking that the propagators remain *on all scales* such that their denominators can be bounded below $\boldsymbol{\omega}\cdot\boldsymbol{\nu}^2$ if $M_0 < 0$ the proof is reduced to Siegel's theorem.

The case $M_0 \ge 0$ (or of undefined sign) is much harder. Already the first resummation changing the propagator from $\boldsymbol{\omega}\cdot\boldsymbol{\nu}^{-2}$ to $\frac{1}{Bo\cdot\boldsymbol{\nu}^2-M_0}$ forces to discard some intervals of ε (which form a dense set because the values of $\boldsymbol{\omega}\cdot\boldsymbol{\nu}^2$ are dense as $\boldsymbol{\nu}$ varies and one has to discard all ε such that $\sqrt{-\varepsilon\mu_j} = \pm\boldsymbol{\omega}\cdot\boldsymbol{\nu}$ as remarked earlier, as well as the values of ε too close to such values). As the scale decreases and the scale n with $2^{-n} \simeq \varepsilon$ is reached the quantity $\boldsymbol{\omega}\cdot\boldsymbol{\nu}^2 - M_0 - M_1 - \ldots - M_{n-1}$ can vanish for more values of ε leading to further values of ε excluded.

The number of excluded ε's is countable and in spite of the fact that it is dense it can have a small measure: however it is clear that the propagators denominators will not be bounded below by $\boldsymbol{\omega}\cdot\boldsymbol{\nu}^2$ but rather by how close $\boldsymbol{\omega}\cdot\boldsymbol{\nu}^2 - M_0 - M_1 - \ldots - M_{n-1}$ is to zero. Therefore Siegel's theorem does not apply directly and one has to devise an extension of it in which the scale of the propagators is measured by the smallest eigenvalue of the matrices $\boldsymbol{\omega}\cdot\boldsymbol{\nu}^2 - M_0 - M_1 - \ldots - M_{n-1}$, recursively constructed, see [GG05].

In the end one finds a convergent series for $\mathbf{h}$ which is a formal resummation of the original series (2.3): it is formal because in obtaining it the prescription $\sum_{k=0}^\infty z^k = (1-z)^{-1}$ for $z \ne 1$ (but $|z| > 1$) has ben used, *infinitely many times*. Therefore *it is not obvious* that the equation (2.3) is solved by the function $\mathbf{h}$ thus constructed. Therefore the last step is to check that inserting $\mathbf{h}$ into (2.3) it turns it into an identity.

The details of the complete analysis can be found in [GG05]. The cusp at the origin in Fig.4 is, however, an improvement implicit in the analysis in [GG05] later noted by the authors.

4. Outlook

The analysis is an instance in which a resummation of self-energy diagrams can be performed rigorously. There are not many such examples. The multiscale analysis is a version of a renormalization group argument: the "flow" of the propagators is analogous to the fow of the running couplings and the analogy could be made much closer by showing that the graphs can be considered as Feynman graphs of a suitable field theory on a torus (although without loops the theory is non trivial because in the graphs vertices can merge as many lines as wished, i.e. the theory is "non polynomial"), see [Ga95],[Ga01].

An important open problem is uniqueness: are the invariant tori constructed through resummations unique? this *seems to be an open problem even in the case of* $s = 0$*, maximal dimension, non resonant tori*: which is a special case of our analysis which generates the usual KAM tori (as a trivial case, see Ch.8,9 in [GBG04]). More precisely: given $\boldsymbol{\omega}$ is it true that in a small enough neighborhood of the unperturbed invariant torus there is no invariant torus with rotation vector $\boldsymbol{\omega}_0 = (\boldsymbol{\omega}, \mathbf{0})$ other than the one constructed in this analysis at least for ε small enough (and, of course, such that the construction works)?

Another question is: can one reconstruct the function $]Vh$ from its Taylor coefficients at the origin via a procedure like Borel summation?

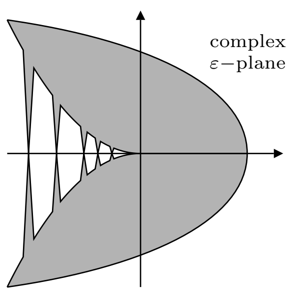

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Illustration of the question about existence of a single holomorphic function generatint at the same time the elliptic and the hyperbolic tori.

A third question is: is there a *single analytic function in the region* $\cup_{\varepsilon\in\mathcal{E}}\mathcal{D})\varepsilon)$, see the theorem at the beginning of Sec. 3 and Figs.4,5, which on the real axis becomes a function $\mathbf{h}$ solution of (1.3)? The region would have the form of Fig.11.

A final question is: what happens if $\det \partial^2_{\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0) = 0$? In the case $s = 1$ *and* if the first non vanishing derivative, at $\boldsymbol{\beta}_0$, is of order $k_0 + 1$ (with $k_0 > 1$) one can expect analyticity in a fractional power of ε (namely $|\varepsilon|^{\frac{1}{k_0}}$), for ε outside some set $\mathcal{E}$ with 0 a density point.

The above seem to me interesting and relevant questions.

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