CHAPTER IV

Markovian pavements

§4.1 Histories compatibility. Markovian pavements

In the previous sections the problem of studying the statistics of motions of a dynamical system (Ω, S), as seen from a partition $\mathcal{P}$, has been shown to be equivalent to studying probability distributions on the space of the sequences of symbols associated with a partition $\mathcal{P}$ (cf. proposition (2.3.2)).

Upon further thought it is, however, clear that the analysis presented so far can be of little help in concrete problems. It is true that on $\{0, \ldots, n\}^{\mathbb{Z}}$ it is possible to study vast classes of ergodic distributions, and such a study can be also developed in a rather detailed and concrete way, but it is hard to give criteria that select those probability distributions (or measures) m that are relevant for the statistical study of motions of (Ω, S). Or, in other words, it is hard to give criteria that guarantee that $m(\widehat{\Omega}) = 1$, cf. proposition (2.3.2), and allow us to identify the possible symbolic motions, *i.e.* to recognize whether $\underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}}$ is the $(\mathcal{P}, S)$–history of some point $x \in \Omega$.

In general the values $\sigma_i(x)$, $i \in \mathbb{Z}$, are linked by very intricate relations and understanding them means a very detailed understanding of the motions structure. On the other hand it is necessarily so: indeed the action of S regarded as an action on the symbolic histories is trivial, being reduced to a mere shift (*i.e.* to a translation). Complexity of a given dynamical system must necessarily be hidden in the map, called *code* in Section §1.4, which associates with every $x \in \Omega$ its $(\mathcal{P}, S)$–history on $\mathcal{P}$; at least in the cases in which $\mathcal{P}$ is generating, *i.e.* it is fine enough to provide a faithful description of the motion.

The simplest compatibility condition between elements of $\underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}}$

is, perhaps, what can be called a *local condition of compatibility*.

D4.1.1 **(4.1.1) Definition:** (Compatibility matrix)
A $(n+1)\times(n+1)$ matrix T with entries $T_{\sigma\sigma'}$ equal to 0 or to 1 will be called a compatibility matrix*. Such a matrix will be called* transitive *if for every pair σ,σ' there is a suitable integer $a_{\sigma\sigma'}$ such that $T^{1+a_{\sigma\sigma'}}_{\sigma\sigma'}>0$. It will be called* mixing *if there is $a\ge 0$ such that $T^{1+a}_{\sigma\sigma'}>0$ for all σ,σ' and a is called the* mixing time *of T.*
*A sequence $\underline{\sigma}\in\{0,\ldots,n\}^{\mathbb{Z}}$ will be called T–*compatible *or* admissible *if and only if every pair of adjacent symbols that appear in $\underline{\sigma}$ is admissible, i.e. $T_{\sigma_i\sigma_{i+1}}=1\,\forall i\in\mathbb{Z}$ or, equivalently, if and only if*

e4.1.1
$$\prod_{i=-\infty}^{+\infty} T_{\sigma_i\sigma_{i+1}}=1. \tag{4.1.1}$$

We shall call $\{0,\ldots,n\}^{\mathbb{Z}}_T\subset\{0,\ldots,n\}^{\mathbb{Z}}$ the (closed and translation invariant) subset of the sequences $\underline{\sigma}$ that verify (4.1.1).
The dynamical system $(\{0,\ldots,n\}^{\mathbb{Z}},\tau)$, with $(\tau\underline{\sigma})_i=\sigma_{i+1}$, is often called a translation *or* shift*, and the dynamical system $(\{0,\ldots,n\}^{\mathbb{Z}}_T,\tau)$ is called a* subshift of finite type.

One could naively think that by suitably selecting $\mathcal{P}$ it should be possible to obtain that the (Ω,S)–histories are all and only those sequences of $\{0,\ldots,n\}^{\mathbb{Z}}$ which verify (4.1.1) for some suitable T. It is however clear that the "totally disconnected" topological structure of $\{0,\ldots,n\}^{\mathbb{Z}}$ can be topologically incompatible with the structure of Ω that, very often, is a Riemannian manifold. In such cases a code that transforms the dynamics into a symbolic dynamics that is a subshift of finite type, even if existent, could not fail to show some pathology (like points not well coded, discontinuities, *etc.*), which in turn we must necessarily expect to produce, sooner or later, difficulties in the theory.
Nevertheless the simplicity of the condition (4.1.1) is so captivating that it is worth looking after systems that admit such a simple symbolic representations.

We shall therefore analyze topological dynamical systems (Ω,S) and we shall try to isolate some further conditions on (Ω,S) that will allow us to describe through simple symbolic dynamics large classes of probability distributions (or measures) and, more precisely, classes of probability distributions like the following ones.

D4.1.2 **(4.1.2) Definition:** (Topological probability distributions, topological pavements)
Given a compact topological space Ω we call topological probability distributions *the probability distributions $\mu\in\mathcal{M}^0(\Omega)$ defined on a σ–algebra of sets containing the Borel sets of Ω which satisfy the condition*

e4.1.2
$$\mu(G)>0 \qquad \text{for all open sets } G. \tag{4.1.2}$$

If (Ω, S) *is a topological dynamical system we denote by* $\mathcal{M}_t(\Omega, S)$ *the topological probability distributions that are* S*–invariant.*
We say that, $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ *is a pavement of* Ω *if it is a covering of* Ω *by closed sets, which are the closures of their internal points and are such that* $Q_i \cap Q_j = \partial Q_i \cap \partial Q_j$ *for all* $i \neq j$.

The key notion that, as we shall see, allows us to use effectively symbolic dynamics for the dynamical systems for which it has a meaning, is that of *Markovian pavement* (also called *Markovian partition*; see remark (4) after the following definition).

D4.1.3 **(4.1.3) Definition:** (Markovian pavements)
Let (Ω, S) *be an invertible topological dynamical system. Given a pavement* $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ *of* Ω *set* $T_{\sigma\sigma'} = 1$ *if* $\mathrm{int}(Q_\sigma) \cap \mathrm{int}(S^{-1}Q_{\sigma'}) \neq \emptyset$ *and* $T_{\sigma\sigma'} = 0$ *otherwise,* $\sigma, \sigma' \in \{1, \ldots, q\}$, $q \geq 2$. *We shall say that* $\mathcal{Q}$ *is* Markovian *if the following holds.*
(i) The set

e4.1.3
$$\mathcal{X}(\underline{\sigma}) = \bigcap_{k=-\infty}^{+\infty} S^{-k} Q_{\sigma_k} \tag{4.1.3}$$

is not empty and consists of a single point $X(\underline{\sigma})$ *for all* $\underline{\sigma}$ *such that*

e4.1.4
$$\prod_{i=-\infty}^{+\infty} T_{\sigma_i \sigma_{i+1}} = 1. \tag{4.1.4}$$

Furthermore, for such $\underline{\sigma}$ *the sets* $\cap_{-N}^{N} S^{-k} Q_{\sigma_k}$ *contain internal points for all* N. *We shall call the matrix* T *the* compatibility matrix for $\mathcal{Q}$, *and the space* $\underline{\sigma} \in \{1, \ldots, q\}_T^{\mathbb{Z}}$ *of the sequences* $\underline{\sigma}$ *satisfying (4.1.4) will be called the space of the* T–compatible *sequences, or simply* compatible *sequences if no confusion is possible.*
(ii) The correspondence $\underline{\sigma} \to X(\underline{\sigma})$ *between* $\{1, \ldots, q\}_T^{\mathbb{Z}}$ *and* Ω *is Hölder continuous, i.e. there exist* $C, a > 0$ *such that*

e4.1.5
$$d(X(\underline{\sigma}), X(\underline{\sigma}')) \leq C\, d(\underline{\sigma}, \underline{\sigma}')^a, \tag{4.1.5}$$

where we define the distance between $\underline{\sigma}$ *and* $\underline{\sigma}'$ *on* $\{1, \ldots, q\}^{\mathbb{Z}}$ *by*

e4.1.6
$$d(\underline{\sigma}, \underline{\sigma}') = \exp\big(-\nu(\underline{\sigma}, \underline{\sigma}')\big), \tag{4.1.6}$$

where $\nu(\underline{\sigma}, \underline{\sigma}')$ *is the largest integer* j *such that* $\sigma_i = \sigma'_i$ *for* $|i| \leq j$.
(iii) There exists an upper bound $M < \infty$ *on the number of compatible sequences mapped into a given point* x*, i.e. the inverse map* X^{-1} *verifies* $\big| X^{-1}(x) \big| \leq M$, *for all* $x \in \Omega$. *The number* M *will be called* multiplicity of the code.
(iv) Setting $\partial_i = \partial Q_i$, $i = 1, \ldots, q$, *and* $\partial = \cup_{i=1}^{q} \partial_i$ *there exist two closed sets* ∂^+ *and* ∂^- *such that*

e4.1.7
$$\partial = \partial^+ \cup \partial^-, \qquad S\partial^- \subset \partial^-, \qquad S^{-1}\partial^+ \subset \partial^+, \tag{4.1.7}$$

i.e. the boundary ∂ can be decomposed into two parts, the first of which (denoted ∂^-) "contracts" under the action of S while the other (denoted ∂^+) "expands".

Remarks: (1) The continuity of $\underline{\sigma} \to X(\underline{\sigma})$ insures that X is a Borel map (*i.e.* the inverse images of the Borel sets are Borel sets).
(2) One has $X(\tau\underline{\sigma}) = SX(\underline{\sigma})$ so that S is coded into the symbolic dynamics.
(3) If $x \in \Omega \setminus \cup_{i\in\mathbb{Z}} S^{-i}\partial$ the $(\mathcal{Q}, S)$–history of x is naturally and unambiguously defined and the correspondence between $\Omega \setminus \cup_i S^{-i}\partial$ and $X^{-1}(\Omega \setminus \cup_i S^{-i}\partial)$ is one-to-one and maps, together with its inverse, Borel sets into Borel sets (by Kuratowsky's theorem, cf. footnote 1, Section §2.3).
(4) Although $\mathcal{Q}$ is not a partition of Ω it is convenient to adopt conventions and notations similar to those used for partitions. We shall denote

*e*4.1.8
$$Q^J_{\underline{\sigma}} = \bigcap_{j\in J} S^{-j} Q_{\sigma_i}, \qquad J \subset \mathbb{Z}, \quad \underline{\sigma} \in \{1, \ldots, q\}^J, \tag{4.1.8}$$

and a cylinder $C^J_{\underline{\sigma}}$ will be called T–compatible if $C^J_{\underline{\sigma}} \cap \{1, \ldots, q\}^{\mathbb{Z}}_T \neq \emptyset$. Note, however, that while the T–compatibility of $C^J_{\underline{\sigma}}$ implies $Q^J_{\underline{\sigma}} \neq \emptyset$ the *vice versa* in general is not true, because a point might belong to the boundary of several Q's.
(5) In analogy to what happens in the case of partitions, sometimes it can be convenient to consider also *non-generating Markovian pavements*: they are defined exactly as in definition (4.1.3), by eliminating only the condition that the set (4.1.3) consists of a single point. Then, if we want to stress the difference with respect to the ones defined in definition (4.1.3), we can refer to the latter as *generating Markovian pavements* (see problems [4.3.6] and [4.3.7] for some examples).
(6) The set $\Omega \setminus \cup_i S^{-i}\partial = \cap_i(\Omega \setminus S^{-i}\partial)$ is an intersection of a countable
*N*4.1.1 family of dense open sets, therefore it is not empty and, in fact, dense.[1] Therefore T "cannot have too many zeroes"; see problem [4.1.3].
(7) It is easy to realize that every point of Ω is the image of some sequence $\underline{\sigma} \in \{1, \ldots, q\}^{\mathbb{Z}}_T$. If $x \in \Omega \setminus \cup_i S^{-i}\partial$ this is obvious. If $x \in \cup_i S^{-i}\partial$ there exists σ such that $x \in Q_\sigma$; then we set $\sigma_0 = \sigma$ and there must exist σ' such that int $(Q_{\sigma_0}) \cap$ int$(S^{-1}Q_{\sigma'}) \neq \emptyset$, and $Sx \in Q_{\sigma'}$ (because $Q_\sigma = \overline{\text{int}\,(Q_\sigma)}$ and this property is also true for $S^{\pm1}Q_\sigma$, since S is a homeomorphism). We shall set then $\sigma_1 = \sigma'$ and, by construction, $T_{\sigma_0\sigma_1} = 1$, *etc.*: in this way one constructs a sequence $\underline{\sigma} \in \{1, \ldots, q\}^{\mathbb{Z}}_T$ such that $x \in S^{-k}Q_{\sigma_k}$ for all $k \in \mathbb{Z}$. Therefore $x = X(\underline{\sigma})$.
(8) If $\mu \in \mathcal{M}^0(\Omega)$ is a Borel probability measure on Ω such that $\mu(S^k\partial) = 0$, for all $k \in \mathbb{Z}$, it is clear that the partition of $\Omega\setminus\cup_i S^{-i}\partial$ generated by $\mathcal{Q}$ is S–separating and in fact S–expansive, cf. (4.1.7), on $\mathcal{Q} \cap (\Omega \setminus \cup_i S^{-i}\partial)$. Hence $\mathcal{Q}$ restricted to $\Omega \setminus \cup_i S^{-i}\partial$ is a generating partition for every topological measure which is invariant and such that $\mu(\partial) = 0$.

[1] This is a consequence of a general Baire's theorem, see [DS58], p. 20.

The following proposition is, essentially, a tautology because definition (4.1.3) originated precisely by an effort to collect hypotheses sufficient to make true the properties that it states.

P4.1.1 **(4.1.1) Proposition:** (Codings of topological dynamical systems into symbolic ones via a Markovian pavement)
Let (Ω, S) be an invertible topological dynamical system which admits a Markovian pavement $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ with a compatibility matrix T.
(i) If the distribution m is in $M_0(\{1, \ldots, q\}_T^{\mathbb{Z}})$ (i.e. $m \in M_0(\{1, \ldots, q\}^{\mathbb{Z}})$ and $m(\{1, \ldots, q\}_T^{\mathbb{Z}}) = 1$) the relation

e4.1.9
$$Am(E) = m(X^{-1}E) \qquad \text{for all } E \in \mathcal{B}(\Omega) \tag{4.1.9}$$

defines a probability measure $Am \in \mathcal{M}^0(\Omega)$. The map A transforms τ–invariant measures in S–invariant measures, τ–ergodic measures in S–ergodic measures, etc. Since Ω is compact A is continuous.
(ii) If $m \in M_e(\{1, \ldots, q\}_T^{\mathbb{Z}})$ is a topological measure, then $Am \in \mathcal{M}_e(\Omega, S)$ and Am is a topological measure.
(iii) The correspondence A between ergodic topological measures on the space of compatible sequences $\{1, \ldots, q\}_T^{\mathbb{Z}}$ and on Ω is, imagining the measures to
N4.1.2 *be completed,*[2] *a correspondence between measures isomorphic mod 0. The dynamical systems $(\{1, \ldots, q\}_T^{\mathbb{Z}}, \tau, m)$ and (Ω, S, Am) are, for such measures m, isomorphic mod 0.*

Remarks: (1) Because of property (iii) it is possible to "reduce" the analysis of the S–invariant ergodic topological measures on Ω to the analysis of the analogous τ–ergodic topological measures on $\{1, \ldots, q\}_T^{\mathbb{Z}}$. Since the latter, as we shall see, can sometimes be studied in detail, this possibility is of great interest. For instance if T is mixing, *i.e.* there exists $N > 0$ such that $(T^N)_{\sigma\sigma'} > 0$ for all σ, σ', cf. definition (4.1.1), it is easy to see that there exist S–ergodic topological measures and this is, by itself, already a nontrivial fact: we shall come back to this point in more detail in the forthcoming sections.
(2) The validity of proposition (4.1.1) rests mainly on the remark that, if $\mu \in \mathcal{M}_e(\Omega, S)$ is a topological ergodic measure, then $\mu(\partial^-) = 0$. In fact
N4.1.3 $\partial^- \supset S\partial^-$ and, hence, $\mu(\partial^-)$ is 0 or 1 by ergodicity:[3] however $\Omega \setminus \partial^-$ is open and hence $\mu(\Omega \setminus \partial^-) > 0$ and $\mu(\partial^-) = 0$. Likewise the ergodicity of μ with respect to S^{-1} implies $\mu(\partial^+) = 0$, hence $\mu(\partial) = 0$; by invariance it follows then that $\mu(\cup_i S^{-i}\partial) = 0$.

Proof: Continuity of the code X implies that the inverse image of a Borel set $E \in \mathcal{B}(\Omega)$ is again a Borel set, so that Am is well defined and one has only to check that it is a measure: this is a general property because the

[2] *i.e.* we imagine extending the σ–algebra of the measurable sets by adding the sets which are not Borel sets but which are contained in zero measure Borel sets.

[3] Since $SE \subset E$ and $\mu(SE) = \mu(E)$ the sets E and SE differ mod 0, *i.e.* E is invariant mod 0 and therefore $\mu(E) = 0, 1$.

X^{-1}–images of the elements of a countable family of pairwise disjoint sets is a countable family of pairwise disjoint sets.

To prove (ii) note that if m is ergodic then also Am is ergodic. In fact if E is an invariant set for S then $X^{-1}E = X^{-1}SE = \tau X^{-1}E$, so if E has Am–measure different from 0 or 1 also $X^{-1}E$ has the same m–measure. By the remark (2) one has $Am(\cup_i S^i\partial) = 0$; this means that $m(X^{-1}(\cup_i S^i\partial)) = 0$ and, hence, X (which is a one-to-one correspondence between $\Omega\setminus\cup_i S^i\partial$ and $\{1,\ldots,q\}_T^{\mathbb{Z}}\setminus X^{-1}(\cup_i S^i\partial)$) is an isomorphism mod 0 between $A\,m$ and m.

Furthermore if G is open in Ω and $x\in G\setminus\cup_i S^{-i}\partial$ let $\underline{\sigma}(x)$ be a compatible sequence coded into x. There must exist N such that $\cap_{-N}^N S^{-i}Q_{\sigma_i(x)}\subset G$; indeed by (4.1.5) the diameter of this set is infinitesimal as $N\to\infty$. But

$$X^{-1}\big(\cap_{-N}^N S^{-i}Q_{\sigma_i(x)}\big)\supset C^{-N\ \ \ldots\ \ N}_{\sigma_{-N}(x)\ldots\sigma_N(x)},$$

and the measure of the latter cylinder is positive because it is T–compatible and m is a topological measure: hence $Am(G) > 0$.

To prove (iii) note that from the ergodicity of $\mu\in\mathcal{M}_e(\Omega,S)$ and from the preceding remark (2) it follows that $\mu(\cup_i S^{-i}\partial) = 0$; then the measure m on $\{1,\ldots,q\}_T^{\mathbb{Z}}$ defined by

e4.1.10 $$m(E) = \mu(X(E\setminus X^{-1}(\cup_i S^{-i}\partial)) \qquad \text{for all } E\in\mathcal{B}(\{1,\ldots,q\}_T^{\mathbb{Z}}) \tag{4.1.10}$$

is isomorphic mod 0 to μ and $Am=\mu$.

If the cylinder $C^{-N\ldots N}_{\sigma_{-N}\ldots\sigma_N}$ is T–compatible we have that

e4.1.11 $$\begin{aligned}&X(C^{-N\ldots N}_{\sigma_{-N}\ldots\sigma_N}\setminus X^{-1}(\cup_i S^{-i}\partial)) =\\ &= \cap_{-N}^N S^{-i}Q_{\sigma_i}\setminus\cup_i S^{-i}\partial = \cap_{-N}^N S^{-i}Q_{\sigma_i} \qquad \text{mod } 0,\end{aligned} \tag{4.1.11}$$

and, therefore, $m(C^{-N\ldots N}_{\sigma_N\ldots\sigma_N}) > 0$ because μ is a topological measure and, by (i), int $\big(\cap_{-N}^N S^{-i}Q_{\sigma_i}\big)\ne\emptyset$. Combining this with property (ii) property (iii) follows. ■

Remarks: (1) In fact the above arguments also prove that A establishes a one-to-one correspondence between measures μ on Ω such that $\mu(\cup_i S^i\partial) = 0$ and measures m on $\{1,\ldots,q\}_T^{\mathbb{Z}}$ such that $m(X^{-1}(\cup_i S^{-i}\partial)) = 0$; furthermore corresponding measures are isomorphic mod 0.

(2) The preceding proposition would remain valid if the condition (4.1.5), in the definition of Markovian pavement, was modified into "$d(X(\underline{\sigma}), X(\underline{\sigma}'))\to 0$ if $d(\underline{\sigma},\sigma')\to 0$". We chose to state the result in a less general form because, as we shall see, Hölder continuity is a very natural regularity property of codes X and it will play an essential role in various applications.

Problems for §**4.1** (*On Perron–Frobenius' theorem and on the structure of Markovian chains)*

Q4.1.1 **[4.1.1]:** Consider the dynamical system $(\{0,\ldots,n\}^{\mathbb{Z}},\tau)$ and show that the pavement $\mathcal{Q} = \{Q_0,\ldots,Q_n\}$ of $\{0,\ldots,n\}^{\mathbb{Z}}$ with $Q_\sigma = \{\underline{\sigma}'\,|\,\underline{\sigma}'_0 = \sigma\}$ is Markovian. (*Hint:* In this case ∂ is empty.)

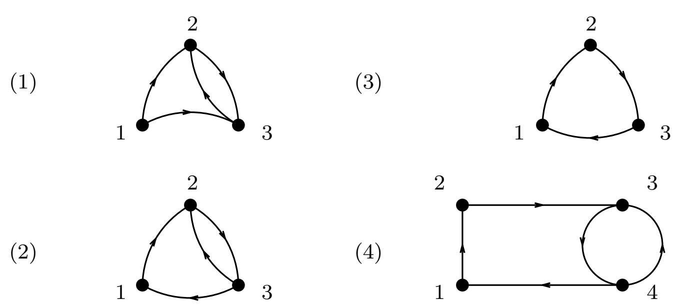
Q4.1.2 **[4.1.2]:** Find examples of dynamical systems with Markovian pavements and matrices of compatibility with some zero entry. (*Hint:* The space of sequences of $0, 1$'s with $T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.)

Q4.1.3 **[4.1.3]:** Check that the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ cannot be a compatibility matrix for a Markovian pavement. (*Hint:* No point would have a compatible symbolic history).

Q4.1.4 **[4.1.4]:** (*Compatibility graphs*)
Let $\sigma, \sigma' = 0, 1, \ldots, n$ and let $T_{\sigma\sigma'} \ge 0$ be a matrix. Let G_T be the graph obtained by connecting all pairs σ, σ' verifying $T_{\sigma,\sigma'} > 0$ by an arrow pointing from σ to σ': we say that σ' follows σ. A symbol σ can follow itself (*i.e.* $T_{\sigma\sigma} > 0$) or it can follow and be followed by a symbol σ' (*i.e.* $T_{\sigma\sigma'} > 0$ and $T_{\sigma'\sigma} > 0$). Two labels σ, σ' will be called *equivalent* if there is a closed loop of coherently oriented arrows in G_T that start at σ and return, proceeding always in the direction of the arrows, to σ after passing through σ'. Show that the set of labels can be divided into the set $I_0 = \mathcal{I}$ of labels that are inequivalent to any other label, that we call *inessential labels*, and into sets $I_1, \ldots, I_a$ (called "classes") such that each I_j contains labels equivalent to any of the other labels in the same I_j but inequivalent to any label in I_k if $k \ne j$. (*Hint:* Try to draw some special cases first, like the ones corresponding to the matrices

$$T_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \; T_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; T_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

and illustrated in Fig. (4.1.1).)



F4.1.1 **Fig.(4.1.1)** The graphs G_T corresponding to the three matrices $T = T_1, T_2, T_3, T_4$ of problem [4.1.4].

Q4.1.5 **[4.1.5]:** In the context of problem [4.1.4] call a semi-infinite sequence $\underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}_+}$ *compatible* if $T_{\sigma_i\sigma_{i+1}} > 0$ for all $i \ge 0$. Likewise call an infinite sequence $\underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}}$ *compatible* if $T_{\sigma_i\sigma_{i+1}} > 0$ for all $i \in \mathbb{Z}$. The spaces of compatible sequences will be denoted $\{0, \ldots, n\}_T^{\mathbb{Z}_+}$ or $\{0, \ldots, n\}_T^{\mathbb{Z}}$: they are *subshifts of finite type* in the sense introduced in this section (see definition (4.1.1). Show that any semi-infinite or infinite compatible sequence contains at most a finite number of inessential labels. Show also that a semi-infinite compatible sequence consists eventually only of labels $\sigma_i \in I_{k_+}$ for some k_+. Likewise an infinite compatible sequence consists, to the right of 0, eventually of labels in some I_{k_+} and, to the left of 0, eventually of labels in some I_{k_-}, for some k_+, k_-. However if $k_+ = k_- = k$ then $\sigma_i \in I_k$ for all i. Furthermore no infinite compatible sequence with $k_- = k_+$ contains inessential labels.

Q4.1.6 **[4.1.6]:** In the context of problems [4.1.4], [4.1.5] let $\{0, \ldots, n\}_T^{\mathbb{Z}}$ be the space of the compatible sequences. Show that if E is a translation invariant Borel set in $\{0, \ldots, n\}_T^{\mathbb{Z}}$ and μ is an ergodic measure such that $\mu(E) = 1$ then μ–almost all elements $\underline{\sigma} \in E$ have symbols σ_i in a *single* I_j. (*Hint:* By problem [4.1.5] all compatible sequences $\underline{\sigma}$ are such that the frequency $\ell_j^+(\underline{\sigma})$ of appearance of symbols in I_j in the "future part $\sigma_0, \sigma_1, \ldots$ of $\underline{\sigma}$" is well defined and it is either 0 or 1. By ergodicity almost all sequences must have the same value of $\ell_j^+(\underline{\sigma})$, so that there will be a single j_0 such that $\ell_{j_0}^+(\underline{\sigma}) = 1$ while for all $j \neq j_0$ one will have $\ell_j^+(\underline{\sigma}) = 0$. But the frequency in the future is equal to the frequency in the past (*i.e.* in the "past part $\ldots, \sigma_{-1}, \sigma_0$ of $\underline{\sigma}$", see problem [2.2.46]), therefore one has $\ell_j^-(\underline{\sigma}) \equiv \ell_j^+(\underline{\sigma})$ μ–almost everywhere: hence the "typical" sequences will have symbols that eventually in the past and the future all lie in I_{j_0}: in the language of problem [4.1.5] one has $k_- = k_+ = j_0$, hence $\sigma_i \in I_{j_0}$ for all i.)

Q4.1.7 **[4.1.7]:** (*Classes and periods of a compatibility matrix*)
If $\sigma \in I_j$ set $d(\sigma)$ = greatest common divisor of the integers $n_\sigma > 0$ such that $(T^{n_\sigma})_{\sigma\sigma} > 0$. Show that $d(\sigma)$ is constant for $\sigma \in I_j$. This allows us to define d_j for the class I_j as $d_j = \{d(\sigma) : \sigma \in I_j\}$; we shall say, for reasons that will be clarified later, that d_j is the *period* of the class I_j. (*Hint:* If s, m, n are such that $T^s_{\sigma\sigma} > 0$, $T^m_{\sigma\sigma'} > 0$, $T^n_{\sigma'\sigma} > 0$, one has $T^{n+m+ks}_{\sigma'\sigma'} \geq T^m_{\sigma'\sigma} T^{ks}_{\sigma\sigma} T^n_{\sigma\sigma'} > 0$, so that $d(\sigma')$ divides $n + m + ks$ for all k, hence it divides s. This implies $d(\sigma') \leq d(\sigma)$; by changing the roles of σ and σ' one finds $d(\sigma) = d(\sigma')$.)

Q4.1.8 **[4.1.8]:** (*Transitivity and mixing of compatibility matrices*)
In the context of the previous problems suppose that the matrix T is such that all labels σ are equivalent (*i.e.* there are no inessential labels and all the labels form a single class I_1). Show that if the period is $d = 1$ then there is p such that $T^p_{\sigma\sigma'} > 0$ for all σ, σ'. If the period is $d \geq 2$ then the set I_1 can be divided into d disjoint subsets $I_{1,1}, \ldots, I_{1,d}$, with $I_{1,d+1} \equiv I_{1,1}$, such that $T_{\sigma\sigma'} > 0$ only if $\sigma \in I_{1,i}$ and $\sigma' \in I_{1,i+1}$ for some i. Furthermore if $p > 0$ is large enough $T^{pd}_{\sigma\sigma'} > 0$ for all $\sigma, \sigma' \in I_{1,i}$ for some i and 0 otherwise, *i.e.* the block of the matrix T^d corresponding to the labels $\sigma, \sigma' \in I_{1,i}$ is *mixing*. (*Hint:* Let $n_{\sigma\sigma'} > 0$ be such that $T^{n_{\sigma\sigma'}}_{\sigma\sigma'} > 0$. Suppose first that there is an element σ_0 such that $T_{\sigma_0\sigma_0} > 0$, hence $n_{\sigma_0\sigma_0} = 1$ and $d_1 = 1$. Then we take $p = \sum_{\sigma\sigma'} n_{\sigma\sigma'}$, $\overline{n} = p - n_{\sigma\sigma_0} - n_{\sigma_0\sigma'}$ and we see that $T^p_{\sigma\sigma'} \geq T^{n_{\sigma\sigma_0}}_{\sigma\sigma_0} T^{\overline{n}}_{\sigma_0\sigma_0} T^{n_{\sigma_0\sigma'}}_{\sigma_0\sigma'} > 0$. Consider next the case in which $d = 1$ and $\sigma = 0, 1$, and let n_{00}, n_{11} be relatively prime integers such that $T^{n_{00}}_{00} > 0, T^{n_{11}}_{11} > 0$. Then any integer k large enough can be *simultaneously* written in the following forms

$$\begin{aligned}
k &= mn_{00} + n_{01} + m'n_{11},\\
k &= \widetilde{m}'n_{11} + n_{10} + \widetilde{m}n_{00},\\
k &= \widehat{m}n_{00} + n_{01} + \widehat{m}'n_{11} + n_{10},\\
k &= \widehat{m}'n_{11} + n_{10} + \widehat{m}n_{00} + n_{01},
\end{aligned}$$

for suitably chosen integers $m, m', \widetilde{m}, \ldots$ because if k is large also $k - \sum n_{\sigma\sigma'}$ is large. Therefore we can write $T^k = (T^{n_{00}})^m T^{n_{01}} (T^{n_{11}})^{m'}$, using the first expression and, in this way, we realize that $T^k_{01} > 0$, *etc.* This clearly implies that $T^k_{\sigma\sigma'} > 0$ for all pairs σ, σ' and for all k large enough. Analogously one discusses the case $d = 1$ with $n > 1$. The general case, $d \geq 2$ and $n \geq 1$, is similar.)

Q4.1.9 **[4.1.9]:** Particularize the results of problem [4.1.7] to the case of the matrices of problem [4.1.4].

Q4.1.10 **[4.1.10]:** (*Iterates of a compatibility matrix*)
Check that the result of the previous problems means that the space $\widetilde{\Omega} = \{0, \ldots, n\}^{\mathbb{Z}}_T$ of compatible infinite sequences can be divided into d disjoint spaces $\Omega_1, \Omega_2, \ldots, \Omega_d$, and the shift maps Ω_j onto Ω_{j+1}, with $\Omega_{d+1} \equiv \Omega_1$. (*Hint:* Note that, by the previous problems, d will turn out to be the period of the class I_1.)

Q4.1.11 **[4.1.11]:** (*Spectral decomposition of subshifts*)
Under the hypotheses of the previous problems and assuming that all labels are essential and belong to the same class I_1 show that the space $\widetilde{\Omega} = \{0, \ldots, n\}^{\mathbb{Z}}_T$ can be split as a union of d disjoint closed sets $\Omega_1, \ldots, \Omega_d$, and the shift τ maps Ω_i into Ω_{i+1}, with

$\Omega_{d+1} \equiv \Omega_1$, so that τ^d maps Ω_i into itself for each i, and given any pair F, G of relatively open sets in Ω_i there is a large enough $q > p$ such that $\tau^{q'\,d} F \cap G \neq \emptyset$ for all $q' > q$. A matrix T with only essential labels and only one class of them is called *transitive matrix*. (*Hint:* One has to note, see problem [4.1.5], that a sequence is in Ω_i if and only if $\sigma_0 \in I_{1,i}$; furthermore any open set can be obtained as a union of cylinders with a long enough *finite* base. The fact that $T^{dp}_{\sigma\sigma'} > 0$ if $\sigma, \sigma' \in I_{1,i}$ means that fixed i we can obtain compatible sequences which have at sites multiple of dp arbitrary labels in Ω_i.)

Q4.1.12 **[4.1.12]:** (*Perron-Frobenius theorem for transitive matrices*)
Under the hypotheses of problems [4.1.11] show that there exist d eigenvectors of the matrix T^d, to be denoted $e^{(0)}, e^{(1)}, \ldots, e^{(d-1)}$, with zero components except those in correspondence of the labels of $I_{1,1}, \ldots, I_{1,d}$, respectively, relative to the matrix T^d. The components of $e^{(i)}$ with labels in $I_{1,i}$ are strictly positive. (*Hint:* Use that T^d is a block matrix which acts in a mixing way in every block, cf. problem [4.1.8], and apply Perron–Frobenius' theorem in its elementary form discussed in problems [2.3.7]÷[2.3.12]).

Q4.1.13 **[4.1.13]:** In the context of problem [4.1.12] show that $Te^{(i)} = \lambda e^{(i+1)}$, for all $i = 0, 1, \ldots, d-1$, if we set $e^{(d)} = e^{(0)}$, and if the eigenvectors are suitably rescaled and $\lambda > 0$ is suitably chosen. (*Hint:* Note that T transforms a vector with nonzero components on the group of labels $I_{1,i}$ into one with components nonzero on the successive group $I_{1,i+1}$, *etc.*)

Q4.1.14 **[4.1.14]:** Deduce from problem [4.1.13] that the eigenvalue of T^d relative to $e^{(i)}$ is $\lambda^d > 0$, independently from i.

Q4.1.15 **[4.1.15]:** Always in the context of problem [4.1.12], let e be an eigenvector for T with eigenvalue $\mu e^{i\varphi}$, $\mu > 0$. Setting $\widehat{e}^{(i)}_j = 0$ if $j \notin I_{1,i}$, and $\widehat{e}^{(i)}_j = e_j$ if $j \in I_{1,i}$, one has $T^d \widehat{e}^{(i)} = (\mu e^{i\varphi})^d \widehat{e}^{(i)}$. If the eigenvalue $\mu e^{i\varphi}$ is, among those of T, one with the largest absolute value, show that $\widehat{e}^{(i)}$ is proportional to $e^{(i)}$ and, furthermore, $(e^{i\varphi})^d = 1$, $\mu = \lambda$.

Q4.1.16 **[4.1.16]:** By making use of the result of problem [4.1.15] show that if the eigenvalue corresponding to e is one with largest absolute value, then it has the form $e = \sum_{j=0}^{d-1} e^{-\frac{2\pi i}{d}\,pj} e^{(j)}$ and this corresponds to the eigenvalue $\lambda e^{\frac{2\pi i}{d}\,p}$.

Q4.1.17 **[4.1.17]:** (*Perron–Frobenius theorem for general matrices*)
Deduce from problems [4.1.10]÷[4.1.16] that if T is a matrix with non-negative entries the eigenvalues of largest absolute value of T are arranged proportionally to the d–th roots of unity on a circle of radius $\lambda \geq 0$. The number d varies, if $\lambda > 0$, in a subset of the set of the periods of the blocks of equivalent labels. In fact, more generally, to each of these blocks I of period d_I correspond d_I eigenvectors of the type $\lambda_I e^{2\pi i p/d_I}$, $p = 0, \ldots, d_I - 1$, with simple multiplicity: between them one finds those of largest absolute value (that are precisely those that maximize λ_I).

Q4.1.18 **[4.1.18]:** (*Errant and non wandering points*)
If (Ω, S) is a topological dynamical system we say that a point $x \in \Omega$ is *errant* or *wandering* if one of its neighborhoods U is such that eventually no point initially in U evolves into point again in U, *i.e.* if there exists an open $U \ni x$ and an integer N_U such that $S^n U \cap U = \emptyset$, for all $n \geq N_U$. Interpret the results of the previous problems [4.1.4] and [4.1.5] as the statement that the set of the nonwandering points of $\{1, \ldots, q\}^{\mathbb{Z}}_T$ is $(I_1)^{\mathbb{Z}}_T \cup \ldots \cup (I_a)^{\mathbb{Z}}_T$, where the subscript T means that one considers only compatible sequences.

Q4.1.19 **[4.1.19]:** (*Topological transitivity and mixing*)
If (Ω, S) is a topological dynamical system with Ω compact we say that (Ω, S) is *topologically transitive* if there exists $x \in \Omega$ such that the set $\cup_{n\geq 0}\{S^n x\}$ is dense in Ω; we say that (Ω, S) is *topologically mixing* if given F, G open there exists N^0 such that $F \cap S^N G \neq \emptyset\ \forall N \geq N^0$. Show that if (Ω, S) is topologically transitive or mixing and if it admits a Markovian pavement with compatibility matrix T, then T is transitive or,

N4.1.4 respectively, mixing.[4]

Q4.1.20 **[4.1.20]:** (*Smale spectral theorem*)
Let (Ω, S) be a topological invertible dynamical system endowed with a Markovian pavement $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$. Show that the set $\Omega_{\rm nw}$ of the nonwandering points of Ω, apart from a set of zero measure for all the ergodic topological measures on Ω, is representable as

$$\Omega_{\rm nw} = \bigcup_{i=1}^{a} \bigcup_{j=1}^{d_i} \Omega_{i,j},$$

where $\Omega_{i,j}$ are closed sets such that $S\Omega_{i,j} = \Omega_{i,j+1}$, with $\Omega_{i,d_i+1} = \Omega_{i,1}$, and S^{d_i} is topologically mixing on $\Omega_{i,j}$. (*Hint:* Use the results of problem [4.1.19] and set $\Omega_{i,j} = X(\widetilde{\Omega}_{i,j})$; see problem [4.1.19] for the notion of topological mixing).

Q4.1.21 **[4.1.21]:** Find the decomposition into equivalence classes of communicating labels of the matrices that follow, compute the periods and determine the subclasses $\Omega_{i,j}$ for every indecomposable block:

$$T_3 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Bibliographical note to §4.1

The abstract notion of Markovian pavement, given here, is inspired by the some of its concrete applications. Other directions in which one can think for an abstract interpretation of the results that lead to the notion of Markovian pavement are possible. See for instance, [Ca76].

The idea of using Markovian pavements to study certain classes of topological measures goes back to the work of Adler and Weiss, [AW68] and to the works of Sinai, [Si68a], [Si68b], with important extensions in [Si72], who proposed a very original method to treat the existence problem and the analysis of the ergodic properties of invariant topological measures associated with a hyperbolic system.

§4.2 Markovian pavements for hyperbolic systems

An interesting class of dynamical systems for which it is possible to construct Markovian pavements $\mathcal{Q}$ consisting of sets of arbitrarily small diameter is the class of the *smooth hyperbolic systems*, also called *Anosov systems*.

The prototype of such systems is among those discussed in Section §1.2: it is the dynamical system (Ω, S), where $\Omega = \mathbb{T}^2$ = bidimensional torus regarded as a Riemannian manifold with the flat metric $ds^2 = d\varphi_1^2 + d\varphi_2^2$ and

e4.2.1
$$S \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \mod 2\pi. \tag{4.2.1}$$

[4] See definition (4.1.1).

This system is hyperbolic in the sense that for every point $\underline{\varphi} \in \Omega$ there exist two manifolds $W^u(\underline{\varphi})$ and $W^s(\underline{\varphi})$ which enjoy the properties that we now list.

$W^s(\underline{\varphi})$ is the straight line directed as the eigenvector $\underline{v}_1$ relative to the eigenvalue $\lambda < 1$ of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ while $W^u(\underline{\varphi})$ is the straight line directed as the eigenvector $\underline{v}_2$ relative to the eigenvalue $\lambda^{-1} > 1$ of the same matrix:

e4.2.2
$$\underline{v}_1 = \begin{pmatrix} 1 \\ \lambda - 1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ \lambda^{-1} - 1 \end{pmatrix}, \quad \lambda = (3 - \sqrt{5})/2, \tag{4.2.2}$$

and, since their slope is irrational, the two straight lines fill densely Ω.

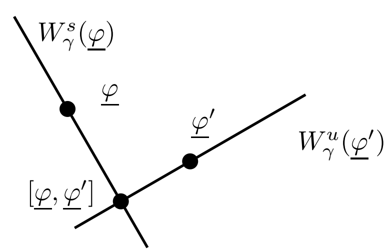
Furthermore the lines $W^a(\underline{\varphi})$ are *covariant*, *i.e.* $SW^a(\underline{\varphi}) = W^a(S\underline{\varphi})$ for $a = u, s$, and any two points $\underline{\psi}$ and $\underline{\psi}'$ on $W^s(\underline{\varphi})$ become close at exponential rate under the action of S, while any two points on $W^u(\underline{\varphi})$ become separated with exponential rate, *i.e.* , for all $n > 0$,

e4.2.3
$$\begin{aligned} d(S^n\underline{\psi}, S^n\underline{\psi}') &\le \lambda^n d(\underline{\psi}, \underline{\psi}') \qquad \text{for all } \underline{\psi}, \underline{\psi}' \in W^s(\underline{\varphi}), \\ d(S^{-n}\underline{\psi}, S^{-n}\underline{\psi}') &\le \lambda^n d(\underline{\psi}, \underline{\psi}') \qquad \text{for all } \underline{\psi}, \underline{\psi}' \in W^u(\underline{\varphi}), \end{aligned} \tag{4.2.3}$$

where d is the distance measured along $W^s(\underline{\varphi})$ or $W^u(\underline{\varphi})$ respectively (and it coincides with the geodesic distance if $\underline{\psi}$, $\underline{\psi}'$ are close enough).

From Fig. (4.2.1) it is also clear that, if $W^u_\gamma(\underline{\varphi})$ and $W^s_\gamma(\underline{\varphi})$ denote the connected parts of $W^u(\underline{\varphi})$ and $W^s(\underline{\varphi})$ containing $\underline{\varphi}$ and contained in a circle
N4.2.1 of small enough radius γ,[1] then there exists $\varepsilon > 0$ such that if $d(\underline{\varphi}, \underline{\varphi}') < \varepsilon$
N4.2.2 it follows that $[\underline{\varphi}', \underline{\varphi}] \stackrel{def}{=} W^u_\gamma(\underline{\varphi}') \cap W^s_\gamma(\underline{\varphi})$ consists of a unique point.[2]

Furthermore if ε is small enough $d(\underline{\varphi}, \underline{\varphi}') < \varepsilon$ implies that $[\underline{\varphi}', \underline{\varphi}]$ depends continuously on $\underline{\varphi}, \underline{\varphi}'$.



F4.2.1 **Fig.(4.2.1)** Representation of the operation that associates $[\underline{\varphi}', \underline{\varphi}]$ with the two points $\underline{\varphi}$ and $\underline{\varphi}'$ as the intersection of a short connected part $W^u_\gamma(\underline{\varphi}')$ of the unstable manifold of $\underline{\varphi}'$ and of a short connected part $W^s_\gamma(\underline{\varphi})$ of the stable manifold of $\underline{\varphi}$. The size γ is short "enough", compared to the diameter of Ω, and it is represented by the segments to the right and left of $\underline{\varphi}$ and $\underline{\varphi}'$.

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[1] The circle is defined in terms of the geodesic distance.

[2] Why is it necessary to require smallness of γ to have, say, uniqueness here?

The manifolds (straight lines, in this case) $W^u(\underline{\varphi})$ and $W^s(\underline{\varphi})$ are called, respectively, the *unstable manifold* and the *stable manifold* of the point $\underline{\varphi}$. Such manifolds are covariant with respect to the action of W, *i.e.*

e4.2.4
$$SW^s(\underline{\varphi}) = W^s(S\underline{\varphi}), \qquad S^{-1}W^u(\underline{\varphi}) = W^u(S^{-1}\underline{\varphi}). \tag{4.2.4}$$

It is natural to call "hyperbolic" the map S: in every point the action of S is strongly unstable analogously to what happens near the unstable equilibrium points called in stability theory "hyperbolic".

The example just discussed is very special and simple because of the local linearity of the map S. However the above described situation is sufficiently simple to admit natural generalizations to the case of more complex maps.

D4.2.1 **(4.2.1) Definition:** (Smooth hyperbolic system or Anosov system)
Let (Ω, S) be a dynamical system on a compact connected Riemannian manifold Ω of class C^∞ with S a diffeomorphism of class C^∞.[3] *Suppose that the system is topologically transitive, i.e. there is a dense orbit.*[4] *Suppose furthermore that there exists a smooth Riemannian metric d (possibly different from the one given on Ω, yet equivalent to it) such that measuring lengths with d the following properties hold.*
N4.2.3
N4.2.4
(i) (Splitting property) There exist two manifolds $\widetilde{W}^u(x)$ and $\widetilde{W}^s(x)$, that we shall suppose of class C^k, with $k > 2$, and with tangent plane at x depending on x with Hölder regularity. Furthermore the manifolds $\widetilde{W}^u(x)$ and $\widetilde{W}^s(x)$ are transversal in x and have complementary positive dimensions.[5]
N4.2.5
(ii) (Covariance property) Calling $\Sigma_\gamma(x)$ the sphere of radius γ centered at x and setting $W^u_\gamma(x) = \{$connected part of $\widetilde{W}^u(x) \cap \Sigma_\gamma(x)$ containing $x\}$ and $W^s_\gamma(x) = \{$connected part of $\widetilde{W}^s(x) \cap \Sigma_\gamma(x)$ containing $x\}$, there exists $\gamma > 0$ such that

e4.2.5
$$SW^s_\gamma(x) \subset W^s_\gamma(Sx), \qquad S^{-1}W^u_\gamma(x) \subset W^u_\gamma(S^{-1}x). \tag{4.2.5}$$

(iii) (Hyperbolicity property) There exists $\lambda < 1$ such that, for all $n \ge 0$,

e4.2.6
$$\begin{aligned} d(S^n y, S^n z) &\le \lambda^n d(y,z) \qquad \text{for all } y,z \in W^s_\gamma(x),\\ d(S^{-n} y, S^{-n} z) &\le \lambda^n d(y,z) \qquad \text{for all } y,z \in W^u_\gamma(x). \end{aligned} \tag{4.2.6}$$

(iv) There exists $\varepsilon > 0$, $\varepsilon < \gamma$ such that, if $x, y \in \Omega$ and $d(x,y) < \varepsilon$, the set $W^u_\gamma(x) \cap W^s_\gamma(y)$ consists in just a single point $[x,y]$ which depends continuously on x and y.
In the above circumstances we say that (Ω, S) is a smooth hyperbolic system *or an* Anosov system *and that the Riemannian metric d is* adapted *to the map S.*

[3] One often requires just class C^2.

[4] Or, equivalently, for all nonempty open $U, V \subset \Omega$ one has $U \cap S^n V \ne \emptyset$ for some n.

[5] The Hölder continuous dependence on x means there is $\delta > 0$ such that in each chart of an atlas for Ω we can find a base of vectors in the plane V^α_x tangent to $\widetilde{W}^a(x)$, $a = u, s$, whose *corresponding* components ξ_x verify $|\xi_x - \xi_y| < Cd(x,y)^\alpha$ if $d(x,y) < \delta$.

The latter definition is formulated so that it will turn out to be useful for the applications but it contains *several elements of redundancy* and it involves conditions and assumptions that appear rather difficult to verify: the following proposition could be used to replace definition (4.2.1) with a "purer" one. One can show that the following holds.

P4.2.1 **(4.2.1) Proposition:** (Anosov)
Let (Ω, S) be a dynamical system on a compact connected C^∞ Riemannian manifold Ω and let S be a C^∞ diffeomorphism. Suppose the following property for the linearization[6] *of S^n hold at the every point $x \in \Omega$.*
N4.2.6
(i) (Splitting) It is possible to decompose (or "split") the tangent space V_x at $x \in \Omega$ into two complementary (non-zero) spaces:

e4.2.7
$$V_x = V_x^s \oplus V_x^u, \tag{4.2.7}$$

*such that $dS^n V_x^s = V_{S^n x}^s$ and $dS^{-n} V_x^u = V_{S^{-n}x}^u$ (*covariance*), and V_x^s and V_x^u depend with continuity on x (*continuity*).*
(ii) (Hyperbolicity) There exist $C > 0$ and $\lambda < 1$ for which, for all $n \ge 0$,

e4.2.8
$$\begin{aligned} \|(dS^n)v\|_{V_{S^n x}} &\le C\lambda^n \|v\|_{V_x} \qquad \text{for all } v \in V_x^s, \\ \|(dS^{-n})v\|_{V_{S^{-n} x}} &\le C\lambda^n \|v\|_{V_x} \qquad \text{for all } v \in V_x^u, \end{aligned} \tag{4.2.8}$$

where the subscripts to the modulus signs indicate that the lengths are measured in the metric used for the tangent vectors at the appropriate points.
(iii) (Transitivity) There is a point with a dense orbit.
Then (Ω, S) is a smooth hyperbolic system. The decomposition of the tangent space into the two spaces V_x^s, V_x^u is called the hyperbolic splitting.

Remarks: (1) *Vice versa*, if (Ω, S) is hyperbolic in the sense of the definition (4.2.1) then it verifies the hypothesis of proposition (4.2.1) and therefore proposition (4.2.1) can be considered as a differential definition of hyperbolicity. Sometimes one defines a hyperbolic dynamical system as a system that verifies the hypothesis of proposition (4.2.1), possibly replacing the regularity requirement in class C^∞ by that of regularity in class C^2.
(2) Definition (4.2.1) brings directly into light the properties which are useful for proving existence of Markovian pavements and it is better suited for further generalizations.
(3) Therefore one can take as definition either the statements of definition (4.2.1) or of proposition (4.2.1). *However* an even weaker and more satisfactory definition, implying all the above statements, can be given. It is discussed in problem [4.2.1] below.
(4) One could define Anosov systems without requiring the existence of a dense trajectory. It is not known whether a system verifying the properties (i) and (ii) of definition (4.2.1) is necessarily transitive.

---

[6] Usually denoted dS^n and thought of as an operator between the tangent space V_x at x and the tangent space $V_{S^n x}$ at $S^n x$.

(5) A proof of a stronger result is presented in the problems at the end of the section.

P4.2.2 **(4.2.2) Proposition:** (Anosov)
Given any $0<\alpha<1$ and under the assumptions of proposition (4.2.1) the fields of spaces V^s_x, V^u_x are Hölder continuous in x with exponent α.

Remarks: (1) See footnote 5 for a definition of Hölder continuity of a field of spaces.
(2) This theorem is particularly important because in a sense it is optimal: there exist analytically smooth Anosov maps whose stable and unstable spaces split the tangent plane in a nonsmooth way (still Hölder continuous, of course). For the proof see problem [4.2.8].

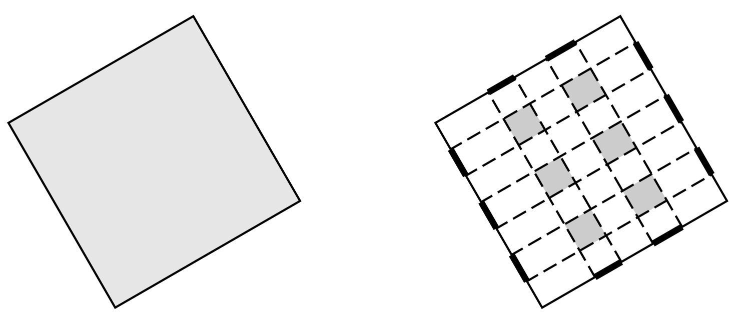
The construction of Markovian pavements for hyperbolic systems is based on the notion of *S–rectangle* or *hyperbolic rectangle*.

D4.2.2 **(4.2.2) Definition:** (S–rectangle)
Using the notations of definition (4.2.1) a set $R\subset\Omega$ is an S–rectangle for the hyperbolic system S, cf. *definition (4.2.1), if*

$$(a)\ R=\overline{\text{int}(R)},\qquad (b)\ \text{diam}(R)<\varepsilon,\qquad (c)\ x,y\in R\Rightarrow[x,y]\in R,$$

with ε small enough, so that property (iv) in definition (4.2.1) (where the operation $[\cdot,\cdot]$ is defined) holds.

In the example at the beginning of this section simple S–rectangles are (small) parallelograms with sides parallel to the eigenvectors $\underline{v}_1$ and $\underline{v}_2$ of the matrix $\begin{pmatrix}1&1\\1&2\end{pmatrix}$.



F4.2.2 **Fig.(4.2.2)** Two examples of S–rectangles, shaded in the figure, for the map of the torus $\mathbb{T}^2$ generated by the matrix $\begin{pmatrix}1&1\\1&2\end{pmatrix}$. In the second case the rectangle is not a connected set and is obtained by letting x run on the disconnected intervals marked on one of the sides of the previous rectangle and y run over a neighbouring side and forming $[x,y]$ defined in definition (4.2.1), (iv), and illustrated in Fig.(4.2.1).

However a more general S–rectangle is, in this case, a union of such parallelograms constructed by assigning a point $\underline{\varphi}$ and drawing from it $W^s(\underline{\varphi})$

and $W^u(\underline{\varphi})$ and, on them, a finite number of closed disconnected intervals and, then, performing the construction in Fig. (4.2.2).

Hence an S–rectangle can be disconnected (nevertheless, in this section, only connected S–rectangles will be considered).

P4.2.3 **(4.2.3) Proposition:** (Existence of Markovian pavements)
If (Ω, S) is an Anosov system and $\delta > 0$, there exists a Markovian pavement $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ of Ω consisting of S–rectangles of diameter less than δ.

Remarks: (1) In the general case the proof that follows becomes somewhat involved because one cannot rely on simple drawings; however with some imagination the two-dimensional analysis carries over essentially unchanged to the higher dimensional cases. Therefore we restrict ourselves, here, to the two–dimensional case, but we do not make use of several simplifications that would make the analysis not immediately extendible to the general case. See problem [4.3.10] for a simpler construction in the two-dimensional case. Our present analysis therefore follows quite closely the work of Bowen, see [Bo75] p.78–83, in which existence of a Markovian pavement for much more general dynamical systems is derived (cf. definition (4.2.3) and proposition (4.2.5) at the end of this section).

(2) For a first reading it may be useful to follow the arguments by imagining that we are dealing with the very special case of Arnold's cat map, *i.e.* of the diffeomorphism of $\mathbb{T}^2$ generated by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. In this case a simpler proof could be devised but, again, it would not be simply extendible to the higher dimensional cases.

(3) Most of what follows is devoted to illustrate a few very simple geometrical constructions. The wording may look intricate but the actual constructions are very simple and easily automatized for use on a digital computer. Therefore drawing what is described in words make the various statements easily intelligible.

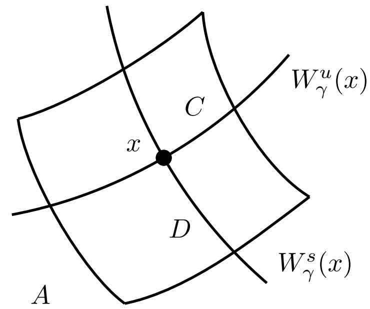
Proof: (case $d = 2$).

(A) *Geometric description of a generic rectangle.*
If A is an S–rectangle and $x \in \mathrm{int}(A)$, set $C = W^u_\gamma(x) \cap A$, $D = W^s_\gamma(x) \cap A$; hence we have

e4.2.9
$$A = [C, D] = \bigcup_{y \in C,\, z \in D} [y, z] \tag{4.2.9}$$

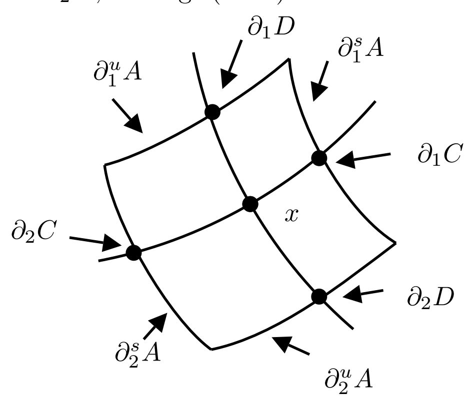
and we call x the *center* of A with respect to the *cross*, or *pair of axes*, C and D (note that any point of A is a center with respect to suitable pair of axes). This is illustrated in Fig.(4.2.3). We shall say that C is an unstable axis and D a stable one; if C, D and C', D' are two pairs of axes for the same rectangle we say that C and C', or D and D', are "parallel"; one has either $C \equiv C'$ or $C \cap C' = \emptyset$.

The boundary of A is composed by four connected sides $\partial^s_\beta A$, $\beta = 1, 2$, $\partial^u_\beta A$, $\beta = 1, 2$; the first two are parallel to the stable axis C and the other two to the unstable axis D, and they can be defined in terms of the boundaries



F4.2.3 **Fig.(4.2.3)** A rectangle A with a pair of axes C, D crossing at the corresponding center x.

∂C and ∂D of C and D considered as subsets of the unstable and stable lines that contain them. Each such boundary consists of two points $\partial_1 C$ and $\partial_2 C$ or $\partial_1 D$ and $\partial_2 D$, see Fig. (4.2.4).



F4.2.4 **Fig.(4.2.4)** The stable and unstable boundaries of a rectangle A and the two pairs of points $\partial_1 C, \partial_2 C$ and $\partial_1 D, \partial_2 D$ that generate them.

We define the stable and unstable parts of the boundary as

e4.2.10 $$\partial^s_\beta A = [D, \partial_\beta C], \qquad \partial^u_\beta A = [\partial_\beta D, C], \qquad \beta = 1, 2. \tag{4.2.10}$$

We shall prove the existence of a pavement $\mathcal{Q}$ of Ω with S–rectangles of diameter $< 2\alpha$, where $\alpha = \delta/2$ is half the preassigned δ, such that for all $\beta = 1, 2$ and for all $Q \in \mathcal{Q}$ one has

e4.2.11 $$S\partial^s_\beta Q \subset \partial^s_{\beta'} Q', \qquad S^{-1}\partial^u_\beta Q \subset \partial^u_{\beta''} Q'', \tag{4.2.11}$$

where β', β'', Q', Q'' depend on β and Q.

It is important to realize immediately that a pavement $\mathcal{Q}$ built with S–rectangles verifying (4.2.11) is Markovian. In fact equation (4.2.11) obviously implies (setting $\partial^s Q = \partial^s_1 Q \cup \partial^s_2 Q$ and $\partial^u Q = \partial^u_1 Q \cup \partial^u_2 Q$)

e4.2.12 $$S\,(\mathop{\cup}_{Q\in\mathcal{Q}} \partial^s Q) \subset \mathop{\cup}_{Q\in\mathcal{Q}} \partial^s Q, \qquad S^{-1}(\mathop{\cup}_{Q\in\mathcal{Q}} \partial^u Q) \subset \mathop{\cup}_{Q\in\mathcal{Q}} \partial^u Q, \tag{4.2.12}$$

N4.2.7 and it is also implied by (4.2.12).[7]

(B) *An equivalent problem.*
It is convenient to suppose that the map S has very large expansion and contraction rates. This is not a loss of generality because the problem can be further reduced to showing existence of a pavement $\mathcal{B}$ of Ω consisting of S–rectangles of diameter $< \alpha$ and such that for some $m > 0$

e4.2.13
$$S^m \bigcup_{B\in\mathcal{B}} \partial^s B \subset \bigcup_{B\in\mathcal{B}} \partial^s B\,, \qquad S^{-m} \bigcup_{B\in\mathcal{B}} \partial^u B \subset \bigcup_{B\in\mathcal{B}} \partial^u B. \tag{4.2.13}$$

Given such a pavement $\mathcal{B}$, a pavement $\mathcal{Q}$ satisfying the previous property (4.2.12) can be constructed as follows. We consider, if $m > 1$, the pavement $\mathcal{Q}$ whose elements are the S–rectangles Q having the form

e4.2.14
$$Q = \bigcap_{k=0}^{m-1} S^k B_k, \tag{4.2.14}$$

with $B_1, \ldots, B_{m-1} \in \mathcal{B}$. Here we use the fact that the intersection between two small S–rectangles *is still an S–rectangle*; it is easy to check that such S–rectangles are a pavement of Ω verifying (4.2.12).
The proof of the existence of $\mathcal{B}$, which is what remains to do, is in fact a description of a nice explicit construction of $\mathcal{B}$.

(C) *Construction of a covering by rectangles.*
We begin by considering a *covering* $\mathcal{A}^0 = \{A_1^0, \ldots, A_r^0\}$ of Ω by means of connected S–rectangles whose internal points cover Ω; this is possible because Ω is a compact connected Riemannian manifold. Furthermore we suppose that such S–rectangles have small diameter $\alpha < \min\{\delta/2, \gamma/2\}$, where δ is the length prefixed in the statement of the proposition and γ is the size of the local portions of stable and unstable manifolds (whose existence is part of the definition of Anosov system).
We imagine that the rectangles A_j^0 are constructed, as discussed in item (A) above, as products of two axes C_j^0 and D_j^0 through a center point x_j:

e4.2.15
$$A_j^0 = [C_j^0, D_j^0]. \tag{4.2.15}$$

(D) *Lebesgue length of the covering.*
A key role will be plaid by a certain length $a > 0$ that is a natural extension of the notion of *Lebesgue length* associated with a covering of a compact space by finitely many open sets. A Lebesgue length of a covering is defined as a length a such that every point x is "well inside" *some* element of the

[7] One checks directly the properties of definition (4.1.3), after having visualized (with the help of a drawing) the geometric meaning of (4.2.11). Use that the two relations in (4.2.12) must be simultaneously valid and that S is a map that transforms points relatively internal to $\partial^s_\beta C$ into points relatively internal to $S\partial^s_\beta C$: assuming that (4.2.11) does not follow from (4.2.12) attempt to represent the situation by a drawing in order to see the arising absurdity.

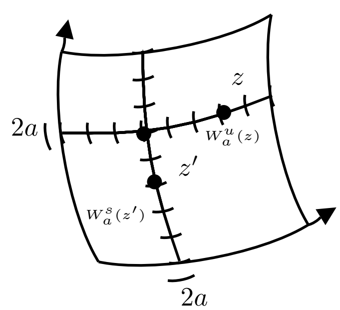
covering in the sense that the entire sphere of radius a around any point x is contained in the interior of some element of the covering.

The extension that we need here is that out of a covering by S–rectangles like $\mathcal{A}^0$ we can always find for each x an element $A_{F(x)} \in \mathcal{A}^0$, with a suitable label $F(x)$, whose boundary parallel to the s–direction stays at a distance $> a$ from $W^s_\gamma(x) \cap A_{F(x)}$, *i.e.* from the stable axis of $A_{F(x)}$ through x; and another one $A_{F'(x)} \in \mathcal{A}^0$, with a suitable label $F'(x)$, whose boundary parallel to the u–direction stays at a distance $> a$ from $W^u_\gamma(x) \cap A_{F'(x)}$, *i.e.* from the unstable axis of $A_{F'(x)}$ through x; actually we can even suppose that $F(x) = F'(x)$. To be precise we should require that, for every $y \in W^s_x \cap A^0_{F(x)}$, the distance between y and the intersection of $W^s_\gamma(y)$ with $\partial^u A^0_{F(x)}$ is greater than a if it is measured along $W^s_\gamma(y)$. Here and in what follows we will always assume that the length α is so small that one can think of $W^s_a(x)$ as a straight segment of length $2a$ centered in x. This simplifies the presentation following argument without any loss of generality.

Given $\mathcal{A}^0$ we shall fix a length $a > 0$ enjoying the above properties and such that $a < \alpha/2$ where $\alpha < \delta/2$ is the above defined maximum diameter of the elements of $\mathcal{A}\mathcal{A}^0$ (the reason for this choice will become clear in following). We call a a Lebesgue length for $\mathcal{A}^0$.

(1) Imagine to have drawn through x the manifolds $W^u_\gamma(x) \cap A^0_{F(x)}$ and $W^s_\gamma(x) \cap A^0_{F(x)}$.

(2) Through each of the points z in $W^u_\gamma(x) \cap A^0_{F(x)}$ and z' in $W^s_\gamma(x) \cap A^0_{F(x)}$ we can draw, respectively, the segments $W^s_a(z)$ and $W^u_a(z')$ of unstable and stable manifold, respectively, with $a < \alpha/2$: by our definitions and our choice of the Lebesgue length a such segments *lie entirely in* $A^0_{F(x)}$, as illustrated in Fig.(4.2.5).



F4.2.5 **Fig.(4.2.5)** Illustration of the check that each point of Ω is far at least as a in the direction of both stable and unstable manifolds from the boundaries of "some" element of the covering $\mathcal{A}^0$. The short segments are portions of stable or unstable manifold of length $2a$, where a is (much) smaller than the diameter α of the rectangle.

(E) *Replacing S by $g = S^m$ with m large.*

Let $m > 0$ be an integer such that the following sum is small enough (*i.e.* as small as indicated)

e4.2.16
$$2 \sum_{k=1}^{\infty} \lambda^{mk} \alpha = 2\lambda^m (1 - \lambda^m)^{-1} \alpha < a/2. \tag{4.2.16}$$

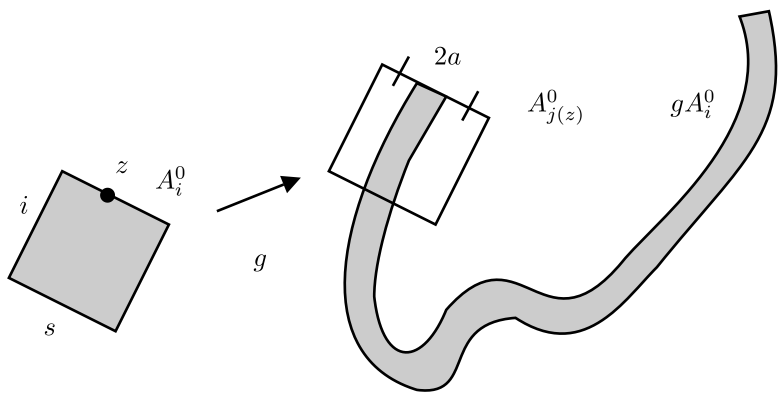
Calling $g \stackrel{def}{=} S^m$ we show that a Markovian pavement $\mathcal{B}$ for g can be constructed starting from a covering $\mathcal{A}$ made of S–rectangles with diameter less than $2\alpha < \delta$ for which $a < \alpha/2$ is a Lebesgue distance and verifying the following properties.

(I) For every $\beta = 1, 2$ and $A \in \mathcal{A}$ there exist $\beta', \beta'' \in \{1, 2\}$, $A', A'' \in \mathcal{A}$ for which

e4.2.17
$$g\, \partial^s_\beta A \subset \partial^s_{\beta'} A', \qquad g^{-1}\, \partial^u_\beta A \subset \partial^u_{\beta''} A''. \tag{4.2.17}$$

(II) Every $z \in \partial^s A_i$ is such that gz is "well inside" a boundary of the same type (*i.e.* stable) of some other rectangle $A_{j(z)}$, *i.e.* $W^s_{a/2}(gz) \subset \partial^s A_{j(z)}$ for a suitable $j(z)$; and, symmetrically, every $z \in \partial^u A_i$ is such that $W^u_{a/2}(g^{-1}z) \subset \partial^u A_{k(z)}$ for a suitable $k(z)$ (see Fig. (4.2.6)).

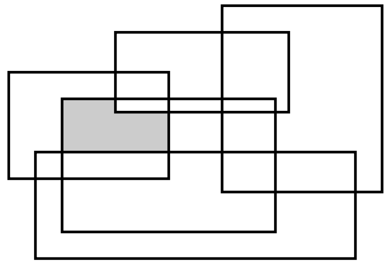
To construct $\mathcal{B}$ starting from $\mathcal{A}$ is the main difficulty of the analysis because realizing properties (I) and (II) will be rather straightforward. We shall see later how to construct the covering $\mathcal{A}$ starting form the initial covering $\mathcal{A}^0$.



F4.2.6 **Fig.(4.2.6)** Geometrical meaning of the properties (I) and (II). The g image of the part of the boundary of the (shaded) rectangle A^0_i on the left containing z is mapped, greatly shortened to a size $< a/2$ at least by (4.2.16), on the upper boundary of another rectangle of the covering $A^0_{j(z)}$; and in fact well in the middle so that even by widening it on either side by $(1 - \frac{1}{2})a = a/2$ it remains in the upper boundary of $A^0_{j(z)}$.

Indeed suppose that the covering $\mathcal{A}$ satisfies the above described properties (I) and (II). We can obtain a finer pavement of Ω by intersecting some of the set of $\mathcal{A}$ with the complements of the others. More precisely for every subset $\mathbf{r} \subset \{1, \ldots, r\}$ we can set $P_\mathbf{r} = (\cap_{i\in\mathbf{r}} A_i) \cap (\cup_{i\notin\mathbf{r}} \Omega\backslash A_i)$. Clearly the
N4.2.8 collection $\mathcal{P}$ of the non-empty $P_\mathbf{r}$ forms a pavement of Ω [8] such that for all $P \in \mathcal{P}$ one has $g\partial^s P \subset \cup_{P'\in\mathcal{P}} \partial^s P'$, $g^{-1}\partial^u P \subset \cup_{P'\in\mathcal{P}} \partial^u P'$, *i.e.* $\mathcal{P}$ is close to satisfy (4.2.13). Moreover remark that the sets $P_\mathbf{r}$ are the connected

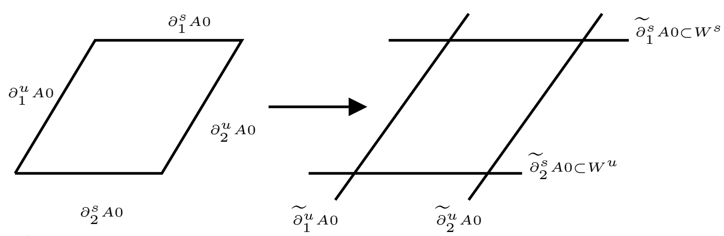
[8] The elements P of $\mathcal{P}$ have boundaries constituted by stable parts ($\partial^s P$) and by unstable parts ($\partial^u P$).



F4.2.7 **Fig.(4.2.7)** The sets obtained by intersecting in all possible ways the elements of the covering $\mathcal{A}$ have interiors which are pairwise disjoint but are not necessarily rectangles, *e.g.* see the gray set. One can call such sets "incomplete" rectangles.

components of $\Omega\setminus\cup_{A\in\mathcal{A}}\,\partial A$. *However* $\mathcal{P}$ has the defect of not necessarily consisting of S–rectangles (see Fig. (4.2.7)).

Nonetheless it is possible to build from $\mathcal{P}$ a pavement consisting of S–rectangles and verifying (4.2.13). One just "continues a little" the sides of every set $A\in\mathcal{A}$ along the stable or unstable manifold that contains them, see Fig. (4.2.8), *until an encounter with a boundary* (of unstable or stable type, respectively, *i.e.* of "opposite" type) is obtained. The boundary where we stop the continuation is the first one meets (this construction depends on the order in which the the sides to be continued are examined, hence it contains some arbitrariness). Since the sets in $\mathcal{A}$ are very small the length of the added parts of lines will be small (*i.e.* not exceeding 2α).



F4.2.8 **Fig.(4.2.8)** Prolonging the sides of the rectangles of the covering $\mathcal{A}$ in order to "complete" the incomplete rectangles like the one shaded in Fig.(4.2.7). The continuation is performed until a line reaches a boundary of a set in $\mathcal{A}$ (see Fig.(4.2.9)) and it means prolonging by a length at most 2α.

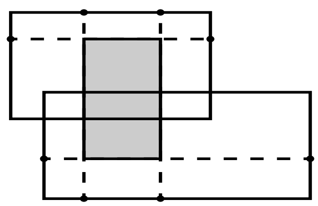
More precisely the continuation can be done by replacing $\partial^s_\beta A,\partial^u_\beta A$ by

e4.2.18
$$\widetilde\partial^s_\beta A=\bigcup_{x\in\partial^s_\beta A}W^s_\gamma(x),\qquad \widetilde\partial^u_\beta A=\bigcup_{x\in\partial^u_\beta A}W^u_\gamma(x),\tag{4.2.18}$$

i.e. we continue $\partial^s_\beta A,\partial^u_\beta A$ on either side by adding a piece of manifold of the same type (*i.e.* stable or unstable) of size γ.

Of course the sets $\widetilde\partial^s_\beta A,\widetilde\partial^u_\beta A$ will go beyond the point where they first meet the boundary of the elements of $\mathcal{A}$ which intersect A. This is so because diam $(A)<2\alpha$ for all $A\in\mathcal{A}$ and $\gamma>2\alpha$.

However we can just delete the parts of such lines that go outside the elements of $\mathcal{A}$ which intersect A. What is left is represented in Fig.(4.2.9) where A_0 is the shaded rectangle and the dashed lines correspond to the part of the continuations of the boundaries of A_0 which have not been deleted: we denote the lines so constructed by $\widehat{\partial}^s_\beta A \subset \widetilde{\partial}^s_\beta A$ and $\widehat{\partial}^u_\beta A \subset \widetilde{\partial}^u_\beta A$. This construction is repeated for each of the rectangles of $\mathcal{A}$ and at the end we shall have a pavement consisting only of S–rectangles, *i.e.* there will be no more "incomplete rectangles" like those appearing in Fig.(4.2.7).



F4.2.9 **Fig.(4.2.9)** Rectangles obtained by the geometric operations illustrated in Fig. (4.2.7), (4.2.8) applied to the shaded rectangle: we see that the number of "incomplete" rectangles becomes smaller (at the expense of an increase in the number of rectangles). The size of the dashed lines is $\le 2\alpha$. The construction has to be repeated for each rectangle, successively.

If $z \notin \cup_j(\widehat{\partial}^s A_j \cup \widehat{\partial}^u A_j)$, *i.e.* if z is not on the boundary of any of the just constructed rectangles, there exists a unique connected component of $\Omega \setminus \cup_j(\widehat{\partial}^s A_j \cup \widehat{\partial}^u A_j)$ that contains z. This component is clearly an open set and its closure is a S-rectangle B. As z varies such S–rectangles describe a finite family $\mathcal{B}$ that is a pavement of Ω with S–rectangles of diameter $< 2\alpha$, being intersections of rectangles with diameter already $< 2\alpha$.

Let $z \in \partial^s_\beta B$, $B \in \mathcal{B}$; then z will not be, in general, on the boundary of some elements of $\mathcal{A}$ but, by construction, there will exist $x \in \{$boundary of a suitable rectangle $A \in \mathcal{A}\}$ such that $z \in W^s_{2\alpha}(x)$. Since $\mathcal{A}$ enjoys, by hypothesis, properties (I) and (II) (cf. (4.2.17)) and $gW^s_{2\alpha}(x) \subset W^s_{a/2}(gx)$ by (4.2.16) we get

e4.2.19
$$g\,x \in \partial^s A_{j(x)} \qquad \text{and} \qquad W^s_{a/2}(g\,x) \subset \partial^s A_{j(x)}. \tag{4.2.19}$$

Noting that the diameter of $g\,W^s_{2\alpha}(x)$ is smaller than $2\lambda^m\alpha < a/2$ by the choice (4.2.16) of m, we see that $gW^s_{2\alpha}(x) \subset W^s_{a/2}(gx)$ so that (4.2.19) implies

e4.2.20
$$g\,\partial^s_\beta B \subset \bigcup_{B' \in \mathcal{B}} \partial^s B', \tag{4.2.20}$$

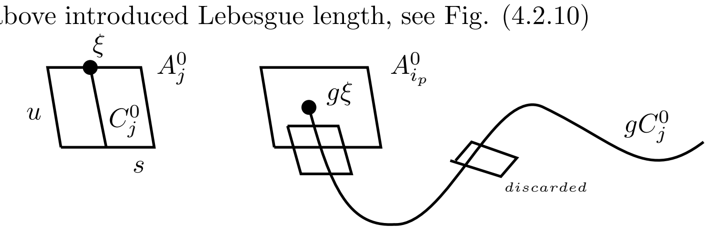
and, likewise, we would evince that

e4.2.21
$$g^{-1}\partial^u_\beta B \subset \bigcup_{B' \in \mathcal{B}} \partial^u B'. \tag{4.2.21}$$

This means that $\mathcal{B}$ satisfies (4.2.13), *i.e.* it is a Markovian pavement for the map g. As we noticed in point (B) it is easy to construct a Markovian pavement $\mathcal{B}'$ for S starting from $\mathcal{B}$, see (4.2.14). Moreover in (4.2.14) the element B_0 of $\mathcal{B}$ appearing in the intersection has diameter less than 2α so that the diameter of the elements of $\mathcal{B}'$ is also smaller that 2α.

Therefore we are reduced to show that if the covering $\mathcal{A}^0$ does not already enjoy properties (I) and (II) stated in connection with (4.2.17) above, it is still possible to modify it slightly so that it becomes a covering $\mathcal{A}$ of the same type which, however, satisfies properties (I) and (II) above. This can be done through an inductive procedure of successive approximations.

(F) *Successive refinements to build a covering verifying properties (I) and (II)*
Consider $g\,C_j^0$, $j = 1, \ldots, r$, and note that it is a "very long curve" (because of our choice of large m and because $g \equiv S^m$). Select a covering $A_{i_1}^0, \ldots, A_{i_{k_j}}^0$ of $g\,C_j^0$ made of elements of $\mathcal{A}^0$ such that gC_j^0 *"passes"* through each of them and *well inside*, *i.e.* at a distance $\geq a$ from the parallel boundary where a is the above introduced Lebesgue length, see Fig. (4.2.10)



F4.2.10 **Fig.(4.2.10)** Here the g-image of C_i^0 is represented as a long curve. The covering of the g-image is realized by discarding the elements of $\mathcal{A}^0$ whose expanding boundaries pass too close to $g\,C_i^0$, *i.e.* closer than the Lebesgue length of $\mathcal{A}^0$, see item (D).

In general some $A_{i_p}^0$, in this covering, are not completely crossed by gC_j^0; see the square $A_{i_p}^0$ in Fig. (4.2.10). Then continue a little C_j^0 so that gC_j^0 crosses *all* the sets $A_{i_1}^0, \ldots, A_{i_k}^0$ of the just considered covering of $g\,A_j^0$: this means replacing A_j^0 with $A_j^1 = [C_j^1, D_j^0]$, where

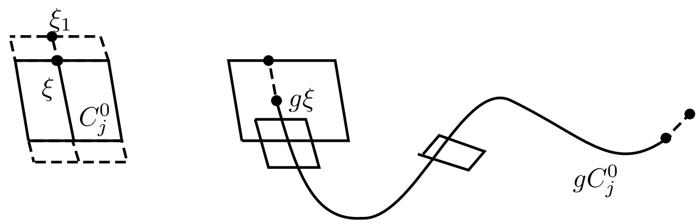
$$C_j^1 = \bigcup_{h=1}^{k_j} g^{-1}\Big(\{\text{continuation of } gC_j^0 \text{ until it crosses all of } A_{i_h}^0\} \cap A_{i_h}^0\Big) = \bigcup_{h=1}^{k_j} g^{-1}([C_{i_h}^0, g\,C_j^0 \cap A_{i_h}^0]);$$

see Fig. (4.2.11).

Inductively we shall set

e4.2.22
$$C_j^n = \bigcup_{h=1}^{k_j} g^{-1}([C_{i_h}^{n-1}, g\,C_j^{n-1} \cap A_{i_h}^{n-1}]). \tag{4.2.22}$$

At each step the lengthening of C_j^n with respect to C_j^{n-1} is by about a factor λ^m shorter so that it is $\leq 2(\lambda^m)^n\alpha$: this is seen by induction, recalling that diam $(A_j^0) \leq \alpha$.



F4.2.11 **Fig.(4.2.11)** Enlarging the manifold C^0_j up to the point ξ_1 so that the g–image of the enlarged manifold ends on the appropriate boundary of the rectangle $A^0_{i_p}$, with the notations of the previous figure.

Hence we can pass to the limit $n\to\infty$ to define $C_j=\mathop{\cup}\limits_{n=0}^{\infty}C^n_j$ and this makes sense if α and λ^m are so small that (see (4.2.16)) $\sum_{n\ge1}^{\infty}(\lambda^m)^n 2\alpha<a<\alpha/2<\gamma/2$.

Likewise one proceeds to lengthen D^0_j to construct D^n_j and to define D_j (using g^{-1} instead of g).

Define then the rectangles A_j, with $j=1,\ldots,r$, as $[C_j,D_j]$, that form a covering $\mathcal{A}$ of Ω with sets of diameter smaller than 2α.

By construction, if $z\in A_j\in\mathcal{A}$, there exist $A_{j'}\in\mathcal{A}$ such that $g\,z\in A_{j'}$ and $g^{-1}z\in A_{j''}$ and

e4.2.23
$$g\,[C_j,z]\supset[C_{j'},g\,z],\qquad g^{-1}[z,D_j]\supset[g^{-1}z,D_{j''}].\tag{4.2.23}$$

Furthermore: $A_{j'}\supset W^s_{\frac{a}{2}}(y)$ for all $y\in[C_{j'},gz]$, and $A_{j''}\supset W^u_{\frac{a}{2}}(y)$ for all $y\in[g^{-1}z,\,D_{j''}]$, provided that, as we can suppose, α and, therefore, the rectangles are small enough so that their sides can be treated as straight in the continuation operations. Drawings can be very useful to visualize the above statements; furthermore diam $(A_j)<2\alpha\le\delta$.

Hence $\mathcal{A}$ satisfies the properties (I) and (II) stated in connection with the (4.2.17). The correspondence that associates a point $x\in\Omega$ with a compatible sequence $\underline{\sigma}$ is Hölder continuous, (4.1.6), and its inverse has finite multiplicity if the value of δ is small enough so that the intersection between any element of $\mathcal{P}$ and the S–image of any other is a connected set, which certainly happens if α is small enough. Hence the proof is complete. ■

A consequence of the definition (4.2.1) is the following result.

P4.2.4 **(4.2.4) Proposition:** (Transitivity and mixing of compatibility matrices for Markovian pavements for Anosov maps)
If $\mathcal{P}$ is a Markovian pavement for an Anosov map, cf. *definition (4.2.1), then its compatibility matrix is transitive and mixing (*cf. *definition (4.1.1)).*

Proof: By definition (4.2.1) and problem [4.1.19] transitivity follows. The mixing property is tightly related to the assumption that Anosov systems are maps of a *connected* phase space Ω. Even dropping connectedness it is still possible to construct a Markovian pavement by the construction of proposition (4.2.3). The transition matrix will have the properties described in the problems [4.1.4] through [4.1.11]. Hence the system can be decomposed into a union of disjoint closed sets $\cup_{i=1}^{a-1} \cup_{j=1}^{d_i} \Omega_{i,j}$ and a "remainder set" Ω' open and invariant (consisting of the wandering points, cf. problem [4.1.16]) by problem [4.1.17] which must be empty by the transitivity assumption: connectedness then implies $a = 1, d = 1$ hence $\Omega = \Omega_{1,1}$ and S acts in a topologically mixing way on Ω so that also T is mixing. ∎

The just described construction of a Markovian pavement is generalizable to a situation that, in some applications, turns out to be quite useful; we shall describe it in the following definition (due to Ruelle, who called the systems that will be defined below an *abstract hyperbolic systems* or *Smale systems*, see p. 125–130 in [Ru78]).

D4.2.3 **(4.2.3) Definition:** (Abstract hyperbolic systems)
Let (Ω, S) be a topological dynamical system. Suppose that (Ω, S) is topologically transitive (i.e. there exists a point x of Ω with a dense orbit $(S^n x)_{n\in\mathbb{Z}^+}$).

The system (Ω, S) is said to be an abstract hyperbolic (transitive) system *if there exist a metric d for the topology of Ω and three constants $\lambda < 1$, $\gamma > 0$, $\varepsilon > 0$, with $\lambda > 0$ and $\gamma > \varepsilon$, and a function $x, y \to [x, y]$ defined on the pairs $\{x, y \,|\, x, y \in \Omega, d(x, y) < \varepsilon\}$ such that*
(1) $[x, x] = x$
(2) $[[x, y], z] = [x, z]$ and $[x, [y, z]] = [x, z]$, whenever the last expressions have a meaning.
(3) $S[x, y] = [Sx, Sy]$, when both sides have a meaning.
(4) Setting

e4.2.24
$$\begin{aligned} W^s_\gamma(x) &= \{u | u = [u, x] \ , \qquad d(u, x) < \gamma\},\\ W^u_\gamma(x) &= \{u | u = [x, u] \ , \qquad d(u, x) < \gamma\}, \end{aligned} \tag{4.2.24}$$

it follows that for all $n > 0$

e4.2.25
$$\begin{aligned} d(S^n y, S^n z) &< \lambda^n d(y, z) \quad \textit{if} \quad y, z \in W^s_\gamma(x),\\ d(S^{-n} y, S^{-n} z) &< \lambda^n d(y, z) \quad \textit{if} \quad y, z \in W^u_\gamma(x). \end{aligned} \tag{4.2.25}$$

Remark: The simplest examples of such systems are naturally the subshifts of finite type with a transitive compatibility matrix. The set $W^s_\gamma(\underline{\sigma})$ consists of all the strings that agree with $\underline{\sigma}$ on the sites with label $i \ge -p$ if $\gamma = e^{-p}$.

Then the following proposition holds.

P4.2.5 **(4.2.5) Proposition:** (Bowen)
Every abstract hyperbolic system admits a Markovian pavement with sets of diameter $\le \delta$ where $\delta > 0$ is arbitrarily prefixed.

The proof of this proposition follows, *grosso modo*, the ideas in the proof of proposition (4.2.3). It presents difficulties of topological character due to the lack of hypotheses of connectedness and differentiability on Ω and

S: this forces us to be careful in stating as obvious certain properties that are such in the case studied in proposition (4.2.3). See, for details, [Bo75], [Ru76].

A remarkable property enjoyed by such systems is the so called *shadowing* or "existence of a shadow motion".

P4.2.6 **(4.2.6) Proposition:** (Shadow symbols)
Let T be a transitive $n\times n$ compatibility matrix. Let $\{\underline{\sigma}_j\}_{j=-\infty}^{\infty}$ be a sequence of elements (sequences) in $\{0,\ldots,n-1\}_T^{\mathbb{Z}}$ such that $d(\tau\underline{\sigma}_j,\underline{\sigma}_{j+1})<\delta$ where τ is the shift; the sequence $\{\underline{\sigma}_j\}$ is called a δ–approximate motion for the dynamical system $(\{0,\ldots,n-1\}_T^{\mathbb{Z}},\tau)$. Then there exists $C>0$ independent of the sequence $\{\underline{\sigma}_j\}_{j=-\infty}^{\infty}$ and $\underline{\sigma}\in\{0,\ldots,n-1\}_T^{\mathbb{Z}}$ such that $d(\tau^j\underline{\sigma},\underline{\sigma}_j)<C\delta$.

Remark: Therefore given a symbolic motion that is at each time perturbed by δ one can find a true motion which remains close to the perturbed motion within $C\delta$ for *all times* $j\ge 0$ where C is a universal constant (*e.g.* $C=e$). In colorful words "the perturbed motion is a shadow of a true motion" (under a slightly trembling light). This is particularly interesting as it implies that the "same" property holds for motions in a more general Smale system, hence for Anosov maps: see proposition (4.2.7) below.

Proof: The distance between elements $\underline{\sigma},\underline{\sigma}'$ of $\{0,\ldots,n-1\}_T^{\mathbb{Z}}$ is the exponential of minus the maximal k such that $\sigma_i=\sigma_i'$ for all $|i|\le k$; see (4.1.6). Let $\delta=e^{-p}$. We define σ_j for $|j|\le p$ by $\sigma_j=(\underline{\sigma}_0)_j$. Then we set $\sigma_{p+1}=(\underline{\sigma}_1)_{p+1}$ and more generally $\sigma_{p+j}=(\underline{\sigma}_j)_{p+j}$ for $j>0$ while $\sigma_{-p+j}=(\underline{\sigma}_j)_{-p+j}$ for $j<0$. Then the result is true with $C=1$. If δ does not have the form e^{-p} with p integer then the same argument can be repeated but one gets that C can be larger than 1 but always $<e$. Hence $C=e$. ∎

P4.2.7 **(4.2.7) Proposition:** (Shadowing)
Let (Ω,S) be an abstract hyperbolic system, in the sense of definition (4.2.3) above. There is a constant C such that if $\ldots,x_{-1},x_0,x_1,\ldots$ is a sequence with $d(Sx_j,x_{j+1})<\delta$ and δ is small enough then there exists a point x such that $d(S^jx,x_j)<C\delta$ for all j.

Proof: The ambiguity in the code X that associates a symbolic motion with a point forbids saying that the proof of the previous proposition implies this result. However we can simply mimic the proof of proposition (4.2.6). We start with x_0 and construct the image Sx_0 which will be very close to x_1 so that a point $\xi_1=[Sx_0,x_1]$ on the unstable manifold of Sx_0 and on the stable of x_1 exists. We have that $S^{-1}\xi_1$ is on the unstable manifold of x_0. Then we construct $S\xi_1$ and define $\xi_2=[S\xi_1,x_2]$ which is a point such that $S^{-2}\xi_2$ is on the unstable manifold of x_0. More generally $\xi_n=[S\xi_{n-1},x_n]$: all points $S^{-i}\xi_i$ are on the unstable manifold of x_0 and all points ξ_i are at distance $<\delta$ from x_i and from the $S\xi_{i-1}$. The limit $\xi=\lim_{n\to\infty}S^{-n}\xi_n$ exists because by construction the variation in position of $S^{-n}\xi_n$ with respect to $S^{-(n-1)}\xi_{n-1}$

has size of order $\lambda^n\delta$ because the points ξ_n lie systematically on the unstable manifold of x_0: hence ξ exists and is on the unstable manifold of x_0. This also implies that there is C such that $d(S^j\xi, x_j) < C\delta$ for $j > 0$. We can repeat this construction using S^{-1} and the points $x_{-1}, x_{-2}, \ldots$ and find a point η on the stable manifold of x_0 such that $d(S^j\eta, x_j) < C\delta$ for $j < 0$. It is now enough to set $x = [\xi, \eta]$. ∎

Remark: It should be noted that the proof of proposition (4.2.6) immediately suggests the proof of proposition (4.2.7) if one takes into account that in symbolic dynamics the unstable "manifold" (quotes used here because it is not a manifold but just a set) consists of the compatible sequences which agree over the labels to the left of a suitable label (*i.e.* for times preceding a certain time).

Problems for the §4.2 (*Existence, regularity, smoothness, uniqueness of the stable and unstable foliations for Anosov maps*)

Q4.2.1 **[4.2.1]:** (*Continuity and invariance of the foliations*, from [AS67])
In the context of proposition (4.2.1) do not assume continuity nor covariance of the fields of spaces V^s_x, V^u_x as functions of x and replace the assumption in (4.2.8) by

$$\begin{aligned} \|(dS^n)v\|_{V_{S^n x}} &\le C\lambda^n\|v\|_{V_x} && \text{for all } v\in V^s_x,\\ \|(dS^{-n})v\|_{V_{S^{-n} x}} &\le C\lambda^n\|v\|_{V_x} && \text{for all } v\in V^u_x,\\ \|(dS^{-n})v\|_{V_{S^{-n} x}} &\ge C\lambda^{-n}\|v\|_{V_x} && \text{for all } v\in V^s_x,\\ \|(dS^n)v\|_{V_{S^n x}} &\ge C\lambda^{-n}\|v\|_{V_x} && \text{for all } v\in V^u_x, \end{aligned} \qquad (*)$$

for all positive n. Show that this implies (a) covariance of the fields, and (b) continuity of the fields. (*Hint:* Continuity at fixed n of $dS^{\pm n}$ implies that for $a = s, u$ if $x_j \to x_0$ and $v_j \in V^a_{x_j} \to v_0$ then $v_0 \in V^a_{x_0}$. This proves "lower semicontinuity" of the function $x \to d^a(x) \stackrel{def}{=}$ dimension of V^a_x. However the dimension is an integer and $d^s(x)+d^u(x)$ is identically equal to the dimension of the manifold. Hence the dimensions of V^a_x are both upper and lower semicontinuous, *i.e.* they are continuous, and being integer valued they are constant. Therefore the above inclusion $\lim V^a_{x_n} \subseteq V^a_{x_0}$ implies continuity of V^a_x.)

Q4.2.2 **[4.2.2]:** Let J be a matrix on $\mathbb{R}^d$ and assume that $\mathbb{R}^d = V^s \oplus V^u$ and that there is $\lambda < 1$ such that for all $n \ge 0$

$$\begin{aligned} |J^n v| &< \lambda^n|v|, & v&\in V^s, & |J^{-n}v| &< \lambda^n|v| & v&\in V^u,\\ |J^{-n}v| &> \lambda^{-n}|v|, & v&\in V^s, & |J^n v| &> \lambda^{-n}|v| & v&\in V^u, \end{aligned} \qquad (**)$$

then V^s, V^u are invariant under the action of $J^{\pm1}$. Furthermore there exist $\alpha, \varepsilon > 0$ such that the cones $\Gamma^{u,\alpha}$ and $\Gamma^{s,\alpha}$ consisting of all the lines forming an angle $\le \alpha$ with the plane V^s or, respectively, V^u are such that

$$J\Gamma^{u,\alpha} \subseteq \Gamma^{u,(1-\varepsilon)\alpha}, \qquad J^{-1}\Gamma^{s,\alpha} \subseteq \Gamma^{s,(1-\varepsilon)\alpha}$$

i.e. the cones $\Gamma^{u,\alpha}$ and $\Gamma^{s,\alpha}$ "shrink by a factor $(1-\varepsilon)$" under the action of J or, respectively, J^{-1}.

Q4.2.3 **[4.2.3]:** If in the right hand side of (∗∗), in problem [4.2.3], one inserts a constant C then the same conclusions hold if J is replaced by J^{n_0} with n_0 large enough.

Q4.2.4 **[4.2.4]:** (*Hadamard–Perron theorem at a hyperbolic fixed point*)
Let S be a C^∞ map of $\mathbb{R}^d$ into itself with a fixed point at the origin and with a Jacobian matrix J at the origin which verifies the property in (∗∗), in problem [4.2.4], for a given

$\lambda < 1$ and for all $n \ge 0$. Given $k > 0$, inside a sphere of radius $\delta_k > 0$, with δ_k small enough, around the origin there exist two surfaces W^s and W^u of class C^k tangent to V^s and V^u at the origin, respectively, and such that $SW^s \subset W^s$ and $S^{-1}W^u \subset W^u$. Furthermore there exists $C > 0$ and $\lambda' < 1$ such that $|S^n x| < C\lambda'^n|x|$ for $x \in W^s$ and $|S^{-n}x| < C\lambda'^n|x|$ for $x \in W^u$. (*Hint:* Let $k = 0$ and let Σ_δ be the sphere in V^s of radius δ and center at the equilibrium point. The results of problems [4.2.2] and [4.2.3] imply that if δ is small enough then a smooth surface whose points have the form $(s, \gamma(s))$, with $s \in V^s \cap \Sigma_\delta$ and $\gamma(s)$ a smooth function, and whose tangent plane forms everywhere an angle with respect to V^s not exceeding α is mapped by S^{-1} into a surface of the same type inside Σ_δ but extends beyond the boundary of Σ_δ. Then one modifies S^{-1} *outside* the sphere Σ_δ. Let (s, u) be a point near the origin and write $S^{-1}(s, u) = (f(s, u), J^{-1}u + g(s, u))$, where, with a slight abuse of notation, we are denoting $J^{-1}(0, u) = (0, J^{-1}u)$: modify the map S^{-1} outside Σ_δ into a new auxiliary map

$$X(s, u) = \Big(\sigma(s, u)f(s, u)\,,\ J^{-1}u + \sigma(s, u)g(s, u)\Big),$$

with $\sigma(s, u) = \tilde\sigma(\rho)$ where $\rho = sqrt u^2 + s^2$ and $\tilde\sigma(\rho)$ is a C^∞ non-negative decreasing function that is identically 1 for $\rho \le \delta$ and is 0 for $\rho > 2\delta$. Check that if δ is small enough then the map X still maps the cone $\Gamma^{s,\alpha}$ into itself *as well as any surface γ tangent to V^s at the origin and forming with V^s and angle $< \alpha$ into a surface of the same type.* Furthermore if $(s, \gamma(s))$ and $(s, \eta(s))$ are two such surfaces and $(s, \gamma'(s))$, $(s, \eta'(s))$ are their X–images, one has $\max_{|s|<2\delta} |\gamma'(s) - \eta'(s)| \le \lambda' \max_{|s|<2\delta} |\gamma(s) - \eta(s)|$, where λ' can be chosen as close as wished to λ provided δ is taken sufficiently small. Therefore if $(s, \gamma_0(s))$ with $\gamma_0(s) \equiv 0$ is a special initial surface Δ_0 the iterates $X^n\Delta_0$ converge exponentially fast to a limit Δ_∞ which has the property of being tangent at the origin to V^s (because such are all $S^n\Delta_0$) and of being invariant (because it is a limit of iterates); it has also the property of contracting exponentially to the origin (because every point in the cone $\Gamma^{s,\alpha} \cap \Sigma_\delta$ gets closer to the origin); furthermore the function $\gamma_\infty(s)$ defining Δ_∞ verifies a Lipshitz property (being the limit of functions with derivatives bounded by α). Hence Δ_∞ is a continuous surface and, in fact, it is even Lipshitz continuous. More generally if we fix $k > 0$ one proceeds in a similar way by controlling also the derivatives of the function $\gamma_\infty(s)$, and one can prove that Δ_∞ is a C^{k-1} regular surface with $(k-1)$–th derivatives verifying a Lipshitz property.)

Q4.2.5 **[4.2.5]:** (*Hadamard–Perron theorem at a hyperbolic point*)
Let S be a system that satisfies point (i) and (ii) of proposition (4.2.1) with C replaced by 1 in (4.2.8). Moreover let x_0 be a point whose trajectory under S is $x_j = S^j x_0$, $j \in \mathbb{Z}$. We imagine to consider a coordinate system that "follows" x_0 in its motion: this means that we consider a sequence of spheres $\Sigma_\delta(S^j x)$ on which a coordinate system with origin at x_j is defined introducing coordinates (s, u) such that $s = 0$ is a surface tangent at the origin to $V^u_{S^j x_0}$ and $u = 0$ is a surface tangent at the origin to $V^s_{S^j x_0}$. One can choose the coordinate systems to be quite similar to each other, *i.e.* one can construct all of them by choosing the origin and a neighborhood of it out of the charts of a finite atlas for Ω (thus the "units of measure" of the lengths cannot vary wildly as j varies). Show that if δ is small enough the cones $\Gamma^{s,\alpha}_{S^j x_0}$ and $\Gamma^{u,\alpha}_{S^j x_0}$ regarded as sets of points of $\Sigma_\delta(S^j x_0)$ with coordinates (s, u) such that $|u| < \alpha|s|$ and, respectively, $|s| < \alpha|u|$, for α small enough, are mapped into $\Gamma^{s,(1-\varepsilon')\alpha}_{S^{j-1}x_0}$ and $\Gamma^{u,(1-\varepsilon')\alpha}_{S^{j+1}x_0}$ respectively by S^{-1} and S, for some $\varepsilon' > 0$. Likewise a surface in $\Sigma_{S^j x_0}$ whose tangent plane forms everywhere angles $< \alpha$ to the axis $V^s_{S^j x_0}$ is mapped into a surface with the same property in $\Sigma_\delta(S^{j-1}x_0)$ by S^{-1}. Furthermore two sequences of surfaces γ^j, η^j in $\Sigma_\delta(S^j x_0)$ with the property of being contained in the cone $\Gamma^{s,\alpha}_{S^j x_0}$ or $\Gamma^{u,\alpha}_{S^j x_0}$ will be mapped by S^{-1} or, respectively, by S into surfaces which are closer by a factor λ' to each other at every point. Conclude that there exist manifolds $W^s(S^j x_0), W^u(S^j x_0)$ in each $\Sigma_\delta(S^j x_0)$ which are tangent to $V^s_{S^j x_0}$ and $V^u_{S^j x_0}$ which verify the properties (4.2.5) and (4.2.6). (*Hint:* Repeat the analysis of problem [4.2.4] carrying along the label j).

Q4.2.6 **[4.2.6]:** Making use of problem [4.2.3] show that one can repeat the analysis in problems [4.2.4] and [4.2.5] with C not equal to one but suitably modifying (4.2.6), *i.e.* substituting

λ^n with $C\lambda^n$. (*Hint:* Check the existence of surfaces W^s, W^u near every point that satisfies (4.2.5) with S replaced by S^{n_0} and n_0 large enough.)

Q4.2.7 **[4.2.7]:** (*Existence of an adapted metric*)
Show that it is always possible to change the Riemannian metric on Ω in such a way that if (4.2.8) holds for the old metric then it holds also for the new metric but with $C = 1$ and a different λ, *i.e.* show the existence of an adapted metric.(*Hint:* Let λ_1 be such that $\lambda < \lambda_1 < 1$. Setting $v = v^s + v^u$ with $v^s \in V^s_x, v^u \in V^u_x$ let $||v|| = \sum_{n=0}^\infty \lambda_1^{-n}||\partial S^n v^s||_{V^s_{S^n x}} + \sum_{n=0}^\infty \lambda_1^{-n}||\partial S^{-n} v^u||_{V^u_{S^n x}}$ The new metric satisfies (4.2.8) with $C = 1$ but with λ_1 replacing λ. However it can fail to be smooth. To avoid this show that it is enough to limit the sum in the definition to a large N, suitably chosen (*i.e.* so that for a suitably large C' one has $\lambda_1(1 + (\lambda/\lambda_1)^N C') < \lambda_2 < 1$), to obtain (4.2.8) with λ_2 replacing λ.)

Q4.2.8 **[4.2.8]:** (*Anosov theorem*)
Show that the spaces V^s_x, V^u_x are not only continuous (as discussed in problem [4.2.1]) but they are also Hölder continuous with an exponent < 1 (arbitrarily prefixed), *i.e.* prove proposition (4.2.2). (*Hint:* Let for simplicity $d = 2$, *i.e.* consider the two dimensional case first. Since we have seen that the surfaces W^u_x, W^s_x are smooth in C^k (for any prefixed k) we only have to see how close is V^s_x to V^s_y when x, y are in the same manifold W^u_x and close enough. We study the various geometrical objects that we introduce in a coordinate system (s, u) in a sphere $\Sigma_\delta(x)$ in which $u = 0$ and $s = 0$ are surfaces tangent to V^s_x, V^u_x. Suppose that there is a sequence $y_n \to x$ with $y_n \in W^u_x$ and $d(y_n, x) = \delta_n$, *but suppose also* that the angle $\alpha(y_n)$ between the stable and unstable manifolds at y_n is such that $\alpha(y_n) - \alpha(x) > \sqrt{\delta_n}$. Then the differential of S at x and the one at y_n differ by $O(\delta_n)$: $dS_{y_n} = dS_x + O(\delta_n)$, and the vector v_n tangent to $W^s_{y_n}$ has a component along V^u_x which by assumption has size $\sqrt{\delta_n}$ at least. Therefore if we apply dS^m to v_n with $m = -\log(\sqrt{\delta_n})$ we get a vector which has lenght $O(\lambda^{-m}\sqrt{\delta_n}) \simeq 1$ times the initial length instead of a vector of length of order λ^m times the initial one: the point is that the difference between dS_{y_n} and dS_x is of order δ_n and therefore dS^m cannot rotate the vector v_n enough to overcome the expansion along the unstable direction at least not if m is not too large. A similar argument works if $\sqrt{\delta}_n$ is replaced by δ_n^α with any $\alpha < 1$: but *not* with $\alpha = 1$. Therefore the V^s_x are Hölder continuous but not necessarily Lipshitz (or smoother): and in fact counterexamples to smoothness can be constructed. This shows that V^s_x is Hölder continuous with exponent $\alpha < 1$ at x. Examining more carefully the argument one concludes that the Hölder continuity constant can be chosen $C = 1$ independently of x for any prefixed $\alpha < 1$: the price that one pays is that the smaller is C or the closer is α to 1 the closer y has to the to x so that $|\xi_y - \xi_x| < Cd(x, y)^\alpha$ holds for a component ξ_y of the tensor that determines the plane V^s_y in a chart on Ω.)

Q4.2.9 **[4.2.9]:** (*Uniqueness of the stable and unstable manifolds*)
Show that if (Ω, S) is a hyperbolic system in the sense of definition (4.2.1) there cannot exist two manifolds for $x \in \Omega$, $\widetilde{W}^s(x)$ and $\widetilde{\widetilde{W}}^s(x)$ of class C^∞ and mutually transversal, such that if $y,\, z \in \widetilde{W}^s(x)$ or $y,\, z \in \widetilde{\widetilde{W}}^s(x)$ one has $d(S^n y, S^n z) \le C\lambda^n d(y, z)$, for all $n \ge 0$. (*Hint:* Consider the following figure which describes an impossible situation if $\widetilde{W} \ne \widetilde{\widetilde{W}}$ and if one imagines to apply to it iterates of the map S.)

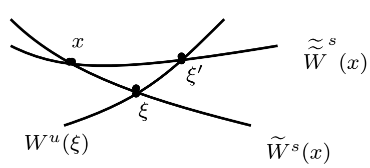


Fig.4.2.12: Illustration of the contradiction in which one would incur if the unstable manifold of x was not unique and two stable manifolds would emerge from x.

Q4.2.10 **[4.2.10]:** (*Mixing and non-transitive hyperbolic systems*)

If the connectedness condition in the definition of Anosov system is dropped then one obtains a hyperbolic system whose phase space Ω can be decomposed into a union of disjoint closed sets $\cup_{i=1}^{a}\cup_{j=1}^{d_i}\Omega_{i,j}$ and a "remainder set" Ω'. The sets $\Omega_{i,j}$ are permuted by the action of S and each of them is invariant under a suitable iterate of S which then acts in a topologically mixing way on it; the points $x\in\Omega'$ admit small neighborhoods U_x whose points evolve under the iterations of S without ever returning to visit U_x and they are called *wandering points*. If (Ω,S) is a connected Anosov system then it is topologically mixing. (*Hint:* It follows from the fact that even dropping connectedness it is still possible to construct a Markovian pavement by the construction of this section. The transition matrix will have the properties described in the problems of Section §4.1 and the result of problems [4.1.19] and [4.1.20] implies the first part. That a connected Anosov system has to be topologically mixing can be seen by noting that if $\Omega'\neq\emptyset$ then there cannot be a dense orbit; while if Ω' is empty then Ω will be the union of a finite number of disjoint sets and therefore it cannot be connected unless there is only one set $\Omega_{i,j}\equiv\Omega$ and in this case Ω will be topologically mixing.)

Q4.2.11 **[4.2.11]:** (*Splitting and hyperbolicity alone imply the Anosov property*)
Let (Ω,S) be a smooth dynamical system which verifies the properties of Anosov systems with the possible exception of the existence of a dense orbit. Show that it is necessarily an Anosov system. (*Hint:* The construction of a Markovian pavement discussed in the proof of proposition (4.2.1) can be performed and it leads to a representation of the points of Ω by symbolic strings with a certain compatibility matrix T: the result of problem [4.1.13] shows that the system can fail to be an Anosov system only if the matrix T is not mixing. However in this case the space Ω is disconnected *or* the matrix T contains inessential labels and correspondingly Ω contains wandering points.)

Q4.2.12 **[4.2.12]:** Find an explicit bound for C and a of (4.1.5) in the pavement constructed in proposition (4.2.3).

General properties of smooth hyperbolic systems discussed without the use of Markovian pavements.

Q4.2.13 **[4.2.13]:** (*Existence of a dense set of periodic orbits*)
Let (Ω,S) be an Anosov system (cf. definition (4.2.1)). The system (Ω,S) admits a dense set of periodic points. (*Hint:* Consider the case $d=2$. Let x_0 generate a dense orbit. Then choose $\overline{x}\in\Omega$ and let P be the rectangle with axes $W^s_\delta(\overline{x})\times W^u_\delta(\overline{x})$ with δ very small. Suppose, without loss of generality that $x_0\in P$. Then there will exist k_0 large such that $S^{k_0}x_0\in P$ again. The image $S^{k_0}P$ will be, if k_0 is large enough long and thin and it will cross P from "left to right" (*i.e.* from one of the two stable boundaries to the other) in a narrow connected band that contains $S^{k_0}x_0$. This band that we can call P_1 will be the image of a very narrow band "from top to bottom" of P (*i.e.* from one of the unstable boundaries of P to the other) containing x_0 and which we shall call P_0. The band P_1 is the connected part of $S^{k_0}P$ that contains $S^{k_0}x_0$ and $P_1=S^{k_0}P_0$. A much narrower band P_{-1} inside P_0 will become under iteration by S^{2k_0} a much narrower band $P_2\subset P_1$, *etc.* In this way we determine a unique point $\xi\subset P_j$, for all j. And one has $S^{k_0}\xi=\xi$: hence there is a periodic point of period k_0 inside P. The latter set was a rectangle of prefixed size of $O(\delta)$ around a prefixed point $\overline{x}$: hence periodic points are dense.)

Q4.2.14 **[4.2.14]:** (*Connectedness and Anosov systems*)
Let (Ω,S) be an Anosov system. Suppose that there is a fixed point $\overline{x}$. Show that $\cup_{i=0}^{\infty}S^iW^u_\delta(\overline{x})$ is dense in Ω. Assuming that (Ω,S) verifies all the properties in definition (4.2.1) *except* the connectedness of Ω show that connectedness follows. (*Hint:* Do not suppose that Ω is connected. Let x_0 generate a dense orbit. Suppose $\Omega\neq A\stackrel{def}{=}\overline{\cup_{i=0}^{\infty}S^iW^{u,\delta}_{\overline{x}}}$. Then there will be a point $y\notin A$ at distance $d(y,A)\stackrel{def}{=}\varepsilon>0$ from A. We can find a point $S^{k_+}x_0$ at distance $<\varepsilon/4$ from y and $k_-<k_+$ such that $S^{k_-}x_0$ is closer than $\varepsilon/4$ to the set A. And ε can be prefixed to be $\varepsilon\ll\delta$. Then the (part of) stable manifold $W^s_\delta(S^{k_-}x_0)$ will intersect the unstable manifold of a point $z\in A$. This will imply that $d(S^{k_+-k_-}z,S^{k_+}x_0)<\varepsilon$ which contradicts $d(y,A)\stackrel{def}{=}\varepsilon$. So far connectedness of Ω has

not been used. Furthermore if $\overline{x}$ is a fixed point then $\cup_{i=0}^{\infty} S^i W_\delta^u(\overline{x})$ is connected and therefore Ω is connected and any two points $z, w \in \Omega$ will be very close to points of A and therefore their stable manifolds will intersect $\cup_{i=0}^{\infty} S^i W_\delta^u(\overline{x})$ so that we can connect by a continuous curve z, w because $\cup_{i=0}^{\infty} S^i W_\delta^u(\overline{x})$ is connected.)

Q4.2.15 **[4.2.15]:** (*Density of stable and unstable manifolds for fixed points in Anosov maps*) Let (Ω, S) be an Anosov system. Let $\overline{x}$ be a fixed point. Then for all $y \in \Omega$ one has $\Omega = \overline{\cup_{i=0}^{\infty} S^i W_\delta^u(y)}$. (*Hint:* Let the point x_0 generate a dense orbit. The points of $W_\delta^u(y)$ will be very close to the points that are obtained by iterating a small segment of $W_\delta^u(\overline{x})$. Therefore it suffices to show that the result is true if $y \in W_{\delta'}^u(\overline{x})$ no matter how small $\delta' > 0$ is. But if $z \notin \overline{\cup_{i=0}^{\infty} S^i W_{\delta'}^u(y)}$ we can find an iterate of x_0 very close to y and a successive iterate very close to z and we are in a situation like the one met in the hint to problem [4.2.14]).

Q4.2.16 **[4.2.16]:** (*Density of stable and unstable manifolds for periodic points in Anosov maps*) Let (Ω, S) be an Anosov system and assume that the point x_0 generates a dense orbit. Let $\overline{x}$ be a periodic point of period $\overline{k}$. Then $\Omega = \overline{\cup_{i=0}^{\infty} S^i W_\delta^u(\overline{x})}$. (*Hint:* Same proof as for problem [4.2.14]: however we could not conclude that Ω is connected if we dropped the connectedness assumption as in problem [4.2.14].)

Q4.2.17 **[4.2.17]:** Let (Ω, S) be an Anosov system. Let $\overline{x}$ be a periodic point of minimal period $\overline{k}$. Then let $\Omega_0 = \overline{\cup_{i=0}^{\infty} S^{i\,\overline{k}}\, W_\delta^u(\overline{x})}$, $\Omega_1 = S\Omega_0, \ldots, \Omega_{\overline{k}-1} = S^{\,\overline{k}-1}\Omega_0$. Show that the sets Ω_j must coincide. Show also that if x_0 generates a dense orbit, then the iterates multiples of $\overline{k}$ of x_0 are dense as well. (*Hint:* Suppose $\overline{k} = 2$ for simplicity. Then if $S^{2i}x_0$ is dense in Ω_0 by repeating the argument in problem [4.2.15] we get that $\Omega_0 = \overline{\cup_{j=-\infty}^{\infty} S^{\,\overline{k}j}\, W_\delta^u(y)}$ if $y \in \Omega_0$. Thus if $y \in \Omega_0 \cap \Omega_1 \neq \emptyset$ one should have $\Omega_0 = \Omega_1$. Therefore we have to check that the even iterates of x_0 are dense in Ω_0 or, if not, then the odd iterates of Sx_0 are dense in Ω_0 and then replace x_0 by Sx_0. Suppose, for instance, that $\overline{x}$ can be approximated by a sequence of even iterates of x_0. Then a sequence of even iterates of x_0 can approximate any point $y \in \cup_{i=0}^{\infty} S^{i\,\overline{k}}\, W_\delta^u(\overline{x})$ or in the closure of the latter set, by the last hint to problem [4.2.15].)

Q4.2.18 **[4.2.18]:** (*Density of the stable and unstable manifolds of all points in Anosov maps*) Let (Ω, S) be an Anosov system. Show that the stable and unstable manifolds of any point, defined as $\cup_{-\infty}^{\infty} S^j W_\gamma^a(x)$, $a = u, s$, are dense in Ω.

Q4.2.19 **[4.2.19]:** (*Non-wandering points in systems with an absolutely continuous invariant measure*) Let (Ω, S) be a smooth dynamical system admitting an invariant measure μ which is absolutely continuous with respect the the volume measure: show that the set of non-wandering points, see problem [4.2.10], is the whole Ω. (*Hint:* Poincaré's recurrence theorem holds.)

Q4.2.20 **[4.2.20]:** (*Non-wandering points and density of periodic points*) Let (Ω, S) be a smooth dynamical system which verifies the properties of Anosov systems with the possible exception of the existence of a dense orbit. Show that if the set of non-wandering points is the whole Ω then the periodic points are dense. (*Hint:* The absence of wandering points allows us to imitate the argument in problem [4.2.12].)

Bibliographical note to §4.2

The proof of the existence theorem of Markovian pavements is taken from the paper of Bowen, see [Bo70], by particularizing its proof to case investigated here. The proof of Bowen widely generalizes and at the same time simplifies the original proof of Sinai, [Si68a], [Si69b]. The first idea and construction of a Markovian pavement appeared in connection with the theory of the automorphism $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ of $\mathbb{T}^2$ in $\mathbb{T}^2$ by Adler and Weiss (see [AW68]): in this work one explicitly and analytically constructs some pavements. This preceded the work of Sinai where the general notion of Markovian pavement for hyperbolic maps is introduced and its existence is shown in full generality.

It is useful to consult also the monograph [Bo75]. The notion of hyperbolic system goes back to Anosov: for elementary as well as for some not elementary properties of such systems see [AS67], (where they are called U–systems), [AA68] (where they are called C–systems), and [Av76]. The abstract version of hyperbolic system is due to Ruelle, see [Ru68].

The problems are classical results, see [AS67], [KH95], and summarize Anosov's extension of the stability theory for fixed points.

§4.3 Coding of the volume measure of smooth hyperbolic systems

In this section we illustrate via an example the method to perform the operations that we have shown to be possible in Section §4.1 for systems that admit Markovian pavements, considering the case of smooth hyperbolic systems.

We shall see that the volume measure on Ω can be described by means of the symbolic code X associated with a Markovian pavement $\mathcal{Q}$.

The fundamental notions for this study are described in the following two definitions: the first of them, setting the notion of conditional probability, will also be fundamental in the coming sections.

D4.3.1 **(4.3.1) Definition:** (Conditional probability)
Let m be a Borel probability measure on the space $\{1,\ldots,q\}^{\mathbb{Z}}$ of the sequences with q symbols and let $J=\{j_1,\ldots,j_s\}\subset\mathbb{Z}$ and $\underline{\sigma}_J=(\sigma_1,\ldots,\sigma_s)\in\{1,\ldots,q\}$.

Consider the σ–algebras $\mathcal{B}(J^c)$ generated by the cylinders $C^{J'}_{\underline{\sigma}'}$, with base $J'\subset J^c=\mathbb{Z}\setminus J$. Given $\underline{\sigma}_J$ we can define on $\mathcal{B}(J^c)$ the measures

e4.3.1
$$E\to m'(E)\stackrel{def}{=}m(E\cap C^J_{\underline{\sigma}_J}),\qquad E\to\overline{m}(E)\stackrel{def}{=}m(E),\tag{4.3.1}$$

for $E\in\mathcal{B}(J^c)$. Therefore the measure m' is absolutely continuous with respect to $\overline{m}$, i.e. m' is proportional to $\overline{m}$ via a suitable function (the Radon–Nykodim derivative $dm'/d\overline{m}$ of m' with respect to $\overline{m}$). Set

e4.3.2
$$m(\underline{\sigma}_J|\underline{\sigma}'_{J^c})=\frac{dm'}{d\overline{m}}(\underline{\sigma}'),\tag{4.3.2}$$

where the notation is correct because $\underline{\sigma}_J$ is fixed and $(dm'/d\overline{m})(\underline{\sigma}')$ is $\mathcal{B}(J^c)$–measurable: hence the latter depends, m–almost everywhere, only on $\underline{\sigma}'_{J^c}$, i.e. $(dm'/d\overline{m})(\underline{\sigma}')=(dm'/d\overline{m})(\underline{\sigma}'_{J^c})$.

The expression (4.3.2) is called probability of $\underline{\sigma}_J$ conditional to $\underline{\sigma}'_{J^c}$*: if we think of it as a function of* $\underline{\sigma}'$ *it is* m*–measurable and* $\overline{m}$*–measurable as well.*

Remarks: (1) Note that $\overline{m}$ is the restriction of m to the sets in $\mathcal{B}(J^c)$.
(2) The notion of conditional probability is important because, as we shall see, in terms of it one can describe in a simple way large classes of nontrivial measures on $\{1,\ldots,q\}^{\mathbb{Z}}$.
(3) The functions $\underline{\sigma}_J$, $\underline{\sigma}'_{J^c} \to m(\underline{\sigma}_J|\underline{\sigma}'_{J^c})$ must verify suitable compatibility conditions implicit in their definition; for instance $\sum_{\underline{\sigma}_J} m(\underline{\sigma}_J|\underline{\sigma}'_{J^c}) = 1$.
(4) The just mentioned compatibility condition shows that in order to define the function in (4.3.2) it is sufficient, if J is fixed, to assign all ratios

e4.3.3 $$m(\underline{\sigma}'_J|\underline{\sigma}_{J^c})/m(\underline{\sigma}''_J|\underline{\sigma}_{J^c}) \quad \text{for all } \underline{\sigma}'_J, \underline{\sigma}''_J, \underline{\sigma}_{J^c}. \tag{4.3.3}$$

Coming back to Anosov systems we note that the stable and unstable manifolds are smooth (see proposition (4.2.2) and problem [4.2.8]). Therefore it makes sense to consider, at x, the map S as a map between a neighborhood of x on W^a_x and a neighborhood of Sx on W^a_{Sx}, $a = u, s$. The Jacobian matrices of these maps are linear maps, *i.e.* matrices, between V^a_x and V^a_{Sx}. We shall call the latter matrices the expanding and the contracting Jacobian matrices at x (their dimensions will be $d^a \times d^a$ if d^a is the dimension of the manifolds W^a, $a = u, s$).

Note *however* that such matrices will be smooth functions of x *along the (smooth) manifolds* $W^a_{x_0}$ of any given point x_0, but they will only be Hölder continuous as functions of x as x varies in Ω. The Hölder continuity exponent can be taken to be $\alpha < 1$ and the Hölder continuity modulus C as well can be taken to be independent of the point x if the comparison involves close enough points (how close may depend on the choice of α).

We consider for simplicity only the two-dimensional case: but most of what we say carries over to the higher dimensional cases simply by replacing the contraction and expansion coefficients by the absolute values of the determinant of the expanding and contracting Jacobian matrices. The latter are Hölder continuous functions on Ω.

D4.3.2 **(4.3.2) Definition:** (Local expansion and contraction coefficients and exponents)
Let (Ω, S) *be a bidimensional Anosov system with a Markovian pavement* $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ *built with* S*–rectangles. Let* T *be its compatibility matrix (see definition (4.1.3)) and call* X *the corresponding code* $X : \{1,\ldots,q\}^{\mathbb{Z}}_T \to \Omega$*. We set*

e4.3.4 $$\lambda_u^{-1}(x) = \Big|\frac{dS^{-1}}{d\xi}(x)\Big|_u, \qquad \lambda_s(x) = \Big|\frac{dS}{d\xi}(x)\Big|_s, \tag{4.3.4}$$

where the subscripts indicate the derivative of S *considered as a map from the stable or unstable manifold at* x *into the corresponding manifold at* $S(x)$ *and* ξ *represent the arc length parameterization of these manifolds. They* *will be called the local* expansion *and* contraction coefficients*, while (the opposite of) their logarithms*

e4.3.5
$$A_u(\underline{\sigma}) = -\log\lambda_u^{-1}(X(\underline{\sigma})), \qquad A_s(\underline{\sigma}) = -\log\lambda_s(X(\underline{\sigma})), \tag{4.3.5}$$

defined on the space of the compatible sequences $\underline{\sigma} \in \{1,\ldots,q\}_T^{\mathbb{Z}}$*, will be called the local* expansion *and* contraction exponents *or* rates*.*

Remarks: (1) For all $x \in \Omega$ one has

e4.3.6
$$\Big|\frac{dS^{-M}}{d\xi}(x)\Big|_u = \prod_{k=0}^{M-1}\Big|\lambda_u(S^{-k}x)\Big|^{-1}, \qquad \Big|\frac{dS^{M}}{d\xi}(x)\Big|_s = \prod_{k=0}^{M-1}\Big|\lambda_s(S^{k}x)\Big|, \tag{4.3.6}$$

by the differentiation rules and by the covariance of the manifolds W_x^s and W_x^u, cf. (4.2.5).
(2) Note that in an adapted metric, see definition (4.2.1), $\lambda_u^{-1}, \lambda_s$ will be everywhere $< \lambda < 1$, and of course > 0. However in general it might well be otherwise: the values of the local expansion and contraction coefficients can be quite arbitrarily changed by changing the metric on Ω. Of course *eventually* for large M both quantities in (4.3.6) will become < 1.
(3) The definition of smooth hyperbolic system implies that the functions (4.3.4) and (4.3.5) are Hölder continuous in $x \in \Omega$. Since the code X is Hölder continuous as well (cf. definition (4.1.3)) we deduce

e4.3.7
$$|A_a(\underline{\sigma}) - A_a(\underline{\sigma}')| \le C\, e^{-\kappa\nu(\underline{\sigma},\underline{\sigma}')}, \qquad a = u, s, \tag{4.3.7}$$

with suitable positive C and κ, if $\nu(\underline{\sigma},\underline{\sigma}') = \max\{j|\sigma_k = \sigma'_k, \text{for all } |k| \le j\}$. In other words $\underline{\sigma} \to A_a(\underline{\sigma})$ is "Hölder continuous" on the space $\{1,\ldots,q\}_T^{\mathbb{Z}}$.
It is convenient to find a representation of A_a in terms of *potentials*.

P4.3.1 **(4.3.1) Proposition:** (Contraction and expansion potentials)
Let (Ω, S) *be an Anosov system and let* $\mathcal{Q} = \{Q_1,\ldots,Q_q\}$ *be a Markovian pavement of* Ω *with compatibility matrix* $T_{\sigma\sigma'}$*. Consider the local expansion and contraction exponents* $A_a(\underline{\sigma})$ *defined by (4.3.5). Then the function*
N4.3.1 $A_a(\underline{\sigma})$ *can be written as a sum of cylindrical functions:*[1]

e4.3.8
$$A_a(\underline{\sigma}) = \mathit{constant} + \sum_{n=0}^{\infty}\Phi^a_{2n+1}(\sigma_{-n},\ldots,\sigma_n), \quad a = u, s, \tag{4.3.8}$$

where the functions Φ^u, Φ^s*, which will be called the* expansion *and* contraction potentials*, are defined on the* T*−compatible strings of* $2n+1$ *symbols, i.e. on* $\{1,\ldots,q\}_T^{2n+1}$*, and decay exponentially in the sense that*

e4.3.9
$$|\Phi^a_{2n+1}(\sigma_{-n},\ldots,\sigma_n)| \le \overline{C}\, e^{-\overline{\kappa}n} \tag{4.3.9}$$

[1] A function on $\{1,\ldots,q\}^{\mathbb{Z}}$ is called *cylindrical* if it depends only on the values of σ_j corresponding to a finite number of labels j.

for suitable positive constants $\overline{C}$, $\overline{\kappa}$.

Proof: To check (4.3.8) and (4.3.9) let $\underline{\sigma}^0 \in \{1,\ldots,q\}^{\mathbb{Z}}_T$ and let $z>0$ be such that $T^z_{\sigma\sigma'}>0$, for all pairs σ,σ': such z exists because of the topological transitivity and mixing properties of Anosov systems (see problems [4.1.19], [4.2.10]). We construct $\underline{\sigma}^0$, $\underline{\sigma}^1$, $\underline{\sigma}^2,\ldots \in \{1,\ldots,q\}^{\mathbb{Z}}_T$ by "chopping" the sequence $\underline{\sigma}$ at sites $\pm j$ and "attaching" to the right and to the left of it the semi-infinite strings read out of the parts of $\underline{\sigma}^0$ with positive or negative labels. This will be done so that the sequence $\underline{\sigma}^1$ has the entry with the label in the site 0 coinciding with the corresponding entry of $\underline{\sigma}$, while it coincides with $\underline{\sigma}^0$ in the sites j with $|j|>z$, $\underline{\sigma}^2$ coincides with $\underline{\sigma}$ in the sites between -1 and 1 and with $\underline{\sigma}^0$ in those with $|j|>1+z$, $\underline{\sigma}^3$ coincides with $\underline{\sigma}$ between -2 and 2 and with $\underline{\sigma}^0$ in the sites with $|j|>2+z$, *etc.*: the $2z$ insertions "between" $\underline{\sigma}$ and $\underline{\sigma}^0$ are each time made arbitrarily (and they are possible because $T^z_{\sigma\sigma'}>0$, by our choice of z). Set *constant* $=A_a(\underline{\sigma}^0)$ and

e4.3.10
$$\Phi^a_{2n+1}(\sigma_{-n},\ldots,\sigma_n)=A_a(\underline{\sigma}^{n+1})-A_a(\underline{\sigma}^n); \tag{4.3.10}$$

then equations (4.3.8) and (4.3.9) follow immediately from (4.3.7), *i.e.* from the Hölder continuity of the symbolic code and of the stable and unstable manifolds. ∎

We can now describe the action of the code X on the (normalized) volume measure μ^0 on Ω (Sinai).

P4.3.2 **(4.3.2) Proposition:** (Symbolic code for the volume measure)
Let (Ω,S) be a bidimensional smooth topologically mixing hyperbolic system and let $\mathcal{Q}=\{Q_1,\ldots,Q_q\}$ be a Markovian pavement with compatibility matrix T (necessarily mixing, cf. *problem [4.1.19]); let $X:\{1,\ldots,q\}^{\mathbb{Z}}_T\to\Omega$ be the corresponding code. Let μ^0 be the volume measure on Ω associated with the metric on Ω and let m_0 be the measure on $\{1,\ldots,q\}^{\mathbb{Z}}_T$ isomorphic to it via X (*cf. *remark (1) following (4.1.11)).*
(i) The measure m_0 has the following conditional probabilities:

e4.3.11
$$\frac{m_0(\sigma'_{-n}\ldots\sigma'_n|\sigma_j,|j|>n)}{m_0(\sigma''_{-n}\ldots\sigma''_n|\sigma_j,|j|>n)}=\frac{\sin\varphi(X(\underline{\sigma}'))}{\sin\varphi(X(\underline{\sigma}''))}\cdot \tag{4.3.11}$$
$$\cdot\exp\Big(-\sum_{k=1}^{\infty}[A_u(\tau^k\underline{\sigma}')-A_u(\tau^k\underline{\sigma}'')+A_s(\tau^{-k}\underline{\sigma}')-A_s(\tau^{-k}\underline{\sigma}'')]\Big),$$

where A_s,A_u are defined in (4.3.5) and satisfy (4.3.7), $x\to\varphi(x)$ is the angle between the stable and unstable manifolds in x, and $\underline{\sigma}'$, $\underline{\sigma}''$ are sequences in $\{1,\ldots,q\}^{\mathbb{Z}}_T$ whose values on the labels j with $|j|>n$ coincide, i.e. $\sigma'_j=\sigma''_j=\sigma_j$ for $|j|>n$.
(ii) The restriction m_0^+ of m_0 to the σ–algebra $\mathcal{B}(\mathbb{Z}^+)$, generated by the

cylinders with base in $\mathbb{Z}^+$*, has conditional probabilities of the form*

e4.3.12
$$\begin{aligned}\frac{m_0^+(\sigma'_0\ldots\sigma'_n|\sigma_i,j>n)}{m_0^+(\sigma''_0\ldots\sigma''_n|\sigma_j,j>n)} &= \frac{f(\sigma')}{f(\underline{\sigma}'')}\cdot\\ &\cdot\exp\Big(-\sum_{k=0}^{\infty}[\widehat{A}_u(\tau^k\underline{\sigma}')-\widehat{A}_u(\tau^k\underline{\sigma}'')]\Big),\end{aligned}\tag{4.3.12}$$

where $f>0$ *and* $\widehat{A}_u$ *are Hölder continuous functions on* $\{1,\ldots,q\}^{\mathbb{Z}}_T$*.*
(iii) Finally there exist $\Phi^a_{2n+1}:\{1,\ldots,q\}^{2n+1}\to\mathbb{R}$ *such that*

e4.3.13
$$A_a(\underline{\sigma})=\sum_{n=0}^{\infty}\Phi^a_{2n+1}(\sigma_{-n}\ldots\sigma_n),\qquad \widehat{A}_a(\underline{\sigma})=\sum_{n=0}^{\infty}\Phi^a_{2n+1}(\sigma_0\ldots\sigma_{2n+1}),\tag{4.3.13}$$

for $a=u,s$ *and there exist constants* $\widetilde{C}$*,* $\widetilde{\kappa}>0$ *for which*

e4.3.14
$$|\Phi^a_{2n+1}(\sigma_{-n}\ldots\sigma_n)|\le\widetilde{C}e^{-\widetilde{\kappa}n}\qquad \textit{for all } n\ge 0.\tag{4.3.14}$$

Remarks: (1) Even though, at first sight, this may appear strange, it is convenient to think of (4.3.13) in the following form. Let $(\Phi^a_X)_{X\subset\mathbb{Z}}$ be a family of functions parameterized by the finite subsets $X\subset\mathbb{Z}$, $\Phi^a_X:\{1,\ldots,q\}^{\mathbb{Z}}\to\mathbb{R}$, such that

e4.3.15
$$\begin{aligned}&\Phi^a_X(\underline{\sigma}_X)=0 \qquad \text{unless } X=(z,z+1,\ldots,z+2n),\\ &\Phi^a_X(\underline{\sigma}_X)=\Phi^a_{2n+1}(\underline{\sigma}_X)\quad \text{if } X=(z,z+1,\ldots,z+2n),\end{aligned}\tag{4.3.15}$$

for $\underline{\sigma}_X\in\{1,\ldots,q\}^X=\{1,\ldots,q\}^{2n+1}$ and some $z\in\mathbb{Z}$. Then one has, or one defines

e4.3.16
$$\begin{aligned}A_a(\underline{\sigma})&\equiv\sum_{X\text{ centered on }0}\Phi^a_X(\underline{\sigma}_X),\qquad a=u,s,\\ \widehat{A}_a(\underline{\sigma})&\equiv\sum_{X\ni 0,X\subset\mathbb{Z}^+}\Phi^a_X(\underline{\sigma}_X),\qquad a=u,s,\\ \overline{A}_a(\underline{\sigma})&\stackrel{def}{=}\sum_{X\ni 0}\frac{\Phi^a_X(\underline{\sigma}_X)}{|X|},\qquad a=u,s.\end{aligned}\tag{4.3.16}$$

The above notations are useful for later use and for a quick check of the following algebraic identity that will be used below:

e4.3.17
$$\begin{aligned}\sum_{k=1}^{\infty}[A_a(\tau^k\underline{\sigma}')-A_a(\tau^k\underline{\sigma}'')]&=\sum_{k=0}^{\infty}[\widehat{A}_a(\tau^k\underline{\sigma}')-\widehat{A}_a(\tau^k\underline{\sigma}'')]+\\ &\qquad+W(\underline{\sigma}')-W(\underline{\sigma}''),\\ \sum_{k=-\infty}^{+\infty}[A_a(\tau^k\underline{\sigma}')-A_a(\tau^k\underline{\sigma}'')]&=\sum_{k=-\infty}^{+\infty}[\widehat{A}_a(\tau^k\underline{\sigma}')-\widehat{A}_a(\tau^k\underline{\sigma}'')]=\\ &=\sum_{k=-\infty}^{+\infty}[\overline{A}_a(\tau^k\underline{\sigma}')-\overline{A}_a(\tau^k\underline{\sigma}'')],\end{aligned}\tag{4.3.17}$$

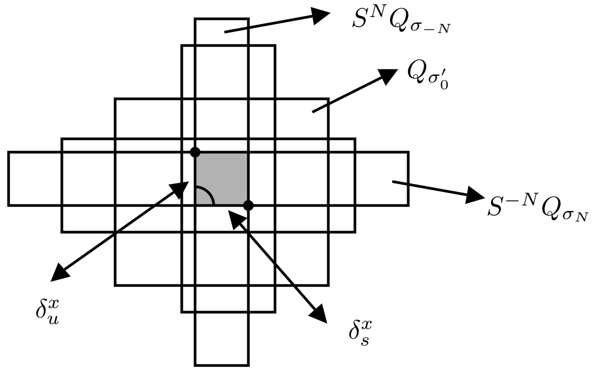
where W is a suitable Hölder continuous function on $\{1,\ldots,q\}^{\mathbb{Z}}_T$ and $\underline{\sigma}'$, $\underline{\sigma}''$ are defined after (4.3.11) (the check is left to the reader).
(2) If one follows the steps of the proof by considering the particular case of the paradigm of the Anosov systems, *i.e.* Arnold's cat map,

e4.3.18
$$\Omega=\mathbb{T}^2,\ S\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\ \mu_0(d\varphi_1 d\varphi_2)=\frac{d\varphi_1 d\varphi_2}{(2\pi)^2},\qquad(4.3.18)$$

with $\mathcal{Q}$ an arbitrary Markovian pavement, one should realize that in this case the proof is very simple because the necessary computation of several limits becomes trivial.

Proof: We prove (4.3.11) in the case $n=0$. The proof however can be immediately adapted to cover the arbitrary $n\ge 0$ case. Let $\underline{\sigma}'$ and $\underline{\sigma}''\in\{1,\ldots,q\}^{\mathbb{Z}}_T$ with $\sigma'_j=\sigma''_j=\sigma_j$ for $|j|>0$, and set $X(\underline{\sigma}')=x$, $X(\underline{\sigma}'')=y$, assuming, furthermore, that $x,y\notin\cup_{i=1}^q\partial Q_i$. By Doob's theorem (cf. the part of the proof of proposition (3.2.1) that follows equation (3.2.25) and, in particular, equation (3.2.28)) one has, m–almost everywhere,

e4.3.19
$$\frac{m_0(\sigma'_0|\sigma_j,|j|>0)}{m_0(\sigma''_0|\sigma_j,|j|>0)}=\lim_{N\to\infty}\frac{m_0(C^{-N\ldots-1\ \ 0\ \ 1\ldots N}_{\sigma_{-N}\ldots\sigma_{-1}\sigma'_0\sigma_1\ldots\sigma_N})}{m_0(C^{-N\ldots-1\ \ 0\ \ 1\ldots N}_{\sigma_{-N}\ldots\sigma_{-1}\sigma''_0\sigma_1\ldots\sigma_N})}.\qquad(4.3.19)$$



F4.3.1 **Fig.(4.3.1)** The angle $\varphi(x')$ is marked in the dashed region (in general it is not 90^o) around the corner denoted x' in the text; the marked corners represent x'' and x''' (with the first up and the latter on the left of x'); the shadowed region represents the intersection $\cap_{-N}^N S^{-j}Q_{\sigma_j}$, with $\sigma_0=\sigma'_0$.

A schematic geometric representation of the numerator of the second member of (4.3.19) is in the drawing in Fig. (4.3.1): the shadowed rectangle is precisely $C^{-N\ldots0\ldots N}_{\sigma_{-N}\ldots\sigma'_0\ldots\sigma_N}$. Note also that such a small rectangle has sides $\le\alpha\lambda^{+N}$, if α is the largest diameter of the elements of $\mathcal{Q}$ and λ is the hyperbolicity parameter entering in the definition of smooth hyperbolic system (cf. definition (4.2.1) in §4.2.)
N4.3.2 The area of the rectangle is [2]

e4.3.20
$$|\delta^x_u|\cdot|\delta^x_s|\sin\varphi(x)+\ \text{higher order infinitesimals},\qquad(4.3.20)$$

[2] By higher order infinitesimal one means $o(\lambda^{2N})$, whereas the first term in (4.3.20) is $O(\lambda^{2N})$.

where, see caption to Fig. (4.3.1), δ_u^x and δ_s^x are the segments $x'x''$ and $x'x'''$, $|\delta_u^x|$ and $|\delta_s^x|$ are the corresponding lengths (in the S–adapted metric considered here), and $\varphi(x)$ is, up to higher order infinitesimals, equal to the angle $\varphi(x')$ between them (always in the considered metric). By definition of m_0 (4.3.20) is also the m_0–measure of the cylinder $C^{-N\ \ldots 0\ \ldots\ N}_{\sigma_{-N}\ldots\sigma_0'\ldots\sigma_N}$.

One repeats an analogous argument for the cylinder $C^{-N\ldots\ 0\ldots\ N}_{\sigma_{-N}\ldots\sigma_0''\ldots\sigma_N}$ and, therefore, the limit (4.3.19) coincides with

e4.3.21
$$\frac{\sin\varphi(x)}{\sin\varphi(y)}\lim_{N\to\infty}\frac{|\delta_u^x|\cdot|\delta_s^x|}{|\delta_u^y|\cdot|\delta_s^y|}. \tag{4.3.21}$$

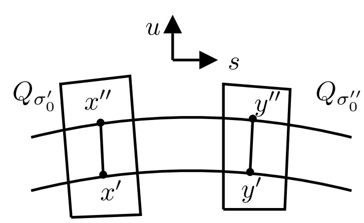
If we call x', x'' the extremes of δ_u^x and y', y'' the extremes of δ_u^y we see that x', x'' are on the same unstable boundary of $S^N Q_{\sigma_{-N}}$ and, likewise, y', y'' are on the same unstable boundary of $S^N Q_{\sigma_{-N}}$, see Fig. (4.3.2), although of course the segments δ_u^x and δ_u^x may be very far from each other as the first is inside $Q_{\sigma_0'}$ and the second inside $Q_{\sigma_0''}$ with $\sigma_0'\neq\sigma_0''$.

We can therefore write, for all k with $0\le k<N$,

e4.3.22
$$\frac{|\delta_u^x|}{|\delta_u^y|}=\frac{|S^{-k}S^k\delta_u^x|}{|S^{-k}S^k\delta_u^y|}=\frac{(\prod_{j=1}^k\lambda_u^{-1}(S^jx'))|S^k\delta_u^x|}{(\prod_{j=1}^k\lambda_u^{-1}(S^jy'))|S^k\delta_u^y|}, \tag{4.3.22}$$

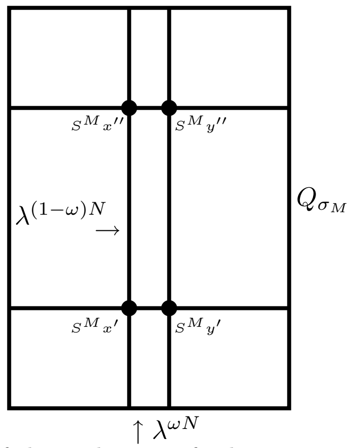
where x' and y' tend, respectively, to x and y for $N\to\infty$. We can remark that k is arbitrary, so that we conveniently choose it to be $k=\omega N$ with $\omega\in(0,1)$ to be fixed below.

Since the points S^jx' and S^jy' are on the same stable manifold they get close exponentially fast and their distance is $\le\lambda^j$, see Fig.(4.3.2).



F4.3.2 **Fig.(4.3.2)** The points x', y' and x'', y'' are, respectively, on the same stable manifolds because their symbols coincide at all sites different from 0 so that they are on the same "vertical" boundary of $S^N Q_{\sigma_{-N}}$. However they lie in different elements of the Markovian pavement (namely $Q_{\sigma_0'}$ and $Q_{\sigma_0''}$).

The Hölder continuity of $\lambda_u(x)$ therefore implies that the ratio of the products in (4.3.22) converges as $k=\omega N\to\infty$. The two arcs $S^{\omega N}\delta_u^x$ and $S^{\omega N}\delta_u^y$ become very close to each other and their distance is of the order of $\le O(\lambda^{\omega N})$ because they are on the same stable manifolds and initially they are far away at a distance of order 1. Considering that the initial length of δ_u^x and δ_u^y is $O(\lambda^N)$ we see that although their extreme points (x', x'' or y', y'') grow quite far from each other these segments stay short,



F4.3.3 **Fig.(4.3.3)** The result of the application of a large iterate $M=\omega N$ of S to the region x',x'',y',y'' in Fig. (4.3.2) if the segments $(x',x'')=\delta^x_u$ and $(y',y'')=\delta^y_u$ are short enough. The stable manifold arc x',y' becomes very short while the arcs x',x'' and y',y'' remain short because they were extremely close and, for large N, $\omega N = M \ll N$. The result is that the S^M image of the rectangle x',x'',y',y'' is small in all directions and if ω is chosen close enough to 1 (but <1) it will look like a rectangle which is very elongated vertically, *i.e.* in the unstable direction: $d(S^Mx',S^My'),d(S^Mx'',S^My'')\ll d(S^Mx',S^Mx''),d(S^My',S^My'')$. In the figure the two dimensions of the rectangle are drawn of the same order of magnitude and the rectangle is drawn so big for illustration purposes but it should be much smaller than the enclosing rectangle Q_{σ_M}.

i.e. $O(\lambda^{(1-\omega)N})$. Therefore we can find $0<\omega<1$ such that for $M=\omega N$ the picture is as in Fig.(4.3.3).

The angles between the four sides of the rectangle vary quite smoothly (*i.e.* Hölder continuously), therefore the ratio between the lengths $|S^M\delta^x_u|$ and $|S^M\delta^y_u|$ gets as close to 1 as wished for $N\to\infty$: this, together with a symmetric argument to study the ration $|\delta^x_s|/|\delta^y_s|$, yields the conclusion

e4.3.23
$$\lim_{N\to\infty}\frac{\delta^x_u}{\delta^y_u}=\prod_{k=1}^{\infty}\frac{\lambda_u^{-1}(S^kx)}{\lambda_u^{-1}(S^ky)},\qquad \lim_{N\to\infty}\frac{\delta^x_s}{\delta^y_s}=\prod_{k=1}^{\infty}\frac{\lambda_s(S^{-k}x)}{\lambda_s(S^{-k}y)}. \tag{4.3.23}$$

This proves property (i).

To show property (ii) one proceeds analogously and the details are omitted.

The statement (iii) is a consequence of (4.3.8) and (4.3.9) and of the explicit equations that are derived to obtain (4.3.12): one thus deduces naturally an expression for the f in (4.3.12) and its Hölder continuity. Details are again omitted: essentially identical arguments will be exposed in detail in Section §(5.3) in connection with a similar problem and the reader can refer to it. ■

A corollary of the above proof, in particular of the argument leading to (4.3.23), is the following version of Fubini's theorem

C4.3.1 **(4.3.1) Corollary:** (Adapted Fubini's theorem)
Let (M,S) be a two-dimensional Anosov map. Given $x_0\in M$ consider

the rectangle $R = W^s_\delta(x_0) \times W^u_\delta(x_0)$ *where* δ *is small enough so that the point* $[x,y]$ *is uniquely defined (*cf. *Fig.(4.2.1)). Then the volume of a subset* $E \subset R$ *is given by*

e4.3.24
$$\mu_0(E) = \int_{W^u_\delta(x_0)} d\sigma_y \int_{W^s_\delta(x_0)} d\sigma_x \cdot \\ \cdot \sin\alpha([y,x]) \prod_{i=1}^{\infty} \frac{\lambda_u^{-1}(S^{-i}[y,x])}{\lambda_u^{-1}(S^{-i}x)} \frac{\lambda_s(S^i[y,x])}{\lambda_s(S^i x)} \chi_E([y,x]), \tag{4.3.24}$$

where $d\sigma_y, d\sigma_x$ *are the area measures (i.e. arc length) on* $W^u_\delta(x_0), W^s_\delta(x_0)$ *and* $\alpha([y,x])$ *is the angle between the stable and unstable manifolds at* $[y,x]$.

This result in fact is general and it holds in any dimension. An interesting similar result will be mentioned in Section §(6.2).

Problems for §4.3

Q4.3.1 **[4.3.1]:** (*Markovian interval maps*)
Consider a continuous map S of $[0,1]$ into itself. Suppose that S is of class $C^{1+\varepsilon}$, $\varepsilon > 0$, in $[a_i, a_{i+1}] = Q_i$, $i = 0,\ldots,n-1$, where $a_0 = 0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = 1$. Suppose that $|S'(x)| \ge \lambda > 1$, for all $x \in [a_i, a_{i+1}]$ (at the extremes one should interpret S' as right or left derivative). Finally assume that S is "Markovian": for each $i = 0,\ldots,n$ there is $j(i)$ such that $Sa_i = a_{j(i)}$. Set $T_{\sigma\sigma'} = 0$ if $S(a_0, a_{\sigma+1}) \cap (a_{\sigma'}, a_{\sigma'+1}) = \emptyset$ and $T_{\sigma\sigma'} = 1$ otherwise. Show that the code that associates with $\underline{\sigma} \in \{0,\ldots,n-1\}^{\mathbb{Z}_+}_T$ the point

$$X(\underline{\sigma}) = \bigcap_{k=0}^{\infty} S^{-k} Q_{\sigma_k}$$

is Hölder continuous in the sense that $|X(\underline{\sigma}) - X(\underline{\sigma}')| < C\ d(\underline{\sigma},\underline{\sigma}')^\alpha$, with $C > 0$ and $\alpha = \log\lambda$.

Q4.3.2 **[4.3.2]:** Denote by $\varphi_\sigma : [Sa_\sigma, Sa_{\sigma+1}] \to [a_\sigma, a_{\sigma+1}]$ the inverse function of S on $[Sa_\sigma, Sa_{\sigma+1}]$ ("σ–th branch of the inverse S^{-1} of S"). Show that the function $\widehat{A}(\underline{\sigma}) = -\log|\varphi'_{\sigma_0}(X(\sigma_1\sigma_2\ldots))|$ is Hölder continuous on $\{0,\ldots,n-1\}^{\mathbb{Z}}_T$.

Q4.3.3 **[4.3.3]:** (*Coding of Lebesgue measure via a Markovian interval map*)
Show that the Lebesgue measure μ_0 on $[0,1]$ is coded by X into a measure m_0 on $\{0,\ldots,n-1\}^{\mathbb{Z}_+}_T$ such that

$$\frac{m_0(\sigma'_0\ldots\sigma'_n|\sigma_{n+1}\ldots)}{m_0(\sigma''_0\ldots\sigma''_n|\sigma_{n+1}\ldots)} = \exp\Big(-\sum_{k=0}^{\infty}[\widehat{A}(\tau^k\underline{\sigma}') - \widehat{A}(\tau^k\underline{\sigma}'')]\Big),$$

where $\tau(\sigma_0\sigma_1\ldots) = (\sigma_1\ldots)$. Note that the sum is, in fact, finite. (*Hint:* One repeats the argument leading to (4.3.23): however in this case it becomes much simpler.)

Q4.3.4 **[4.3.4]:** (*Expansive maps and Markovian pavements*)
Give a definition of Markovian pavement for an expansive map of a compact topological metric space inspired by the previous problems and by the definition (4.1.3). Here we say that S is expansive on Ω (a metric compact space) if there exist $\lambda < 1$ and $\varepsilon > 0$ such that $d(x,y) \le \lambda \cdot d(Sx,Sy)$, for every pair x, y such that $d(x,y) < \varepsilon$; and furthermore the equation $Sy = y'$ has only one solution y such that $d(x,y) < \varepsilon/2$ for every x such that $d(Sx,y') < \varepsilon/2$.

Q4.3.5 **[4.3.5]:** (*Interval maps and invariant absolutely continuous measures*)
Show, in the context of problem [4.3.3], that the condition for the existence of an S–invariant measure $\overline{\mu} = \overline{h}\mu_0$ absolutely continuous with respect to μ_0 is that there exists

a solution to the equation in $L_1(m_0)$

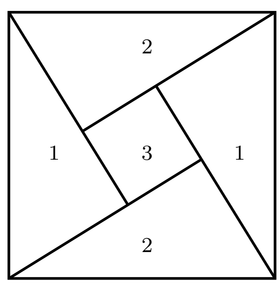
$$h(\sigma_0\sigma_1\sigma_2\ldots) = \sum_{\sigma'=0}^{n-1} e^{-\widehat{A}(\sigma'\sigma_0\sigma_1\ldots)} h(\sigma'\sigma_0\sigma_1\ldots)$$

m_0–a.e, if we set $h(\underline{\sigma}) = \overline{h}(X(\underline{\sigma}))$. (*Hint:* Write the condition $\overline{\mu}(E) = \overline{\mu}(S^{-1}E)$, that is

$$\overline{h}(x) = \sum_{\sigma'=0}^{n-1} |\varphi'_{\sigma'}(x)|\,\overline{h}(\varphi_{\sigma'}(x)) \qquad x \in L_1(\mu_0)$$

in the symbolic variables $\underline{\sigma}$.)

Q4.3.6 **[4.3.6]:** (*A non-generating Markovian pavement for the square root of Arnold's cat map*)
Consider the map of $\mathbb{T}^2$ into itself defined by the matrix $S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and let $\underline{v}^{\pm}$ be the two eigenvectors of S with components $(1, \lambda_+ - 1)$ and $(1, \lambda_- - 1)$ where $\lambda_{\pm} = (3 \pm \sqrt{5})/2$ are the eigenvalues of S. Construct the pavement formed by three rectangles with sides parallel to $\underline{v}^{\pm}$ as in Fig. (4.3.4).



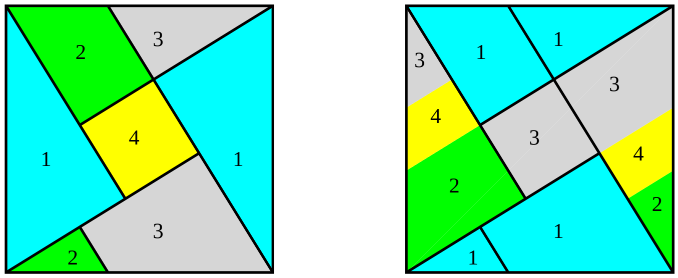
F4.3.4 **Fig.(4.3.4)** The pavement of problem [4.3.6] with three rectangles whose sides lie on two *connected* portions of stable and unstable manifold of the fixed point at the origin.

Check that it verifies the property (4.1.7). Show that nevertheless *it is not* a Markovian pavement according to definition (4.1.3) (see remark (5) after that definition). Show that the same pavement is also not Markovian for the map of $\mathbb{T}^2$ generated by the matrix $S_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and note that $S_0^2 \equiv S$. Compute the transition matrix for the latter map showing that it is

$$T = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

and that $T^5_{ij} > 0$ for all $i, j = 1, 2, 3$. (*Hint:* The property (4.1.7) follows simply because the unions of the boundaries consisting of stable or of unstable manifolds of the origin are connected and therefore invariant under S or S^{-1} respectively. It is not Markovian because the sets in (4.1.3) do not consist of a single point, so that the property (i) in definition (4.1.3) is not fulfilled. The image of the rectangle labeled 1 (for instance) crosses the rectangle labeled 2 in two sets with disconnected interiors.)
grestore

Q4.3.7 **[4.3.7]:** (*A generating Markovian pavement of the square root of Arnold's cat map*)
Consider the partition in Fig.(4.3.5) and show that it is a Markovian partition for the



F4.3.5 **Fig.(4.3.5)** A Markovian pavement (left) for the square root S_0 of Arnold's cat map. It is obtained from the partition in Fig. (4.3.4) by continuing a little further the stable manifold of the origin breaking into two parts the rectangle labeled 2 (whose large size was responsible for the non-Markovian nature of the pavement in Fig. (4.3.4)). The images under S_0 of the pavement rectangles is shown in the right figure: corresponding rectangle are marked by the same colors.

square root S_0 of Arnold's cat map with transition matrix

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

and check that $T^5_{\sigma\sigma'} > 0$ for all $\sigma, \sigma' = 1, 2, 3, 4$.

Q4.3.8 **[4.3.8]:** (*An example of a hyperbolic algebraic map on* $\mathbb{T}^3$)
The compatibility matrix of a Markovian pavement may define a hyperbolic algebraic map on a torus of dimension equal to the number of elements of the pavement. Check that this is the case for the matrix T in problem [4.3.6]. Check also that this is not the case for the matrix T, which is the compatibility matrix of a generating pavement, in problem [4.3.7]. (*Hint:* The characteristic equation for the eigenvalues of the matrix T of problem [4.3.6] is $\lambda^3 = \lambda + 1$ and the eigenvector corresponding to the largest eigenvalue λ (*spiral mean*) is $(1, \lambda, \lambda^{-1})$. This is a vector with rationally independent components, see problems [8.1.2], [8.1.4] and [8.1.4] below.)

Q4.3.9 **[4.3.9]:** (*A simple construction of Markovian pavements for two-dimensional Anosov systems*)
Show that the construction of Markovian pavements in two dimensional Anosov systems admitting a fixed point can be easily obtained by generalizing the construction in problem [4.3.6], *i.e.* by drawing a connected part of the stable and unstable manifolds of the fixed point and letting them "go around" until they form a net whose elements have a diameter smaller than a prefixed δ (using the density of the stable and unstable manifolds, see problem [4.2.18]) and stopping the drawing of the stable manifold when its extremes cross the unstable manifold and viceversa, as done in the illustration in Fig.(4.3.4).

Q4.3.10 **[4.3.10]:** By problem [4.2.18]) also the stable and unstable manifolds of a periodic point are dense. Furthermore Anosov systems admit a dense set of periodic points, as problem [4.2.13] shows. Show that this implies that the construction in problem [4.3.9] can be estended to Anosov systems which have no fixed point. (*Hint:* Let x_0 be a periodic point for S with period p. Then x_0 is a fixed point for the Anosov map S^p and the stable and unstable manifolds for S^p and for S are the same. Then we construct a Markovian pavement $\mathcal{P}_0$ with the method of problem [4.3.9] and, by the argument discussed in the proof of proposition (4.2.1), at item (C), $\cap_{j=0}^{p-1} S^j \mathcal{P}_0$ is a Markovian pavement for S.)

Bibliographical note to §4.3

The idea of using Markovian pavements to code symbolically topological measures associated with an algebraic hyperbolic map of the torus is due to Adler and Weiss, [AW68], and Sinai, [Si68a], [Si68b], [Si72], who proved and applied various versions of proposition (4.3.2). Another interesting application to the theory of systems endowed of an attractor that verifies the axiom A of Smale is in [Ru76]. Markovian maps of the interval have been studied by several authors, see for early contributions [Ru77], [PY79] and [CE80a].