CHAPTER X

Special ergodic theory problems in chaotic dynamics

§10.1 Perturbing Arnold's cat map

A general theorem on Anosov maps allows us to say that in a certain sense Anosov maps that are close enough in C^2 can be considered as derived one from the other by a "change of coordinates", which, however, is not really smooth. This is the theorem of structural stability of Anosov that can be formulated as follows.

*P*10.1.1 **(10.1.1) Proposition:** (Structural stability) *Let S, S' be two Anosov diffeomorphisms of a manifold Ω. If they are close enough together with their*
*N*10.1.1 *first two derivatives*[1] *then there exists a homeomorphism $H : \Omega \longleftrightarrow \Omega$ such that*

*e*10.1.1
$$S \circ H = H \circ S'. \tag{10.1.1}$$

The homeomorphism is Hölder continuous but, in general, not differentiable.

Remarks: (1) The lack of real smoothness, and even of differentiability, is however a considerable obstacle to arguments based on the naive interpretation of the theorem suggesting that S and S' "just" differ by the change of coordinates H.
(2) It is important to realize that if μ is an invariant measure for S then $\mu' = H\mu$ is an invariant measure for S'. Notwithstanding this nice property, in general, the image under H of the SRB distribution μ_{SRB} of S is not the SRB distribution μ'_{SRB} of S'. Indeed $H\mu_{SRB}$ is the weak limit under the

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[1] One also says "close enough in the C^2 topology"

action of S' of $H\lambda$ where λ is the volume measure on Ω. Due to the above remarks $H\lambda$ is, in general, singular with respect to λ.

We suggest a proof for the simple two-dimensional case in the problems at the end of this section. Here we discuss how to construct concretely the map H in a class of special cases. An explicit construction of H as well as of the stable and unstable manifolds of an Anosov system and a detailed study of the SRB distribution is of course a very difficult task. Nevertheless it can be performed with remarkable depth in some special cases.

Here we consider maps that are perturbations of Arnold's cat map and the following proposition (or better its proof) is the key result [BFG03].

P10.1.2 **(10.1.2) Proposition:** (Arnold's cat map perturbations) *Let $\underline{f}(\underline{\varphi})$ be a real trigonometric polynomial, $\underline{f}(\underline{\varphi}) = \sum_{\underline{\nu}\in\mathbb{Z}^2,|\underline{\nu}|\le N} e^{i\underline{\nu}\cdot\underline{\varphi}}\underline{f}_{\underline{\nu}}$, defined on the two-dimensional torus $\mathbb{T}^2$ and let*

e10.1.2
$$S_\varepsilon\underline{\varphi} = S_0\underline{\varphi} - \varepsilon\underline{f}(\underline{\varphi}), \qquad \textit{with} \qquad S_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{10.1.2}$$

For $\beta \in (0,1)$ there exist $C(\beta) < \infty$ and $\varepsilon_0(\beta) > 0$ such that for $|\varepsilon| < \varepsilon_0(\beta)$ the equation

e10.1.3
$$H \circ S_0 = S_\varepsilon \circ H \tag{10.1.3}$$

defines a unique homeomorphism $\underline{\varphi} \to H(\underline{\varphi})$ which is analytic in ε in the complex disk $|\varepsilon| < \varepsilon_0(\beta)$ and Hölder continuous with exponent at least as large as β and with Hölder continuity modulus bounded by $C(\beta)$.

Remarks: (1) Proposition (10.1.2) tells us that we can conjugate the map S_ε to the map S_0 via a function H that is analytic in ε. However, in general *it is not true that we can conjugate S_0 to S_ε via an analytic homeomorphism.* Indeed the equation

e10.1.4
$$\widetilde{H} \circ S_\varepsilon = S_0 \circ \widetilde{H} \tag{10.1.4}$$

cannot be studied with the method developed in the proof below because it would require an expansion of $\tilde{H}$ in power of ψ.
(2) Clearly, in the hypotheses of proposition (10.1.2), $\widetilde{H}$ exists and is the inverse of H. Conversely, (10.1.4) can be solved explicitly by using the special properties of S_0. Moreover, if S_ε is Anosov, the solution is a homeomorphism and is Hölder continuous in ε, as can be directly checked from $\widetilde{H} \circ H = \mathrm{Id}$. See problems [10.1.6], [10.1.9] for a more detailed discussion.

Proof: We shall write $\underline{\varphi} = H(\underline{\psi}) = \underline{\psi} + \underline{h}(\underline{\psi})$, $\underline{\psi} \in \mathbb{T}^2$; then the relation (10.1.3) becomes an equation for $\underline{h}$, namely

e10.1.5
$$S_0\underline{h}(\underline{\psi}) - \underline{h}(S_0\underline{\psi}) = \varepsilon\underline{f}(\underline{\psi} + \underline{h}(\underline{\psi})), \tag{10.1.5}$$

and the analogies with (8.1.12) or (9.3.4) suggest employing the same method to solve it. Hence we look for a solution which is analytic in ε:

$\underline{h}(\underline{\psi}) = \varepsilon \underline{h}^{(1)}(\underline{\psi}) + \varepsilon^2 \underline{h}^{(2)}(\underline{\psi}) + \cdots$ with $\underline{h}^{(k)}$ an ε–independent function. For instance the equation for the first order is

e10.1.6
$$S_0 \underline{h}^{(1)}(\underline{\psi}) - \underline{h}^{(1)}(S_0 \underline{\psi}) = \underline{f}(\underline{\psi}). \tag{10.1.6}$$

We call $\underline{v}_+, \underline{v}_-$ the two normalized eigenvectors of S_0 relative to the eigenvalues $(1 \pm \sqrt{5})/2$ and we call λ the inverse of the largest one ($\lambda = (\sqrt{5}-1)/2$), so that $\lambda_+ = \lambda^{-1}, \lambda_- = -\lambda$: although the fact that λ is the golden mean might be intellectually nice in the following the only property that we shall use is $\lambda < 1$.

The functions $\underline{f}, \underline{h}$ can be split into two components along the vectors $\underline{v}_\pm$:

e10.1.7
$$\begin{aligned} \underline{f}(\underline{\psi}) &= f_+(\underline{\psi})\underline{v}_+ + f_-(\underline{\psi})\underline{v}_-, \\ \underline{h}(\underline{\psi}) &= h_+(\underline{\psi})\underline{v}_+ + h_-(\underline{\psi})\underline{v}_-, \end{aligned} \tag{10.1.7}$$

and the equations (10.1.6) for $h_\pm^{(1)}$ are

e10.1.8
$$\begin{aligned} \lambda_+ h_+^{(1)}(\underline{\psi}) - h_+^{(1)}(S_0\underline{\psi}) &= f_+(\underline{\psi}), \\ \lambda_- h_-^{(1)}(\underline{\psi}) - h_-^{(1)}(S_0\underline{\psi}) &= f_-(\underline{\psi}). \end{aligned} \tag{10.1.8}$$

The equations (10.1.8) can be solved by simply setting

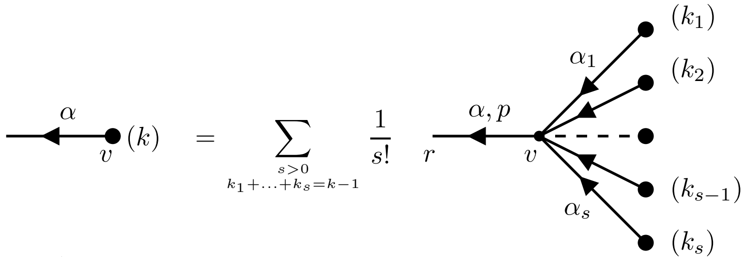
e10.1.9
$$h_\alpha^{(1)}(\underline{\psi}) = -\sum_{p \in \mathbb{Z}_\alpha} \alpha \lambda_\alpha^{-|p+1|\alpha} f_\alpha(S_0^p \underline{\psi}), \qquad \alpha = \pm, \tag{10.1.9}$$

where $\mathbb{Z}_+ = [0,\infty) \cap \mathbb{Z}$ and $\mathbb{Z}_- = (-\infty, 0) \cap \mathbb{Z}$ are subsets of the integers $\mathbb{Z}$ and advantage is taken of the inequality $\lambda = \lambda_+^{-1} = |\lambda_-| < 1$ to ensure convergence.

Therefore the equations for $h_\pm^{(k)}$ become

e10.1.10
$$\begin{aligned} h_\alpha^{(k)}(\underline{\psi}) = \sum_{s=0}^\infty \frac{1}{s!} &\sum_{\substack{k_1+\cdots+k_s=k-1,\, k_i \ge 0 \\ \alpha_1,\ldots,\alpha_s=\pm}} \sum_{p \in \mathbb{Z}_\alpha} \alpha \lambda_\alpha^{-|p+1|\alpha} \cdot \\ &\cdot \Big(\prod_{j=1}^s (\underline{v}_{\alpha_j} \cdot \partial_{\underline{\varphi}}) \Big) f_\alpha(S_0^p \underline{\psi}) \Big(\prod_{j=1}^s h_{\alpha_j}^{(k_j)}(S_0^p \underline{\psi}) \Big), \end{aligned} \tag{10.1.10}$$

and proceeding as in Section §8.1 we obtain a similar graphical representation



F10.1.1 **Fig.(10.1.1)** Graphical interpretation of (10.1.10) for $k \ge 1$.

where the l.h.s. represents $h^{(k)}_\alpha(\underline{\psi})$. Representing again, in the same way, the graph elements that appear on the r.h.s. one obtains an expression for $h^{(k)}_\alpha(\underline{\psi})$ in terms of *trees*, oriented toward the root, just like we already saw in Section §8.1, cf. Fig.(8.2.1), (8.2.2) and (8.3.1), in the KAM theory.

N10.1.2 A tree ϑ with k nodes will carry on the branches[2] ℓ a pair of labels α_ℓ, p_ℓ, with $p_\ell \in \mathbb{Z}$ and $\alpha_\ell \in \{-,+\}$, and on the nodes v a pair of labels α_v, p_v, with $\alpha_v = \alpha_{\ell_v}$ and $p_v \in \mathbb{Z}_{\alpha_v}$ such that

e10.1.11
$$p(v) \equiv p_{\ell_v} = \sum_{w \succeq v} p_v, \tag{10.1.11}$$

where the sum is over the nodes following v (*i.e.* over the nodes along the path connecting v to the root), ℓ_v denotes the branch $v'v$ exiting from the node v, and to each tree we shall assign a *value* given by

e10.1.12
$$\mathrm{Val}(\vartheta) = \prod_{v\in V(\vartheta)} \frac{\alpha_v}{s_v!} \lambda_{\alpha_v}^{-|p_v+1|\alpha_v} \Big(\prod_{j=1}^{s_v} \partial_{\alpha_{v_j}}\Big) f_{\alpha_v}(S_0^{p(v)}\underline{\psi}), \tag{10.1.12}$$

where $\partial_\alpha \overset{def}{=} \underline{v}_\alpha \cdot \underline{\partial}_{\underline{\varphi}}$, $V(\vartheta)$ is the set of nodes in ϑ, the nodes $v_1, \ldots, v_{s_v}$ are the s_v nodes preceding v (if v is a top node then the derivatives are simply missing). If $\Theta_{k,\alpha}$ denotes the set of all trees with k nodes and with label α associated to the root line, then one has

e10.1.13
$$h_\alpha(\underline{\psi}) = \sum_{k=1}^\infty \varepsilon^k \sum_{\vartheta\in\Theta_{k,\alpha}} \mathrm{Val}(\vartheta), \tag{10.1.13}$$

and the "only" problem left is to estimate the radius of convergence of the above formal power series. For this purpose it is convenient to study the Fourier transform of the function $h_\alpha(\underline{\psi})$. This is easily done graphically because it is enough to attach a label $\underline{\nu}_v \in \mathbb{Z}^2$ to each node and define the momentum that flows on the tree branch $v'v$ as in Section §8.1, *i.e.* $\underline{\nu}_{\ell_v} \overset{def}{=} \sum_{w\preceq v} \underline{\nu}_w$, see (8.2.4).

Then (10.1.13) becomes

e10.1.14
$$h_\alpha(\underline{\psi}) = \sum_{k=1}^\infty \varepsilon^k \sum_{\underline{\nu}\in\mathbb{Z}^2} e^{i\underline{\nu}\cdot\underline{\psi}}\, h^{(k)}_{\alpha,\underline{\nu}}, \tag{10.1.14}$$

with

e10.1.15
$$\begin{aligned} h^{(k)}_{\alpha,\underline{\nu}} = &\sum_{\vartheta\in\Theta_{k,\underline{\nu},\alpha}} \sum_{p_v\in\mathbb{Z}_{\alpha_v}} \Big(\prod_{v\in V(\vartheta)} \frac{\alpha_v}{s_v!} \lambda_{\alpha_v}^{-|p_v+1|\alpha_v} f_{\alpha_v, S_0^{-p(v)}\underline{\nu}_v} \Big) \cdot \\ &\cdot \prod_{\substack{v\in V(\vartheta)\\ v'\neq v_0}} \big(- S_0^{-p(v')} \underline{\nu}_{v'} \cdot \underline{v}_{\alpha_v} \big), \end{aligned} \tag{10.1.15}$$

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[2] We do not use the letter λ as in Section §8.1 to denote the lines in order to avoid confusion with the parameter λ introduced after (10.1.6).

where $\Theta_{k,\underline{\nu},\alpha}$ denotes the set of all trees with k nodes and with labels $\underline{\nu}$ and α associated with the root line.

Calling $F=\max_{\underline{\nu}}|f_{\underline{\nu}}|$ we can estimate $\sum_{\underline{\nu}}|\underline{\nu}|^{\beta}|h^{(k)}_{\alpha,\underline{\nu}}|$. The only problem is given by the presence of the factor $|\underline{\nu}|^{\beta}$. In fact consider first the case $\beta=0$: since we are assuming that $\underline{f}$ is a trigonometric polynomial there are only $(2N+1)^2$ possible choices for each $\underline{\nu}_v$, given p_v, such that $|S_0^{-p_v}\underline{\nu}_v|\le N$. Hence fixed ϑ, $\{\alpha_v\}_{v\in V(\vartheta)}$ and $\{p_v\}_{v\in V(\vartheta)}$ the remaining sum of products in (10.1.15) is bounded by (if $\lambda\equiv\lambda_+^{-1}\equiv-\lambda_-$)

e10.1.16
$$(3N)^{2k}N^kF^k\prod_{v\in V(\vartheta)}\frac{\lambda^{|p_v|}}{s_v!}. \tag{10.1.16}$$

The sum over the p_v is a geometric series bounded by $(2/(1-\lambda))^k$.

The combinatorial problem is identical to the one discussed in Section §8.1 and in the other sections of Chapter VIII, so that the factor $\prod_v(1/s_v!)$ becomes, after summing over all the trees, simply bounded by 2^{3k}, (2^{2k} due to the number of trees for fixed labels, see Section §8.1, and 2^k due to the sum of labels α_v), noting that in the present problem all the intricacies due to the small divisors are just absent. In fact we have not used the $(s_v!)^{-1}$ and we have bounded them by 1: they would be necessary if we had supposed f to be only analytic rather than a trigonometric polynomial; see problems [10.1.3], [10.1.4] and [10.1.5], and bear in mind the problems of Section §8.4.

Therefore for $\beta=0$ we have proved that the conjugating function H exists and that inside the complex domain $|\varepsilon|<\varepsilon_0(0)\overset{def}{=}(3N)^{-3}F^{-1}2^{-4}(1-\lambda)$ it is uniformly continuous and uniformly bounded with a uniformly summable Fourier transform.

Taking $\beta>0$ requires estimating $|\underline{\nu}|^{\beta}$: we bound it by $\sum_v|\underline{\nu}_v|^{\beta}$. Then we can make use of the fact that $|S_0^{-p(v)}\underline{\nu}_v|\le N$ to infer that $|\underline{\nu}_v|\le\lambda^{-|p(v)|}BN$, where $B\ge1$ is a suitable constant; see problem [10.1.2]. The sum $\sum_v|\underline{\nu}_v|^{\beta}$ is over k terms which can be estimated separately so that we can write $\sum_v|\underline{\nu}_v|^{\beta}\le k|\underline{\nu}_{\overline{v}}|^{\beta}$ where $|\underline{\nu}_{\overline{v}}|=\max_v|\underline{\nu}_v|$. This can be taken into account by multiplying (10.1.16) by an extra factor $(BN)^{\beta}\lambda^{-\beta|p(\overline{v})|}\le BN\lambda^{-\beta\sum_v|p_v|}$. Therefore if $\beta<1$ the bound that we found for $\beta=0$ is modified into

e10.1.17
$$\varepsilon_0(\beta)=(3N)^{-3}F^{-1}(1-\lambda^{1-\beta})2^{-5}. \tag{10.1.17}$$

This shows that $H(\underline{\psi})$ analytic in ε in the disk with radius $\varepsilon_0(\beta)$. Furthermore, since in (10.1.17) we inserted (for simplicity) an extra factor 2^{-1} in excess of the result obtained by the procedure described, the Hölder modulus is also uniformly bounded by a suitable function $C(\beta)$ of β. Note that $\varepsilon_0(\beta)\to0$ for $\beta\to1$.

The map H is a homeomorphism. In fact it is one-to-one because if $\underline{\varphi}=\underline{\psi}_i+\underline{h}(\underline{\psi}_i)$ for $i=1,2$ and $\underline{\psi}_1\ne\underline{\psi}_2$ one would have $S_0^k\underline{\psi}_1+\underline{h}(S_0^k\underline{\psi}_1)=S_0^k\underline{\psi}_2+\underline{h}(S_0^k\underline{\psi}_2)$ for all k, which is impossible being incompatible with the hyperbolicity of S_0 (as a matrix). Furthermore given any $\underline{\varphi}$ there is a $\underline{\psi}$ such that $H(\underline{\psi}) = \underline{\varphi}$: this is a general property of injective maps of the torus $\mathbb{T}^d$ of the form $\underline{\varphi} = \underline{\psi} + \underline{h}(\underline{\psi})$, with $\underline{h}$ Hölder continuous. Indeed if $d = 1$ we see that as ψ runs around the circle the point $\varphi = \psi + h(\psi)$ follows and must pass through all points of the circle as well: hence if ψ is given there is a φ and a function $k(\varphi)$ such that $\psi = \varphi + k(\varphi)$ and the function k is continuous. If $d = 2$ one repeats twice the argument: first we compute the partial inverse of $\varphi_1 = \psi_1 + h_1(\psi_1, \psi_2)$ by fixing ψ_2 and determining $k'(\varphi_1, \psi_2)$ such that $\psi_1 = \varphi_1 + k'(\varphi_1, \psi_2)$, with k' continuous; then one considers the map $\varphi_2 = \psi_2 + h_2(\varphi_1 + k'(\varphi_1, \psi_2), \psi_2)$ and repeats the argument obtaining $\psi_2 = \varphi_2 + k_2(\varphi_1, \varphi_2)$. One finally sets $\psi_1 = \varphi_1 + k_1(\varphi_1, \varphi_2)$ with $k_1(\varphi_1, \varphi_2) = k'(\varphi_1, \varphi_2 + k_2(\varphi_1, \varphi_2))$. ■

Problems for §10.1

Q10.1.1 **[10.1.1]:** (*A property of the golden mean*)
Prove that since λ is Diophantine the correlations of Arnold's cat map decay superexponentially. (*Hint:* Given two functions f and g, analytic on $\mathbb{T}^2$, write the correlations in Fourier space, $S = \sum_{\underline{\nu}} f_{\underline{\nu}} g_{-S^{-p}\underline{\nu}} \le FG \sum_{\underline{\nu}} e^{-\kappa|\underline{\nu}|} e^{-\kappa'|S^{-p}\underline{\nu}|}$, where F, G, κ, κ' are constants depending on f, g. Then use the Diophantine condition to deduce the inequality $|S^{-p}\underline{\nu}| > C\lambda^p / |\underline{\nu}|^{-\tau}$, for suitable constants C and τ. If λ is the golden mean one can take $\tau = 1$.)

Q10.1.2 **[10.1.2]:** Show that if $|S_0^p \underline{\nu}| \le N$, then there is a constant $B = O(N)$ such that $|\underline{\nu}| \le \lambda^{-|p|} B$ and $B = N$ is a possible choice. (*Hint:* Call the eigenvectors $\underline{v}_\alpha, \alpha = \pm$, and let $2B = \max_{\underline{\mu} \in \mathbb{Z}^2, |\underline{\mu}| \le N, \alpha = \pm} |\underline{v}_\alpha \cdot \underline{\mu}|$; then note that, by the spectral decomposition of the matrix S_0, one has $|\underline{\nu}| = |S_0^{-p} \underline{\mu}| \le 2|\lambda|^{-|p|} |\underline{\mu}| \le B|\lambda|^{-|p|}$.)

Q10.1.3 **[10.1.3]:** (*Alternative to the convergence proof for H*)
Show without using the Fourier transform that the series for H defined by (10.1.12) converges. (*Hint:* Bound the s-th derivatives by the maximum of F of the Fourier coefficients of $\underline{f}$ times N^s.)

Q10.1.4 **[10.1.4]:** (*Homeomorphism in the case of analytic $\underline{f}$*)
Show that the series for H defined by (10.1.12) converges under the only assumption that $\underline{f}$ is analytic. (*Hint:* Bound the m-th derivatives by the maximum F_∞ of $|\underline{f}(\underline{\varphi})|$ in a strip $|\mathrm{Im}\, \varphi_j| < \xi$ (for some $\xi > 0$) times $m! \xi^{-m_v} F_\infty$. The factorial is compensated by the $s_v!$'s in (10.1.12) and the estimate proceeds as in the trigonometric polynomial case.)

Q10.1.5 **[10.1.5]:** (*Hölder continuity in the case of analytic $\underline{f}$*)
Show that the series for H defined by (10.1.12) is Hölder continuous under the only assumption that f is analytic. (*Hint:* Make use of the Fourier transform and do more carefully the same bounds.)

Q10.1.6 **[10.1.6]:** (*The inverse conjugation $\widetilde{H}$*)
Show that the solution of (10.1.4) can be written as

$$\tilde{h}_\alpha(\underline{\psi}) = \varepsilon \sum_{p \in \mathbb{Z}_\alpha} \alpha \lambda_\alpha^{-|p+1|\alpha} f_\alpha(S_\varepsilon^p \underline{\psi}) \qquad \alpha = \pm$$

where $\widetilde{H}(\underline{\psi}) = \underline{\psi} + \tilde{h}(\underline{\psi})$. Show that if we assume that S_ε is still Anosov then $\widetilde{H}$ is a homeomorphism. What can you say of the regularity of $\widetilde{H}$ as a function of $\underline{\varphi}$ and ε? (*Hint:* Writing $\widetilde{H}(\underline{\varphi}) = \underline{\varphi} + \underline{\tilde{h}}(\underline{\varphi})$ we get that $\underline{\tilde{h}}(\underline{\varphi})$ satisfies $S_0 \underline{\tilde{h}}(\underline{\varphi}) - \underline{\tilde{h}}(S_\varepsilon(\underline{\varphi})) = \varepsilon \underline{f}(\underline{\varphi})$.

This equation look very much like (10.1.6) and the linearity of S_0 allows us to write the solution as in (10.1.8) with S_ε in the place of S_0. Invertibility of $\widetilde{H}(\underline{\varphi})$ follows from an argument similar to the one used for H.)

Q10.1.7 **[10.1.7]:** (*Linear part of a homeomorphism on the torus*)
Consider a homeomorphism S of $\mathbb{T}^2$. Show that it can be written in a unique way has $S = L + f$ where L is a linear homeomorphism, *i.e.* is given by a invertible 2×2 matrix with integer entries, and f is a periodic function on $\mathbb{R}^2$. We call L the linear part of S.(*Hint:* consider S as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ and use periodicity.)

Q10.1.8 **[10.1.8]:** (*Anosov system with the same linear part are conjugated*)
Show that if S_1 and S_2 are two Anosov homeomorphisms of $\mathbb{T}^2$ with the same linear part L and L is also Anosov, then there exist $\widehat{H}$ such that $\widehat{H} \circ S_1 = S_2 \circ \widehat{H}$. (*Hint:* Use the result of problem [10.1.6] to construct $\widetilde{H}_i$ such that $\widetilde{H} \circ S_i = L \circ \widetilde{H}$. Then $\widehat{H} = \widetilde{H}_2 \circ \widetilde{H}_1^{-1}$.)

Q10.1.9 **[10.1.9]:** (*Domain of existence of $\widetilde{H}$ and H: a simple case*)
Assume that $\underline{f}(\underline{\varphi}) = \underline{v}_+ g(\underline{\varphi})$ where g is a trigonometric polynomial. Find a condition on ε that assure that $S_\varepsilon = S_0 + \varepsilon \underline{v}_+ g$ is still Anosov. This will give a condition for the existence and invertibility of $\widetilde{H}$ in problem [10.1.6]. Compare it with (10.1.17). (*Hint:* The differential of S_ε has the same eigendirections of S_0. The relative eigenvalues are $\lambda_{\varepsilon,-}(\underline{\varphi}) \equiv \lambda_-$ and $\lambda_{\varepsilon,+}(\underline{\varphi})$. If $|\lambda_{\varepsilon,+}(\underline{\varphi})| > 1$ problem [10.1.6] allows us to construct $\widetilde{H}$ and shows that it is invertible. The Anosov property for S_ε follows from proposition (4.2.1) where points (i) and (ii) are evident and point (iii) follows from the existence of $\widetilde{H}$.)

Q10.1.10 **[10.1.10]:** (*Domain of existence of $\widetilde{H}$ and H: general case*)
Generalize problem [10.1.9] to a generic trigonometric polynomial $\underline{f}(\underline{\varphi})$.(*Hint:* The lines defined by the two vectors $\underline{c}_\pm = \underline{v}_+ \pm \underline{v}_-$ split $\mathbb{R}^2$ in two cones $\Gamma_\pm$ with the property that $DS_0\Gamma_\pm$ is well inside $\Gamma_\pm$, see problem [4.2.2] for a precise formulation. Write a condition on $\varepsilon \underline{f}(\underline{\varphi})$ so that $DS_\varepsilon \Gamma_\pm$ is still well inside $\Gamma_\pm$. Compare it with (10.1.17).)

Q10.1.11 **[10.1.11]:** (*Continuity of Markov pavements in* $d = 2$)
Consider a two-dimensional Anosov map (Ω, S) and assume that it admits a fixed point x_0. Let S_ε be a small perturbation of S (in class C^∞) depending on a parameter ε. Show that the map S_ε admits a Markov pavement $\mathcal{P}_\varepsilon = \{P_{0\varepsilon}, \ldots, P_{n\varepsilon}\}$ with the same compatibility matrix T as that of $\mathcal{P}_0$. (*Hint:* Construct a Markov pavement $\mathcal{P} = \{P_0, \ldots, P_n\}$ for (Ω, S) by the method of problem [4.3.9]. Note that S_ε will have a fixed point x_ε which merges differentially with x_0 as $\varepsilon \to 0$ together with any finite portion of its stable and unstable manifolds. Note that the construction of $\mathcal{P}$ in problem [4.3.9] is based on two *finite* connected portions of the stable and of the unstable manifolds of x_0.)

Q10.1.12 **[10.1.12]:** (*Anosov structural stability in* $d = 2$)
In the context of problem [10.1.11] show that there is a Hölder continuous map H which conjugates S and S_ε as in (10.1.1) if ε is small enough. (*Hint:* Define a map of Ω into itself by setting $Hx = x'$ if $x' = X_\varepsilon(\underline{\sigma}(x))$ where $\underline{\sigma}(x)$ is the symbolic history of x on the pavement $\mathcal{P}$ under the map S and X_ε is the map that associates with a compatible sequence $\underline{\sigma}$ a point $X_\varepsilon(\underline{\sigma}) \in \Omega$, see definition (4.1.3). Hölder continuity follows from the Hölder continuity of the correspondence between points and symbolic histories on Markov pavements, see proposition (4.1.1), (4.1.6). In other words x, x' are mapped into each other by H if they have the same symbolic representation in the pavements $\mathcal{P}, \mathcal{P}_\varepsilon$ under the maps S, S' respectively.)

§10.2 Extended systems. Lattices of Arnold's cat maps

Let $\Lambda = [-L/2, L/2]^d$ be the cube of side L in $\mathbb{Z}^d$. If $\Omega_\Lambda \overset{def}{=} (\mathbb{T}^2)^\Lambda$ we call

$\underline{\varphi}\in\Omega_\Lambda$, $\underline{\varphi}=(\underline{\varphi}_\xi)_{\xi\in\Lambda}$ a *microstate* of the *lattice system* Ω_Λ. The microstates $\underline{\varphi}$ are considered with *periodic boundary conditions*, *i.e.* $\underline{\varphi}=(\underline{\varphi}_\xi)_{\xi\in\mathbb{Z}^d}$ with $\underline{\varphi}_{\xi+\underline{d}_i L}=\underline{\varphi}_\xi$ for every ξ and i, where $\underline{d}_i$ is the unit vector in the direction i in $\mathbb{Z}^d$. We define a map $\mathcal{S}_0:\Omega_\Lambda\longleftrightarrow\Omega_\Lambda$

e10.2.1
$$(\mathcal{S}_0(\underline{\varphi}))_\xi=S_0\underline{\varphi}_\xi,\qquad S_0=\begin{pmatrix}1&1\\1&0\end{pmatrix},\tag{10.2.1}$$

and we call $\mathcal{S}_0$ the *unperturbed evolution map* of the microstates.

Let $\underline{f}(\underline{\varphi}_{nn})$ be a $\mathbb{R}^2$-valued function of $2d+1$ arguments in $\mathbb{T}^2$: we label the $2d+1$ arguments $\underline{\varphi}_{nn}=(\underline{\varphi}_0,\underline{\varphi}_1,\cdots,\underline{\varphi}_{2d})$. We imagine that $\underline{\varphi}_0$ is associated with the origin in $\mathbb{Z}^d$ and that the remaining $2d$ arguments are associated with the lattice sites which are nearest neighbors of the origin ordered in a arbitrary way, *e.g.* lexicographically. We shall call $nn(\xi)$ the set formed by ξ and by its nearest neighbors so that $\underline{f}(\underline{\varphi}_{nn(\xi)})$ make sense and depend only on the values of the microstate $\underline{\varphi}$ at ξ and at its neighboring sites.

We shall suppose that we are given a $\underline{f}(\underline{\varphi}_{nn})$ which is a trigonometric polynomial of degree $\le N$ with values in $\mathbb{T}^2$, *i.e.*

e10.2.2
$$\underline{f}(\underline{\varphi}_{nn})=\sum_{\underline{\nu}_0,\underline{\nu}_1,\ldots,\underline{\nu}_{2d},\,|\underline{\nu}_j|\le N}\underline{f}_{\underline{\nu}_0,\underline{\nu}_1,\ldots,\underline{\nu}_{2d}}\,e^{i\sum_{j=0}^{2d}\underline{\nu}_j\cdot\underline{\varphi}_j}.\tag{10.2.2}$$

We shall call $\underline{f}$ a *nearest neighbor interaction* and define

e10.2.3
$$(\mathcal{S}_\varepsilon\underline{\varphi})_\xi=S_0\underline{\varphi}_\xi-\varepsilon\underline{f}(\underline{\varphi}_{nn(\xi)}),\tag{10.2.3}$$

which maps $\Omega_\Lambda\longleftrightarrow\Omega_\Lambda$ if ε is small enough. We call $\mathcal{S}_\varepsilon$ the *perturbed evolution map.*

For small ε the system will be an Anosov system by the general structural stability theorem in proposition (10.1.1) and it will be conjugated with the unperturbed system $(\Omega_\Lambda,\mathcal{S}_0)$. However in order to insure that this happens it might be necessary to take ε smaller and smaller as the *infrared cutoff* Λ tends to ∞. Therefore it is important to note that it is not so: *the system remains chaotic enough no matter how spatially extended it is.*

P10.2.1 **(10.2.1) Proposition:** (Uniform structural stability of lattices of maps) *Given a nearest neighbors interaction $\underline{f}$ as above and $\beta\in(0,1)$, there exist $\varepsilon_0(\beta)>0$ and $C(\beta)<\infty$ such that for all $\Lambda\subset\mathbb{Z}^d$ and for all $|\varepsilon|<\varepsilon_0(\beta)$ there is a homeomorphism $H:\Omega_\Lambda\longleftrightarrow\Omega_\Lambda$ and a constant $\kappa=\kappa(\beta,\varepsilon)$ such that*

e10.2.4
$$H\circ\mathcal{S}_0=\mathcal{S}_\varepsilon\circ H,\tag{10.2.4}$$

where H is analytic in the disk $|\varepsilon|<\varepsilon_0(\beta)$ and Hölder continuous with range $\kappa(\beta,\varepsilon)^{-1}$ and modulus $C(\beta)$, in the sense that if $\underline{\varphi},\underline{\varphi}'\in\Omega_\Lambda$ and $\underline{\varphi}_\xi\equiv\underline{\varphi}'_\xi$ for all ξ but for ξ' then

e10.2.5
$$|(H(\underline{\varphi}))_\xi-(H(\underline{\varphi}'))_\xi|\le C(\beta)e^{-\kappa|\xi-\xi'|}|\underline{\varphi}_{\xi'}-\underline{\varphi}'_{\xi'}|^\beta\tag{10.2.5}$$

for all ε in the complex disk $|\varepsilon| < \varepsilon_0(\beta)$. The range constant $\kappa = \kappa(\beta, \varepsilon)$ can be taken $-(2d)^{-1} \log(|\varepsilon|/2\varepsilon_0(\beta))$.

Remarks: (1) As we did in Section §10.1 we can study the equation for $\widetilde{H} = H^{-1}$ and obtain directly a solution that turns out to be invertible if we assume that $\mathcal{S}_\varepsilon$ is still an Anosov system, see problem [10.2.1].
(2) Using an argument similar to the one used in problem [10.1.10] one can prove that $\mathcal{S}_\varepsilon$ is Anosov uniformly in Λ for ε small enough so that $\widetilde{H}$ exists and is invertible, see problem [10.2.2]. As before this construction is not suitable to study the regularity of H as a function of ε. Moreover our proof give us a detailed description of H that will be fundamental in the construction of the SRB measure, see corollary (10.2.1).

Proof: The proof of this proposition is essentially a repetition of the corresponding "single site" $(L = 0, \Lambda = \{0\}, \Omega = \mathbb{T}^2)$ case discussed in proposition (10.1.1). We write H as $(H(\underline{\psi}))_\xi = \underline{\psi}_\xi + (\underline{h}(\underline{\psi}))_\xi$ and decompose $\underline{f}_\xi, \underline{h}_\xi$ along the eigenvectors $\underline{v}_\pm$ of S_0, see (10.1.7). The equations become

e10.2.6
$$\begin{aligned} \lambda_+ h_+(\underline{\psi})_\xi - h_+(S_0\underline{\psi})_\xi &= f_+((\underline{\psi} + \underline{h}(\underline{\psi}))_{nn(\xi)}), \\ \lambda_- h_-(\underline{\psi})_\xi - h_-(S_0\underline{\psi})_\xi &= f_-((\underline{\psi} + \underline{h}(\underline{\psi}))_{nn(\xi)}). \end{aligned} \tag{10.2.6}$$

We can solve the equation (10.2.6) by power series in ε. The first order equation has a solution similar to the corresponding (10.1.9) and the general order recursion can also be written in a form similar to (10.1.10), namely

e10.2.7
$$\begin{aligned} h^{(k)}_{\xi,\alpha}(\underline{\psi}) = \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{\substack{k_1+\cdots+k_s=k-1,\, k_i\ge 0 \\ \alpha_1,\ldots,\alpha_s=\pm;\ \xi_1,\ldots,\xi_s\in nn(\xi)}} \sum_{p\in\mathbb{Z}_\alpha} \alpha \lambda_\alpha^{-|p+1|\alpha} \cdot \\ \cdot \Big(\prod_{j=1}^{s} (\partial_{(\alpha_j,\xi_j)}) \Big) f_\alpha((S_0^p \underline{\psi})_{nn(\xi)}) \cdot \Big(\prod_{j=1}^{s} h^{(k_j)}_{\xi_j,\alpha_j}(S_0^p\underline{\psi}) \Big), \end{aligned} \tag{10.2.7}$$

where $\partial_{(\alpha,\xi)} \overset{def}{=} \underline{v}_\alpha \cdot \partial_{\underline{\psi}_\xi}$.

Therefore we can represent the complete $\underline{h}^{(k)}_{\xi,\alpha}(\underline{\psi})$ again in terms of the same tree graphs introduced in Section §10.1 with a few more labels attached to the branches. Namely we must add to each node v a new label ξ_v with the restriction that if the line $\lambda = v'v$ emerging from v enters a node $v' \succ v$ then $\xi_v \in nn(\xi_{v'})$ (which reflects the nearest neighbor property of the interaction). We add also a label ξ_ℓ to the line $\ell = v'v$ by setting $\xi_\ell = \xi_v$.
Hence the *value* of a tree ϑ will be given, see (10.1.12) for comparison, by

e10.2.8
$$\mathrm{Val}(\vartheta) = \prod_{v\in V(\vartheta)} \frac{\alpha_v}{s_v!} \lambda_{\alpha_v}^{-|p_v+1|\alpha_v} \Big(\prod_{j=1}^{s_v} \partial_{(\alpha_{v_j},\xi_{v_j})} \Big) f_{\alpha_v}((S_0^{p(v)} \underline{\psi})_{nn(\xi_v)}), \tag{10.2.8}$$

and it will be a contribution to $h^{(k)}_{\xi,\alpha}$ if α, ξ are the labels attached to the root branch of ϑ and k is the number of nodes of ϑ.

Remark: By construction the sets $nn(\xi_v)$ must all intersect $nn(\xi_{v'})$ if v' is the node immediately following v. Hence the set $\cup_v nn(\xi_v)$ is connected (by nearest neighbors) and, in fact, its connectivity reflects that of the tree to which it is associated.

The estimates are done as in Section §10.1 provided that one takes care of the fact that for each site ξ there are $2d+1$ points in $nn(\xi)$ rather than just 1 as in the case of Section §10.1. This gives an extra factor $(2N+1)^{2(2d+1)k}(2d+1)^k$, that we shall bound (for simplicity) by $(3Ne)^{2(2d+1)k}$, so that, in Fourier space,

e10.2.9
$$\begin{aligned}\sum_{\vartheta\in\Theta_{k,\underline{\nu},\alpha}}\ \sum_{v\in V(\vartheta)}|\underline{\nu}_v|^{\beta}|\mathrm{Val}(\vartheta)| &\le\\ &\le (3Ne)^{2(2d+1)k}N^kF^k(2/(1-\lambda^{1-\beta}))^k 2^{3k}\end{aligned}\tag{10.2.9}$$

will be a bound on the sum of the values of the trees of order k (*i.e.* with k nodes): in fact the combinatorics is again the same as in Section §10.1 and it is accounted by the last factor 2^{3k}.

We have obtained a rather explicit expression for $\underline{h}$ which can be expressed as

e10.2.10
$$h_{\xi,\alpha}(\underline{\psi})=\sum_{X\ni\xi}\Phi_{X,\alpha}(\underline{\psi}_X),\tag{10.2.10}$$

where X is a subset, connected by nearest neighbors, of Λ, and Φ_X is a function of $\underline{\psi}_X$ which is translation invariant in the sense that $\Phi_{X,\alpha}(\underline{\psi}_X)=\Phi_{X+\eta,\alpha}(\underline{\psi}_X)$ for all $\eta\in\mathbb{Z}^d$ (the translations must be considered modulo L of course, because of the periodic boundary conditions on Λ). This is because (10.2.8) has the structure of (10.2.10) if $X=\cup_{v\in V(\vartheta)}nn(\xi_v)$.

By the estimate in (10.2.9) we conclude that the functions $\Phi_{X,\alpha}(\underline{\psi}_X)$ are analytic in ε in the complex disk with radius twice the quantity

e10.2.11
$$\varepsilon_0(\beta)=(3Ne)^{-2(2d+1)}N^{-1}F^{-1}(1-\lambda^{1-\beta})2^{-5},\tag{10.2.11}$$

and that they verify the bounds

e10.2.12
$$\begin{aligned}&\max_{|\varepsilon|\le\varepsilon_0(\beta),\,X,\,\underline{\psi}_X}|\Phi_{X,\alpha}(\underline{\psi}_X)|<B(\beta),\\ &\max_{|\varepsilon|\le\varepsilon_0(\beta),\,X,\,\underline{\psi}_{X\setminus\xi'}}|\Phi_{X,\alpha}(\underline{\psi}_X)-\Phi_{X,\alpha}(\underline{\psi}'_X)|<B(\beta)\,|\underline{\psi}_{\xi'}-\underline{\psi}'_{\xi'}|^{\beta},\end{aligned}\tag{10.2.12}$$

if $\underline{\psi}_X,\underline{\psi}'_X$ coincide at all points but at $\xi'\in\Lambda$, with $B(\beta)$ finite because of
N10.2.1 the extra factor $1/2$ we inserted in (10.2.11).[1]

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[1] Note that (10.2.9) says that the analyticity disk can be taken to be the r.h.s. of (10.2.11) with 2^{-4} instead of 2^{-5}: it is convenient to give up a factor 2 in the size of $\varepsilon_0(\beta)$ in order to have uniform bounds in the disk of radius $\varepsilon_0(\beta)$.

A further key remark, that follows immediately from the formula (10.2.8) and from the remark on connectivity following it, is that the Taylor coefficient $\Phi^{(k)}_{X,\alpha}(\underline{\psi}_X)$ of order k in ε of $\Phi_{X,\alpha}(\underline{\psi}_X)$ verifies

e10.2.13
$$\Phi^{(k)}_{X,\alpha}(\underline{\psi}_X) \equiv 0 \qquad \text{if } k \le \delta(X)/2d, \tag{10.2.13}$$

where $\delta(X)$ is the tree length of the set X, see definition (7.1.3); therefore if $\kappa(\beta,\varepsilon) = -(2d)^{-1}\log(|\varepsilon|/2\varepsilon_0)$ the bound (10.2.12) can be improved into

e10.2.14
$$\begin{aligned}&\max_{|\varepsilon|\le\varepsilon_0(\beta),\,\underline{\psi}_X} |\Phi_{X,\alpha}(\underline{\psi}_X)| < B(\beta)\, e^{-\kappa(\beta,\varepsilon)\delta(X)} \\ &\max_{|\varepsilon|\le\varepsilon_0(\beta),\,\underline{\psi}_{X\setminus\xi'}} |\Phi_{X,\alpha}(\underline{\psi}_X) - \Phi_{X,\alpha}(\underline{\psi}'_X)| < B(\beta)\, e^{-\kappa(\beta,\varepsilon)\delta(X)}\, |\underline{\psi}_{\xi'} - \underline{\psi}'_{\xi'}|^\beta,\end{aligned} \tag{10.2.14}$$

as a consequence of the maximum principle for holomorphic functions. Since $\kappa(\beta,\varepsilon) \to 0$ for $|\varepsilon| \to \varepsilon_0(\beta)$ in order to obtain the result stated in the proposition we have to reduce by an extra factor 2 the value of $\varepsilon_0(\beta)$ obtained in (10.2.11) to obtain the quantity named $\varepsilon_0(\beta)$ in the statement of proposition (10.2.1). ■

The above analysis has led to a result that it is convenient to state separately.

C10.2.1 **(10.2.1) Corollary:** (Conjugation potentials) *In the context of proposition (10.1.1) the homeomorphism H can be written as*

e10.2.15
$$H(\underline{\psi})_\xi = \underline{\psi}_\xi + \sum_{X\ni\xi} \underline{\Phi}_X(\underline{\psi}_X), \tag{10.2.15}$$

where the sum is over connected sets X, and $\Phi_X(\underline{\psi}_X)$ are translation invariant *functions (i.e. $\Phi_X(\underline{\psi}_X) = \Phi_{X+\eta}(\underline{\psi}_X)$ for $\eta \in \Lambda$) analytic in ε in the complex disk $|\varepsilon| < \varepsilon_0(\beta)$ and verifying, for a suitably chosen constant $B(\beta)$, the bounds*

e10.2.16
$$\begin{aligned}&\max_{|\varepsilon|\le\varepsilon_0(\beta),\,\underline{\psi}_X} |\underline{\Phi}_X(\underline{\psi}_X)| < B(\beta)\Big(\frac{|\varepsilon|}{2\varepsilon_0(\beta)}\Big)^{\frac{\delta(X)}{2d}}, \\ &\max_{|\varepsilon|\le\varepsilon_0(\beta),\,\underline{\psi}_{X\setminus\xi}} |\underline{\Phi}_X(\underline{\psi}_X) - \underline{\Phi}_X(\underline{\psi}'_X)| < B(\beta)\Big(\frac{|\varepsilon|}{2\varepsilon_0(\beta)}\Big)^{\frac{\delta(X)}{2d}} |\underline{\psi}_\xi - \underline{\psi}'_\xi|^\beta,\end{aligned} \tag{10.2.16}$$

*if $\underline{\psi}'_X$ differs from $\underline{\psi}_X$ only at the site ξ and $\delta(X)$ is the tree length of X, for all X (*cf.* definition (7.1.3)).*

Remarks: (1) We can summarize the above analysis by saying that the lattice of coupled maps remains an Anosov system if the perturbation is small enough and the allowed maximum size of ε does not depend on the size of the lattice Λ "containing" the system (proposition (10.2.1)).

(2) The result of corollary (10.2.1) falls short of exhibiting a key property of the conjugation. Suppose that instead of the sequence $\{\underline{\psi}_\xi\}_{\xi\in\Lambda}$ we represent each coordinate $\underline{\psi}_\xi$ by an infinite symbolic sequence $\sigma_{\xi,t}$, $\xi\in\Lambda, t\in\mathbb{Z}$ where $\{\sigma_{\xi,t}\}_{t\in\mathbb{Z},\xi\in\mathbb{Z}^d}$ is the symbolic sequence that represents $H^{-1}(\underline{\psi}_\xi)$ on a Markovian pavement of S_0. Then we can use the corollary to obtain a representation of H in terms of the symbolic dynamics $\{\sigma_{\xi,t}\}_{t\in\mathbb{Z}}$ representing $\underline{\psi}$ as a sequence on a *space–time lattice*, see also section §(10.4) above proposition (10.4.2) for more details on this contraction. By following the "telescopic" procedure to express a Hölder continuous function in terms of potentials, cf. proposition (4.3.1), we find easily that H can be written

e10.2.17
$$H(\underline{\psi})_\xi-\underline{\psi}_\xi=\sum_{\mathbb{Z}^{d+1}\supset X\ni(\xi,0)}\Phi_X(\underline{\sigma}_X),\tag{10.2.17}$$

where the summation is restricted to sets X which are rectangles on the $(d+1)$–dimensional lattice and there are constants F,κ such that $|\Phi_X(\underline{\sigma}_X)|\le Fe^{-\kappa\,\mathrm{diam}(X)}$. We see that H can be expressed in this way in terms of potentials which however do not decay exponentially as the tree length but "only" as the diameter. This would eventually create unsurmountable difficulties when we shall try to derive analyticity of the SRB distribution.

(3) However one can get a much better representation of the form (10.2.17) in which sets more general than rectangles appear in the sum, but the decay rate will be exponential in the tree length $\delta(X)$ rather than in the diameter. This is in fact implicit in the expression (10.2.8) and one just has to read it out: see the proof of proposition (10.4.2), where this is discussed and needed.

The above results will allow us to define a Markov pavement for the extended system quite easily, as we shall show in Section §(10.4), and a detailed construction of the SRB distribution. We proceed to perform the construction by first considering the case of a perturbation of a single map.

Problems for §10.2

Q10.2.1 **[10.2.1]:** (*The inverse conjugation in* $d\ge1$)
Show that the equation $\mathcal{S}_\varepsilon\circ\widetilde{H}=\widetilde{H}\circ\mathcal{S}_\varepsilon$ can be solved uniformly in L if one assumes that $\mathcal{S}_\varepsilon$ is Anosov uniformly in Λ. (*Hint:* See problem [10.1.6].)

Q10.2.2 **[10.2.2]:** (*Anosov property for slow decaying interaction*)
Give a condition on ε and $\underline{f}$ such that $\mathcal{S}_\varepsilon$ is Anosov uniformly in Λ. (*Hint:* Let w be a tangent vector to Ω_Λ. Define $|w|_+=\sup_\xi|(w_\xi,v_+)|$ and $|w|_-=\sup_\xi|(w_\xi,v_-)|$ and $|w|_\infty=\max\{|w|_+,|w|_-\}$. Proceed like in problem [10.1.10] using the cones $\Gamma_+=\{w\,|\,|w|_-\le\alpha|w|_+\}$ and $\Gamma_-=\{w\,|\,|w|_+\le\alpha|w|_-\}$ with $\alpha<1$. Show that if $|D\underline{f}w|_\infty\le C|w|_\infty$, where $D\underline{f}$ is the differential of $\underline{f}$, then it is possible to find ε such that $D\mathcal{S}_\varepsilon\Gamma_+$ is well inside Γ_+ and similarly for Γ_-.)

Q10.2.3 **[10.2.3]:** (*Homeomorphism in the case of analytic* $\underline{f}$)
Show that the series for H defined by (10.2.7) converges under the only assumption that f is analytic. In particular show that corollary (10.2.1) is still valid. (*Hint:* See [10.1.4]).

Q10.2.4 **[10.2.4]:** (*Exponential decay for* $\underline{f}$)

Let $\underline{f}$ be a bounded function from $(\mathbb{T}^2)^{\mathbb{Z}^d}$ in $\mathbb{T}^2$ that can be extended to a bounded analytic function on the complex neighbor of $(\mathbb{T}^2)^{\mathbb{Z}^d}$ defined by $|\mathrm{Im}\,\underline{\varphi}_\xi| < \gamma e^{\omega|\xi|}$. Show that for such an $\underline{f}$ the series for H defined by (10.2.7) converges. Is corollary (10.2.1) still valid? (*Hint:* : Use Cauchy estimate to bound the derivatives of $\underline{f}$ with respect to $\underline{\varphi}_\xi$).

Q10.2.5 **[10.2.5]:** (*Tree decay for* $\underline{f}$)
Consider a function $\underline{f}$ from $(\mathbb{T}^2)^{\mathbb{Z}^d}$ in $\mathbb{T}^2$ that can be written as

$$\underline{f}(\underline{\varphi}) = \sum_{X \ni 0} \underline{f}_X(\underline{\varphi}_X),$$

with the functions $\underline{f}_X(\underline{\varphi}_X)$ analytic in the complex neighbor of $(\mathbb{T}^2)^X$ defined by $|\mathrm{Im}\,\underline{\varphi}_\xi| < \gamma e^{\omega\delta(X)}$ and there bounded by a common constant C. Show that for such an $\underline{f}$ the series for H defined by (10.2.7) converges and that H can still be written as in (10.2.10) and Φ_X satisfy (10.2.14) with a suitable decay rate κ. (*Hint:* Use Cauchy estimate to bound the derivatives of $\underline{f}_X$ with respect to $\underline{\varphi}_\xi$).

§10.3 Chaos in time: an SRB distribution

We shall use the notations of Section §10.1, where we have constructed the conjugation H that transforms a perturbed cat map S_ε of the torus $\mathbb{T}^2$ into a "free" cat map S_0. The conjugation is not differentiable (in general) and we could only prove that it can be taken to be Hölder continuous with a prefixed exponent $\beta < 1$ for a perturbation strength that is suitably small.

In spite of the lack of regularity we can still use the homeomorphism $\underline{\varphi} = H(\underline{\psi})$ to construct the dynamics as well as the stable and unstable manifolds of each point. If $\underline{\varphi} = H(\underline{\psi})$ the latter manifolds are given by parametric equations of the form

e10.3.1
$$\underline{\varphi}(t) = H(\underline{\psi} + t\underline{v}_\alpha) \qquad t \in \mathbb{R}, \alpha = \pm, \tag{10.3.1}$$

where $t \to \underline{\psi} + t\,\underline{v}_\alpha$ is the unstable or stable manifold for the unperturbed map S_0 through $\underline{\psi}$ if $\alpha = +$ or $\alpha = -$.

However the above parameterization is not very useful because the function $H(\underline{\psi} + t\underline{v}_\alpha)$ is not regular as a function of t. This can be seen already from the fact that the first order term in its expansion in powers of ε cannot be (in general) differentiated with respect to t at $t = 0$. Indeed we see from (10.1.9) that the component $h^{(1)}_-(\underline{\psi})$ can be differentiated in the direction $\underline{v}_+$ because term by term differentiation enhances convergence since $p < 0$; on the other hand the component $h^{(1)}_+(\underline{\psi})$ cannot be differentiated in the direction $\underline{v}_+$ (unless special cancellations occur) because the convergence factor $\lambda_+^{-(p+1)}$ in (10.1.9) is compensated by the λ_+^p that the differentiation along $\underline{v}_+$ brings out since $p \geq 0$. To second order in ε not even the t–derivative of the component $h^{(2)}_-(\underline{\psi} + \underline{v}_+ t)$ can be shown to exist.

To construct the stable and unstable manifolds as well as the other necessary ingredients to define the SRB distribution we apply once more the technique discussed in the previous two sections. Calling $\widehat{\Omega}$ the (*non compact*) space $\mathbb{T}^2 \times \mathbb{R}^2$ we define the dynamical system[1] N10.3.1

e10.3.2
$$\widehat{S}_0(\underline{\varphi}, \underline{v}) = (S_0\underline{\varphi}, S_0\underline{v}). \tag{10.3.2}$$

This is a system that fails to be an Anosov system because the phase space is not compact. We can nevertheless consider its perturbation

e10.3.3
$$\widehat{S}_\varepsilon(\underline{\varphi}, \underline{v}) = (S_0\underline{\varphi} + \varepsilon \underline{f}(\underline{\varphi}), S_0\underline{v} + \varepsilon(\underline{v} \cdot \partial_{\underline{\varphi}})\underline{f}(\underline{\varphi})), \tag{10.3.3}$$

and we can attempt at finding an isomorphism between $\widehat{S}_\varepsilon$ and $\widehat{S}_0$ or, since this turns out to be in general impossible (as it will be implicit in what follows), between $\widehat{S}_\varepsilon$ and $\widehat{S}_{0,\varepsilon}$ defined by

e10.3.4
$$\widehat{S}_{0,\varepsilon}(\underline{\varphi}, \underline{v}) = (S_0\underline{\varphi}, (S_0 + \Gamma_\varepsilon(\underline{\varphi}))\underline{v}), \qquad (10.3.4).$$

with $\Gamma_\varepsilon(\underline{\varphi})$ a matrix *diagonal* on the basis $\underline{v}_\pm$ (on which S_0 is diagonal too). Therefore we look for a map $\widehat{H}$ of a simple form and such that $\widehat{S}_\varepsilon \circ \widehat{H} = \widehat{H} \circ \widehat{S}_{0,\varepsilon}$, *i.e.*

e10.3.5
$$\widehat{H} \,:\, (\underline{\psi}, \underline{w}) \longleftrightarrow (\underline{\varphi}, \underline{v}) = (\underline{\psi} + \underline{h}(\underline{\psi}), \underline{w} + K(\underline{\psi})\underline{w}), \tag{10.3.5}$$

as we already know how to conjugate S_ε with S_0, from the analysis of Section §10.1.

Remarks: (1) Let $\mathcal{K}_\varepsilon(\varphi) = 1 + K(\varphi)$ and $\mathcal{L}_\varepsilon(\varphi) = S_0 + \Gamma_\varepsilon(\varphi)$, the above conjugation is equivalent to the following equation

e10.3.6
$$DS_\varepsilon(H(\underline{\varphi}))\mathcal{K}_\varepsilon(\underline{\varphi})\underline{v} = \mathcal{K}_\varepsilon(S_\varepsilon\underline{\varphi})\mathcal{L}_\varepsilon(\underline{\varphi})\underline{v}. \tag{10.3.6}$$

This implies that the vector $\underline{w}_\pm(\underline{\varphi}) = \mathcal{K}_\varepsilon(\underline{\varphi})\underline{v}_\pm$ satisfies:

e10.3.7
$$DS_\varepsilon(H(\underline{\varphi}))\underline{w}_\pm(\underline{\varphi}) = \lambda_\pm(\underline{\varphi})\underline{w}_\pm(S_\varepsilon\underline{\varphi}) \tag{10.3.7}$$

where $\lambda_\pm(\underline{\varphi})$ are the diagonal element of $\mathcal{L}_\varepsilon(\underline{\varphi})$.
(2) A naive attempt to construct the stable and unstable directions for S_ε would lead to the equation:

e10.3.8
$$DS_\varepsilon(\underline{\varphi})\underline{\tilde{w}}_\pm(\underline{\varphi}) = \tilde{\lambda}_\pm(\underline{\varphi})\underline{\tilde{w}}_\pm(S_\varepsilon\underline{\varphi}) \tag{10.3.8}$$

Indeed (10.3.8) can be considered as a definition of the stable and unstable directions.
(3) From the general theory we know that $\underline{\widetilde{w}}_\pm(\underline{\varphi})$ are, in general, only

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[1] Note that, by allowing the space to be non compact, we are using a definition of dynamical system slightly more general of that given in Section §1.2 and used so far.

Hölder continuous as function of $\underline{\varphi}$, so that the solution of (10.3.8) cannot be analytic in ε, unless special cancellations occur. On the other hand, if $\widetilde{w}_\pm(\varphi)$ is a solution of (10.3.8) then $\underline{w}_\pm(\underline{\varphi}) = \underline{\widetilde{w}}_\pm(H(\underline{\varphi}))$ is a solution of (10.3.7) and is, as we shall see shortly, analytic in ε. In conclusion we can say that the conjugation defined in (10.3.5) is the right one to look at the stable and unstable directions and its solution give the stable and unstable directions as functions of the "unperturbed" point $H^{-1}(\underline{\varphi})$, cf. the corresponding remarks to proposition (10.1.1).

(4) Equation (10.3.7) does not determine $\Gamma_\varepsilon(\underline{\varphi})$ and $\mathcal{K}(\underline{\psi})$ uniquely. Indeed, if $l(\underline{\varphi})$ is a non zero function from $\mathbb{T}^2$ to $\mathbb{R}$ and $\lambda_\pm(\underline{\varphi})$, $\underline{w}_\pm(S_\varepsilon\underline{\varphi})$ solve (10.3.7), then also $\bar{\lambda}_\pm(\underline{\varphi}) = \frac{l(S_\varepsilon\underline{\varphi})}{l(\underline{\varphi})}\lambda_\pm(\underline{\varphi})$ and $\underline{\bar{v}}_\pm(\underline{\varphi}) = l(\underline{\varphi})\underline{w}_\pm(\underline{\varphi})$ solve it.[2]
N10.3.2 To fix this ambiguity we will require that the diagonal elements of $\mathcal{K}(\underline{\varphi})$, on the basis $\underline{v}_\pm$, are equal to 1, *i.e.* the matrix $K(\underline{\varphi})$ is completely off diagonal.

The equation that the matrix $K(\underline{\psi})$ has to verify is

e10.3.9
$$\begin{aligned}&(S_0K(\underline{\psi}) - K(S_0\underline{\psi})S_0)_{ij} = -\varepsilon\partial_{\varphi_j} f_i(\underline{\psi} + \underline{h}(\underline{\psi}))-\\ &-\varepsilon\partial_{\varphi_s} f_i(\underline{\psi} + \underline{h}(\underline{\psi}))K(\underline{\psi})_{sj} + \Gamma_\varepsilon(\underline{\psi})_{ij} + (K(S_0\underline{\psi})\Gamma_\varepsilon(\underline{\psi}))_{ij},\end{aligned}\tag{10.3.9}$$

where $\partial_{\underline{\varphi}}$ denotes a derivative of f with respect to its original argument and repeated indices mean implicit summation (to abridge notations).

We write the above matrix equation on the basis in which S_0 and Γ_ε are diagonal, *i.e.* on the basis formed by the two eigenvectors $\underline{v}_\pm$ of S_0 in which the matrices K, Γ have been assumed to take the form

e10.3.10
$$\Gamma(\underline{\psi}) = \begin{pmatrix}\gamma_+(\underline{\psi}) & 0\\ 0 & \gamma_-(\underline{\psi})\end{pmatrix}, \qquad K(\underline{\psi}) = \begin{pmatrix}0 & k_+(\underline{\psi})\\ k_-(\underline{\psi}) & 0\end{pmatrix}. \tag{10.3.10}$$

If $\alpha = \pm$ and $\beta = -\alpha$ (10.3.9) becomes, by setting $\partial_\alpha = \underline{v}_\alpha \cdot \partial_{\underline{\varphi}}$,

e10.3.11
$$\begin{aligned}&0 = -\varepsilon\partial_\alpha f_\alpha(\underline{\varphi}) - \varepsilon K_{\beta\alpha}(\underline{\psi})\partial_\beta f_\alpha(\underline{\varphi}) + \gamma_\alpha(\underline{\psi}),\\ &(\lambda_\alpha K_{\alpha\beta}(\underline{\psi}) - \lambda_\beta K_{\alpha\beta}(S_0\underline{\psi})) =\\ &\qquad = -\varepsilon\partial_\beta f_\alpha(\underline{\varphi}) - \varepsilon K_{\alpha\beta}(\underline{\psi})\partial_\alpha f_\alpha(\underline{\varphi}) + K_{\alpha\beta}(S_0\underline{\psi})\gamma_\beta(\underline{\psi}),\end{aligned}\tag{10.3.11}$$

where $\underline{\varphi}$ means $\underline{\psi} + \underline{h}(\underline{\psi})$, $\lambda_+ = \lambda^{-1}, \lambda_- = -\lambda$ are the eigenvalues of S_0 (with $\lambda = (\sqrt{5} - 1)/2$), and $\gamma_\alpha(\underline{\psi}) = \underline{v}_\alpha \cdot \Gamma(\underline{\psi})\underline{v}_\alpha$.

Calling $k_\alpha(\underline{\psi}) = K_{\alpha\beta}(\underline{\psi})$, with $\alpha = \pm$ and $\beta = -\alpha$, we can rewrite the equations (10.3.11) as

e10.3.12
$$\begin{aligned}&\gamma_\alpha(\underline{\psi}) = \varepsilon\,\partial_\alpha f_\alpha(\underline{\varphi}) + \varepsilon\, k_\beta(\underline{\psi})\partial_\beta f_\alpha(\underline{\varphi}),\\ &k_\alpha(\underline{\psi}) + \lambda^2 k_\alpha(S_0^\alpha\underline{\psi}) =\\ &\qquad \alpha\lambda\big(-\varepsilon\partial_\beta f_\alpha(\underline{\varphi}^\alpha) - \varepsilon k_\alpha(\underline{\psi}^\alpha)\partial_\alpha f_\alpha(\underline{\varphi}^\alpha) + k_\alpha(S_0\underline{\psi}^\alpha)\gamma_\beta(\underline{\psi}^\alpha)\big),\end{aligned}\tag{10.3.12}$$

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[2] $l(\underline{\varphi})$ is sometime called a *cocycle*. Observe that the expansion coefficient on the unstable manifold with respect to the parameterization defined by $\underline{\bar{w}}_+$ is linked to that of $\underline{w}_+$ by $l(\underline{\varphi})(\underline{w}_+(\underline{\varphi}) \cdot \underline{\partial})S_\varepsilon^n(\underline{\varphi}) = (\underline{\bar{w}}_+(\underline{\varphi}) \cdot \underline{\partial})S_\varepsilon^n(\underline{\varphi})l(S_\varepsilon^n(\underline{\varphi}))$.

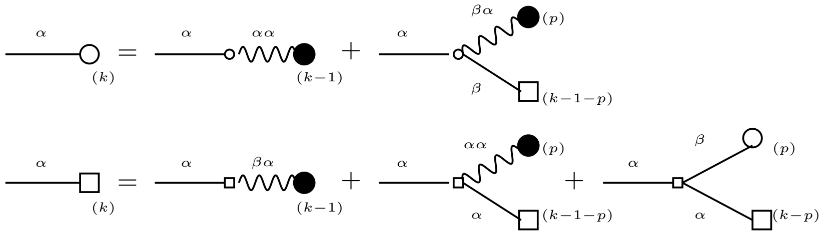
where $\alpha=\pm$ and $\underline{\varphi}$ must be thought of as denoting $\underline{\psi}+\underline{h}(\underline{\psi})$ (hence $S_\varepsilon^{-1}\underline{\varphi}$ means $S_0^{-1}\underline{\psi}+\underline{h}(S_0^{-1}\underline{\psi})$), and we have set $\underline{\psi}^\alpha=S_0^{-(1-\alpha)/2}\underline{\psi}$ and likewise $\underline{\varphi}^\alpha=S_\varepsilon^{-(1-\alpha)/2}\underline{\varphi}$. These equations are in a form suitable for a recursive solution in powers of ε. For instance the first order is

e10.3.13
$$\begin{aligned}&\gamma_\alpha^{(1)}(\underline{\psi})=\partial_\alpha f_\alpha(\underline{\psi}),\qquad \alpha=\pm,\\&k_+^{(1)}(\underline{\psi})+\lambda^2k_+^{(1)}(S_0\underline{\psi})=-\lambda\,\partial_-f_+(\underline{\psi}),\\&k_-^{(1)}(\underline{\psi})+\lambda^2k_-^{(1)}(S_0^{-1}\underline{\psi})=\lambda\,\partial_+f_-(S_0^{-1}\underline{\psi}),\end{aligned}\tag{10.3.13}$$

which has the solution

e10.3.14
$$\begin{aligned}&\gamma_\alpha^{(1)}(\underline{\psi})=\partial_\alpha f_\alpha(\underline{\psi})\qquad \alpha=\pm\\&k_+^{(1)}(\underline{\psi})=-\lambda\sum_{n=0}^\infty(-1)^n\lambda^{2n}\partial_-f_+(S_0^n\underline{\psi}),\\&k_-^{(1)}(\underline{\psi})=\lambda\sum_{n=0}^\infty(-1)^n\lambda^{2n}\partial_+f_-(S_0^{-(n+1)}\underline{\psi}).\end{aligned}\tag{10.3.14}$$

The equations (10.3.12) can be represented in graph form by suitably modifying the similar representation derived for $\underline{h}^{(k)}$ in Section §10.1:



F10.3.1 **Fig.(10.3.1)** Here $\alpha=\pm$ and $\beta=-\alpha$. All the lines have to imagined to carry arrows (not drawn) pointing toward the root. The line carrying a label α and emerging from a circle or a square with label k denotes $\gamma_\alpha^{(k)}$ or $k_\alpha^{(k)}$, respectively. The wavy line emerging from a bullet with label p, with $1\le p<k-1$, carrying a pair of labels γ,δ, represents $[\partial_\gamma f_\delta]^{(p)}$, the p-th order in the power expansion in ε of $\partial_\gamma f_\delta$ (evaluated at a point dependent on ε, see below). The small circle or square in the the last node (*i.e.* in the node closest to the root) expresses that $\underline{\varphi}$ (circle) or $S_\varepsilon^{\widetilde{q}\alpha}\underline{\varphi}^\alpha$ (square) is the argument in which the functions $\partial_\gamma f_\delta$ are computed, with $\widetilde{q}=q$ if $\delta=+$ and $\widetilde{q}=q+1$ if $\delta=-$, or it expresses that $\underline{\psi}$ (circle) or $S_0^{\widetilde{q}\alpha}\underline{\psi}^\alpha$ (square) is the argument in which the functions γ_α,k_α are computed, with $\widetilde{q}=q$ if $\alpha=+$ and $\widetilde{q}=q+1$ if $\alpha=-$ (for the square in the last graph contributing to $k_\alpha^{(k-1-p)}(\underline{\psi})$ there is an extra S_0 in the argument). Furthermore a summation over $q=0,1,\ldots$ and a multiplication by $-\alpha(-1)^q\lambda^{1+2q}$ is understood to be performed over the nodes represented as small squares.

The representation is drawn in figure (10.3.1) and the symbols are explained in the corresponding caption: the reader will recognize in them a pictorial rewriting of (10.3.12).

In this case too we can continue the expansion until in the r.h.s. of (10.3.9) all endpoints of the graph are either squares or circles carrying a label (1) (*i.e.* they represent a first order contribution to Γ or to K (circle or square) bullets representing either $\partial_\alpha f_\beta(\underline{\varphi})$ or $\partial_\alpha f_\beta(\widetilde{S_0^{q\alpha}}\underline{\varphi}^\alpha)$. Of course the latter quantities can themselves be represented by the tree expansion discussed in Section §10.1: if we do so then we obtain a full expansion in powers of ε in which the wavy lines with label p are replaced by a tree with p nodes.

The rule to construct the value of each tree graph is easily read from (10.3.9) and from the rules discussed above and in Section §10.1 to build the value of trees representing $\underline{h}$.

The estimate of the k–th order contribution is given by (10.1.16) with an extra factor N^k to take into account the extra derivatives due to the lines with two labels. Also the counting of the trees has to be modified but at the end result will be that K is expressed by a convergent series in ε for $|\varepsilon| < \varepsilon_0(\beta)$, where $\varepsilon_0(\beta)$ can be taken of the form (10.1.17) with a different numerical factor and with N replaced by N^2. The above analysis yields the following result.

$P10.3.1$ **(10.3.1) Proposition:** (Symbolic representation of the expansion rate) *Given a Markovian pavement $\mathcal{P}_0 = \{P_1, \ldots, P_n\}$ of $\mathbb{T}^2$ for S_0, let $\underline{\sigma}$ be the symbolic representation with respect to the Markov partition $\mathcal{P}_\varepsilon = H(\mathcal{P}_0)$ of a point $\underline{\varphi} = X_\varepsilon(\underline{\sigma})$. There exists $\varepsilon_0(\beta)$ such that the expansion rate $\lambda_u(\underline{\sigma})$ of S_ε along the unstable manifold of $\underline{\varphi}$, see definition (4.3.2), is defined and holomorphic in ε in the disk $|\varepsilon| < \varepsilon_0(\beta)$. As a function of $\underline{\sigma}$ it is Hölder continuous of exponent β and modulus $C(\beta)$.*

Proof: One just notes that if $\underline{\varphi} = H(\underline{\psi})$ then the unstable direction at $\underline{\varphi}$ will be $\underline{w}_+(\underline{\psi}) = \underline{v}_+ + K(\underline{\psi})\underline{v}_+$. Furthermore a Markov pavement for S_ε will be the image of a Markov pavement for S_0 under the map $\underline{\psi} \to H(\underline{\psi})$; therefore $\underline{\psi}$ evolved with S_0 and $\underline{\varphi}$ evolved with S_ε will have the *same symbolic history* $\underline{\sigma}$ on such pavements. Hence $X_\varepsilon(\underline{\sigma}) = \underline{\varphi}$ and $X_0(\underline{\sigma}) = \underline{\psi}$ if $\underline{\varphi} = \underline{\psi} + \underline{h}(\underline{\psi})$. The expansion rate along the unstable manifold will be, following the notations of definition (4.3.2), $\lambda_u(\underline{\sigma}) = e^{A_u(\underline{\sigma})}$ with

$e10.3.15$
$$A_u(\tau\underline{\sigma}) = \log\left(\big(\lambda_+ + \gamma_+(\psi)\big)\frac{|\underline{w}_+(S_0\underline{\psi})|}{|\underline{w}_+(\underline{\psi})|}\right) \tag{10.3.15}$$

where $|\underline{w}_+(\underline{\psi})| = \sqrt{1 + k_+(\psi)^2}$. The functions γ_+, k_+ are analytic in ε and Hölder continuous in $\underline{\psi}$ with a uniformly bounded modulus $C(\beta)$ if $\beta < 1$ and ε is small enough and since $\underline{\sigma}$ fixed means $\underline{\psi}$ fixed (because $\underline{\sigma}$ is the history of a point $\underline{\psi}$ on a Markov pavement for the ε–independent map S_0) we see that $A_u(\underline{\sigma})$ is analytic in ε at fixed $\underline{\sigma}$ and Hölder continuous in $\underline{\sigma}$ as well as in $\underline{\psi}$ and therefore in $\underline{\varphi}$. Hence the function $A_u(\underline{\sigma})$ generates a potential of Fisher type (cf. (4.3.16) and propositions (4.3.1) and (4.3.2) apply). ■

Remark: The above proposition allows us to conclude that the SRB distribution μ_ε will be a Gibbs state for the energy function $A_u(\underline{\sigma})$ and, therefore, it will mix at an exponential rate any pair of Hölder continuous function. Observe that if G is a Hölder continuous function then $G \circ H^{-1}$ has a representation on the Markov pavement $\mathcal{P}_\varepsilon$ that is independent of ε. Moreover Gibbs states with exponentially decaying Fisher potential have the property that the expectation values of observables depend analytically on the parameters on which the potential itself depends analytically, cf. proposition (7.3.1) ([CO81]). This can be summarized in the following corollary.

$C10.3.1$ **(10.3.1) Corollary:** (Mixing for SRB distributions)
Let F and G be two Hölder continuous "observables" (i.e. functions on $\mathbb{T}^2$). Then if μ_ε denotes the SRB distribution for S_ε the following properties hold.
(i) The expectation values $\mu_\varepsilon(F)$ and $\mu_\varepsilon(G)$ are defined and Hölder continuous in ε. Moreover the expectation values $\mu_\varepsilon(F \circ H^{-1})$ and $\mu_\varepsilon(G \circ H^{-1})$ are analytic function in ε for $|\varepsilon| \le \varepsilon_0$, with ε_0 independent of F or G.
(ii) If F is an analytic observable then the expectation value $\mu_\varepsilon(F)$ is analytic in ε for $\varepsilon \le \varepsilon_F$, with ε_F dependent on F.
(iii) The functions F, G mix at an exponential rate in the sense that the difference between the l.h.s. and the r.h.s. of

e10.3.16
$$\mu_\varepsilon((S_\varepsilon^n F)G) \equiv \int \mu_\varepsilon(d\underline{\varphi}) F(S^n \underline{\varphi}) G(\underline{\varphi}) \xrightarrow[n\to\pm\infty]{} \mu_\varepsilon(F)\mu_\varepsilon(G) \tag{10.3.16}$$

tends to 0 bounded by a constant (depending of F, G) times $e^{-\kappa_{F,G} n}$, for a suitable constant $\kappa_{F,G} > 0$.
(iv) The volume distribution $\mu_0(d\underline{\varphi}) = d\underline{\varphi}/(2\pi)^2$ "mixes with the SRB distribution" exponentially fast in the sense that given the functions F, G the difference between the l.h.s. and the r.h.s. of

e10.3.17
$$\mu_0((S_\varepsilon^n F)G) \equiv \int \mu_0(d\underline{\varphi}) F(S^n \underline{\varphi}) G(\underline{\varphi}) \xrightarrow[n\to+\infty]{} \mu_\varepsilon(F)\mu_0(G) \tag{10.3.17}$$

tends to 0 bounded by a constant (depending of F, G) times $e^{-\kappa_{F,G} n}$.
The latter limit has, in general, a different value if $n \to -\infty$ and one has to replace the SRB distribution μ_ε with the one for S_ε^{-1} (which essentially has the same properties as μ_ε).

We can therefore appreciate the power of the perturbation expansion method which allows us to obtain rather detailed and precise informations about the Markov partition (which of course can be defined as the H–image of the trivial partition for Arnold's cat map S_0); and, what is more important, it provides us with an analytic description of the SRB distribution (in the considered small perturbation problem).

In particular it is interesting to compute the derivatives of $\mu_\varepsilon(F)$ for $\varepsilon = 0$. Since $\mu_\varepsilon = \lim_{N\to\infty} \mu_\varepsilon^{(N)}$, with $\mu_\varepsilon^{(N)}$ being the finite volume Gibbs distribution in volume $\Lambda_N = [-N, N]$ with potential energy function $A(\underline{\sigma}) =$

$\lambda_u(\underline{\sigma})$, cf. (6.1.15), with Δ_{Λ_N} chosen to correspond to periodic boundary conditions; then the derivative at $\varepsilon = 0$ of $\mu_\varepsilon^{(N)}(F)$ is

e10.3.18
$$\begin{aligned}&\mu_0^{(N)}(\underline{\partial}F\cdot\underline{h}^{(1)}) - \lambda_+\sum_{k=-N}^{N}(\mu_0^{(N)}(-(\partial_+f_+)\circ S_0^kF)-\\&-\mu_0^{(N)}(-(\partial_+f_+)\circ S_0^k)\mu_0^{(N)}(F)),\end{aligned}\tag{10.3.18}$$

because the first order of the derivative of $\lambda_u(\underline{\sigma})$ is (in the $\underline{\psi}$ coordinates $\partial_+f_+(\underline{\psi})$ (cf. (10.3.12))). But $\mu_0^{(N)}(-(\partial_+f_+)\circ S_0^k)\xrightarrow[N\to\infty]{}0$ because $\mu_0(\partial_+f_+)\equiv 0$. Hence, up to an interchange of limits which it is not difficult to justify, the derivative of $\mu_\varepsilon(F)$ is given by $\mu_0(\underline{h}^{(1)}\cdot\underline{\partial}F) + \lambda_+\sum_{k=-\infty}^{\infty}\mu_0(\partial_+f_+F\circ S_0^k)$. Note that

e10.3.19
$$\begin{aligned}\mu_0(\underline{h}^{(1)}\cdot\underline{\partial}F) =& \sum_{-\infty}^{k=0}\mu_0(\lambda_+^{k-1}f_+\partial_+F\circ S_0^k) - \sum_{k=1}^{\infty}\mu_0(\lambda_-^{k-1}f_-\partial_-F\circ S_0^k)\\=&\sum_{-\infty}^{k=-1}\mu_0(f_+\circ S_0^{-1}\partial_+(F\circ S_0^k))-\\&\sum_{k=0}^{\infty}\mu_0(f_-\circ S_0^{-1}\partial_-(F\circ S_0^k))\end{aligned}\tag{10.3.19}$$

We can now write $\sum_{k=-\infty}^{\infty}\mu_0(\partial_+f_+F\circ S_0^k) = \sum_{k=-\infty}^{\infty}\mu_0(\partial_+(f_+F\circ S_0^k)) - \sum_{k=-\infty}^{\infty}\mu_0(f_+\partial_+(F\circ S_0^k))$. After further elaboration and taking into account that $\mu_0(\partial_+G) = 0$ for every differentiable G, we get $\lambda_+\sum_{k=-\infty}^{\infty}\mu_0(\partial_+f_+F\circ S_0^k) = \sum_{k=-\infty}^{\infty}\mu_0(f_+\circ S_0^{-1}\partial_+(F\circ S_0^k))$ so that

e10.3.20
$$\partial_\varepsilon\mu_\varepsilon(F)\big|_{\varepsilon=0} = -\sum_{k=0}^{\infty}\mu_0(\underline{\partial}(F\circ S_0^k)\cdot\underline{f}\circ S_0^{-1}).\tag{10.3.20}$$

This formula was proved, in a much more general setting, by Ruelle, see [Ru97], and is very interesting for applications because it closely resembles the standard Green-Kubo formula.[GR97]

Remark: We note that once $\underline{w}_+(\underline{\varphi})$ is known we can construct the unstable manifold of a point $\underline{\varphi}_0$ solving the differential equation

e10.3.21
$$\begin{cases}\dot{\mathcal{X}}(t,\underline{\varphi}_0) = \widetilde{\underline{w}}_+(\mathcal{X}(t,\underline{\varphi}_0)) = \underline{w}_+(H^{-1}(\mathcal{X}(t,\underline{\varphi}_0))),\\ \mathcal{X}(0,\underline{\varphi}_0) = \underline{\varphi}_0,\end{cases}\tag{10.3.21}$$

where $\mathcal{X}(t,\underline{\varphi}_0)$ is a parameterization of the unstable manifold $W^u(\underline{\varphi}_0)$. Clearly this parameterization is differentiable in t. From Section §4.2 we know that $W^u(\underline{\varphi}_0)$ is at least a C^∞ manifold so that it should be possible to find a parameterization much smoother than the one given by (10.3.21).

Moreover (10.3.21) does not allow us to discuss the smoothness of $\mathcal{X}(t,\underline{\varphi})$ as a function of ε.

A general theory for $\mathcal{X}(t,\underline{\varphi})$ on the lines developed in this section can be achieved by generalizing equation (10.3.4) or (10.3.7), see problem [10.3.6].

Problems for §10.3

Q10.3.1 **[10.3.1]:** (Regularity of the SRB measure for analytic f.)
Extend the result of this section to the case of an analytic $\underline{f}$. (*Hint:* See problem [10.1.4]).

Q10.3.2 **[10.3.2]:** (*Generalization of (10.3.5) to the nonlinear part of* $W^+(\underline{\psi})$*: an attempt.*) Let $\widetilde{S}_\varepsilon$ be defined on $\mathbb{T}^2\times\mathbb{R}^2$ by

$$\widetilde{S}_\varepsilon(\underline{\varphi},\underline{v}) = (S_\varepsilon(\underline{\varphi}), S_\varepsilon(\underline{\varphi}+\underline{v}) - S_\varepsilon(\underline{\varphi}))$$

and $\widehat{S}_{0,\varepsilon}$ by (10.3.4). We can look for a conjugation $\widetilde{H}$ of the form

$$\widetilde{H}\,:\,(\underline{\psi},\underline{w}) \longleftrightarrow (\underline{\varphi},\underline{v}) = (\underline{\psi}+\underline{h}(\underline{\psi}), \mathcal{X}(\underline{\psi},\underline{w})). \qquad (*)$$

Show that, if $\widetilde{H}$ satisfies $\widetilde{H}\circ\widehat{S}_{0,\varepsilon} = \widetilde{S}_\varepsilon\circ\widetilde{H}$ then, $\underline{\partial}\mathcal{X}(\underline{\psi},0)\underline{w} = \underline{w} + K(\underline{\psi})\underline{w}$ with $K(\underline{\psi})$ defined by (10.3.5).

Q10.3.3 **[10.3.3]:** (*Failure of the attempt in problem [10.3.2].*)
Write an equation for the $n+1$-order tensor $\underline{\partial}^n\mathcal{X}(\psi,0)$ for $n=2$, 3. Expand this equation in series of ε and consider the equation for $\underline{\partial}^n\mathcal{X}^{(0)}(\psi,0)$. Are these equations solvable? (*Hint:* Check the equation for $\partial_+^2\partial_-\mathcal{X}_+(\psi,0)$.)

Q10.3.4 **[10.3.4]:** (*Tree expansion for the nonlinear part of* $W^+(\psi)$.)
Let $\overline{S}_{0,\varepsilon}$ be the restriction of $\widetilde{S}_\varepsilon$ to $\mathbb{T}^2\times\mathbb{R}v_+$, *i.e.*

$$\overline{S}_{0,\varepsilon}(\underline{\varphi},t) = (S_0(\underline{\varphi}), (\lambda_+ + \gamma_+(\underline{\varphi}))t)$$

and analogously

$$\overline{H}(\underline{\psi},t) = (\underline{\psi}+\underline{h}(\underline{\psi}), \mathcal{X}(\underline{\psi},t)),$$

where we want that

$$\overline{H}\circ\overline{S}_{0,\varepsilon} = \widetilde{S}_\varepsilon\circ\overline{H}.$$

Note that both sides of the equation represent a function from $\mathbb{T}^2\times\mathbb{R}v_+$ to $\mathbb{T}^2\times\mathbb{R}^2$. Assume that

$$\mathcal{X}(\underline{\psi},t) = \sum_{n=1}^\infty\sum_{k=0}^\infty \mathcal{X}^{(n,k)}(\underline{\psi})t^n\varepsilon^k.$$

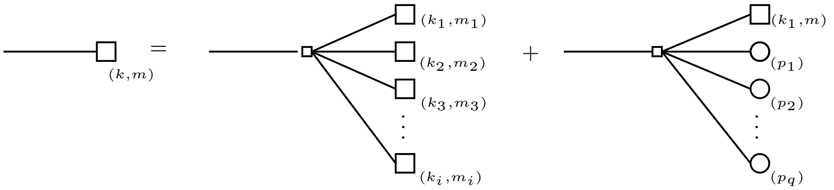
Write a tree expansion for $\mathcal{X}^{(n,k)}(\underline{\psi})$ and prove that $\mathcal{X}(\underline{\psi},t)$ is Hölder continuous of exponent β in $\underline{\psi}$ and analytic in t and ε for $|t|\le t_0$ and $|\varepsilon|\le\varepsilon(\beta)$.(*Hint:* We have that $\mathcal{X}(\underline{\psi},t)$ satisfies the equation:

$$S_\varepsilon(H(\underline{\psi})+\mathcal{X}(\underline{\psi},t)) = H(S_0(\underline{\psi})) + \mathcal{X}(S_0(\underline{\psi}),\lambda_{+,\varepsilon}(\underline{\psi})t).$$

Expanding in serie of ε and t we get:

$$\begin{aligned}
&\lambda^m\mathcal{X}^{(m,k)}(\underline{\psi}) - \mathcal{X}^{(m,k)}(S_0(\underline{\psi})) = \\
&\qquad \lambda^m\sum_i \frac{D^i f(\underline{\psi})}{i!} \sum_{\substack{m_1+m_2+\cdots+m_i=m\\ k_1+k_2+\cdots+k_i=k-1}} \mathcal{X}^{(m_1,k_1)}(\underline{\psi})\mathcal{X}^{(m_2,k_2)}(\underline{\psi})\cdots\mathcal{X}^{(m_i,k_i)}(\underline{\psi})+ \\
&\qquad +\sum_{q\le m}\lambda^q\sum_p \mathcal{X}^{(m,k-p)}(\underline{\psi}) \sum_{p_1+p_2+\cdots+p_q=p} \lambda_+^{(p_1)}(\underline{\psi})\lambda_+^{(p_1)}(\underline{\psi})\cdots\lambda_+^{(p_q)}(\underline{\psi})
\end{aligned}$$

where $k_i \geq 1$, $m_i \geq 0$, $p_i \geq 0$. Finally $\lambda_+^{(0)}(\underline{\psi}) = \lambda_+$ and $\mathcal{X}^{(0,k)}(\underline{\psi}) = h^{(k)}(\underline{\psi})$. Moreover a proper contraction of the index of the tensor $D^i f(\underline{\psi})$ with the component of $\mathcal{X}^{(m_i,k_i)}(\underline{\psi})$ in understood. This can be represented graphically as in the following figure.



F10.3.2 **Fig.(10.3.2)** Graphical representation of the expansion for $\mathcal{X}^{(m,k)}(\underline{\psi})$. We call the second graphical element on the right hand side a *counterterm vertex*.

The representation for $\lambda_+^{(p)}$ is similar to the one in figure (10.3.1) where we must replace the bullet with label p with a square with label $(p,0)$, the wavy line with a line and the square as to be considered with label $(p,1)$.

We can immagine that the small square in the vertex carry a label m equal to the sum of the m labels of the square entering in it. With this convention the line exiting from a small square with label $m \neq 1$ represent the operator $\lambda^m(\mathbf{T}_m^{-1})$ where

$$\mathbf{T}_m \mathcal{X}^{(m)}(\underline{\psi}) = \lambda^m \mathcal{X}^{(m)}(\underline{\psi}) - \mathcal{X}^{(m)}(S_0(\underline{\psi}))$$

The inverse of $\mathbf{T}_m$ for $m = 0$ was computed while studying the conjugation H while the cases $m \geq 2$ can be easily solved and do not require to divide $\mathcal{X}^{(m)}$ in its $+$ and $-$ components.

The case $m = 1$ need a particular treatment. In this case the line exiting from a small square with label 1 represent the operator $\lambda \mathbf{T}_1^{-1}\mathbf{P}_+$ where $\mathbf{P}_+$ is the projection on the $+$ direction. In the same way the line exiting from a small circe (that has necessarily a label 1) represent $\lambda \mathbf{P}_-$, see figure (10.3.1). We can now iterate the graphical equation of figure (10.3.2) untill all square have label $(m,1)$ or all circle have label 1. The value of the square with label $(m,1)$ for $m =$ 0,1 and for the circle with $p = 1$ has already been computed. For $m \geq 2$ it easy to find that

$$\mathcal{X}^{(m,1)}(\underline{\psi}) = \sum_{n=0}^{\infty} \lambda^{mn} S_0^n \partial_+^m f(S_0^{-n-1}(\underline{\psi})).$$

The analysis of the convergence goes has in the case of the conjugation. Note that the a counterterm vertex has no ε factor associated but does not contribute a $C^q q!$ to the estimates because it bear no derivative of f. Moreover the number of operator $\mathbf{T}_m^{-1}$ associated with a tree is equal to the number of vertex with a small square plus the number of final points, *i.e.* it is equal to k if the tree contributes to $\mathcal{X}^{(m,k)}$. Finally it is easy to check that $\|\mathbf{T}_m^{-1}\| \leq CK^m$ for suitable constant C and K. The only thing left is to estimate the number of tree that contributes to $\mathcal{X}^{(m,k)}$. To do this one can immagine that to every final point with label $(m,1)$ are attached m incoming lines with a final points with label $(1,0)$. This reduces the estimates on the number of tree to the case already studied.)

Q10.3.5 **[10.3.5]:** Show that equation $(*)$ in problem [10.3.2] is equivalent to the equation

$$S_\varepsilon(H(\underline{\varphi}) + \mathcal{X}(\underline{\varphi},t)) = H(S_0\underline{\varphi}) + \mathcal{X}(S_0\underline{\varphi}, \lambda_+(\underline{\varphi})t), \qquad (**)$$

where $\lambda_+(\underline{\varphi}) = \lambda_+ + \gamma_+(\underline{\varphi})$. A natural definition of the unstable manifold would be given by

$$S_\varepsilon(\underline{\varphi} + \widetilde{\mathcal{X}}(\underline{\varphi},t)) = S_0\underline{\varphi} + \mathcal{X}(S_0\underline{\varphi}, \widetilde{\lambda}_+(\underline{\varphi})t).$$

What is the relation between $\widetilde{\mathcal{X}}(\underline{\varphi},t)$ and $\mathcal{X}(\underline{\varphi},t)$?

Q10.3.6 **[10.3.6]:** (*Extension of the parametrization of* $W^+(\psi)$.)
Use equation $(**)$ of problem [10.3.5] to show that $\mathcal{X}(\underline{\varphi},t)$ can be extended to an analytic function of t in a complex strip around the real axis. How large is this strip? (*Hint:* Use $(**)$ to extend the domain of analyticity of $\mathcal{X}(\underline{\varphi},t)$ by a factor $\lambda_+(\underline{\varphi})$. Note that $\mathcal{X}(\underline{\varphi},t)$ should be in the domain of analyticity of $\underline{f}(\underline{\varphi})$.)

§10.4 Chaos in space–time and SRB distributions

The theory of Section §10.2 can be extended along the lines of Section §10.3 to lattices of "coupled maps" thereby answering various natural questions.

Adopting the notations of Section §10.2 we consider a lattice of Arnold's cat maps. Following the method of Section §10.3 we can construct the unstable manifold by studying the dynamical system (on a noncompact phase space $\widetilde{\Omega}_\Lambda = (\mathbb{T}^2\times\mathbb{R}^2)^\Lambda$, see footnote 1)

e10.4.1 $$(\mathcal{S}_\varepsilon(\underline{\varphi},\underline{v}))_\xi = \big(S_0\underline{\varphi}_\xi - \varepsilon\underline{f}(\underline{\varphi}_{nn(\xi)}), S_0\underline{v}_\xi - \varepsilon\underset{\sim}{v}_\xi\cdot\partial_{\underline{\varphi}_\xi}\underline{f}(\underline{\varphi}_{nn(\xi)})\big). \tag{10.4.1}$$

Therefore a point $\underline{\varphi}$ in $(\mathbb{T}^2)^\Lambda$ can be identified by assigning for each $\xi\in\Lambda$ a pair of coordinates $\underline{\varphi}_\xi$ labeled by ξ which identify a point in $\mathbb{T}^2$; and a vector $\underline{w}$ in $(\mathbb{R}^2)^\Lambda$ can be identifies by assigning $|\Lambda|$ two-component vectors $\underline{w}_\xi$ labeled by a point in Λ.

We construct functions $\underline{h}(\underline{\psi})$ and $K(\underline{\psi})$ so that

e10.4.2 $$\begin{aligned}\underline{\varphi}_\xi =& \underline{\psi}_\xi + \underline{h}_\xi(\underline{\psi}) \overset{def}{=} H_\xi(\underline{\psi}),\\ \underline{v}_\xi =& \underline{w}_\xi + (K(\underline{\psi})\underline{w})_\xi\end{aligned} \tag{10.4.2}$$

defines a map $\widetilde{H}$ of $\widetilde{\Omega}_\Lambda$ into itself, where now $\underline{h}(\underline{\psi})$ is a function defined on $(\mathbb{T}^2)^\Lambda$ with values in $(\mathbb{T}^2)^\Lambda$ and K is defined on $(\mathbb{T}^2)^\Lambda$ with values in the $2|\Lambda|\times 2|\Lambda|$ matrices mapping $(\mathbb{R}^2)^\Lambda$ into itself. The construction is achieved by imposing that the map $\widetilde{H}$ transforms $\widetilde{\mathcal{S}}_\varepsilon$ into $\widetilde{\mathcal{S}}_{0,\varepsilon}$, *i.e.*

e10.4.3 $$\begin{aligned}&\widetilde{\mathcal{S}}_\varepsilon\circ\widetilde{H} = \widetilde{H}\circ\widetilde{\mathcal{S}}_{0,\varepsilon}, \qquad \text{with}\\ &\widetilde{\mathcal{S}}_{0,\varepsilon}(\underline{\psi},\underline{w})_\xi = \big(S_0\underline{\psi}_\xi, S_0\underline{w}_\xi + (\Gamma(\underline{\psi})\underline{w})_\xi\big),\end{aligned} \tag{10.4.3}$$

with $\Gamma(\underline{\psi})$ depending on ε. Let $\underline{B}$ be the basis formed by the vectors $(\underline{0},\ldots,\underline{v}_\pm,\ldots,\underline{0})$ in the $2|\Lambda|$–dimensional space $(\mathbb{R}^2)^\Lambda$. We suppose that, in the basis $\underline{B}$, the matrix $K(\underline{\psi})_{\xi i,\eta j}$ has vanishing matrix elements except for $K_{\xi+,\eta-}(\underline{\psi}) \overset{def}{=} k_{\xi\eta,+}(\underline{\psi})$ and $K_{\xi-,\eta+}(\underline{\psi}) \overset{def}{=} k_{\xi\eta,-}(\underline{\psi})$. The matrix Γ will be supposed to have zero matrix elements except for $\Gamma_{\xi\alpha,\eta\alpha}(\underline{\psi}) = \gamma_{\xi\eta,\alpha}$, $\alpha=\pm$. This implies that e10.4.4

$$\begin{aligned}(K(\underline{\psi})\underline{w})_{\xi,\pm} &= \sum_\eta k_{\xi\eta,\pm}(\underline{\psi})\underline{w}_{\eta,\mp}\\ (\Gamma(\underline{\psi})\underline{w})_{\xi,\pm} &= \sum_\eta \gamma_{\xi\eta,\pm}(\underline{\psi})\underline{w}_{\eta,\pm},\end{aligned}\tag{10.4.4}$$

i.e. for each ξ,η, Γ is an ε–dependent diagonal matrix on the basis $\underline{v}_\pm$ of the eigenvectors of $\mathcal{S}_0$ and K is an ε–dependent off-diagonal matrix on the same basis. Inspired by the results of Section §10.3 we shall also try to show that $\widetilde{H}(\underline{\psi})$ be expressible as

e10.4.5

$$\begin{aligned}\underline{h}_\xi(\underline{\psi}) &= \sum_{X\ni\xi}\Phi_X(\underline{\psi}_X),\\ (K(\underline{\psi})\underline{w})_{\xi,\pm} &= \sum_{\xi'}\sum_{X\ni\xi,\,\xi'}\gamma_{X,\pm}(\underline{\psi}_X)(\underline{w})_{\xi',\pm}\\ (K(\underline{\psi})\underline{w})_{\xi,\pm} &= \sum_{\xi'}\sum_{X\ni\xi,\,\xi'}k_{X,\pm}(\underline{\psi}_X)(\underline{w})_{\xi',\mp},\end{aligned}\tag{10.4.5}$$

where, of course, we have already determined $\underline{h}$ in Section §10.2. The equations that should be obeyed by $\widetilde{H}$ become, cf. (10.3.9),

e10.4.6

$$\begin{aligned}&(\mathcal{S}_0K(\underline{\psi})-K(\mathcal{S}_0\underline{\psi})\mathcal{S}_0)_{\xi i,\eta j} = -\varepsilon\partial_{\varphi_{\eta,j}}f_i(\underline{\varphi}_{nn(\xi)})-\\ &-\varepsilon\partial_{\underline{\varphi}_{\rho,s}}f_i(\underline{\varphi}_{nn(\xi)})K(\underline{\psi})_{\rho s,\eta j}+\Gamma(\underline{\psi})_{\xi i,\eta j}+(K(\mathcal{S}_0\underline{\psi})\Gamma(\underline{\psi}))_{\xi i,\eta j},\end{aligned}\tag{10.4.6}$$

where $\underline{\varphi}=\underline{\psi}+\underline{h}(\underline{\psi})$ and summation over the repeated indices ρ,s is understood. The labels i,j have values $\pm$ as we imagine that the equation is written in the basis $\underline{B}$.

The equations can be solved by writing them by components; defining

e10.4.7

$$\begin{aligned}&\underline{\psi}^{(\alpha)}\overset{def}{=}\mathcal{S}_0^{-(1-\alpha)/2}\underline{\psi},\quad \underline{\psi}'^{(\alpha)}=\mathcal{S}_0^{(1+\alpha)/2}\underline{\psi}, &&\alpha=\pm,\\ &\underline{\varphi}^{(\alpha)}=\mathcal{S}_\varepsilon^{-(1-\alpha)/2}\underline{\varphi}, &&\alpha=\pm,\end{aligned}\tag{10.4.7}$$

one finds, for $\alpha=\pm,\beta=-\alpha$,

e10.4.8

$$\begin{aligned}\gamma_{\xi\eta,\alpha}(\underline{\psi}) =&\varepsilon\partial_{\underline{\varphi}_{\eta,\alpha}}f_\alpha(\underline{\varphi}_{nn(\xi)})+\varepsilon\partial_{\underline{\varphi}_{\rho,\beta}}f_\alpha(\underline{\varphi}_{nn(\rho)})k_{\rho\eta,\beta}(\underline{\psi}),\\ k_{\xi\eta,\alpha}(\underline{\psi}) =&-\lambda^2k_{\xi\eta,\alpha}(\mathcal{S}_0^\alpha\underline{\psi})-\alpha\lambda\left[\varepsilon\partial_{\varphi_{\eta,\beta}}f_\alpha(\underline{\varphi}^{(\alpha)}_{nn(\xi)})+\right.\\ &+k_{\xi\rho,\alpha}(\underline{\psi}'^{(\alpha)})\gamma_{\rho\eta,\beta}(\underline{\psi}^{(\alpha)})+\\ &\left.+\varepsilon k_{\rho\eta,\beta}(\underline{\psi}^{(\alpha)})\partial_{\varphi_{\rho,\beta}}f_\alpha(\underline{\varphi}^{(\alpha)}_{nn(\xi)})\right],\end{aligned}\tag{10.4.8}$$

where $\lambda=\lambda_+^{-1}=-\lambda_-=(\sqrt{5}-1)/2<1$. Here the derivatives of $\underline{f}$ are meant to be performed with respect to the argument $\underline{\varphi}$ of $\underline{f}$ (*i.e.* to be precise one should everywhere write $(\partial_{\underline{\varphi}_{\xi,\alpha}}f_{\alpha'})$ rather than writing $\partial_{\underline{\varphi}_{\xi,\alpha}}f_{\alpha'}$ without the parentheses). This has the consequence that values $\gamma_{\xi\eta},k_{\xi\eta}$ will be of order $O(\varepsilon^{|\xi-\eta|})$, because the recursion for γ, k involves only nearest neighbors.

Proceeding as in Section §10.3 we can interpret (10.4.8) in terms of suitable tree graphs by simply adding a few more labels to the graphs in Fig. (10.3.1) and we obtain the following result.

P10.4.1 **(10.4.1) Proposition:** (Stable and unstable planes for lattices of cat maps)
Given a nearest neighbors interaction $\underline{f}$ *as above and* $0<\beta<1$ *there exist* $\varepsilon_0(\beta)>0$, $C(\beta)<\infty$ *and* $\kappa(\beta)>0$ *as in proposition (10.2.1), and functions* Φ_X *with values in* $(\mathbb{T}^2)^\Lambda$ *and* $\gamma_{X,\alpha}$, $k_{X,\alpha}$ *with values in the* $2|\Lambda|\times 2|\Lambda|$ *matrices such that defining* $\underline{h}$, K *and* Γ *by (10.4.5) and therefore* $\widetilde{H}$ *by (10.4.2) the following results hold.*
(i) Equation (10.4.3) holds, i.e. $\widetilde{H}$ *conjugates* $\widetilde{\mathcal{S}}_\varepsilon$ *with* $\widetilde{\mathcal{S}}_0$.
(ii) The functions Φ_X *verify the bounds in (10.2.16) and the functions* $\gamma_{X,\alpha}$ *and* $k_{X,\alpha}$ *verify, if* $\kappa_{X,\alpha}$ *is either* $\gamma_{X,\alpha}$ *or* $k_{X,\alpha}$*:*

e10.4.9
$$\max_{\substack{|\varepsilon|\le\varepsilon_0(\beta)\\ \underline{\psi}_X}} |\kappa_{X,\alpha}(\underline{\psi}_X)| < B(\beta)\Big(\frac{|\varepsilon|}{2\varepsilon_0(\beta)}\Big)^{\frac{\delta(X)}{2d}}, \tag{10.4.9}$$
$$\max_{|\varepsilon|\le\varepsilon_0(\beta),\underline{\psi}_{X\setminus\xi}} |\kappa_X(\underline{\psi}_X)-\kappa_X(\underline{\psi}'_X)| < B(\beta)\Big(\frac{|\varepsilon|}{2\varepsilon_0(\beta)}\Big)^{\frac{\delta(X)}{2d}} |\underline{\psi}_\xi-\underline{\psi}'_\xi|^\beta,$$

where $\underline{\psi}_X, \underline{\psi}'_X$ *differ only in the site* $\xi\in X$ *and* $\delta(X)$ *denotes the tree length of the set* X.

It is now immediate to define a basis of tangent vectors to the unstable manifold at a point $\underline{\varphi}=\underline{\psi}+\underline{h}(\underline{\psi})$. If $\underline{v}_{\xi,+}=(\underline{v}_+\delta_{\xi'\xi})_{\xi'\in\Lambda}$ is, as ξ varies in Λ, a basis of vectors for the unstable manifold of the unperturbed system then a basis for the perturbed system will be given by the $|\Lambda|$ vectors

e10.4.10
$$(\underline{w}_{\xi,+}(\underline{\psi}))_{\xi'}=\underline{v}_+\delta_{\xi\xi'}+(K(\underline{\psi})\underline{v}_{\xi,+})_{\xi'},\qquad \xi\in\Lambda. \tag{10.4.10}$$

The situation is similar to the one discussed in Section §10.3: a Markovian pavement is constructed as the H–image of a fixed Markovian pavement for $\mathcal{S}_0$. The latter is simply obtained by fixing a generating Markov pavement $\mathcal{P}$ for the simple Arnold's cat map (see Section §4.2) and then defining the pavement $\mathcal{P}^\Lambda$ obtained as "product" of copies of the pavement $\mathcal{P}$ on each factor of the product $(\mathbb{T}^2)^\Lambda$. If $\underline{\varphi}=\underline{\psi}+\underline{h}(\underline{\psi})$ then $\underline{\varphi}$ and $\underline{\psi}$ have symbolic histories $X_\varepsilon(\underline{\varphi})$ and $X_0(\underline{\psi})$ which coincide, by construction:

e10.4.11
$$\underline{\varphi}=X_\varepsilon(\underline{\sigma}),\qquad \underline{\psi}=X_0(\underline{\sigma}),\qquad \underline{\varphi}=\underline{\psi}+\underline{h}(\underline{\psi}). \tag{10.4.11}$$

Therefore it is now interesting to evaluate the volume of the parallelepiped spanned by the $|\Lambda|$ vectors $\underline{w}_{\xi,+}(\underline{\psi})$ as ξ varies in Λ. The latter is

e10.4.12
$$D(\underline{\psi})=|\mathrm{Vol}(\underline{w}_{\xi_1,+}(\underline{\psi}),\ldots,\underline{w}_{\xi_{|\Lambda|},+}(\underline{\psi}))|=\det\Big((1+K_{++})(1+K_{++}^T)\Big)^{\frac12}, \tag{10.4.12}$$

where K_{++} denotes the linear operator $K(\underline{\psi})$ regarded as a map between the linear space spanned by the vectors $\underline{v}_{\xi,+}$, *i.e.* the tangent plane P_+ to the unstable manifold of $\mathcal{S}^0$ and its image $1+K(\underline{\psi}))P_+$, and K^T_{++} denotes its adjoint.

The volume of the image under $\widetilde{\mathcal{S}}_\varepsilon$ of the latter plane P_+ is by (10.4.3) the volume $D'(\underline{\psi})$ of the parallelepiped spanned by the vectors $\underline{w}'_{\xi,+} \overset{def}{=} (1+K(\mathcal{S}_0\underline{\psi}))(\lambda_+ + \Gamma_+(\underline{\psi}))\underline{v}_{\xi,+}$, hence[1] N10.4.1

e10.4.13
$$\begin{aligned}D'(\underline{\psi}) &= |\mathrm{Vol}(\underline{w}'_{\xi_1,+}(\underline{\psi}),\ldots,\underline{w}'_{\xi_{|\Lambda|},+}(\underline{\psi}))| =\\ &= D(\mathcal{S}_0\underline{\psi})\det\Big((\lambda_+ + \Gamma^T_+(\underline{\psi}))(\lambda_+ + \Gamma_+(\underline{\psi}))\Big)^{\frac{1}{2}},\end{aligned} \tag{10.4.13}$$

where $\Gamma^T_+(\underline{\psi})$ is the transpose of $\Gamma_+(\underline{\psi})$ defined as the restriction to the plane P_+ of the matrix $\Gamma(\underline{\psi})$ (quite simple as the latter matrix is $\Gamma_{\xi\alpha,\eta\beta} = \Gamma_{\xi,\eta,\alpha}\delta_{\alpha\beta}$ (*i.e.* diagonal in the local label $\alpha=\pm$) so that $\mathcal{S}_0+\Gamma^T$ is a $|\Lambda|\times|\Lambda|$ matrix. The matrix $\widetilde{M} = ((1+\lambda_+^{-1}\Gamma^T)((1+\lambda_+^{-1}\Gamma))^{\frac{1}{2}}$ has the form, as it is implied by (10.4.9),

e10.4.14
$$\widetilde{M}_{\xi,\xi'}(\underline{\psi}) = \sum_{X\subset\Lambda\,:\,\xi,\xi'\in X} \widetilde{m}_X(\underline{\psi}), \tag{10.4.14}$$

with the functions $m_X(\underline{\psi})$ admitting the bounds (10.4.9) with k_X replaced with m_X.

By construction the differential of $\mathcal{S}_\varepsilon(\underline{\psi}+\underline{h}(\underline{\psi})))$ on the bases generated by $\underline{w}_{\xi,+}(\underline{\psi})$ and $\underline{w}_{\xi,+}(\mathcal{S}_0\underline{\psi})$ is given by $\lambda_+ + \Gamma_+(\underline{\psi})$ so that the expansion coefficient of the surface area of the unstable manifold is given by

e10.4.15
$$\lambda_+^{|\Lambda|}\frac{D(\mathcal{S}_0\underline{\psi})}{D(\underline{\psi})}D_0(\underline{\psi}). \tag{10.4.15}$$

where $D_0(\underline{\psi}) = \det\widetilde{M}(\underline{\psi}) \equiv \det(1+\widetilde{\Gamma}_+(\underline{\psi}))$, and $\widetilde{\Gamma}_+(\underline{\psi})$ can be written in a form analogous to (10.4.14).

For the purpose of constructing the SRB distribution we can ignore the "cocycle" ratio $\frac{D(\mathcal{S}_0\underline{\psi})}{D(\underline{\psi})}$ and the potential energy for the SRB distribution is simply $A(\underline{\psi}) = -\log\det(1+\widetilde{\Gamma}_+(\underline{\psi}))$. Equations (10.4.9) imply that the determinant $D_0(\underline{\psi})$ can be expressed as

e10.4.16
$$\exp\Big(\sum_{X\subset\Lambda} p_X(\underline{\psi}_X)\Big). \tag{10.4.16}$$

Moreover $p_X(\underline{\psi}_X)$ still verify the bounds (10.4.9) with κ_X replaced by $p_X(\underline{\psi}_X)$. This follows from the form of the matrices K, Γ in (10.4.14) and from the following lemma.

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[1] Here we note that the n–dimensional volume of the parallelepiped generated by n vectors $\underline{w}_1,\ldots,\underline{w}_n$ in $\mathbb{R}^m$, $m\ge n$, is the square root of the determinant of the matrix $(\underline{w}_i\cdot\underline{w}_j), i,j=1,\ldots,n$.

L10.4.1 **(10.4.1) Lemma:** (Cluster expansion of a determinant)
Let $m_X(\underline{\psi}_X)$ *be a* $2|\Lambda|\times 2|\Lambda|$ *matrix verifying the bounds (10.4.9) with* m_X *replacing* κ_X*. Then if* $M=\sum_{X\subset\Lambda} m_X(\underline{\psi}_X)$ *there exist scalars* $n_X(\underline{\psi}_X)$ *such that*

e10.4.17
$$\det(1+M)=\exp\Big(\sum_{X\subset\Lambda} n_X(\underline{\psi}_X)\Big), \tag{10.4.17}$$

and the functions $n_X(\underline{\psi}_X)$ *verify the same bounds with a different constant* B' *instead of* B*.*

Remark: To check this property note that the determinant of $1+M$ can be written

e10.4.18
$$\det(1+M)=e^{\mathrm{Tr}\,\log(1+M)}=e^{\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\mathrm{Tr}\,M^k}, \tag{10.4.18}$$

and $\mathrm{Tr}\,M^k=\sum_{X_1,\ldots,X_k}\mathrm{Tr}(m_{X_1}\cdots m_{X_k})$ has the desired form provided one collects together all terms whose X_j's form a connected set X and consider the result as a contribution to the value of n_X. The sum over k gives no problem because the terms of order k that arise in this way and which have the same X have size bounded at least by $(B''|\varepsilon|)^k$ for a suitable B'' because of the bounds (10.4.9).

Note that the sum in the exponent (10.4.17) is to be expected to be of size of the order of $|\Lambda|$ in spite of the bounds on n_X. And the matrix M is also to be expected to have large size. Since the (10.4.18) is a power series expansion one may be worried that the expression (10.4.17) is only formal and that it is affected by convergence problems. Imagine that instead of $\det(1+M)$ we consider $\det(1+\vartheta M)$ with ϑ a parameter. Then for ϑ small enough (at Λ fixed) the above formal calculation applied to $\det(1+\vartheta M)$ is certainly correct and n_X will become ϑ–dependent remaining bounded by $2^{\delta(X)}$ times the same quantities in (10.4.9) (with a B' replacing B) not only for ϑ small but also for $|\vartheta|\le 2$ and complex. Hence the relation (10.4.17) equates a polynomial in ϑ to the exponential of a function which is analytic for $|\vartheta|<2$: therefore setting $\vartheta=1$ we see that (10.4.17) is rigorously established.

We now have all the ingredients to discuss the SRB distribution for the coupled system. If $\mathcal{P}=\{P_1,\ldots,P_n\}$ is a Markov pavement for the map S_0 then we can construct a Markov partition for $\mathcal{S}_0$ simply considering, for every $\underline{\tau}\in\{1,\ldots,n\}^{\mathbb{Z}^d}$, the square $Q_{\underline{\tau}}=\times_{\xi\in\mathbb{Z}^d}P_{\underline{\tau}_\xi}$. Associated with this Markov partition $\mathcal{Q}$ there is a symbolic representation $X(\underline{\sigma})$ where now $\underline{\sigma}$ can be naturally thought as a point in $\{0,\ldots,n\}^{\mathbb{Z}^{d+1}}$. We indicate the coordinate on $\mathbb{Z}^{d+1}$ with (ξ,t) and call the ξ coordinates *spatial* or *horizontal* and the t coordinate *temporal* or *vertical.* As in the single map case the pavement $H(\mathcal{Q})$ is a Markov pavement for $\mathcal{S}_\varepsilon$ with a symbolic representation given by $X_\varepsilon(\underline{\sigma})=H(X(\underline{\sigma}))$. As an energy for the SRB distribution we can use $\lambda_u(\underline{\psi})$ given by (10.4.15). Note that because of proposition (6.4.1) we can use also $\lambda_u(\underline{\psi})=D_0(\underline{\psi})$.

We obtain the following key result (for more elementary statements, see[BK95], or alternative ones, see [JP99], can be found in the literature; see also problems [10.4.1], [10.4.2]).

P10.4.2 **(10.4.2) Proposition:** (SRB distribution potential for lattices of cat maps)
Fixed Λ *for all* $\beta \in (0,1)$ *there exist a* Λ*–independent constant* $\varepsilon_0(\beta) > 0$ *and a Markovian pavement* $\mathcal{Q} = \{Q_{\underline{\tau}}\}_{\tau \in \mathbb{Z}^d}$ *of* Ω *such that, for* $|\varepsilon| < \varepsilon_0(\beta)$*, the expansion rate* $\lambda_u(\underline{\sigma})$ *of* $\mathcal{S}_\varepsilon$ *along the unstable manifold of a point* $x = X_\varepsilon(\underline{\sigma})$ *that has a symbolic representation* $\underline{\sigma}$ *is Hölder continuous in* $\underline{\sigma}$ *and has the form*

e10.4.19
$$\lambda_u(\underline{\sigma}) = \sum_{X \in \Lambda} \Phi_X(\underline{\sigma}_X), \tag{10.4.19}$$

where X *are sets in* $\mathbb{Z}^{d+1}$ *and* $\Phi_X(\underline{\sigma}_X)$ *is holomorphic in* ε *in the disk* $|\varepsilon| < \varepsilon_0(\beta)$*. The potential* Φ *verifies the bound*

e10.4.20
$$|\Phi_X(\underline{\sigma}_X)| < C\Big(\frac{|\varepsilon|}{\varepsilon_0(\beta)}\Big)^{\frac12 \delta_\perp(X) + \frac12 n(X)} e^{-\kappa \delta_\parallel(X)}, \tag{10.4.20}$$

where $\delta_\parallel(X), \delta_\perp(X)$ *denote the tree length of the projection of the set* X *on the horizontal plane* $t = 0$ *and, respectively, the sum of the tree lengths of the intersections of* X *with the vertical lines;* $n(X)$ *denotes the number of the timelike intervals whose union is* X (cf. *remarks to definition (7.3.3) and equation (7.3.26)). The constant* C *is* Λ*–independent. Furthermore* Φ_X *is translation invariant. Note that the* $\frac12$ *in the exponent of (10.4.20) arises because by construction the number* $n(X)$ *of vertical intervals building* X *is necessarily* $n(X) \ge \delta_\perp(X)$ *so that the natural bound would be the stronger one with* $\frac{|\varepsilon|}{\varepsilon_0(\beta)}\Big)^{\frac12\delta_\perp(X)+\frac12 n(X)}$ *replaced by* $\frac{|\varepsilon|}{\varepsilon_0(\beta)}\Big)^{\delta_\perp(X)}$.
The SRB distribution μ_ε *is a Gibbs state for a system on a* $(d+1)$*-dimensional lattice with a nearest neighbor hard core interaction in the direction of "time" and a longer range exponentially decreasing "many-body" potential* Φ *with* $||\Phi||_\kappa$*,* cf. *(7.2.6), small with* ε.

Proof: For simplicity we shall give the proof by supposing that the lattice (*i.e.* spatial) dimension is $d = 1$.

In the present case the representation in (10.2.8) yields a representation of $\underline{h}$ with the desired properties.

One starts from (10.2.8) in order to obtain a representation of $\underline{h}$ in terms of potentials. Proceeding as in the proof of propositions (10.4.1) and lemma (10.4.1) one represents, with the same technique, the conjugating map $\underline{h}(\underline{\psi})$ (see below), the stable and unstable planes which split the tangent plane, cf. proposition (4.2.1), then the determinant of the map along the unstable manifold and then its logarithm. At each step the above quantities are expressed as functions of the $d+1$–dimensional symbolic representations $\underline{\sigma}$ of the points $\underline{\psi}$ in the form analogous to (10.4.19) together with bounds like (10.4.20). We discuss only the representation of the map $\underline{h}$ leaving the remaining analogous representations of the stable and unstable planes and of the determinant of the map along the unstable manifold.

Consider a single tree: its value is given by (10.2.8). We represent the quantity $\prod_j \partial_{(\alpha_{v_j},\xi_{v_j})} f_{\alpha_v}(\underline{\psi}_{nn(\xi_v)})$ via the symbols $\underline{\sigma}_{\xi,s}$ as a sum of "potentials" $\Phi^{\mathbf{x}}_v(\{\sigma_{\xi,t}\}) = \sum_{t=0}^{\infty} \varphi^{\mathbf{x}}_t(\{\sigma_{\xi,s}\}_{\xi\in nn(\xi_v),|s|<t})$ by means of the telescopic method used several times (*e.g.* to derive (4.3.8), see (4.3.10)). Here $bfx = \{(\alpha_i,\xi_i)\}$ indicates the derivative of the function f we are considering. The potentials φ will be bounded by $C_v e^{-\kappa t}$ where κ can be taken $\frac{1}{2}\log\lambda^{-1}$ (because the symbols sequences determine the corresponding points ψ "at rate" λ in the cat map we consider) and C_v is a constant.

Therefore we can represent $\lambda_{\alpha_v}^{-|p_v+1|\alpha_v} \prod_{j=1}^{s_v} \partial_{(\alpha_{v_j},\xi_{v_j})} f(S_0^{p(v)} \underline{\psi}_{nn(\xi_v)})$ as a sum of

e10.4.21
$$\lambda_{\alpha_v}^{-|p_v+1|\alpha_v} \varphi^{\mathbf{x}}_{t_v}(\{\sigma_{\xi,s}\}_{\xi\in nn(\xi_v),|s-p(v)|<t_v}) \tag{10.4.21}$$

over t_v. Repeating the same considerations for the other nodes v of the tree we represent the value of the tree ϑ of order k in (10.2.8) as a product of factors like (10.4.21) and, after properly multiplying it by ε^k, we can interpret it as a contribution to the total potential for the representation (10.2.10) of the conjugating function relative to the space–time set

e10.4.22
$$X = (\{\xi_{v_0}\}\times\{0\}) \cup \cup_{v\in V(\vartheta)\backslash v_0} \big(nn(\xi_v)\times[p(v)-t_v, p(v)+t_v]\big), \tag{10.4.22}$$

if v_0 is the first node of the tree ϑ. The contribution is bounded by $\varepsilon^k\, C^k \lambda^{-\sum_v (t_v+p(v))}$ for a suitable $C>0$ (note that the $\sup_v C_v$ can be bounded uniformly in all the labels attached to the node v because f contains only finitely many Fourier modes). Although this might seem awkward, it happens that $(\{\xi_{v_0}\}\times\{0\}) \in X$, with the above definition, but $\Phi_X(\underline{\sigma}_X)$ does not necessarily depend on $\sigma_{(\xi_0,0)}$.

Since $k \ge c\delta_{||}(X)$ because the interaction involves only nearest neighbors this is a bound of the type (10.4.20). The convergence of the summation over the tree values implies that the bound holds, with different constants, also for the total potential Φ_X. Note that the set X has a rather arbitrary shape, unlike the simpler shapes that are considered in [BS88], [JM96], [BK94], [BK96], [BK97] and [JP99].

The expansion illustrated in Fig.(10.3.1) shows that the same representation can be given to the planes tangent to the unstable (and to the stable) manifolds. And continuing with the same arguments one gets, from (10.4.12) and from (10.4.18), the representation of $\lambda_u(\underline{\sigma})$ in the form (10.4.20).

This leads (after decimation to eliminate the hard core in the time direction by proposition (7.3.3) and in the remarks preceding and following it) to the analyticity result. We omit further details (for which we refer to [BFG03]). ■

Remarks: (1) The potentials Φ_X in (10.4.19) are "space–time" potentials because X is now a subset $\mathbb{Z}^{d+1}$.

(2) The representation of $\underline{h}$ in terms of potentials is not unique: proceeding more naively one obtains a potential in which the sets X are space–time "rectangles" which are therefore simpler: however the potentials decay less fast, essentially with the diameter rather than with the tree length (see problem [10.4.2]). This is not really useful if one looks for analyticity results, but it is already sufficient to get smoothness results ([JM96], [BK95], [BK96] and [BK97]).

(3) In our case aside from the hard core the unperturbed potential is exactly 0 so that the potentials can be made as small as wished and with range as short as wished, as shown by the bounds (10.4.9),(10.4.20), by taking ε small. Nevertheless we cannot apply the results of Section §7.2 immediately. In fact our potentials still contain a hard core so that proposition (7.2.3) cannot be applied. In absence of hard cores, however, it would solve the problem.

(4) Therefore one has first to eliminate the hard core by a decimation procedure. In fact by the above remark (3) the assumptions of proposition (7.3.3) can be satisfied with κ growing arbitrarily large for $\varepsilon \to 0$ (one can in fact take $\kappa = O(\log\log\varepsilon^{-1})$). The decimation argument, *i.e.* the proposition (7.3.3), implies several consequences: we explicitly list some among them because they are the conclusion of the theory of lattices of Arnold's cat maps envisaged in this book.

C10.4.1 **(10.4.1) Corollary:** (Regularity of SRB distributions) *Let F and G be two Hölder continuous "observables" (i.e. functions on $\Omega = (\mathbb{T}^2)^\Lambda$) and suppose that they depend on $\underline{\varphi}$ only through the $\underline{\varphi}_\xi$ with ξ in a finite region V. If μ_ε^Λ denotes the SRB distribution for $\mathcal{S}_\varepsilon$ the following properties hold.*

(i) The expectation values $\mu_\varepsilon^\Lambda(F)$ is well defined and Hölder continuous in ε for $|\varepsilon|$ small enough, uniformly in Λ, V. Moreover the expectation values $\mu_\varepsilon^\Lambda(F \circ H^{-1})$ is analytic in ε for $|\varepsilon|$ small enough, uniformly in Λ, V.

(ii) If F is an analytic observable the expectation values $\mu_\varepsilon^\Lambda(F)$ is analytic in ε for $|\varepsilon| \le \varepsilon_F$

(iii) Any two functions F, G mix at an exponential rate, i.e. there exists two positive constants κ and ℓ_0, depending on F and G, such that, if $\mu_\varepsilon^\Lambda((\mathcal{S}_\varepsilon^n F)G) \overset{def}{=} \int \mu_\varepsilon^\Lambda(d\underline{\varphi}) F(\mathcal{S}_\varepsilon^n \underline{\varphi}) G(\underline{\varphi})$ and $\|F\|, \|G\|$ denote the maximum modulus of F, G, one has

e10.4.23
$$|\mu_\varepsilon^\Lambda((\mathcal{S}_\varepsilon^n F)G) - \mu_\varepsilon^\Lambda(F)\mu_\varepsilon^\Lambda(G)| \le \|F\|\,\|G\| e^{-\kappa_{F,G}(n-\ell_0)} \tag{10.4.23}$$

for all Λ, V and, therefore, for the limit μ_ε as $\Lambda \to \infty$ of μ_ε^Λ.

(iv) The volume distribution $\mu_0(d\underline{\varphi}) = d\underline{\varphi}/(2\pi)^2$ "mixes with the SRB distribution" exponentially fast the functions F, G in the sense that $\mu_0((\mathcal{S}_\varepsilon^n F)G) \overset{def}{=} \int \mu_0(d\underline{\varphi}) F(\mathcal{S}_\varepsilon^n \underline{\varphi}) G(\underline{\varphi})$ verifies, for all Λ,

e10.4.24
$$|\mu_0((\mathcal{S}_\varepsilon^n F)G) - \mu_\varepsilon^\Lambda(F)\mu_0(G)| \le \|F\|\,\|G\| e^{-\kappa_{F,G}(n-\ell_0)}, \qquad n > 0. \tag{10.4.24}$$

For $n < 0$, in general, (10.4.23) and (10.4.24) hold with a different distribution μ'_ε, which is the SRB distribution for S_ε^{-1} (which essentially has the same properties as μ_ε).

Remarks: (1) One could make the statements above without introducing the constant ℓ_0 and writing (10.4.23) and (10.4.24) with an extra positive constant B in front of the right hand sides and with no ℓ_0 in the exponent, provided that n is chosen large enough. Of course the two formulations are quite equivalent.
(2) Statement (iii) requires a new analysis based on the representation for the volume distribution μ_0 for Anosov systems discussed in Section §4.3. We saw, in fact, that in a single Anosov system the SRB distribution μ and the volume μ_0 are related because the restriction of μ_0 to the functions depending only on the symbols with time label ≥ 0 is absolutely continuous with respect to the restriction of μ to the same functions. For lattices of Anosov systems this is still true but the ratio $d\mu_0/d\mu$ is not uniformly finite, away from 0 and ∞, in the spatial size $|\Lambda|$ of the lattice $\Lambda \subset \mathbb{Z}^d$. However a careful examination of the above proofs implies that if one further restricts the distributions μ, μ_0 to functions which are spatially local in a region V and depend only on the symbols with positive time label then one gets two probability distributions $\mu^{+,V}, \mu_0^{+,V}$ which are absolutely continuous with respect to each other. And $d\mu_0^{+,V}/d\mu^{+,V} = \rho(\underline{\sigma})$ is a Hölder continuous function so that the statement (iii) follows from (ii).
(3) The study the SRB distribution for $\mathcal{S}_\varepsilon^{-1}$ is not immediate due to the fact that $\mathcal{S}_\varepsilon^{-1}$ in general is not given by a nearest neighbor perturbation of $\mathcal{S}_0^{-1}$. A very simple way to solve this problem is to observe that the expansion rate on the unstable manifold for $\mathcal{S}_\varepsilon^{-1}$ is the inverse of the contraction rate on the stable manifold for $\mathcal{S}_\varepsilon$, that has a representation similar to the one discussed in the proof of proposition (10.4.2). Moreover the Markov partition and the symbolic code for $\mathcal{S}_\varepsilon^{-1}$ are strictly related to those for for $\mathcal{S}_\varepsilon$. One can also prove this directly constructing $\mathcal{S}_\varepsilon^{-1}$ as a perturbation of $\mathcal{S}_0^{-1}$ with a perturbation that still decay fast enough in space and time, see problem [10.4.3].

In the above analysis the potentials $m_X(\underline{\psi}_X)$ do depend on Λ. However from the trees expansion formulae we see that the dependence of Φ_X, G_X and therefore all the other potentials we have introduced, in particular n_X, consists of a term which is Λ independent as soon as $\Lambda \supset X$ plus a small correction that while verifying uniformly in Λ the general bounds like (10.4.9) contains corrections of size of order $e^{-\mathrm{const}\,L}$. In our situation in which Gibbs states depend continuously on the potential this implies not only the mentioned *existence of the thermodynamic limit* for the SRB distribution but also a wealth of results that can be derived from the well established theory of Gibbs states with weak coupling. Here we mention only the following.

$P10.4.3$ **(10.4.3) Proposition:** (Space-time chaos in cat maps lattices) *Let μ_ε^Λ be* *the SRB distribution for the lattice of Arnold's cat maps considered above. For all* $F(\underline{\varphi})$ *defined on* $(\mathbb{T}^2)^\Lambda$ *which are analytic and dependent only on the microstates* $\underline{\varphi}_\xi$ *with* $\xi\in\Lambda_0$ *where* Λ_0 *is a finite region, the limit*

e10.4.25
$$\mu_\varepsilon(F)=\lim_{\Lambda\to\infty}\mu_\varepsilon^\Lambda(F) \tag{10.4.25}$$

exists and is analytic in ε *for* $|\varepsilon|<\varepsilon_F$. *Functions* F *which depend only on the microstates* $\underline{\varphi}_\xi$ *with* ξ *contained in a given finite region are called* spatially local.
(ii) If F,G *are two smooth enough (e.g. Hölder continuous with some exponent* $\beta>0$*) functions which are local in space then*

e10.4.26
$$\mu_\varepsilon((\tau^\xi\mathcal{S}_\varepsilon^tF)G)\xrightarrow[|\xi|+|t|\to\infty]{}\mu_\varepsilon(F)\mu_\varepsilon(G), \tag{10.4.26}$$

where τ^ξ *denotes the lattice translation by* $\xi\in\mathbb{Z}^d$ *and the limit is on* t *or on* ξ *or both.*

(iii) If F,G *are two smooth enough (e.g. Hölder continuous with some exponent* $\beta>0$*) functions which are local in space then*

e10.4.27
$$\mu_0((\mathcal{S}_\varepsilon^tF)G)\xrightarrow[t\to\infty]{}\mu_\varepsilon(F)\mu_0(G), \tag{10.4.27}$$

where τ^ξ *denotes the lattice translation by* $\xi\in\mathbb{Z}^d$.
(iv) If F *is a smooth enough observable (e.g. Hölder continuous with some exponent* $\beta>0$*) which is local in space then*

e10.4.28
$$\lim_{t\to+\infty}\frac{1}{t}\sum_{j=0}^{t}F(\mathcal{S}_\varepsilon^j\underline{\varphi})=\int F(\underline{\varphi})\mu_\varepsilon(d\underline{\varphi}) \tag{10.4.28}$$

for $\mu_0(d\underline{\varphi})$*–almost all* $\underline{\varphi}\in(\mathbb{T}^2)^{\mathbb{Z}^d}$.

Remarks: (1) The above result is sometimes read as saying "lattices of Anosov maps weakly coupled via short range interactions" show spatio–temporal chaos.
(2) The method of proof followed here can be made very elementary in the case of weakly interacting lattices $\Lambda\subset\mathbb{Z}^d$ of expansive maps S_0 of the interval like the ones considered in proposition (5.4.1) (strictly expansive and surjective) or of expansive maps of the circle: the theory is very similar but the lattice spin system that they generate is a seminfinite lattice system with spins located on $\Lambda\times\mathbb{Z}_+^d$. The great simplification is that there is no hard core in the spin interactions because of the Markovian assumption of strict surjectivity. In this case proposition (7.4.1) immediately applies and gives smoothness as well as mixing, cf. [BK95] and problem [10.4.1]. It does *not* however show the analyticity that follows from an (equally immediate) application of proposition (7.3.2).
(3) Smoothness in the case of lattices of cat maps can also be proved by a

method close to the one presented here (although it does not yield analyticity), cf. [JP99].

Problems for §10.4

Q10.4.1 **[10.4.1]:** (*Nonanalytic proof of regularity of SRB distributions for lattices of circle maps*, from [BK95].)
Consider a chain of circle maps or of interval maps instead of Arnold cats small perturbation of independent maps S_0 acting on each site variable. Assume the expansiveness for S_0 in the sense of Section §5.4 in the interval case and assume that $S_0\varphi_\xi = 2\varphi_\xi$ for $\xi \in \Lambda$ in the case of circle maps (for simplicity). Check that in this case one can transform the problem of determining a SRB distribution into a problem of a lattice spin system on a semi-infinite lattice $\Lambda \times \mathbb{Z}_+$ without hard core conditions. Check that the general uniqueness and smoothness results of Section §7.4 applies immediately, given the result of proposition (10.4.2), (10.4.20), to obtain the results of corollary (10.4.1) with analyticity replacing smoothness. The advantage of this approach is that it is elementary and one does not need the cluster expansion theory of Chapter VII. (*Hint:* The proposition (7.4.1) applies directly because of lack of hard cores. The seminfinite lattice is due to the fact that the maps in question are described symbolically by semiinfinite sequences of symbols, being not invertible).

Q10.4.2 **[10.4.2]:** (*Alternative potentials*)
Potentials $\Phi_X(\underline{\sigma}_X)$ can be immediately derived from the potentials $n_X(\underline{\psi}_X)$ of (10.4.17) by the method of Section §10.3: for fixed X we consider the sequence $\underline{\sigma}_X$ corresponding to $\underline{\psi}_X$ on the Markov pavement and call $\underline{\sigma}_{X,h}$ the sequence obtained from $\underline{\sigma}_X$ by truncating $\underline{\sigma}_X$ beyond the heights $[-h, h]$ and replacing the deleted $\sigma_{\xi,t}$ with $|t| > h$ by standard compatible sequences as done in the proof of proposition (4.3.1). Check that the potentials thus obtained are not zero only for sets of the form $H_0 \times I$ with $H_0 \subset \mathbb{Z}^d$ and $I \subset \mathbb{Z}$. Check that the bounds on $n_X(\underline{\psi}_X)$, $X \in \mathbb{Z}^d$ and the Hölder continuity of $n_X(\underline{\psi}_X)$ imply bounds on the potential that decays exponentially as $e^{-\kappa\delta(H_0)+|I|}$, *i.e.* if $d = 2$ as the diameters of the sets $H = H_0 \times I$.

Q10.4.3 **[10.4.3]:** (*Tree expansion for* $\mathcal{S}_\varepsilon^{-1}$.)
Show that $\mathcal{S}_\varepsilon^{-1}(\underline{\psi})$ can be written as $\mathcal{S}_0^{-1}(\underline{\psi}) + \varepsilon \underline{g}(\varepsilon, \underline{\psi})$ where

$$g_\xi(\varepsilon, \underline{\psi}) = \sum_{X \ni \xi} \Psi_X(\underline{\psi}_X)$$

where X is a connected set in $\mathbb{Z}^d$ and Ψ_X is of order $\varepsilon^{\frac{\delta(X)}{2d}}$. (*Hint:* write an series expansion for $\underline{g}(\varepsilon, \underline{\psi})$ as a function of ε and use $\mathcal{S}_\varepsilon^{-1} \circ \mathcal{S}_\varepsilon(\underline{\psi}) = \underline{\psi}$ to compute recursively the coefficients. The decay follows from an argument very similar to the one used for H.)

Q10.4.4 **[10.4.4]:** Extend the result of problem [10.2.5] to the construction of the unstable direction and of the expansion rate on the unstable direction.

Q10.4.5 **[10.4.5]:** Refine the result of problem [10.4.3] to show that the symbolic representation $\underline{\tilde{g}}(\sigma)$ of $\underline{g}(\varepsilon, \underline{\psi})$ can be written as

$$\tilde{g}_\xi(\underline{\sigma}) = \sum_{X \ni \xi} \widetilde{\Psi}_X(\underline{\sigma}_X)$$

where now X is a set in $\mathbb{Z}^{d+1}$ such that its projection on the temporal coordinate is connected and the intersection of X with any vertical line is a segment centered at 0. Moreover we have:

$$|\widetilde{\Psi}_X(\underline{\sigma}_X)| \le C e^{-\kappa_0 \delta_\perp(X)} e^{-\kappa_1 \delta_\parallel(X)}.$$

(*Hint:* simply write $\underline{\varphi}$ as a sum of potential in the expansion studied in problem [10.4.3].

Q10.4.6 **[10.4.6]:** Show that this is enough to obtain again estimate (10.4.20) for the potential of the SRB distribution of $\mathcal{S}_\varepsilon^{-1}$.

Bibliographical note §10.4

The theory of Sections §10.2 and §10.4 was initiated by [BS88]. The main technical tool in our approach is the cluster expansion discussed in Chapter VII in the form introduced for the decimation problem in [CO82]. The advantage over the original approaches to spatio–temporal chaos theory started in [BS88], and essentially completed in [JM96] is that here one obtains analyticity instead of infinite differentiability of the SRB distribution. Subsequent papers eliminated technical restrictions present in [JM96], at first at the price of restricting the theory to lattices of coupled expanding maps of the circle (which exhibit the important simplification of having a symbolic dynamics representation without hard cores, see [BK95], [BK96] and [BK97]). Later the restriction to expanding maps has been eliminated, [JP99], [BK96] and [JP99], by further developing the ideas in [JM96]. The approach discussed in Section §10.4, due to [BFG03], goes a little beyond the results in the just quoted papers as it also gives analyticity in the dependence of the Gibbs distribution on the perturbation size. The latter result is discussed in [BFG03] where also the theory of Sections §10.1 and §10.3 were developed. For an example of application of the above results see [Ga99].

§10.5 Isomorphisms

We can ask in which cases two Gibbs states μ and μ', one on $\{0,\ldots,n\}^{\mathbb{Z}}_T$ and the other on $\{0,\ldots,n'\}^{\mathbb{Z}}_T$, with respective potentials Φ and Φ' are isomorphic in the sense that the dynamical systems $(\{0,\ldots,n\}^{\mathbb{Z}}_T,\tau,\mu)$ and $(0,\ldots,n')^{\mathbb{Z}}_T,\tau,\mu')$ are isomorphic $\bmod\ 0$.

P10.5.1 **(10.5.1) Proposition:** (Isomorphims of Gibbs distributions with fast decreasin potential)
If T and T' are two mixing compatibility matrices, $(n+1)\times(n+1)$ and $(n'+1)\times(n'+1)$ respectively, and if Φ and Φ' are two potentials for $\{0,\ldots,n\}^{\mathbb{Z}}_T$ and for $\{0,\ldots,n'\}^{\mathbb{Z}}_T$ such that

e10.5.1
$$\sum_{X\ni 0}\frac{1+\mathrm{diam}(X)}{|X|}\|\Psi_X\|<+\infty,\qquad \text{if } \Psi=\Phi \text{ or } \Phi', \tag{10.5.1}$$

with $\|\Phi_X\|=\sup_{\underline{\sigma}_X}\|\Phi_X(\underline{\sigma}_X)\|$, then the dynamical systems $(\{0,\ldots,n\}^{\mathbb{Z}}_T,\tau,$ $\mu)$ and $(\{0,\ldots,n\}^{\mathbb{Z}}_T,\tau,\mu)$ are isomorphic $\bmod\ 0$ if and only if their entropies are equal: $s(\mu)=s(\mu')$.

Remarks: (i) The average entropy is therefore a complete invariant for the isomorphisms $\bmod\ 0$ in a rather wide class of dynamical systems.

(ii) The proof of this theorem reaches already its maximal difficulty in the case in which $\Phi = \Phi' = 0, T_{\sigma\sigma'} = T'_{\sigma\sigma} = 1$: this is the case of the Bernoulli schemes.

(iii) The problem of the isomorphisms between Bernoulli schemes is a famous problem solved at the end of the 1960's by Ornstein who also established new techniques to attack and solve the problem of the isomorphism between large classes of dynamical systems. Proposition (10.5.1) gives us an example of a problem that can be solved with the ideas and the techniques of Ornstein.

Ornstein's theorem was preceded by another important result of Sinai that established the weak equivalence $\bmod\ 0$ (see below for a precise definition) between two Bernoulli schemes of equal entropy. It was followed by several deep explicit constructions of the codes that realize the isomorphisms. The first one, due to Monroy and Russo, [MR75], concerned a very special case, but it introduced, in that case, some new ideas: such a construction has been improved by the constructions of Keane and Smorodinski, who, with the use of new and deep ideas, explicitly realized the code of the isomorphism between two isentropic Bernoulli schemes, see [KS79]. Such codes are "constructive" in the sense that it is possible to construct an arbitrarily *prefixed* number of values of the elements of the sequence $\underline{\sigma}'$ image, in the isomorphism in question, of a sequence $\underline{\sigma}$ by making use of an algorithm that can be implemented on a computer, so that it requires a finite time for almost all the sequences $\underline{\sigma}$ (randomly chosen with respect to the measure of one of the two Bernoulli schemes).

From a practical point of view the description of the algorithm of Keane and Smorodinski is certainly the fastest way to achieve the proof of the isomorphism between isoentropic Bernoulli schemes. Nevertheless Ornstein's proof remains a monument and an instrument for whoever wants to study in more detail the abstract and conceptual aspects of the isomorphism theory and of the meaning of entropy. Furthermore it provides us with the means to deal with isomorphism problems between dynamical systems that until now, were not otherwise soluble.

(iv) We shall not discuss here the proof of Ornstein's theorem, and the reader should consult for this purpose the book by Ornstein, [Or74], and then meditate on the codes of Monroy-Russo and Keane-Smorodinski. It seems difficult to present Ornstein's theory in a more compact or simpler way than the one he himself chose and likewise it is difficult to present the constructions of Monroy-Russo and of Kean-Smorodinski without repeating word for word the relatively short and brilliant original works.

For completeness it is convenient to give a precise definition of weak isomorphism $\bmod\ 0$ and to state the theorem of Sinai.

D10.5.1 **(10.5.1) Definition:** (Weak isomorphism)
If $(\{0,\ldots,n\}^{\mathbb{Z}},\tau,\mu)$ *and* $(\{0,\ldots,n'\}^{\mathbb{Z}},\tau,\mu')$ *are two ergodic metric dynamical systems, they will be said to be weakly isomorphic* $\bmod\ 0$ *if there is a measurable partition* $\mathcal{P}=\{P_0,\ldots,P_{n'}\}$ *of* $\{0,\ldots,n\}^{\mathbb{Z}}$ *and a measurable*

partition $\mathcal{P}'=\{P'_0,\ldots,P'_n\}$ *of* $\{0,\ldots,n'\}^{\mathbb{Z}}$ *such that*

e10.5.2
$$\begin{aligned}\mu(C^{0\ldots M}_{\sigma_0\ldots\sigma_M})&=\mu'(P'^{0\ldots M}_{\sigma_0\ldots\sigma_M}) \qquad \textit{for all } M\ge 0,\underline{\sigma}\in\{0,\ldots,n\}^{M+1},\\ \mu'(C^{0\ldots M}_{\sigma'_0\ldots\sigma'_M})&=\mu(P^{0\ldots M}_{\sigma'_0\ldots\sigma'_M}) \qquad \textit{for all } M\ge 0,\underline{\sigma'}\in\{0,\ldots,n'\}^{M+1},\end{aligned} \tag{10.5.2}$$

where $P'^{0\ldots M}_{\sigma_0\ldots\sigma_M}=\cap_{j=0}^{M}\tau^{-j}P'_{\sigma_j}$ *and* $P^{0\ldots M}_{\sigma'_0\ldots\sigma'_M}=\cap_{j=0}^{M}\tau^{-j}P_{\sigma'_j}$*. This means that the "two dynamical systems are weakly isomorphic if each contains a copy of the other".*

An important result (due to Sinai) is the following one.

P10.5.2 **(10.5.2) Proposition:** (Sinai's theorem)
Let $(\{0,\ldots,n\}^{\mathbb{Z}},\tau,\mu)$ *and* $(\{0,\ldots n'\}^{P}\mathbb{Z},\tau,\mu'\}$ *be two Bernoulli schemes of equal entropy. Then they are weakly isomorphic* mod 0*.*

We conclude this very brief introduction to the isomorphisms theory with a comment and an interesting definition.

One of the key remarks in Ornstein's theory is the importance of the notion of *finite determination* of a dynamical system $(\{0,\ldots,\}^{\mathbb{Z}},\tau,\mu)$. From a mathematical point of view the interest of this notion lies in the fact that, as shown by Ornstein, a system that enjoys this property is isomorphic mod 0 to a Bernoulli scheme and *vice versa*. From a"physical" point of view it is a property that has a very interesting interpretation that we illustrate after giving the precise notion of *finite determination* (due to Ornstein, see [Or74]).

D10.5.2 **(10.5.2) Definition:** (Finitely determined systems)
A dynamical system $(\{0,\ldots,n\}^{\mathbb{Z}},\tau,\mu)$ *is* finitely determined *if, given* $\varepsilon>0$*, there exist* $\delta_\varepsilon,N_\varepsilon$ *such that every other ergodic dynamical system* $(\{0,\ldots,n\}^{\mathbb{Z}},\tau,\mu')$*, for which*

e10.5.3
$$\begin{aligned}&\sum_{\sigma_0\ldots\sigma_{N_\varepsilon}}|\mu(C^{0\ldots N_\varepsilon}_{\sigma_0\ldots\sigma_{N_\varepsilon}})-\mu'(C^{0\ldots N_\varepsilon}_{\sigma_0\ldots\sigma_{N_\varepsilon}})|<\delta_\varepsilon,\\ &|s(\mu)-s(\mu')|<\varepsilon,\end{aligned} \tag{10.5.3}$$

is such that we can construct a code $I_\varepsilon:\{0,\ldots,n\}^{\mathbb{Z}}\longleftrightarrow\{0,\ldots,n\}^{\mathbb{Z}}$ *such that* $\mu(I_\varepsilon(E))=\mu'(E)$ *for all* μ'*–measurable set* $E\subset\{0,\ldots,n\}^{\mathbb{Z}}$ *with the property*

e10.5.4
$$\limsup_{N\to\infty}(2N)^{-1}\sum_{i=-N}^{N-1}|I_\varepsilon(\underline{\sigma})_i-\sigma_i|,\le\varepsilon \qquad \mu'-\text{almost everywhere.} \tag{10.5.4}$$

We shall say, briefly, that μ *is finitely determined if* $(\{0,\ldots,n\}^{\mathbb{Z}},\tau,\mu)$ *is such.*

Remarks: (i) This means that if μ is a finitely determined every measure $\mu'\in\mathcal{M}(\{0,\ldots,n\}^{\mathbb{Z}})$ "close to it in entropy and in distribution" (cf. problem

[3.3.9])) produces "the same set of typical configurations up to errors that occur with a small density in time".

(ii) It is easy to construct ergodic distributions that have preassigned probability distributions for the cylinders of preassigned base length and having, furthermore, preassigned entropy (within a preassigned approximation), (cf. problem [3.3.9]). We then see that if μ is finitely determined such "approximate models" of μ can be used to generate sequences (via random extractions) that differ from those that one would obtain by using μ itself only in a fraction ε of sites. In other words by examining the statistics of short sequences it is possible to infer properties of infinitely long sequences, if such sequences are generated at random with a finitely determined distribution.

(iii) If μ_Φ is a Gibbs state with a potential Φ and if Φ varies in an open region $\Sigma \subset B$ where there are no phase transitions (*i.e.* for all $\Phi \in \Sigma, G(\Phi)$ contains a single element) then, from the variational principle (corollary (6.1.1)), from the convexity and continuity of $P(\Phi)$ and from the interpretation of μ_Φ as tangent plane to the graph of P, it follows that both $s(\mu_\Phi)$ and μ_Φ vary with continuity (with respect to the weak topology for the distributions). This means that as Φ varies in Σ the Gibbs state changes by a small amount in distribution and entropy if Φ changes by little.

Therefore if every Gibbs process μ_Φ with $\Phi \in B$ was finitely determined it would follow, from this observation, that also the typical configurations vary "with continuity": note that for the validity of such a property in general the continuity of μ_Φ in Φ would not be enough if the distributions μ_Φ were not finitely determined.

(iv) it appears difficult to think that there can exist Gibbs states with potential $\Phi \in B$ that are not finitely determined, because of the physical meaning of this property, that emerges from the remarks (iii) and (ii).

Hence we see the interest of proposition (10.5.1) that goes in the direction of a confirming the latter idea. And we also understand the interest of studying what happens in the case in which $\Phi \in B$ but it does not verify (10.5.1).

In reality no examples of Gibbs states that are not finitely determined (with $\Phi \in B$) are known; and in some cases in which one could perhaps have of doubts and that refer to the analogous problems for systems on lattices of dimension ≥ 2 finite determinacy has been established (an unpublished work of Ornstein-Weiss shows the finite determination of the Gibbs state for the 2–dimensional Ising model at the critical point).

(v) The property of finite determination allows us to speak of certain global properties of typical configurations in terms of local properties. It can therefore be used to formulate in a mathematically precise way various notions of physical nature that concern global properties of typical (*i.e.* randomly chosen) configurations. This fact han not until now been really exploited in the Physics literature. An example of an attempt to exploit this idea can be found in [EG73] in connection with a probabilistic interpretation of the "approximate symmetries".

Bibliographical note

Ornstein's theory is exposed in [Sh73] and in the book by D. Ornstein, [Or74]. Proposition (10.5.1) is based on the derivation of certain estimates needed to check a sufficient criterion in order that a dynamical system is isomorphic to a Bernoulli scheme and it can be found in [Ga73], [Le73]. The code of Monroy-Russo is found in [MR75]; the code of Kean-Smorodinski is found in [KM79]. The quoted work D. of Ornstein and B. Weiss is unfortunately still unpublished, [OW74].

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