CHAPTER I

General qualitative properties

§1.1 Historical note

We begin with a few comments on the meaning of the word *ergodic*.

At the beginnings of Statistical Mechanics a "collection of time evolution invariant probability distributions on phase space" or a "collection of *stationary distributions* on phase space" for a given Hamiltonian system, was called by Boltzmann, see [Bo09] and pag. 85 of [Bo96], a *monode*: the modern abbreviated locution describing this notion is a "statistical ensemble" (or simply an "element of a stationary ensemble").

The word *monode*, one among many coined by Boltzmann who clearly loved to invent this kind of words, is of greek origin: it is composed by μόνος , unique, and by εἶδος , aspect, appearance. Probably because this suggests the image of a collection of copies of the considered system which keeps, as a collection, its appearance as time evolves: each copy being subjected to a motion whose only global effect is a permutation of the various copies.

Boltzmann, then, called *ergode* a monode characterized by a uniform distribution on a surface of constant energy: a monode is *ergodic* if it is an ergode.

Hence it looks difficult or, better, impossible to attribute a literal meaning to *ergodic theory*, a locution that has not been used by Boltzmann. The origin of the word seems to go back to the Ehrenfests who, in the their important work of interpretation and popularization of Boltzmann's ideas, called "ergodic" any mechanical systems whose surfaces of constant energy consist of a single trajectory (which had been named, instead, *isodic* by

Boltzmann, from ἴσος , same, ὁδός , road, path), see footnote 93 in [EE59]. They called *quasi-ergodic* any mechanical system whose trajectories invaded densely the surface Σ_E of constant energy E, see footnote 98/99 in [EE59], and concluded with words of discomfort and doubt on the actual existence of systems endowed with the ergodic property, see pag. 25, line 15, of [EE59]. In this respect see also the footnote 97 in [EE59], where one cannot avoid being surprised by the depth of Boltzmann's intuitions. The meaning of these intuition, we feel, still escaped the Ehrenfests in 1912. Indeed Boltzmann chooses, in quoting an example of an ergodic system, a system which is still today considered a possible example. The work of the Ehrenfests generated renewed efforts, by many mathematicians, to interpret formally Boltzmann's ideas. Thus the true and proper ergodic theory began and the first fundamental result has been the classical formulation by Birkhoff of the *ergodic hypothesis*: a precise mathematical translation of a nice formula by Boltzmann, see formula (34) at pag. 25 of [EE59].

Such a hypothesis also called *metric transitivity hypothesis*, supposes that for "interesting" mechanical systems the trajectories of almost all points of the surface Σ_E invade it densely and, furthermore, spend in each of its parts $\Lambda \subset \Sigma_E$ a fraction of time proportional to its Liouville measure:

e1.1.1
$$\frac{\int_\Lambda \delta(H(\underline{p},\underline{q})-E)d\underline{p}d\underline{q}}{\int_{\Sigma_E} \delta(H(\underline{p},\underline{q})-E)d\underline{p}d\underline{q}} \tag{1.1.1}$$

having denoted with $\underline{p}$, $\underline{q}$ the canonical coordinates of the system and with $H(\underline{p},\underline{q})$ its Hamiltonian function.

Ergodic theory thus acquired the precise meaning of "theory of the ergodic hypothesis": *i.e.* it became the complex of mathematical propositions connected, or held as connected, with attempts of showing the validity (or the falsity) of the hypothesis in the case of various mechanical systems.

As a curiosity it can be interesting to note that, oddly, the Ehrenfests derive the ethimology of the word *ergode* from ἔργον , energy, and ὁδός , road, rather than from ἔργον and εἶδος , as it seems to be beyond doubt: by reading Boltzmann the word *ergode* comes out as a natural abbreviation of the word (more clearly explicative of the concept but lengthier hence, perhaps, less satisfactory) *ergomonode*, see footnote 93 in [EE59].

Mathematicians and physicists made efforts (with increasing vigor, particularly after the 1950's) directed to obtain a proof of the validity of the ergodic hypothesis in particular mechanical systems: although the efforts did not lead to a solution of the original problem, at least not in a generality relevant for applications, they led to extensions of the problem and to a vast class of mathematical results that encompass fields which at a first sight are rather inhomogeneous (like number theory or information theory). It is to this set of results, a mixture of results on diverse fields and in continuous expansion, that we refer today when we talk of ergodic theory which is, therefore, a name characterizing rather unprecisely each of its arguments.

In this book we shall select and illustrate several different problems that represent aspects of ergodic theory, trying to stress the common features that justify their classification as elements of a single mosaic.

Note to §1.1

(1) The use of the *axiom of choice* seems to have influenced in an essential way the interval of time that has been necessary for the correct mathematical formulation of the metric transitivity hypothesis: this appears quite clearly from the footnote 87 of the quoted book of the Ehrenfests, [EE59], that seems to consider as an insuperable obstacle to the otherwise natural formulation of the hypothesis a (probably unconscious) application of the axiom.

Indeed the $\sigma_0(p,q)$ of the quoted footnote would be, in a metrically transitive system, a nonmeasurable function whenever really varying from a G–path to another and it would not be possible to construct it without using the axiom of choice, that is therefore implicitly used, see problem [2.2.54] for an example of the construction of such a function along the argument of [EE59].

This instance is likely to be the only one in which the axiom of choice has exerted its sinister influence over a fundamental question of physical and applicative interest.

(2) A detailed critical analysis, and an exegesis in contemporary language, of Boltzmann's work would be very interesting. Until now such an enterprise has not been really undertaken, obviously because of the prohibitive amount of work that it implies. Nevertheless the literature on Boltzmann is large and rich of ideas and proposals for a deeper understanding, as the fascination of his personality allows anyone (even a profane) to predict, see [Ce99]. A first bibliography of studies on Boltzmann is given here although it is possibly seriously incomplete.

Bibliographical note to §1.1

On the foundations of statistical mechanics many of the original works are interesting. Among these we mention those of Boltzmann, Gibbs, Maxwell; see [Bo09], [Bo02], [Bo03],[Gi60], [Ma65]

Among the works of critique on the foundations see for instance Ehrenfest and Ehrenfest, [EE59], and Krylov, [Kr79]. More recent discussions can be found in [Br99], [Ga00].

The life and scientific achievements of Boltzmann are molded together in the books [Ce99], [Li01]: the first being more informative on the scientific aspects of Boltzmann's figure while the second gives a very clear picture of the aspects of his personality that made his life look to himself rather unhappy.

§1.2 Examples and some definitions

We begin our analysis by illustrating simple examples of "*dynamical systems*": dynamical systems, indeed, constitute the fundamental mathematical entity of ergodic theory, [AA68].

E1.2.1 *Example (1.2.1)* : Let $\underline{f} \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ be a function on $\mathbb{R}^d$ with values in $\mathbb{R}^d$ that we shall suppose to be uniformly bounded together with its first derivatives. We denote by $t \to \underline{x}(t) \equiv S_t(\underline{\xi})$ the solution of the differential equation

e1.2.1
$$\dot{\underline{x}} = \underline{f}(\underline{x}) \qquad \underline{x}(0) = \underline{\xi}\ , \qquad \underline{\xi} \in \mathbb{R}^d. \tag{1.2.1}$$

Then $(S_t)_{t\in\mathbb{R}}$ is a group of maps of class C^∞ of $\mathbb{R}^d$ into itself and $(\mathbb{R}^d, (S_t)_{t\in\mathbb{R}})$ is a dynamical system called *the flow on* $\mathbb{R}^d$ *generated by the differential equation (1.2.1)* or *by the vector field* $\underline{f}$.

Often $\underline{f}$ has other remarkable properties. For example sometimes $\underline{f}$ has *zero divergence*:

e1.2.2
$$\sum_{i=1}^d \frac{\partial f_i}{\partial \xi_i}(\underline{\xi}) = 0 \qquad \text{for all } \underline{\xi} \in \mathbb{R}^d, \tag{1.2.2}$$

if $\underline{f} = (f_1, \ldots, f_d)$. If λ denotes the Lebesgue measure on $\mathbb{R}^d$ one has, in the latter case,

e1.2.3
$$\lambda(E) = \lambda(S_t E) \qquad \text{for all } t \in \mathbb{R}, \tag{1.2.3}$$

for every λ–measurable set E: *i.e.* the map S_t *preserves the Lebesgue measure.*

Sometimes $\underline{f}$ generates a group that leaves a closed set Ω invariant. For instance this happens if Ω is a sphere or a torus or, more generally, a domain with C^∞–regular boundary $\partial\Omega$ and, denoting by $\underline{n}(\underline{\xi})$ the external normal to $\partial\Omega$ in $\underline{\xi}$,

e1.2.4
$$\underline{f}(\underline{\xi}) \cdot \underline{n}(\underline{\xi}) \le 0 \qquad \text{for all } \underline{\xi} \in \partial\Omega, \tag{1.2.4}$$

holds everywhere on $\partial\Omega$. In such a case $S_t(\Omega) \subset \Omega$, for all $t > 0$, and one can consider the *flow generated, for* $t \ge 0$, *by equation (1.2.1) in* Ω, that will be denoted $(\Omega, (S_t)_{t\in\mathbb{R}_+})$.

Invariant sets can also be constructed from *first integrals*, also called *constants of motion*, of equation (1.2.1), when they exist. Indeed if $F \in C^\infty(\mathbb{R}^d, \mathbb{R})$ is a first integral for equation (1.2.1), *i.e.* if $F(S_t\underline{\xi}) \equiv F(\underline{\xi})$, for all $\underline{\xi} \in \mathbb{R}^d$ and for all $t \in \mathbb{R}$, given $\gamma \in \mathbb{R}$ the sets

e1.2.5
$$\Omega_\gamma^< = \{\underline{\xi} | F(\underline{\xi}) < \gamma\},\ \Omega_\gamma^> = \{\underline{\xi} | F(\underline{\xi}) > \gamma\},\ \Omega_\gamma = \{\underline{\xi} | F(\underline{\xi}) = \gamma\} \tag{1.2.5}$$

are obviously invariant under the flow generated by equation (1.2.1).

From example (1.2.1) that we have just discussed one derives as particular cases, or as generalizations, the examples below (1.2.2) (Hamiltonian flows) or (1.2.3) (flows on differentiable manifolds), respectively.

E1.2.2 *Example (1.2.2)* : (*Hamiltonian flows*) Let $H \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$ and $E \in \mathbb{R}$ be such that

(i) $\Omega_E^{\leq} = \{(\underline{p}, \underline{q}) | (\underline{p}, \underline{q}) \in \mathbb{R}^{2N},\ H(\underline{p}, \underline{q}) \leq E\}$ is a bounded set.
(ii) $\Omega_E = \{(\underline{p}, \underline{q}) | (\underline{p}, \underline{q}) \in \mathbb{R}^{2N},\ H(\underline{p}, \underline{q}) = E\}$ is a bounded smooth surface of dimension $2N - 1$ with no *equilibrium points*, *i.e.* with no points such that $\underline{\partial} H = 0$, where $\underline{\partial}$ denotes the gradient.

Under such conditions the Hamilton equations

e1.2.6
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad i = 1, \ldots, N, \tag{1.2.6}$$

allow us to define a group $(S_t)_{t \in \mathbb{R}}$ of maps of class C^∞ on $\mathbb{R}^{2N}$ such that for all $E \in \mathbb{R}$:
(1) the sets $\Omega_E^{\leq}$ and Ω_E are invariant.
N1.2.1 (2) the maps preserve the Lebesgue measure λ restricted to $\Omega_E^{\leq}$, or to Ω_E.[1]

The *dynamical systems* $(\Omega_E^{\leq}, (S_t)_{t \in \mathbb{R}})$, $(\Omega_E, (S_t)_{t \in \mathbb{R}})$ are called *Hamiltonian flow* with energy $\leq E$ or with fixed energy E, respectively.
The hypotheses (i) and (ii) allow us to define the normalized Lebesgue measures $\overline{\lambda}_E$ on $\Omega_E^{\leq}$ or on Ω_E obtained by normalizing to 1 the restriction of the Lebesgue measure to $\Omega_E^{\leq}$ or to Ω_E: the normalization factor turns out to be, indeed, $< +\infty$.

The pairs $(\Omega_E^{\leq}, \overline{\lambda}_E)$ or $(\Omega_E, \overline{\lambda}_E)$ are, in the language of probability theory, *probability distributions* on $\Omega_E^{\leq}$ and on Ω_E. Such distributions are invariant with respect to the Hamiltonian flows on $\Omega_E^{\leq}$ or on Ω_E, because the Hamilton equations have zero divergence. When considered together with the corresponding Hamiltonian flows, they are denoted with the symbol $(\Omega_E^{\leq}, \overline{\lambda}_E^{\leq}, (S_t)_{t \in \mathbb{R}})$ or $(\Omega_E, \overline{\lambda}_E, (S_t)_{t \in \mathbb{R}})$, respectively, and their collections, as the energy E varies, constitute two examples of *invariant statistical ensembles* or *stationary ensembles*. In Boltzmann's nomenclature they are an example of a *monode* (the first collection) and of *ergode* (the second), see Sec.1.1.

E1.2.3 *Example (1.2.3)* : Let V be a compact differentiable Riemannian manifold of class C^∞ and let $\underline{f}$ a vector field tangent to V. The differential equation

e1.2.7
$$\dot{\underline{x}} = \underline{f}(\underline{x}), \qquad \underline{x}(0) = \underline{\xi}, \tag{1.2.7}$$

[1] The Lebesgue measure restricted to Ω_E is, the measure $\lambda_E(d\underline{p}d\underline{q}) = \delta(H(\underline{p}, \underline{q}) - E)d\underline{p}d\underline{q} = \frac{d\sigma_E}{|\underline{\partial} H|}$ if $d\sigma_E$ is the surface element on Ω_E. This is locally finite in all dimensions $N > 1$ if $\underline{\partial} H$ does not vanish to a too high order at the equilibrium points, if any. See also Appendix (1.2).

allows us to define a group of maps $(S_t)_{t\in\mathbb{R}}$ of V into itself as we did in the example (1.2.1). Such maps are of class C^∞ and the pair $(V,(S_t)_{t\in\mathbb{R}})$ will be the *flow generated on* V *by* (1.2.7).

We shall say that $(S_t)_{t\in\mathbb{R}}$ preserves the volume on V if the volume measure μ on V is invariant under the action of S_t, *i.e.* if for each $E\in\mathcal{B}(V)=\{$Borel sets of $V\}$ (see Appendix (1.2)) one has

e1.2.8
$$\mu(E)=\mu(S_tE)\qquad \text{for all}\, t\in\mathbb{R}. \tag{1.2.8}$$

The measure μ will be always considered normalized (note that the volume of V is certainly finite) and the probability distribution (V,μ) will be a stationary distribution for the flow $(V,(S_t)_{t\in\mathbb{R}})$ when (1.2.8) holds. In the last case μ will be called a *stationary distribution* for the flow $(V,(S_t)_{t\in\mathbb{R}})$.

A concrete case associated with the above example is described in the following one.

E1.2.4 *Example (1.2.4)* : (*Quasi-periodic flows*) Let $\mathbb{T}^d$ be the standard d–dimensional torus (*i.e.* $\mathbb{T}^d$ is $[0,2\pi]^d$ with "opposite sides identified" or, more precisely, $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$) and consider the differential equation on $\mathbb{T}^d$

e1.2.9
$$\underline{\dot\varphi}=\underline{\omega}, \tag{1.2.9}$$

where $\underline{\omega}\in\mathbb{R}^d$. It defines the group $(S_t)_{t\in\mathbb{R}}$:

e1.2.10
$$S_t\underline{\varphi}=\underline{\varphi}+\underline{\omega}t=(\varphi_1+\omega_1t,\ldots,\varphi_d+\omega_dt)\ \mathrm{mod}\ 2\pi \tag{1.2.10}$$

The flow $(\mathbb{T}^d,(S_t)_{t\in\mathbb{R}})$ is called a *rotation flow* or a *quasi-periodic flow* of the torus $\mathbb{T}^d$ with *velocities* $\underline{\omega}\in\mathbb{R}^d$. It preserves the volume measure λ on $\mathbb{T}^d$ (thought of as a flat Riemannian manifold with the natural metric)

e1.2.11
$$\lambda(d\underline{\varphi})=\frac{d\underline{\varphi}}{(2\pi)^d}. \tag{1.2.11}$$

If $\underline{\rho}\in\mathbb{R}^d$ the map

e1.2.12
$$S\underline{\varphi}=\underline{\varphi}+\underline{\rho}=(\varphi_1+\rho_1,\ldots,\varphi_d+\rho_d)\ \mathrm{mod}\ 2\pi \tag{1.2.12}$$

will be called a *rotation map* of $\mathbb{T}^d$ with *rotation vector* $\underline{\rho}$. Obviously $S=S_1$ if $(S_t)_{t\in\mathbb{R}}$ is the rotation flow of $\mathbb{T}^d$ with velocities $\underline{\rho}$.

Many important examples of maps S of manifolds do not correspond to differential equations on the same manifold (in the sense that $S=S_1$ if $(S_t)_{t\in\mathbb{R}}$ is the flow associated with a suitable differential equation). A classical case
N1.2.2 is the following.[2]

[2] Another one which is widely studied in literature is the *standard map* which will be discussed in Sec. (9.2).

E1.2.5 *Example (1.2.5)* : (*Arnold's cat map*) Let S be a $d \times d$ matrix with integer entries and define, for all $\underline{\varphi} \in \mathbb{T}^d$,

e1.2.13
$$(S\underline{\varphi})_i = \sum_{j=1}^{d} S_{ij}\varphi_j \mod 2\pi, \qquad i = 1, \ldots, d. \tag{1.2.13}$$

Then S maps the torus $\mathbb{T}^d$ into itself and is of class C^∞. Furthermore, if $\det S = \pm 1$ then S is invertible and preserves the volume.

In general even if $\det S = 1$, it is not possible to find a differential equation on $\mathbb{T}^d$ generating a flow $(S_t)_{t\in\mathbb{R}}$ that interpolates S (*i.e.* such that $S_1 = S$).

A typical and important example is provided by the map of $\mathbb{T}^2$ associated, via (1.2.13), with the matrix:

e1.2.14
$$S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}; \tag{1.2.14}$$

see problem [1.2.9].

If the map S defined by (1.2.13) has $\det S \neq \pm 1$ then it is not invertible: every $\underline{\psi} \in \mathbb{T}^d$ is an image of $|\det S|$ pairwise distinct points φ of $\mathbb{T}^d$. In this case the pair $(\mathbb{T}^d, S)$ is a *noninvertible dynamical system.*

E1.2.6 *Example (1.2.6)* : (*Lorenz' equation*) Noninvertible systems have usually origin in connection with the theory of equations, or maps, that model dissipative phenomena. Consider the equation in $\mathbb{R}^3$

e1.2.15
$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= -\sigma x - y - xz, \\ \dot{z} &= -bz + xy - \alpha; \end{aligned} \tag{1.2.15}$$

then the solutions of this equation exist globally for $t \geq 0$ for every initial datum if, as we shall suppose, σ, b, $\alpha > 0$. If $(S_t)_{t\geq 0}$ is the semigroup that solves the (1.2.15) it is, furthermore, true that the sphere

e1.2.16
$$\Omega_0 = \Big\{\underline{\xi} \mid \underline{\xi} \in \mathbb{R}^3,\ |\underline{\xi}| \leq \rho_0 = \frac{2\alpha}{\min\{1, \sigma, b\}}\Big\} \tag{1.2.16}$$

is invariant (as a consequence of (1.2.4), see also problem [1.2.11]): $S_t\Omega_0 \subset \Omega_0$, for all $t \geq 0$. The pair $(\Omega_0, (S_t)_{t\geq 0})$ is a noninvertible dynamical system or a noninvertible flow on Ω_0, because S_t fails to be surjective from Ω_0 to Ω_0. Note however that S_t is invertible as a map from $\mathbb{R}^3$ into itself, see problem [1.2.11].

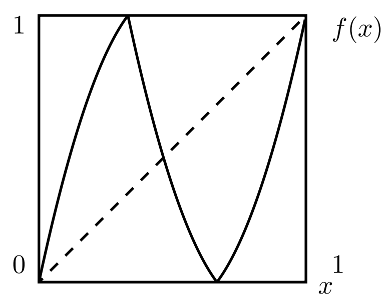
Note that the divergence of the second member of (1.2.15) is $-(1+b+\sigma) < 0$ and, hence, the measure of every (measurable) set A contracts according to

e1.2.17
$$\lambda(S_t A) = e^{-(1+b+\sigma)t}\lambda(A) \qquad t \geq 0 \tag{1.2.17}$$

i.e. the Lebesgue measure is not invariant under the flow $(S_t)_{t\ge0}$ and there
N1.2.3 can be no invariant measure on Ω_0 equivalent to it.[3] If a measure μ is absolutely continuous with respect to the Lebesgue measure or with respect to the volume measure on a manifold one often omits the reference measure and one says simply that μ is absolutely continuous.

The equation (1.2.15) is usually written using $x, y, z + r + \sigma$, instead of x, y, z, and is called *Lorenz' equation*, see equations (25),(26),(27) in [Lo63].

E1.2.7 *Example (1.2.7)* : (*Interval maps*) Another example of noninvertible dynamical system is provided by a continuous, piecewise C^∞, map $S : [0, 1] \to [0, 1]$ with a graph of the form



F1.2.1 **Fig.(1.2.1)** : *A map of the interval* $[0, 1]$.

In general the map S will not be invertible and the pair $([0, 1], S)$ will form, therefore, a noninvertible dynamical system.

The above are examples of the mathematical entities called *dynamical systems*. Because of their obvious interest, they motivate the following definitions that summarize and put in abstract form their main properties.

D1.2.1 **(1.2.1) Definition:** (Topological dynamical systems and metric dynamical systems)
Let Ω be a compact separable metric space and let S be a continuous map of Ω into itself. The pair (Ω, S) will be said a discrete topological dynamical system on Ω. *(Ω, S) will be said* invertible *if S has a continuous inverse S^{-1}.*

Let μ be a complete probability measure defined on a σ–algebra $\mathcal{B}$ of sets of Ω and let $N \in \mathcal{B}$ be a set with zero μ–measure, and suppose that S is a map measurable with respect to $\mathcal{B}$ outside N, see Appendix (1.2). If

e1.2.18 $$\mu(A) = \mu(S^{-1}A) \qquad \textit{for all } A \in \mathcal{B} \cap N^c, \tag{1.2.18}$$

*where $N^c = \Omega/N$ is the complement of N, we shall say that (Ω, S) is μ–*preserving *or that μ is* S–invariant. *The triple (Ω, S, μ) will be called a*

[3] Given two measures μ and ν defined on the same σ–algebra $\mathcal{B}$, see Appendix (1.2), ν is absolutely continuous with respect to μ if there exists $f \in L_1(\mu)$ such that $\nu = f\mu$. The measures μ and ν are equivalent if μ is absolutely continuous with respect to ν and ν is absolutely continuous with respect to μ.

(discrete) metric dynamical system mod 0 *defined outside* N. *If* $N = \emptyset$ *the triple* (Ω, S, μ) *will be called a* (discrete) metric dynamical system.

Often the set N of "singularities of S" is not explicitly mentioned and one simply talks of a metric dynamical system (Ω, S, μ). We shall try to avoid this practice because of the risks of confusion it generates. Analogously:

$D1.2.2$ **(1.2.2) Definition:** (Topological flows and metric flows)
Let Ω *be a compact separable metric space. Let* $(S_t)_{t\in\mathbb{R}}$ *or* $(S_t)_{t\in\mathbb{R}_+}$ *be a group or a semigroup, homomorphic to* $\mathbb{R}$ *or to* $\mathbb{R}_+$*, of maps that act with*
$N1.2.4$ *continuity on* Ω.[4]

The pair $(\Omega, (S_t)_{t\in\mathbb{R}})$ *or* $(\Omega, (S_t)_{t\in\mathbb{R}_+})$ *will be called respectively an* invertible topological flow *or a* topological flow *on* Ω.

Let μ *be a complete probability measure on a* σ*–algebra* $\mathcal{B}$ *and let* $(S_t)_{t\in\mathbb{R}}$ *be a group of* μ*–measurable maps homomorphic to* $\mathbb{R}$ *and preserving* μ*. Suppose that the function* $(t, x) \to S_t x$ *defined on* $\mathbb{R} \times \Omega$ *with values in* Ω *is* μ*–measurable with respect to the* σ*–algebra generated by the sets in* $\mathcal{B}(\mathbb{R}) \times \mathcal{B}$*. Then the flow* $(\Omega, (S_t)_{t\in\mathbb{R}}, \mu)$ *will be called a* invertible metric flow *on* Ω.

If, mutatis mutandis, we replace $(S_t)_{t\in\mathbb{R}}$ *by a semigroup* $(S_t)_{t\in\mathbb{R}_+}$, $\mathcal{B}(\mathbb{R}_+)\times$ $\mathcal{B}$*–measurable, we obtain the notion of* metric flow.

Flows will also be called, sometimes, *continuous dynamical systems* because their time "elapses continuously".

Speaking of dynamical systems one often omits the qualifications (topological, metric, discrete, continuous, invertible,...) that are usually supposed to be understandable from the context. One can (generously) even consider the (here seldom used) notions of *abstract discrete dynamical system* (Ω, S) with Ω being a "space" and S a "map" acting on it, as well as the similar notion of *abstract flow*.

Before proceeding to analyze the structure of certain classes of dynamical systems it is convenient to set up the notion of *isomorphism* between dynamical systems.

$D1.2.3$ **(1.2.3) Definition:** (Isomorphisms)
If (Ω, S) *and* (Ω', S') *are two abstract discrete dynamical systems, we shall say that they are* isomorphic*, or* conjugated*, when there exists an invertible map* $I : \Omega \leftrightarrow \Omega'$ *such that*

$e1.2.19$
$$IS = S'I. \tag{1.2.19}$$

If the systems (Ω, S) *and* (Ω', S') *are discrete topological dynamical systems we shall say that they are topologically isomorphic if they are isomorphic and if the isomorphism* I *can be chosen bicontinuous.*

Let (Ω, S, μ) *and* (Ω', S', μ') *be two discrete metric dynamical systems mod* 0*, defined outside the sets* N, N' *of zero* μ, μ' *measures and defined on the* σ*–algebras* $\mathcal{B}_\mu$ *and* $\mathcal{B}_{\mu'}$ *respectively. If* $(\Omega/N, S)$ *and* $(\Omega'/N', S')$ *are*

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[4] This means that the functions $(t, x) \to S_t x$ of $\mathbb{R} \times \Omega$ or of $\mathbb{R}_+ \times \Omega$ into Ω are continuous.

isomorphic and if I can be chosen bimeasurable outside N, N' with respect to the σ–algebras $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu'}$ and if, furthermore,

e1.2.20

$$\mu'(IA) = \mu(A) \qquad \text{for all } A \in \mathcal{B}_\mu \tag{1.2.20}$$

we shall say that the systems are isomorphic mod 0.

In the latter case there exist two zero measure sets $N \in \mathcal{B}$ and $N' \in \mathcal{B}'$ such that $S^{-1}N = SN = N$ and $S'^{-1}N' = S'N' = N'$ and furthermore $(\Omega/N, S, \mu)$ and $(\Omega'/N', S', \mu')$ are isomorphic in the sense that there is a bimeasurable map I of Ω/N to Ω'/N' verifying $IS = S'I$ on such sets and mapping μ onto μ'. One can formulate, in an analogous fashion, various notions of isomorphism between flows.

Appendix 1.2: *Basic definitions of measure theory*

We recall here some basic defintitions of measure theory. A collection of sets among which is the empty set and that is closed under the operations of complementation and of finite union is called an *algebra*; if it is also closed under countable union is called a *σ–algebra*. The smallest σ–algebra containing a given family $\mathcal{F}$ of sets is called the σ–algebra generated by $\mathcal{F}$. The σ–algebra generated by the open sets of a topological space Ω is called the *Borel σ–algebra* $\mathcal{B}(\Omega)$ and its elements are called *Borel sets*. A *measure* μ on a σ–algebra $\mathcal{B}$ is a countably additive non-negative function of the sets of $\mathcal{B}$, which are called *μ–measurable* sets. Given a σ–algebra $\mathcal{B}$ and a measure μ we can consider the σ–algebra $\mathcal{B}_\mu$ generated by $\mathcal{B}$ and all the sets contained in a μ–measurable set with zero μ measure. The σ–algebra $\mathcal{B}_\mu$ is called the *μ–completion* of $\mathcal{B}$. A measure μ can always be extended uniquely to a measure on the σ–algebra $\mathcal{B}_\mu$. If $\mathcal{B} = \mathcal{B}_\mu$ the measure μ is called *complete*. A measure μ which is defined on the Borel sets of a topological space is called a *Borel measure*, while a measure that is the completion of its restriction to the Borel sets is called a *complete Borel measure*.

The usual Riemann measure on $\mathbb{R}^d$ is defined over the algebra of sets which are approximable from inside and from outside by unions of rectangles with an arbitrarily small volume in between. It is not countably additive but it can be extended to the Borel sets and then completed defining in this way the Lebesgue measure. On Riemannian manifolds the Riemann measure is defined in the same way by using local charts and the volume form generated by the metric. The usual Lebesgue measure and, more generally, the volume measures λ on Riemannian manifolds will be regarded as complete measures defined on the Borel sets $\mathcal{B}(\Omega)$ and extended to $\mathcal{B}_\lambda(\Omega)$, see problem [1.2.19].

Given a function S from Ω to Ω' the inverse image $S^{-1}(E)$ of a set $E \subset \Omega'$ is the set of all $x \in \Omega$ such that $S(x) \in E$. A function (map) S between two spaces Ω and Ω' is measurable with respect to the σ–algebras $\mathcal{B}$ and $\mathcal{B}'$, defined respectively on Ω and Ω', if the inverse image of every set $E \in \mathcal{B}'$ is an element of $\mathcal{B}$. A function S measurable with respect to the completion $\mathcal{B}_\mu$ of $\mathcal{B}$ relative to a measure μ is called μ–measurable. A function S is called *bimesurable* if S^{-1} exists and if both S and S^{-1} are measurable. If $\Omega = \Omega'$ and $\mathcal{B} = \mathcal{B}'$ we simply say that S is measurable with respect to $\mathcal{B}$. Given a set N in $\mathcal{B}$ we call $\mathcal{B}(\Omega) \cap N^c$ the σ-algebra generated by the sets $A\underline{\nu}$ for $A \in \mathcal{B}$. If N, N' are sets in $\mathcal{B}, \mathcal{B}'$ and the map S of $\Omega \setminus N$ into $\Omega' \setminus N'$ is such that $S^{-1}E' \in \mathcal{B}(\Omega) \cap N^c$ for all $E' \in \mathcal{B}(\Omega') \cap N'^c$ then S is said to be measurable with respect to $\mathcal{B}, \mathcal{B}'$ outside N, N'. If both S and S^{-1} are defined and both are $\mathcal{B}, \mathcal{B}'$ measurable outside N, N' then the map S is said *bimeasurable* with respect to $\mathcal{B}, \mathcal{B}'$ outside N, N'.

Problems for §1.2

Q1.2.1 **[1.2.1]:** Show that the differential equation (1.2.1) admits, as stated in example (1.2.1), global solutions under the assumptions considered there.

Q1.2.2 **[1.2.2]:** (*Liouville theorem*)
Show that equation (1.2.2) implies equation (1.2.3).

Q1.2.3 **[1.2.3]:** Show that equation (1.2.4) implies the invariance of Ω.

Q1.2.4 **[1.2.4]:** Construct an example of a non-Hamiltonian differential equation in $\mathbb{R}^2$ possessing first integrals in the sense of example (1.2.1), in $C^\infty(\mathbb{R}^2)$.

Q1.2.5 **[1.2.5]:** Show that $\dot{\underline{x}} = -\underline{x}$ is a differential equation with no non-constant C^∞ first integrals.

Q1.2.6 **[1.2.6]:** Consider a differential equation in $\mathbb{R}^d$ of the type considered in example (1.2.1) and suppose that for every $\underline{x} \in \mathbb{R}^d$ the limit $\lim_{t\to+\infty} S_t\underline{x}$ exists and takes only a finite number of values as $\underline{x}$ varies in $\mathbb{R}^d$. Show that the differential equation does not admit non-constant C^∞ first integrals.

Q1.2.7 **[1.2.7]:** Find some other criterion of non-existence of first integrals for a differential equation inspired by that of problems [1.2.5], [1.2.6].

Q1.2.8 **[1.2.8]:** Show that, in the case of equation (1.2.13), if $\det S = \pm 1$ then S is invertible and it is of class C^∞ together with its inverse.

Q1.2.9 **[1.2.9]:** (*A map not embeddable into a flow*)
Show the non-existence of a differential equation on the torus $\mathbb{T}^2$ such that $S_1 = S$ where S is the map of $\mathbb{T}^2$ into itself defined by (1.2.14). (*Hint:* Show that if $\dot{\underline{\varphi}} = \underline{f}(\underline{\varphi})$ were such an equation we would have: $\sum_j S_{ij} f_j(\underline{\varphi}) \equiv f_i(S\underline{\varphi})$ and, more generally: $\sum_j (S^n)_{ij} f_j(\underline{\varphi}) \equiv f_i(S^n\underline{\varphi})$ for all $n \in \mathbb{Z}$. This is absurd. In fact since the matrix S is Hermitian with eigenvalues $(3 \pm \sqrt{5})/2$, one greater and one smaller than 1, $\limsup_{|n|\to\infty} |S^n\underline{v}| = +\infty$, for all $\underline{v} \in \mathbb{R}^2 \setminus \{\underline{0}\}$, while $|\underline{f}(S^n\underline{\varphi})| \le \max_{\underline{\psi}} |\underline{f}(\underline{\psi})| < \infty$).

Q1.2.10 **[1.2.10]:** (*Toral maps*)
Show that if S is a matrix with integer entries and if $N = |\det S| \neq 0$ then S is a continuous map of the torus into itself and every point of $\mathbb{T}^d$ is the image of N points of $\mathbb{T}^d$. Furthermore S is of class C^∞ on $\mathbb{T}^d$.

Q1.2.11 **[1.2.11]:** (*Lorenz' equation*)
Show that Lorenz' equation (of example (1.2.6)) admits global solutions (in the past and in the future); and show that Ω_0 is invariant only in the future. (*Hint:* Multiply scalarly the (1.2.15) by (x, y, z) and deduce an *a priori* bound on the solutions of (1.2.15) by noting that the cubic terms obtained in this way disappear. Use criterion (1.2.4) to conclude.)

Q1.2.12 **[1.2.12]:** (*Expanding interval maps*)
Let $Sx = 2x$, $0 \le x \le 1/2$, $Sx = 2(1-x)$, $1/2 \le x \le 1$, cf. example (1.2.7). Show that

the Lebesgue measure λ on $[0,1]$ is S–invariant ($\lambda(E)=\lambda(S^{-1}E)$, for all $E\in\mathcal{B}([0,1])$).

Q1.2.13 **[1.2.13]:** (*The equation for invariant density*)
Consider a continuous piecewise smooth (in general noninvertible) map T of $[0,1]$ and suppose that the measure $\mu_T(dy)=f(y)dy$, absolutely continuous with respect to the Lebesgue measure, is T–invariant, see definition (1.2.1): then the equation for its density is $f(y)=\sum_{z:\,Tz=y}|T'(z)|^{-1}f(z)$.

Q1.2.14 **[1.2.14]:** Let S and T be two smooth maps of $[0,1]$ which are *conjugated* or *isomorphic*, *i.e.* such that there exists an invertible $C^1([0,1])$ map $F:[0,1]\to[0,1]$ such that $Ty=FSF^{-1}y$, and suppose that $\mu_T(dy)=f(y)dy$ is T–invariant (see problem [1.2.13]). Check that setting $\mu_S(E)=\mu_T(F(E))$, *i.e.* $\mu_S(dx)=f(F(x))|F'(x)|dx$, yields an S–invariant measure μ_S.

Q1.2.15 **[1.2.15]:** (*Ulam–von Neumann map*)
Show that the Ulam-von Neumann map $x\to T(x)=4x(1-x)$ and the map $x=Sx$, where S is defined as in problem [1.2.12], are conjugated via the map $y=F(x)=(2/\pi)\arcsin\sqrt{x}$. The map $T(x)=ax(1-x)$, with $a\in\mathbb{R}$, is sometimes referred to as the *logistic map*.

Q1.2.16 **[1.2.16]:** It is known that every compact C^∞ Riemannian manifold V can be smoothly immersed into an Euclidean space of suitably large dimension n. Denoting by Y such an immersion show that there exists a differential equation $\dot{\underline{\xi}}=\underline{\varphi}(\underline{\xi})$ in $\mathbb{R}^n$, of class C^∞, that admits the manifold $\widetilde V=Y(V)$ as an invariant manifold and that induces on $\widetilde V$ a flow $(\widetilde S_t)_{t\in\mathbb{R}}$ that is the image of the flow on V generated by equation (1.2.7) so that, denoting the latter by $(S_t)_{t\in\mathbb{R}}$, it is $\widetilde S_tY\underline{x}=YS_t\underline{x}$, $t\in\mathbb{R}$, $\underline{x}\in V$.

Q1.2.17 **[1.2.17]:** (*Geodesic flow*)
Let V be a C^∞ Riemannian manifold, let g be its metric tensor and let W be the *tangent bundle*, *i.e.* the manifold consisting of the points (x,v) with $x\in V$ and with v tangent to V in x. Define on W the *geodesic flow* $(S_t)_{t\in\mathbb{R}}$ that associates with (x,v) the point $S_t(x,v)=(x_t,v_t)$ obtained by constructing the geodesic that starts in x in the direction v and by running on it with uniform velocity given by $|v|=\sqrt{\sum_{ij}g_{ij}v^iv^j}$ on a portion of length $|v|\,t$ (in the metric g) reaching in this way a point x_t with velocity v_t.

It is convenient to describe the geodesic flow as a flow on the space $\widehat W$ of the pairs (x,p) of points of V and of vectors *cotangent* to V in x. Recall that the vector p cotangent in x and corresponding to the vector v tangent in x is, by definition:

$$p_i=\sum_{j=1}^d g_{ij}(x)v^j,\qquad i=1,\ldots,d.$$

The geodesic flow on $\widehat W$ is naturally described by the correspondence $(x,p)\to(x_t,p_t)$ built by
(1) starting with $(x,p)\in\widehat W$ and constructing, via the preceding construction, the vector v tangent in x, namely $v=g^{-1}(x)p$,
(2) associating with (x,v) the point $S_t(x,v)=(x_t,v_t)$ and then
(3) defining $(p_t)_i=\sum_{j=1}^d g_{ij}(x_t)v_t^j$.

A celebrated proposition of geometry and of mechanics states that the geodesic flow is described on $\widehat W$, in every chart, by the Hamiltonian differential equations with Hamiltonian

$$H(x,p)=\frac12\sum_{j=1}^d(g^{-1}(x))^{ij}p_ip_j.$$

Deduce that the geodesic flow on W preserves the measure that on $\widehat W$ coincides with the Lebesgue measure $dx\,dp$. Deduce that the geodesic flow on W conserves a measure that is absolutely continuous with respect to the volume measure on W considered as a Riemannian manifold with the natural metric $\Big(\sum_{j=1}^{d}(g_{ij}(x)dx^i dx^i + g_{ij}(x)dv^i dv^j)\Big)^{\frac12}$ and express the density with respect to the measure that in a natural chart for W becomes the measure $dx\,dv$. Find its expression in terms of the metric tensor.

Q1.2.18 **[1.2.18]:** Deduce the differential equation of the geodesics on V and hence that of the geodesic flow on W and $\widehat W$ by using the definition of geodesic as an extremal curve for the line element $ds^2 = \sum_{i,j} g_{ij}(x)dx^i dx^j$ and the principles of Mechanics. (*Hint:* The geodesic flow takes place at constant speed and takes place over curves that minimize or make stationary $\int_{P_1}^{P_2}\sqrt{\sum_{i,j} g_{ij}dx^i dx^j}$ over all lines joining P_1, P_2.).

Q1.2.19 **[1.2.19]:** (*Incompleteness of the restriction of Lebesgue measure to Borel sets*)
Assuming that there exist sets in $[0,1]$ that are not Lebesgue measurable show that there exist sets in $\mathbb{R}^2$ which are not Borel sets but that are contained in sets of 0–Lebesgue measure. Show in fact that if L is a not Lebesgue–measurable set of the segment $[0,1]$ then the set $L\times\{0\}\subset[0,1]\times[0,1]$ is an example of a set which is not a Borel set of the square $[0,1]^2$ and it is nevertheless contained inside the set $[0,1]\times\{0\}$ with 0 Lebesgue measure. (*Hint:* Just show that $L\times\{0\}$ cannot be a Borel set as a subset of the square $[0,1]\times[0,1]$: if so it could be obtained by a transfinite induction via operations of countable unions and intersections starting from open sets of the square. But the same transfinite construction performed with the intersections of the open sets with the segment $[0,1]\times\{0\}$ would lead to the set L which would therefore be a Borel subset of $[0,1]$.)

Bibliographical note to §1.2

Examples and definitions are taken from [AA68], [Si77], [Av76] where several other examples can be found.

§1.3 Harmonic oscillators and integrable systems as dynamical systems

We shall consider a few simple dynamical systems which provide us with a first illustration of the just introduced notions and establish a relation between the examples (1.2.2), (1.2.3), (1.2.4) of Section §1.2.

P1.3.1 **(1.3.1) Proposition:** (Harmonic oscillations)
Let $G=(g_{ij})$, $V=(v_{ij})$ be two positive definite symmetric matrices and let $G^{-1}=(g^{-1}_{ij})$ be the inverse matrix of G. Consider the function

e1.3.1
$$H = H(\underline{p},\underline{q}) = \frac12\Big(\sum_{i,j=1}^{r} g^{-1}_{ij}p_i p_j + \sum_{i,j=1}^{r} v_{ij}q_i q_j\Big) \tag{1.3.1}$$

N1.3.1 *on $\mathbb{R}^{2r}$, and let $\eta^{(1)},\ldots,\eta^{(r)}$ be G–orthonormal vectors*[1] *verifying*

e1.3.2
$$(-\omega_k^2 G + V)\underline{\eta}^{(k)} = \underline{0}, \qquad k=1,\ldots,r, \tag{1.3.2}$$

[1] *i.e.* such that $(G\underline{\eta}^{(i)}\cdot\underline{\eta}^{(j)}) = \sum_{h,k=1}^{r} g_{hk}\eta^{(i)}_h\eta^{(j)}_k = \delta_{ij}$.

where $\omega_1^2,\ldots,\omega_r^2$ *are the roots, ordered and repeated according to multiplicity, of the secular equation* $\det(-\omega^2 G+V)=0$.

Define $\underline{x}$ *and* $\underline{\dot x}$ *in* $\mathbb{R}^r$ *so that*

e1.3.3
$$\underline{q}=\sum_{i=1}^r x^{(i)}\underline{\eta}^{(i)},\qquad G^{-1}\underline{p}=\sum_{i=1}^r \dot x^{(i)}\underline{\eta}^{(i)},\tag{1.3.3}$$

and consider the motion associated with the Hamiltonian H *in* $\mathbb{R}^{2r}$ *with given initial data* $\underline{p}$, $\underline{q}$. *This motion is described by* $t\to S_t(\underline{p},\underline{q})\equiv(\underline{p}(t),\underline{q}(t))$, *with*

e1.3.4
$$\begin{aligned}\underline{q}(t)=&\sum_{i=1}^r\Big(x^{(i)}\cos\omega_i t+\frac{\dot x^{(i)}}{\omega_i}\,\sin\omega_i t\Big)\ \underline{\eta}^{(i)},\\ \underline{p}(t)=&G\frac{d\underline{q}}{dt}\,(t)\equiv G\underline{\dot q}(t).\end{aligned}\tag{1.3.4}$$

The proof of this well known proposition is, for instance, a simple check by substitution of (1.3.4) into the equations of motion.

Remark: It is convenient to remark that the parameters $\underline{x}$ and $\underline{\dot x}$ are determined by equation (1.3.3) simply by using the G–orthonormality of the vectors $\underline{\eta}^{(1)},\ldots,\underline{\eta}^{(r)}$

e1.3.5
$$\begin{aligned}x^{(i)}&=G\underline{\eta}^{(i)}\cdot\underline{q}\equiv(\sqrt{G}\underline{\eta}^{(i)}\cdot\sqrt{G}\underline{q}),\\ \dot x^{(i)}&=\ \ \underline{\eta}^{(i)}\cdot\underline{p}\equiv(\sqrt{G}\underline{\eta}^{(i)}\cdot\sqrt{G^{-1}}\underline{p}),\end{aligned}\tag{1.3.5}$$

In other words $(x^{(i)},\dot x^{(i)})$ are the components of $\sqrt{G}\underline{q}$ and of $\sqrt{G^{-1}}\underline{p}$ on the orthonormal basis in $\mathbb{R}^r$ formed by the vectors $\sqrt{G}\underline{\eta}^{(i)}$. Note that the map
N1.3.2 $\underline{p}\to\underline{p}'=(\sqrt{G})^{-1}\underline{p}$ and $\underline{q}\to\underline{q}'=(\sqrt{G})\underline{q}$ is a linear canonical map.[2]

From (1.3.5) and from the above proposition an easy, but remarkable, corollary follows.

C1.3.1 **(1.3.1) Corollary:** (Action–angle coordinates for harmonic oscillations) *Under the assumptions of proposition (1.3.1) let, for* $(\underline{p},\,\underline{q})\in\mathbb{R}^{2r}$

e1.3.6
$$\begin{aligned}A_i&=\Big(x^{(i)2}+\big(\dot x^{(i)}/\omega_i\big)^2\Big)^{1/2},\\ \cos\varphi_i&=\frac{x^{(i)}}{A_i},\qquad \sin\varphi_i=\frac{\dot x^{(i)}}{A_i\omega_i},\end{aligned}\tag{1.3.6}$$

---

[2] We recall that a $2n\times 2n$ matrix $L=\begin{pmatrix}A&B\\C&D\end{pmatrix}$ is called *canonical* (or also *symplectic*) if $L^{-1}=\begin{pmatrix}D^T&-B^T\\-C^T&A^T\end{pmatrix}$, where T denotes transposition: a map defined on an open domain of $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$ is called canonical (or symplectic) if its Jacobian matrix is canonical at every point of the domain.

where $x^{(i)}$ *and* $\dot{x}^{(i)}$ *are determined by (1.3.5).*

Let $W \subset \mathbb{R}^{2r}$ *be the set of the points* $(\underline{p},\ \underline{q}) \in \mathbb{R}^{2r}$ *for which* $A_i \neq 0$, $i = 1, \ldots, r$. *Define on* W, *via (1.3.6), the map* $I(\underline{p}, \underline{q}) = (\underline{A}, \underline{\varphi}) = (A_1, \ldots, A_r, \varphi_1, \ldots, \varphi_r) \in (\mathbb{R}_+ \setminus \{\underline{0}\})^r \times \mathbb{T}^r$.

(1) The map I *is invertible as a map between* W *and* $IW = (\mathbb{R}_+ \setminus \{0\})^r \times \mathbb{T}^r$; *furthermore it has a Jacobian determinant which does not vanish on* W,

e1.3.7
$$\Big|\frac{\partial I^{-1}(\underline{A}, \underline{\varphi})}{\partial \underline{A}\ \partial \underline{\varphi}}\Big| = (\det G) \prod_{i=1}^{r} \omega_i A_i, \tag{1.3.7}$$

and, therefore, I *is analytic and invertible on* W.

(2) In the coordinates $(\underline{A}, \underline{\varphi})$ *the motion (1.3.4) becomes*

e1.3.8
$$I(S_t(p, q)) = (\underline{A}, \underline{\varphi} + \underline{\omega} t) \bmod 2\pi. \tag{1.3.8}$$

Remarks: (1) For $\underline{a} = (a_1, \ldots, a_r) \in (\mathbb{R}_+ \setminus \{0\})^r$ we set

e1.3.9
$$\Omega_{\underline{a}} = \{(\underline{p}, \underline{q}) \in \mathbb{R}^{2n} | A_1 = a_1, \ldots, A_r = a_r\}, \tag{1.3.9}$$

and we denote with

e1.3.10
$$\lambda_{\underline{a}}(d\underline{p}d\underline{q}) = \frac{\prod_{i=1}^{r} \delta(A_i(\underline{p}, \underline{q}) - a_i) d\underline{p}d\underline{q}}{\text{“normalization to 1”}} = \prod_{i=1}^{r} \frac{d\varphi_i}{2\pi} \tag{1.3.10}$$

the Lebesgue measure $d\underline{p}\, d\underline{q}$ restricted to $\Omega_{\underline{a}}$ and normalized to 1. If S_t denotes the Hamiltonian evolution on $\mathbb{R}^{2r}$ corresponding to (1.3.4), we can say that $\Omega_{\underline{a}}$ is invariant, that $\lambda_{\underline{a}}$ is invariant and that the metric flow $(\Omega_{\underline{a}}, (S_t)_{t\in\mathbb{R}}, \lambda_{\underline{a}})$ is isomorphic to the rotation of $\mathbb{T}^r$ with velocity $\underline{\omega}$, *i.e.* to the dynamical system $(\mathbb{T}^r, (\widetilde{S}_t)_{t\in\mathbb{R}}, \lambda)$ with $\widetilde{S}_t \underline{\varphi} = \underline{\varphi} + \underline{\omega} t (\bmod 2\pi)$ and $\lambda(d\underline{\varphi}) = \prod_i \frac{d\varphi_i}{2\pi}$.

(2) This is an isomorphism to which one refers by stating that the Hamiltonian system (1.3.1) is a system of harmonic oscillators whose trajectories develop with uniform velocity on r–dimensional invariant tori parameterized by r parameters (each of which, of course, is a first integral).

(3) One says that the phase space W is *foliated* into analytic r–dimensional invariant tori: this refers to the relation $W = \cup_{\underline{a}\in(\mathbb{R}_+\setminus\{0\})^r} \Omega_{\underline{a}}$ and to the fact that $\Omega_{\underline{a}}$ topologically is a torus, which by (1.3.6), depends analytically on r parameters.

The property of oscillators systems established by corollary (1.3.1) lead to their natural generalization expressed by the following definition that isolates a class of Hamiltonian systems whose phase space can be thought of as foliated into r–dimensional invariant tori parameterized by r parameters and such that the Hamiltonian flow on each of them reduces to a constant velocity flow, *i.e.* to a quasi-periodic motion.

D1.3.1 **(1.3.1) Definition:** (Hamiltonian integrability)
A Hamiltonian system of class C^∞ *in* $\mathbb{R}^{2r}$ *is called* integrable *in an open region* $W \subset \mathbb{R}^{2r}$ *of phase space if there exists a regular change of coordinates*
N1.3.3 [3] *I that transforms* W *into* $V \times \mathbb{T}^r$*, with* V *open subset of* $\mathbb{R}^r$*, and, denoting by* $(S_t)_{t\in\mathbb{R}}$ *the Hamiltonian flow on* W *and setting*

e1.3.11
$$(\underline{A}, \underline{\varphi}) = I(\underline{p}, \underline{q}), \tag{1.3.11}$$

one has

e1.3.12
$$I(S_t(\underline{p}, \underline{q})) = (\underline{A}, \underline{\varphi} + \underline{\omega}(\underline{A})t) \bmod 2\pi \tag{1.3.12}$$

where $\underline{\omega}(\underline{A}) = (\omega_1(\underline{A}), \ldots, \omega_r(\underline{A}))$ *are* r *functions of class* C^∞ *on* V*, called* velocities.
If the Hamiltonian is analytic and the map I *and the functions* $\underline{A} \to \underline{\omega}(\underline{A})$ *are analytic on the respective domains of definition the system is called* analytically integrable.
Finally if I *is a canonical map, see footnote 2 above, we shall say that the system is* canonically integrable.

Remark: (1) The linear oscillators are an *analytically integrable* system in the region W defined by proposition (1.3.1). The velocities $\underline{\omega}$ are in this case A–independent: this is the phenomenon of *isochrony* of harmonic oscillations.
(2) There exist several other examples, although not really many, of physically or mathematically interesting integrable systems. We refer to treatises on Rational Mechanics for their list and analysis (such systems are classically called *systems integrable by quadratures*). Here we mention only the system

e1.3.13
$$H(\underline{p}, \underline{q}) = \frac{1}{2}\,\underline{p}^2 - g/|\underline{q}| \qquad \underline{p}, \underline{q} \in \mathbb{R}^d \times \mathbb{R}^d, \qquad d = 2, 3, \tag{1.3.13}$$

known as the *Kepler system*. This system is integrable in the region in which $H < 0$ and $|\underline{q} \wedge \underline{p}| \neq 0$, but it is not isochronous (indeed the third Kepler's law shows that the period of a motion depends on parameters of the motion itself).
We shall come back later to integrable systems to analyze some of their aspects that are more interesting from a technical and conceptual point of view: the above brief introduction is motivated by the simplicity of their motions and their relevance for the discussion of complexity of a dynamical system. The latter notion, of obvious interest, will be the first, among several notions typical of ergodic theory: integrable systems will be the prototype of systems exhibiting simple motions. They are well suited to be discussed, from this point of view, as an introduction to the theory of more complex motions.

[3] This means that the change of coordinates is of class C^∞, that it is invertible and that it has non-zero Jacobian determinant.

In the next section we begin this analysis by studying the simplest property of regularity of a motion: namely the property of visiting given regions with well defined frequency.

Problems for §1.3

Q1.3.1 **[1.3.1]:** (*Analytical integrability of harmonic oscillations*)
Show that the system of harmonic oscillators of proposition (1.3.1) is also analytically canonically integrable (and not just analytically integrable): the analytic and canonical map that integrates it on W is $(\underline{p},\underline{q}) \to (\underline{B},\underline{\varphi})$ with $\underline{\varphi}$ defined by equation (1.3.6) and $B_i = \omega_i A_i^2/2$. In the new coordinates one has $H(\underline{p},\underline{q}) = \sum_{i=1}^{r} \omega_i B_i$. (*Hint:* (a) The map $(\underline{p},\underline{q}) \to (\underline{\dot{x}},\underline{x})$ is canonical as remarked after proposition (1.3.1); hence the problem is reduced to the case $r = 1$; (b) the map $(\dot{x}_i, x_i) \to (B_i, \varphi_i)$ is canonical for $i = 1,\ldots,r$
N1.3.4 because it has as generating function $F_0(x_i,\varphi_i) = \frac{1}{2}\omega_i x_i^2 \tan\varphi_i$.[4], cf. [Ga82].)

Q1.3.2 **[1.3.2]:** (*Analytical integrability of Kepler's system*)
Show that equation (1.3.13) is analytically integrable via a map which is canonical in the region $W = \{\underline{p},\underline{q}|\ |\underline{q}\wedge\underline{p}| \neq 0,\ H(\underline{p},\underline{q}) < 0\}$. Compute explicitly the variables $\underline{A}$, $\underline{\varphi}$ in the bidimensional case. (*Hint:* Just suitably interpret the classical results of the two-body Kepler problem).

§1.4 Frequencies of visit

Let (Ω, S) be a (discrete) invertible topological dynamical system. Motions $i \to S^i x$, $i \in \mathbb{Z}$, beginning in $x \in \Omega$, will be usually described by selecting a certain number of possible properties of a point of Ω can enjoy and, then, by listing which of such property is actually possessed by the points successively visited by x in its motion.

The mathematical model associated with the latter description is the *history* $\underline{\sigma}(x)$ of x on a partition $\mathcal{P} = \{P_0, P_1, \ldots, P_n\}$ of Ω into $n+1$, $n \geq 1$, pairwise disjoint sets where $\underline{\sigma}(x) = \{\sigma_i(x)\}_{i\in\mathbb{Z}} \in \{0,\ldots,n\}^{\mathbb{Z}}$ is the sequence such that

e1.4.1
$$S^i x \in P_{\sigma_i(x)} \qquad \text{for all } i \in \mathbb{Z}. \tag{1.4.1}$$

The set P_i, $i = 1,2,3,\ldots,n$, is the collection of the points of Ω that enjoy the property indicated with the label i, so that the union $\bigcup_{i=1}^{n} P_i$ is the set of points that enjoy anyone of the properties labeled with $1,2,\ldots,n$ while $P_0 = \Omega \setminus \cup_{i=1}^{n} P_i$ is the set of the points that do not enjoy any of them.

We will always require that the elements of the partition $\mathcal{P}$ (often called *atoms* of $\mathcal{P}$) be Borel sets in $\mathcal{B}(\Omega)$ (see Appendix 1.2): one says that such a partition is a *Borel partition*. For the time being, we do not impose further regularity requirements on the atoms of $\mathcal{P}$.

[4] We recall that a sufficient condition for canonicity of a map $I : W \longleftrightarrow V \times \mathbb{T}^r$ is that for all $(\underline{p}_0,\underline{q}_0) \in W$ there exist a neighborhood U_0 of $(\underline{A}_0,\ \underline{q}_0)$ and a function $F_0 : (\underline{A},\underline{q}) \to F_0(\underline{A},\underline{q})$ in $C^\infty(\mathbb{R}^{2r})$ such that for all $(\underline{A},\underline{q}) \in U_0$ one has: $\underline{p} = \frac{\partial F_0}{\partial \underline{q}}(\underline{A},\underline{q})$ and $\underline{\varphi} = \frac{\partial F_0}{\partial \underline{A}}(\underline{A},\underline{q})$ if $(\underline{A},\underline{\varphi}) = I(\underline{p},\underline{q})$.

It is nevertheless clear that in the applications we shall be interested only on rather reasonable partitions. For example we often consider the following types of partitions

D1.4.1 **(1.4.1) Definition:** (Topological, analytically regular, C^∞-regular partitions)
(i) Given a compact metric space Ω we call topological partitions *$\mathcal{P}$ those in which each P_i is either open or closed for all $i = 0, 1, \ldots, n$, and, furthermore, the closure $\overline{P}_i$ of P_i and the closure of its interior $\overline{int\, P_i}$ coincide: $\overline{P}_i = \overline{int\, P_i}$.*
(ii) When Ω is a class C^∞ or analytic manifold one often considers topological partitions $\mathcal{P}$ with C^∞–regular or, respectively, analytically regular atoms, cf. *Appendix (1.4).*
We often briefly call any such partition a regular partition.

E1.4.1 *Example (1.4.1)* : Let $\mathbb{T}$ be the unit circle and S the rotation by an angle $\rho \in (0, 2\pi)$ (one also says a rotation with *rotation number* $\rho/2\pi$). We can consider the partion $\mathcal{P}$ for the dynamical system $(\mathbb{T}, S)$ formed by the sets $P_0 = [0, \pi/2]$, $P_1 = (\pi/2, \pi)$, $P_2 = [\pi, 3\pi/2]$ and $P_3 = (3\pi/2, 2\pi)$. This simple example will be used in the problems to illustrate some of the concepts discussed in what follows.

Before formalizing with a mathematical definition the notion of "observation" of a motion on a given partition of (Ω, S) and before introducing other possible properties and restrictions on $\mathcal{P}$ it is convenient to introduce some philosophical considerations that should elucidate such a notion avoiding a too abstract tone and form.

It is natural to think of the result of "real observations" of a motion starting at x as sequences $\underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}}$ generated by making large the observation time N during which the motion visits successively the elements P_{σ_i}, $i = -N, \ldots, N$. For all N the string $\underline{\sigma}^{(N)} \in \{0, \ldots, n\}^{[-N,N]}$ has the property

e1.4.2
$$\bigcap_{k=-N}^{N} S^{-k} P_{\sigma_k} \neq \emptyset, \tag{1.4.2}$$

because $S^k x \in P_{\sigma_k}$ for all k, so that the intersection in (1.4.2) certainly contains at least x.

Therefore a motion, or better an observation of a motion, will appear as a sequence $\underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}}$ that enjoys the *finite intersection property*, in the sense that

e1.4.3
$$\bigcap_{j\in J} S^{-j} P_{\sigma_j} \neq \emptyset \qquad \text{for all } J \subset \mathbb{Z}\,,\ |J| < \infty, \tag{1.4.3}$$

where $|J|$ indicates the number of elements of J.

Hence it will be natural to identify the space of the motions *observed on* $\mathcal{P}$ with the set of infinite sequences

e1.4.4
$$\widehat{\Omega} = \big\{\underline{\sigma} \,|\, \underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}},\ \bigcap_{j\in J} S^{-j} P_{\sigma_j} \neq \emptyset \text{ for all } J \subset \mathbb{Z},\ |J| < \infty\big\}. \tag{1.4.4}$$

Furthermore, when possible, it will be convenient that the partition $\mathcal{P}$ be fine enough to be *S–separating*. This means that if x, x' have the same history $\underline{\sigma}$ then $x = x'$: if $\mathcal{P}$ is a separating partition it will be possible to identify the points of Ω with their histories.

We note, however, that even if $\mathcal{P}$ is S–separating it will not in general be possible to identify $\widehat{\Omega}$ with Ω: setting aside the important but trivial case in which $P_0, \ldots, P_n$ are all *closed and pairwise disjoint* and $\mathcal{P}$ is separating we must expect that $\widehat{\Omega}$ contains sequences that are not histories of points of Ω.

In fact it is quite generally possible to find $\underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}}$ for which there is no $x \in \Omega$ with $\underline{\sigma} = \underline{\sigma}(x)$ but such that for any arbitrarily long prefixed time T we can find a point x whose history coincides with $\underline{\sigma}$ between $-T$ and T. The sequence $\underline{\sigma}$ is thus in $\widehat{\Omega}$ but is not the history of any point in Ω. The set of sequences in $\widehat{\Omega}$ that are not histories of points in Ω is, however, generally negligible is a sense that will be made clear in the following.

The above remark serves to clarify the interest of the space $\widehat{\Omega}$ and why in the study of the motions of (Ω, S) observed on $\mathcal{P}$ it is, in a certain sense, natural to identify motions with sequences of $\widehat{\Omega}$ rather than with trajectories of points of Ω.

In applications the requirement that $\mathcal{P}$ be separating appears to be often imposed via the requirement of *expansivity of S on $\mathcal{P}$*: S is *expansive* with respect to $\mathcal{P}$ if for all $\underline{\sigma} \in \widehat{\Omega}$ one has

e1.4.5
$$\lim_{N\to\infty} \big(\mathrm{diam} \bigcap_{j=-N}^{N} S^{-j} P_{\sigma_j}\big) = 0. \tag{1.4.5}$$

In this case it is clear that if x, x' have the same history then they must coincide.

The above analysis is conveniently summarized into a precise definition that will be useful as a reference and as a basis for the future study of the structural properties of dynamical systems.

D1.4.2 **(1.4.2) Definition:** (Symbolic motions)
Let (Ω, S) be an invertible topological dynamical system and let $\{P_0, \ldots, P_n\} = \mathcal{P}$ be a partition of Ω into $n+1$, $n \ge 1$, Borel sets.
N1.4.1 *(1) Consider the set* [1]

e1.4.6
$$\widehat{\Omega} = \Big\{\underline{\sigma} \,|\, \underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}}\ , \quad \bigcap_{j=-N}^{N} S^{-j} P_{\sigma_j} \ne \emptyset \textit{ for all } N\Big\}. \tag{1.4.6}$$

We shall call $\widehat{\Omega}$ the set of $(\mathcal{P}, S)$–histories of the symbolic motions generated by S on Ω as seen from $\mathcal{P}$. *When $\mathcal{P}$ and S are clearly implicit in the text*

[1] Remark that (1.4.6) is equivalent to (1.4.4)

we shall simply call $\widehat{\Omega}$ *the* set of symbolic motions *(seen from* $\mathcal{P}$*).*
If $x \in \Omega$ *we shall call* $(\mathcal{P}, S)$–history of x *the element* $\underline{\sigma}(x)$ *of* $\widehat{\Omega}$ *such that*

e1.4.7 $$x \in S^{-j} P_{\sigma_j(x)} \quad \text{for all } j \in \mathbb{Z}. \tag{1.4.7}$$

(2) If the relation $\underline{\sigma}(x) = \underline{\sigma}(x')$ *implies* $x = x'$ *we shall say that* $\mathcal{P}$ is S–separating*. If* $\mathcal{P}$ *and* S *verify (1.4.5) we shall say that* S is $\mathcal{P}$–expansive *or that it is* expansive on $\mathcal{P}$.
(3) The correspondence defined by the map $\Sigma : \Omega \to \widehat{\Omega}$ *that associates with every* $x \in \Omega$ *the sequence* $\underline{\sigma}(x) \in \widehat{\Omega}$ *will be called the* code of the symbolic dynamics of S with respect to $\mathcal{P}$.
(4) Finally when considering sets $J = (j_1, \ldots, j_q) \subset \mathbb{Z}$*, and sequences* $\underline{\sigma}_J \in \{0, \ldots, n\}^J$ *we shall often employ the notation*

e1.4.8 $$P^J_{\underline{\sigma}_J} \equiv P^{j_1 \ldots j_q}_{\sigma_1 \ldots \sigma_q} = \bigcap_{j \in J} S^{-j} P_{\sigma_j} \tag{1.4.8}$$

to denote the points whose history in J is specified by $\underline{\sigma}_J$.

Remarks: (1) If $\Omega = \{0, \ldots, n\}^{\mathbb{Z}}$, $S = \tau$ is the translation on $\{0, \ldots, n\}^{\mathbb{Z}}$, and $P_i = \{\underline{\sigma} \,|\, \sigma_0 = i\}$, $i = 0, 1, \ldots, n$, the sets $P^J_{\underline{\sigma}}$ will be denoted also $C^J_{\underline{\sigma}}$ and will be called *cylinders* of $\{0, \ldots, n\}^{\mathbb{Z}}$ with "base J and specification $\underline{\sigma}$", *i.e.*

e1.4.9 $$C^J_{\underline{\sigma}} = \{\underline{\sigma}' \,|\, \underline{\sigma}' \in \{0, \ldots, n\}^{\mathbb{Z}}, \sigma'_{j_k} = \sigma_k \quad \text{for all } k = 1, \ldots, q\}, \tag{1.4.9}$$

if $J = (j_1, \ldots, j_q)$ and $\underline{\sigma} = (\sigma_1, \ldots, \sigma_q) \in \{0, \ldots, n\}^J$.
(2) If $\Omega = \{0, \ldots, n\}^{\mathbb{Z}}$ one has $\widehat{\Omega} = \Sigma(\Omega)$, but this is essentially the only case; if Ω is a connected space it will be, in general, $\widehat{\Omega} \supset \Sigma(\Omega)$ and $\widehat{\Omega} \neq \Sigma(\Omega)$.
(3) It is natural to consider the set of symbolic motions $\widehat{\Omega}$ as a topological space with the topology that it inherits as a subset of $\{0, \ldots, n\}^{\mathbb{Z}}$ which, in turn, is always considered with the product topology of the discrete topologies on the factors $\{0, \ldots, n\}$. The set $\widehat{\Omega}$ is closed in $\{0, \ldots, n\}^{\mathbb{Z}}$, see problem [1.4.2].

If (Ω, S) is a topological dynamical system and $\mathcal{P}$ is an expansive partition we can "invert" the coding map Σ. Given a point $\underline{\sigma} \in \widehat{\Omega}$ we can consider the set $\mathcal{X}(\underline{\sigma}) \in \Omega$ defined by

e1.4.10 $$\mathcal{X}(\underline{\sigma}) = \bigcap_{j=-\infty}^{+\infty} S^{-j}\, \overline{P}_{\sigma_j}. \tag{1.4.10}$$

where $\overline{P}_j$ is the closure of P_j. The set $\mathcal{X}(\underline{\sigma})$ is not empty because Ω is compact.

P1.4.1 **(1.4.1) Proposition:** (Symbolic codes)
Let (Ω, S) *be an invertible topological dynamical system (*cf. *definition*

(1.2.1)); let $\mathcal{P} = \{P_0, \ldots, P_n\}$ be a topological partition of Ω on which S is expansive.
(i) The set $\mathcal{X}(\underline{\sigma})$ defined by (1.4.10) contains one and only one point so that we can define the map $X : \widehat{\Omega} \to \Omega$ by setting $X(\underline{\sigma})$ to be the unique point in $\mathcal{X}(\underline{\sigma})$ or, with a slight abuse of notation,

e1.4.11
$$\underline{\sigma} \to X(\underline{\sigma}) = \bigcap_{j=-\infty}^{+\infty} S^{-j}\, \overline{P}_{\sigma_j}. \tag{1.4.11}$$

(ii) X is a continuos map;
(iii) $\Sigma^{-1} X^{-1}(x) = x$ for every x of Ω;
(iv) X and Σ are the inverse of each other if considered as maps between Ω and $\Sigma\Omega$ and, respectively, between $\Sigma\Omega$ and Ω;
(v) $S(x) = X(\tau\Sigma(x))$, where $(\tau\underline{\sigma})_i = \sigma_{i+1}$ is the translation, by one time unit to the left, of the sequence $\underline{\sigma}$.

Remark: Note also that $\tau\widehat{\Omega} = \widehat{\Omega}$, hence $(\widehat{\Omega}, \tau)$ as well is an invertible topological dynamical system (indeed $\widehat{\Omega}$ is closed in $\{0, \ldots, n\}^{\mathbb{Z}}$ and τ is a continuous map).

Proof: It is clear that the set $\mathcal{X}(\underline{\sigma})$ defined in (1.4.10) is not empty because Ω is compact and the closed sets $\overline{P_{\sigma_j}}$ form a family with the property of non empty finite intersection, if $\underline{\sigma} \in \widehat{\Omega}$.
The S–expansivity on $\mathcal{P}$ guarantees that $\mathcal{X}(\underline{\sigma})$ consists of a single point; indeed, cf. definition (1.4.2) and (1.4.5), by assumption for all $\underline{\sigma} \in \widehat{\Omega}$:

e1.4.12
$$\text{diam}\,\Big(\bigcap_{-N}^{N} S^{-j}\overline{P_{\sigma_j}}\Big) \equiv \text{diam}\,\Big(\bigcap_{-N}^{N} S^{-j} P_{\sigma_j}\Big) \xrightarrow[N\to\infty]{} 0. \tag{1.4.12}$$

The continuity of $\underline{\sigma} \to X(\underline{\sigma})$ also follows from the S–expansivity. If $\underline{\sigma}_n \xrightarrow[n\to\infty]{} \overline{\underline{\sigma}}$ we have that for every T there exists N such that $\underline{\sigma}_n$ coincides with $\overline{\underline{\sigma}}$ from $-T$ to T, if $n > N$. This implies that eventually $X(\underline{\sigma}_n)$ is in $\cap_{j=-T}^{T} S^{-j}\, \overline{P_{\sigma_j}}$. The fact that the diameter of this set tend to 0 complete the proof of (ii).
Since the history of x is uniquely determined by x and it determines x the validity of (iii) follows. We can say that, among the various sequences that determine x via the (1.4.10), only one is in $\Sigma\Omega$. The validity of (iv) and (v) is also clear. ∎

One among the remarkable properties that a "simple" motion should have is that of spending a well determined fraction of time visiting an arbitrary "reasonable" set $E \subset \Omega$. This means that for the motion $(S^i x)_{i\in\mathbb{Z}}$ to be "simple enough" the limit

e1.4.13
$$\lim_{N\to\infty} N^{-1} \sum_{j=0}^{N-1} \chi_E(S^j x) = \nu_x(E) \tag{1.4.13}$$

should exist (here χ_E denotes the characteristic function of E): this is, by definition, the *frequency of visit* to E by the motion originating in x.

If E is an element of a partition $\mathcal{P}=\{P_0,\ldots,P_n\}$ of Ω, the existence of the limit (1.4.13) is directly deducible from the $(\mathcal{P},S)$–history of x: if $E=P_j$ (1.4.13) becomes

e1.4.14
$$\nu_x(E)=\lim_{N\to\infty}N^{-1}\quad\{\text{number of labels } h \text{ between } 0 \text{ and } N-1 \text{ such that } \underline{\sigma}_h(x)=j\}. \tag{1.4.14}$$

From the history $\underline{\sigma}(x)$ of x one can obtain a more general information: one can indeed deduce the frequency with which certain groups of p symbols $\sigma_1,\ldots,\sigma_p$ appear in the history $\underline{\sigma}(x)$ of $x\in\Omega$, in sites which are translates of given sites $j_1,\ldots,j_p\in\mathbb{Z}$. Such frequency, if defined, is the limit for $N\to\infty$ of

e1.4.15
$$N^{-1}\{\text{number of labels } h \text{ between } 0 \text{ and } N-1 \text{ such that } \underline{\sigma}_{j_1+h}(x)=\sigma_1,\ \underline{\sigma}_{j_2+h}(x)=\sigma_2,\ldots,\underline{\sigma}_{j_p+h}(x)=\sigma_p\}, \tag{1.4.15}$$

or, equivalently, it is the limit

e1.4.16
$$p\left(\begin{matrix} j_1\ldots j_p\\ \sigma_1\ldots\sigma_p\end{matrix}\,\middle|\,\underline{\sigma}(x)\right)=\lim_{N\to\infty}N^{-1}\sum_{h=0}^{N-1}\chi_{P^{j_1\ldots j_p}_{\sigma_1\ldots\sigma_p}}(S^h x). \tag{1.4.16}$$

The number defined inside curly brackets in (1.4.15) will be called the *number of the strings homologue to* $\begin{pmatrix} J\\ \underline{\sigma}\end{pmatrix}\equiv\begin{pmatrix} j_1\ldots j_p\\ \sigma_1\ldots\sigma_p\end{pmatrix}$ *that appear in* $\underline{\sigma}(x)$ *between 0 and* $N-1$ and the value of the limit (1.4.15) will be the *frequency of appearance of the portion of history* $\begin{pmatrix} J\\ \underline{\sigma}\end{pmatrix}$ *in* $\underline{\sigma}(x)$.

Therefore we set a definition that will be useful in the following and that fixes more precisely the above notion.

D1.4.3 **(1.4.3) Definition:** (Sequences with defined frequencies)
Let $\underline{\widehat{\sigma}}\in\{0,\ldots,n\}^{\mathbb{Z}}$, $\{j_1,\ldots,j_p\}\subset\mathbb{Z}$ *and* $\sigma_1,\ldots,\sigma_p\in\{0,\ldots,n\}$.
(i) We define the number of strings homologue to $\begin{pmatrix} j_1\ldots j_p\\ \sigma_1\ldots\sigma_p\end{pmatrix}$ appearing in $\underline{\widehat{\sigma}}$ between 0 and $N-1$ *as the number of labels h between* 0 *and* $N-1$ *such that* $\widehat{\sigma}_{j_1+h}=\sigma_1,\ldots,\widehat{\sigma}_{j_p+h}=\sigma_p$. *We denote such number* $\mathcal{N}_N\left(\begin{matrix} j_1\ldots j_p\\ \sigma_1\ldots\sigma_p\end{matrix}\,\middle|\,\underline{\widehat{\sigma}}\right)$.
(ii) We define the frequency of appearance in $\underline{\widehat{\sigma}}$ of the string homologue to $\begin{pmatrix} j_1\ldots j_q\\ \sigma_1\ldots\sigma_q\end{pmatrix}$ *as the limit, when it exists,*

e1.4.17
$$\lim_{N\to\infty}N^{-1}\mathcal{N}_N\left(\begin{matrix} j_1\ldots j_p\\ \sigma_1\ldots\sigma_p\end{matrix}\,\middle|\,\underline{\widehat{\sigma}}\right)=p\left(\begin{matrix} j_1\ldots j_p\\ \sigma_1\ldots\sigma_p\end{matrix}\,\middle|\,\underline{\widehat{\sigma}}\right). \tag{1.4.17}$$

(iii) A sequence $\widehat{\underline{\sigma}}$ *will be said* with defined frequencies *if the limit (1.4.17) exists for all* $j_1, \ldots, j_p \in \mathbb{Z}$*, for all* $\sigma_1, \ldots, \sigma_p \in \{0, \ldots, n\}$*, and for all* $p = 1, 2, \ldots$.

In the following chapter we shall show that having defined frequencies is a "not too rare" property for the sequences $\underline{\sigma}(x)$ that are $(\mathcal{P}, S)$–histories of $x \in \Omega$, for the dynamical system (Ω, S).

Appendix 1.4: *Analytically regular sets in* $\mathbb{R}^n$ *and* $\mathbb{T}^n$

An *analytic system of coordinates* defined on the open set $U \subset \mathbb{R}^n$ is a pair (Ω, Ξ), where Ω is an open set of $\mathbb{R}^n$ and Ξ is an invertible function from Ω to U, analytic together with its inverse.

If $x = \Xi(\underline{b})$, $\underline{b} \in \Omega$, we shall say that $\underline{b} = (b_1, \ldots, b_n)$, are the coordinates of x in (Ω, Ξ).

A surface $M \subset \mathbb{R}^n$ will be called *locally analytic in* U if $M \cap U$ can be covered by a finite family $(U_a)_{a \in A}$ of open set of $\mathbb{R}^n$ endowed with analytic systems of coordinates (Ω_a, Ξ_a) such that the points of $M \cap U_a$ have coordinates $b_i = \bar{b}_i$ for $i = 1, \ldots, s$, with $s > 0$ and $\bar{b}_1, \ldots, \bar{b}_s$ given; s is the *codimension* of M, *i.e.* $n - s$ is the dimension of M.

In the same way we can define a C^∞ *system of coordinates* (Ω, Ξ) and a *locally* C^∞ *surface* M of dimension k in $\mathbb{R}^n$.

A closed set $G \subset \mathbb{R}^n$ is said to be *locally analytic* if ∂G is a locally analytic surface.

A set $G \subset \mathbb{R}^n$ is said *analytically regular* in U if it can be constructed via a *finite number of operations of union and intersection* starting with sets which are locally analytic in U.

The torus $\mathbb{T}^n$ can be thought of as an analytically regular set in $\mathbb{R}^{2n}$ via the coordinate system:

eA1.4.1

$$\begin{aligned}&(\rho_1, \ldots, \rho_n, \phi_1, \ldots, \phi_n) \to (\rho_1 e^{i\phi_1}, \ldots, \rho_n e^{i\phi_n}) \to \\ &\to (\rho_1 \cos\phi_1, \rho_1 \sin\phi_1, \ldots, \rho_n \cos\phi_n, \rho_n \sin\phi_n)\end{aligned} \tag{A1.4.1}$$

by fixing $\rho_1 = \ldots = \rho_n = 1$. Then the analytically regular subsets of $\mathbb{T}^n$ can be naturally defined as the intersections between analytically regular subsets of $\mathbb{R}^{2n}$ and $\mathbb{T}^n$.

Analytically regular sets have, by definition, the property of being stable with respect to operations of union and intersection. The most relevant consequence of this is that the intersection of any pair E, F of them has the property of being Riemann–measurable with respect to the Riemann measure restricted to E or F: thus the intersection of an analytically regular set with an analytically regular surface is Riemann measurable with respect to the Riemann area measure on the surface. Replacing, in the previous definitions, "analyticity" with "C^∞–differentiability" the last property would not be true in general: see problems [1.4.12] and [1.4.13].

Therefore we shall prefer to define, inductively, C^∞*–regular of dimension* k a set G in $\mathbb{R}^n$ if:

(i) it is contained in a locally C^∞ surface M of dimesion k;
(ii) its closure $\overline{G}$ in M coincides with the closure of its interior $\overline{\text{int } G}$, relative to the topology on M;
(iii) its boundary ∂G in M is the union of a finite number $\widetilde{G}_i$, $i = 1, \ldots, n$ of C^∞–regular sets of dimension $k-1$, *i.e.* $\partial G = \cup_{i=1}^n \widetilde{G}_i$;
(iv) $\overline{G}/G = \cup_{j\in\mathcal{N}} \widetilde{G}_j$ where $\mathcal{N} \subset \{1 \ldots, n\}$.

Finally we shall call C^∞–regular of dimension 0 a finite collection of points.

Remark that this class of sets is not closed with respect to operation of finite intersection, cf. problems [1.4.12], [1.4.13]. A C^∞–regular set G is Riemann measurable with respect to the Riemann measure restricted to the locally C^∞ surface $M \supset G$. However problem [1.4.13] shows that if G and G' are two C^∞–regular sets contained in two surfaces M and M' then $G \cap G'$ needs not to be measurable with respect to the Riemann measure on $M \cap M'$. This is the main reason why we shall try to *avoid* the use of C^∞–regular sets.

Problems for §1.4

Q1.4.1 **[1.4.1]:** (*Permutations as dynamical systems*)
Let $W = \{1, \ldots, n\}$ and let y be a permutation $i \to y(i)$ of the n elements. Let P be the partition of W into its points. Show that the (P, S)–history of every point of W is periodic and, hence, with defined frequencies. What is the meaning of the frequency in terms of the cycles that represent y?

Q1.4.2 **[1.4.2]:** ($\widehat{\Omega}$ *is closed*)
Show explicitly that $\widehat{\Omega}$ is a closed set in $\{0, \ldots, n\}^{\mathbb{Z}}$, for any choice of dynamical system (Ω, S) and of partition $\mathcal{P}$.

Q1.4.3 **[1.4.3]:** (*Non-expanding partitions*)
Find an example of a dynamical system (Ω, S) with $S \neq$ identity and such that no partition is S–separating for it.

Q1.4.4 **[1.4.4]:** Let $(\mathbb{T}, S)$ and $\mathcal{P}$ be the dynamical system and the partition of example (1.4.1). Show that $\mathcal{P}$ is S–separating if and only if the components of the vector $(\rho, 2\pi)$ are rationally independent (*i.e.* if $\rho/2\pi$ is irrational).

Q1.4.5 **[1.4.5]:** ($\widehat{\Omega} \neq \Sigma\Omega$)
Under the hypotheses of the previous problem with the components of the vector $(\rho, 2\pi)$ rationally independent find a sequence $\underline{\sigma} \in \widehat{\Omega}$ that is not the history on any point x in Ω. (*Hint:* Let $\underline{\overline{\sigma}}$ be the history of the point π. Clearly $\overline{\sigma}_0 = 1$. Consider the sequence $\underline{\sigma}$ identical to $\underline{\overline{\sigma}}$ but for $\sigma_0 = 2$.)

Q1.4.6 **[1.4.6]:** (*Arnold's cat map expansivity*)
Consider the dynamical system $(\mathbb{T}^2, S)$ defined in the example (1.2.5), cf. (1.2.13). Show that every partition $\mathcal{P}$ of $\mathbb{T}^2$ into Borel sets of the torus and with diameter small enough is such that S is expansive on $\mathcal{P}$.

Q1.4.7 **[1.4.7]:** (*Sequences with undefined frequencies*)
Find an example of a sequence $\underline{\sigma} \in \{0, \ldots, n\}^{\mathbb{Z}}$ that does not have defined frequencies. (*Hint:* 0 followed by ten 1's followed by hundred 0's followed by thousand 0's,...)

Q1.4.8 **[1.4.8]:** Adapt this section definitions of $(\mathcal{P}, S)$–histories, frequencies of visit *etc* to the case of non-invertible dynamical systems. (*Hint:* "It suffices to replace $\mathbb{Z}$ with $\mathbb{Z}^+$".)

Q1.4.9 **[1.4.9]:** (*Interval maps and cylinders*)
Consider example (1.2.7) assuming that $[0, 1]$ can be thought of as the union $[0, 1] =$

$\cup_{\sigma=0}^{n-1}[a_\sigma, a_{\sigma+1}]$ where $0 = a_0 < a_1 < \ldots < a_n = 1$ and that S is strictly monotonic on $[a_\sigma, a_{\sigma+1}]$, for all $\sigma = 0, 1 \ldots, n-1$. Consider the partition $P_0 = [a_0, a_1), \ldots, P_{n-1} = [a_{n-1}, 1]$ of $[0, 1]$. Show that the sets $P^{0,\ldots,N}_{\sigma_0,\ldots,\sigma_N}$ are intervals, for all N.

Q1.4.10 **[1.4.10]:** (*Tent map and binary expansion*)
If $S(x) = 2x$, $0 \le x \le 1/2$, and $S(x) = 2(1-x)$, $1/2 \le x \le 1$, consider $\mathcal{P} = \{[0, 1/2), [1/2, 1]\}$ and find the relation between the history of x, $\underline{\sigma}(x)$, on $\mathcal{P}$ and the sequence of digits of the binary development of $x : x = \sum_{k=1}^{\infty} \gamma_k 2^{-k}$, $\gamma_k = 0, 1$.

Q1.4.11 **[1.4.11]:** (*Expansive interval maps*)
Under the hypotheses of problem [1.4.9] suppose S of class C^1 in every interval $(a_\sigma, a_{\sigma+1})$, $\sigma =, 0, \ldots, n-1$, and suppose $|S'(x)| \ge \lambda > 1$, for all $x \in \cup_\sigma (a_\sigma, a_{\sigma+1})$. Show that $\mathcal{P}$ is S–separating and that S is expansive on $\mathcal{P}$ and, furthermore,

$$\operatorname{diam}\left(\bigcap_{i=0}^{N-1} S^{-i} P_{\sigma_i}\right) \le \lambda^{-N} \;, \quad \text{for all } \sigma_0, \ldots, \sigma_{N-1} \;, \quad \text{for all } N$$

Q1.4.12 **[1.4.12]:** Let $x_1, x_2, \ldots$ be an enumeration of the rationals in $[0, 1]$. For every x_k consider the open interval of length 2^{-1-k} and center x_k. Show that the union A of such intervals is not Riemann measurable. (*Hint:* It has external measure ≥ 1 and internal measure $\le 1/2$.)

Q1.4.13 **[1.4.13]:** (*A C^∞ regular set with non-Riemann measurable intersection with another C^∞ regular set*)
With the notation of problem [1.4.12], consider a function $g(x)$ which is $C^\infty(\mathbb{R})$ and positive in the open interval $(-\frac{1}{2}, \frac{1}{2})$ and zero elsewhere. Set $f(x) = \sum_{k=1}^{\infty} k!^{-1} g(2^{k+1}(x - x_k))$. Show that f is C^∞ and that it is positive in A and vanishes elsewhere. Show that the set $\{x, y \,|\, y \ge f(x)\} \subset \mathbb{R}^2$ is a C^∞–regular set but its intersection with the x–axis is A which is not only not C^∞–regular but it is not even Riemann–measurable with respect to the Riemann measure on the x–axis ([Ga82], p. 337).

Bibliographical note to §1.4

The idea of studying motions by means of symbolic dynamics was essentially born with ergodic theory: it can be found, for instance, in [Mo21], [Bi35].

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