CHAPTER V

Gibbs distributions

§5.1 Gibbs distributions

The study of the general structure of dynamical systems, begun in the previous sections, could continue and constitutes one of the directions in which ergodic theory can be developed. We shall, however, look in a somewhat different direction dedicating attention to a few concrete problems that do not belong to the general theory. The more concrete studies involve analytic work of "classical" type and are more directly related to the applications.

In the previous sections, for instance in proposition (4.3.2), we faced for the first time, in implicit form, the following problem: given a probability distribution on $\{1,\ldots,q\}^{\mathbb{Z}}_T$ described by its family of conditional probabilities, under which conditions do the latter determine the probability distribution uniquely?

If the conditional probabilities are assigned in a form similar to (4.3.11) or (4.3.12) with the functions A_a having the form (4.3.13) or (4.3.16) the problem is known as the "determination of a Gibbs distribution from its potential".

To proceed orderly it is convenient to set up a general definition and an appropriately suggestive nomenclature, in spite of a few repetitions of notions and definitions already discussed previously in different contexts.

Let T be a $(n+1)\times(n+1)$ compatibility matrix, with entries equal to 0 or 1, and let $\Omega=\{0,\ldots,n\}^{\mathbb{Z}}_T=\{\underline{\sigma}\,|\,\underline{\sigma}\in\{0,\ldots,n\}^{\mathbb{Z}},\ \prod_{-\infty}^{+\infty}T_{\sigma_i\sigma_{i+1}}=1\}$ be the space of the T–compatible sequences, cf. definition (4.1.1).

We shall say that the cylinder $C^J_{\underline{\sigma}}$ with base J and specification $\underline{\sigma}$ ($J =$

$\{j_1, \ldots, j_q\} \subset \mathbb{Z}$, $\underline{\sigma} \in \{0, \ldots, n\}^J$) is a T–compatible cylinder if $C^J_{\underline{\sigma}} \cap \{0, \ldots, n\}^{\mathbb{Z}}_T \neq \emptyset$.

If $\Lambda \subset \mathbb{Z}$ is a set, finite or not, $\mathcal{B}(\Lambda)$ will be the σ–algebra generated by the cylinders with base in Λ. If Λ is finite with $|\Lambda|$ elements then $\mathcal{B}(\Lambda)$ is a finite σ–algebra with at most $|\Lambda|^{n+1}$ atoms.

The matrix T is said *mixing*, cf. (4.1.1), if there exists an integer $z \geq 0$ such that $T^{z+1}_{\sigma\sigma'} > 0$ for all $\sigma, \sigma' \in \{0, \ldots, n\}$: the minimum value of such z will be denoted $a(T)$ and it will be called *mixing time* (or *mixing length*) of T.

If T is mixing the dynamical system $(\{0, \ldots, n\}^{\mathbb{Z}}_T, \tau)$ will be topologically
N5.1.1 mixing.[1]

N5.1.2 If m is a probability distribution on Ω and if $\Lambda \subset \mathbb{Z}$ is a finite set,[2] we can define, for every T–compatible cylinder $C^{\Lambda}_{\underline{\sigma}_\Lambda}$ with base Λ, the probability distribution on $\mathcal{B}(\Lambda^c)$

e5.1.1
$$m'(E) = m(C^{\Lambda}_{\underline{\sigma}_\Lambda} \cap E), \qquad E \in \mathcal{B}(\Lambda^c). \tag{5.1.1}$$

Such a probability distribution is obviously absolutely continuous with respect to the restriction of m to $\mathcal{B}(\Lambda^c)$, which we shall still denote with m, and the Radon–Nykodim derivative of m' with respect to m will be a $(m, \mathcal{B}(\Lambda^c))$–measurable function:

e5.1.2
$$\underline{\sigma}' \to m(\underline{\sigma}_\Lambda | \underline{\sigma}'_{\Lambda^c}) = \frac{dm'}{dm}\,(\underline{\sigma}'), \tag{5.1.2}$$

where the notation is admissible because $\frac{dm'}{dm}\,(\underline{\sigma}')$, being $(m, \mathcal{B}(\Lambda^c))$–measurable, depends on $\underline{\sigma}'$ only via the restriction to Λ^c of $\underline{\sigma}'$: $\underline{\sigma}'_{\Lambda^c} = (\sigma'_j)_{j \in \Lambda^c}$.

The (5.1.2) is the *probability of the event* $\underline{\sigma}_\Lambda$ *in* Λ *conditional to the event* $\underline{\sigma}'_{\Lambda^c}$ *in* Λ^c.

Extending a convention, employed so far, we shall consider pairs J_1 and $J_2 \subset \mathbb{Z}$, $J_1 \cap J_2 = \emptyset$, and if $\underline{\sigma}_{J_1} \in \{0, \ldots, n\}^{J_1}$ and $\underline{\sigma}_{J_2} \in \{0, \ldots, n\}^{J_2}$ then $\underline{\sigma}_{J_1}\underline{\sigma}_{J_2}$ will denote the element $\underline{\sigma}'_{J_1 \cup J_2} \in \{0, \ldots, n\}^{J_1 \cup J_2}$ such that $\sigma'_i = \sigma_i$, for all $i \in J_1 \cup J_2$.

The following general definition of *Gibbs distribution* or *Gibbs measure* or *Gibbs state* is obviously inspired by (4.3.11), (4.3.12) and (4.3.16); making use of the notions and notations previously introduced we set the following definition.

D5.1.1 **(5.1.1) Definition:** (Potentials for symbolic dynamics)
Let T be a mixing compatibility matrix for the sequences in $\{0, \ldots, n\}^{\mathbb{Z}}$ and let $a(T)$ be the mixing time of T, cf. *definition (4.1.1). Let B the space*

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[1] Given two open sets F and G there exists N_0 such that $S^N F \cap G \neq \emptyset$ for all $N \geq N_0$.

[2] We shall say that Λ is an interval or, equivalently, that it is connected if Λ is given by the intersection of $\mathbb{Z}$ with an interval of $\mathbb{R}$.

of the sequences $\Phi = \{\Phi_X\}_{X\subset\mathbb{Z}}$ *parameterized by the finite subsets of* $\mathbb{Z}$, *consisting in the functions*

e5.1.3
$$\Phi_X : \{0,\ldots,n\}^X \to \mathbb{R} \tag{5.1.3}$$

such that, having set $\|\Phi_X\| = \max_{\underline{\sigma}\in\{0,\ldots,n\}^X} |\Phi_X(\underline{\sigma})|$, *one has*
(i) (Shift invariance) Φ *is invariant under translations of* $\mathbb{Z}$, *i.e. for all* $\sigma_1,\ldots,\sigma_p$

e5.1.4
$$\Phi_X(\sigma_1,\ldots,\sigma_p) = \Phi_{\tau X}(\sigma_1,\ldots,\sigma_p), \tag{5.1.4}$$

if $X = (\xi_1,\ldots,\xi_p)$ *and* $\tau X = (\xi_1+1,\ldots,\xi_p+1)$.
(ii) (Stability) Φ *is "summable":*

e5.1.5
$$\|\Phi\| \equiv \sum_{X\ni 0} \frac{\|\Phi_X\|}{|X|} < +\infty. \tag{5.1.5}$$

We shall say that B *is the space of the* potentials *on* $\{0,\ldots,n\}^{\mathbb{Z}}_T = \Omega$. *The function on* $\{0,\ldots,n\}^{\mathbb{Z}}_T$ *(*cf. *the third of (4.3.16)) defined by*

e5.1.6
$$A_\Phi(\underline{\sigma}) = \sum_{X\ni 0} \frac{\Phi_X(\underline{\sigma}_X)}{|X|} \tag{5.1.6}$$

will be called potential energy per site *or* energy function *associated with* Φ.

Remark: In classical potential theory the energy of a configuration of points on $\mathbb{Z}$ interacting via a potential $\Phi(x,y)$ is written as

e5.1.7
$$\sum_{(x,y)\in\mathbb{Z}^2} \sigma_x\sigma_y\Phi(x,y) = \sum_{x\in\mathbb{Z}} \Big(\sum_{\substack{y\in\mathbb{Z}\\ y\neq x}} \frac{\sigma_x\sigma_y\Phi(x,y)}{2} \Big), \tag{5.1.7}$$

where $\sigma_x = 0$ if the site x is empty and $\sigma_x = 1$ if it is occupied. This explains the name given to (5.1.6) and why Φ is called a *many-body potential* or, more properly, a collection of many-body potentials. In the case of (5.1.7) one can imagine that $\Phi_X = 0$ unless $X = \{x,y\}$ and $\Phi_{\{x,y\}}(\sigma,\sigma') = \sigma\sigma'\Phi(x,y)$: this is the case in which the potential is a *two-body potential*.

In terms of potentials it is possible to give a general enough notion of Gibbs distribution.

D5.1.2 **(5.1.2) Definition:** (Gibbs distributions, DLR equations)
In the context of definition (5.1.1) we shall say that $m \in M^0(\{0,\ldots,n\}^{\mathbb{Z}}_T)$ *is a Gibbs distribution on* $\{0,\ldots,n\}^{\mathbb{Z}}_T$ *with potential* $\Phi \in B$ *if it is a probability distribution on the Borel sets of* $\{0,\ldots,n\}^{\mathbb{Z}}_T$ *whose conditional probabilities*
N5.1.3 $m(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda_c})$ *are,* m*–almost everywhere, such that* [3]

e5.1.8
$$\frac{m(\underline{\sigma}'_\Lambda|\underline{\sigma}_{\Lambda^c})}{m(\underline{\sigma}''_\Lambda|\underline{\sigma}_{\Lambda^c})} = \exp\Big(- \sum_{k=-\infty}^{+\infty} \{A_\Phi(\tau^k\underline{\sigma}') - A_\Phi(\tau^k\underline{\sigma}'')\}\Big) \tag{5.1.8}$$

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[3] One refers to (5.1.8) by calling it *Dobrushin–Lanford–Ruelle relations* or simply *DLR relations*; see also remark (5) after this definition.

for every interval $\Lambda\subset\mathbb{Z}$ *longer than* $a(T)$ *and for* $\underline{\sigma}'=(\underline{\sigma}'_\Lambda\underline{\sigma}_{\Lambda^c})$, $\underline{\sigma}''=$
N5.1.4 $(\underline{\sigma}''_\Lambda\underline{\sigma}_{\Lambda^c})\in\{0,\ldots,n\}^{\mathbb{Z}}_T$.[4]

Equivalently: m *is a Gibbs distribution with potential* Φ *if for every long enough interval* Λ *one has,* m*–almost everywhere,*

e5.1.9
$$m(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda^c})=\frac{\exp\big(-\sum_{R\cap\Lambda\ne\emptyset}\Phi_R(\underline{\sigma}_R)\big)}{\sum_{\underline{\sigma}'_\Lambda}\exp\big(-\sum_{R\cap\Lambda\ne\emptyset}\Phi_R(\underline{\sigma}'_R)\big)},\tag{5.1.9}$$

where (5.1.9) are interpreted as zero if $\underline{\sigma}\notin\{0,\ldots,n\}^{\mathbb{Z}}_T$.

The set of the Gibbs distributions with potential Φ *will be denoted* $G^0(\Phi)$, $G(\Phi)\subset G^0(\Phi)$ *will be the set of the Gibbs distributions on* $\{0,\ldots,n\}^{\mathbb{Z}}_T$
N5.1.5 *which are invariant under translations,*[5] $G_e(\Phi)\subset G(\Phi)$ *will be the set of the ergodic Gibbs distributions with potential* Φ, *etc.*

Remarks: (1) It is necessary to associate with this definition an existence theorem. Indeed it is by no means clear that Gibbs distributions exist, as it is not obvious that translation invariance of the potential implies translation invariance of the relative Gibbs distributions. In fact shift invariance is not in general a consequence of the shift invariance of the potential. We shall not meet such "pathologies" in what follows because the potentials we consider will have further properties which allow to exclude them.

(2) Equivalence between (5.1.8) and (5.1.9) follows from (5.1.6) and from the observation that

e5.1.10
$$\sum_{k=-\infty}^{\infty}\{A_\Phi(\tau^k\underline{\sigma}')-A_\Phi(\tau^k\underline{\sigma}'')\}=\sum_{R\cap\Lambda\ne\emptyset}\{\Phi_R(\underline{\sigma}'_R)-\Phi_R(\underline{\sigma}''_R)\},\tag{5.1.10}$$

that is obtained from the definitions. Furthermore in (5.1.10) the sum can be decomposed, as shown by the right hand side, into 2 absolutely convergent sums. This shows immediately that $m(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda^c})$ must be proportional to $\exp(-\sum_{R\cap\Lambda\ne\emptyset}\Phi_R(\underline{\sigma}_R))$. The denominator in (5.1.9) is precisely the normalization coefficient determined by the condition $\sum_{\underline{\sigma}_\Lambda}m(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda^c})=1$, which holds because $m(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda^c})$ is a conditional probability.

What said so far is correct for every connected $\Lambda\subset\mathbb{Z}$ provided all the elements of T are positive. In the general case, if T is only mixing, it is necessary to consider in the previous argument only T–compatible sequences. This does not present particular difficulties provided Λ is a large enough interval (at least longer than $a(T)$). If Λ is too short and T has many zeroes it could happen that the denominator of (5.1.9) is zero for some $\underline{\sigma}$ because

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[4] More precisely: chosen $\underline{\sigma}\in\{0,\ldots,n\}^{\mathbb{Z}}_T$ m–almost everywhere and chosen $\underline{\sigma}'_\Lambda$, $\underline{\sigma}''_\Lambda\in\{0,\ldots,n\}^{\Lambda}_T$ then (5.1.8) holds provided Λ is connected and of length larger than $a(T)$, where $a(T)$ is the mixing time of T. The latter condition is needed to make sure that the denominator in (5.1.8) or (5.1.9) does not vanish.

[5] We call a distribution m on $\{0,\ldots,n\}^{\mathbb{Z}}_T$ *invariant under translation* if $\tau m=m$ where the action of τ on the probability distributions is obvious: we set $(\tau m)(E)=m(\tau^{-1}E)$, for all $E\in\mathcal{B}$.

there might be no possibility to match the sequence $\underline{\sigma}_{\Lambda^c}$ outside Λ with at least one sequence $\underline{\sigma}_\Lambda$ internal to Λ so that $\underline{\sigma}_\Lambda\underline{\sigma}_{\Lambda^c}$ is T–compatible. In the latter case the definition becomes more involved as one has to appeal to the fact that sequences $\underline{\sigma}$ with probability 1 will be compatible: we simply state as a part of the definition that Λ be connected and long enough to avoid having to deal with the problem, as it is not necessary to do so. Of course once the conditional probabilities relative to an interval Λ are known the ones relative to shorter intervals $\Lambda' \subset \Lambda$ are also determined so that requiring property (5.1.9) for long intervals L is not restrictive.

(3) The reader familiar with the theory of Markov processes will recognize without difficulty that the case $\Phi_X = 0$ if $|X| > 2$ or if $X = \{x, y\}$ is not a pair of nearest neighbors corresponds to the case of a mixing Markov process: hence *Gibbs processes*, *i.e.* Gibbs distributions, constitute a (non-trivial) generalization of Markov processes.

(4) We shall denote the expression (5.1.9) with the symbol $p_\Phi(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda^c})$ and its denominator will be called simply the "normalization".

(5) The properties (5.1.8) and the equivalent (5.1.9) are called *DLR equations.* Here they are taken as defining properties of Gibbs distributions. However in other approaches the Gibbs distributions are defined in a different way and the DLR relations become theorems. See also Section §(6.1).

Before discussing some properties of Gibbs distributions it is convenient to set the following definition.

D5.1.3 **(5.1.3) Definition:** (Bulk and surface energies)
In the context of the definitions (5.1.1) and (5.1.2), let $\Phi \in B$, $\Omega \in \{0,\ldots,n\}^{\mathbb{Z}}_T$, $\Lambda \subset \mathbb{Z}$, $|\Lambda| < +\infty$, *and set*

e5.1.11
$$\begin{aligned} &\textit{(i)}\quad U^0_\Lambda(\underline{\sigma}) = \sum_{R\subset\Lambda} \Phi_R(\underline{\sigma}_R),\\ &\textit{(ii)}\quad U_\Lambda(\underline{\sigma}) = \sum_{R\cap\Lambda\neq\emptyset} \Phi_R(\underline{\sigma}_R),\\ &\textit{(iii)}\quad E_\Lambda(\underline{\sigma}) = \sum_{j\in\Lambda} A_\Phi(\tau^j\underline{\sigma}),\end{aligned} \tag{5.1.11}$$

which we call, respectively, the energy of $\underline{\sigma}_\Lambda$ in Λ, *the* energy of $\underline{\sigma}_\Lambda$ in Λ with boundary condition $\underline{\sigma}_{\Lambda^c}$ outside Λ, *and the* contribution of Λ to the energy of the configuration $\underline{\sigma}$. *Furthermore we set*

e5.1.12
$$U_\Lambda(\underline{\sigma}) = E_\Lambda(\underline{\sigma}) + D_{\Lambda,1}(\underline{\sigma}), \qquad U^0_\Lambda(\underline{\sigma}) = E_\Lambda(\underline{\sigma}) + D_{\Lambda,2}(\underline{\sigma}), \tag{5.1.12}$$

and, finally,

e5.1.13
$$\varepsilon(\Lambda) = \sup_{\underline{\sigma}\in\{0,\ldots,n\}^{\mathbb{Z}}_T} \Big(|D_{\Lambda,1}(\underline{\sigma})| + |D_{\Lambda,2}(\underline{\sigma})|\Big), \tag{5.1.13}$$

e5.1.14
$$\varepsilon_{N,a} = \varepsilon([a, a+1, a+2, \ldots, a+N-1]), \tag{5.1.14}$$

and $D_{\Lambda,1}(\underline{\sigma}), D_{\Lambda,2}(\underline{\sigma})$ *will be called* surface corrections *or* surface terms.

Remarks: (1) The names come from the form and the interpretation of equations (5.1.11)÷(5.1.14) in the case of the theory of the potential discussed in the remark to definition (5.1.1): the reader should write the various quantities just defined in that case.
(2) Note that $|U^0_\Lambda(\underline{\sigma})|$, $|U_\Lambda(\underline{\sigma})|$, $|E_\Lambda(\underline{\sigma})| \le ||\Phi||\,|\Lambda|$.
(3) With the new notations the r.h.s. of (5.1.9) is written as

*e*5.1.15
$$p_\Phi(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda^c}) = \frac{\exp\big(-U_\Lambda(\underline{\sigma})\big)}{\text{normalization}}, \tag{5.1.15}$$

and it is an expression that *makes sense without referring to any distribution m*. It will be called the "probability of the configuration $\underline{\sigma}_\Lambda$ in presence of the external configuration $\underline{\sigma}_{\Lambda^c}$" and we shall say that it is "proportional to the exponential of the opposite of its energy". In Statistical Mechanics the probability distributions, on the space of the configurations of a system, which associate with a configuration a probability proportional to the exponential of $-U$, if U is the energy of the configuration, are called *Boltzmann–Gibbs distributions* and have great importance for Physics.
(4) An immediate consequence of the finiteness of the norm $\|\Phi\|$ is that

*e*5.1.16
$$\lim_{N\to\infty} \varepsilon_{N,a}/N = 0, \tag{5.1.16}$$

that explains the name of "surface corrections" used for $D_{\Lambda,1}(\underline{\sigma}), D_{\Lambda,2}(\underline{\sigma})$. The reader should check that if $B^0 \subset B$ is the space of the *finite range potentials*, *i.e.* of the potentials $\Phi \in B$ with $\Phi_X = 0$ except for a finite number of sets X, then

*e*5.1.17
$$\varepsilon_{N,a} \le \|\Phi\| R_\Phi, \tag{5.1.17}$$

where R_Φ is defined by

*e*5.1.18
$$R_\Phi = \max_{X,\Phi_X\ne 0} \operatorname{diam}(X), \tag{5.1.18}$$

and is called the *range* of Φ.

We conclude this section by proving the following proposition.

*P*5.1.1 **(5.1.1) Proposition:** (Existence of Gibbs states)
If T is a mixing compatibility matrix with labels $0,\ldots,n$ and if Φ is a potential on $\{0,\ldots,n\}^{\mathbb{Z}}$ then the sets $G^0(\Phi)$ and $G(\Phi)$ are not empty: therefore there exists at least one translation invariant Gibbs distribution with potential $\Phi \in B$, cf. *definition (5.1.1).*

Proof: To check the proposition we begin with a general remark on probability distributions on spaces of sequences: as implied by the definition of conditional probability a function $\underline{\sigma}' \to m(\underline{\sigma}_\Lambda|\underline{\sigma}'_{\Lambda^c})$ is the conditional probability of the event $C^\Lambda_{\underline{\sigma}_\Lambda}$ with respect to the σ–algebra $\mathcal{B}(\Lambda^c)$ and to the probability distribution m if and only if, given arbitrarily two functions $\underline{\sigma} \to f(\underline{\sigma}_\Lambda)$ and $\underline{\sigma} \to g(\underline{\sigma}_{\Lambda^c})$ continuous and, respectively, $\mathcal{B}(\Lambda)$ and $\mathcal{B}(\Lambda^c)$–measurable, the following *Fubini's conditional integration* holds

e5.1.19
$$\int f g dm \equiv \sum_{\underline{\sigma}_\Lambda} \int f(\underline{\sigma}_\Lambda) g(\underline{\sigma}'_{\Lambda^c}) m(\underline{\sigma}_\Lambda|\underline{\sigma}'_{\Lambda^c}) m(d\underline{\sigma}'_{\Lambda^c}), \tag{5.1.19}$$

where, for ease of notations, we do not use a different symbol for the probability distribution m on $\mathcal{B}$ and for its restriction to the σ–algebra $\mathcal{B}(\Lambda^c)$.

Note, furthermore, that $\Phi \in B$ (so that $||\Phi|| < \infty$, see definition (5.1.1)) implies that $\underline{\sigma} \to p_\Phi(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda^c})$, defined by the (5.1.15) with Λ large enough, is a continuous function of the variable $\underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}}_T$.

We first show that $G^0(\Phi)$ is a convex and compact (possibly empty) set if it is regarded as a subset of the space of the probability distributions $\mathcal{M}^0(\{0, \ldots, n\}^{\mathbb{Z}}_T)$ thought of, as usual, as a topological space with the weak topology induced on it by the continuous functions on $\{0, \ldots, n\}^{\mathbb{Z}}_T$.

Indeed (5.1.19) written for m_1 and m_2 in $G^0(\Phi)$, *i.e.* with $m(\underline{\sigma}_\Lambda \,|\, \underline{\sigma}_{\Lambda_c}) = p_\Phi(\underline{\sigma}_\Lambda \,|\, \underline{\sigma}_{\Lambda_c})$ in both cases, implies that also $\alpha m_1 + (1-\alpha) m_2$, $\alpha \in (0,1)$, is in $G^0(\Phi)$ (as we see by performing the suitable linear combination of (5.1.19) written for m_1 and m_2). Furthermore, if we consider a sequence of probability distributions $m_k \in G^0(\Phi)$ weakly converging to m and we write (5.1.19) for m_k we see that, due to the continuity of f, g and p_Φ as
N5.1.6 functions of $\underline{\sigma}$, also $m \in G^0(\Phi)$, *i.e.* that $G^0(\Phi)$ is closed (hence compact).[6]

Another preliminary observation is that if $\Phi^k \in B$, $k = 1, 2, \ldots$ is a sequence of potentials in B which converges in the norm (5.1.5) to $\Phi \in B$ and if $m_k \in G^0(\Phi^k)$, $k = 1, \ldots$, is a corresponding sequence of Gibbs distributions converging to a probability distribution m, then $m \in G^0(\Phi)$ (*continuity of Gibbs states as functions of their potential*). In fact the convergence of Φ^k to a Φ in B implies, as it is immediate to see, that for every $\Lambda = \{-N, \ldots, N\}$, $N > a(T)$, the limit

e5.1.20
$$\lim_{k \to \infty} p_{\Phi^k}(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda^c}) = p_\Phi(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda^c}) \tag{5.1.20}$$

takes place uniformly in $\underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}}_T$. Therefore, due to the uniformity in $\underline{\sigma}$ of the limit (5.1.20), equation (5.1.19) written for m_k implies that $m \in G^0(\Phi)$ (by taking the limit $k \to \infty$).

From the latter considerations and from the metric compactness of the space $M^0(\{0, \ldots, n\}^{\mathbb{Z}}_T)$ of the probability distributions on $\{0, \ldots, n\}^{\mathbb{Z}}_T$ it follows that it will suffice to show that $G^0(\Phi) \neq \emptyset$, for all $\Phi \in B^0$, where $B^0 \subset B$ is an arbitrary subset of B which is dense in B in the norm (5.1.5).

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[6] We always consider the weak topology on the space of probability distributions, *i.e.* two probability distributions are close to each other if the integrals that they attribute to a finite large enough family of continuous functions are close to each other, see footnote 1, Section §2.3. In this topology the space of the probability distributions on a compact separable space is compact.

This will also show that $G(\Phi) \neq \emptyset$: indeed if $G^0(\Phi) \neq \emptyset$ we shall be able to consider, given $m_0 \in G^0(\Phi)$, the average over translations

e5.1.21

$$m_0^{(N)}(E) = N^{-1} \sum_{k=0}^{N-1} m_0(\tau^k E), \qquad E \in \mathcal{B}(\{0,\ldots,n\}_T^{\mathbb{Z}}), \tag{5.1.21}$$

which is a convex combination of elements in $G^0(\Phi)$ because if $m \in G^0(\Phi)$ also $E \to m_0(\tau E)$ is a probability distribution in $G^0(\Phi)$ (as follows from the translation invariance of Φ and p_Φ and from (5.1.19)).

Every limit point of the sequence $m_0^{(N)}$ defined in (5.1.21) is in $G^0(\Phi)$, since $G^0(\Phi)$ is closed, and furthermore it is obviously τ–invariant, *i.e.* it is in $G(\Phi)$, which, therefore, is not empty.

It remains to show that $G^0(\Phi) \neq \emptyset$ for Φ in a dense set $B^0 \subset B$: it is natural to select the set B^0 to be the set of the finite range potentials (cf. remark (4) to definition (5.1.3)). If $\Phi \in B^0$ we denote by R_Φ its range, as in (5.1.18), and define a sequence of "approximate" Gibbs distributions via the integrals that they assign to continuous functions.

Let $\widehat{\underline{\sigma}}$ be a configuration arbitrarily chosen in $\{0,\ldots,n\}_T^{\mathbb{Z}}$ (which will be used to define a boundary condition) and let $\Lambda_k = [-k,k]$, $k > a(T)$. We define

e5.1.22

$$\int f(\underline{\sigma}) m_{k,\widehat{\underline{\sigma}}}(d\underline{\sigma}) \overset{def}{=} \sum_{\underline{\sigma}'_{\Lambda_k}} f(\underline{\sigma}'_{\Lambda_k} \widehat{\underline{\sigma}}_{\Lambda_k^c}) \frac{e^{-U_{\Lambda_k}(\underline{\sigma}'_{\Lambda_k} \widehat{\underline{\sigma}}_{\Lambda_k^c})}}{\sum_{\underline{\sigma}''_{\Lambda_k}} e^{-U_{\Lambda_k}(\underline{\sigma}''_{\Lambda_k} \widehat{\sigma}_{\Lambda_k^c})}}, \tag{5.1.22}$$

where the two sums are over the configurations $\underline{\sigma}'_{\Lambda_k}$ and $\underline{\sigma}''_{\Lambda_k} \in \{0,\ldots,n\}^{\Lambda_k}$ compatible with $\widehat{\underline{\sigma}}$, *i.e.* such that $\underline{\sigma}'_{\Lambda_k} \widehat{\underline{\sigma}}_{\Lambda_k^c}$ as well as $\underline{\sigma}''_{\Lambda_k} \widehat{\sigma}_{\Lambda_k^c}$ are sequences in $\{0,\ldots,n\}_T^{\mathbb{Z}}$. The sums in the denominator involve at least one positive element because $k > a(T)$. It should be noted that $m_{k,\widehat{\underline{\sigma}}}$ only depends on $\widehat{\underline{\sigma}}$ via $\widehat{\underline{\sigma}}_{\Lambda_k^c}$: in other words fixing $\widehat{\underline{\sigma}}$ actually is a convenient way to fix a sequence of boundary conditions on the intervals Λ_k.

We can compute the conditional probability $m_k(\underline{\sigma}_\Lambda | \underline{\sigma}_{\Lambda^c})$ of the probability distributions m_k just defined, finding

e5.1.23

$$m_k(\underline{\sigma}_\Lambda | \underline{\sigma}_{\Lambda^c}) = p_\Phi(\underline{\sigma}_\Lambda | \underline{\sigma}_{\Lambda^c}) \qquad \text{for all } \underline{\sigma} \in \{0,\ldots,n\}_T^{\mathbb{Z}} \tag{5.1.23}$$

for every interval $\Lambda = [a,b]$ such that $|\Lambda| > a(T)$ and Λ is "well inside" Λ_k, *i.e.* $\Lambda^{R_\Phi} = [a - R_\Phi, b + R_\Phi] \subset \Lambda_k$.

Therefore m_k fails to verify (5.1.9), *i.e.* fails to be in $G^0(\Phi)$, "only" because its validity is restricted to Λ well inside Λ_k. Hence it is natural to take the limit $k \to \infty$: let $m_{k_i} \to m$ be a convergent subsequence extracted from the sequence $\{m_k\}_{k \in \mathbb{N}}$.

Imagine to write (5.1.19) for cylindrical functions f and g (*i.e.* for functions dependent on $\underline{\sigma}$ via the values of σ_j corresponding to a finite number of $j \in \mathbb{Z}$); then the k–independence of the r.h.s. of (5.1.23) (for $\Lambda_k \supset \Lambda$)

implies that the limit m, as $k \to \infty$ of the selected subsequence verifies (5.1.19) for all pairs of cylindrical functions f, g.

By the density of the cylindrical functions in the continuous functions and by (5.1.23), (5.1.19) extends to all pairs of continuous functions so that $m \in G^0(\Phi)$ and $G^0(\Phi)$, hence $G(\Phi)$ as well, is not empty. ■

In fact the method of the above proof gives us also an approximation procedure for the actual construction of an element of $G^0(\Phi)$.

C5.1.1 **(5.1.1) Corollary:** (Continuity of Gibbs states with respect to the potential)

Under the hypotheses of proposition (5.1.1) one has:

(i) $G^0(\Phi)$ and $G(\Phi)$ are not empty, convex, compact and τ–invariant.

(ii) If $\Phi^{(k)} \xrightarrow[k\to\infty]{} \Phi$ is a convergent sequence of potentials in B (convergence in the norm (5.1.5) of B) and if $m_k \in G^0(\Phi^{(k)})$ and $m_k \xrightarrow[k\to\infty]{} m$, then $m \in G^0(\Phi)$; symbolically we shall write

e5.1.24
$$\lim_{\Phi\to\overline{\Phi}} G^0(\Phi) \subset G^0(\overline{\Phi}). \tag{5.1.24}$$

(iii) If $\Phi^{(k)} \xrightarrow[k\to\infty]{} \Phi$ in B and if $m_k \in G(\Phi^{(k)})$ and $m_k \xrightarrow[k\to\infty]{} m$, then $m \in G(\Phi)$; symbolically

e5.1.25
$$\lim_{\Phi\to\overline{\Phi}} G(\Phi) \subset G(\overline{\Phi}). \tag{5.1.25}$$

Proof: (i) and (ii) have been proved within the proof of proposition (5.1.1) and (iii) is a consequence of (ii). ■

Problems for §5.1

Q5.1.1 **[5.1.1]:** (*Bernoulli shifts and Gibbs states*)

Study the Gibbs state on $\{0,1\}^{\mathbb{Z}}$ with potential $\Phi_X = 0$ if $|X| \neq 1$ and $\Phi_{\{x\}}(\sigma_x) = h(\sigma_x)$ and prove that it is a Bernoulli scheme.

Q5.1.2 **[5.1.2]:** (*A Markov process*)

Study the Gibbs state on $\{0,1,2\}^{\mathbb{Z}}_T$ with $\Phi = 0$ and $T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and prove that is a Markov process. Compute its transition probabilities in terms of the spectral properties of the matrix T.

Q5.1.3 **[5.1.3]:** (*Gibbs distributions on higher dimensional lattices*)

Generalize the notion of Gibbs distribution to the case of the space of d-dimensional sequences $\{0,1\}^{\mathbb{Z}^d}$, without compatibility conditions, replacing the translation τ with the group of the translations of $\mathbb{Z}^d$ so that the statements analogous to proposition (5.1.1) and to corollary (5.1.1) remain valid. (*Hint:* Just suppose that Λ is a finite cubic region and require that (5.1.15) gives the conditional probabilities of configurations in Λ when outside of Λ there is a fixed configuration.)

Q5.1.4 **[5.1.4]:** (*Gibbs states and Markov processes in $\mathbb{Z}^d$*)

Generalize the notion of Markov process to the space of the sequences $\{0,1\}^{\mathbb{Z}^d}$. (*Hint:* See problem [5.1.3] and suppose that the range of Φ is 1, *i.e.* identify Markov processes with nearest neighbour interaction Gibbs states.).

Q5.1.5 **[5.1.5]:** (*Averintzev–Spitzer theorem*)
Consider the most general probability distribution on $\{0,1\}^{\mathbb{Z}^2}$ with conditional probability $m(\sigma_0|\underline{\sigma}') = p(\sigma_0|\sigma'_1\sigma'_2\sigma'_3\sigma'_4)$ where $1,2,3,4$ are the 4 nearest neighbour sites of 0 and $\sigma_0\underline{\sigma}' \in \{0,1\}^{\mathbb{Z}^2}$. Suppose, furthermore, that m is invariant under rotations (of $\pi/2$) and that $p(.|\ldots) > 0$. Show that m is a Gibbs state with a potential Φ which can be chosen so that $\Phi_X = 0$ if $|X| > 2$, $\Phi_{\{i,j\}}(\sigma_i,\sigma_j) = 0$ if $|i-j| > 1$, $\Phi_{\{i,j\}}(\sigma_i,\sigma_j) = J\sigma_i\sigma_j$ if $|i-j| = 1$ and $\Phi_{\{0\}}(\sigma_0) = h\sigma_0$.

Q5.1.6 **[5.1.6]:** (*An equivalence condition for potentials*)
If B is the space of potentials on $\{0,\ldots,n\}^{\mathbb{Z}}_T$, show that if Φ and $\Psi \in B$ and if there exists $C \in \mathbb{R}$ and a continuous function F over $\{0,\ldots,n\}^{\mathbb{Z}}_T$ such that $A_\Phi(\underline{\sigma}) = C + A_\Psi(\underline{\sigma}) + F(\underline{\sigma}) - F(\tau\underline{\sigma})$, then $G(\Phi) = G(\Psi)$. (*Hint:* Use (5.1.8)).

Q5.1.7 **[5.1.7]:** (*Potentials and Hölder continuous functions*)
If $A \in C(\{0,\ldots,n\}^{\mathbb{Z}}_T)$ is a Hölder continuous function and if T is a mixing compatibility matrix then there exist $\Phi \in B$ such that $A = A_\Phi$. (*Hint:* See proposition (4.3.2) and equation (4.3.13).)

Q5.1.8 **[5.1.8]:** (*Particle potentials*)
Let B^0 be the space of finite range potentials. Under the hypotheses of problem [5.1.6] with $n = 1$, $T_{\sigma\sigma'} > 0$, show that if $\Phi \in B^0$ there always exists $\Psi \in B^0$ such that $G(\Phi) = G(\Psi)$ and $\Psi_X(\sigma_X) = 0$ unless $\sigma_\xi \neq 0$, for some $\xi \in X$. We say that Ψ is a potential equivalent to Φ and with $\underline{\sigma} = \underline{0}$ as *vacuum configuration.* (*Hint:* Define Ψ recursively keeping in mind the equivalence condition in problem [5.1.6].)

Q5.1.9 **[5.1.9]:** (*Existence of particle potentials*)
In the context of the problem [5.1.8] find some sufficient conditions in order that $\Phi \notin B^0$ admits an equivalent potential Ψ (*i.e.* such that $G(\Phi) = G(\Psi)$) with the property of the Ψ in problem [5.1.8]. (*Hint:* The recursion suggested in the hint to problem [5.1.8] leads to an expression for Ψ in terms of sums over values of Φ over various configurations. One just makes sure that the sums involved are absolutely convergent, including the sum that provides an estimate for the norm $||\Psi||$.)

Q5.1.10 **[5.1.10]:** (*Open boundary conditions*)
Assume that the compatibility matrix T has no vanishing entries. Replace in the r.h.s. of (5.1.22) $U_{\Lambda_k}(\underline{\sigma}_{\Lambda_k}\widehat{\underline{\sigma}}_{\Lambda_k^c})$ with $U^0_{\Lambda_k}(\underline{\sigma}_{\Lambda_k})$ and $f(\underline{\sigma}_{\Lambda_k}\underline{\sigma}_{\Lambda_k^c})$ with $f(\underline{\sigma}_{\Lambda_k}\underline{0})$ where $\underline{0}$ denotes the sequence of symbols identically 0. This is equivalent to replacing m_k with

$$\widetilde{m}_k = \sum_{\underline{\sigma}_{\Lambda_k}} \frac{\exp\big(-U^0_{\Lambda_k}(\underline{\sigma}_{\Lambda_k})\big)}{\sum_{\underline{\sigma}'_{\Lambda_k}} \exp\big(-U^0_{\Lambda_k}(\underline{\sigma}'_{\Lambda_k})\big)}\,\delta_{(\underline{\sigma}_{\Lambda_k}\underline{0})},$$

where $\delta(\underline{\sigma}_{\Lambda_k}\underline{0})$ is the Dirac probability distribution concentrated on the configuration that inside Λ_k coincides with $\underline{\sigma}_{\Lambda_k}$ and outside of Λ_k is identically zero. The distribution $\widetilde{m}_k$ is called a *finite volume Gibbs distribution with open boundary conditions* when $\underline{0}$ is the "vacuum" for the potential Φ in the sense of problem [5.1.8]. Show that every limit point of the sequence $\widetilde{m}_k$ is in $G^0(\Phi)$ if $\Phi \in B^0$, where B^0 is space of finite range potentials. (*Hint:* Repeat word by word the proof of proposition (5.1.1).)

Q5.1.11 **[5.1.11]:** Same as problem [5.1.10] but replacing B^0 with the set B (in which B^0 is dense).

Q5.1.12 **[5.1.12]:** Adapt definitions and results of Section §5.1 and the corresponding problems to the case in which the matrix T is just transitive (rather than mixing).

Bibliographical note to §5.1 and §(5.2)

The notion of Gibbs distribution represents an interesting product of the close interaction that took place between Mathematical Physics and Theoretical Physics in the 1960's. It was due to two main reasons. On the one hand a large number of scientists with a basic formation and research activity experience in High Energy Physics became interested in the mathematical problems connected with their previous works. This was done within the framework of a general rethinking on the foundations of a theory that seemed to undergo a deep methodology crisis (field theory, *i.e.* relativistic quantum mechanics, after the failure of its naive application to the theory of strong interactions). On the other hand the need by several condensed matter physicists to refine the theoretical prediction instruments of statistical mechanics in order to interpret the experimental results on phase transitions (that were produced in great abundance thanks to the substantial progress of the experimental techniques). The interest into rigorous results developed mainly for the purposes of having reliable terms of comparison to check the reliability of hitherto uncontrolled approximations needed to solve delicate theoretical problems like the theoretical computation of the critical exponents (made possible, to a previously unimaginable extent, by the progress of electronic computational machines)

The notion of Gibbs distribution was developed independently in the West (thanks mainly to the works of Ruelle, Fisher, Griffiths, Lanford, *etc.*) and in the East (thanks mainly to the works of Dobrushin, Minlos, Sinai, etc.).

The more or less definitive formulation of the notion and of the basic properties of a Gibbs distribution can be found in the classical papers [Do68a], [Do68b], [Do68c], [Do69], [Ru69] and [LR69].

§5.2 Properties of Gibbs distributions

For a better understanding of the nature and properties of Gibbs distributions we shall discuss an important uniqueness criterion and some of its simple consequences.

*P*5.2.1 **(5.2.1) Proposition:** (Uniqueness of Gibbs distributions)
Let T a mixing compatibility matrix with labels $0,\ldots,n$ and with mixing time $a(T)$. Let $\Phi\in B$ be a potential (see definition (5.1.1)) on $\{0,\ldots,n\}_T^{\mathbb{Z}}$ and suppose

e5.2.1
$$\|\Phi\|_1=\sum_{X\ni 0}\frac{(1+\mathrm{diam}(X))}{|X|}\,\|\Phi_X\|<+\infty, \tag{5.2.1}$$

(i) Every T–compatible cylinder has positive m–measure for all probability distributions $m\in G(\Phi)$ (hence every set of m–probability 1 is dense in $\{0,\ldots,n\}_T^{\mathbb{Z}}$).
(ii) $G^0(\Phi)$ contains a unique element m.

Remarks: (1) Condition (5.2.1) says that "the interaction energy between the left half of a configuration $\underline{\sigma}$ and the other half is finite, uniformly in

$\underline\sigma$": indeed the interaction energy of the left half of the configuration $\underline\sigma$ with the right half is naturally defined by

e5.2.2
$$\sum_{R\cap\mathbb{Z}_+\ne\emptyset,\,R\cap\mathbb{Z}_-\ne\emptyset}\Phi_R(\underline\sigma_R)=W(\underline\sigma^-,\underline\sigma^+),\tag{5.2.2}$$

having performed the division of $\underline\sigma$ in the left half $\underline\sigma^-\equiv(\sigma_j)_{j<0}$ and the right half $\underline\sigma^+=(\sigma_j)_{j\ge0}$. Then (5.2.2) is bounded above by (5.2.1) as

e5.2.3
$$|W(\underline\sigma^-,\underline\sigma^+)|\le\|\Phi\|_1;\tag{5.2.3}$$

furthermore there exists potentials $\Phi\in B$ and sequences $\underline\sigma\in\{0,\ldots,n\}^{\mathbb{Z}}_T$ for which equality sign holds.
(2) Technically (5.2.1) or (5.2.3) will be employed to compare conditional probabilities. For example if $\underline\sigma'=(\ldots\widetilde\sigma_{a-1}\sigma'_a\ldots\sigma'_b\widetilde\sigma_{b+1}\ldots)$ and $\underline\sigma''=(\ldots\widetilde\sigma_{a-1}\sigma''_a\ldots\sigma''_b\widetilde\sigma_{b+1}\ldots)$ are in $\{0,\ldots,n\}^{\mathbb{Z}}_T$ and if we set $\Lambda=[a,b]$, one has

e5.2.4
$$\begin{aligned}\frac{p_\Phi(\sigma'_a\ldots\sigma'_b|\ldots\widetilde\sigma_{a-1}\widetilde\sigma_{b+1}\ldots)}{p_\Phi(\sigma''_a\ldots\sigma''_b|\ldots\widetilde\sigma_{a-1}\widetilde\sigma_{b+1}\ldots)}&=e^{-\sum_{R\cap\Lambda\ne\emptyset}(\Phi_R(\underline\sigma'_R)-\Phi_R(\underline\sigma''_R))}\le\\ &\le e^{4\|\Phi\|_1}e^{-\sum_{R\subset\Lambda}(\Phi_R(\underline\sigma'_R)-\Phi_R(\underline\sigma''_R))},\end{aligned}\tag{5.2.4}$$

as it follows immediately from (5.2.1) and from the translation invariance of Φ.
Likewise if $(\ldots\widetilde\sigma_{a-1}\sigma_a\ldots\sigma_b\widetilde\sigma_{b+1})$ and $(\ldots\widehat\sigma_{a-1}\sigma_a\ldots\sigma_b\widehat\sigma_{b+1}\ldots)$ are in $\{0,\ldots,n\}^{\mathbb{Z}}_T$ one has, if $b-a>a(T)$,

e5.2.5
$$\frac{p_\Phi(\sigma_a\ldots\sigma_b|\ldots\widetilde\sigma_{a-1}\widetilde\sigma_{b+1}\ldots)}{p_\Phi(\sigma_a\ldots\sigma_b|\ldots\widehat\sigma_{a-1}\widehat\sigma_{b+1}\ldots)}\le e^{8\|\Phi\|_1}(n+1)^{a(T)},\tag{5.2.5}$$

as we see starting from the explicit expression for p_Φ, see (5.1.9) and (5.1.15), and paying attention to the normalization factor.
(3) Note that (5.2.4) and (5.2.5) also imply lower bounds with $\|\Phi\|_1$ replaced by $-\|\Phi\|_1$ and $(n+1)^{a(T)}$ by $(n+1)^{-a(T)}$, because of the arbitrariness of $\underline\sigma'$, $\underline\sigma''$, $\underline{\widetilde\sigma}$, $\underline{\widehat\sigma}$.

Proof: In the proof of the second statement we shall suppose, for simplicity, $a(T)=0$, *i.e.* $T_{\sigma\sigma'}\equiv1$.
Note that if $m,\,m_1\in G(\Phi)$ then m is absolutely continuous with respect to m_1, and viceversa, with Radon–Nykodim derivative between $\exp(-8\|\Phi\|_1)$ and $\exp(8\|\Phi\|_1)$. Indeed one has

e5.2.6
$$\begin{aligned}&m(C^{a\ldots b}_{\sigma_a\ldots\sigma_b})\equiv\int m(C^{a\ldots b}_{\sigma_a\ldots\sigma_b})\,m_1(d\underline{\widetilde\sigma})=\\ &=\int m_1(d\underline{\widetilde\sigma})\int m(d\underline{\widehat\sigma})p_\Phi(\sigma_a\ldots\sigma_b|\ldots\widehat\sigma_{a-1}\widehat\sigma_{b+1}\ldots)\le\\ &\le e^{8\|\Phi\|_1}\int m_1(d\underline{\widetilde\sigma})\int m(d\underline{\widehat\sigma})p_\Phi(\sigma_a\ldots\sigma_b|\ldots\widetilde\sigma_{a-1}\widetilde\sigma_{b+1}\ldots)\equiv\\ &\equiv e^{8\|\Phi\|_1}\int m_1(d\underline{\widetilde\sigma})p_\Phi(\sigma_a\ldots\sigma_b|\ldots\widetilde\sigma_{a-1}\widetilde\sigma_{b+1}\ldots)\equiv e^{8\|\Phi\|_1}m_1(C^{a\ldots b}_{\sigma_a\ldots\sigma_b}),\end{aligned}\tag{5.2.6}$$

having used inequality (5.2.5) in the third step.

The arbitrariness of a, b and $\sigma_a \ldots \sigma_b$ allow us to deduce from (5.2.6) that for all $E \in \mathcal{B}$

e5.2.7
$$m(E) \le e^{8\|\Phi\|_1} m_1(E), \tag{5.2.7}$$

that shows that m is absolutely continuous with respect to any probability distribution m_1 with Radon–Nykodim derivative f that, by the symmetric role of m and m_1, is in $L_\infty(m_1)$ and

e5.2.8
$$e^{-8\|\Phi\|_1} \le f(\underline{\sigma}) \le e^{8\|\Phi\|_1} \qquad m_1 - \text{almost everywhere.} \tag{5.2.8}$$

One has $m = f\, m_1$ and, at the same time, m and m_1 have the same conditional probability $p_\Phi(\underline{\sigma}_\Lambda | \underline{\sigma}_{\Lambda^c})$, for all $\Lambda = [a, b]$: this implies that for every $\Lambda = [a, b]$ and every $\underline{\sigma}^1$, $\underline{\sigma}^2 \in \{0, \ldots, n\}^\Lambda$ one has

e5.2.9
$$\frac{f(\underline{\sigma}^1_\Lambda \underline{\sigma}_{\Lambda^c})}{f(\underline{\sigma}^2_\Lambda \underline{\sigma}_{\Lambda^c})} \equiv 1 \qquad m - \text{almost everywhere in } \underline{\sigma}_{\Lambda^c}, \tag{5.2.9}$$

N5.2.1 which from a formal point of view is an obvious relation.[1]

If we set for $k \ge 0$

e5.2.10
$$r(k) \equiv \int f(\underline{\sigma})^k \, m_1(d\underline{\sigma}), \tag{5.2.10}$$

it follows, also, that $r(k)^{-1} f^k m_1$ is (for every $k \in \mathbb{Z}$) a probability distribution of $G^0(\Phi)$, since (5.2.9) remains valid if we replace f with the powers of f itself. By (5.2.8) applied by selecting $m' = r(k)^{-1} f^k m_1$ instead of m one has

e5.2.11
$$e^{-8\|\Phi\|_1} \le r(k)^{-1} f^k(\underline{\sigma}) \le e^{8\|\Phi\|_1} \qquad m_1 - \text{almost everywhere,} \tag{5.2.11}$$

for all $k \in \mathbb{Z}$.

The relation (5.2.11) and the arbitrariness of k imply that f is constant m_1–almost everywhere, *i.e.* $m = m_1$. Hence $G^0(\Phi)$ consists of a single point and, therefore, $G^0(\Phi) = G(\Phi)$. This proves the statement (ii).

To show that $m(C^{a\,\cdots\,b}_{\sigma_a \cdots \sigma_b}) > 0$ if $C^{a\,\cdots\,b}_{\sigma_a \cdots \sigma_b}$ is T–compatible we treat the general case, since the case $T_{\sigma\sigma'} = 1$ is easy but too special. Having set $J = [a - a(T), b + a(T)]$, let $\overline{\underline{\sigma}}_J$ be a string such that $m(C^J_{\overline{\underline{\sigma}}_J}) > 0$ (which certainly exists because otherwise m itself would vanish). Then

e5.2.12
$$m(C^J_{\overline{\underline{\sigma}}_J}) = \int p_\Phi(\overline{\underline{\sigma}}_J | \underline{\sigma}'_{J^c}) m(d\underline{\sigma}') > 0, \tag{5.2.12}$$

and, hence, there exists a set D of configurations $\underline{\sigma}'$ such that $m(D) > \varepsilon$ and $p_\Phi(\overline{\underline{\sigma}}_J | \underline{\sigma}'_{J^c}) > \varepsilon'$, for $\underline{\sigma}' \in D$, for some ε, $\varepsilon' > 0$. One can therefore construct a configuration $\widehat{\underline{\sigma}}_J$, T–compatible, that coincides with $\overline{\underline{\sigma}}_J$ on the

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[1] A rigorous check of (5.2.9) passes through Doob's theorem already cited in Section §3.2 after (3.2.25): we leave this to the reader although it is somewhat subtle.

extreme sites $a-a(T)$ and $b+b(T)$ and with $\sigma_a,\ldots,\sigma_b$ in the sites in $[a,b]$: it is clear that $\widehat{\underline{\sigma}}_J$ is T–compatible with $\underline{\sigma}'_{J^c}$ for all $\underline{\sigma}'$ in D. Hence (5.2.4), the mentioned properties of $m(D)$ and the strict positivity of $p_\Phi(\underline{\sigma}_J|\underline{\sigma}'_{J^c})$ whenever $\underline{\sigma}_J$ and $\underline{\sigma}'_{J^c}$ are T–compatible immediately imply that also $m(C^J_{\widehat{\underline{\sigma}}_J})>0$. Since the cylinder $C^{a\,\ldots\,b}_{\sigma_a\ldots\sigma_b}$ that we are considering contains, by construction, $C^J_{\underline{\sigma}_J}$ the result follows.

Note the great simplification of the above proof in the case $a(T)=0$, *i.e.* in the case in which all sequences are compatible. ■

$C5.2.1$ **(5.2.1) Corollary:** (Ergodicity and mixing of Gibbs states)
Under the hypotheses of proposition (5.2.1) one has $G^0(\Phi)=G(\Phi)=G_e(\Phi)=G_m(\Phi)$.

Remark: Hence if $\|\Phi\|_1<+\infty$ the corresponding Gibbs distribution m is ergodic, and in fact mixing.

Proof: Let m be an arbitrary probability distribution on $\{0,\ldots,n\}^{\mathbb{Z}}_T$, possibly not in $G(\Phi)$. Let $\mathcal{B}(\infty)_m$ be the m–complete σ–algebra generated by the functions of $L_1(m)$ that are $\mathcal{B}([-N,N]^c)$–measurable for all $N>0$.

The latter functions are often called *functions measurable at infinity* because their values do not change by changing any finite number of labels in their argument $\underline{\sigma}$. They form an algebra that is called the *algebra at infinity of the probability distribution m.*

$N5.2.2$ If $\mathcal{B}([-N,N]^c)_m$ is the completion**[2]** with respect to m of $\mathcal{B}([-N,N]^c)$ one has, by definition

$e5.2.13$
$$\mathcal{B}(\infty)_m=\bigcap_{N>0}\mathcal{B}([-N,N]^c)_m. \tag{5.2.13}$$

If $f>0$ is a $\mathcal{B}(\infty)_m$–measurable function and if it is m–summable with integral 1, then the probability distribution $m_1=fm$ has the same conditional probabilities, *i.e.* $m(\underline{\sigma}_\Lambda|\underline{\sigma}_{\Lambda^c})$, of the probability distribution m. Indeed such probabilities would be in general given by the left hand side expression in the relation

$e5.2.14$
$$\frac{f(\underline{\sigma}'_\Lambda\underline{\sigma}_{\Lambda^c})m(\underline{\sigma}'_\Lambda|\underline{\sigma}_{\Lambda^c})}{f(\underline{\sigma}''_\Lambda\underline{\sigma}_{\Lambda^c})m(\underline{\sigma}''_\Lambda|\underline{\sigma}_{\Lambda^c})}=\frac{m(\underline{\sigma}'_\Lambda|\underline{\sigma}_{\Lambda^c})}{m(\underline{\sigma}''_\Lambda|\underline{\sigma}_{\Lambda^c})}, \tag{5.2.14}$$

where the equality takes place because, if f is $\mathcal{B}(\infty)$–measurable, $f(\underline{\sigma}'_\Lambda\underline{\sigma}_{\Lambda^c})=f(\underline{\sigma}''\underline{\sigma}_{\Lambda^c})$ since f must assume the same value on configurations that differ only in a finite number of sites.

It follows that if $m\in G(\Phi)$ one must have that $\mathcal{B}(\infty)_m$ is a trivial σ–algebra: if indeed nonconstant $\mathcal{B}(\infty)_m$–measurable functions f existed then there would exist positive ones among them and bounded away from 0 and $+\infty$ (*e.g.* if f was such a function one could take $(1+|f(\cdot)|)/(2+|f(\cdot)|)$). It could therefore be possible, by multiplying m by any of the latter, to construct a Gibbs distribution fm different from m itself, against the uniqueness shown in proposition (5.2.1)

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[2] See appendix 1.2.

This implies that $m \in G_m(\Phi)$, *i.e.* that m is mixing. Indeed let $f,\, g \in C(\{0,\ldots,\}^{\mathbb{Z}}_T)$; consider the sequence of functions $\underline{\sigma} \to g(\tau^k \underline{\sigma})$, $k = 0, 1, \ldots$: by the compactness of the spheres of $L_\infty(m)$ with respect to the weak topology induced by $L_1(m)$ one can extract from the sequence a subsequence $g(\tau^{k_i}\underline{\sigma})$ converging, always in the weak topology, as $i \to \infty$ to a limit $\widetilde{g}(\underline{\sigma}) \in L_\infty(m)$ such that $\|\widetilde{g}\|_{L_\infty(m)} \le \max_{\underline{\sigma}} |g(\underline{\sigma})|$.

It is however clear that $\widetilde{g}$ is $\mathcal{B}(\infty)$–measurable because

e5.2.15
$$\lim_{k\to\infty} |g(\tau^k \underline{\sigma}') - g(\tau^k \underline{\sigma}'')| = 0, \tag{5.2.15}$$

if $\underline{\sigma}'$ and $\underline{\sigma}''$ differ only on a finite number of labels (recall that g is assumed continuous). It follows that $\widetilde{g}$ is constant m–almost everywhere, hence

e5.2.16
$$\widetilde{g} = \lim_{i\to\infty} m(1 \cdot \tau^{k_i} g) \equiv m(g), \tag{5.2.16}$$

i.e. $\widetilde{g}$ does not depend on the choice of the subsequence. Therefore we deduce that $g(\tau^k \cdot) \xrightarrow[k\to+\infty]{} m(g)$ in the $L_1(m)$–topology of $L_\infty(m)$. In other words, and in particular,

e5.2.17
$$\lim_{k\to\infty} m(f \tau^k\, g) = m(f)\, m(g) \tag{5.2.17}$$

for every pair of continuous functions f and g and, hence, by density it follows that m is mixing. ■

Remarks: (1) We have also shown that $\mathcal{B}(\infty)_m$ (and hence also $\mathcal{B}(-\infty)_m = \cap_{N>0}\mathcal{B}([-\infty,-N])_m \subseteq \mathcal{B}(\infty)_m$) is a trivial σ–algebra in the sense that all functions measurable with respect to it are necessarily constant. The latter property is, in general, stronger than mixing: the invertible dynamical systems $(\{0,\ldots,n\}^{\mathbb{Z}}, \tau, m)$ such that $\mathcal{B}(\infty)_m$ is trivial are called *systems with trivial algebra at infinity*, while those for which "already" $\mathcal{B}(-\infty)_m$ is trivial are called *K–systems* or *systems with trivial remote past.*

(2) The notion of K–system is naturally generalizable to invertible metric dynamical systems (Ω, S, μ): let $\mathcal{P} = \{P_0, \ldots, P_n\}$ be a nontrivial partition of Ω into μ–measurable sets (*i.e.* $n \ge 1$ and $\mu(P_i) > 0$ for all i) and let $(\{0,\ldots,n\}^{\mathbb{Z}}, \tau, m)$ be the symbolic dynamical system associated with the $(S,\mathcal{P})$–histories by (2.3.19) and proposition (2.3.1); *we shall say that* (Ω, S, μ) *is a K–system* if $\mathcal{B}(-\infty)_m$ is trivial for all choices of $\mathcal{P}$.

(3) An interesting property, among many, of K–systems is that one can show (Sinai) that the triviality of $\mathcal{B}(-\infty)_m$ is equivalent to the requirement that, for all nontrivial μ–measurable partitions, one has $s(\mathcal{P}, S, \mu) > 0$, cf. definition (3.3.2). We shall not enter into the discussion of the properties of K–systems.

Problems for §5.2

Q5.2.1 **[5.2.1]:** (*Fisher potential*)
Consider the potential on $\{0,1\}^{\mathbb{Z}}$ defined as follows:

(i) $\Phi_X(\underline{\sigma}_X) = 0$ unless $X = [a,b]$, $b \ge a$, and $\underline{\sigma}_X = (1,1,\ldots,1)$,
(ii) if $X = \{a, a+1, a+2, \ldots, a+n\}$ and $\sigma_a = \sigma_{a+1} = \ldots = \sigma_{a+n} = 1$

$$\Phi_X(\underline{\sigma}_X) = -\Phi_n.$$

Show that $\|\Phi\| = \sum_{n=0}^{\infty} |\Phi_n|$, $\|\Phi\|_1 = \sum_{n=0}^{\infty}(n+1)|\Phi_n|$, and check directly the inequality $|W(\underline{\sigma}^-, \underline{\sigma}^+)| \le \|\Phi\|_1$.

§5.3 Gibbs distributions on $\mathbb{Z}^+$

Let T be a $(n+1) \times (n+1)$ mixing compatibility matrix with mixing time $a(T)$ (*i.e.* $T^{a(T)+1}_{\sigma\sigma'} > 0$) and let $\Phi \in B$ be a potential on $\{0,\ldots,n\}^{\mathbb{Z}}_T$. Let $\{0,\ldots,n\}^{\mathbb{Z}^+}_T$ be the space of sequences $\underline{\sigma} \in \{0,\ldots,n\}^{\mathbb{Z}^+}_T$ such that $T_{\sigma_i\sigma_{i+1}} = 1$, $i = 0,1,\ldots$, *i.e.* the space of the "unilateral" sequences with labels in $\mathbb{Z}^+$.

If $m \in \mathcal{M}^0(\{0,\ldots,n\}^{\mathbb{Z}^+}_T)$ and if $\Lambda = [a, a+1, \ldots, b-1, b] \subset \mathbb{Z}^+$, $\underline{\sigma}_\Lambda \in \{0,\ldots,n\}^{\Lambda}_T$, we called the function $\underline{\sigma}' \to m(\underline{\sigma}_\Lambda | \underline{\sigma}'_{\Lambda^c})$, defined by $\frac{dm'}{dm}(\underline{\sigma}')$, where $m'(E) = m(E \cap C^\Lambda_{\underline{\sigma}_\Lambda})$, the "probability of $\underline{\sigma}_\Lambda$ conditional to $\underline{\sigma}'$ with respect to the probability distribution m" (cf. definition (5.1.2)).

D5.3.1 **(5.3.1) Definition:** (Semiinfinite Gibbs distributions)
Let $\Phi \in B$ be a potential for $\{0,\ldots,n\}^{\mathbb{Z}}_T$ (see definition (5.1.1)), where T is mixing with mixing time $a(T) \ge 0$; we shall say that $m \in \mathcal{M}^0(\{0,\ldots,n\}^{\mathbb{Z}^+}_T)$ is a semiinfinite Gibbs distribution *or a* Gibbs distribution on $\mathbb{Z}^+$ *with potential Φ if the conditional probabilities of m verify*

e5.3.1
$$\frac{m(\sigma'_0,\ldots,\sigma'_a|\sigma_{a+1},\ldots)}{m(\sigma''_0,\ldots,\sigma''_a|\sigma_{a+1},\ldots)} = \exp\Big(-\sum_{\substack{X\cap[0,a]\neq\emptyset\\ X\subset\mathbb{Z}^+}} [\Phi_X(\underline{\sigma}'_X) - \Phi_X(\underline{\sigma}''_X)]\Big) \qquad (5.3.1)$$

for every choice of $a \ge a(T)$ and of the sequences $\underline{\sigma}' = (\sigma'_0 \ldots \sigma'_a \sigma_{a+1} \ldots)$, $\underline{\sigma}'' = (\sigma''_0 \ldots \sigma''_a \sigma_{a+1}, \ldots) \in \{0,\ldots,n\}^{\mathbb{Z}^+}_T$.

We define the shift τ *on $\{0,\ldots,n\}^{\mathbb{Z}^+}_T$ as*

e5.3.2
$$\tau(\sigma_0, \sigma_1, \ldots) = (\sigma_1, \sigma_2, \ldots), \qquad (5.3.2)$$

and the semiinfinite energy per site *of Φ as*

e5.3.3
$$\widehat{A}(\underline{\sigma}) = \sum_{X\ni 0,\, X\subset\mathbb{Z}^+} \Phi_X(\underline{\sigma}_X) \qquad \underline{\sigma} \in \{0,\ldots,n\}^{\mathbb{Z}}_T. \qquad (5.3.3)$$

In terms of $\widehat{A}$ *we can equivalently say that* m *is a Gibbs distribution on* $\mathbb{Z}^+$ *with potential* $\Phi \in B$ *if*

e5.3.4
$$\frac{m(\sigma'_0 \ldots \sigma'_a|\sigma_{a+1}\ldots)}{m(\sigma''_0 \ldots \sigma''_a|\sigma_{a+1}\ldots)} = \exp\Big(-\sum_{k=0}^{\infty}(\widehat{A}(\tau^k\underline{\sigma}') - \widehat{A}(\tau^k\underline{\sigma}''))\Big), \tag{5.3.4}$$

with the notations in (5.3.1), (5.3.2) and (5.3.3), (note that the series in (5.3.4) is, in fact, a finite sum of at most $a+1$ *terms). We call* $G^0_+(\Phi)$ *the set of Gibbs distributions on* $\mathbb{Z}^+$ *with potential* $\Phi \in B$*, and* $G_+(\Phi)$ *the set of Gibbs distributions on* $\mathbb{Z}^+$ *with potential* $\Phi \in B$ *which are invariant under translation.*

Proceeding exactly as in Sections §5.1 and §5.2 one shows the following proposition, analogous to proposition (5.2.1).

P5.3.1 **(5.3.1) Proposition:** (Uniqueness of semiinfinite Gibbs distributions) *Under the hypotheses of proposition (5.1.1) one has*
(i) $G_+(\Phi)$ *contains at least one element for all* $\Phi \in B$*, and it is convex and compact;*
(ii) if $\|\Phi\|_1 < +\infty$, cf. *(5.2.1),* $G_+(\Phi)$ *contains a unique element* $\widetilde{m}_+$*;*
(iii) the distribution $\widetilde{m}_+$ *attributes positive probability to all* T*–compatible cylinders (hence every set of* $\widetilde{m}_+$*–probability* 1 *is dense in* $\{0,\ldots,n\}^{\mathbb{Z}^+}_T$ *);*
(iv) if $\|\Phi\|_1 < +\infty$ *and if* $\mathcal{B}(+\infty)_{\widetilde{m}^+} = \cap_{N>0}\mathcal{B}([N,+\infty])_{\widetilde{m}^+}$ *then* $\mathcal{B}(+\infty)_{\widetilde{m}^+}$ *is a trivial* σ*–algebra in the sense that if* f *is* $\mathcal{B}(+\infty)_{\widetilde{m}^+}$*–measurable then* f *is constant* $\widetilde{m}_+$*–almost everywhere;*
v) if $\Phi^{(k)} \to \Phi$ *in* B *then* $G_+(\Phi^{(k)}) \to G_+(\Phi)$ *in the sense analogous to that discussed in item (ii) of corollary (5.1.1).*

Analogous definitions can be given for *Gibbs distributions on* $\mathbb{Z}^-$: formulations are left to the reader.

The interest of the semiinfinite Gibbs distributions lies in the following remark. Let $\Lambda_k = [-k,k]$ and, for simplicity, let $T_{\sigma\sigma'} \equiv 1$. Then, if $\underline{\sigma} \in \{0,\ldots,n\}^{\mathbb{Z}}$ we can regard $\underline{\sigma}$ as composed by its "left and right parts" $\underline{\sigma}^- = (\sigma_i)_{i\in\mathbb{Z}^-}$ and $\underline{\sigma}^+ = (\sigma_i)_{i\in\mathbb{Z}^+}$, $\underline{\sigma} = (\underline{\sigma}^-, \underline{\sigma}^+)$; then (cf. second equation in (5.1.11))

e5.3.5
$$\begin{aligned} U_{\Lambda_k}(\underline{\sigma}_{\Lambda_k}, \underline{\sigma}_{\Lambda^c_k}) = &\sum_{\substack{X\subset\mathbb{Z}^-\\ X\cap\Lambda_k\neq\emptyset}} \Phi_X(\underline{\sigma}^-_X) + \sum_{\substack{X\subset\mathbb{Z}^+\\ X\cap\Lambda_k\neq\emptyset}} \Phi_X(\underline{\sigma}^+_X) + W_\Phi(\underline{\sigma}^-, \underline{\sigma}^+) = \\ &\overset{def}{=} U^-_k(\underline{\sigma}^-) + U^+_k(\underline{\sigma}^+) + W_\Phi(\underline{\sigma}^-, \underline{\sigma}^+), \end{aligned} \tag{5.3.5}$$

where W_Φ is defined in (5.2.2) with the further constraint $R \cap \Lambda_k \neq \emptyset$ (and it verifies (5.2.3)), and $U^\pm_k$ are implicitly defined in (5.3.5).

Equation (5.3.5), inserted in (5.1.23), shows that the "finite volume" approximation m_k, defined there, to the Gibbs distribution with potential Φ can be expressed as

e5.3.6
$$m_k(d\underline{\sigma}) = \frac{\widetilde{m}^+_k(d\underline{\sigma}^+)\,\widetilde{m}^-_k(d\underline{\sigma}^-)\,e^{-W_\Phi(\underline{\sigma}^+,\underline{\sigma}^-)}}{c(k)}, \tag{5.3.6}$$

where $c(k) > 0$ is a normalization constant and $\widetilde{m}_k^+$, $\widetilde{m}_k^-$ are two suitable distributions on $\{0,\ldots,n\}^{\mathbb{Z}^+}$ and on $\{0,\ldots,n\}^{\mathbb{Z}^-}$. For example $\widetilde{m}_k^+$ is defined by

e5.3.7
$$\int f(\underline{\sigma})\,\widetilde{m}_k^+(d\underline{\sigma}) = \sum_{\underline{\sigma}\in\{0,\ldots,n\}^{k+1}} \frac{f(\underline{\sigma}\,\underline{\widetilde{\sigma}}_{\Lambda_k^{+,c}})\,e^{-U_k^+(\underline{\sigma}\,\underline{\widetilde{\sigma}}_{\Lambda_k^{+,c}})}}{\sum\limits_{\underline{\sigma}'\in\{0,\ldots,n\}^{k+1}} e^{-U_k^+(\underline{\sigma}'\,\underline{\widetilde{\sigma}}_{\Lambda_k^{+,c}})}} \tag{5.3.7}$$

for every $f \in C(\{0,\ldots,n\}_T^{\mathbb{Z}^+})$, if $\Lambda_k^+ = [0,k]$, hence $\Lambda_k^{+,c} = [k+1,\ldots,+\infty)$ and if $\underline{\widetilde{\sigma}} \in \{0,\ldots,n\}_T^{\mathbb{Z}_+}$ is an arbitrarily fixed reference configuration (a device already repeatedly used, *e.g.* in (5.1.22)): for instance one could take $\widetilde{\sigma}_j \equiv 0$ for all $j \geq 0$, in the case $T_{\sigma\sigma'} \equiv 1$.

The following proposition appears, therefore, quite natural.

P5.3.2 **(5.3.2) Proposition:** (Relation between infinite and semiinfinite Gibbs distributions)
Under the hypotheses of proposition (5.2.1) there exist two distributions $\widetilde{m}^+$, *on* $\{0,\ldots,n\}_T^{\mathbb{Z}^+}$, *and* $\widetilde{m}^-$, *on* $\{0,\ldots,n\}_T^{\mathbb{Z}^-}$, *such that*

e5.3.8
$$m(d\underline{\sigma}) = C^{-1}\widetilde{m}^+(d\underline{\sigma}^+)\widetilde{m}^-(d\underline{\sigma}^-)\chi(\underline{\sigma}^-,\underline{\sigma}^+)e^{-W_\Phi(\underline{\sigma}^-,\underline{\sigma}^+)}, \tag{5.3.8}$$

where $C > 0$ *is a normalization constant,* $\chi(\underline{\sigma}^-,\underline{\sigma}^+) = 1$ *if the bilateral sequence* $\underline{\sigma}^-\underline{\sigma}^+$, *obtained by appending* $\underline{\sigma}^+$ *to the right of* $\underline{\sigma}^-$, *is in* $\{0,\ldots,n\}_T^{\mathbb{Z}}$ *and* $\chi(\underline{\sigma}^-,\underline{\sigma}^+) = 0$ *otherwise, and* m *is the Gibbs distribution (unique by proposition (5.2.1)) with potential* Φ.

Furthermore $\widetilde{m}^+$ *is the Gibbs distribution on* $\mathbb{Z}^+$ *with potential* Φ *and* $\widetilde{m}^-$ *is the analogous Gibbs distribution on* $\mathbb{Z}^-$, cf. *definition (5.3.1).*

Proof: For simplicity we consider only the case $T_{\sigma\sigma'} = 1$. A computation identical to that leading from (5.1.22) to (5.1.23) shows that if Φ has finite range R then m_k^+ has conditional probability such that

e5.3.9
$$\frac{m_k^+(\sigma_0'\ldots\sigma_a'|\sigma_{a+1}\ldots)}{m_k^+(\sigma_0''\ldots\sigma_a''|\sigma_{a+1}\ldots)} = \exp\Big(-\sum_{h=0}^{\infty}[\widehat{A}(\tau^h\underline{\sigma}') - \widehat{A}(\tau^h\underline{\sigma}'')]\Big), \tag{5.3.9}$$

if $a + R < k$, and m_k^+ is defined in (5.3.7).

Assuming that Φ is in the space B_0 of the finite range potentials and proceeding as in the proof of proposition (5.1.1), we see that m_k^+ converges to the semiinfinite Gibbs state $\widetilde{m}^+$. A similar argument holds for m_k^-; therefore (5.3.6) and the continuity of the function $(\underline{\sigma}^-,\underline{\sigma}^+) \to W_\Phi(\underline{\sigma}^-,\underline{\sigma}^+)$ imply (5.3.8) in the limit as $k \to \infty$, with $\chi = 1$ (because under our hypotheses $T_{\sigma\sigma'} = 1$).

If $\Phi \notin B_0$ but $\|\Phi\|_1 < +\infty$ there exists a sequence $\Phi^{(k)} \in B_0$, $k = 1, 2, \ldots$, such that $\|\Phi^{(k)} - \Phi\|_1 \xrightarrow[k\to\infty]{} 0$. Then from (5.3.8) written for $\Phi^{(k)}$ and from the uniform continuity, with respect to Φ, of the function $(\underline{\sigma}^-,\underline{\sigma}^+)\to W_\Phi(\underline{\sigma}^-,\underline{\sigma}^+)$ the validity of (5.3.8) for Φ follows in the limit $k\to\infty$. ■

The relation (5.3.8) and propositions (5.2.1) and (5.3.1) imply the following corollary; if $a(T)>0$ one must also make use of (5.2.5).

*C*5.3.1 **(5.3.1) Corollary:** (Absolute continuity of the restriction to $\mathcal{B}(\mathbb{Z}_+)$ of a Gibbs state with respect to the corresponding semiinfinite Gibbs state)
If $\Phi\in B$ and $\|\Phi\|_1<+\infty$, if m^+ denotes the restriction of the Gibbs distribution $m\in G(\Phi)$ to the σ–algebra $\mathcal{B}(\mathbb{Z}^+)$ and if $\widetilde m^+$ denotes the element of $G_+(\Phi)$, one has

*e*5.3.10
$$\left(\frac{dm^+}{d\widetilde m^+}\right)^{\pm1}\le e^{4\|\Phi\|_1}((1+n)e^{2\|\Phi\|})^{a(T)}. \tag{5.3.10}$$

Furthermore the function $h\overset{def}{=}\frac{dm^+}{d\widetilde m^+}$ is continuous: $h\in C(\{0,\ldots,n\}_T^{\mathbb{Z}^+})$.

Proof: This follows immediately from the integration of (5.3.6) with respect to $\underline{\sigma}^-$ and from proposition (5.3.2): by expressing $\widetilde m_k^-$ with the formula analogous to (5.3.7) one has to use that, for any string $\ldots,\sigma_{-a-2},\sigma_{-a-1}$ there is at least one string $\underline{\sigma}_0\in\{0,\ldots,n\}^a$ such that $\ldots,\sigma_{-a-2},\sigma_{-a-1},\underline{\sigma}_0,\underline{\sigma}^+$ is a compatible string, while the number of such connection strings is at most
*N*5.3.1 $(n+1)^a$.[1] ■

It is interesting to remark that the semiinfinite Gibbs states solve certain eigenvalue problems.

*P*5.3.3 **(5.3.3) Proposition:** (Gibbs distributions as solutions of eigenvalue problems)
Let Φ be a potential for $\{0,\ldots,n\}_T^{\mathbb{Z}}$ with $\|\Phi\|_1<\infty$ and T a mixing compatibility matrix; using the notations of corollary (5.3.1), proposition (5.3.2) and definition (5.3.1), set

*e*5.3.11
$$(Lf)(\sigma_0\sigma_1\ldots)=\sum_{\sigma=0}^{n}e^{-\widehat A(\sigma\sigma_0\sigma_1\ldots)}f(\sigma\sigma_0\sigma_1\ldots), \tag{5.3.11}$$

where $(\sigma_0\sigma_1\ldots)\in\{0,\ldots,n\}_T^{\mathbb{Z}^+}$, $f\in C(\{0,\ldots,n\}_T^{\mathbb{Z}^+})$, $\widehat A$ is defined in (5.3.3) and, finally, the sum runs over $\sigma\in\{0,\ldots,n\}$ such that $(\sigma\sigma_0\sigma_1,\ldots)\in\{0,\ldots,n\}_T^{\mathbb{Z}^+}$.

Then L is a continuous operator on $C(\{0,\ldots,n\}_T^{\mathbb{Z}^+})$ and its norm $||L||$ is bounded by $(n+1)e^{\|\Phi\|}$. Denote L^* the adjoint operator of L acting on $M^0(\{0,\ldots,n\}_T^{\mathbb{Z}})$. There exists $\lambda>0$ such that

*e*5.3.12
$$L\,h=\lambda h,\qquad L^*\widetilde m^+=\lambda\widetilde m^+,\qquad \lambda>0. \tag{5.3.12}$$

---

[1] To obtain the upper bound in (5.3.10) the factor $(n+1)$ could be replaced with 1: the expression in (5.3.10) has the advantage of being symmetric for the upper and lower bounds.

Furthermore the equations (5.3.12), regarded as eigenvalue problems in the unknowns $h > 0$, $h \in C(\{0,\ldots,n\}_T^{\mathbb{Z}^+})$, $\lambda > 0$, *and in the unknowns* $\widetilde{m}^+ \in M^0(\{0,\ldots,n\}_T^{\mathbb{Z}^+})$, $\lambda > 0$, *admit as unique solution* h *and* $\widetilde{m}^+$ *defined in proposition (5.3.2) and corollary (5.3.1).*

Remark: The operator L is called the *transfer operator*. The proof below derives the results from the previously studied existence and uniqueness theorems for infinite and semiinfinite Gibbs states. A proof entirely based on the spectral theory of the transfer operator is also possible, and classical since [Ru67]: see problems below.

Proof: Consider for simplicity only the case $a(T) = 0$ (*i.e.* $T_{\sigma\sigma'} \equiv 1$). Note that if $k > a(T)$ the equation $(L^*)^k \,\overline{m} = \overline{\lambda}^k \,\overline{m}$, $\overline{\lambda} > 0$, $\overline{m} \in M^0(\{0,\ldots,n\}_T^{\mathbb{Z}^+})$ is a different way of writing the condition that $\overline{m}$ has conditional probability verifying (5.3.4): hence $\widetilde{m}^+$ verifies equation (5.3.12) and it is its only solution because any other solution would be an element of $G_+(\Phi)$ that, instead, contains only one element.

Evidently $\lambda = (L^*\widetilde{m}^+)(1)$.

By proposition (5.3.2) one has $h\widetilde{m}^+ = m^+$ for a suitable continuous h: in fact $h(\underline{\sigma}_+)$ is proportional to $\int \widetilde{m}^-(d\underline{\sigma}_-)e^{-W(\underline{\sigma}_-,\underline{\sigma}_+)}$ for $\underline{\sigma}_+ \in \{0,\ldots,n\}^{\mathbb{Z}_+}$ by integrating (5.3.8) over $\underline{\sigma}_-$.

Furthermore m^+ is the restriction of the probability distribution m to $\mathcal{B}(\mathbb{Z}^+)$ and m is τ–invariant. Hence for all $f \in C(\{0,\ldots,\}_T^{\mathbb{Z}^+})$

e5.3.13

$$\begin{aligned}\int f\,dm^+ &= \int f(\sigma_0\sigma_1\ldots)h(\sigma_0\sigma_1\ldots)\widetilde{m}^+(d\sigma_0\sigma_1\ldots) = \\ &= \int f(\sigma_1\sigma_2\ldots)h(\sigma_0\sigma_1\ldots)\widetilde{m}^+(d\sigma_0\sigma_1\ldots).\end{aligned} \tag{5.3.13}$$

Using in the right hand side the relation $\widetilde{m}^+ = \lambda^{-1}L^*\widetilde{m}^+$, we see that

e5.3.14

$$\begin{aligned}\int fhd\widetilde{m}^+ &= \sum_{\sigma_0=0}^{n}\int f(\sigma_1\sigma_2\ldots)\lambda^{-1}h(\sigma_0\sigma_1\ldots)e^{-\widehat{A}(\sigma_0\sigma_1\ldots)}\widetilde{m}^+(d\sigma_1\sigma_2\ldots) \\ &= \int f(\sigma_1\sigma_2\ldots)(\lambda^{-1}Lh)(\sigma_1\sigma_2\ldots)\widetilde{m}^+(d\sigma_1\sigma_2\ldots) = \\ &= \int f\cdot\lambda^{-1}Lh\,d\widetilde{m}^+,\end{aligned} \tag{5.3.14}$$

hence by the arbitrariness of f we deduce that $\lambda^{-1}Lh = h$, $\widetilde{m}^+$–almost everywhere. However, as already observed in proposition (5.3.1), (iii), every set with m^+–measure 1 is dense so that the equality $\lambda^{-1}Lh = h$, holding on a dense set and involving continuous functions, must hold everywhere.

To prove uniqueness of h suppose that there exists another $\overline{h} > 0$, $\overline{h} \in C(\{0,\ldots,n\}_T^{\mathbb{Z}})$, such that $L\,\overline{h} = \overline{\lambda}\,\overline{h}$, $\overline{\lambda} > 0$. Then we begin by remarking that by integrating the latter relation with respect to $\widetilde{m}^+$ and using $L^*\widetilde{m}^+ =$

$\lambda\widetilde{m}^+$ we see that $\lambda\widetilde{m}^+(\overline{h}) = \overline{\lambda}\widetilde{m}^+(\overline{h})$, *i.e.* $\lambda = \overline{\lambda}$ (because $\overline{h} > 0$ and is continuous). We can and shall suppose $\widetilde{m}^+(\overline{h}) = 1$.

Following backwards (5.3.14), (5.3.13) we deduce that the probability distribution $(\lambda^{-1}L\,\overline{h})\widetilde{m}^+$ is identical to $\overline{h}\widetilde{m}^+$ and it is τ–invariant in the sense that

e5.3.15

$$\begin{aligned}&\int f(\sigma_0\sigma_1\ldots)\,\overline{h}(\sigma_0\sigma_1\ldots)\widetilde{m}^+(d\sigma_0\sigma_1\ldots) =\\ &= \int f(\sigma_1\sigma_2\ldots)\,\overline{h}(\sigma_0\sigma_1\ldots)\widetilde{m}^+(d\sigma_0\sigma_1\ldots).\end{aligned} \tag{5.3.15}$$

It is then possible to define a τ–invariant distribution $\overline{m}$ on $\{0,\ldots,n\}^{\mathbb{Z}}$ such that its restriction to $\mathcal{B}(\mathbb{Z}^+)$ is precisely $\overline{h}\widetilde{m}^+$: indeed we shall set, for $f \in \mathcal{B}([-N,+\infty))$–measurable,

e5.3.16

$$\int f(\sigma)d\overline{m} = \int f(\sigma_{-N}\sigma_{-N+1}\ldots)\,\overline{h}(\sigma_{-N}\sigma_{-N+1}\ldots)\widetilde{m}(d\sigma_{-N}\ldots), \tag{5.3.16}$$

where $\underline{\sigma} = (\sigma_{-N}\sigma_{-N+1}\ldots) \in \{0,\ldots,n\}^{[-N,+\infty)}$ is regarded as an element $\underline{\sigma}'$ of $\{0,\ldots,\}^{\mathbb{Z}^+}$ by setting $\sigma'_j = \sigma_{j-N}$, for all $j \ge 0$.

The compatibility problems that must be faced to check that this is a definition of a linear, continuous and positive functional on $C(\{0,\ldots,n\}^{\mathbb{Z}^+})$ can be solved via (5.3.15).

We proceed to compute the conditional probability of the probability distribution $\overline{m}$ in order to show that $\overline{m} = m$.

By Doob's theorem one has, $\overline{m}$–almost everywhere in $\underline{\sigma}$,

e5.3.17

$$\begin{aligned}&\frac{\overline{m}(\sigma'_{-a}\ldots\sigma'_a|\sigma_j,|j|>a)}{\overline{m}(\sigma''_{-a}\ldots\sigma''_a|\sigma_j,|j|>a)} =\\ &= \lim_{N\to\infty}\frac{\overline{m}(\sigma'_{-a}\ldots\sigma'_a|\sigma_{-N}\ldots\sigma_{-a-1}\sigma_{a+1}\ldots\sigma_N)}{\overline{m}(\sigma''_{-a}\ldots\sigma''_a|\sigma_{-N}\ldots\sigma_{-a-1}\sigma_{a+1}\ldots\sigma_N)} =\\ &= \lim_{N\to\infty}\frac{\overline{m}(\sigma_{-N}\ldots\sigma_{-a-1}\sigma'_{-a}\ldots\sigma'_a|\sigma_{a+1}\ldots\sigma_N)}{\overline{m}(\sigma_{-N}\ldots\sigma_{-a-1}\sigma''_{-a}\ldots\sigma''_a|\sigma_{a+1}\ldots\sigma_N)} =\\ &= \lim_{N\to\infty}\frac{\overline{h}(\sigma_{-N}\ldots\sigma_{-a-1}\sigma'_{-a}\ldots\sigma'_s\sigma_{a+1}\ldots\sigma_N)}{\overline{h}(\sigma_{-N}\ldots\sigma_{-a-1}\sigma''_{-a}\ldots\sigma''_a\sigma_{a+1}\ldots\sigma_N)}\cdot\\ &\quad\cdot\frac{\widetilde{m}^+(\sigma_{-N}\ldots\sigma_{-a-1}\sigma'_{-a}\ldots\sigma'_a|\sigma_{a+1}\ldots\sigma_N)}{\widetilde{m}^+(\sigma_{-N}\ldots\sigma_{-a-1}\sigma''_{-a}\ldots\sigma''_a|\sigma_{a+1}\ldots\sigma_N)},\end{aligned} \tag{5.3.17}$$

where, in the last expression, the sequences $\underline{\sigma}' = (\sigma_{-N}\ldots\sigma_{-a-1}\sigma'_{-a}\ldots\sigma'_a\,\sigma_{a+1}\ldots)$ and $\underline{\sigma}'' = (\sigma_{-N}\ldots\sigma_{-a-1}\sigma''_{-a}\ldots\sigma''_a\sigma_{a+1}\ldots)$ are thought of as elements $\widehat{\underline{\sigma}}'$ and $\widehat{\underline{\sigma}}''$ of $\{0,\ldots,\}^{\mathbb{Z}^+}$ setting, as above, $\widehat{\sigma}'_j = \sigma'_{j-N}$, $\widehat{\sigma}''_j = \sigma''_{j-N}$, $j \ge 0$.

Since the distance between $\widehat{\sigma}'$ and $\widehat{\sigma}''$ as elements of $\{0,\ldots,n\}^{\mathbb{Z}^+}$ is $\le e^{-(N-a)}$ and since $\overline{h}$ is uniformly continuous on $\{0,\ldots,n\}^{\mathbb{Z}^+}$ the first ratio in the limit in (5.3.17) tends to 1.

The second ratio is, by (5.3.1) (recall that $\widetilde{m}^+ \in G^0_+(\Phi)$),

e5.3.18

$$\begin{aligned}
&\exp\Big(-\sum_{\substack{X\subset[-N,+\infty)\\ X\cap[-N,a]\ne\emptyset}}[\Phi_X(\widehat{\underline{\sigma}}'_X)-\Phi_X(\widehat{\underline{\sigma}}''_X)]\Big)=\\
&=\exp\Big(-\sum_{\substack{X\subset[-N,+\infty)\\ X\cap[-a,a]\ne\emptyset}}[\Phi_X(\underline{\sigma}'_X)-\Phi_X(\underline{\sigma}''_X)]\Big)\xrightarrow[N\to\infty]{}\\
&\xrightarrow[N\to\infty]{}\exp\Big(-\sum_{X\cap[-a,a]\ne\emptyset}[\Phi_X(\underline{\sigma}'_X)-\ \Phi_X(\underline{\sigma}''_X)]\Big)=\\
&=\exp\Big(-\sum_{k=-\infty}^{\infty}[A_\Phi(\tau^k\underline{\sigma}')-A_\Phi(\tau^k\underline{\sigma}'')]\Big),
\end{aligned}\tag{5.3.18}$$

showing that $\overline{m}$ has the correct conditional probabilities to say that it is the Gibbs distribution $m \in G(\Phi)$. It follows that $\overline{h}\widetilde{m}^+ = h\widetilde{m}^+$, *i.e.* $\overline{h} = h$, $\widetilde{m}^+$–almost everywhere. This means that $\overline{h} = h$ everywhere because $\overline{h}$ and h are continuous and furthermore, as already observed, the sets of m^+–probability 1 are dense.

Problems for §5.3 (*Spectral theory of the transfer operator and Gibbs states*)

Remark: The spectral theory of the operator L on the space of the continuous functions $C(\{0,\ldots,n\}^{\mathbb{Z}})$ is made easy by the results of proposition (5.3.3) proving existence of positive eigenfunctions $h, \widetilde{m}_+$ in (5.3.12) relative to a positive eigenvalue λ. However one can prove the existence of a positive solution to the eigenvalue problems posed by (5.3.12) independently of proposition (5.3.3). Therefore we present problems which can be solved by *assuming* that (5.3.12) has positive solutions $h, \widetilde{m}_+$ with a positive eigenvalue λ. The independent proof of their existence (and therefore an alternative proof of proposition (5.3.3)) will be discussed in problems [5.3.17], [5.3.18] and [5.3.19].

Q5.3.1 **[5.3.1]:** (*Positivity of the transfer operator*)
Let L be the operator defined in (5.3.11), where Φ is a potential for $\{0,\ldots,n\}^{\mathbb{Z}}$ and $\|\Phi\|_1 < +\infty$ and let $h, \widetilde{m}_+, \lambda$ be as in (5.3.12) and assume the normalization $\widetilde{m}^+(h) = 1$. Show that the functions $(\lambda^{-1}L)^k f$ are equicontinuous and equibounded with respect to k, if f is positive and continuous. (*Hint:* Consider first the case $f = 1$, *i.e.* $f(\underline{\sigma}) \overset{def}{=} 1(\underline{\sigma}) \equiv 1$ and proceed by comparing with 1 the ratios $(\lambda^{-1}L)^k 1(\underline{\sigma})/(\lambda^{-1}L)^k 1(\underline{\sigma}')$; for this purpose write explicitly the ratios and use the uniform boundedness to deduce the result from the constancy of $\widetilde{m}^+$–integral of the functions $(\lambda^{-1}L)^k 1(\underline{\sigma})$ as k varies.)

Q5.3.2 **[5.3.2]:** In the context and under the hypotheses of problem [5.3.1], given a continuous non-negative function f show that there exists N_f such that for all $k \ge N_f$ one has $(\lambda^{-1}L)^k f > 0$. (*Hint:* First note that this is obvious for cylindrical functions. Then just take a very good cylindrical approximation of f and then k larger than the size of the base of the cylinder on which f is cylindrical.)

Q5.3.3 **[5.3.3]:** In the context of problem [5.3.1], if $f \ge 0$ is continuous and $\mathcal{B}([0,N])$–measurable there exists N_f such that for every $k \ge N_f$ one has $(\lambda^{-1}L)^k f \ge e^{-2\|\Phi\|_1}\widetilde{m}^+(f)$, where $\widetilde{m}_+$ is defined as in (5.3.12). (*Hint:* Same argument as the one suggested in problem [5.3.1] making use of $\widetilde{m}^+((\lambda L)^k f) \equiv \widetilde{m}^+(f)$; $N_f = N$.)

Q5.3.4 **[5.3.4]:** (*Contractivity of iterates of* $\lambda^{-1}L$; see [Ru67])
In the context of problem [5.3.1], let f be continuous, $\mathcal{B}([0,N])$–measurable and such

that $\widetilde{m}^+(f) = 0$. There exists N_f such that for all $k \geq N_f$ one has

$$\widetilde{m}^+(|(\lambda^{-1}L)^k f|) \leq (1 - e^{-2\|\Phi\|_1})\, \widetilde{m}^+(|f|).$$

(*Hint:* Let $f^+ = (|f|+f)/2$ and $f^- = (|f|-f)/2$, then $\widetilde{m}^+(f^+) = \widetilde{m}^+(f^-)$. Furthermore by using the result of problem [5.3.3] for k large enough one has

$$\begin{aligned}&|(\lambda^{-1}L)^k f| = |(\lambda^{-1}L)^k f^+ - (\lambda^{-1}L)^k f^-| = \\ &|(\lambda^{-1}L)^k f^+ - e^{-2\|\Phi\|_1}\widetilde{m}^+(f^+) - (\lambda^{-1}L)^k f^- + e^{-2\|\Phi\|_1}\widetilde{m}^+(f^-)| \leq \\ &\leq (\lambda^{-1}L)^k f^+ - e^{-2\|\Phi\|_1}\widetilde{m}^+(f^+) + (\lambda^{-1}L)^k f^- - e^{-2\|\Phi\|_1}\widetilde{m}^+(f^-).\end{aligned}$$

Then integrate both sides with respect to $\widetilde{m}^+$ and note that $\widetilde{m}^+$ is an eigenvector for $(\lambda^{-1}L^*)$ with eigenvalue 1, and that $\widetilde{m}^+(|f|) = \widetilde{m}^+(f^+) + \widetilde{m}^+(f^-)$.)

Q5.3.5 **[5.3.5]:** From the result in problem [5.3.4] show that if f is continuous and $\widetilde{m}^+(f) = 0$ then the limit as $k \to \infty$ of $(\lambda^{-1}L)^k f$ is 0 (in the the uniform convergence topology).

Q5.3.6 **[5.3.6]:** (*Convergence of* $(\lambda^{-1}L)^k f$)
Deduce from the result of problem [5.3.5] that if f is continuous the limit for $k \to \infty$ of $(\lambda^{-1}L)^k f$ is $h\,\widetilde{m}^+(f)$, uniformly if $h > 0$ is a normalized eigenfunction $Lh = \lambda h$. Check that such an eigenfunction exists because of (5.3.12). *Existence of h can be obtained independently of proposition (5.3.3): see problems [5.3.18] and [5.3.19].* (*Hint:* : Write $f = h\widetilde{m}^+(f) + (f - h\widetilde{m}^+(f))$).

Q5.3.7 **[5.3.7]:** (*Exponential convergence*; see [Ru67] and [GL70])
Deduce from the results of problems [5.3.3], [5.3.4] and [5.3.5] that if for some $\kappa > 0$ the sum $\sum_{X \ni 0} e^{\kappa\,(\mathrm{diam}X)}\|\Phi_X\| < \infty$ then $|(\lambda^{-1}L)^n 1(\underline{\sigma}) - h(\underline{\sigma})| \leq Ce^{-\kappa' n}$, for $0 < \kappa' < \kappa$.

Q5.3.8 **[5.3.8]:** (*Case of subshifts*; see [GM70])
Solve the problems analogous to the previous ones with $\{0,\ldots,n\}^{\mathbb{Z}}$ replaced by $\{0,\ldots,n\}^{\mathbb{Z}}_T$ under the only assumption that T is mixing.

Q5.3.9 **[5.3.9]:** (*Decimation of finite range Gibbs states*)
Let $\Phi \in B$ be a potential on $\{0,\ldots,n\}^{\mathbb{Z}}_T$ with T a mixing compatibility matrix and with finite range R, see definition (5.1.1); denote by $\alpha = 0, 1, \ldots, M$ the elements of $\{0,\ldots,n\}^{a(T)+R}_T$. Consider the restriction of $m \in G(\Phi)$ to the σ–algebra generated by the cylinders with base given by $\cup_i \Lambda_{k_i}$ where $k_i \in \mathbb{Z}$ and $\Lambda_k = \{0 + 2k(a(T)+R), 1 + 2k((a(T)+R), \ldots, a(T)+R-1+2k(a(T)+R)\}$ *i.e.* the cylinders “measurable on the blocks of $a(T)+R$ sites spaced by $(a(T)+R)$”. Show that it is interpretable as a Gibbs state on $\{0,\ldots,M\}^{\mathbb{Z}}$ with a “nearest neighbours” potential $\widetilde{\Phi}$ (*i.e.* $\widetilde{\Phi}_X = 0$ unless X consists of only one or two points and, in the latter case, it is also zero unless that the two points are adjacent). Find a possible expression for $\widetilde{\Phi}$.

Q5.3.10 **[5.3.10]:** (*Decimation of a Markov process*)
Let m be a Markov process on $\{0,\ldots,n\}^{\mathbb{Z}}$. Show that the restriction of m to the σ–algebra generated by the cylinders with base on the even sites is still a Markov process and that its potential $\Phi^{(1)}$ can be chosen by defining suitably a map $\Theta : \Phi \to \Theta\Phi = \Phi^{(1)}$. Such map can be chosen so that one has (having set $\Phi^{(n)} = \Theta^n\Phi$)

$$\Phi^{(n)}_{\{x\}}(\sigma_x) \xrightarrow[n\to\infty]{} w(\sigma_x), \qquad |\Phi^{(n)}_{\{x,x+1\}}(\sigma,\sigma')| \leq C\varepsilon^{2^n},$$

with w suitable, $C > 0$ and $\varepsilon < 1$.

Q5.3.11 **[5.3.11]:** (*Transfer matrix*)
Check that in the finite range potential case the operator L becomes “in a certain sense” a finite matrix (called *transfer matrix*) and the problems of existence of $\widetilde{m}_+, h$ reduce to the Perron–Frobenius theorem for matrices.. (*Hint:* Note that the equations for h,

$\lambda^{-1}Lh = h$ can be interpreted, if Φ has finite range, as an equation for a positive eigenfunction of a finite mixing matrix with non-negative entries.)

Q5.3.12 **[5.3.12]:** (*Finite range potentials and algebraic spectral problem for the transfer operator*)
Check that from the proof of the result of problem [5.3.11] it also follows that the determination of $m(C^{0\ldots p}_{\sigma_0\ldots\sigma_p})$ is, if $R<\infty$, an "algebraic" problem (*i.e.* it is reduced to the study of the eigenvalues and eigenvectors of a suitable finite matrix).

Q5.3.13 **[5.3.13]:** (*Decimation theorem*)
If Φ is a potential on $\{0,\ldots,n\}^{\mathbb{Z}}_T$, with T mixing, such that $\|\Phi\|_1<\infty$, the restriction of $m\in G(\Phi)$ to the algebra generated by the cylinders measurable on the sites $k\,n$, $n\in\mathbb{Z}$ is, for k large, a Gibbs state (in a natural sense) with potential $\Phi^{(k)}$ that can be chosen such that $\sum_{X\ni 0,|X|\ge 2}\|\Phi^{(k)}_X\|\le\varepsilon_\Phi(k)$, where $\varepsilon_\Phi(k)\xrightarrow[k\to\infty]{}0$, while $\Phi^{(k)}_{\{0\}}(\sigma)$ is a convergent sequence as $k\to\infty$. Finally the $\varepsilon_\Phi(k)$ is continuous, in Φ at fixed k, with respect to the norm $\|\Phi\|_1$. Try to prove this result in the case proposed in the following problem and refer to [CO81], where it is proved.

Q5.3.14 **[5.3.14]:** (*Transfer operator for Fisher potentials*)
Check the statement in problem [5.3.13] in the case of the potential of problem [5.2.1].

Q5.3.15 **[5.3.15]:** Via the arguments necessary to solve problem [5.3.7] and under the same assumptions deduce the existence of $C_k,\alpha>0$ for which

$$|h(\sigma'_0\ldots\sigma'_k\;\sigma_{k+1}\ldots)-h(\sigma''_0\ldots\sigma''_k\sigma_{k+1}\ldots)|\le C_k e^{-\alpha k}.$$

Q5.3.16 **[5.3.16]:** (*Exponential mixing rates*)
By the arguments necessary to solve problems [5.3.7] and [5.3.15] and under the same assumptions deduce that the distribution $\widetilde{m}^+$ has the property of "exponential mixing on the cylinders":

$$\begin{aligned}&|\widetilde{m}^+(C^{0\ldots N}_{\sigma_0\ldots\sigma_N}\cap C^{j\ldots j+M}_{\sigma'_0\ldots\sigma'_M})-\widetilde{m}^+(C^{0\ldots N}_{\sigma_0\ldots\sigma_N})(h\widetilde{m}^+)(C^{0\ldots M}_{\sigma'_0\ldots\sigma'_M})|\le\\ &\le K\,\widetilde{m}^+(C^{0\ldots N}_{\sigma_0\ldots\sigma_N})(h\widetilde{m}^+)(C^{0\ldots M}_{\sigma'_0\ldots\sigma'_M})\,\min\{1,\,e^{-\kappa\,(j-N)}\}\end{aligned}$$

for suitably chosen $K,\,\kappa>0$. An identical property holds if in this relation we replace $\widetilde{m}^+$ with $h\widetilde{m}^+$ everywhere it appears alone (*i.e.* except where already appears $h\widetilde{m}^+$).

Q5.3.17 **[5.3.17]:** (*Alternative proof of existence of eigenfunction of L^**)
Let L^* be the adjoint of the operator L on $C(\{0,\ldots,n\}^{\mathbb{Z}})$: write it explicitly as an operator on the measures on $\{0,\ldots,n\}^{\mathbb{Z}}$. Define $m\to\frac{L^*m}{L^*m(1)}$ on the space of the probability distributions m on $\{0,\ldots,n\}^{\mathbb{Z}}$ where 1 denotes the function identically 1 on $\{0,\ldots,n\}^{\mathbb{Z}}$. This is a continuous map of $M(\{0,\ldots,n\}^{\mathbb{Z}})$ into itself in the natural weak topology induced by $C(\{0,\ldots,n\}^{\mathbb{Z}})$. Show that there is a fixed point $\widetilde{m}_+$ such that $\widetilde{m}_+=\frac{L^*\widetilde{m}_+}{L^*\widetilde{m}_+(1)}\equiv\lambda^{-1}L^*\widetilde{m}_+$ where $\lambda=L^*\widetilde{m}_+(1)>0$. (*Hint:* The space $M(\{0,\ldots,n\}^{\mathbb{Z}})$ is compact and convex so that by the fixed point theorem there is a fixed point.)

Q5.3.18 **[5.3.18]:** (*Equiboundedness and equicontinuity of $(\lambda^{-1}L)^k1$*)
Show that $(\lambda^{-1}L)^k1(\sigma_0,\sigma_1,\ldots)$ is uniformly bounded and uniformly continuous in $\underline{\sigma}$. (*Hint:* If $U(\eta_0,\ldots,\eta_{k-1}\,|\,\sigma_0,\sigma_1,\ldots)\overset{def}{=}\sum_{X\cap 0,\ldots,k-1\ne\emptyset}\Phi_X((\underline{\eta}\underline{\sigma})_X)$ it is, comparing ratios of terms corresponding to the same $\underline{\eta}$ –labels,

$$e^{-2||\Phi||_1}\frac{\sum_{\eta_1,\ldots,\eta_k}e^{-U(\eta_1,\ldots,\eta_k\,|\,\sigma_0,\sigma_1,\ldots)}}{\sum_{\eta_1,\ldots,\eta_k}e^{-U(\eta_1,\ldots,\eta_k\,|\,\sigma'_0,\sigma'_1,\ldots)}}\le e^{2||\Phi||_1}$$

However the ratio is just $\frac{L^k 1(\underline{\sigma})}{L^k 1(\underline{\sigma}')} \equiv \frac{\lambda^{-k} L^k 1(\underline{\sigma})}{\lambda^{-k} L^k 1(\underline{\sigma}')}$ and there must exist a point $\underline{\sigma}$ where $\lambda^{-k} L^k 1(\underline{\sigma}) \le 1$ and one point where $\lambda^{-k} L^k 1(\underline{\sigma}) \ge 1$ because $\widetilde{m}_+(\lambda^{-k} L^k 1) = \widetilde{m}_+(1) = 1$. Hence equiboundedness follows. Likewise one sees equicontinuity if the sequences $(\sigma_0, \sigma_1, \ldots)$ are close *i.e.* $(\sigma'_0, \sigma'_1, \ldots)$ are such that $\sigma_i = \sigma'_i$ for many $i's$.)

Q5.3.19 **[5.3.19]:** (*Alternative proof of existence of an eigenfunction h for $\lambda^{-1}L$*)
Consider the sequence $N^{-1} \sum_{k=0}^{N-1} (\lambda^{-1} L)^k 1(\underline{\sigma})$: show that the problem [5.3.18] implies that this sequence is equicontinuous and equibounded . Therefore it admits a uniformly convergent subsequence (by Ascoli–Arzelà theorem): show that its limit is a positive function h with $\lambda^{-1} L h = h$.

Bibliographical note to §5.3

The results of this section and of Section §5.2 are due to Ruelle,[Ru67], who substantially extended previous results and conjectures (by Van Hove, [VH50]). The choice of the arguments of this section and of the previous, §5.1 and §5.2, is inspired by the book [Ru78] although the exposition of this section is somewhat different from Ruelle's original.

§5.4 An application: expansive maps of [0,1]

As an application of the results of Section §5.3 we shall discuss a theory of invariant probability distributions for the simplest expansive maps of $[0,1]$, cf. example (1.2.7).

Let $f_0,\ f_1, \ldots, f_n$ be $(n+1)$ functions defined on the intervals $[a_\sigma, a_{\sigma+1}]$, respectively, with

e5.4.1
$$0 = a_0 < a_1 < \ldots < a_n < a_{n+1} = 1, \tag{5.4.1}$$

and with values in $[0,1]$, such that each f_σ is of class $C^{1+\varepsilon}([a_\sigma, a_{\sigma+1}])$, $\varepsilon > 0$. Let S be the map of $[0,1]$ into itself defined by

e5.4.2
$$x \to Sx = f_\sigma(x) \quad \text{if} \quad x \in (a_\sigma, a_{\sigma+1}), \tag{5.4.2}$$

and defined in 0 as $f_0(0)$, in 1 as $f_{n+1}(1)$, and, arbitrarily, in a_σ as $f_\sigma(a_\sigma)$ or $f_{\sigma-1}(a_\sigma)$, for $\sigma = 1, \ldots, n$.

The (noninvertible) dynamical system $([0,1], S)$ is well defined but in general the Lebesgue measure $\mu_0(dx) = dx$ is *not S–invariant* .

If one chooses randomly a point $x \in [0,1]$ with distribution μ_0 the asymptotic behavior of the sequence $n \to F(S^n x)$, where $F \in C^\infty([0,1])$ can, sometimes, be described by means of a Borel probability distribution μ on $[0,1]$, such that

e5.4.3
$$N^{-1} \sum_{j=0}^{N-1} F(S^j x) \xrightarrow[N \to \infty]{} \int_0^1 \mu(dx') F(x') \tag{5.4.3}$$

for μ_0–almost all points $x \in [0,1]$.

This case, obviously, occurs when there exists a probability distribution μ absolutely continuous with respect to μ_0, with a positive density h, and which is S–invariant and S–ergodic:

$e5.4.4$
$$\mu(dx)=h(x)dx=h(x)\mu_0(dx),\qquad \mu(E)=\mu(S^{-1}E). \tag{5.4.4}$$

The first question that can be asked is, thus, to find conditions for the existence of a measure μ which is S–invariant, S–ergodic and absolutely equivalent[1] or, at least, absolutely continuous with respect to μ_0. $N5.4.1$

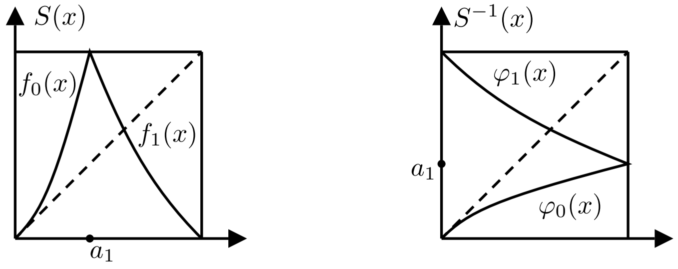
We shall not discuss the motivations of such questions which, at least for what concerns their interest for Physics, rest on the necessity of simple mathematical models to classify the phenomenology of the asymptotic behaviour of the solutions of nonlinear ordinary differential equations or of sequences of points obtained by iterating a map, starting from a given randomly chosen initial point.

A certain success has been achieved in the interpretation of experimental results in terms of simple objects like, in particular, maps of the interval $[0,1]$: above all famous is the interpretation of certain turbulence phenomena by Lorenz, [Lo63]. A kind of "experimental mathematics" has been (and is being) developed with the aim of achieving an orderly classification of the wealth of results that computer simulations or actual experiments produce.

We shall consider here the simplest case in which the map S is *strictly expansive and surjective*; this corresponds to the following conditions:

(1) $|f'_\sigma(x)|\ge\lambda>1$ for all $x\in[a_\sigma,a_{\sigma+1}]$,
(2) $f_\sigma:[a_\sigma,a_{\sigma+1}]\leftrightarrow[0,1]$ for all $\sigma=0,1,\ldots,n$.

An illustration is in Fig. (5.4.1).



$F5.4.1$ **Fig.(5.4.1)** Graphs of an expansive and surjective map S and of its inverse.

The following proposition holds.

$P5.4.1$ **(5.4.1) Proposition:** (Mixing maps of the interval)
Let S be a strictly expansive surjective map of class $C^{1+\varepsilon}$, $\varepsilon>0$, of the interval $[0,1]$ *into itself.*[2] $N5.4.2$

---

[1] μ is said to be absolutely equivalent to μ_0 if $\mu=h\mu_0$ and $h,h^{-1}\in L_1(\mu_0)$, *i.e.* μ and μ_0 are absolutely equivalent if μ is absolutely continuous with respect to μ_0 and μ_0 is absolutely continuous with respect to μ.

[2] This means that $f_\sigma\in C^{1+\varepsilon}([a_\sigma,a_{\sigma+1}])$, for all $\sigma=0,1,\ldots,n$.

(i) There exists an S–invariant probability distribution μ which is absolutely equivalent to the Lebesgue measure.
(ii) Such a probability distribution μ is mixing and one has

e5.4.5
$$\int_0^1 F(S^m x)G(x)\,\mu(dx) \xrightarrow[m\to\infty]{} \Big(\int_0^1 F(x)\mu(dx)\Big)\Big(\int_0^1 G(x)\mu(dx)\Big) \tag{5.4.5}$$

exponentially fast if $F,G\in C^\alpha([0,1])$, $\alpha>0$. This means that $([0,1],S,\mu)$ is a mixing dynamical system that "mixes at an exponential rate the Hölder continuous functions".

Remark: This theorem is a particular case of a general theorem of Lasota–Yorke, [LY73]. We discuss it here by using a different technique of proof based on the theory of Gibbs distributions developed in the previous sections.

Proof: Consider $\{0,\ldots,n\}^{\mathbb{Z}^+}$ and the symbolic code $X:\underline{\sigma}\to X(\underline{\sigma})=\cap_{j=0}^{+\infty}S^{-j}P_{\sigma_j}$, if we denote by P_σ the interval $P_\sigma=[a_\sigma,a_{\sigma+1}]$ and $\underline{\sigma}=(\sigma_j)_{j\in\mathbb{Z}^+}$, $\sigma_j=0,\ldots,n$.
One deduces, by induction, calling $\varphi_\sigma:[0,1]\leftrightarrow[a_\sigma,a_{\sigma+1}]$ the inverse function of f_σ (cf. Fig. (5.4.1)) that

e5.4.6
$$\begin{aligned}&P_{\sigma_0}=P^0_{\sigma_0}=\varphi_{\sigma_0}([0,1])\ ,\quad P^{0\,1}_{\sigma_0\sigma_1}=P_{\sigma_0}\cap S^{-1}P_{\sigma_1}=\varphi_{\sigma_0}\varphi_{\sigma_1}([0,1]),\\ &P^{0\ldots N}_{\sigma_0\ldots\sigma_N}=P_{\sigma_0}\cap S^{-1}P_{\sigma_1}\cap\ldots\cap S^{-N}P_{\sigma_N}=\varphi_{\sigma_0}\varphi_{\sigma_1}\ldots\varphi_{\sigma_N}([0,1]).\end{aligned} \tag{5.4.6}$$

Furthermore φ_σ transforms intervals with interior points into intervals with interior points, so that $P^{0\ldots N}_{\sigma_0\ldots\sigma_N}$ is an interval with interior points. Hence $X(\underline{\sigma})\neq\emptyset$.
However φ_σ "contracts" (see Fig.(5.4.1)) by a scale factor which is at least λ^{-1}, because f_σ expands by at least λ, hence the length $|\varphi_{\sigma_0}\ldots\varphi_{\sigma_N}([0,1])|$ of the interval $\varphi_{\sigma_0}\varphi_{\sigma_1}\ldots\varphi_{\sigma_N}([0,1])$ is

e5.4.7
$$|\varphi_{\sigma_0}\ldots\varphi_{\sigma_N}([0,1])|\le\lambda^{-(N+1)}. \tag{5.4.7}$$

so that $X(\underline{\sigma})$ consists of a single point. Therefore the code X is Hölder continuous and by (4.1.6)

e5.4.8
$$d(X(\underline{\sigma}),X(\underline{\sigma}'))\le d(\underline{\sigma},\underline{\sigma}')^{\log\lambda}. \tag{5.4.8}$$

The probability distribution μ_0 is coded by X into a probability distribution $\overline{\mu}_0$ on $\{0,\ldots,n\}^{\mathbb{Z}^+}$ by setting

e5.4.9
$$\overline{\mu}_0(E)\overset{def}{=}\mu_0(X(E))\qquad\text{for all } E\in\mathcal{B}(\{0,\ldots,n\}^{\mathbb{Z}^+}) \tag{5.4.9}$$

N5.4.3 isomorphic to it mod 0.[3]

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[3] Indeed X is continuous and invertible outside a set of measure 0,

$$\cup_{j=-\infty}^{+\infty}S^{-j}\{a_0,\ldots,a_{n+1}\}=N\subset[0,1],$$

and it is therefore bimeasurable, see Kuratowsky's theorem quoted in the proof of proposition (2.3.3), as a map between $[0,1]\setminus N$ and $X([0,1]\setminus N)$.

The probability distribution $\overline{\mu}_0$ is now easily identified as the Gibbs distribution on $\mathbb{Z}^+$ with potential Φ such that

e5.4.10

$$\begin{aligned}&\Phi_X(\underline{\sigma}_X)=0\quad\text{if}\quad X\ne\{a,\ldots,a+p\}\text{ and for all }a,\ \text{for all}\,p\ge0,\\&\Phi_{\{a,\ldots,a+p\}}(\sigma_0\ldots\sigma_p)=\hat{A}(\sigma_0\ldots\sigma_p000\ldots)-\hat{A}(\sigma_0\ldots\sigma_{p-1}000\ldots),\end{aligned}\tag{5.4.10}$$

where

e5.4.11

$$\hat{A}(\underline{\sigma})=-\log|\varphi'_{\sigma_0}(X(\sigma_1\sigma_2\ldots))|.\tag{5.4.11}$$

Indeed, if $\underline{\sigma}'=(\sigma'_0\ldots\sigma'_N\ \sigma_{N+1}\ldots)$, $\underline{\sigma}''=(\sigma''_0\ldots\sigma''_N\sigma_{N+1}\ldots)$ and $x'=X(\underline{\sigma}')$, $x''=X(\underline{\sigma}'')$, one has

e5.4.12

$$\begin{aligned}&\frac{\overline{\mu}_0(\sigma'_0\ldots\sigma'_N|\sigma_{N+1}\ldots)}{\overline{\mu}_0(\sigma''_0\ldots\sigma''_N|\sigma_{N+1}\ldots)}=\lim_{M\to\infty}\frac{\overline{\mu}_0(\sigma'_0\ldots\sigma'_N|\sigma_{N+1}\ldots\sigma_M)}{\overline{\mu}_0(\sigma''_0\ldots\sigma''_N|\sigma_{N+1}\ldots\sigma_M)}=\\&=\lim_{M\to\infty}\frac{\overline{\mu}_0\Big(C^{0\ldots N\ N+1\ldots M}_{\sigma'_0\ldots\sigma'_N\ \sigma_{N+1}\ldots\sigma_M}\Big)}{\overline{\mu}_0\Big(C^{0\ldots N\ N+1\ldots M}_{\sigma''_0\ldots\sigma''_N\ \sigma_{N+1}\ldots\sigma_M}\Big)}=\\&=\lim_{M\to\infty}\frac{|\varphi_{\sigma'_0}\cdots\varphi_{\sigma'_N}\varphi_{\sigma_{N+1}}\cdots\varphi_{\sigma_M}([0,1])|}{|\varphi_{\sigma''_0}\cdots\varphi_{\sigma''_N}\varphi_{\sigma_{N+1}}\cdots\varphi_{\sigma_M}([0,1])|}=\\&=\lim_{M\to\infty}\frac{|\varphi_{\sigma'_0}\cdots\varphi_{\sigma'_N}(\beta)-\varphi_{\sigma'_0}\cdots\varphi_{\sigma'_N}(\alpha)|}{|\varphi_{\sigma''_0}\cdots\varphi_{\sigma''_N}(\beta)-\varphi_{\sigma''_0}\cdots\varphi_{\sigma''_N}(\alpha)|},\end{aligned}\tag{5.4.12}$$

where $[\alpha,\beta]=\varphi_{\sigma_{N+1}}\cdots\varphi_{\sigma_M}([0,1])$ is an interval of size at most $\lambda^{-(M-N)}$ around the point $X(\sigma_{N+1}\ldots)=\xi$. Hence

e5.4.13

$$\frac{\overline{\mu}_0(\sigma'_0\ldots\sigma'_N|\sigma_{N+1}\ldots)}{\overline{\mu}_0(\sigma''_0\ldots\sigma''_N|\sigma_{N+1}\ldots)}=\frac{\Big|\Big[\frac{d}{dx}\,\varphi_{\sigma'_0}\cdots\varphi_{\sigma'_N}(x)\Big]_{x=\xi}\Big|}{\Big|\Big[\frac{d}{dx}\,\varphi_{\sigma''_0}\cdots\varphi_{\sigma''_N}(x)\Big]_{x=\xi}\Big|},\tag{5.4.13}$$

and, by the composition of derivatives, this ratio is

e5.4.14

$$\prod_{j=0}^{N}\frac{|\varphi'_{\sigma'_j}(\varphi_{\sigma'_{j+1}}\cdots\varphi_{\sigma'_N}(\xi))|}{|\varphi'_{\sigma''_j}(\varphi_{\sigma''_{j+1}}\cdots\varphi_{\sigma''_N}(\xi))|}.\tag{5.4.14}$$

Noting that $\varphi_{\sigma'_{j+1}}\cdots\varphi_{\sigma'_N}(\xi)\equiv\varphi_{\sigma'_{j+1}}\cdots\varphi_{\sigma'_N}(X(\sigma_{N+1}\ldots))$ so that $\varphi_{\sigma'_{j+1}}\cdots\varphi_{\sigma'_N}(\xi)$ is just $X(\sigma'_{j+1}\ldots\sigma'_N\ \sigma_{N+1}\ldots)$ we find

e5.4.15

$$\begin{aligned}&\frac{\overline{\mu}_0(\sigma'_0\ldots\sigma'_N|\sigma_{N+1}\ldots)}{\overline{\mu}_0(\sigma''_0\ldots\sigma''_N|\sigma_{N+1}\ldots)}=\prod_{j=0}^{N}\frac{|\varphi'_{\sigma'_j}(X(\sigma'_{j+1}\ldots))|}{|\varphi'_{\sigma''_j}(X(\sigma''_{j+1}\ldots))|}=\\&=\exp\Big[-\sum_{j=0}^{N}\Big(-\log|\varphi'_{\sigma'_j}(X(\sigma'_{j+1}\ldots))|+\log|\varphi'_{\sigma''_j}(X(\sigma''_{j+1}\ldots))|\Big)\Big]=\\&=\exp\Big[-\sum_{j=0}^{N}\Big(\hat{A}(\tau^j\underline{\sigma}')-\hat{A}(\tau^j\underline{\sigma}'')\Big)\Big]=\exp\Big[-\sum_{j=0}^{\infty}\Big(\hat{A}(\tau^j\underline{\sigma}')-\hat{A}(\tau^j\underline{\sigma}'')\Big)\Big],\end{aligned}\tag{5.4.15}$$

from (5.4.11) and because $\tau^M\underline{\sigma}'=\tau^M\underline{\sigma}''$, if $M>N$. Therefore if we note that

e5.4.16
$$\begin{aligned}&\hat A(\underline{\sigma})=\hat A(000\ldots)+\sum_{k=0}^{\infty}\Phi_k(\sigma_0\ldots\sigma_k),\\ &\Phi_k(\sigma_0\ldots\sigma_k)=\hat A(\sigma_0\ldots\sigma_k000\ldots)-\hat A(\sigma_0\ldots\sigma_{k-1}000\ldots),\end{aligned}\tag{5.4.16}$$

we deduce that

e5.4.17
$$\frac{\overline{\mu}_0(\sigma_0'\ldots\sigma_N'|\sigma_{N+1}\ldots)}{\overline{\mu}_0(\sigma_0''\ldots\sigma_N''|\sigma_{N+1}\ldots)}=e^{-\sum_{j,k=0}^{\infty}\{\Phi_k(\sigma_j'\ldots\sigma_{j+k}')-\Phi_k(\sigma_j''\ldots\sigma_{j+k}'')\}}.\tag{5.4.17}$$

Hence by comparing (5.4.17) with (5.3.1) (or by comparing (5.4.15) with (5.3.4)), we see that $\overline{\mu}_0$ is the Gibbs distribution with potential Φ given by (5.4.10), provided no convergence problems arise in summing the series in (5.4.16).

If $k>2$

e5.4.18
$$\begin{aligned}&|\Phi_k(\sigma_0\ldots\sigma_k)|\equiv|\hat A(\sigma_0\ldots\sigma_k000\ldots)-\hat A(\sigma_0\ldots\sigma_{k-1}000\ldots)|\equiv\\ &\equiv\Big|\log\frac{|\varphi_{\sigma_0}'(X(\sigma_1\sigma_2\ldots\sigma_k000\ldots))|}{|\varphi_{\sigma_0}'(X(\sigma_1\sigma_2\ldots\sigma_{k-1}000\ldots))|}\Big|\le\\ &\le(\inf_{x\in[0,1],\sigma}|\varphi_\sigma'(x)|)^{-1}\\ &\qquad\Big||\varphi_{\sigma_0}'(X(\sigma_1\sigma_2\ldots\sigma_k000\ldots))|-|\varphi_{\sigma_0}'(X(\sigma_1\sigma_2\ldots\sigma_{k-1}000\ldots))|\Big|\le\\ &\le\frac{\sup_\sigma C_{\varphi_\sigma}}{\inf|\varphi_\sigma'(x)|}\Big|X(\sigma_1\sigma_2\ldots\sigma_k000\ldots)-X(\sigma_1\sigma_2\ldots\sigma_{k-1}000\ldots)\Big|^{\varepsilon}\le\\ &\le\frac{\sup_\sigma C_{\varphi_\sigma}}{\inf|\varphi_\sigma'(x)|}\quad\lambda^{-\varepsilon(k-1)}\le C\lambda^{-\varepsilon k},\end{aligned}\tag{5.4.18}$$

if C_{φ_σ} is the Hölder continuity modulus in $C^\varepsilon([0,1])$ of $x\to|\varphi_{\sigma_0}'(X)|$ (we recall that $f_\sigma\in C^{1+\varepsilon}([a_\sigma,a_{\sigma+1}])$ and $|f_\sigma'|\ge\lambda>1$ by assumption) and C is a suitable constant. Hence not only the series in (5.4.16) is totally convergent with respect to the variations of $\underline{\sigma}$, but also one has

e5.4.19
$$\begin{aligned}&\|\Phi\|_1<+\infty,\qquad\text{and}\\ &\sum_{X\ni 0}e^{\kappa\,(\text{diam }X)}\|\Phi_X\|<+\infty\text{ for all }\kappa<\varepsilon\log\lambda.\end{aligned}\tag{5.4.19}$$

From item (ii) in proposition (5.3.1) it follows that $\overline{\mu}_0$ is uniquely determined by its conditional probability (5.4.17).

Furthermore from corollary (5.3.1) it follows that the Gibbs distribution $\overline{\mu}$ with potential Φ, on $\{0,\ldots,n\}^{\mathbb{Z}}$ (*i.e.* on the *bilateral* sequences), restricted to the σ–algebra $\mathcal{B}(\mathbb{Z}^+)$ is absolutely continuous (in fact equivalent) with respect to $\overline{\mu}_0$ and it is τ–invariant:

e5.4.20
$$\overline{\mu}(E)=\overline{\mu}(\tau^{-1}E)\qquad\text{for all }E\in\mathcal{B}(\mathbb{Z}^+),\tag{5.4.20}$$

as a consequence of the τ–invariance of $\overline{\mu}$.

The triad $([0,1], S, \mu)$ with μ defined by

e5.4.21
$$\mu(E) = \overline{\mu}(X^{-1}(E)) \qquad \text{for all } E \in \mathcal{B}([0,1]) \tag{5.4.21}$$

is a dynamical system in which μ is S–invariant and S–mixing: this follows from the corresponding property of $\overline{\mu}$ that, in turn, follows from the results of Section §5.2 on Gibbs distributions on $\{0,\ldots,n\}^{\mathbb{Z}}$.

The bound on the exponential mixing rate follows from the property of $\overline{\mu}$

e5.4.22
$$\begin{aligned}|\,\overline{\mu}(C^{0\ldots N}_{\sigma_0\ldots\sigma_N} \cap C^{j\ldots j+M}_{\sigma'_0\ldots\sigma'_M}) - \overline{\mu}(C^{0\ldots N}_{\sigma_0\ldots\sigma_N})\,\overline{\mu}(C^{0\ldots M}_{\sigma'_0\ldots\sigma'_N})| \le\\ \le K\,\overline{\mu}(C^{0\ldots N}_{\sigma_0\ldots\sigma_N})\,\overline{\mu}(C^{0\ldots M}_{\sigma'_0\ldots\sigma'_M})\min\{1, e^{-\kappa\,(j-N)}\},\end{aligned} \tag{5.4.22}$$

where $\kappa > 0$, $K > 0$; the latter property follows from the results discussed in the problems [5.3.7], [5.3.15] and [5.3.16] and the second of (5.4.20).

Every $F \in C^{\alpha}([0,1])$, $\alpha > 0$, with a Hölder continuity modulus C_F can be expressed in the coordinates $\underline{\sigma}$ by setting

e5.4.23
$$\overline{F}(\underline{\sigma}) = F(X(\underline{\sigma})) = F(X(000\ldots)) + \sum_{k=0}^{\infty} \psi_k(\sigma_0\ldots\sigma_k), \tag{5.4.23}$$

where

e5.4.24
$$\begin{aligned}&\psi_k(\sigma_0\ldots\sigma_k) = F(X(\sigma_0\ldots\sigma_k 000\ldots)) - F(X(\sigma_0\ldots\sigma_{k-1}000\ldots)),\\ &|\psi_k(\sigma_0\ldots\sigma_k)| \le C_F|X(\sigma_0\ldots\sigma_k 000\ldots) - X(\sigma_0\ldots\sigma_{k-1}000)|^{\alpha} \le C'\lambda^{-\alpha k},\end{aligned} \tag{5.4.24}$$

for a suitable $C' > 0$. Therefore we can remark that, having set $\psi_{-1}(\sigma_{-1}) = F(X(000\ldots))$,

e5.4.25
$$\begin{aligned}&\int F(S^j x)\, F(x)\mu(dx) = \int \overline{F}(\tau^j\underline{\sigma})\;\overline{F}(\underline{\sigma})\,\overline{\mu}(d\underline{\sigma}) =\\ &\sum_{h,k=-1}^{\infty}\sum_{\sigma}\psi_k(\sigma_0\ldots\sigma_k)\psi_h(\sigma_j\ldots\sigma_{j+h})\,\overline{\mu}(C^{0\ldots k}_{\sigma_0\ldots\sigma_k} \cap C^{j\ldots j+h}_{\sigma_j\ldots\sigma_{j+h}}),\end{aligned} \tag{5.4.25}$$

where $\sum_{\sigma}$ denotes summation over $\sigma_0\ldots\sigma_k, \sigma_j\ldots\sigma_{j+h}$. From (5.4.22) and (5.4.23) we see that the (5.4.25) converges to the square of

$$\sum_{k=-1}^{\infty}\sum_{\sigma_0\ldots\sigma_k}\psi_k(\sigma_0\ldots\sigma_k)\,\overline{\mu}(C^{0\ldots k}_{\sigma_0\ldots\sigma_k})$$

with exponential rate, as $j \to \infty$, and the constant κ' of the exponential convergence can be chosen arbitrarily provided $\kappa' < (\min\{\alpha, \varepsilon\})\log\lambda$. ■

Problems for §5.4

Q5.4.1 **[5.4.1]:** (*Markovian maps of the interval*)
If S is strictly expansive but not surjective (cf. proposition (5.4.1) and Fig. (5.4.1))

check that the same arguments of the proof of proposition (5.4.1) can be repeated under the hypothesis that S is *Markovian i.e.* $\cup_{\sigma=0}^{n+1} S^{\pm} a_\sigma \subseteq \{0, a_1, \ldots, a_{n+1}\}$, where $S^{\pm}(a_\sigma)$ denotes the right and left limits of $S(x)$ as $x \to a_\sigma$. (*Hint:* We must replace $\{0, \ldots, n\}^{\mathbb{Z}^{\pm}}$ with $\{0, \ldots, n\}_T^{\mathbb{Z}^{\pm}}$, where T is a suitable compatibility matrix (which?), see problems of §5.3.)

Q5.4.2 **[5.4.2]:** (*Symbolic theory of invariant distributions for Markovian maps of the interval*)
Show that the equation for the density h of the probability distribution $\overline{\mu}$ with respect to $\overline{\mu}_0$ is, by proposition (5.3.3), the positive solution of the equation $Lf = f$ on $C(\{0, \ldots, n\}^{\mathbb{Z}})$ with

$$(Lf)(\underline{\sigma}) = \sum_{\sigma=0}^{n} e^{-\widehat{A}(\sigma\sigma_0\sigma_1\ldots)} f(\sigma\sigma_0\sigma_1\ldots).$$

Q5.4.3 **[5.4.3]:** (*Density for the invariant distribution for expansive Markovian maps of the interval*)
Show that the equation of problem [5.4.2] is the symbolic version of the equation, on $L_\infty([0,1])$, $Lg = g$ with:

$$(Lg)(x) = \sum_{\sigma=0}^{n} |\varphi_0'(x)| g(\varphi_\sigma(x)).$$

Give a geometric interpretation of this equation.

Q5.4.4 **[5.4.4]:** (*Existence of coordinates in which the Lebesgue measure is invariant for an expansive Markovian map of the interval*)
If the interval map S admits μ as invariant probability distribution $\mu(dx) = h(x)dx$ with h continuous and positive and if we set $y = \pi(x) = \int_0^x h(\xi)d\xi$ we define a change of coordinates $x \leftrightarrow y$ such that $\overline{S} = \pi S \pi^{-1}$, image of S in the new coordinates, is an expansive map (not necessarily strictly such) that admits Lebesgue measure as invariant measure.

Q5.4.5 **[5.4.5]:** Interpret the result of problem [5.4.4] as the statement that "a necessary condition in order that S admits an invariant measure absolutely continuous with respect to Lebesgue measure, with continuous and positive density h, is that there exists a system of coordinates on $[0,1]$ in which S appears as expansive", [Pi79].

Q5.4.6 **[5.4.6]:** (*About Ulam–Von Neumann map*)
Strict expansivity is not a necessary condition in order that a sufficiently regular surjective map admits an invariant probability distribution which is absolutely continuous with respect to Lebesgue measure. For an example check that the map $S(x) = 4x(1-x)$ admits the measure with density $(\pi^2 x(1-x))^{-1/2}$ as an invariant probability distribution.

Q5.4.7 **[5.4.7]:** (*Markovian quadratic maps of the interval and Ruelle's points*)
The map $S(x) = Ax(1-x)$, $A < 4$ is Markovian in the sense of problem [5.4.1], with respect to suitable decompositions $a_0 = 0 < a_1 < \ldots < a_{n+1} = 1$ of $[0,1]$, for infinitely many values of A (however the decomposition can depend on A): find one such value of A. The latter values of A are called "Ruelle's points". (*Hint:* If A is such that $S^3(1/2) = x_A$, with $x_A =$ non-zero fixed point of S, then the decomposition 0, 1/2, x_A, 1 is Markovian, [Ru77], [Pi79], [Pi80] and [Pi81].)

Bibliographical note to §5.4

The theory of the maps of $[0,1]$ into itself has a long history. Its origin is independent from the theory of Gibbs distributions and it often deals with cases that are not easily reduced to problems relative to Gibbs distributions: see for instance the works of Renij, [Re57], Ulam and von Neumann, [UV47], and Lasota and Yorke, [LY73]. The method of symbolic dynamics, is easily applicable only in the very special "Markovian" cases (*i.e.* when there exists a finite decomposition of $[0,1]$ into intervals such that the points on the boundaries of the intervals are transformed by the map into a subset of themselves, and, furthermore, in every interval of the decomposition the map is monotonic), has been employed in [La78], [Ru77] and [Bo75]. Other more recent applications often using other methods are in [Mi79], [CE80a], [Pi79], [Pi80], [Pi??], [PY79], [Fe78], [Fe79] and [CEL80].