CHAPTER IX

Some special topics in KAM theory

§9.1 Resummation and renormalized series for invariant tori

In Chapter VIII the proof of the KAM theorem was based on diagrams and notions becoming visually clear if the diagrammatic interpretation of the various terms building the Lindstedt series was kept in mind. The analogy with the diagrams used in perturbation theory in Quantum Field Theory and in Statistical Mechanics is, we feel, quite striking. Therefore one can wonder whether one could go ahead and apply other techniques widely employed in those fields.

One key technique is that of transforming power series solutions to perturbation problems parameterized by a small parameter ε into series which are no longer power series but which are series of functions that depend in a nontrivial way on ε. This is interesting particularly in the cases (virtually all cases in the quoted fields) in which the power series solutions are either very unlikely to converge even for small ε or are even known to diverge: although they often are asymptotic series. The new series in terms of functions of the parameters are usually obtained by collecting together parts of *different* orders in ε identified by suitable properties of graphs that are used to represent them: in the best cases the new series may even be absolutely convergent and from them one can obtain informations about the nature of the singularity at $\varepsilon = 0$, as well as much simpler expressions for the solutions to perturbation problems.

In the case of the KAM theory however we have seen that the results are analytic in ε and no resummation is, strictly speaking, necessary. Nevertheless the analogies allow us to apply ideas that had so much impact in the understanding of quantum and statistical theories to the "simpler" case of KAM.

This is interesting enough in itself, and its interest becomes even more manifest if one attempts studying the invariant tori of dimension lower than the maximal one (*i.e.* lower than the number of degrees of freedom). Such tori exist and are conjectured to have parametric equations that are non analytic functions of the perturbation parameter. This will be discussed briefly later: we show here how *resummations* of the Lindstedt series can be actually performed in the case of the KAM theory leading to simpler *renormalized* series, for instance, for the parametric equations of the invariant tori $\underline{h}, \underline{H}$ considered in proposition (8.1.1).

We consider the case of perturbations which are even trigonometric polynomials depending only on the angle variables, and impose the strong Diophantine condition, definition (8.3.2), with scale factor $\gamma \equiv 2$ on the rotation vector $\underline{\omega}_0$ of the invariant torus that we study. In this case we know from the analysis in Chapter VIII that the power expansion in ε for the parametric equation $\underline{h}(\underline{\psi})$ converges for small ε. However, conceptually, *the analysis here is independent of the one developed in Chapter VIII* and it develops an independent proof of the KAM theorem, which at the same time leads to a natural approach to the study of some invariant tori of dimension lower than the maximal one, namely the hyperbolic ones.

In terms of the power expansion envisaged in the previous Chapter, we can define a solution "approximated to order k" as

*e*9.1.1
$$\underline{h}^{(\le k)}(\underline{\psi},\varepsilon)=\sum_{\underline{\nu}\in\mathbb{Z}^\ell} e^{i\underline{\nu}\cdot\underline{\psi}}\underline{h}^{(\le k)}_{\underline{\nu}}(\varepsilon),\qquad \underline{h}^{(\le k)}_{\underline{\nu}}(\varepsilon)=\sum_{k'=1}^{k}\varepsilon^{k'}\underline{h}^{(k')}_{\underline{\nu}},\tag{9.1.1}$$

where $\underline{h}^{(k)}_{\underline{\nu}}$ is defined in (8.2.5) and $\underline{h}^{(k)}_{\underline{0}}=\underline{0}$.

The propagators G_λ introduced in (8.2.3) are matrices (proportional to the identity). Given a branch $\lambda = v'v$ it carries two tensor labels $j'_{v'}, j_v$ associated with the nodes v' and v, respectively. The propagator labels are contracted with the tensor labels of the line λ_v which are then summed over

*N*9.1.1 all the ℓ possible values.[1]

Note in particular that the propagators $G_\lambda \equiv G(\underline{\omega}_0\cdot\underline{\nu}_\lambda)$ satisfy (trivially) the relations

*e*9.1.2
$$G^T(-x)=G^\dagger(x)=G(x),\tag{9.1.2}$$

which will play a crucial rôle in the following; here and henceforth superscripts ^{T} and † on a matrix denote, respectively, the transposed and the adjoint (*i.e.* the conjugated transposed) of a matrix.

We define scales and clusters exactly as in Chapter VIII using $\gamma \equiv 2$ as a scale factor.

---

[1] Since G_λ is diagonal this means the we have to take $j_v = j'_{v'}$ and sum over this common value.

If T is a self-energy graph in a tree ϑ, we denote by $V(T)$ the set of nodes in T, by $\Lambda(T)$ the set of branches in T (see definition (8.3.3) for details), by k_T the number of nodes in T (*i.e.* $k_T = |V(T)|$), and by λ_T^1 and λ_T^2 the branches, respectively, exiting and entering T.

Let ϑ be a tree $\vartheta \in \Theta_{k,\underline{\nu},j}$, cf. item (vii) in definition (8.2.1), with a self-energy subgraph T. Define $\vartheta_0 = \vartheta \setminus T$ as the set of nodes and branches of ϑ outside T (of course ϑ_0 is not a tree), we define $V(\vartheta_0) = V(\vartheta) \setminus V(T)$ and $\Lambda(\vartheta_0) = \Lambda(\vartheta) \setminus \Lambda(T)$. Consider simultaneously all trees such that the structure ϑ_0 outside of the self-energy graph is the same, while the self-energy graph itself can be arbitrary, *i.e.* T can be replaced by any other self-energy graph T' with $k_{T'} \geq 1$. This allows us to define as a formal power series the matrix

e9.1.3
$$M(\underline{\omega}_0 \cdot \underline{\nu}; \varepsilon) = \sum_{\vartheta = \vartheta_0 \cup T'} \mathcal{V}_{T'}(\underline{\omega}_0 \cdot \underline{\nu}), \qquad \text{where}$$
$$\mathcal{V}_T(\underline{\omega}_0 \cdot \underline{\nu}) \stackrel{def}{=} \varepsilon^{k_T} \Big(\prod_{v \in V(T)} F_v \Big) \Big(\prod_{\lambda \in \Lambda(T)} G_\lambda \Big), \tag{9.1.3}$$

where the sum is over all trees ϑ such that $\vartheta \setminus T$ is fixed to be ϑ_0 and the mode labels of the nodes $v \in V(T)$ have to satisfy the conditions defining the self-energy graphs (see (8.3.3)).

Recall that a self-energy graph T has *height* $p = 0$ if it does not contain any other self-energy graphs, and that it has height $p \in \mathbb{N}$, recursively, if it contains maximal self-energy graphs with height $p - 1$ (see definition following (8.4.1)).

The following property holds as an algebraic identity between formal power series.

L9.1.1 **(9.1.1) Lemma:** (Symmetries of the propagators)
The following two properties hold:
(1) $(M(x;\varepsilon))^T = M(-x;\varepsilon)$, *and*
(2) $(M(x;\varepsilon))^\dagger = M(x;\varepsilon)$;
The latter means that the matrix $M(x;\varepsilon)$ *is self-adjoint.*

Proof: Consider a graph computed with propagators verifying the properties (9.1.2), which is trivially valid in our case since the propagator is a real and diagonal matrix, cf. (8.2.3). Given a self-energy graph T with momentum $\underline{\nu}$ flowing through the entering branch λ_T^2, call $\mathcal{P}$ the path connecting the
N9.1.2 exiting branch λ_T^1 to the entering branch λ_T^2.[2] Then consider also the self-energy graph T' obtained by taking λ_T^1 as entering branch and λ_T^2 as exiting branch and by taking $-\underline{\nu}$ as momentum flowing through the (new) entering branch λ_T^1: in this way the arrows of all branches along the path $\mathcal{P}$ change orientation, while all the subtrees (internal to T) having the root in $\mathcal{P}$ are

[2] *i.e.* the minimal connected set of branches joining the branches λ_T^1 and λ_T^2 along the lines of the graph T.

left unchanged. This implies that the momenta of the branches belonging to $\mathcal{P}$ change sign, while all the other momenta do not change. Since all propagators G_λ corresponding to the branches $\lambda \in \mathcal{P}$ are transformed into G_λ^T, the property (9.1.2) implies that the entry ij of the matrix $M(\underline{\omega}_0 \cdot \underline{\nu}; \varepsilon)$ corresponding to the self-energy graph T is equal to the entry ji of the matrix $M(-\underline{\omega}_0 \cdot \underline{\nu}; \varepsilon)$; then the property (1) is proved.

Given a self-energy graph T, consider also the self-energy graph T' obtained by reversing the sign of the mode labels of the nodes $v \in V(T)$, and by swapping the entering branch with the exiting one. In this way the arrows of all the branches along the path $\mathcal{P}$ joining the two external branches are reversed, while all the subtrees (internal to T) having the root in $\mathcal{P}$ are left unchanged (as before). One then realizes that the complex conjugate of $\mathcal{V}_{T'}(\underline{\omega}_0 \cdot \underline{\nu})$ equals $\mathcal{V}_T(\underline{\omega}_0 \cdot \underline{\nu})$, by using the form of the node factors in (8.2.8), and the fact that one has $f^*_{\underline{\nu}} = f_{-\underline{\nu}}$ (as $V(\underline{\alpha})$ is real) and $G^\dagger(\underline{\omega}_0 \cdot \underline{\nu}) = G(\underline{\omega}_0 \cdot \underline{\nu})$; this proves the property (2). ∎

Remark: The lemma has been proved without making use of the exact form of the propagator, but only exploiting the fact that it satisfies the property (9.1.2). Therefore it has a more general validity, a fact that will be exploited below.

The function $M(\underline{\omega}_0 \cdot \underline{\nu}; \varepsilon)$ depends on ε but, by construction, it is independent of ϑ_0: hence we can rewrite (9.1.3) as

*e*9.1.4
$$M(\underline{\omega}_0 \cdot \underline{\nu}; \varepsilon) = \sum_{T'} \mathcal{V}_{T'}(\underline{\omega}_0 \cdot \underline{\nu}), \tag{9.1.4}$$

where the sum is over all self-energy graphs of order $k \geq 1$ with external branches with momentum $\underline{\nu}$.

Note that, by setting $\underline{\omega} = 2^\tau C \underline{\omega}_0$, in (9.1.4), if $2^{n-1} \leq |\underline{\omega} \cdot \underline{\nu}| < 2^n$, the sum is restricted to the self-energy graphs T' on scale $n_{T'} \geq n+3$. If we define n^0_λ so that, if $\underline{\nu}^0_\lambda$ is the momentum which *would* flow on the line λ *if* the entering line had momentum $\underline{\nu} = \underline{0}$ (cf. comment following ♣8.3.5),

*e*9.1.5
$$2^{n^0_\lambda - 1} \leq \left|\underline{\omega} \cdot \underline{\nu}^0_\lambda\right| < 2^{n^0_\lambda}, \tag{9.1.5}$$

then, for all branches $\lambda \in \Lambda(\vartheta)$ one has $n_\lambda = n^0_\lambda$ because, for the same reasons discussed, in Sections §8.1 and §8.2, when shifting the branches external to the self-energy graphs of a tree ϑ, the scale labels n_λ of all branches $\lambda \in \Lambda(\vartheta)$ do not change.

So far we considered formal power expansions in ε. Instead of the function in (9.1.1), we can define a different sequence $\{\overline{\underline{h}}^{[d]}(\underline{\psi}, \varepsilon)\}_{d\in\mathbb{N}}$ of *approximating functions* converging to the solution (as we shall see below), by defining it iteratively as follows.

*D*9.1.1 **(9.1.1) Definition:** (Renormalized graphs and clusters)
Denote by $\Theta^{\mathcal{R}}_{k,\underline{\nu},j}$ *the set of all trees of order* k *without self-energy graphs*

and with labels $\underline{\nu}_{\lambda_0} = \underline{\nu}$ *and* $j_{\lambda_0} = j$ *associated with the root branch; we shall call* $\Theta^{\mathcal{R}}_{k,\underline{\nu}}$ *the set of* renormalized trees *of order* k *and with label* $\underline{\nu}$ *associated with the root branch. Given a tree* $\vartheta \in \Theta^{\mathcal{R}}_{k,\underline{\nu}}$ *and a cluster* T *in* ϑ, *by extension we shall say that* T *is a* renormalized cluster.
We can also consider a self-energy graph which does not contain any other self-energy graph: we shall say that such a self-energy graph is a renormalized self-energy graph*; of course no one of such clusters can appear in any tree in* $\Theta^{\mathcal{R}}_{k,\underline{\nu},j}$.

For a renormalized tree ϑ of arbitrary order k, define

e9.1.6
$$\overline{\mathrm{Val}}^{[d]}(\vartheta) \stackrel{def}{=} \Big(\prod_{v\in V(\vartheta)} F_v\Big)\Big(\prod_{\lambda\in\Lambda(\vartheta)} \overline{G}^{[d-1]}_{\lambda}\Big), \tag{9.1.6}$$

with the *dressed propagators* given by the matrices

e9.1.7
$$\begin{cases} \overline{G}^{[0]}_{\lambda} = (\underline{\omega}_0\cdot\underline{\nu}_\lambda)^{-2}\,, \\ \overline{G}^{[d]}_{\lambda} = \Big[(\underline{\omega}_0\cdot\underline{\nu}_\lambda)^2 - M^{[d]}(\underline{\omega}_0\cdot\underline{\nu}_\lambda;\varepsilon)\Big]^{-1}, \quad \text{for } d\ge 1\ , \end{cases} \tag{9.1.7}$$

and the sequence $\{M^{[d]}(\underline{\omega}_0\cdot\underline{\nu};\varepsilon)\}_{d\in\mathbb{N}}$ is iteratively defined as sum of the values of all renormalized self-energy graphs that can be inserted on a line of momentum $\underline{\nu}$ and that are computed by using the propagators $\overline{G}^{[d-1]}_{\lambda}$, *i.e.* as

e9.1.8
$$\begin{aligned} M^{[d]}(\underline{\omega}_0\cdot\underline{\nu};\varepsilon) &= \sum_{\text{renormalized}\, T} \mathcal{V}^{[d]}_T(\underline{\omega}_0\cdot\underline{\nu}), \\ \mathcal{V}^{[d]}_T(\underline{\omega}_0\cdot\underline{\nu}) &= \varepsilon^{k_T}\Big(\prod_{v\in V(T)} F_v\Big)\Big(\prod_{\lambda\in\Lambda(T)} \overline{G}^{[d-1]}_{\lambda}\Big); \end{aligned} \tag{9.1.8}$$

We set $M^{[0]}(\underline{\omega}_0\cdot\underline{\nu};\varepsilon)\equiv 0$. If a line λ has scale $-d_\lambda$ the self energy graphs that can be inserted on λ necessarily consist of lines with scales $< -d_\lambda$ *therefore* $M^{[d]}(\underline{\omega}\cdot\underline{\nu}_\lambda) \equiv M^{[d_\lambda]}(\underline{\omega}\cdot\underline{\nu})$ for $d > d_\lambda$.
To avoid confusing the value of a renormalized tree with the tree value introduced in (8.2.7), we shall call (9.1.6) the *renormalized value* of the (renormalized) tree. Then we shall write

e9.1.9
$$\begin{aligned} &\overline{\underline{h}}^{[d]}(\underline{\psi},\varepsilon) = \sum_{\underline{\nu}\in\mathbb{Z}^\ell} e^{i\underline{\nu}\cdot\underline{\psi}}\,\overline{\underline{h}}^{[d]}_{\underline{\nu}}(\varepsilon), \\ &\overline{\underline{h}}^{[d]}_{\underline{\nu}}(\varepsilon) = \sum_{k'=1}^{\infty} \varepsilon^{k'}\,\overline{\underline{h}}^{[d,k]}_{\underline{\nu}}(\varepsilon), \qquad \overline{\underline{h}}^{[d,k]}_{\underline{\nu}}(\varepsilon) = \sum_{\vartheta\in\Theta^{\mathcal{R}}_{k,\underline{\nu}}} \overline{\mathrm{Val}}^{[d]}(\vartheta), \end{aligned} \tag{9.1.9}$$

where the last formula holds for $\underline{\nu}\neq\underline{0}$, because for $\underline{\nu}=\underline{0}$ one has $\overline{\underline{h}}^{[d,k]}_{\underline{0}} \equiv 0$, cf. (9.1.1).

Remark: Note that if we expand the quantity $M^{[d]}(\underline{\omega}_0\cdot\underline{\nu};\varepsilon)$ in powers of ε, by expanding the propagators $\overline{G}^{[d-1]}_{\lambda}$, we reconstruct the sum of the values of all self-energy graphs containing only self-energy graphs with height $p < d$ (recall the definition of height given after (9.1.3)). Therefore if we expand $M^{[d+1]}(\underline{\omega}_0 \cdot \underline{\nu}; \varepsilon)$ in powers of ε we obtain the same terms that we would obtain by expanding $M^{[d]}(\underline{\omega}_0 \cdot \underline{\nu}; \varepsilon)$, plus the sum of the values of all the self-energy graphs containing also self-energies graphs with height d, which are absent in the self-energy graphs contributing to $M^{[d]}(\underline{\omega}_0 \cdot \underline{\nu}; \varepsilon)$. Such a result can be used in order to prove that the power series defining the functions $\overline{\underline{h}}_{\underline{\nu}}^{[d]}(\varepsilon)$, truncated at order $k < d$, coincide with the functions $\underline{h}_{\underline{\nu}}^{(\le k)}(\varepsilon)$ given by (9.1.1), as it is not difficult to check (see problem [8.3.1]).

In the coming section we shall derive the following renormalization result.

P9.1.1 **(9.1.1) Proposition:** (Resummation and renormalization of Lindstedt series)
(i) The matrices $M^{[d]}(x,\varepsilon)$ defined in (9.1.8) admit a limit $M^{[\infty]}((x,\varepsilon)$ as $d \to \infty$ which is analytic in ε and is defined for all real $x \ne 0$ and verifies, for all ε small enough

e9.1.10
$$\|M^{[d]}(x,\varepsilon)\| \le C\,\varepsilon^2\,x^2 \tag{9.1.10}$$

with an ε–independent constant C.
(ii) The series obtained by considering all renormalized tree graphs (i.e. all graphs without any self–energy subgraph) and computing their values by the rules of definition (8.2.1) but replacing the propagator $\delta_{j'j}(\underline{\omega}_o \cdot \underline{\nu})^{-2}$ by the matrix

e9.1.11
$$((\underline{\omega}_0 \cdot \underline{\nu})^2 + M^{[\infty]}(\underline{\omega}_0 \cdot \underline{\nu}, \varepsilon))^{-1} \tag{9.1.11}$$

is convergent for $|\varepsilon|$ small.
(iii) The functions $t \to \underline{\psi} + \underline{\omega}_0 t + \underline{h}^{[\infty]}(\underline{\psi} + \underline{\omega}_0 t)$, where $\underline{h}^{[\infty]}$ is the sum of the series in item (ii), is analytic in ε for ε small and in $\underline{\psi} \in \mathbb{T}^\ell$ and solve the equations of motion for the Hamiltonian (8.1.1).
(iv) The function $\underline{h}^{[\infty]}$ defines the parametric equations for an invariant torus with rotation vector $\underline{\omega}_0$. Therefore it coincides with the function $\underline{h}$ considered in proposition (8.1.1).

Remark: This means that the resummation procedure leads to a renormalized series with propagators (9.1.11) which is a representation of the function $\underline{h}$ whose existence is proved by the KAM theorem. At the same time this yields an independent proof of the KAM theorem. We devote Section §(9.2) to a proof of the above proposition.

§9.2 Bounds on renormalized series. Convergence

Let $\|\cdot\|$ be an algebraic matrix norm (*i.e.* a norm which verifies $\|AB\| \le \|A\|\,\|B\|$ for all matrices A and B); for instance $\|\cdot\|$ can be $\|A\| =$

$\max_w \|Aw\|$, with the maximum taken over all vectors w of norm 1, *i.e.* $||w||^2 = \sum_j |w_j|^2 = 1$.

L9.2.1 **(9.2.1) Lemma:** (Uniform bounds on resummed series)
Assume that the propagators $\overline{G}^{[d]}_\lambda \equiv \overline{G}^{[d]}(\underline{\omega}\cdot\underline{\nu}_\lambda;\varepsilon)$ *satisfy*

e9.2.1
$$\Big(\overline{G}^{[d]}(x;\varepsilon)\Big)^T = \overline{G}^{[d]}(-x;\varepsilon), \qquad \big\|\overline{G}^{[d]}(x;\varepsilon)\big\| < \frac{2}{x^2} \tag{9.2.1}$$

for all $|\varepsilon|<\varepsilon_0$, *if* ε_0 *is small enough. Then given* $\kappa>0$ *there is a constant* B_f *such that one has, setting* $M(\vartheta)=\sum_{v\in V(\vartheta)}|\underline{\nu}_v|$,

e9.2.2
$$\begin{aligned}&\overline{\underline{h}}^{[d,V]}_{\underline{\nu}} \le \frac{1}{V!}\sum_{\vartheta\in\Theta^{\mathcal{R}}_{V,\underline{\nu}}}\Big|\overline{\mathrm{Val}}^{[d]}(\vartheta)\Big| \le \left(|\varepsilon|B_f\right)^V,\\ &\frac{1}{V!}\sum_{\substack{\vartheta\in\Theta^{\mathcal{R}}_{V,\underline{\nu}}\\ M(\vartheta)=s}}\Big|\overline{\mathrm{Val}}^{[d]}(\vartheta)\Big| \le \left(|\varepsilon|B_f\right)^V e^{-\kappa s/4},\end{aligned} \tag{9.2.2}$$

for all $s>0$.

Proof: The hypothesis (9.2.1) implies that for all propagators $\overline{G}^{[d]}_\lambda$ one has

e9.2.3
$$\Big\|\overline{G}^{[d]}_\lambda\Big\| \le C_1 2^{-2n_\lambda}, \qquad C_1 = 2\left(2^{\tau+2}C\right)^2. \tag{9.2.3}$$

where C is the constant in definition (8.3.2). Therefore the contribution from a single tree is bounded for all $n_0\le 0$ by

e9.2.4
$$|\varepsilon|^V\left(C_1\,2^{-2n_0}\right)^V \prod_{v\in V(\vartheta)}\Big[N^{m_v+1}f_0\Big(\prod_{n=-\infty}^{n_0} C_1 2^{-ck2^{n/\tau}}\Big)\Big], \tag{9.2.4}$$

where f_0 is a bound of the size of the Fourier coefficients of the perturbation, cf. (8.4.4), $V=|V(\vartheta)|$, having used that, for all trees $\vartheta\in\Theta^{\mathcal{R}}_{k,\underline{\nu},j}$, the number $N_n(\vartheta)$ of branches with scale n in ϑ satisfy the bound

e9.2.5
$$N_n(\vartheta)\le c\,k\,2^{n/\tau} \tag{9.2.5}$$

for some constant c (see lemma (8.3.1) where c is shown to be bounded by $4\,E^{-1}$.

We then can simply proceed as we did in the analysis of the convergence of the Lindstedt series when the tree values of trees not verifying property [P], introduced before lemma (8.2.2), were ignored. The only difference being that the constant C_1 which bounds the propagators is now different from the constant C and is given by (9.2.3).

Finally the last factor in the second of Eq. (9.2.2) can be inserted by further increasing the constant that multiplies ε because $\sum_v|\underline{\nu}_v|\le kN$ as

f is a trigonometric polynomial of degree N and $1 = e^{\kappa\sum_v |\underline{\nu}_v|} e^{-\kappa\sum_v |\underline{\nu}_v|} \le e^{N\,\kappa\,k} e^{-\kappa\sum_v |\underline{\nu}_v|}$. ■

Remark: Note that, although the propagators are no longer diagonal, they still satisfy the same property as (9.1.2), which was the crucial one which is used in order to prove the formal cancellations between tree values.

L9.2.2 **(9.2.2) Lemma:** (Symmetry and cancellations properties)
The matrices $M^{[d]}(\underline{\omega}_0\cdot\underline{\nu};\varepsilon)$ satisfy the relation

e9.2.6
$$\Big(M^{[d]}(x;\varepsilon)\Big)^T = M^{[d]}(-x;\varepsilon). \tag{9.2.6}$$

Moreover the matrix $M^{[-q]}(x;\varepsilon)$ is the restriction to the x's of the form $x=\underline{\omega}\cdot\underline{\nu}$ with $\underline{\nu}$ of scale $\le q$ of an analytic function of x in the disk $|x|\le 2^q$ if ε_0 is small enough; it satisfies the bound

e9.2.7
$$\Big\|M^{[d]}(x;\varepsilon)\Big\| \le D x^2|\varepsilon|^2, \tag{9.2.7}$$

for all $d\in\mathbb{N}$ and for a suitable d–independent constant D. As a consequence $\overline{G}^{[d]}_\lambda$ verify (9.2.1) for all $d\ge 1$, and therefore (9.2.2) holds for all $d\ge 1$.

Proof. We consider the matrices $M^{[d]}$ defined in (9.1.8) and suppose *inductively* that $M^{[d]}$ verifies (9.2.7) and the analyticity property preceding it for $0\le d\le p-1$; note that the assumption holds trivially for $d=0$. Note also that (9.2.6) imply that the propagators $\overline{G}^{[d]}_\lambda$ verify (9.2.1) for ε_0 small enough.

To define $M^{[p]}$ we must consider the renormalized self-energy graphs T and evaluate their values by using the propagators $\overline{G}^{[p-1]}_\lambda$, according to (9.1.8).

Given $\underline{\omega}\cdot\underline{\nu}$ such that $2^{q-1}\le|\underline{\omega}\cdot\underline{\nu}|<2^q$ for some $q\le 0$, the propagators $\overline{G}^{[p-1]}_\lambda$ have an analytic extension to the disk $|\underline{\omega}\cdot\underline{\nu}|<2^{q+2}$ and, under the hypotheses (9.2.6) and (9.2.7), verify the symmetry property and the bound in (9.2.1).

We have

e9.2.8
$$M^{[p]}(x;\varepsilon) = \sum_{h=q+3}^{0}\ \sum_{\substack{\text{renormalized } T\\ n_T=h}} \mathcal{V}^{[p]}_{T,h}(x), \tag{9.2.8}$$

where by appending the label h to $\mathcal{V}^{[p]}_T(x)$ we distinguish the contributions to $M^{[p]}(x;\varepsilon)$ coming from self-energy graphs T on scale h (which is constrained to be $\ge q+3$).

The value $\mathcal{V}^{[p]}_T(\underline{\omega}_0\cdot\underline{\nu})$ is analytic in $\underline{\omega}\cdot\underline{\nu}$ for $|\underline{\omega}\cdot\underline{\nu}|\le 2^{h+2}$, and the sum over all the self-energy graphs T with V nodes is bounded by

e9.2.9
$$\sum_{\substack{T\\ k_T=V}} \Big|\mathcal{V}^{[p]}_{T,h}(\underline{\omega}_0\cdot\underline{\nu})\Big| \le \frac{(|\varepsilon|B_f)^V}{1-e^{-\kappa/8}} e^{-\kappa 2^{-h/\tau}/8}, \tag{9.2.9}$$

because the mode labels $\underline{\nu}_v$ of the nodes $v \in V(T)$ must satisfy $\sum_{v\in V(T)} |\underline{\nu}_v| > 2^{-h/\tau}$ (recall that we are dealing with renormalized trees, so that for all clusters T one has $\widetilde{T} = T$, *i.e.* T contains no self-energy graph, cf. definition (8.3.5)).

Since the symmetry property expressed by (9.2.1) for $d = p-1$ is implied by(9.2.6) and this is the only property of the propagators that one needs in order to check the algebraic cancellations, we can conclude that the same cancellation mechanisms extend to the renormalized self-energy values $\mathcal{V}_T^{[p]}(\underline{\omega}_0 \cdot \underline{\nu})$. Therefore we see that $\mathcal{V}_{T,h}^{[p]}(x)$ will vanish at $x = 0$ to order 2.

By the analyticity in $\underline{\omega}\cdot\underline{\nu}$ for $|\underline{\omega}\cdot\underline{\nu}| \le 2^{h+2}$ and by the maximum principle we deduce from (9.2.9) that one has

e9.2.10
$$\sum_{\substack{T\\ k_T=V}} \Big|\mathcal{V}_{T,h}^{[p]}(x)\Big| \le \frac{(|\varepsilon|B_f)^V}{1-e^{-\kappa/8}} e^{-\kappa 2^{-h/\tau}/8} \left(\frac{2^\tau C x}{2^{h+2}}\right)^2, \tag{9.2.10}$$

Therefore we can use that $\sum_{h=q+3}^{0} e^{-\kappa 2^{-h/\tau}} 2^{-2h} < B_1 < \infty$ and that $V \ge 2$, and the proof is complete. ■

It also follows that there exists the limit (reached at a finite value of d if x is fixed because $M^{[d]}(x;\varepsilon)$ becomes identically equal to $M^{[d_0]}(x;\varepsilon)$ if if $x = \underline{\omega}\cdot\underline{\nu}_\lambda$ with λ on scale $-d_0$))

e9.2.11
$$\lim_{d\to\infty} M^{[d]}(x;\varepsilon) = M^{[\infty]}(x;\varepsilon), \tag{9.2.11}$$

with $M^{[\infty]}(x;\varepsilon)$ analytic in ε for $|\varepsilon| < \varepsilon_0$: in fact the following result holds.

L9.2.3 **(9.2.3) Lemma:** (Further uniform bounds on resummed Lindstedt series) *For all $d \ge 1$ one has*

e9.2.12
$$\Big\|M^{[d+1]}(x;\varepsilon) - M^{[d]}(x;\varepsilon)\Big\| \le \hat{B}_1 \hat{B}_2^d \varepsilon^{2d} x^2, \tag{9.2.12}$$

for some constants $\hat{B}_1$ and $\hat{B}_2$ and for $|\varepsilon| < \varepsilon_0$ with ε_0 small enough.

Proof: For $-d <$ scale of x the difference is 0. For $-d \ge$ scale of x this is implied by (9.2.10). ■

We can now define the "fully renormalized" expansion of the parametric equations of the invariant torus as the sum of the values of the renormalized trees evaluated according to (9.1.8) with $\overline{G}_\lambda^{[d-1]}$ replaced by

e9.2.18
$$\overline{G}^{[\infty]}(x;\varepsilon) = \Big(x^2 - M^{[\infty]}(x;\varepsilon)\Big)^{-1}, \qquad x = \underline{\omega}_0\cdot\underline{\nu}_\lambda. \tag{9.2.13}$$

The above discussion shows that the series

e9.2.19
$$\begin{aligned}\underline{h}^{[\infty]}(\underline{\psi},\varepsilon) &= \sum_{k=1}^{\infty}\sum_{\underline{\nu}\in\mathbb{Z}^\ell} \varepsilon^k e^{i\underline{\nu}\cdot\underline{\psi}} \sum_{\vartheta\in\Theta^{\mathcal{R}}_{k,\underline{\nu}}} \underline{\mathrm{Val}}^{[\infty]}(\vartheta),\\ \underline{\mathrm{Val}}^{[\infty]}(\vartheta) &= \Big(\prod_{v\in V(\vartheta)} F_v\Big)\Big(\prod_{\lambda\in\Lambda(\vartheta)} \overline{G}_\lambda^{[\infty]}\Big),\end{aligned} \tag{9.2.14}$$

converges for $|\varepsilon| < \varepsilon_0$ and that it coincides with the limit for $d \to \infty$ of $\overline{\underline{h}}^{[d]}(\underline{\psi};\varepsilon)$, which therefore exists. Moreover the function $\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)$, which we have just shown to be well defined and analytic for ε small enough, solves the equations of motion, as the following lemma shows.

L9.2.4 **(9.2.4) Lemma:** Convergence of renormalized series)
One has, formally (i.e. order by order in the expansion in ε around $\varepsilon = 0$)

e9.2.20
$$\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon) \equiv \lim_{d\to\infty} \overline{\underline{h}}^{[d]}(\underline{\psi};\varepsilon) = \underline{h}(\underline{\psi};\varepsilon), \tag{9.2.15}$$

where $\underline{h}(\underline{\psi};\varepsilon)$ is the formal power series which solves the equations of motion. The function $\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)$ solves, therefore, the equations of motion.

Remark: (1) In the formulation of the above lemma we combine the information about the existence and analyticity of the function $\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)$ which has been proved in the previous lemmata and the existence of the formal power series for $\underline{h}(\underline{\psi},\varepsilon)$, about whose convergence *we make no statement here* as we do not want to make use of the KAM result of Section §8.1, but we want to derive it again by the alternative approach that we are following.
(2) If we show that the analytic function in the l.h.s. of (9.2.15) has a power series at $\varepsilon = 0$ whose coefficients coincide with those of the Lindstedt series (which is a formal series) for $\underline{h}(\underline{\psi})$ then it follows that the Lindstedt series necessarily converges and its sum coincides with $\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)$. Furthermore the function $\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)$ solves the equations of motion in the sense that $t \to \underline{\alpha}(t) = \underline{\psi} + \underline{\omega}t + \underline{h}^{[\infty]}(\underline{\psi} + \underline{\omega}t, \varepsilon)$ is a solution of the equations of motion $\underline{\ddot{\alpha}} = -\underline{\partial}V(\underline{\alpha})$: here we use that if two analytic functions (in our case the two sides of (8.1.12) evaluated with $\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)$ in place of $\underline{h}(\underline{\psi},\varepsilon)$) have equal derivatives of all orders at $\varepsilon = 0$ then they coincide.
(3) The above remarks show the validity of the lemma: it is however interesting to check its validity directly by substituin of the series for $\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)$ into the equation (8.1.12) that it solves. Therefore we discuss how to perform the check. This provides a useful method in similar cases in which it is not known or it is not true that generalizations of Lindstedt series converge and their solution is produced by summation rules of divergent series: see Appendix 9.2 for an example in which this situation arises.

Proof: We denote by $\overline{G}^{[\infty]}$ the operator with kernel $\overline{G}^{[\infty]}(\underline{\omega}_0 \cdot \underline{\nu};\varepsilon)$ in Fourier space, and we represent (9.2.14), in a more compact notations, as

e9.2.21
$$\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon) = \sum_{\vartheta\in\Theta^{\mathcal{R}}} \overline{\underline{\rm Val}}^{\mathcal{R}}(\vartheta;\underline{\psi},\varepsilon), \tag{9.2.16}$$

where $\Theta^{\mathcal{R}}$ is the set of all renormalized trees, and, for $\vartheta \in \Theta^{\mathcal{R}}_{k,\underline{\nu}} \subset \Theta^{\mathcal{R}}$, we have defined

e9.2.22
$$\overline{\underline{\rm Val}}^{\mathcal{R}}(\vartheta;\underline{\psi},\varepsilon) = \varepsilon^k e^{i\underline{\nu}\cdot\underline{\psi}} \overline{\underline{\rm Val}}^{[\infty]}(\vartheta). \tag{9.2.17}$$

The function $\underline{h}(\underline{\psi},\varepsilon)$ solving the equations of motion is formally defined as the solution of the functional equation

e9.2.23
$$\underline{h}(\underline{\psi},\varepsilon) = G\partial_{\underline{\psi}} f\left(\underline{\psi} + \underline{h}(\underline{\psi},\varepsilon)\right), \tag{9.2.18}$$

where $G = (i\underline{\omega}_0 \cdot \partial)^{-2} = \overline{G}^{[0]}$ is the operator with kernel $G(x) = x^{-2}$.
We have (see problem [8.3.1])

e9.2.24
$$G(x)\left(M^{[\infty]}(x;\varepsilon) + (G^{[\infty]}(x;\varepsilon))^{-1}\right) = 1. \tag{9.2.19}$$

Then it is easy to realize that the function $\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)$ solves the equation of motions; one can reason as follows. One has

e9.2.25
$$\begin{aligned}
G\partial_{\underline{\psi}} f\left(\underline{\psi} + \overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)\right) &= G\sum_{m=0}^{\infty}\frac{1}{m!}\partial_{\underline{\psi}}^{m+1} f(\underline{\psi})\left(\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)\right)^m \\
&= G\sum_{m=0}^{\infty}\frac{1}{m!}\partial_{\underline{\psi}}^{m+1} f(\underline{\psi})\sum_{\vartheta_1\in\Theta^{\mathcal{R}}}\overline{\underline{\mathrm{Val}}}^{\mathcal{R}}(\vartheta_1;\underline{\psi},\varepsilon)\ldots\sum_{\vartheta_m\in\Theta^{\mathcal{R}}}\overline{\underline{\mathrm{Val}}}^{\mathcal{R}}(\vartheta_m;\underline{\psi},\varepsilon) \\
&= G\left(\overline{G}^{[\infty]}\right)^{-1}\sum_{\vartheta\in\Theta^{\mathcal{R}}_*}\overline{\underline{\mathrm{Val}}}^{\mathcal{R}}(\vartheta;\underline{\psi},\varepsilon),
\end{aligned} \tag{9.2.20}$$

where $\Theta^{\mathcal{R}}_*$ differs from $\Theta^{\mathcal{R}}$ as it contains also trees which can have only one self-energy graph with exiting branch λ_0, if, as usual, λ_0 denotes the root branch of ϑ; the operator $G(\overline{G}^{[\infty]})^{-1}$ takes into account the fact that, by construction, to the root branch λ_0 an operator G is associated, while in $\overline{\underline{\mathrm{Val}}}^{\mathcal{R}}(\vartheta;\underline{\psi},\varepsilon)$, by definition, a propagator $\overline{G}^{[\infty]}$ is associated.
Then we can write (9.2.20), by explicitly separating the trees containing such a self-energy graph from the others,

e9.2.26
$$\begin{aligned}
&G\partial_{\underline{\psi}} f\left(\underline{\psi} + \overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)\right) \\
&\quad = G\left(\overline{G}^{[\infty]}\right)^{-1}\left(\overline{G}^{[\infty]}M^{[\infty]}\sum_{\vartheta\in\Theta^{\mathcal{R}}}\overline{\underline{\mathrm{Val}}}^{\mathcal{R}}(\vartheta,\underline{\psi},\varepsilon) + \sum_{\vartheta\in\Theta^{\mathcal{R}}}\overline{\underline{\mathrm{Val}}}^{\mathcal{R}}(\vartheta,\underline{\psi},\varepsilon)\right) \\
&\quad = G\Big(M^{[\infty]}\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon) + (\overline{G}^{[\infty]})^{-1}\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon)\Big) \\
&\quad = G\Big(M^{[\infty]} + (\overline{G}^{[\infty]})^{-1}\Big)\overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon) = \overline{\underline{h}}^{[\infty]}(\underline{\psi},\varepsilon),
\end{aligned} \tag{9.2.21}$$

where the property (9.2.19) has been used in the last line.
Note that at each step only absolutely converging series have been dealt with; the lemma is thus proved. ■

This completes the alternative proof of proposition (8.1.1). We can take full advantage of such a different approach when dealing with hyperbolic lower-dimensional tori (see Appendix (9.2)).

Appendix 9.2: Resonances and low dimensional invariant tori

Consider the Hamiltonian

eA9.2.1
$$\mathcal{H}(\underline{A},\underline{\alpha})=\underline{\omega}_0\cdot\underline{A}+\frac{1}{2}\underline{A}\cdot\underline{A}+\frac{1}{2}\underline{B}\cdot\underline{B}+\varepsilon f(\underline{\alpha},\underline{\beta}),\tag{A9.2.1}$$

where $(\underline{\alpha},\underline{A})\in\mathbb{T}^r\times\mathbb{R}^r$ and $(\underline{\beta},\underline{B})\in\mathbb{T}^s\times\mathbb{R}^s$ are conjugated variables, and $\underline{\omega}_0$ is a vector in $\mathbb{R}^r$ satisfying the Diophantine property $C\,|\underline{\omega}_0\cdot\underline{\nu}|>|\underline{\nu}|^{-\tau}$, $\forall\underline{\nu}\in\mathbb{Z}^r\setminus\{\underline{0}\}$, with $C>0$ and $\tau\ge r-1$, and $f(\underline{\alpha},\underline{\beta})=\sum_{\underline{\nu}\in\mathbb{Z}^r}e^{i\underline{\nu}\cdot\underline{\alpha}}f_{\underline{\nu}}(\underline{\beta})$.

The motions $\underline{\alpha}(t)=\underline{\alpha}_0+\underline{\omega}_0 t$, $\underline{\beta}(t)=\underline{\beta}_0$, $\underline{A}=\underline{0}$, $\underline{B}=\underline{0}$ define, if $\varepsilon=0$, an invariant torus of dimension $r+s$ which is called *resonant* and which is foliated into invariant tori of dimension r on which the motion is quasi-periodic and ergodic (parameterized by the s angles $\underline{\beta}_0$).

The question is whether the resonant tori continue to exist, slightly deformed, when ε is >0 and small. This means that we ask whether there are solutions of the form

eA9.2.2
$$\begin{cases}\underline{\alpha}(t)=\underline{\psi}(t)+\underline{a}(\underline{\psi}(t),\underline{\beta}_0;\varepsilon),\\ \underline{\beta}(t)=\underline{\beta}_0+\underline{b}(\underline{\psi}(t),\underline{\beta}_0;\varepsilon),\end{cases}\tag{A9.2.2}$$

for some functions $\underline{a}$ and $\underline{b}$, real analytic and 2π-periodic in $\underline{\psi}\in\mathbb{T}^r$, such that the motion in the variable $\underline{\psi}$ is $\dot{\underline{\psi}}=\underline{\omega}_0$.

The problem can be naturally studied via the resummation method used for the maximal tori: however not all invariant tori will still exist for $\varepsilon>0$, at least not in general. Only invariant tori which are close to the unperturbed ones which have $\underline{\beta}_0$ coinciding with a stationary point for the function $f_{\underline{0}}(\underline{\beta})=(2\pi)^{-r}\int d\underline{\alpha}\,f(\underline{\alpha},\underline{\beta})$ and which correspond to strict maxima of $f_{\underline{0}}(\underline{\beta})$ can be shown to exist for $\varepsilon>0$ small and to be analytically reducible to the unperturbed ones.

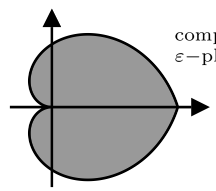
P9.2.1 **(9.2.1) Proposition:** (Hyperbolic tori, [GG02])
Consider the equations of motion $\ddot{\underline{\alpha}}=-\underline{\partial}_{\underline{\alpha}}f(\underline{\alpha},\underline{\beta})$, $\ddot{\underline{\beta}}=-\underline{\partial}_{\underline{\beta}}f(\underline{\alpha},\underline{\beta})$ (the Hamiltonian equations for (A9.2.1)), and suppose $\underline{\omega}_0$ to satisfy (8.1.6) and $\underline{\beta}_0$ to be such that

eA9.2.3
$$\partial_{\underline{\beta}}f_{\underline{0}}(\underline{\beta}_0)=\underline{0},\qquad \partial^2_{\underline{\beta}}f_{\underline{0}}(\underline{\beta}_0)\ \textit{is negative definite.}\tag{A9.2.3}$$

There exist a constant $\varepsilon_0>0$ and, for all $\varepsilon\in(0,\varepsilon_0)$, two functions $\underline{a}(\underline{\psi},\underline{\beta}_0;\varepsilon)$ and $\underline{b}(\underline{\psi},\underline{\beta}_0;\varepsilon)$, real analytic and 2π-periodic in $\underline{\psi}\in\mathbb{T}^r$, such that (A9.2.2) is a solution of (A9.2.1) with $\dot{\underline{\psi}}=\underline{\omega}_0$. Moreover $\underline{a}(\underline{\psi},\underline{\beta}_0;\varepsilon)$ and $\underline{b}(\underline{\psi},\underline{\beta}_0;\varepsilon)$ are analytic in ε for $\varepsilon\in(0,\varepsilon_0)$.

Remarks: (1) The solutions whose existence is stated by the theorem do not seem to be analytic in ε at $\varepsilon=0$: however they are certainly analytic

for $\varepsilon > 0$ and small: it appears to be an open problem to prove their actual divergence (or convergence) for $|\varepsilon|$ small. Furthermore, if the second condition in (A9.2.3) is replaced by $\partial^2_{\underline{\beta}} f_{\underline{0}}(\underline{\beta}_0)$ *is positive definite* then the same conclusions hold for $\varepsilon \in (-\varepsilon_0, 0)$.
(2) The proof based on the resummation methods is not the only possible one, nor it has been the first in historical order, however it yields rather detailed information on the regularity of the considered tori. In particular the analyticity domain is much larger, see the heart-like domain D_0 in Fig.(9.2.1).
(3) From the point of view of stability theory the negative definiteness assumption in (A9.2.3) and $\varepsilon > 0$ imply that the invariant torus with $\underline{\beta}$ close to $\underline{\beta}_0$ is unstable and has positive Lyapunov exponents which, to first order, coincide with the eigenvalues of $-\varepsilon\partial^2_{\underline{\beta}} f(\underline{\beta}_0)$: for this reason we call them *hyperbolic tori*.



F9.2.1 **Fig.(9.2.1)** Analyticity domain D_0 for the hyperbolic invariant torus. The cusp at the origin is a second order cusp. The figure corresponds to the case in (A9.2.3) of the proposition (9.2.1).

A tempting conjecture is that the analytic functions $\underline{a}(\underline{\psi}), \underline{b}(\underline{\psi})$ admit a more extended domain of analyticity which touches the negative ε axis on a set Δ which has 0 as a density point and such that the limiting value, as ε approaches points on the negative real axis in Δ, of the functions $\underline{a}(\underline{\psi}), \underline{b}(\underline{\psi})$ is real and describes the parametrization of an "*elliptic invariant torus*". The conjectured form of the extended analyticity domain is represented in figure (9.2.2).

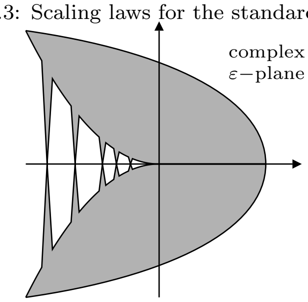
The above proposition is proved by the method of resummations in [GG02].

Problems for §9.2

Q9.2.1 **[9.2.1]:** Check that for any self-energy graph T with self-energy-scale n, one has $n_T \geq n+3$. (*Hint:* Writing, for any branch $\lambda \in \Lambda(T')$, $\underline{\nu}_\lambda$ as $\underline{\nu}_\lambda = \underline{\nu}^0_\lambda + \sigma_\lambda \underline{\nu}$, one has $|\underline{\omega}\cdot\underline{\nu}^0_\lambda| > 2^\tau C^{-1} C |\underline{\nu}^0_\lambda|^{-\tau} \geq 2^\tau (\sum_{v\in V(\widetilde{T})} |\underline{\nu}_v|)^{-\tau} \geq 2^\tau 2^{n+3}$, while $|\underline{\omega}\cdot\underline{\nu}| < 2^n$, so that one obtains $|\underline{\omega}\cdot\underline{\nu}_\lambda| > 2^\tau 2^{n+3} - 2^n > 2^{n+2}$, which implies $n_\lambda \geq n+3$.)

Q9.2.2 **[9.2.2]:** Check that the power series defining the functions $\overline{\underline{h}}^{[d]}_{\underline{\nu}}(\varepsilon)$, truncated at any order $k < d$, coincide with the functions $\underline{h}^{(\leq k)}_{\underline{\nu}}(\varepsilon)$ given by (9.1.1). (*Hint:* Compare the diagrammatic representations of the two functions.)

Q9.2.3 **[9.2.3]:** Prove (9.2.19). (*Hint:* One has $\overline{G}^{[\infty]}(x;\varepsilon) = (G^{-1}(x) - M^{[\infty]}(x;\varepsilon))^{-1}$, so that $G^{-1}(x) = (\overline{G}^{[\infty]}(x;\varepsilon))^{-1} + M^{[\infty]}(x;\varepsilon)$.)



F9.2.2 **Fig.(9.2.2)** The domain D_0 of Figure 1 can be further extended? The conjecture above asks whether the extended analyticity domain could possibly be represented (close to the origin) as here: the domain reaches the real axis at cusp points which are in a set Δ which has 0 as a density point: the points of Δ correspond to elliptic tori which are analytic continuations of the hyperbolic tori in the complex ε-plane. The analytic continuation would be continuous through the real axis at the points of Δ. The cusps would be at least quadratic.

Q9.2.4 **[9.2.4]:** (*Lindstedt series for resonant invariant tori*)
Look for an expansion

$$\underline{a}(\underline{\psi};\varepsilon)=\sum_{k=1}^{\infty}\varepsilon^k\underline{a}^{(k)}(\underline{\psi})=\sum_{\underline{\nu}\in\mathbb{Z}^r}e^{i\underline{\nu}\cdot\underline{\psi}}\underline{a}_{\underline{\nu}}(\varepsilon)=\sum_{k=1}^{\infty}\varepsilon^k\sum_{\underline{\nu}\in\mathbb{Z}^r}e^{i\underline{\nu}\cdot\underline{\psi}}\,\underline{a}_{\underline{\nu}}^{(k)},$$
$$\underline{b}(\underline{\psi};\varepsilon)=\sum_{k=1}^{\infty}\varepsilon^k\underline{b}^{(k)}(\underline{\psi})=\sum_{\underline{\nu}\in\mathbb{Z}^r}e^{i\underline{\nu}\cdot\underline{\psi}}\underline{b}_{\underline{\nu}}(\varepsilon)=\sum_{k=1}^{\infty}\varepsilon^k\sum_{\underline{\nu}\in\mathbb{Z}^r}e^{i\underline{\nu}\cdot\underline{\psi}}\,\underline{b}_{\underline{\nu}}^{(k)},$$

of the parametric equations for the resonant invariant tori in the context of proposition (9.2.1): the dependence on $\underline{\beta}_0$ has not been explicitly written. Check that to order k the equations of motion for the Hamiltonian (A9.2.1) have solutions that, written in the form (A9.2.2), become

$$(\underline{\omega}\cdot\underline{\nu})^2\,\underline{a}_{\underline{\nu}}^{(k)}=\Big[\partial_{\underline{\alpha}}f\Big]_{\underline{\nu}}^{(k-1)},$$
$$(\underline{\omega}\cdot\underline{\nu})^2\,\underline{b}_{\underline{\nu}}^{(k)}=\Big[\partial_{\underline{\beta}}f\Big]_{\underline{\nu}}^{(k-1)},$$

and show that they can be uniquely solved to all orders, recursively, under the only assumption that $\underline{\beta}_0$ is a *non degenerate* stationarity point for $f_{\underline{0}}(\underline{\beta})$ (*i.e.* $\det\partial^2_{\underline{\beta}}f_{\underline{0}}(\underline{\beta}_0)\neq 0$) and that $\underline{\alpha}^{(k)}$ has $\underline{0}$ average over $\underline{\psi}$. (*Hint:* it is essential that one does not assume that $\underline{\beta}^{(k)}$ has zero average over $\underline{\psi}$ and therefore also its average is determined (uniquely) recursively; the definiteness positive or negative of $\underline{\partial}^2_{\underline{\beta}}f(\underline{\beta}_0)$ is *not necessary* for this result.)

§9.3 Scaling laws for the standard map

In this section we shall consider another case in which the diagrammatic formalism developed in Chapter VIII can be naturally applied. The problem consists in considering what happens in some simple dynamical systems for complex values of the rotation vector; for simplicity one can take systems with two degrees of freedom, and with one component of the rotation vector fixed to some value (for instance 1), so that only one free component ω is left, and one can refers to it as the rotation number.

The main motivation comes from studying the optimal arithmetic dependence on the rotation number in order to have an invariant curve. It is known that, for analytic perturbations of two-dimensional integrable area-preserving maps, the best condition one can impose on the rotation number ω, whose continued fraction convergents are p_k/q_k (see problem [2.2.3]), is that the function $B_1(\omega) = \sum_{k=0}^{\infty} q_k^{-1} \log q_{k+1}$ is finite: if $B_1(\omega) < \infty$ one says that ω satisfies the Bryuno condition (which is weaker than the Diophantine condition). The function $B_1(\omega)$ is related to the *Bryuno function* $B(\omega)$ introduced by Yoccoz in studying Siegel's problem [Yo95]: the difference $B_1(\omega) - B(\omega)$ is an essentially bounded function.

Then one asks if it is possible to interpolate the radius of convergence $\rho(\omega)$ of the Lindstedt series for the standard map in terms of the Bryuno function, in the sense that there exists $\beta \in \mathbb{R}^+$ such that the quantity $Q_\beta(\omega) \equiv \log \rho(\omega) + \beta B(\omega)$ remains uniformly bounded, independently of ω (provided that ω satisfies the Bryuno condition, so that both $\rho(\omega)$ and $B(\omega)$ are well defined).

The possibility that $Q_2(\omega)$ be uniformly bounded is related, see Remark (1) to proposition (9.3.1), to the divergence rate of the conjugating function for $\omega = p/q + i\eta$, with $\gcd(p, q) = 1$ and $\eta \neq 0$. This is the problem that we shall consider in this and next sections.

The standard map is a discrete-time, one-dimensional dynamical system generated by the iteration of the area-preserving (symplectic) map of the cylinder into itself, $T_\varepsilon : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, given by

e9.3.1
$$T_\varepsilon : \begin{cases} x' = x + y + \varepsilon \sin x, \\ y' = y + \varepsilon \sin x. \end{cases} \tag{9.3.1}$$

It was first introduced by Greene [Gr79] and Chirikov [Ch79], and it can be considered as one of the simplest Hamiltonian dynamical systems with a nontrivial behaviour (see problem [9.3.1] for the interpretation of the standard map as the Poincaré map of a continuous-time Hamiltonian system).

The homotopically non-trivial invariant curves $\mathcal{C}_{\varepsilon,\omega}$ with rotation number ω of the map T_ε may be determined by changing coordinates on $\mathbb{T} \times \mathbb{R}$,

e9.3.2
$$\begin{cases} x = \alpha + u(\alpha, \varepsilon, \omega), \\ y = 2\pi\omega + v(\alpha, \varepsilon, \omega), \end{cases} \tag{9.3.2}$$

and imposing that the dynamics induced in the variables (α, ω) is given by the unperturbed map

e9.3.3
$$\begin{cases} \alpha' = \alpha + 2\pi\omega, \\ \omega' = \omega. \end{cases} \tag{9.3.3}$$

By (9.3.1) one has $y' = x' - x$ so that $v(\alpha, \varepsilon, \omega) = u(\alpha, \varepsilon, \omega) - u(\alpha - 2\pi\omega, \varepsilon, \omega)$; we can therefore consider only the function u.

The coordinate transformation (9.3.2) *conjugates* the dynamics on the invariant curve to a rotation, and the function u is called the *conjugating function*. It satisfies the functional equation

e9.3.4
$$D_\omega^2 u(\alpha,\varepsilon,\omega)=\varepsilon\sin(\alpha+u(\alpha)),\tag{9.3.4}$$

where the operator D_ω^2 acts on functions of α as follows:

e9.3.5
$$D_\omega^2\phi(\alpha)=\phi(\alpha+2\pi\omega)-2\phi(\alpha)+\phi(\alpha-2\pi\omega).\tag{9.3.5}$$

Assuming that equations (9.3.4) admit ε-analytic solutions near $\varepsilon=0$ and imposing that the average of u over α be 0 one checks that the Taylor coefficients in ε of $u(\alpha,\varepsilon,\omega)$ are uniquely determined and are odd as functions of α. The proofs of such statements can be carried out as in Section §8.1. However, as in the cases considered in Chapter VIII one needs to study, and prove, the convergence of the series.

To each smooth solution to (9.3.4) corresponds an invariant curve $\mathcal{C}_{\varepsilon,\omega}$ whose parametric equations are

e9.3.6
$$\mathcal{C}_{\varepsilon,\omega}:\begin{cases}x=\alpha+u(\alpha,\varepsilon,\omega),\\ y=2\pi\omega+u(\alpha,\varepsilon,\omega)-u(\alpha-2\pi\omega,\varepsilon,\omega),\end{cases}\tag{9.3.6}$$

and it has the same smoothness properties as those of $u(\cdot,\varepsilon,\omega)$.

To simplify the notations, we shall drop the dependence on ω and write just $u(\alpha,\varepsilon)$. The expansion in powers of ε of the conjugating function u (that we naturally call *Lindstedt series* because it is derived by following arguments analogous to those repeatedly discussed in Chapter VIII) has the form

e9.3.7
$$u(\alpha,\varepsilon)=\sum_{\nu\in\mathbb{Z}}u_\nu(\varepsilon)\,e^{i\nu\alpha}=\sum_{k\ge1}u^{(k)}(\alpha)\varepsilon^k=\sum_{k\ge1}\sum_{\nu\in\mathbb{Z}}u_\nu^{(k)}e^{i\nu\alpha}\,\varepsilon^k.\tag{9.3.7}$$

By inserting in (9.3.4) the series (9.3.7) and equating the Fourier and Taylor coefficients of both sides (see Section 8.1), one finds that the coefficients $u_\nu^{(k)}$ are defined by the recursion relations

e9.3.8
$$u_\nu^{(k)}=\frac{1}{\gamma(\nu)}\sum_{m\ge0}\frac{1}{m!}\sum_{\substack{\nu_0+\ldots+\nu_m=\nu\\ k_1+\ldots+k_m=k-1}}\frac12(-i\nu_0)(i\nu_0)^m\prod_{j=1}^m u_{\nu_j}^{(k_j)},\tag{9.3.8}$$

with $\nu_0=\pm1$ and

e9.3.9
$$\gamma(\nu)=2\left(\cos 2\pi\omega\nu-1\right)=-4\sin^2\pi\omega\nu,\tag{9.3.9}$$

for $\nu\ne0$, while $u_0^{(k)}=0$ for all $k\ge1$. The case $m=0$ in (9.3.8) has to be interpreted as $u_\nu^{(k)}=(-i\nu_0)/\gamma(\nu)$, which imposes $k=1$ and $\nu=\nu_0$.

The radius of convergence of the Lindstedt series is defined as

e9.3.10
$$\rho = \inf_{\alpha\in\mathcal{T}} \left(\limsup_{k\to\infty} \left| u^{(k)}(\alpha) \right|^{1/k} \right)^{-1} . \tag{9.3.10}$$

Of course the Lindstedt series is plagued by the *small divisors* problem, due to the fact that $\gamma(\nu)$ can be arbitrarily close to 0 for $\omega \in \mathbb{R}\setminus\mathbb{Q}$ and can be 0 for $\omega\in\mathbb{Q}$ (here $\mathbb{Q}$ denotes the real rational numbers). If $\omega\in\mathbb{R}$ satisfies a Diophantine property and ε is sufficiently small, ρ is strictly greater than 0 and therefore we have an analytic invariant curve and an analytic conjugation to a circle rotation. Here we are interested in the behavior of ρ as the rotation number ω tends to a *resonant* value, *i.e.* $\omega \to p/q$, with $p,q\in\mathbb{Z}$ and $\gcd(p,q)=1$. The behavior of the radius of convergence near a resonant value of the rotation number is related to the problem of *Bryuno's interpolation* and to the problem of determining the optimal arithmetic condition on ω to have an analytic invariant curve, see below. We consider

e9.3.11
$$\omega = \frac{p}{q} + i\eta, \tag{9.3.11}$$

with $p,q\in\mathbb{Z}$, $\gcd(p,q)=1$ and $\eta\in\mathbb{R}$, in the limit $\eta\to 0$. Later on we shall show how to extend our results to the case in which ω tends to p/q *along any path on the complex ω plane non-tangential to the real axis.*

We are interested in the exact (asymptotic) dependence of ρ on η, in the limit $\eta\to 0$. In particular, we shall prove the following result.

P9.3.1 **(9.3.1) Proposition:** (Asymptotics near resonances in the standard map) *Consider the standard map (9.3.1) with $\omega = p/q+i\eta$, $p,q\in\mathbb{Z}$, $\gcd(p,q)=1$ and $\eta\in\mathbb{R}$. Then the following results hold.*
(i) For fixed $\eta\neq 0$ the function $u(\alpha,\varepsilon)$, defined by (9.3.2), is divisible by ε and jointly analytic in (α,ε) in the product of a strip of width $\delta_0>0$ around the real axis in the complex α-plane and a neighborhood $|\varepsilon|<\varepsilon_0$ of the origin in the complex ε-plane, with $\varepsilon_0 = O(\eta^{2/q}e^{-\delta_0})$.
(ii) The function $U(\alpha,\varepsilon)\equiv u(\alpha,(2\pi\eta)^{2/q}\varepsilon)$ is well defined for $\eta\to 0$ and converges to a function $\bar u(\alpha,\varepsilon)$, divisible by ε^q and analytic in ε^q in a neighborhood of the origin. Furthermore $\overline{u}$ is $2\pi/q$ periodic and solves the differential equation

e9.3.12
$$\frac{\mathrm{d}^2\bar u(\alpha)}{\mathrm{d}\alpha^2} = C_{p/q}\varepsilon^q \sin\left(q(\alpha+\bar u(\alpha))\right), \tag{9.3.12}$$

with boundary conditions $\bar u(0)=\bar u(2\pi)=0$, for some nonvanishing explicitly computable constant $C_{p/q}$.

Remarks: (1) The bound $\varepsilon_0 = O(\eta^{2/q})$ of proposition (9.2.1) *is consistent* with the law $|Q_2(\omega)|<C$, because the logarithm of the complex extension of the Bryuno function diverges as $\eta\to 0$ in the same way as the $\log\varepsilon_0$, [Yo95]; but of course it does not implies such a law. The proof of the latter bound can be found in [BG01].
(2) The analyticity strip in α width can be prefixed arbitrarily.

The proof will be completed in Section §(9.4). Here we derive explicit expressions for $u_\nu^{(k)}$ as sums of suitable quantities and exhibit cancellations that occur if the quantities used to express the value of $u_\nu^{(k)}$ are suitably collected showing the validity of item (i) of proposition (9.3.1).

We can envisage a tree expansion for the conjugating function as in Section §8.1, with some simplifications, due to the fact that mode labels take only the values $\nu_v = \pm 1$ (because in (9.3.1) the non linear terms, *i.e.* $\sin x$ have only harmonics ± 1) has to be associated with the nodes v of the trees. Then the momentum flowing through a branch λ_v will be defined as

e9.3.13
$$\nu_{\lambda_v} = \sum_{w \le u} \nu_w, \qquad \nu_w = \pm 1; \tag{9.3.13}$$

and the condition $u_0^{(k)} = 0$ implies that no branch λ can have momentum $\nu_\lambda = 0$.

So we can write

e9.3.14
$$u_\nu^{(k)} = \frac{1}{2^k} \sum_{\vartheta \in \Theta_{k,\nu}} \mathrm{Val}(\vartheta), \qquad \mathrm{Val}(\vartheta) = -i \prod_{v \in V(\vartheta)} \frac{1}{m_v!} \frac{\nu_v^{m_v+1}}{\gamma(\nu_{\lambda_v})}, \tag{9.3.14}$$

where $\Theta_{k,\nu}$ is (unlike in the previous Sections) the set of *non-numbered trees* with k nodes and momentum $\nu_{\lambda_{v_0}} = \nu$, if v_0 is the last node of the tree, and such that $\nu_\lambda \neq 0$ for all branches $\lambda \in \Lambda(\vartheta)$ (see Section §8.1).

Remark: Unlike what was done in the previous Sections, we consider here non-numbered trees. This means that we do not introduce the number labels for the branches: in order to count the number of non-numbered trees of a given order k, one can imagine to assign to the m_v subtrees with root on a node v a number from 1 to m_v from up to down, and to consider as distinct only trees which cannot be overlapped by permuting such subtrees (for instance if two subtrees with numbers $1 \le m < m' \le m_v$ are equal to each other, then by exchanging the two subtrees we obtain no further tree). This corresponds to the the factorials $m_v!$ associated, in evaluating tree values, with the nodes of the trees. The number of elements in $\Theta_{k,\nu}$ is now bounded by 2^{2k} (see problem [8.2.2]). The reason to proceed in this way shall become clear in the following where the new combinatorial factors will turn out to be quite convenient to prove that the limit function $\bar{u}(\alpha)$
N9.3.1 solves the differential equation (9.3.12).[1]

For $\eta \neq 0$ the power series (9.3.7) is well defined and no small divisors appear: the denominators in $\mathrm{Val}(\vartheta)$ are all bounded from below by $|\eta|^2$. Nevertheless the convergence radius ρ is not uniform in η, and it shrinks to 0 when $\eta \to 0$. The small divisor $\gamma(\nu)$ satisfies the bound

e9.3.15
$$|\gamma(\nu)| \ge \begin{cases} c|\nu\eta|^2, & \text{for } \nu \in q\mathbb{Z} \setminus \{0\}, \\ cq^{-2}, & \text{otherwise}, \end{cases} \tag{9.3.15}$$

[1] The enumeration of trees considered in Chapter VIII and in Sections 9.1, 9.2 will still be useful later in checking cancellations. See also [Ga94b].

for some positive constant c; one can take $c = 2\pi^2$.

Given a tree ϑ, we can associate with each branch λ of ϑ a scale label n_λ, setting $n_\lambda = 0$ if its momentum ν_λ is a multiple of q, and $n_\lambda = 1$ otherwise. Hence in the present analysis *only two scales* arise, unlike the infinitely many arising in the previous sections and in Chapter VIII. Given a tree ϑ, a *cluster* T of ϑ is a maximal connected set of branches on scale $n = 1$; we shall say that such branches are *internal* to T, and write $\lambda \in \Lambda(T)$, so denoting by $\Lambda(T)$ the set of branches in T. A node v will be considered internal to T if the exiting branch or at least one of its entering branches is in T, and we shall write $v \in V(T)$ so denoting by $V(T)$ the set of nodes in T. The branches outside the clusters are all on scale $n = 0$, and each cluster has an arbitrary number $m_T \geq 0$ of entering branches but only one exiting branch.

A cluster T will be called a *null graph* if

*e*9.3.16
$$\sum_{v \in V(T)} \nu_v = 0, \tag{9.3.16}$$

At least one entering branch *must* be present otherwise the exiting branch or the root branch would have momentum $\nu = 0$. For the same reason at least one entering branch must exist. In such a case, the exiting branch of the cluster T will be called a *null branch*; we also denote by k_T the number of nodes internal to T.

Remark: Note that the definition of null graph given here is *not* equivalent to that of self-energy graph of the previous Sections, because a null graph can have an arbitrary number of entering branches, and no constraint is imposed on the number of internal nodes. Here we shall call *null branch* a branch entering a null graph.

A null graph T in a tree graph ϑ_0 determines a subtree ϑ attached to the rest of the tree graph by its entering lines and its exiting line: we can define its *null graph factor* $\mathcal{V}_T(\vartheta)$ as

*e*9.3.17
$$\mathcal{V}_T(\vartheta) = \Big(\prod_{v \in V(T)} \frac{1}{m_v!} \Big) \Big(\prod_{v \in V(T)} \nu_v^{m_v + 1} \Big) \Big(\prod_{\lambda \in \Lambda(T)} \frac{1}{\gamma(\nu_{\lambda_v})} \Big) \tag{9.3.17}$$

the null graph factor will depend on ϑ_0 only through the momenta of the incoming branches of T. By (9.3.15), we have the bound

*e*9.3.18
$$|\mathcal{V}_T(\vartheta)| \leq c^{-k_T} q^{2k_T}, \tag{9.3.18}$$

because all branches inside T are on scale $n = 1$.

Let $N_n(\vartheta_0)$, $n = 0, 1$, denote the number of branches in a tree graph ϑ_0 which have scale n: using again (9.3.15) it follows that for a fixed tree ϑ_0 one has

*e*9.3.19
$$|\mathrm{Val}(\vartheta_0)| \leq c^{-k} q^{2N_1(\vartheta_0)} |\eta|^{-2N_0(\vartheta_0)}. \tag{9.3.19}$$

Let $N_0^*(\vartheta_0)$ be the number of branches on scale $n = 0$ which are not null branches in a given tree ϑ. We then have the bound

e9.3.20
$$N_0^*(\vartheta_0) \le \left[\frac{k}{q}\right], \tag{9.3.20}$$

where k is the order of the tree and $[x]$ is the largest integer smaller or equal to x. This is a statement analogous to the one that in Chapter VIII was called "Siegel–Bryuno" bound and its check is very similar too, see problem [9.3.4].

We now show how to construct a suitable partial resummation of the Lindstedt series. This means that we shall define families of trees to be grouped and bounded together to obtain "extra" η factors in the bounds of the sums of their values.

Given m trees $\vartheta_1, \ldots, \vartheta_m$ with root lines $\lambda_1 \ldots \lambda_m$ on scale 0 and a tree $\overline{\vartheta}$ which has among its lowest lines (in its partial order) a line $\overline{\lambda}$ on scale 0 as well, we can imagine to construct a tree ϑ_0 which contains a null graph T with k_T nodes, $m_T \equiv m$ entering lines $\lambda_1 \ldots \lambda_{m_T}$ and with $\overline{\lambda}$ as exiting line. The k_T nodes $v_1, \ldots, v_{k_T}$ carry modes $\nu_{v_1}, \ldots, \nu_{v_{k_T}}$ such that $\sum \nu_{v_i} = 0$.

We shall call $\mathcal{F}$ the family of trees formed by all trees that can be built in this way (it depends on $\overline{\vartheta}, \vartheta_1, \ldots, \vartheta_m$). Each element of $\mathcal{F}$ is entirely defined by the tree ϑ determined by the null graph T (see comments before (9.3.17)) and its value is the product of a ϑ–independent factor times the null graph factor $\mathcal{V}_T(\vartheta)$ defined in (9.3.17).

It is now convenient, for enumeration purposes, to strip $\mathcal{V}_T(\vartheta)$ of the first factor $\prod_v \frac{1}{m_v!}$ and to attach a label $1, 2, \ldots, k_T + m$ to the $k_T + m$ lines forming ϑ. We obtain a family of "numbered null graphs" which contain many more graphs: we want to regard them as distinct unless they can be overlapped (number labels included) by pivoting, around their ending nodes, the lines ending on the nodes of ϑ.

If we define the value of the new numbered null graphs as in (9.3.17) with $\prod_v \frac{1}{m_v!}$ replaced by $\frac{1}{(k_T+m)!}$ we realize that the sum of the values of the distinct numbered null graphs equals the sum of the values of the null graphs previously considered in the family $\mathcal{F}$.

The momentum flowing through a branch λ_v inside a null graph will depend on the modes of the nodes $w \in T$ such that $w \preceq v$ and on the momenta of the entering branches of the null graph only if the latter end into nodes which precede v; we call L_v the set of such branches.

For any λ_v internal to a null graph T we have

e9.3.21
$$\nu_{\lambda_v} = \sum_{\substack{w \in V(T) \\ w \le v}} \nu_w + \sum_{\lambda' \in L_v} \nu_{\lambda'}, \tag{9.3.21}$$

and, in the corresponding propagator, we can write the argument of the cosine as

e9.3.22
$$\frac{2\pi p}{q} \sum_{\substack{w \in V(T) \\ w \le v}} \nu_w + 2\pi i \Big[\sum_{\substack{w \in V(T) \\ w \le v}} \eta \nu_w + \sum_{\lambda' \in L_v} \eta \nu_{\lambda'} \Big]. \tag{9.3.22}$$

because the lines entering the graph have scale 0 so that $\sum_{\lambda'\in L_v}\nu_{\lambda'}$ is a multiple of q.

We can consider the null graph factor $\mathcal{V}_T(\vartheta)$ as a function of the quantities $\mu_1 \equiv \eta\nu_{\lambda_1}$, …, $\mu_{m_T} \equiv \eta\nu_{\lambda_{m_T}}$ of the incoming branches λ_1, …, λ_{m_T}, *i.e.* $\mathcal{V}_T(\vartheta) \equiv \mathcal{V}_T(\vartheta;\eta\nu_{\lambda_1},\ldots,\eta\nu_{\lambda_{m_T}})$. For any $v \in V(T)$, we have $L_v \subseteq \{\lambda_1,\ldots,\lambda_{m_T}\}$. We can write

e9.3.23
$$\begin{aligned}
&\mathcal{V}_T(\vartheta;\eta\nu_{\lambda_1},\ldots,\eta\nu_{\lambda_{m_T}}) = \mathcal{V}_T(\vartheta;0,\ldots,0)\\
&\quad+\sum_{m=1}^{m_T}\eta\nu_{\lambda_m}\frac{\partial}{\partial\mu_m}\mathcal{V}_T(\vartheta;0,\ldots,0)\\
&\quad+\sum_{m,m'=1}^{m_T}\eta^2\nu_{\lambda_m}\nu_{\lambda_{m'}}\int_0^1 \mathrm{d}t\,(1-t)\frac{\partial^2}{\partial\mu_m\partial\mu_{m'}}\mathcal{V}_T(\vartheta;t\eta\nu_{\lambda_1},\ldots,t\eta\nu_{\lambda_{m_T}}),
\end{aligned}\tag{9.3.23}$$

where $[\partial\mathcal{V}_T/\partial\mu_m](\vartheta;0,\ldots,0)$ denotes the first derivative of $\mathcal{V}_T(\vartheta;\mu_1,\ldots,\mu_{m_T})$ with respect to the argument μ_m, computed in $\mu_1=\ldots=\mu_{m_T}=0$, while the term in the second line is the integral interpolation formula for the second order remainder.

We call $\mathcal{F}_T(\vartheta)$ the collection of all trees which are in the family $\mathcal{F}$ defined above and which contain in the null graph T a tree ϑ' of topological shape which can differ from that of ϑ because of the shift of the entering lines of
N9.3.2 T, and with given mode labels $\{\nu_v\}_{v\in\vartheta}$ assigned to the nodes $v\in\vartheta$;[2] we further enlarge the family $\mathcal{F}_T(\vartheta)$ by adding to it the trees in which the signs of the mode labels are all *simultaneously* reversed. Therefore the elements of the family $\mathcal{F}_T(\vartheta)$ differ from each other because the entering lines can be attached in various ways to the nodes of ϑ or because the signs of the node modes are reversed simultaneously, *i.e.* the trees of $\mathcal{F}$ are obtained by "shifting the external lines entrance nodes" in the null graph T or by the "reversing all signs of the node modes".

For the same reasons discussed in Section 8.3 we see that the sum of all the numbered null graphs in $\mathcal{F}_T(\vartheta)$ is an *even* function of η (because if the modes ν_v can be attached to the nodes of ϑ also the modes $-\nu_v$ can) which vanishes as $\eta\to 0$ to order η^{m_T}: therefore it vanishes to order η^{m_T} if m_T
N9.3.3 is even and η^{m_T+1} if m_T is odd.[3] Hence, since $m_T\ge 1$, it vanishes at least to second order in η so that the zero-th and first orders in η in the r.h.s. of (9.3.23) vanish. This implies the following lemma which is the crucial one, where cancellations between trees in the same family are exploited. The lemma will be immediately used to prove the main estimate on the radius of convergence of the Lindstedt series.

L9.3.1 **(9.3.1) Lemma:** (Cancellations : single null graph case)
Given a tree ϑ_0, with a null graph T with k_T nodes containing a tree of the

[2] Two numbered trees have the same topological shape if, disregarding the number labels, they can be overlapped by pivoting the branches around the nodes into which they enter.

[3] The higher order of the zero in η is due to the fact that one can independently shift the m_T entering lines, while in Section 8.3 there was only one entering line.

topological shape of some ϑ, let $\nu_1, \ldots, \nu_{m_T}$ be the momenta flowing through the entering branches $\lambda_1, \ldots, \lambda_{m_T}$ of T. We consider the family $\mathcal{F}_T(\vartheta)$: we have the bound

e9.3.24
$$\frac{1}{\mathcal{F}_T(\vartheta)}\Big|\sum_{\vartheta'\in\mathcal{F}_T(\vartheta)} \mathcal{V}_T(\vartheta')\Big| \le c^{-k_T} q^{2k_T} D_0 k_T^2 \sum_{m,m'=1}^{m_T} \big|\nu_m \nu_{m'} \eta^2\big| , \qquad (9.3.24)$$

for some constant D_0.

Remark: The improved bound (9.3.24) is due to the above discussed remarkable cancellations which extend those considered in Section 8.3. Without exploiting such cancellations, the value of a null graph could only be bounded through (9.3.18). The cancellations produce the extra factors appearing in the sum: the factor k_T^2 is due to the fact that in order to exhibit the cancellation we use an interpolation formula, see (9.3.23), which involves a second order derivative with respect to the k_T momenta of the entering lines of the null graph. The latter is proportional to the product of k_T propagators.

Proof: The first two terms in (9.3.23) give a vanishing contribution to the sum over $\mathcal{F}_T(\vartheta)$, as discussed above; the integral appearing in the third term in (9.3.23) gives, for each $\vartheta' \in \mathcal{F}_T(\vartheta)$, a contribution bounded by $d_0 q^2$ times the original bound on $|\mathcal{V}_T(\vartheta)|$, for a suitable d_0. In fact the propagators of all branches inside V are bounded by $c^{-1}q^2$, and their first and second derivatives, respectively, by $d_1 q^3$ and $d_2 q^4$ for some constants d_1 and d_2 (such branches remain on scale $n = 1$: the incoming branches contribute a quantity that modifies only the imaginary part of the momenta of branches inside V). Then the lemma follows, with $D_0 = \max\{c d_1^2, d_2\} q^2$. ∎

However a typical tree may contain more than one null graph: hence we shall need the following easy corollary of the above lemma.

C9.3.1 **(9.3.1) Corollary:** (Cancellations: several null graphs case)
Given a tree ϑ_0 with null graphs $T_1, \ldots, T_s$, containing numbered trees with shapes $\vartheta_1, \ldots, \vartheta_s$; consider the families $\mathcal{F}_1 = \mathcal{F}_{T_1}(\vartheta_0)$, $\mathcal{F}_2 = \cup_{\vartheta'\in\mathcal{F}_1} \mathcal{F}_{T_2}(\vartheta')$ and so on recursively till $\mathcal{F}_s$; the number of trees in the family $\mathcal{F}$ is given by $k_{\mathcal{F}_s} = \prod_{i=1}^s |\mathcal{F}_{T_i}(\vartheta_i)|$, and

e9.3.25
$$\frac{1}{k_{\mathcal{F}_s}}\Big|\sum_{\vartheta'\in\mathcal{F}_s} \mathrm{Val}(\vartheta')\Big| \le c^{-k} q^{2N_1(\vartheta_0)} |\eta|^{-2N_0(\vartheta_0)} D_0^s k^2 |\eta|^{2s}, \qquad (9.3.25)$$

with a suitable D_0.

Proof: One remarks that the cancellation mechanisms operating for each null graph do not interfere with each other (*i.e.* there is no cancellations overlap because here we have only two scales), as noted (in a more general context) in Section §8.1. ∎

The above corollary implies the main result on the convergence radius of the Lindstedt series (9.3.7), which can be formulated as follows.

Let $N_0^R(\vartheta_0)$ be the number of null branches (necessarily on scale $n = 0$), so that $N_0(\vartheta_0) = N_0^*(\vartheta_0) + N_0^R(\vartheta_0)$. For the null graphs an overall gain $D_0^s k^2 |\eta|^{2s}$ is obtained, by (9.3.25). Then we have

e9.3.26
$$\begin{aligned}\frac{1}{k_{\mathcal{F}_s}}\Big|\sum_{\vartheta'\in\mathcal{F}_s}\mathrm{Val}(\vartheta')\Big| &\le c^{-k} q^{2N_1(\vartheta_0)} |\eta|^{-2N_0^*(\vartheta_0)} |\eta|^{-2N_0^R(\vartheta_0)} D_0^{N_0^R(\vartheta_0)} k^4 |\eta|^{2N_0^R(\vartheta_0)} \\ &\le \left(c^{-1} e^2 D_0 q^2\right)^k |\eta|^{-2k/q},\end{aligned} \tag{9.3.26}$$

as $k^2 \le e^{2k}$ and the number of null graphs is equal to the number of null branches, *i.e.* $s = N_0^R(\vartheta_0)$. Therefore, writing the sum over all trees as

e9.3.27
$$\sum_{\vartheta\in\Theta_{\nu,k}} \mathrm{Val}(\vartheta) = \sum_{\vartheta\in\Theta_{\nu,k}} \frac{1}{k_{\mathcal{F}_s}} \sum_{\vartheta'\in\mathcal{F}_s} \mathrm{Val}(\vartheta'), \tag{9.3.27}$$

we can conclude that one has, for α real,

e9.3.28
$$\left|u_\nu^{(k)}\right| \le D_1^k |\eta|^{-2k/q}, \qquad \Big|\sum_{\nu\in\mathbb{Z}} e^{i\nu\alpha} u_\nu^{(k)}\Big| \le D_1^k |\eta|^{-2k/q}, \tag{9.3.28}$$

for some constants D_1. The above analysis gives $D_1 = 4cD_0q^2$; this implies that $\rho(\eta) \ge r|\eta|^{2/q}$, for some explicitly computable constant r.

Since the momentum ν of a k–th order tree is $|\nu| \le k$ we see that analyticity in the strip $|\mathrm{Im}\,\alpha| < \delta_0$ has also been obtained in the domain $|\varepsilon| < \rho(\eta)e^{-\delta_0}$ because $|e^{i\alpha\nu}| \le e^{\delta_0 k}$. This proves item (i) of proposition (9.3.1). Item (ii) will be studied in Section §(9.4).

Problems for §9.3

Q9.3.1 **[9.3.1]:** (*Kicked rotator*)
Show that (9.3.1) can be seen as the map at time 1 of the continuous-time Hamiltonian system (known in Physics literature as the *kicked rotator*) described by the singular Hamiltonian

$$2\pi A + \frac{I^2}{2} + 2\pi\varepsilon \sum_{\nu\in\mathbb{Z}} \delta(\alpha - 2\pi n)\,(\cos\varphi - 1)\,,$$

where δ is the delta function. (*Hint:* Simply write down the Hamilton equations and integrate them between $t = 0$ and $t = 1$, then define $x = \varphi$ and $y = I - \varepsilon\sin\varphi/2$.)

Q9.3.2 **[9.3.2]:** (*Kicked rotator and KAM*)
Prove the existence of the analytical invariant curves (9.3.6) for the standard map, for ε small enough. (*Hint:* Simply adapt the proof of KAM theorem in Section §8.1 to the standard map by using the tree expansion (9.3.14)).

Q9.3.3 **[9.3.3]:** Prove the bound (9.3.15). (*Hint:* Use that $|\cos z - 1| \ge |z|^2/4$, and $|\cos z - 1| \ge |\mathrm{Im}\,z|^2/2$ for $|\mathrm{Re}\,z| \le \pi/4$, while $|\cos z - 1| \ge 1/2$ for $|\mathrm{Re}\,z \bmod 2\pi| \ge \pi/4$.)

Q9.3.4 **[9.3.4]:** Prove the bound (9.3.20). (*Hint:* The cases $q = 1$ and $q \ge 2$, $k \le q$ are trivial. So consider the case $k > q$. If the tree ϑ has the root branch either on scale $n = 1$ or on scale $n = 0$ and resonant, then the bound $N_0^*(\vartheta) \le k/q$ follows inductively. If the root branch is on scale $n = 0$ and non-resonant, then the branches entering v_0 cannot be all on scale $n = 0$, otherwise $\nu_{v_0} = 0$ (as $q > 1$), which is not allowed. Then at least one branch will have scale $n = 1$: let T be the cluster containing it. The cluster T will have

m_T entering branches, with $m_T \geq 0$, and it is not a null graph; then $\sum_{v \in V(T)} \nu_v \neq 0$, so there must be at least q nodes, hence $q-1$ branches, inside T. The subtrees entering into T will have, respectively, $k_1, \ldots, k_{m_T}$ nodes, with $q + \sum_{j=1}^{m_T} k_j \leq k$ so that, again inductively, one obtains $N_0^*(\vartheta) = 1 + \sum_{j=1}^{m_T} N_0^*(\vartheta_j) \leq 1 + (k-q)/q \leq k/q$. Then N_0^* will be bounded by k/q. As N_0^* has to be an integer number, the assertion follows.)

Bibliographical note to §9.3

The possible relations between radius of convergence of the conjugating function and some function depending only on the arithmetic properties of the rotation number was first considered by Yoccoz for Siegel's problem, [Yo95], who also introduced the Bryuno function $B(\omega)$ as the solution of a suitable functional equation. In particular he considered the problem of the optimal dependence of the radius of convergence on the Bryuno function. The analogous problem for the *semistandard map*, which is a simplified model sharing with the standard maps many interesting features, was considered by Davie, [Da94], who found that, by denoting with $\rho_0(\omega)$ the radius of convergence of the function which takes the role of the conjugating function for the standard map, a bound $|\log \rho_0(\omega) + \beta B(\omega)| < C$ holds with $\beta = 2$. A natural question is then whether a bound of the same kind can be obtained also for the standard map, *i.e.* if, by defining $Q_\beta(\omega) \equiv \log \rho(\omega) + \beta B(\omega)$, there exists a universal constant C such that $|Q_\beta(\omega)| < C$ for some value of β and for all irrational numbers ω such that $B(\omega) < \infty$ (the so called *Bryuno numbers*). Davie's result implies the upper bound $Q_2(\omega) < C$

The results stated in proposition (9.2.1) were first conjectured in [BM94], supported by numerical results, and a proof was given for the resonances $p/q = 0/1$ and $p/q = 1/2$. Validity of Bryuno's interpolation formula $|Q_2(\omega)| < C$ was recently proved in [BG01], where a bound $Q_2(\omega) > C$ was found via a refinement of the techniques discussed in the present section.

§9.4 Scaling laws for the standard map

In Section §9.3 we have proved that at least a rescaling $\varepsilon \to (2\pi\eta)^{2/q}\varepsilon$ is needed in order to obtain a well defined limit of the conjugating function as $\eta \to 0$. Without exploiting any cancellations a rescaling $\varepsilon \to (2\pi\eta)^2\varepsilon$ might seem to be necessary, but of course, by taking into account the cancellations *a posteriori*, one sees that such a rescaling would have produced a vanishing function. The analysis performed so far, however, does not exclude that further cancellations are still possible, so that it could happen that the limit of the rescaled conjugating function is still zero: in other words it could happen that we are still rescaling "too much". In this section we want to show that this is not the case, and, furthermore, we want to provide an explicit expression for the rescaled function.

Let us consider the coefficient $u_\nu(\varepsilon)$ in (9.3.7). By definition of momentum, only trees of order $k \ge |\nu|$ can contribute to $u_\nu(\varepsilon)$, so that $u_\nu^{(k)} = 0$ for $k < |\nu|$. Therefore we can write

e9.4.1 $$u_\nu(\varepsilon) = \sum_{k\ge|\nu|} \varepsilon^k u_\nu^{(k)} = \varepsilon^{|\nu|} u_\nu^{(|\nu|)} + \sum_{k=|\nu|+1}^{\infty} \varepsilon^k u_\nu^{(k)}, \tag{9.4.1}$$

and use the first bound in (9.3.28) for $u_\nu^{(k)}$, $k > |\nu|$, in order to bound the last sum in (9.4.1) by

e9.4.2 $$\Big| \sum_{k=|\nu|+1}^{\infty} \varepsilon^k u_\nu^{(k)} \Big| \le 2 \left(D_1 |\eta|^{-2/q} \varepsilon \right)^{|\nu|+1}, \tag{9.4.2}$$

provided that $|\varepsilon| < D_1^{-1}|\eta|^{2/q}/2$. The coefficient $u_\nu^{(|\nu|)}$ in (9.4.1) can be expressed in terms of trees having all the modes $\nu_v = \sigma$, where $\sigma = \operatorname{sign}\nu$. From the definition of tree value it is easy to see that such trees have the same product $\prod_{v\in V(\vartheta)} \nu_v^{m_v+1}$ appearing in $\mathrm{Val}(\vartheta)$, which is given by $\sigma^{2|\nu|+1} = \sigma$. Furthermore among such trees there will also be trees having $[|\nu|/q]$ propagators with momentum respectively q, $2q$, …, $[|\nu|/q]q$: for instance the linear tree (*i.e.* the tree whose nodes have all only one entering branch). Therefore there will be trees whose value will be bounded *from below* by $O\left(|\eta|^{-2[|\nu|/q]}\right)$. When the limit $\eta \to 0$ is taken, remarking that the quantities $\gamma(\nu)$ for $\eta = 0$ are not positive, cf. (9.3.9), each of such trees ϑ has a value which is:

e9.4.3 $$\begin{aligned} &\mathrm{Val}(\vartheta) = -i\sigma B(\vartheta) \eta^{-2[|\nu|/q]}, \\ &B(\vartheta) = (-1)^k \Big(\prod_{v\in V(\vartheta)} \frac{1}{m_v!} \Big) \Big(\prod_{\substack{\lambda\in\Lambda(\vartheta) \\ n_\lambda=0}} \frac{1}{(2\pi\nu_\lambda)^2} \Big) \Big(\prod_{\substack{\lambda\in\Lambda(\vartheta) \\ n_\lambda=1}} \frac{1}{|\gamma(\nu_\lambda)|} \Big), \end{aligned} \tag{9.4.3}$$

because the propagators $\gamma(\nu)^{-1}$ diverge as $(2\pi i\nu\eta)^{-2}$ for $\eta \to 0$.
This means that no cancellation is possible among them, so that

e9.4.4 $$u_\nu^{(|\nu|)} = A_\nu |\eta|^{-2[|\nu|/q]}, \qquad |A_\nu| > 0. \tag{9.4.4}$$

If we want that, in the limit $\eta \to 0$, the coefficient $u_\nu(\varepsilon)$ non only does not diverge but also does not vanish, we have to impose that

e9.4.5 $$0 < \lim_{\eta\to 0} \varepsilon^{|\nu|} u_\nu^{(|\nu|)} < \infty, \tag{9.4.5}$$

so that (9.4.4) implies that ε has to be taken of order $O(|\eta|^{2/q})$. In such a way all the limits (9.4.5) exist, for any ν, and they are vanishing but for $|\nu|$ multiple of q.

Moreover, when we compute $u_\nu(\varepsilon)$, with ν multiple of q, only the coefficients $u_\nu^{(k)}$ with k multiple of $|\nu|$ will contribute to the limit $\eta \to 0$, because all other contributions arise from trees containing null graphs, and the analysis above shows that the propagators corresponding to the resonant branches do not introduce new denominators small in η, while each new node contributes a factor $|\eta|^{2/q}$, by (9.4.4) and (9.4.5).

The function $U(\alpha,\varepsilon) = u(\alpha,(2\pi\eta)^{2/q}\varepsilon)$ admits a Taylor series in ε convergent for $|\varepsilon| < \varepsilon_0 = O(1)$ uniformly in η. The k-th order Fourier coefficients of U are defined by

e9.4.6
$$U_\nu^{(k)} = u_\nu^{(k)}(2\pi\eta)^{2k/q}, \tag{9.4.6}$$

so that, if we introduce the function

e9.4.7
$$\bar u(\alpha,\varepsilon) = \lim_{\eta\to 0} U(\alpha,\varepsilon), \tag{9.4.7}$$

we have that

e9.4.8
$$\begin{aligned} \bar u(\alpha,\varepsilon) &= \sum_{k=1}^{\infty}\sum_{\nu\in\mathbb{Z}} \varepsilon^k e^{i\nu\alpha}\bar u_\nu^{(k)}, \\ \bar u_\nu^{(k)} &= \lim_{\eta\to 0} u_\nu^{(k)}(2\pi\eta)^{2k/q}, \end{aligned} \tag{9.4.8}$$

and the first series in (9.4.8) converges absolutely by (9.3.28): this means that the coefficients $\bar u_\nu^{(k)}$ are well defined, and, from the analysis above, we know that only the Fourier coefficients with modes multiples of q survive when the limit $\eta \to 0$ is taken; furthermore, the function $\bar u(\alpha,\varepsilon)$ is analytic in ε for ε small enough and η independent, and periodic in α with period $2\pi/q$. Summarizing, we have

e9.4.9
$$\bar u_\nu^{(k)} = \begin{cases} -\dfrac{(2\pi)^{k/q}}{2^k} \displaystyle\sum_{\vartheta\in\Theta_{\nu,k}}{}' \mathrm{Val}'(\vartheta), & \text{if } \nu\in q\mathbb{Z}\setminus\{0\} \text{ and } k\in q\mathbb{Z}\setminus\{0\}\ , \\ 0, & \text{otherwise}\ , \end{cases} \tag{9.4.9}$$

where $\sum'$ means that only trees without null graphs have to be summed over and $\mathrm{Val}'(\vartheta)$ differs from $\mathrm{Val}(\vartheta)$ inasmuch as, for ν multiple of q, the denominator $\gamma(\nu)$ has to be replaced with $(2\pi i\nu)^2$.

Let us now consider all trees of order q contributing to $\nu = \sigma q$, with $\sigma = \operatorname{sign}\nu$: the values of such trees can be read from (9.3.14), and one sees that the numerator is identically $-i\sigma$, so that

e9.4.10
$$\sum_{\vartheta\in\Theta_{\sigma q,q}} \mathrm{Val}'(\vartheta) = (2\pi)^2 \sum_{\vartheta\in\Theta_{\sigma q,q}} (-i\sigma) \prod_{v\in V(\vartheta)} \frac{1}{m_v!}\frac{1}{\gamma(\nu_{\lambda_v})} = \frac{1}{(i\nu)^2}(-i\sigma)S_{p/q}, \tag{9.4.10}$$

thus defining the expression $S_{p/q}$ (which does not depends on σ): the factor $(i\nu)^{-2}$ arises from $(2\pi)^2$ times the propagator $\gamma(\nu)$, which appears in all tree values of trees in $\Theta_{\sigma q,q}$ so it can be factored out. For instance, for $p/q=0/1$, $p/q=1/2$ and $p/q=1/3$, by explicit computation, we find

e9.4.11
$$\begin{aligned} &S_{0/1}=1,\\ &S_{1/2}=\frac{1}{2(\cos\pi-1)}=-\frac{1}{4},\\ &S_{1/3}=\frac{1}{2(\cos(4\pi/3)-1)}\frac{1}{2(\cos(2\pi/3)-1)}+\frac{1}{2}\frac{1}{[2(\cos(4\pi/3)-1)]^2}=\frac{1}{6}, \end{aligned} \tag{9.4.11}$$

as one checks from the definition (9.4.10).

We have the following result.

L9.4.1 **(9.4.1) Lemma:** (Limit function near the resonances)
The coefficients $\bar u_\nu^{(k)}$ satisfy the recursion relation

e9.4.12
$$\bar u_\nu^{(k)}=\frac{1}{2^{q-1}}\frac{1}{(i\nu)^2}\sum_{m=1}^{\infty}\frac{1}{m!}\frac{S_{p/q}}{2}(-i\sigma)(i\sigma q)^m\sum_{\substack{\sigma q+\nu_1+\ldots+\nu_m=\nu\\ k_1+\ldots+k_m=k-q}}\prod_{i=1}^{m}\bar u_{\nu_i}^{(k_i)}, \tag{9.4.12}$$

where the constant $S_{p/q}$ is defined in (9.4.10).

Proof: A generic tree of order $k=\kappa q$, $\kappa\ge1$, and momentum $\nu=nq$, $n\ge1$, can be obtained starting from a tree ϑ_0 of order q by attaching to its nodes $m\ge0$ trees $\vartheta_1,\ldots,\vartheta_m$ of orders $k_1=\kappa_1 q,\ldots,k_m=\kappa_m q$, with $\kappa_1+\ldots+\kappa_m=\kappa-1$ and total momenta $\nu_1=n_1q,\ldots,\nu_m=n_mq$ with $n_1+\ldots+n_m=n-1$. Each tree can be attached to any node of ϑ_0 so that the combinatorial factor associated to any node $v\in V(\vartheta_0)$will be $m_v!^{-1}$, with $m_v=s_v+r_v$, if s_v is the number of branches connecting v to other nodes of ϑ_0 and r_v is the number of trees attached to v. Then by construction $\sum_{v\in V(\vartheta_0)}r_v=m$. Note that two trees in which one of the first s_v subtrees (with root branch belonging to ϑ_0) is permuted with one of the remaining r_v cannot be identical, as they have a different number of nodes: only the latter will have a number of nodes which is multiple of q.

If we sum together all trees which can be obtained from each other by choosing in a different way the s_v subtrees with root branch belonging to ϑ_0 and the remaining r_v subtrees, we have $\prod_{v\in V(\vartheta_0)}\frac{m_v!}{s_v!\,r_v!}$ terms, so that, by taking into account that (see (9.4.10))

e9.4.13
$$\mathrm{Val}'(\vartheta_0)=(2\pi)^2(-i\sigma)\prod_{v\in V(\vartheta_0)}\frac{1}{s_v!}\frac{1}{\gamma(\nu_{\lambda_v})}. \tag{9.4.13}$$

Furthermore, by shifting the subtrees attached to the nodes of ϑ_0, the momentum flowing through any branch of ϑ_0 can vary by an amount proportional to a multiple of q, so that the corresponding propagator does not change, we can write

*e*9.4.14
$$\mathrm{Val}'(\vartheta)=\sum_{\vartheta_0\in\Theta_{\sigma q,q}}\mathrm{Val}(\vartheta_0)\sum_{m=0}^{\infty}\sum_{\substack{\{r_v\ge 0\}\\ \sum_{v\in V(\vartheta_0)}r_v=m}} \prod_{v\in V(\vartheta_0)}\frac{1}{r_v!}(i\sigma)^m \qquad \sum_{\vartheta_1,\dots,\vartheta_m}{}' \prod_{i=1}^{m}\mathrm{Val}'(\vartheta_i), \tag{9.4.14}$$

where $'$ recalls the constraint on the trees $\vartheta_1,\ldots,\vartheta_p$ described above. Since

*e*9.4.15
$$\prod_{\substack{\{r_v\ge 0\}\\ \sum_{v\in V(\vartheta_0)}r_v=m}}\frac{1}{r_v!}=\frac{q^m}{m!}, \tag{9.4.15}$$

we deduce from (9.4.14) that

*e*9.4.16
$$\mathrm{Val}'(\vartheta)=\sum_{\vartheta_0\in\Theta_{\sigma q,q}}\mathrm{Val}'(\vartheta_0)\sum_{m=0}^{\infty}\frac{1}{m!}(i\sigma q)^m\sum_{\vartheta_1,\dots,\vartheta_m}{}'\prod_{i=1}^{m}(i\sigma q)\mathrm{Val}'(\vartheta_i). \tag{9.4.16}$$

Then from (9.4.10) and (9.4.16) we read that

*e*9.4.17
$$\bar u_\nu^{(k)}=\frac{1}{2^q}\sum_{\vartheta_0\in\Theta_{q,q}}\mathrm{Val}'(\vartheta_0)\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{\substack{k_1+\ldots+k_m=k-q\\ \sigma+\nu_1+\ldots+\nu_m=\nu}}\prod_{i=1}^{m}(i\sigma q)\bar u_{\nu_i}^{(k_i)}. \tag{9.4.17}$$

and using (9.4.10) we obtain (9.4.12). ∎

The above results yield that the function $\bar u(\alpha)\equiv\bar u(\alpha,\varepsilon)$ in (9.4.7) satisfies the differential equation

*e*9.4.18
$$\frac{\mathrm{d}^2\bar u(\alpha)}{\mathrm{d}\alpha^2}=C_{p/q}\varepsilon^q\sin(q(\alpha+\bar u(\alpha))), \tag{9.4.18}$$

with boundary conditions $\bar u(0)=\bar u(2\pi)=0$, and $C_{p/q}$ given by

*e*9.4.19
$$C_{p/q}=2^{-(q-1)}S_{p/q}=2^{-(q-1)}\sum_{\vartheta\in\Theta_{q,q}}(2\pi i\nu)^2\prod_{v\in V(\vartheta)}\frac{1}{m_v!}\frac{1}{\gamma(\nu_{\lambda_v})}, \tag{9.4.19}$$

where the factor $(2\pi i\nu)^2$ simply cancels the propagator of the root branch of ϑ. This is seen by a straightforward check: write (9.4.18) in Fourier space, and write the recursion relations defining the coefficients, by taking into account that only orders and momenta multiples of q can occur (this can be seen inductively), obtaining

*e*9.4.20
$$\bar u_\nu^{(k)}=\frac{1}{(i\nu)^2}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{C_{p/q}}{2}(-i\sigma)(i\sigma q)^m\sum_{\substack{\sigma q+\nu_1+\ldots+\nu_m=\nu\\ k_1+\ldots+k_m=k-q}}\prod_{i=1}^{m}\bar u_{\nu_i}^{(k_i)}, \tag{9.4.20}$$

with $\sigma = \pm 1$. This is the same expression as (9.4.12) provided the constant $\mathcal{C}_{p/q}$ is chosen as (9.4.19).

Now we come back to the proof of proposition (9.3.1), and, to conclude the proof, we collect the results obtained so far.

Since for $\eta \neq 0$ there are no small divisors and the convergence of the Lindstedt series can be proved by elementary means, the only non obvious part of item (i) is the behavior of the radius of convergence as $\eta \to 0$. This has been studied in detail at the end of Section 9.3, and the conclusion is that the radius of convergence is strictly positive; to exclude that the radius of convergence could be infinite (see also problem [9.4.2]) one can consider the limit function, whose existence is stated in item (ii), and use the properties of the elliptic functions to conclude that its radius of convergence is finite
N9.4.1 (see [BM94] for an explicit discussion).[1]

Remark: The restriction on the path over which we take the limit $\omega \to p/q$ in the complex plane (*i.e.* approaching the real axis perpendicularly) is taken only for the sake of simplicity: in fact, it is easy to modify the proofs in such a way that any path in the complex plane, *provided it is not tangent to the real axis*, can be taken. More precisely, let

e9.4.21
$$\omega = \frac{p}{q} + \zeta + i\eta, \tag{9.4.21}$$

with $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$, and $\zeta, \eta \in \mathbb{R}$, with

e9.4.22
$$|\eta| \geq a|\zeta|, \qquad a > 0, \tag{9.4.22}$$

in the limit $\eta \to 0$. Condition (9.4.22) defines a cone in the complex ω plane, with its vertex in p/q and its slope equal to a: any path inside this cone tends to p/q non-tangentially.

First we show that inequalities like (9.3.15) can be derived under the condition (9.4.22), with the only difference that now $c = 2\pi^2 a^{-2}$.

Since the analysis above was based on the inequalities (9.3.15) and on the definition of null graph, it can be repeated essentially unchanged in the case (9.4.21), and the same results hold. In fact, the proof of lemma (9.3.1) can be carried out in a similar manner, by expressing the null graph value as a function of the quantities $\xi\nu_{\lambda_1}, \ldots, \xi\nu_{\lambda_{m_T}}$, with $\xi = \zeta + i\eta$; then by taking into account that, for any ν such that $\nu = 0 \pmod q$, one has $|\gamma(\nu)| \geq 4\pi^2|\nu\eta|^2$, and $|\nu\zeta| \leq |\nu|\sqrt{\zeta^2+\eta^2} \leq |\nu\eta|(1+a^{-1})$, one sees that the cancellation mechanisms operate exactly in the same way as before, and the second order terms can be dealt with as before, with the only difference that now $D_0 = \max\{cd_1^2, d_2\}q^2a^{-2}(1+a^{-1})$.

Once the perturbation parameter ε has been scaled to $(2\pi\xi)^{2/q}\varepsilon$, the surviving terms are exactly the same as before, so that all of the above discussions

[1] Alternatively one can use an argument due to Davie [Da94]; see also [BG01] for the implementation of such an idea in a more general context.

apply *verbatim*; in fact

e9.4.23
$$\lim_{\eta\to 0}\gamma(\nu)[2\pi(\zeta+i\eta)]^{-2}=1, \tag{9.4.23}$$

for ν multiple of q and ζ, η satisfying (9.4.22).

Our main results therefore still apply provided the path taken by ω while tending to p/q is not tangential to the real axis, so that (9.4.22) applies for some a.

Problems for §9.4

Q9.4.1 **[9.4.1]:** Show that for rotation numbers ω satisfying (9.4.21) and (9.4.22), the bound (9.3.15) still holds, for a different constant c. (*Hint:* The first inequality holds as, for ν multiple of q, we can write $\cos(2\pi(p/q+\zeta+i\eta)\nu)=\cos(2\pi(\zeta+i\eta)\nu)$, so that $2|\cos(2\pi(p/q+\zeta+i\eta)\nu)-1|\ge 4\pi^2|\nu\eta|^2$. If $\nu\ne 0 \pmod q$, we can deduce that, by denoting $x=(p/q+\zeta)\nu$, one has $|\gamma(\nu)|\ge 1/2$ for $2\pi|x|\ge\pi/4$. Then, if $2\pi|x|\le\pi/4$ we have $|\gamma(\nu)|\ge 2\pi^2(|x|^2+|\eta\nu|^2)$, so that, if $|x|\le(2q)^{-1}$, one has $|\zeta\nu|\ge(2q)^{-1}$ as $|p\nu/q|\ge 1$, hence $|\nu|\ge(2q|\zeta|)^{-1}$, *i.e.* $|\eta\nu|\ge(2q)^{-1}|\eta/\zeta|$, which implies $|\gamma(\nu)|\ge\pi^2q^{-2}a^2/2$, while, if $|x|\ge(2q)^{-1}$, then $|\gamma(\nu)|\ge\pi^2q^{-2}/2$.)

Q9.4.2 **[9.4.2]:** Provide an example of a power series in ε depending on a parameter η such that, for $\eta\to 0$, there is only one possible rescaling of ε as a function of η making the radius of convergence different from zero such that the rescaled function is different from zero and its radius of convergence id infinite. (*Hint:* Consider $\sum_{k=1}(\varepsilon/\eta)^k/k!$.)