CHAPTER VII

Statistical properties of turbulence

§7.1 Viscosity, reversibility and irreversible dissipation.

It is now convenient to reexamine some questions of fundamental nature with the purpose of analyzing the possible consequences of Ruelle's principle introduced in §5.7. We shall make frequently reference to the general description of motions given in Chap. 5 in a context in which we imagine that the considered motions are attracted by some attracting set in phase space, which will have zero volume when energy dissipation occurs in the system. The main purpose of this section and of the following ones is to analyze consequences of Ruelle's principle, see §5.7, with particular attention to fluid motions.

(A): Reversible equations for dissipative fluids.

First of all we must stress (again) that the derivation of the NS equations presented in §1.1, §1.2 was based on the empirical assumption that there was a viscous force opposing gliding of adjacent layers of fluid (*c.f.r.* the tensor denoted, in (1.1.17), as $\underset{\approx}{\tau}'$).

It is difficult to imagine how the reversible microscopic dynamics could generate a macroscopic dynamics in which time reversal symmetry is completely absent. We recall, *c.f.r.* remark (xv) to theorem I in §5.4, that if we consider a time evolution described by a differential equation of which S_t is the solution flow (so that $t \to S_t x$ is the motion with initial datum x, *c.f.r.* §5.3 Definition 3) or a discrete evolution S (associated with a timed observation, *c.f.r.* §5.2) a *time reversal symmetry* is (any) isometric map i such that

$$i^2 = 1 \quad and \quad iS_t = S_{-t}i \qquad or \qquad Si = S^{-1}i \tag{7.1.1}$$

respectively.

Note that this definition is *more general* than the often used and more common definition which takes i to be the "*velocity reversal with unchanged positions*" map, which in the case of simple fluids becomes simply velocity reversal and which for clarity of exposition will be called here, perhaps be more appropriately, *velocity reversal* symmetry. It is clear that while Newton's equations are reversible in the latter sense the NS equations do not have such velocity reversal symmetry and for this reason they are called irreversible. In principle a system may admit time reversal symmetry in the sense of (7.1.1) even though it does not admit the special velocity reversal symmetry: we shall see some interesting examples below.

The negation of above notion of reversibility is not "irreversibility": *it is instead the property that a map i does not verify* (7.1.1). This is likely to generate misunderstandings as the word irreversibility usually refers to lack of velocity reversal symmetry in systems whose microscopic description is or should be velocity reversal symmetric

To understand how it is possible that a reversible microscopic dynamics, in the sense of velocity reversal or in the more general sense in (7.1.1), is compatible with irreversible macroscopic equations (as the NS equations manifestly are) we must think that several scales of time and of space are relevant to the problem.

The macroscopic equations are approximations apt to describe properties on "large spatial scale" and "large time scale"[1] of the solutions of reversible equations. The typical phenomenon of reversibility (*i.e.* the indefinite repetition, or "*recurrence*", of "*impossible*" states) should indeed manifest itself, but on time scales much longer and/or on scales of space much smaller than those interesting for the class of motions considered here: which must be motions in which the system could be considered as a continuous fluid.

We have already seen in §1.3 and mainly in §1.5 how the equations can change aspect if one is interested in studying a property that becomes manifest in particular regimes. For example in the theory of the Rayleigh equations of §1.5 we have seen that "in the Rayleigh regime" the general equations (1.2.1) simplify and become the equations (1.5.14) in which the generation of heat due to viscous friction between layers of fluid (last term in (1.5.6)) is absent, *c.f.r.* the third of the (1.5.14) or the comment I preceding (1.5.8).

This does not mean that friction does not generate heat. It only means, as it turned out from the analysis of §1.5, that on a time scale in which the (1.5.14) can be considered a good approximation (*c.f.r.* (1.5.12)) the quantity of heat generated is *negligible*.

The same mechanism is, or at least is believed to be, at the basis of the derivation of the equations (1.2.1) from atomic dynamics, [EM94].

This immediately makes us understand that it should be possible to express[2] the phenomenological coefficients of viscosity or thermal conductivity in terms of *averages*, over time and space, of microscopic quantities which are more or less rapidly fluctuating.

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[1] Compared to the atomic scales where every motion is reversible.

[2] As indeed elementary kinetic theory strongly suggests that this is possible, *c.f.r.* problems [1.1.4], [1.1.5].

We deduce that the transport coefficients (such as viscosity or conductivity or other) *do not have a fundamental nature*: they must be rather thought of as macroscopic parameters that measure the disorder at molecular level.

Therefore it should be possible to describe in different ways the same systems, simply by replacing the macroscopic coefficients with quantities that vary in time or in space but rapidly enough to make it possible identifying them with some average values (at least on suitable scales of time and space). *The equations thus obtained would then be equivalent to the previous.*

Obviously we can *neither hope nor expect* that by modifying the equations (1.2.1) into equations in which various constant are replaced by variable quantities we shall obtain simpler or easier equations to study (on the contrary!). However imposing that equations that should describe the same phenomena do give, actually, the same results can be expected to lead to *nontrivial relations* between properties of the solutions (of both equations).

This is a phenomenon quite familiar in statistical mechanics of equilibrium where one can think of describing a gas in equilibrium at a certain temperature as a gas enclosed into an adiabatic container with perfect walls or as enclosed in a thermostat at the same temperature.

The two situations are described respectively by the microcanonical distribution and by the canonical distribution. Such distributions are *different*: for example the first is concentrated on a surface of given energy and the other on the whole phase space. A sharp difference, indeed, being the surfaces of constant energy sets with zero volume.

Nevertheless the physical phenomena predicted in the two descriptions must be the same: it is well known that from this Boltzmann derived the *heat theorem*, [Bo84],[Ga95c] p. 205,[Ga99a], *i.e.* a proof of the second law of equilibrium thermodynamics, and the general theory of the *statistical ensembles*.

Hence providing different descriptions of the same system is not only possible but it can even lead to laws and deductions that would be impossible (or at least difficult) to derive if one did confine himself to consider just a single description of the system.

What just said *has not been systematically applied to the mechanics of fluids*, although by now there are several deductions of macroscopic irreversible equations starting from microscopic velocity reversible dynamics, to begin with Lanford's derivation of the Boltzmann equation, [La74]. In the remaining sections I try to show that the above viewpoint is at least promising in view of its possible applications to the theory of fluids.

It is well known that Boltzmann was drawn into disputes, [Bo97], occasionally quite animated, to defend his theory of irreversibility. His point has been that one should make a distinction between reversibility of motion and irreversibility of the phenomena that the accompany it.

The works of Sinai on Anosov systems, *c.f.r.* [Si94], [Ru79], show why it is not necessary, in order to make this distinction clear, to deal with systems of very many particles, to which Boltzmann was always making reference. Systems with few degrees of freedom, even 2 degrees of freedom, are sufficient at least for illustration purposes and show irreversible phenomena although their motions are governed by a reversible dynamics.

This phenomenon has been empirically rediscovered, independently by several experimental physicists:[3] some hailed it as the "solution of the Loschmidt paradox",[HHP87], [Ho99], correctly seeing its relation with the Boltzmanian polemics (without perhaps taking into account that, at least in the case of Boltzmann, the question had been already solved and precisely in the same terms posed by Boltzmann himself, in papers that few had appreciated, [Bo84],[Bo97], see [Le93], [Ga99a]).

Therefore keeping in mind the above considerations we shall imagine other equations that should be "equivalent" to the Navier–Stokes incompressible equation (in a container Ω with periodic boundary conditions).

Note that a forced fluid has an average energy and an average dissipation that rapidly end up fluctuating around an average value depending only on the acting force (I simplify here a little, to avoid trivial digressions needed to take into account situations in which there are important *hysteresis phenomena*, *i.e.* several attracting sets, *e.g.* [FSG79], [FT79]).

In situations in which viscosity is small (*i.e.* the Reynolds number is large) the theory K41 suggests that essentially the fluid flows subject to Euler's equations (*i.e.* with zero viscosity acting on the "important" degrees of freedom in number of $O(R^{9/4})$, *c.f.r.* §6.2,) but *with energy dissipation* rate constant in time. The rate of energy dissipation in an incompressible Navier–Stokes fluid[4] is

$$\eta = \nu \int_\Omega (\underline{\partial} \wedge \underline{u})^2 d\underline{x} \tag{7.1.2}$$

(this quantity is called ε in §6.2). Clearly the way in which dissipation takes place is not properly accounted for by Euler's equation and the phenomenon of heat production due to friction can possibly be well described only from a really microscopic viewpoint, *c.f.r.* the problems of the §1.1. However apart from the heat production the equations of Euler supplemented with some mechanism that takes away the energy supplied to the system by the forcing forces should properly describe the flows, *c.f.r.* §6.2.

Then if the only difference between the Euler equations and the NS equations at low viscosity is the existence of dissipation we can imagine another equation that has the same properties; *i.e.* the Euler equation with the addition of a force that performs work on the system but in such a way to absorb (in the average) a constant quantity of energy per unit time.

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[3] Long after the early works on the SRB distributions, [Lo63], [Si68], [Bo70], [Ru76].

[4] In which (7.1.2) equals $\nu \int_\Omega dx\, (\underset{\sim}{\underline{\partial u}} + \underset{\sim}{\partial}\underline{u})^2/2$, *c.f.r.* (1.2.7).

It is interesting that already Gauss posed the problem of which would be the *minimum* force necessary to impose a constraint (whether *holonomic* or *anholonomic*). The principle of Gauss, *c.f.r.* problems, applied to an ideal fluid subject to the constraint of dissipating energy at constant rate leads to the following equations

$$\dot{\underline{u}} + \underset{\sim}{u} \cdot \underset{\sim}{\partial}\, \underline{u} = -\frac{1}{\rho}\underline{\partial} p + \underline{g} + \alpha\, \Delta\, \underline{u}, \qquad \underline{\partial} \cdot \underline{u} = 0 \tag{7.1.3}$$

where $\alpha = \alpha(\underline{u})$ *is not* the viscosity but, rather, it is the multiplier necessary to impose the constraint that (7.1.2) is a constant of motion for the equation (7.1.3). A simple computation provides us with:

$$\alpha(\underline{u}) = \frac{\int \big(\underline{\partial} \wedge \underline{g} \cdot \underline{\omega} + \underline{\omega} \cdot\, (\underset{\sim}{\omega} \cdot \underset{\sim}{\partial}\, \underline{u})\big)\, d\underline{x}}{\int (\underline{\partial} \wedge \underline{\omega})^2\, d\underline{x}} \tag{7.1.4}$$

The idea that is suggested by the analysis developed until now is precisely that the equations (7.1.3), (7.1.4) are *equivalent* to the NS equation. Suppose that the solutions of the NS equation with given viscosity and force admit an attracting set on which the average energy dissipation has a certain value $\eta(\nu)$ and imagine that the r.h.s. of the relation (7.1.2) is fixed to be *precisely equal* to η via a Gaussian constraint of the type described by (7.1.3) and (7.1.4). Then the average values of the observables *may turn out to be the same* with respect to the statistics of the motions on the attracting set for the NS equations and for the equations (7.1.3) and (7.1.4). Such identity will certainly be approximate if the Reynolds number, *i.e.* the intensity of the force, is finite but one can conjecture that it can become more and more exact as R increase.

We shall call the (7.1.3) and (7.1.4) *GNS equations*, or *Gaussian Navier–Stokes equations.* And the just proposed conjecture is, formally

Equivalence conjecture: *Consider the GNS equations (7.1.3), (7.1.4) with initial data, in which the quantity* η *in (7.1.2) is fixed equal to the average value of the same quantity with respect to the SRB distribution, that we denote* $\mu_{\nu,ns}$*, for the NS equations with viscosity* ν*. Let* $\mu_{\eta,gns}$ *be the SRB distribution for the GNS equations thus defined.*
(i) Then the distributions $\mu_{\nu,ns}$ *and* $\mu_{\eta,gns}$ *assign, in the limit in which the Reynolds number tends to infinity,*[5] *equal values to the same observables* $F(\underline{u})$ *that are "local" in the momenta, i.e. that depend only on the Fourier components of the velocity field* $\underline{\gamma}_{\underline{k}}$ *with* $\underline{k}$ *in a finite interval of values of* $|\underline{k}|$*.*
(ii) In such conditions $\langle\alpha\rangle_{\eta,gns} = \nu$*, if* $\langle\cdot\rangle_{\eta,gns}$ *and* $\langle\cdot\rangle_{\nu,ns}$ *denote the average values with respect to the distributions* $\mu_{\eta,gns}$ *and* $\mu_{\nu,ns}$ *respectively.*

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[5] *e.g.* $\nu \to 0$ at fixed external force density and at fixed container size.

Remarks:

(1) The conjecture is closely related, and in a way it is a natural extension, of similar conjectures, [Ga99a], of equivalence between different thermostats in particle systems. For the latter systems it was clearly formulated and pursued, in particle systems, in several papers by Evans and coworkers starting witht he early 1980's: for a more recent review see [ES93] where a proof under suitable assumptions is presented. For fluids it was proposed in [Ga96] but in a sense it was already clear from the paper [SJ93] and it is somewhat close to ideas developed in [Ge86], see also [GPMC91] and the review [MK00].
(2) This conjecture proposes therefore that *for the purpose of computing average values* the NS and GNS equations are *equivalent* provided, of course, the free parameters in the two equations are chosen in a suitable relation.
(3) It will not escape to the reader that the described correspondence is very analogous to the equivalence, so familiar in statistical mechanics, between the statistical ensembles that describe equilibrium of a single system,[Ga95c], [Ga95b], [Ga99a].
(4) We see therefore how this conjecture expresses that different equations can describe the same phenomena: and in particular the *reversible* GNS equation and the NS equation (much better known and *irreversible*), describe the same physical phenomenon at least for R large.
(5) A formal way to express the point (i) of the conjecture is that

$$\lim_{R\to\infty} \frac{\langle F\rangle_{\nu,ns}}{\langle F\rangle_{\eta(\nu),gns}} = 1 \tag{7.1.5}$$

for all local observables $F(\underline{u})$ whose average $\langle F\rangle_{\nu,ns}$ does not tend to 0 as $R\to\infty$.

(B) Microscopic reversibility and macroscopic irreversibility.

The question is then how could this dual reversible and irreversible nature of the phenomena be possible?
The crucial point to remark is *that there is no relation between irreversibility, understood in the common sense of the word, and lack of reversibility of the equations that describe motions.*
At first sight this appears paradoxical: but this is a fact that becomes substantially already clear to anyone who studies, even superficially, the disputes on irreversibility between Boltzmann and his critics, [Bo97],[Ga95c],[Ga95b]. We therefore examine this "paradox" in more detail.

A reversible dynamical system (M,S), see (7.1.1), in general, shall have attracting sets A that can *fail to be invariant under time reversal*: $iA\neq A$.

The case in which the sets A and iA are really different is clearer and we shall discuss it first, keeping in mind, however, that it is a case that arises in systems that are strongly out of equilibrium. Indeed for systems in equilibrium we imagine, at least when their evolution is chaotic, that the attracting set is the whole phase space (a consequence of the "ergodic hypothesis"): and this remains true also if the system is subject to small external forces that keep it out of equilibrium, (this is a *structural stability* theorem in the case of simple chaotic motions, *c.f.r.* §5.7).

Let us suppose, for simplicity, that there is a unique attracting set A. Then the set iA is a *repelling set*, *i.e.* it is an attracting set for motions observed *backwards in time* (described by S^{-1}).

Motions starting in the basin of attraction of A develop, after an initial transient, essentially on the attracting set A. Hence they *no longer exhibit any symmetry with respect to time reversal.* By "motions" we mean here "typical' motions", *i.e.* all of them except a set of 0–volume in phase space or except a set of 0–probability with respect to a distribution μ_0 which is "absolutely continuous" with respect to volume.

Strictly speaking such motions do not start, with μ_0–probability 1, do not start exactly on A but get close to A with exponential rapidity, as time runs past, *without ever getting into it.* If we could really distinguish the point x reached after a long time from a point x' *near it but located exactly* on the attracting set then, by proceeding backwards in the time, the point x will go back and in the long run it will get, with exponential rapidity, very close to iA, *i.e.* to the repelling set, because the latter is an attracting set for the motions that are observed backwards in time. A completely different behavior will be that of the point x', once close to x *but lying on the attractor*, because it will stay forever on A.

However the time reversal invariance *does not refer to the difference between these two motions* followed backwards and foward in time, rather it usually "only" says that if we inverted all velocities of the points of the system (such is the effect of the map i in most cases) then the system would go through a trajectory that, *as time increases*, coincides in position space with that followed until the instant of the reversal, while in velocities space it has velocities systematically opposite to those previously assumed when occupying the same positions.

By applying the time reversal map to a state x of the system very near to A (*but not on* A) one finds a state ix very close to iA (*but not on* iA) and hence "atypical":[6] but proceeding in the motion, as the time n increases, *also* $S^n\,i\,x$ will *again* evolve getting close to A, very rapidly so: hence the initial state ix would appear as an atypical fluctuation.

This *would not happen* if instead of considering a state x very near A one considered one *exactly* on A:[7] the state ix would be exactly on iA and

[6] For example, in a system of charged particles which is out of equilibrium because of an field electric acting on it, such state would have a current opposite to field.

[7] A 0–probability event if initial data are randomly chosen with a distribution μ_0 abso-

then, proceeding in time *both towards the future and towards the past*, this point ix would move on iA without ever leaving it. And this evolution, *anomalous for an observer unaware of the particularity of the initial state*, would indefinitely continue on $i\,A$.

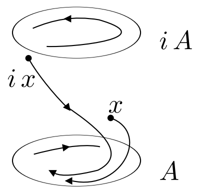

Fig. (7.1.1): *Illustration of a "strongly forced" system with attracting set A different from its time reversal image iA. The forward evolution of a point x and of its time reversal image ix. Both x and ix eventually approach A in their forward evolution. Motions starting on A or iA stay there in the forward as well as in the backwards evolutions.*

An observer could say that the system behaves in a irreversible way in which starting from a configuration x it reaches asymptotically a stationary state μ, always the same whatever the initial state is (with μ_0–probability 1) and *the same* for x and ix. Furthermore proceeding backwards in time the system reaches *instead* (always with μ_0–probability 1) a stationary state μ_- different[8] from the state μ but still *the same* for x and ix.

Starting instead with an initial datum randomly chosen precisely on the 0–volume attracting set A and *with the asymptotic and stationary statistics* μ one generates a motion that, *both towards the future and towards the past*, has the same statistics μ: *i.e.* starting with an initial datum typical of the stationary state and proceeding backwards *there is no way to reach* states which are not typical of the stationary state (such as the states close to the repelling set iA).

The question is, as we see, rather delicate: macroscopic irreversibility is, in this case, the manifestation of the existence of an attracting set $A \neq iA$ while the microscopic reversibility only implies that to every attracting set must correspond a repelling one that is substantially a copy of it. This implies that except for an initial transient the property of the motions towards the future are the same for all x's and $i\,x$'s (unless one succeeds in choosing an initial datum *exactly* on the sets of zero phase space volume A or iA).

But if one was able to measure and fix exactly all coordinates of the system then the transient time could be made as long as wanted: it would be enough to invert exactly all the velocities at a given time and the motion of a system that has been observed as absolutely "normal" for a prefixed time T would develop *in a way now absolutely strange* "backwards" for the same time T: and then, at least if the initial data were randomly chosen with a distribution

lutely continuous with respect to the volume

[8] Although "isomorphic" to μ because of time reversal symmetry, *i.e.* a state that can be transformed into μ via the map i.

μ_0 proportional to the volume of phase space, it would proceed again in a normal way *and for ever so.*

What said would seem to give rise to problems in the case in which the attracting set A is the whole space (accessible compatibly with the initial value of the possible constants of motion) and hence it coincides with iA: in fact this happens in systems in thermodynamical equilibrium or slightly away from it.

In reality no new problem arises because, upon an attentive exam, the attracting set A whether equal or different from iA will be such that, obviously, not all points of A are equally probable with respect to the statistics μ on A.

We have seen in §5.5 that the SRB statistics is "in some sense" concentrated on the more stable motions: (5.5.8) says indeed that, if the system is "approximated" by the set of its periodic motions of period n with n very large, then the statistical weight, for the purpose of the SRB averages, of the periodic motions is inversely proportional to their "instability", *i.e.* to the product of the eigenvalues larger than 1 (in average) of their stability matrix (*c.f.r.* the factor $\Lambda_e(x)^{-1}$ in eq. (5.5.8)).

Hence even when it is $A = iA$ it remains nevertheless possible that the properties of the motions towards the future are, *for possibly very long transient times*, very different from the average ones and this can be seen by the behavior of the fluctuations and is essentially the only form in which irreversibility can manifest itself in systems in equilibrium.

What can then be said about the cases in which $A = iA$ but the system *is not* in equilibrium? (Think of a gas of charged particles in a toroidal container subjected to an axially directed weak, but not vanishingly small, electric field).

(C) Attractors and attractive sets.

In reality the notion of attracting set, as a closed set A to which the motions starting near enough get nearer and nearer, is too rough to describe what happens and it is not adequate when $A = i\,A$.

In §5.5 we introduced the distinction between attractive set and attractor for data randomly chosen with distribution μ_0 with density with respect to volume on phase space, defining (we recall):

Definition: *(reminder of the notion of attractor):*
Given the dynamical system (M, S) an attractor for motions with initial data chosen with a distribution μ_0 in the vicinity of an attracting set A is any invariant set $A_0 \subset A$ which has probability 1 with respect to the statistics μ of these motions and at the same time has minimum Hausdorff dimension. The value of this minimum is the "information dimension" of the attractor. If μ_0 is absolutely continuous with respect to volume the information dimension is also called the dimension of the system (M, S) on

A.

This is a much more precise notion than that of attracting set and it allows us to distinguish A_0 from iA_0 also when the closures A and iA of these sets are the same. It allows us, furthermore, to define naturally the dimension of the attractor contained in A, so that this notion is not trivial even when A_0 is dense in phase space and A is therefore the whole phase space.

For example in conservative systems it is usually (believed to be) true that the attracting set is the entire surface of given energy ("*ergodic hypothesis*"). And in such cases it also happens that there exists *only one* attracting set for the motion towards the future and towards the past of data chosen randomly with a distribution proportional to the Liouville measure μ_0.

Forcing these systems and providing, at the same time, also some dissipation mechanism allowing them to keep constant (or bounded) the energy and hence to reach a stationary state, we obtain systems that still have the *whole phase space* as an *attracting set* at least if the force is not too large (structural stability of chaotic motions, *c.f.r.* §5.7): but this time the *attractor* for the motion towards the future and that towards the past will be different.

In the sense that it will be possible to find two sets, *both dense on the full phase space*, A_0 and A_0' to which the SRB statistics μ_+ and μ_-, for the motions towards the future and those towards the past, attribute probability $\mu_+(A_0) = 1 = \mu_-(A_0')$ while $\mu_+(A_0') = \mu_-(A_0) = 0$.

The above analysis of the distinction between microscopic reversibility and macroscopic irreversibility can be repeated and remains substantially unchanged. *Hence also the discussion made in point (B) can be essentially repeated with A_0 and A_0' playing the role of A and iA*: we see that microscopic reversibility and macroscopic irreversibility are compatible also in this case.

In strongly out of equilibrium systems (*e.g.* a fluid at large Reynolds number) we expect (as said above) that the attracting sets for motions towards the future and towards the past are different: this can be interpreted as a *spontaneous breaking* of time reversal symmetry and, as discussed in (B), it provides us with a simpler version of the mechanism that shows the compatibility between microscopic reversibility and macroscopic irreversibility. A mechanism that, as we have seen, is more hidden in the cases of equilibrium or close to equilibrium because of the identity of the closures of the attractor and of the repeller.

We conclude the discussion by discussing, to provide a concrete example, various aspects of the abiove analysis in the case of the well known case of the expansion of a gas in a (perfect) container of which it initially occupies only a half, we can distinguish the adiabatic expansion from the isothermal expansion (*i.e.* with the system in thermal contact with a heat reservoir keeping fixed its temperature identified with its average kinetic energy).

In the first case the system is Hamiltonian and it will evolve towards an attractor A_0 that is dense on the whole surface of energy E (with E equal to the initial energy of the gas) and, in the statistics μ of the motion, the initial state will appear as a very rare fluctuation. In the second case, instead, the attractor will be determined by a microscopic mechanism of interaction between gas and thermostat (and it will appear to have an energy which will be distributed with very unlikely fluctations around its *average* determined by the temperature of the heat reservoir).

In both cases the attractor will still be symmetric with respect to time reversal and dense on the available phase space; the initial state will still appear as a rare fluctuation.

If now the gas is imagined to be made of charged particles on which an electric nonconservative field (*i.e.* an electromotive force) acts, the system will reach a stationary state only in presence of a mechanism of interaction with a thermostat or with external bodies that absorb the energy generated by the work of the field. If the field is different from zero the *attractors* A_0 and $A'_0 = iA_0$ *become different* and of 0–volume in phase space: and if the field is sufficiently strong then it will become possible that *even* the attracting set A, closure of A_0, becomes smaller than the whole phase space and different from A', closure of A'_0. This would be a case of "*spontaneous breakdown*" of time reversal symmetry: it will be discussed again later.

Problems:

[7.1.1]: Let $\varphi(\underline{\dot{x}},\underline{x}) = 0$, $\underline{x} = \{\underline{\dot{x}}_j,\underline{x}_j\}$ be a general anholonomic constraint for a mechanical system. Let $\underline{R}(\underline{\dot{x}},\underline{x})$ be the constraint reaction and $\underline{F}(\underline{\dot{x}},\underline{x})$ be the active force. Consider all possible accelerations compatible with the constraints when the system is in the state $\underline{\dot{x}},\underline{x}$. We say that $\underline{R}$ is *ideal* or *verifies the principle of least effort* if the actual acceleration due to the forces $\underline{a}_i = (\underline{F}_i + \underline{R}_i)/m_i$ minimizes the *effort*: $\mathcal{E} = \sum_{i=1}^N \frac{1}{m_i}\underline{R}_i^2 \equiv \sum_{i=1}^N (\underline{F}_i - m_i\underline{a}_i)^2/m_i$, *i.e.*

$$\sum_{i=1}^N (\underline{F}_i - m_i\underline{a}_i)\cdot\delta\underline{a}_i = 0$$

for all the possible variations of the accelerations $\delta\underline{a}_i$ compatible with the constraints φ at *fixed* velocities and positions. Show that the possible accelerations, in the configuration $\underline{\dot{x}},\underline{x}$, are those such that: $\sum_{i=1}^N \partial_{\underline{\dot{x}}_i}\varphi(\underline{\dot{x}},\underline{x})\cdot\delta\underline{a}_i = 0$.

[7.1.2]: Show that, thanks to the observations in [7.1.1], the condition of minimum constraint becomes:

$$\underline{F}_i - m_i\underline{a}_i - \alpha\,\partial_{\underline{\dot{x}}_i}\varphi(\underline{\dot{x}},\underline{x}) = \underline{0}$$

$$\alpha = \frac{\sum_i (\underline{\dot{x}}_i\cdot\partial_{\underline{x}_i}\varphi + \frac{1}{m_i}\underline{F}_i\cdot\partial_{\underline{\dot{x}}_i}\varphi)}{\sum_i m_i^{-1}(\partial_{\underline{\dot{x}}_i}\varphi)^2}$$

which is the analytic expression of Gauss' principle, *c.f.r.* [LA27], [Wi89]. Of course the definition of effort $\mathcal{E}$ is quite arbitrary and modifying it leads to different analytic expressions.

f**[7.1.3]:** Check that if the constraints are holonomic then Gauss' principle reduces to the principle of D'Alembert. (*Idea:* Note that the velocities permitted by the holonomic constraint $\varphi(\underline{x}) = 0$ are $\underline{\dot{x}}\cdot\underline{\partial}\varphi(\underline{x}) = 0$ and hence a holonomic constraint can be thought of as a constraint anholonomic having the form special: $\underline{\dot{x}}\cdot\underline{\partial}\varphi(\underline{x}) = 0$.)

[7.1.4]: Consider a system of N particles subjected to a conservative force with potential energy V. Consider the system of points subject to the force $\underline{f} = -\underline{\partial} V$ and to a Gaussian constraint imposing that $T = const$ (where T is the kinetic energy $T = \sum_i \underline{p}_i^2/2m$). Verify that the equations of motion are (by [7.1.3]):

$$m\dot{\underline{x}}_i = \underline{p}_i,\ \dot{\underline{p}}_i = -\underline{\partial}_{\underline{q}_i} V - \alpha\,\underline{p}_i \stackrel{def}{=} \underline{F}_i, \qquad \alpha = -\frac{\sum_i \underline{\partial}_{\underline{q}_i} V \cdot \underline{p}_i}{\sum_i \underline{p}_i^2}$$

Show that for arbitrary choices of the function $r(T)$ the probability distribution on phase space with density: $\rho(\underline{p},\underline{q}) = r(T) e^{-\beta V(\underline{q})}$ is invariant if, defined ϑ as $3N\,k_B\vartheta/2 \stackrel{def}{=} T$: $\beta = 3N - 1/(3Nk_B\,\vartheta)$. (*Idea:* The continuity equation is indeed $\partial_t \rho + \sum_i \underline{\partial}_{\underline{p}_i}(\rho \underline{F}_i) + \sum_i \underline{\partial}_{\underline{q}_i}(\rho\,\underline{p}_i/m) = 0$).

For Gauss principle applications to fluid mechanical equations see problems in §7.4.

Bibliography: [GC95a], [GC95b],[Ga95],[Ga95b],[Ga96],[LA27].

§7.2 Reversibility, axiom C, chaotic hypothesis.

> *I do not intend to claim that the description that I have given of the soul and of its functions is exactly right – a wise man could not possibly say that. But I claim that, once immortality is accepted as proved, one can think, not improperly nor lightheartedly, that someting like that is true.*
> *(*words of Socrates *(in Phaedon)).*

To study in more detail some of the problems posed in the discussion in §7.1 we shall refer, assuming it *a priori*, to Ruelle's principle of Sec. §5.7.

(A) The SRB distribution, and other invariant distributions.

Suppose that the dynamical system (M, S) has an attracting set A verifying axiom A (*c.f.r.* definition 2, §5.4): it will then be possible to describe the points of A via the *symbolic dynamics* associated with a Markovian pavement $\mathcal{P}$, *c.f.r.* §5.7 (C).
If the logarithm of the determinant $\Lambda_e(x)$ of the matrix ∂S^n, thought of as a map acting on the unstable manifold of $x \in A$, is considered as a function $\lambda_e(\underline{\sigma})$ of the history $\underline{\sigma}$ of x then, since the function $\Lambda_e(x)$ is

"regular" (Hölder continuous), the dependence of $\lambda_e(\underline{\sigma})$ on the "far" digits of $\underline{\sigma}$ is also exponentially small.[1]

In this context we can make use of the expression (5.7.8) for the SRB distribution in terms of the code $x = X(\underline{\sigma})$ between points and histories.

If ϑ denotes the translation to the left of the histories and if μ is the SRB distribution on A the equations (5.7.7), (5.7.8) yield the following expression for μ:

$$\int f(y)\mu(dy) = \lim_{N\to\infty} \frac{\sum_{\sigma_{-N/2},\ldots,\sigma_{N/2}} e^{-\sum_{j=-N/2}^{N/2} \lambda_e(\vartheta^j \underline{\sigma})} f(X(\underline{\sigma}))}{\sum_{\sigma_{-N/2},\ldots,\sigma_{N/2}} e^{-\sum_{j=-N/2}^{N/2} \lambda_e(\vartheta^j \underline{\sigma})}} \qquad (7.2.1)$$

where $\underline{\sigma}$ is an infinite compatible sequence in the sense of the histories on Markovian pavements, *c.f.r.* §5.7, (C), obtained extending the string of digits $\sigma_{-\frac{1}{2}N}, \ldots, \sigma_{\frac{1}{2}N}$ in "a standard way", *c.f.r.* §5.7, (D).

If we replace, in formula (7.2.1), the function $\lambda_e(\underline{\sigma})$ with an *arbitrary* function $\rho(\underline{\sigma})$ that has a very weak dependence on the digits σ_i with large label i (for example it depends only on the digit with label 0) the new formula still defines an invariant distribution μ' on A which, however, in general is *completely different from* μ.[2]

Just as the probability distributions of two different Bernoulli schemes can be chosen[3] to be different even if they have the same space of states: in fact their attractors consist in the sequences that have given frequencies of appearance of given symbols; but such frequencies are *different* in the two cases: hence the attractors are *different and disjoint* sets. Nevertheless both attractors are dense in the space of all sequences[4] and the space of all sequences is therefore the attracting set for both cases, while the attractors are different (and can be chosen so that they do not have points in common).

This example lets us well appreciate the difference between the two possible notions of attracting set and of attractor. It makes us, furthermore, see that in a system with an attracting set A verifying axiom A *there exist infinitely many other invariant distributions besides* the SRB distribution which can be very different from, or just about equal to, the SRB. In fact

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[1] Indeed the digits of the history $\underline{\sigma}$ determine the point x with exponential rapidity, *i.e.* the distance between two points whose histories coincide between $-N$ and N tends to zero as $e^{-\lambda N}$, *c.f.r.* §5.7: hence the values of $\lambda_e(\underline{\sigma})$ and $\lambda_e(\underline{\sigma}')$ differ by $O(e^{-\alpha\lambda N})$, if the sequences $\underline{\sigma}$ and $\underline{\sigma}'$ coincide between $-N$ and N and if α is the exponent of Hölder continuity of $\Lambda_e(x)$.

[2] The (7.2.1), even if modified in this way, can be interpreted as the definition of the thermodynamic limit of an unidimensional "Ising model" with a short range interaction, see also §9.6 in [Ga99a]; hence this property is a well known result: the technique for the proof is illustrated in detail in the problems [5.7.1]÷[5.7.16] of §5.7 taken from [Ga81].

[3] We recall that while an attracting set is closed and uniquely characterized by the dynamics, an attractor is only defined up some (trivial) ambiguity, *c.f.r.* definition in (C) of §7.1 and §5.7.

[4] If the distance between two sequences is, as usual, defined as e^{-N}, where N is the largest value for which $\sigma_i = \sigma_i'$ for $|i| \le N/2$.

given an arbitrary *finite number* of observables one can define an invariant probability distribution μ' concentrated on A (*i.e.* giving probability 1 to A: $\mu'(A) = 1$) and attributing to the chosen observables average values close to those attributed by the SRB distribution within a prefixed approximation (one obtains μ' by simply modifying by a small quantity the functions $\lambda_e(\underline{\sigma})$ in (7.2.1)).

(B) Attractors and reversibility. Unbreakability of the time reversal symmetry.

Dynamical systems can be reversible, and in general their attracting sets A differ from the (repelling) sets iA that are their images under the time reversal map i, *c.f.r.* §7.1.

This can be interpreted, as already observed in §7.1, as a phenomenon of *spontaneous symmetry breaking*: one can think that phase space is precisely the attracting set A and ignore, for the purpose of studying the statistical properties of motions, the points outside of A. Limiting ourselves, for simplicity, to the case of an attracting set A that is a regular surface, we see that as far as asymptotic (in the future) observations are concerned the dynamical system is *de facto* the system (A, S).

Obviously this system *is no longer reversible* if time reversal is performed with the map i (which *cannot* even be thought of as a map of A into itself, because $iA \neq A$).

We shall see however, in what follows, that reversible systems with an attracting set that coincides with the whole space have extremely interesting properties. In this respect we can ask whether we could define another map $i^* : A \longleftrightarrow A$ that anticommutes with the evolution, *i.e.* $i^*S = S^{-1}i^*$ *while leaving invariant the attracting set* A and squaring to the identity, ${i^*}^2 = 1$.

If, with some generality, it was possible to define i^* we could say that the dynamical system is still reversible, although the time reversal symmetry is now i^* and not the original i. In this way, in the same generality, reversible systems endowed with an attractive set that is *smaller* than the entire phase space could also be considered as systems enjoying time reversal symmetry and endowed with an attractive set that *coincides* with the whole phase space.

In other words time reversal symmetry would be unbreakable. When spontaneously violated it would spawn an analogous symmetry on every non symmetric attracting set!

A first example of unbreakability of time reversal symmetry, or at least an example of a very similar phenomenon, can be found even in fundamental Physics. In relativistic quantum theories that should describe our Universe time reversal symmetry T, that we could think as valid at a fundamental level, is spontaneously broken, as it is well known, *c.f.r.* [Al93] p. 241. But a symmetry which anticommutes with time evolution continues to exist as a symmetry of the dynamics of our Universe: it is the TCP symmetry.

By applying to our universe the map T we obtain another universe, absolutely different from ours, but equally possible.

One can think that the fundamental equations are symmetric with respect to the map T, *i.e.* reversible, but dissipative. Therefore our Universe would evolve towards an attracting set smaller than the whole phase space and it would no longer be time reversal invariant. Nevertheless if time reversal symmetry was actually unbreakable (in the above sense) then the motion would be still symmetric with respect *to another operation* that inverts the sign of time, that could be TCP.

To support what just said one can fear that it would be necessary to think that there exists a dissipative mechanism in the dynamics of the Universe: nothing more unsatisfactory. However the dissipation of which we talk here *would not be* the empirical dissipation to which we are perhaps used: *because the fundamental equations would remain reversible.* One could rather think of a level higher than the one accessible directly to us, a "Universe of Universes", that acts on our evolution like a reversible thermostat acts on the evolution of a gas or of a fluid out of equilibrium (*i.e.* as a force that absorbs heat without breaking time reversal, like the forces that impose the constant energy or constant dissipation in the ED or GNS equations in (7.1.1)). This would be sufficient and it would allow us to think that the Universe evolved rapidly, ending up on an attracting set that does not have any more the symmetry T but "only" TCP.

Or, with a larger conceptual economy and without crossing (as done above) the border of science fiction, one could think that it is our same Universe to act, in a reversible though dissipative way, as a thermostat on the world of elementary particles generating the symmetry breaking of their dynamics that we observe experimentally. The asymmetry observed in the weak interactions could be a trace in the subatomic world of the asymmetry that we observe between past and future, at macroscopic level. It would certainly be important to produce a concrete and credibile model for the mechanism of interaction between the macroscopic (atomic) world and the microscopic (subatomic) one: by imagining a lagrangian for the description of the weak interactions, *c.f.r.* [Al93], that is *a priori* not time reversal invariant we, perhaps, only take into account the atomic–subatomic interaction phenomenologically. This vision does not seem absurd to me, although admittedly is very daring.

(C) An example.

Coming back to our much more modest analysis of the motion of a gas or of a fluid, let us consider the structure of the motions of a reversible system endowed with an attracting hyperbolic set A, that we think as a *regular* surface of dimension lower than that of phase space.

The points of A will have stable and unstable manifolds: the unstable will be entirely contained in A, and the stable will consist of a part on A and a part *outside* A, precisely because A is attractive.[5]

We shall assume, for simplicity, that the dynamical system has only two invariant closed sets of *nonwandering* points, A and its image iA, that we shall call *poles*, *c.f.r.* §5.4 observation (2) to definition 2.

The stable manifold of the points of A extends out of A reaching the set iA: *i.e.* if x is a point near A (*but not on it*) and if we follow it backwards in time we see that $S^{-n}x$ tends to the set iA, that repells, *i.e.* that attracts motions seen backwards in time.

Therefore the stable manifold in question extends until iA and we must expect that it is "dense" on this set (meaning that the closure of such manifold will contain iA). It can *a priori* behave, in its vicinity, in several ways: for example it could *wrap* around iA.

Nevertheless the *simplest geometric hypothesis* is that the manifold reaches the surface $i\,A$ "cutting" this surface in a "transversal" way: this is possible even though, obviously, no point of the manifold can belong to $i\,A$.

To understand how to interpret the latter geometric property we first discuss a paradigmatic example, see Fig.(7.2.1) below, with the aim of a later abstraction of a general formulation.

Referring to Fig.(7.2.1) below, the poles $A_+ = A$ and $A_- = iA$ are, in the example in Fig.(7.2.1), proposed below, two regular closed and bounded surfaces identical, copies of a surface M^*.

The map S on the *whole* phase space will be defined by

$$S(x,z) = (S_* x, \tilde{S} z) \tag{7.2.2}$$

where the generic phase space point is a pair (x,z) with $x \in M^*$ and z is a set of transversal coordinates that tell us how far away the point (x,z) is from the attracting set.

The point z will be imagined as a point on a smooth manifold Z and $\tilde{S}$ will be an evolution on Z that has two fixed points, one z_- (unstable) and another z_+ (stable). Furthermore $\tilde{S}$ will be supposed to evolve all points $z \neq z_\pm$ in such a way that all motions tend to z_+ in the future and to z_- in the past. This can be realized in many ways and we select arbitrarily one of them. By construction the sets A_+ and A_- are the sets with $z = z_+$ and, respectively, $z = z_-$.

The coordinate x identifies a point on the compact surface M^* on which a reversible map S_* acts: we suppose the system (M^*, S_*) is an Anosov

[5] This means that at every point of A the stable manifold has a tangent plane which can be decomposed into a direct sum of two independent planes one of which is tangent to A and one which is transversal to it. Since A *in general is not a smooth surface*, and it might have fractal structure, this has to be made more precise; supposing for simplicity that A is a set immersed in R^d, for some d, the "tangent" part of the stable plane at x will be such that the maximum distance to A of points y on it and with $d(y,x) \le r$ is $< O(r^2)$ while the maximum distance to A of points y on the part of the stable plane outside A will be $> O(r)$. The second might be empty.

system. For example M^* can be the bidimensional torus $\mathcal{T}^2$ and S_* could be the "Arnold cat", *c.f.r.* §5.3

$$S_*(\psi_1\,,\,\psi_2) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{mod } 2\pi \tag{7.2.3}$$

which is reversible if one defines time reversal as the map that permutes the coordinates of $\underline{\psi}$ and changes the sign to the first, *i.e.* $i'x = i'(\psi_1, \psi_2) = (-\psi_2, \psi_1)$.

For instance Z could be a circle $z = (v, w)$ with $v^2 + w^2 = 1$. The evolution $\tilde S$ could be the map that one obtains by considering at time $t = 1$ the point $\tilde S(v, w)$ into which (v, w) evolves according to the differential equation $\dot v = -\alpha v,\ \dot w = E - \alpha w$ with $\alpha = Ew$. Such evolution has precisely the property $\tilde S^n z \xrightarrow[n\to\pm\infty]{} z_\pm$ with $z_+ = (v_+, w_+) \equiv (0, 1)$ and $z_- = (v_-, w_-) \equiv (0, -1)$. With the above choices for $S_*, \tilde S$ the map S, in (7.2.2), is *reversible* if we define time reversal by $i(x, z) = (i'x, -z)$.

Therefore we see that the system $(M^* \times Z, S)$ is a system endowed with an attracting set and a repulsive set both hyperbolic, respectively given by $M^* \times \{z_+\}$ and $M^* \times \{z_-\}$.

The two poles $A_\pm = M^* \times \{z_\pm\}$ are transformed into each other by the symmetry i that, obviously, *is not* a symmetry for motions that develop on them.

The dynamical system is chaotic, having an attractive set $(M^*, \{z_+\})$ on which the evolution enjoys the Anosov property: in the full system (M, S) time reversal symmetry is spontaneously violated (in the above sense).

Nevertheless we see that if one defines the map $i^* A_+ \longleftrightarrow A_+$ as: $i^*(x, z_+) = (i'x, z_+)$ then i^* anticommutes with the evolution S *restricted* to A_+ (and a map i^* can be defined also on A_- in a analogous way and, analogously, it anticommutes with S on A_-).

Therefore in this case i^* is a "time reversal" map defined only "locally" (*i.e.* on the poles of the system) "inherited" from the global symmetry i: the symmetry i, however, is not a local symmetry, *i.e.* it cannot be restricted to the poles, because the poles are not symmetric and they are not i–invariant.

We now examine in which cases the construction just described is generalizable.

(D) The axiom C.

First of all we give a formal description of the geometric property introduced in [BG97] and called there *axiom C* property of a dynamical system endowed with hyperbolic poles.

In the observations to definition 2 of §5.4 we noted that any axiom A dynamical system (M, S) is a system whose nonwandering points can be decomposed into a finite number of sets, called *basic sets* or *poles*, densely covered by periodic orbits and on which there exists a dense orbit (this is a theorem by Smale, *c.f.r.* §5.4).

Not all poles are attracting sets: if a system is reversible then every pole A of attraction has a time reversed "image" $i\,A$ that is a repulsive pole.

Given a pole Ω (attractive, repulsive or other) one defines $W^s(\Omega)$ as the set of the points that evolve towards Ω for $t \to +\infty$ and $W^e(\Omega)$ as the set of points that evolve towards Ω for $t \to -\infty$.

When Ω is attractive the set $W^s(\Omega)$ is the basin of attraction of Ω, while $W^e(\Omega)$ is Ω itself. In general Ω is neither attractive nor repulsive and the two sets $W^s(\Omega)$ and $W^e(\Omega)$ are both nontrivial. For simplicity we restrict the following discussion to the case in which the system has at most two poles Ω_+, Ω_-, one attractive and one repulsive.

It is convenient to define the "distance" $\delta(x)$ of a point x from the poles. If d_0 is the diameter of the phase space M and $d_\Omega(x)$ is the ordinary distance (in the metric of M) of the point x from the pole Ω

$$\delta(x) = \min_{i=\pm} \frac{d_{\Omega_i}(x)}{d_0} \tag{7.2.4}$$

We say that two manifolds intersect transversally if the plane spanned by their tangent planes at a point of intersection has dimension equal to that of the whole phase space, *c.f.r.* §5.4. The latter notion of transversality is then useful to fix the notion of "*axiom B system*" or of system that "*verifies axiom B*". It is (rephrasing here definition 4 of §5.4) a system that verifies the axiom A with the further property that if $W^s(\Omega_i)$ has a point y in common with $W^e(\Omega_j)$, hence $y \in W^s_x \cap W^e_{x'}$ for some pair $x \in \Omega_i$ and $x' \in \Omega_j$, then the intersection between W^s_x and $W^e_{x'}$ is *transversal* in y, *c.f.r.* §5.4 observation 3 to the definition 2.

The structures now described are interesting because the systems that verify them are *stable*: if a system verifies axiom B then, by perturbing the map S in class C^∞, one generates a new system that, via a *continuous* (but, in general, not differentiable and hence not necessarily regular) coordinates change can be transformed into the original one.

The latter is a deep result (*Robbin theorem*), [Ru89b] p. 170. The converse statement is a conjecture (conjecture of Palis–Smale); "in class C^r" for $r \ge 1$ and for $r = 1$ it is already a theorem (*Mañé theorem*), *c.f.r.* [Ru89b], p. 171 for a precise formulation: see, also, the comments to definition 4 in §5.4.

The example given at point (C) obviously verifies the axioms A and B. It verifies furthermore the property that in [BG97] has been called axiom C

Definition *(axiom C): A dynamical system $(\mathcal{C}, S)$ verifies axiom C if it is a mixing Anosov system or at least it verifies axiom B and if in the latter case*

(i) It admits only one attractive pole and only one repulsive pole, A_+ and A_-, with basins of attraction for A_+ and of repulsion for A_- open and with complement with zero volume, (globality property of the attracting and of repulsive set). The poles are, furthermore, regular *surfaces on which S acts in topologically mixing way (hence $(A_\pm, S)$ are mixing Anosov systems).*

(ii) For every $x \in M$, the tangent plane T_x admits a Hölder–continuous decomposition as a sum of three planes T_x^u, T_x^s, T_x^m such that[6]

$$\begin{array}{llll} a) & dS\,T_x^\alpha & = T_{Sx}^\alpha & \alpha = u, s, m \\ b) & |dS^n w| & \le C e^{-\lambda n}|w|, & w \in T_x^s,\ n \ge 0 \\ c) & |dS^{-n} w| & \le C e^{-\lambda n}|w|, & w \in T_x^u,\ n \ge 0 \\ d) & |dS^n w| & \le C\delta(x)^{-1} e^{-\lambda |n|}|w|, & w \in T_x^m,\ \forall n \end{array}$$

where the dimensions of T_x^u, T_x^s, T_x^m are > 0 and $\delta(x)$ is defined in (7.2.4).
(iii) if x is on the attractive pole A_+ then $T_x^s \oplus T_x^m$ is the tangent plane to the stable manifold in x; viceversa if x is on the repulsive pole A_- then $T_x^u \oplus T_x^m$ is the tangent plane to the unstable manifold in x.

Remarks:

(1) Although T_x^u and T_x^s are not uniquely determined for general x's the planes $T_x^s \oplus T_x^m$ and $T_x^u \oplus T_x^m$ are uniquely determined for all $x \in A_+$ and, respectively, for all $x \in A_-$.
(2) It is clear that an axiom C system verifies also, necessarily, axiom B. The possibility that every axiom B reversible system that has only two poles, one attractive and one repulsive, verifies necessarily axiom C is not remote, possibly with the help of some additional ("natural") hypothesis.
(3) (3) If we drop the condition that the poles are smooth and just require that motion on them is topologically mixing then the resulting weaker notion is structurally stable: this is implied by the quoted theorem by Robbin, [BG97].
(4) The hypothesis that there are two poles is posed here only for simplicity and probably one could dispense of it, [BG97].
(5) Also the requirement that the poles be regular surfaces, is probably not always necessary for the purposes of the discussions that follow.

It is possible to visualize the axiom C property via the following figure (commented below)

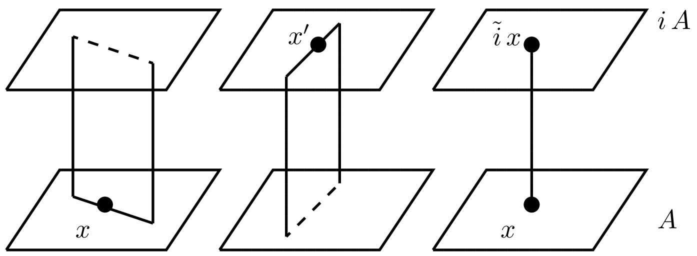

Fig. (7.2.1): *Illustration of the example, preceding the definition, of axiom C system. The vertical direction represents z on which the map $\tilde S$ of the example acts; the horizontal plane is a symbolic representation of the torus on which S_* acts; see (7.2.2). See below for a detailed interpretation of the figure.*

[6] One could prefer C^∞ or C^p regularity, with $1 \le p \le \infty$: but this would exclude most cases. On the other hand Hölder continuity could be equivalent to simple continuity C^0, as in the case of Anosov systems, *c.f.r.* [AA68], [Sm67].

Axiom C property, [BG97], is stronger than axiom B property and it is a *structurally stable* notion, *i.e.* it remains a true property also if the systems obtained with small enough perturbations from an axiom C system still verify axiom C.

Informally, in systems endowed only with an attracting and a repelling set, without other invariant sets, axiom C says that the stable manifold of the points on A_+ arrives "*transversally*" on the repelling set A_-, rather than "*wrapping around it*" and, furthermore, $A_\pm$ are smooth surfaces.

The first figure in Fig.(7.2.1) illustrates a point $x \in A_+$ and a local part of its stable manifold that extends until the set A_- intersecting it in the hatched line (that is a stable manifold for the motion *restricted* to the surface A_- but that *is not part* of the stable manifold of x). Likewise the second figure describes a point x' on A_- with a local part of its unstable manifold.

The third figure in Fig.(7.2.1) shows the intersection between the stable manifold of a point $x \in A_+$ and the unstable manifold of the point $\tilde{x} \in A_-$: in the figure such intersection is a unidimensional curve that connects x with $\tilde{x}$ (that is uniquely determined by x, *c.f.r.* following) *establishing the correspondence defining* $\tilde{i}$.

In this case the stable manifold of x is the sum $T^s_x \oplus T^m_x$ if T^m_x is the vertical direction, while the unstable manifold of x' is $T^u_x \oplus T^m_x$. Here T^s_x and $T^u_{x'}$ are parallel to the stable and unstable manifolds (the solid horizontal lines) of the map S_*. The intersection of the two manifolds is a line T^m_x in the vertical direction.

The points "between the two surfaces" $A_\pm$ represent most of the points of phase space, but they are *wandering points*, *c.f.r.* §5.4.

Let $\delta_\pm$ the dimension of the surfaces $A_\pm$ and let $u_\pm, s_\pm$ the dimension of the stable and unstable manifolds of the dynamical systems $(A_\pm, S)$ respectively $(\delta_\pm = u_\pm + s_\pm)$. It is $s_+ = u_-, u_+ = s_-$ and hence $\delta_+ = \delta_- = \delta$ and the total dimension of phase space is $\delta + m$ with $m > 0$.

The dimension of the stable manifold of $x \in A_+$ in the original dynamical system (*i.e.* describing dynamics not restricted to A_+) is, therefore, $m+s_+$, because such manifold "sticks outside of A_+" (because A_+ attracts) and that of the unstable manifold of $x' \in A_-$ is $m+u_-$. Hence the dimension of their intersections is m and such surface intersects A_+ and A_- in *two points* that we can call x and $\tilde{x}$, thus allowing us to define $\tilde{i}x = \tilde{x}$. We see that such surface is a *wire* that joins points of A_+ to points of A_- (defining $\tilde{i}$): hence the representation in Fig.(7.2.1) is an accurate representation, in spite of it being schematic.

Axiom C, that forbids the stable manifold of the points of A_+ and the unstable manifold of the points of A_- to "wrap" around A_- or, respectively, around A_+, can be seen as a hypothesis of maximum simplicity on the geometry of the system. The interest of this geometric notion lies in the

Theorem *(axiom C and time reversal stability): Let the dynamical system* (M, S) *be reversible and verifying axiom C. Then there exists a map* i^*

defined on the poles of the system that leaves them invariant, squares to the identity, and that anticommutes with the time evolution.

In the case illustrated in figure the i^* is indeed the composition $i\cdot\tilde{i}$, [BG97]. Hence this theorem shows that in systems verifying axiom C time reversal symmetry is *unbreakable*: if its spontaneous breaking occurs as some parameters of the system vary (with appearance of attractive sets smaller than the whole phase space, and not invariant for the global time reversal i) the attractive pole always admits a symmetry i^* that inverts time, *i.e.* that anticommutes with time evolution. The unsatisfactory aspect of the above analysis is that in general a perturbation of a system verifying Axiom C will have all the properties of the unperturbed system with the possible exception that the poles might lose the property of being smooth surfaces and a better understanding of this is desirable.

(E) The chaotic hypothesis.

The structural stability properties of axioms A, B and C systems has been one of the reasons of the following reinterpretation and extension of Ruelle' principle of §5.7, called *chaoticity principle* or the *chaotic hypothesis*. It has not been, however, the main reason because the chaoticity principle has been reality "derived" on the basis of the interpretation of experimental results, [ECM93], [GC95].

Chaotic Hypothesis: *A mechanical system, be it a particles system or a fluid, in a chaotic stationary state* [7] *can be considered as a mixing Anosov system* for the purpose of the computation of the macroscopic properties. *In the case of reversible systems a map i^* of the attractive sets into themselves exists which anticommutes with the evolution and squares to the identity.*

In the case the system has an attracting set A which is smaller than phase space the hypothesis has to be interpreted as saying that the attracting set A is a smooth surface and the time evolution map S is such that (A, S) is a mixing Anosov system which enjoys a time reversal symmetry: both assumptions are true if the system verifies axiom C, by the theorem in subsection (D) above. Therefore it is not restrictive to suppose that the attracting set is the entire phase space and that on it a time reversal symmetry exists.

This principle, together with the considerations developed at the point (A), will allow us to propose rather detailed properties of the Navier–Stokes equation, in the §7.4.

We conclude with some comments on the meaning of the chaotic hypothesis. The hypothesis has to be interpreted in the same way one interprets

[7] We shall understand by *chaotic* any stationary distribution with *at least* one positive Lyapunov exponent which is not close to zero: this is necessary, like in equilibrium, because near a non chaotic system one may have phenomena reminiscent of those that arise in equilibrium theory with systems that are close to integrable ones.

the ergodic hypothesis in statistical mechanics. *One must not intend that a system of physical interest "really" verifies axiom C*, rather one must intend that this property holds *only for the purpose* of the computation of the average values of the few interesting observables with respect to the distribution SRB, *i.e.* the average with respect to the statistics of motions that follow initial data randomly chosen in phase space with some distribution proportional to volume, *c.f.r.* §5.7,.

We must also remark that this principle is stronger than the ergodic hypothesis: indeed it applies to non equilibrium systems and one can show, [Si94], that if the dynamical system is Hamiltonian and if it is a mixing Anosov system, then the SRB distribution is precisely the *Liouville distribution* on the constant energy surface: hence the ergodic hypothesis holds.

The ergodic hypothesis implies classical thermodynamics, even when applied to systems that manifestly are not ergodic, like the perfect gas. Likewise one has to understand that chaotic hypothesis *cannot be generally true*, strictly speaking, for many systems of interest for physics: sometimes because of the trivial reason that the evolution of these systems is described by maps S that *are not regular everywhere* but only piecewise so, *c.f.r.* Definition 5 in §5.4.

The idea is that the chaotic hypothesis could allow us to establish *relations* between physical quantities without really computing the value of any of them. In the same way as Boltzmann deduced the heat theorem (*i.e.* the equality of the derivatives of the functions of state expressing that the differential form $(dU + pdV)/T$ is exact) from a formal expression for the equilibrium distribution of a gas.

Now the role of the formal expressions for the Gibbs distributions in equilibrium statistical mechanics will be plaid, for the SRB distributions, by the formula of Sinai (7.2.1).

Bibliography: [Ru79], [Si94], [Ga81], [Ga95], [GC95].
In the original work [GC95] the chaotic hypothesis has been formulated by requiring that the dynamical system be a mixing Anosov system (that in the notations of the work was, somewhat improperly with respect to the current terminology, called transitive). As explained in [BGG97] this statement has to be interpreted, to be in agreement with experiments relative to situations *strongly* outside of equilibrium, in the sense that the system is of Anosov mixing type if *restricted* to the attracting set (*i.e.* the attracting set must be a regular surface on the which S acts in a mixing way). But in this last case it also became necessary to add the hypothesis that the time reversal was "unbreakable". The search of a geometric condition that guaranteed *a priori* the unbreakability of time reversal and that was *a priori* stable led in [BG97] to formulate axiom C.

The end point of this chain of refinements, and in a certain sense, of simplifications of the original hypothesis, *still requires* in order to "be completely satisfactory" the elimination of the hypothesis that the poles are regular manifold.

§7.3 Chaotic hypothesis, fluctuation theorem and Onsager reciprocity, entropy driven intermittency.

We consider now a rather general dynamical system: however we keep in mind the reversible NS equations introduced in §7.1 as the example to which we would like to apply the following ideas. We study motions on a regular surface V described by an equation

$$\underline{\dot{x}} = \underline{f}(\underline{x}) \tag{7.3.1}$$

whose solutions $t \to \underline{x}(t) = S_t\underline{x}$, with initial datum $\underline{x}$ admit a time reversal symmetry i. This means, see §7.1, that S_t anticommutes with the isometric operation[1] i: $iS_t = S_{-t}i$, and $i^2 = 1$.

First of all we look at motions through timed observations. This means that we imagine that our system is observed at discrete times, *c.f.r.* §5.2. Although not really necessary, this simplifies a little the discussion and reduces by one unit the dimension of phase space and allows us to consider the evolution as a map S. The phase space $M \subset V$ on which S acts can be considered as a piecewise regular surface, possibly made of various connected parts and everywhere transversal to the trajectories of the solutions of (7.3.1), see [Ge98].

The map S is related to the flow S_t by the relation $Sx = S_{t(x)}x$ if $t(x)$ is the time that elapses between the "timing event" $x \in M$ and the successive one.

The dynamical system that we study will then be (M, S) and we shall call, as usual, μ_0 a probability distribution endowed with a density with respect to the volume measure on M. Time reversal invariance becomes $iS = S^{-1}i$, with i isometry of M and $i^2 = 1$: if the continuous time evolution S_t is time reversal symmetric then the timed observations will also be such for a suitably defined i.[2]

(A) The fluctuation theorem.

On the basis of the chaotic hypothesis we imagine that the attractive set for the evolution S is the whole phase space M, without loss of generality.

[1] Supposing, under our hypotheses, that i is an isometry is not restrictive: it suffices to redefine suitably the metric so that the distance $d(x, y)$ between two close points becomes $(d(x, y) + d(ix, iy))/2$.

[2] For instance if the system is a "billiard" and the observations are timed at the collisions with the obstacles then a possible time reversal maps a collision c into the new collision ic obtained by considering the result c' of the collision c, which is no longer a collision "being a vector that comes out of the obstacle", and changing the sign of the velocity obtaining again a collision, *i.e.* "a vector entering the obstacle", that defines ic.

The distribution SRB shall have the form (7.2.3): the important point is that the combination of (7.2.3) with the time reversal symmetry is rich of consequences, surprising at least at first sight.

Note that we can suppose without loss of generality that the Markovian pavement $\mathcal{P}$, used to represent the SRB distribution via Sinai's formula, *c.f.r.* (7.2.1), could be chosen *invariant under time reversal*: *i.e.* such that if $\mathcal{P} = (P_1, \ldots, P_n)$ then iP_σ is still an element $P_{i\sigma}$ of $\mathcal{P}$ with $i\sigma = \sigma'$ suitable. In fact from the definition of Markovian pavement, *c.f.r.* §5.7 (C), it follows that

(1) by intersecting two Markovian pavements $\mathcal{P}$ and $\mathcal{P}'$ we obtain a third Markovian pavement: *i.e.* the pavement whose elements are the sets $P_i \cap P'_j$ is still Markovian.
(2) by applying the map i to the elements of a Markovian pavement $\mathcal{P}$ we obtain a Markovian pavement $i\mathcal{P}$: this is so because time reversal i transforms the stable manifold $W^s(x)$ and the unstable manifold $W^e(x)$ into $W^e(ix)$ and $W^s(ix)$, respectively.

Hence intersecting $\mathcal{P}$ and $i\mathcal{P}$ we obtain a time reversal symmetric pavement. If $iP_\sigma \stackrel{def}{=} P_{i\sigma}$ is the correspondence between elements of the pavement established by the action of i we see that i is therefore represented as the map that acts on the sequence of symbols $\underline{\sigma} = \{\sigma_k\}$ by transforming it into $\underline{\sigma}' = \{\sigma'_k\}$ with $\sigma'_k = i\,\sigma_{-k}$.

Furthermore it is not difficult to verify that this implies that a standard extension of the compatible strings $\sigma_{-\tau/2}, \ldots, \sigma_{\tau/2}$, *c.f.r.* (7.2.1) and §5.7, can be performed so that *if x_j is the center of $E_j = \cap_{k=-\tau/2}^{\tau/2} S^{-k} P_{\sigma_k}$ then ix_j is the center of iE_j.*

Let us denote by $J_\tau(x)$ the Jacobian matrix of the map S^τ, where τ is an even integer, as a map of $S^{-\tau/2}x$ to $S^{\tau/2}x$; and denote with $J_{e,\tau}(x)$ and $J_{s,\tau}(x)$ the Jacobian matrices of the same maps thought of as maps of $W^e_{S^{-\tau/2}x}$ to $W^e_{S^{\tau/2}x}$ or, respectively, of $W^s_{S^{-\tau/2}x}$ to $W^s_{S^{\tau/2}x}$. Then one can establish simple relations between the determinants of these matrices.

If $\alpha(x)$ is the angle formed, in x, between the stable and the unstable manifolds[3] and if we denote, respectively, $\Lambda_\tau(x) = |\det J_\tau(x)|$, $\Lambda_{s,\tau}(x) = |\det J_{s,\tau}(x)|$, $\Lambda_{e,\tau}(x) = |\det J_{e,\tau}(x)|$ then, noting that such determinants are related to the expansion or contraction of the elements of surface of the manifolds M, W^s_x and, respectively, W^e_x, it follows that

$$\Lambda_\tau(x) = \Lambda_{s,\tau}(x)\Lambda_{e,\tau}(x)\frac{\sin\alpha(S^{\tau/2}x)}{\sin\alpha(S^{-\tau/2}x)} \qquad (7.3.2)$$

[3] The angle between two planes that have in common only one point can be defined as the minimum angle between non zero vectors lying on the two planes attached to the common point. The angle between two manifolds that locally have only one point in common is defined as the angle between their tangent planes.

Time reversal symmetry (and its isometric character) implies that

$$\Lambda_\tau(ix) = \Lambda_\tau(x)^{-1}, \qquad \Lambda_{e,\tau}(ix) = \Lambda^{-1}_{s,\tau}(x), \qquad \Lambda_{s,\tau}(ix) = \Lambda^{-1}_{e,\tau}(x) \quad (7.3.3)$$

and if $\lambda_\tau(x) = -\log \Lambda_\tau(x)$, $\lambda_{e,\tau}(x) = -\log \Lambda_{e,\tau}(x)$, $\lambda_{s,\tau}(x) = -\log \Lambda_{s,\tau}(x)$, therefore

$$\lambda_\tau(ix) = -\lambda_\tau(x), \quad \lambda_{e,\tau}(x) = -\lambda_{s,\tau}(ix), \quad \lambda_{s,\tau}(x) = -\lambda_{e,\tau}(ix) \qquad (7.3.4)$$

note that the quantity $\lambda_\tau(x)$ is simply related to the divergence $\delta(x)$ of the equation (7.3.1), $\delta(x) = -\sum_j \partial_j f_j(x)$. If, as above, $t(x)$ denotes the time interval between the timed observation producing the result x and the next one it is $\lambda(x) = \int_0^{t(x)} \delta(S_t x)dt$. Then

$$\lambda_\tau(x) = \sum_{j=-\frac12\tau}^{\frac12\tau-1} \lambda(S^j x) \qquad (7.3.5)$$

We shall call *entropy creation* on τ timing events the contraction of the volume of phase space (that could *also be negative*, *i.e.* in fact an expansion), which is the quantity

$$\sigma_\tau(x) = \sum_{j=-\frac12\tau}^{\frac12\tau-1} \lambda(S^j x) \stackrel{def}{=} \tau\,\langle\lambda\rangle_+ p \qquad \text{if } \langle\lambda\rangle_+ \ne 0 \qquad (7.3.6)$$

where $\langle\lambda\rangle_+$ denotes the average value of the function $\lambda(x)$ with respect to the SRB distribution of the system (M, S), *c.f.r.* §7.2, §5.7 and p is a variable (that depends on τ and x) on the phase space M and that we can call the (adimensional) average *rate of creation of entropy* on τ events around x.

Whether the name of *entropy*, [An82] and Sec. 9.7 in [Ga99a], is properly used here, or not, is debatable. In reality we are interested in cases in which the quantity $\langle\lambda\rangle_+$ is not zero: such cases will be called *dissipative*. In this respect one should note a theorem that says that if such average is not zero then it is necessarily positive and this is a property that, without doubt, is certainly desired from a definition of rate of entropy creation, [Ru96].

Here we cannot invoke, to justify the use of the name "entropy", independent definitions of such notion: simply because the notion of entropy *has never been well defined* in cases of systems outside of equilibrium.

We shall adopt this name also because we shall see that this quantity has various other desirable properties that help making the notion a satisfactory one, see [An82],

The first important property is that the dimensionless rate of entropy creation p, *c.f.r.* (7.3.6), is a variable that has a probability distribution $\pi_\tau(p)$ with respect to the stationary statistics SRB that describes the asymptotic properties of motions. By definition $\langle p\rangle_+\equiv 1$ (in the dissipative cases, of course).

We shall set:

$$\zeta(p)=-\lim_{\tau\to\infty}\frac{1}{\tau}\log\pi_\tau(p) \tag{7.3.7}$$

where the existence of the function $\zeta(p)$ (in the case of a mixing Anosov system, *i.e.* if the chaotic hypothesis holds) is proved by a theorem of Sinai, [Si72], [Si77]. Then the function $\zeta(p)$ verifies, [GC95a], [GC95b],the following fluctuation theorem

I Theorem *(fluctuation theorem): The "rate function" $\zeta(p)\ge-\infty$ has odd part verifying*

$$\zeta(-p)=\zeta(p)-\langle\lambda\rangle_+\,p \tag{7.3.8}$$

for all p.

An illustration is provided by the Fig. (7.3.1).

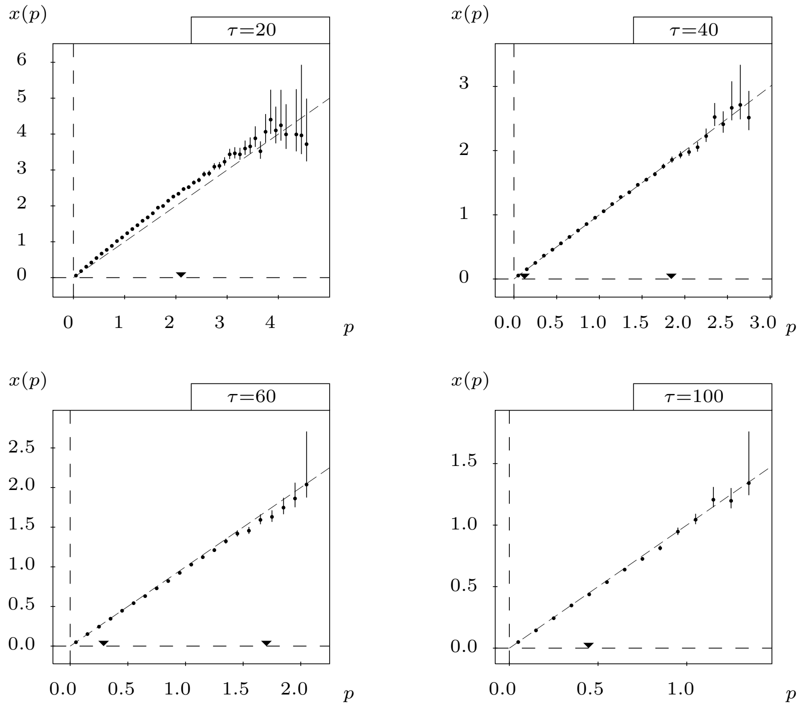

Fig. (7.3.1) *Illustration of the fluctuation theorem of §7.3 which gives $x(p)=\frac{\zeta(-p)-\zeta(p)}{\langle\sigma\rangle_+}=p$ in the limit $\tau\to\infty$, for an electrical conduction model in a very strong electromotive field, taken from the experiment in [BGG97]. The dashed graph is $x(p)=p$ while the four graphs correspond to the choices $\tau=20,40,80,100$.*

The proof is simple: informally it is the following. One can compute the ratio $\pi_\tau(p)/\pi_\tau(-p)$ by (7.2.1)

$$\frac{\pi_\tau(p)}{\pi_\tau(-p)} = \frac{\sum_{\sigma_\tau(x_j)=\langle\lambda\rangle_+\tau p} \Lambda_{e,\tau}(x_j)^{-1}}{\sum_{\sigma_\tau(x_j)=-\langle\lambda\rangle_+\tau p} \Lambda_{e,\tau}(x_j)^{-1}} \tag{7.3.9}$$

where the sum runs over the centers x_j of the partition elements $\mathcal{E}_\tau = \cap_{-\tau/2}^{\tau/2} S^h\mathcal{E}$, *c.f.r.* (5.7.8) (*i.e.* on the points whose history, with respect to the compatibility matrix of the Markovian pavement, is a compatible sequence $\sigma_{-\tau/2},\ldots,\sigma_{\tau/2}$ extended in "standard way" to an infinite compatible sequence, *c.f.r.* (7.2.1), (5.7.8)).

In the discussion that follows *we do not take into account* that (7.3.9) is not correct and that the correct formula, *c.f.r.* (5.7.8), should be

$$\frac{\pi_\tau(p)}{\pi_\tau(-p)} = \lim_{T\to\infty} \frac{\sum_{\sigma_\tau(x_j)=\langle\lambda\rangle_+\tau p} \Lambda_{e,T}(x_j)^{-1}}{\sum_{\sigma_\tau(x_j)=-\langle\lambda\rangle_+\tau p} \Lambda_{e,T}(x_j)^{-1}} \tag{7.3.10}$$

where the sum runs on the elements of the partition $\cap_{-T/2}^{T/2} S^{-k}\mathcal{E}$, *i.e.* on the sets $E_j = E_{\sigma_{-T/2},\ldots,\sigma_{T/2}}$. In other words we should first let T to ∞ and then $\tau\to\infty$. *The eq. (7.3.9), instead, considers* $T=\tau$.

Evidently by using (7.3.9) instead of the correct (7.3.10) one commits *errors* that we could fear to be unrepairable. But it is not so and the error can be bounded, *c.f.r.* [GC95a],[GC95b], and one can show (easily) that the correct ratio between $\pi_\tau(p)$ and $\pi_\tau(-p)$ is bounded from above and from below by the r.h.s. of equation (7.3.9) respectively *multiplied or divided by a factor* a *which is* τ*–independent*. Since we are only interested in the limit (7.3.7) we see that such an error has no influence on the result.

The possibility of this bound is obviously *essential* for the discussion: it is actually easy, but it rests on the deep structure of symbolic dynamics and on well known properties of probability distributions on spaces of compatible sequences, the reader is referred to [Ga95a] or [Ru99c].

Accepting (7.3.9) one remarks that the sum in the denominator can be rewritten by making use of the fact that if x_j is the center of E_j and it has adimensional rate of entropy creation p, then ix_j is center of iE_j (*c.f.r.* observations at the beginning of the section) and by time reversal symmetry, see (7.3.3), rate $-p$: hence the (7.3.9) is rewritten as

$$\frac{\pi_\tau(p)}{\pi_\tau(-p)} = \frac{\sum_{\sigma_\tau(x_j)=\langle\lambda\rangle_+\tau p} \Lambda_{e,\tau}(x_j)^{-1}}{\sum_{\sigma_\tau(x_j)=\langle\lambda\rangle_+\tau p} \Lambda_{e,\tau}(ix_j)^{-1}} \tag{7.3.11}$$

We can now remark that (7.3.3) allows us to rewrite this identity as

$$\frac{\pi_\tau(p)}{\pi_\tau(-p)} = \frac{\sum_{\sigma_\tau(x_j)=\langle\lambda\rangle_+\tau p} \Lambda_{e,\tau}(x_j)^{-1}}{\sum_{\sigma_\tau(x_j)=\langle\lambda\rangle_+\tau p} \Lambda_{s,\tau}(x_j)} \tag{7.3.12}$$

so that numerator and denominator are sums over an equal number of terms. We also see that corresponding terms (*i.e.* terms with the same label j) have ratio $\Lambda_{e,\tau}(x_j)^{-1}\Lambda_{s,\tau}(x_j)^{-1}$ *and this ratio is equal to* $\Lambda_\tau(x_j)$ apart from the factors that come from the ratios of the sines of the angles, *c.f.r.* (7.3.2), $\alpha(S^{-\tau/2}x_j)$ and $\alpha(S^{\tau/2}x_j)$: such ratios will however be bounded from below and from above by a^{-1} and a, for a suitable a (because the angles $\alpha(x)$ are bounded away from 0 and π by the assumed hyperbolicity of A, *c.f.r.* §5.4, and furthermore by taking the products of the fractions in (7.3.2) all sines simplify "telescopically" and one is left only with the first numerator and the last denominator).

We note that what said is rigorously correct only if the system (M,S), restricted to the attracting set, is really a mixing Anosov system (or just only transitive, *c.f.r.* §5.4).[4] We make use of the chaotic hypothesis when we suppose that the properties used are "in practice" true at least for the purposes of computing quantities of interest, like precisely $\zeta(p)$.

By definition of p it is $\Lambda_\tau^{-1}(x_j)=e^{p\,\tau\,\langle\lambda\rangle_+}$, for all choices of j, so that

$$\frac{1}{a}e^{-p\tau\langle\lambda\rangle_+}<\frac{\pi_\tau(p)}{\pi_\tau(-p)}<a e^{-p\tau\langle\lambda\rangle_+} \tag{7.3.13}$$

and (7.3.8) follows in the limit $\tau\to\infty$.

More generally let $\kappa_1(x),\ldots,\kappa_n(x)$ are n functions on phase space such that

$$\kappa_j(ix)=-\kappa_j(x) \tag{7.3.14}$$

i.e. they are *odd* under time reversal, define

$$\kappa_{\tau,j}(x)\stackrel{def}{=}\sum_{r=-\tau/2}^{\tau/2}\kappa_j(S^r x)=q_j\,\tau\,\langle\kappa_j\rangle_+ \tag{7.3.15}$$

Consider the joint probability, with respect to the SRB statistics, of the event in which the variables $p,q_1,\ldots,q_n$ have given values, and denote such probability by $\pi_\tau(p,q_1,\ldots,q_n)$. Let

$$\zeta(p,q_1,\ldots,q_n)=-\lim_{\tau\to\infty}\frac{1}{\tau}\log\pi_\tau(p,q_1,\ldots,q_n) \tag{7.3.16}$$

then the same argument exposed above to prove theorem I implies (obviously)

II Theorem: *(extended fluctuation theorem): The large deviations functions* $\zeta(p,q_1,\ldots,q_n)$ *verify*

$$\zeta(-p,-q_1,\ldots,-q_n)=\zeta(p,q_1,\ldots,q_n)-\tau\,p\,\langle\lambda\rangle_+ \tag{7.3.17}$$

[4] A property used several times, when using the formula (7.2.1) for the SRB distribution and now when saying that the ratios between the sines of the angles above introduced are uniformly bounded from below and from above.

if $p, q_1, \ldots, q_n$ *is a point internal to the domain* D *in which the variables* $p, q_1, \ldots, q_n$ *can assume values.*

It is very interesting, as we shall see in the successive point (B), that the last term in (7.3.17) *does not depend on the variables* q_j.

Remark: It is important to note that the fluctuation theorem can also be formulated in terms of properties of the not–discretized system, *i.e.* in terms of the quantity $\delta(x) = -\delta(ix)$, divergence of the equations of the motion, that we call rate of entropy creation per unit of time (instead of "per timing event"). And possibly in terms of the other quantities $\gamma_j(x) = -\gamma_j(ix)$ odd with respect to time reversal. The theorems are stated in the same way provided we modify the definitions of p, q_j by replacing the sums in (7.3.15) and (7.3.6) as

$$\sigma_t = p\langle\delta\rangle_+ \int_{-t/2}^{t/2} dt'\, \delta(S_{t'}x) \qquad \kappa_{t,j} = q\,\langle\gamma\rangle_+ \int_{-t/2}^{t/2} dt'\, \gamma_j(S_{t'}x) \qquad (7.3.18)$$

The extension requires a detailed analysis, *c.f.r.* [Ge98].

The above fluctuation theorems are remarkable because they can be considered *laws of large deviations* in the probabilistic sense of the term (namely they give a property of the probabilities of deviations away from the average of a sum of τ random variables and such deviations have magnitude $2p\tau$ or $2q_j\tau$: hence they have *order of magnitude much larger than the "normal size"* of fluctuations *i.e.* $\gg \sqrt{\tau}$, if p or q_j are close to their typical value ~ 1).
It is a result that can be accessible to experimental checks in many non-trivial cases: indeed the experimental observation, *c.f.r.* [ECM93], of the validity of the (7.3.8) in a special case has been the origin and the root of the development of the chaotic hypothesis and of the derivation of the above theorems. Successively is has been reproduced in several different experiments, *c.f.r.* [BGG97], [BCL98].

One should be careful to understand properly the nature of the fluctuation theorem: it involves considering non trivial limits which *cannot* be light-heartedly interchanged as discussions in the literature have amply shown, [CG99].

The interest of the relations (7.3.8) and (7.3.17) is increased because it has been noted that can they can be considered a *generalization to systems outside equilibrium* of Onsager's reciprocity relations, *c.f.r.* §1.1, and of the Green–Kubo formulae for transport coefficients, *c.f.r.* [Ga96a].

(B) Onsager's reciprocity and the chaotic hypothesis.

We shall study a typical system of N particles subjected to internal and external conservative forces, with potential $V(\underline{q}_1, \ldots, \underline{q}_N)$, and to not conservative external forces $\{\underline{F}_j\}$, $j = 1, \ldots, N$, of intensity measured by parameters $\{G_j\}$, $j = 1, \ldots, N$. Furthermore the system will be subject also to forces $\{\underline{\varphi}_j\}$, $j = 1, \ldots, N$, that have the role of absorbing the energy given to system by the nonconservative forces and, hence, of allowing the attainment of a stationary state. Let μ_+ be the statistics of the motion of an initial datum randomly chosen with a distribution μ_0, with density with respect to volume in phase space.

The equations of motion will have, therefore, the form

$$m\dot{\underline{q}}_j = \underline{p}_j, \qquad \dot{\underline{p}}_j = -\partial_{\underline{q}_j} V(\underline{q}_j) + \underline{F}_j(\{G\}) + \underline{\varphi}_j \tag{7.3.19}$$

with m = mass of the particles: they will be supposed reversible for all $\underline{G}$.

If $O(\{\underline{q}, \dot{\underline{q}}\})$ is an observable and if S_t is the map that describes the evolution, the distribution μ_+ is defined by

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T O(S_t x)\, dt = \int_M O(y)\mu_+(dy) \stackrel{def}{=} \langle O\rangle_+ \tag{7.3.20}$$

for all $x \in M$ except a set of zero μ_0–volume on M.

We shall suppose also that the rate of entropy generation $\delta(x)$, *c.f.r.* (7.3.5), which however we shall continue to denote $\sigma(x)$ (but recall that it now represents a rate of entropy generation per unit time and not per timing event) has the form:

$$\sigma(x) = \sum_{i=1}^{s} G_i J_i^0(x) + O(G^2) \tag{7.3.21}$$

an assumption that, in fact, only sets the restriction that in absence of not conservative forces it is $\sigma = 0$.

Following Onsager, one defines the *thermodynamic current* associated with the force G_i as $J_i(x) = \partial_{G_i}\sigma(x)$. Onsager's relations concern the *transport coefficients* defined by

$$L_{ij} = \partial_{G_i}\langle J_j\rangle_+\big|_{\underline{G}=0} \tag{7.3.22}$$

and establish the *symmetry* of the matrix L.

We want to show that the fluctuation theorem (7.3.8), (7.3.17) can be considered an extension to nonzero values $\underline{G}$ of the external forces (*"thermodynamic forces"*) of the reciprocity relations.

This will be obtained by computing $\zeta(p), \zeta(p, q_1, \ldots)$ for $\underline{G}$ small, up to infinitesimals of order larger than $O(|\underline{G}|^3)$ (and it will result that $\zeta(p)$ is an infinitesimal of second order in $\underline{G}$, so that in the computation infinitesimals of the third order will be neglected). The expression obtained will be compared with the fluctuation theorem and the *relations of Onsager* will follow, together with the *formulae of Green–Kubo* (also called at times *fluctuation dissipation theorem*) that imply them.

Therefore we shall say that the fluctuation theorem is a proper extension to nonzero fields of Onsager's reciprocity relations, because indeed it is valid without the conditions $\underline{G} = \underline{0}$ characteristic of the classical Onsager relations and *exactly* (if one supposes the chaotic hypothesis): a property that is also characteristic of Onsager's relations.

For simplicity we shall refer to the continuous time version of the fluctuation theorem described in connection with equation (7.3.18)..

The proof of the above statements is put in the appendix being of rather technical nature.

(C) A fluidodynamic application.

As an application we can consider the equation

$$\dot{\underline{u}} + (\underset{\sim}{u} \cdot \underset{\sim}{\partial}\, \underline{u})^{(\kappa)} = -\underline{\partial}\, p \tag{7.3.23}$$

with $\underline{u} = \sum_{\kappa<|\underline{k}|<2\kappa} \underline{u}_{\underline{k}} e^{i\underline{k}\cdot\underline{x}}$, $\underline{u}_{\underline{k}} = \overline{\underline{u}}_{-\underline{k}}$, $\underline{k}\cdot\underline{u}_{\underline{k}} = 0$; and with $f^{(\kappa)}$ we denote the truncation of the Fourier series of the function f to the modes $\underline{k}$ such that $\kappa < |\underline{k}| < 2\kappa$. We can interpret this equation as an equation describing the motion of a "single inertial shell" of Fourier modes in the Navier Stokes equation in the sense of §6.2,§6.3.

Let us suppose also that the energy $E = L^3 \sum_{\kappa<|\underline{k}|<2\kappa} |\underline{u}_{\underline{k}}|^2$ (conserved by the dynamics of (7.3.23)) is $E = C\kappa^{-2/3}$: *i.e.* is given by the energy content, in the Kolmogorov distribution, of the shell of momenta in $(\kappa, 2\kappa)$: *c.f.r.* (6.2.8) with C correspondent to a given value of ε.

We now ask which is the response of the system to the switching on of of an infinitesimal force $\underline{g}_{\underline{k}}$ acting on the mode $\underline{k}$, while the system is kept at *constant energy* E by means of a force defined by the Gauss' principle, *i.e.* assuming that the system is governed (in presence of forces) by an equation:

$$\dot{\underline{u}} + (\underset{\sim}{u} \cdot \underset{\sim}{\partial}\, \underline{u})^{(\kappa)} = -\underline{\partial} p - \kappa^2\alpha\underline{u} + \underline{g}_{\underline{k}} \tag{7.3.24}$$

with

$$\alpha = \frac{1}{\kappa^2}\frac{\sum_{\underline{k}} \overline{\underline{u}}_{\underline{k}}\cdot \underline{g}_{\underline{k}}}{\sum_{\underline{k}} |\underline{u}_{\underline{k}}|^2} \tag{7.3.25}$$

This equation is reversible and is forced by the external force $\underline{g}$. The entropy production in this equation is zero if $\underline{g} = \underline{0}$ and hence we are in the situation of the point (B).

It follows that in this regime we shall have

$$L_{\underline{k},\beta;\underline{k}',\beta'} = \partial_{g_{\underline{k},\beta}} \langle \gamma_{\underline{k}',\beta'}\rangle_+|_{\underline{g}=\underline{0}} = L_{\underline{k}',\beta';\underline{k},\beta} \tag{7.3.26}$$

because from the (7.3.25) we see that $\partial_{g_{\underline{k},\beta}}\sigma = \gamma_{\underline{k},\beta}$.

This also shows that if the conjecture of §7.1 on the equivalence of the statistical ensembles could be interpreted in a "suitably wide" sense one could perhaps deduce reciprocity relations for Navier–Stokes fluids in a regime of developed turbulence, *c.f.r.* [Ga97]. And it appears even possible that such relations could be experimentally checked both in real and numerical experiments. But setting these predictions in a mathematically and physically cleaner form, susceptible of checks, requires further analysis and ideas. I shall try to expose them in the §7.4.

(D) Physical interpretation of the fluctuation relations. Onsager–Machlup fluctuations. Entropy driven intermittency.

An important question is, naturally, "which is the physical interpretation" of the fluctuation theorem?

A simple extension of the theorem holds under the same hypotheses (*i.e.* chaotic hypothesis and time reversibility). It can be regarded as an extension of the Onsager–Machlup theory of fluctuation patterns, [OM53]. Let F, G be observables that, for simplicity, we suppose odd under time reversal, *i.e.* such that: $F(ix) = -F(x), G(ix) = -G(x)$; and let $h, k : [-T/2, T/2] \to R^1$ be two real valued functions or "*patterns*".

We call $h'(t) = -h(-t)$, $k'(t) = -k(-t)$ the "*time–reversed patterns*" or "*antipatterns*" of the patterns h, k. If $F(S_t x) = h(t)$ for $t \in [-T/2, T/2]$ we say that F follows the pattern h around the reference point x in the time interval $[-T/2, T/2] \stackrel{def}{=} W_T$. Then the following theorem can be proved in the same way as the above theorems I, II, see [Ga99c],

Theorem III *(extension of Onsager–Machlup fluctuations theory): The probabilities of the patterns h, k conditioned to a T–average dimensionless entropy production p, see (7.3.6), denoted $\pi\big(F(S_t\cdot)\big) = h(t),\, t \in W_T \,\big|\, p\big)$ and $\pi\big(G(S_t\cdot)\big) = k(t),\, t \in W_T \,\big|\, p\big)$ respectively, verify*

$$\frac{\pi\big(F(S_t\cdot) = h(t),\, t \in W_T \,\big|\, p\big)}{\pi\big(F(S_t\cdot) = -h(-t),\, t \in W_T \,\big|\, -p\big)} = 1 \tag{7.3.27}$$

and (consequently)

$$\frac{\pi\big(F(S_t\cdot) = h(t),\, t \in W_T \,\big|\, p\big)}{\pi\big(G(S_t\cdot) = k(t),\, t \in W_T \,\big|\, p\big)} = \frac{\pi\big(F(S_t\cdot) = -h(-t),\, t \in W_T \,\big|\, -p\big)}{\pi\big(G(S_t\cdot) = -k(-t),\, t \in W_T \,\big|\, -p\big)}$$

Hence relative probabilities of patterns observed in a time interval of size T and in presence of an average entropy production p are the same as those of the corresponding antipatterns in presence of the opposite average entropy production rate.

Remark: an equivalent way to write (7.3.27) is

$$\frac{\pi\big(F(S_t\cdot) = h(t),\, t \in W_T,\, p\big)}{\pi\big(F(S_t\cdot) = -h(-t),\, t \in W_T,\, -p\big)} = e^{-p\langle\lambda\rangle_+ t} \tag{7.3.28}$$

expressing the same relation as (7.3.27) in terms of joint probabilities rather than probabilities conditined to the event that in the time interval it is $\sigma_t = \langle\lambda\rangle_+ \, p\, T$

In other words it "suffices" to change the sign of the entropy production to reverse the arrow of time. In a reversible system the quantity $\zeta(p)$ measures the degree of irreversibility of a motion observed to have the value p of dimensionless entropy creation rate during an observation time of size T: if we observe patterns over time intervals of size T then the fraction of such intervals in which we shall see an entropy production p rather than 1 (which is the most probable value) will be proportional to

$$e^{(\zeta(p)-\zeta(1))T} \tag{7.3.29}$$

This tells us that normally we shall see an entropy production $p = 1$ but occasionally, with a frequency in time proportional to

$$e^{-\langle\lambda\rangle_+ T} \tag{7.3.30}$$

an entropy production $p = -1$ will be seen, *c.f.r.* (7.3.8): and it will be accompanied by very unexpected behavior of the time evolution of most observables, *c.f.r.* (7.3.28).

This is an *entropy driven intermittency* phenomenon. "Intermittency" seems to be a notion defined on a case by case basis (we have met it in a sense similar to the present in §4.3 and also in a somewhat different sense in §6.2) and we interpret here as a the phenomenon of rare, randomly spaced, interval of time during which some observables behave in avery different way with respect to their average.

In applying the above analysis to a fluid other difficulties arise due to the fact that we expect that $\langle\lambda\rangle_+$ becomes very large when the forcing, hence the Reynolds number, becomes large so that such strong fluctuations become practically unobservable.

However one can regard continua as composed by small macroscopic systems which are identified with volume elements of the continuum so that one can imagine that each such small system is a system in a stationary state transported around by the fluid motion. Therefore by making local observations on small volume elements we can imagine that fluctuations are more frequent there than in the whole fluid and that in each volume element the evolution takes place as a thermostatted evolution with chaotic fluctuations governed by a fluctuation theorem in which the rate function $\zeta_V(p)$ and the local entropy creation rate $\langle\lambda\rangle_+$ are proportional to the volume under observation $\zeta_V(p) = V\,\overline{\zeta}(p)$ and $\langle\lambda\rangle_+ = V\,\overline{\sigma}_+$ if p is the dimensionless local entropy creation rate and $\overline{\sigma}_+$ is the entropy production rate per unit volume. In other words we want think of the fluid in a stationary state as an ensemble of copies of small systems in the same conditions each of which can be described as a system evolving in presence of a thermostat.

It is difficult to get a more precise picture of the above situation and to check it in concrete model. So far there are examples of systems in which a local entropy creation rate $p\,\overline{\sigma}_+$ per unit volume and a local rate function $\overline{\zeta}(p)$ can be defined and verify a local fluctuation theorem in the sense that the rate function *per unit volume* can be properly defined and verifies

$$\overline{\zeta}(-p)=\overline{\zeta}(p)-p\,\overline{\sigma}_+ \tag{7.3.31}$$

In such systems one has a "*spatio–temporal intermittency* in the sense that the fraction of time intervals of size T in which we shall observe p in a given box of size V_0 will be $e^{(\,\overline{\zeta}(p)-\overline{\zeta}(1))\,V_0\,T}$. This same quantity will be the fraction of boxes in which we shall observe, within a given time interval of size T, entropy production p.

Normally we shall see $p=1$ in a fixed box V_0 but "seldom" we shall see $p=-1$ and then, by the above extension of the Onsager–Machlup theory, *everything will look wrong*: every improbable pattern will appear as frequently as we would expect its (probable) antipattern to appear. This will last only for a moment and then things will return normal for a very long time (as the fractions of time in which this can happen in a given box is $e^{-\,\overline{\sigma}_+V_0T}$). Furthermore, fixed a time interval of size T, we shall also see intermittency, in the form of a reversed time arrow, *happening somewhere* in a small volume V_0 in the volume V of the system, provided

$$V\,{V_0}^{-1}e^{-\,\overline{\sigma}_+V_0\,T}\simeq 1 \tag{7.3.32}$$

Hence, in such cases, there will be a simple relation between fraction of volumes and fraction of times where time reversal occurs: namely they are equal and directly measured by $e^{-\,\overline{\sigma}_+V_0\,T}$, *i.e.* by the average entropy creation rate. And the situation looks more promising from an experimental viewpoint (on real fluids) because we can imagine taking V_0 and T not too large so that the fluctuations will not be so rare to be unobservable and we find ourselves in a situation analogous to the one we meet when we try to observe density fluctuations in a rarified gas. The latter can be seen only in small volumes and intermittently in space: but the rate function that controls the fluctuations is proportional to the volume in which they are observed. The above considerations are at the basis of attempts to intepret certain experimental results, [CL98], [Ga00].

Appendix: Onsager reciprocity as a consequence of the fluctuation theorem.

The computation of $\zeta(p)$ for $\underline{G}$ small will be performed by means of a development in series. As often in statistical mechanics it is useful to first compute the Laplace transform of the probability distribution $\pi_\tau(p)=e^{-\tau\zeta(p)}$:

$$\begin{aligned}e^{\tau\lambda(\beta)}=&\int e^{\beta\tau\,(p-1)\langle\sigma\rangle_+-\tau\zeta(p)}dp=\\=&\int d\mu_+(x)e^{\beta\sum_{-\tau/2}^{\tau/2}(\sigma(S^jx)-\langle\sigma\rangle_+)}\end{aligned} \tag{7.3.33}$$

(where now τ is a continuous variable) and then deduce $\zeta(p)$ via a suitable anti-transform. Which is, as it is well known, the Legendre transform of the function λ: $\zeta(p) = \max_\beta (\beta(p-1) - \lambda(\beta))$.

Taking the logarithm of (7.3.33) and developing in series the result one finds

$$\lambda(\beta) = \frac{1}{2!}\beta^2 C_2 + \frac{1}{3!}\beta^3 C_3 + \ldots \tag{7.3.34}$$

where the coefficients C_j are combinations of average values of products of $\sigma(S^j x)$ computed at various values of j. In the limit $\tau \to \infty$, and provided the integrals

$$C_j = \int_{-\infty}^{\infty} \langle \sigma(S_{t_1}\cdot)\sigma(S_{t_2}\cdot)\ldots\sigma(S_{t_{j-1}}\cdot)\sigma(\cdot)\rangle_+^T \, dt_1 \ldots dt_{j-1} \tag{7.3.35}$$

converge absolutely, if $\langle\ldots\rangle_+^T$ denote precisely the "suitable combinations of products". Such combinations are called *cumulants* of the distribution of $\sigma(\cdot)$ and for example (as one verifies directly):

$$C_2 = \int_{-\infty}^{\infty} (\langle \sigma(S_t\cdot)\sigma(\cdot)\rangle_+ - \langle\sigma(\cdot)\rangle_+\langle\sigma(\cdot)\rangle_+)\, dt \tag{7.3.36}$$

The convergence of the integrals is a consequence of the chaotic hypothesis that implies that the dynamical system (M, S, μ_+) is mixing and mixes with exponential velocity the correlations between regular observables (as the $\sigma(x)$).

Hence the computation to second order, for which the (7.3.36) suffices, tells us that $\lambda(\beta) = \frac{1}{2}\beta^2 C_2$ and hence, inverting the transform as said after (7.3.33), we deduce an expression for $\zeta(p)$

$$\zeta(p) = \frac{1}{2}\frac{\langle\sigma\rangle_+^2}{C_2}(p-1)^2 + O((p-1)^3 G^3) \tag{7.3.37}$$

Comparing with the fluctuation theorem (*i.e.* $\zeta(-p) - \zeta(p) = p\langle\sigma\rangle_+$) and imposing the compatibility between the two relations we get

$$\langle\sigma\rangle_+ = \frac{1}{2}C_2 + O(G^3) \tag{7.3.38}$$

If we now develop the left hand side in series of $\underline{G}$ around $\underline{G} = \underline{0}$ one finds, *to second order in* $\underline{G}$ and abridging from now on ∂_{G_i} with ∂_i:

$$\langle\sigma\rangle_+ = \frac{1}{2}\sum_{ij} G_i G_j [\partial_i\partial_j\langle\sigma\rangle_+]_{\underline{G}=\underline{0}} \tag{7.3.39}$$

But the quantity $\partial_i\partial_j\langle\sigma\rangle_+$ is the sum of three terms

$$\begin{aligned} &\int \mu_+(dx)\Big(\partial_i\partial_j\sigma(x)\Big) + \int \Big(\partial_i\partial_j\mu_+(dx)\Big)\,\sigma(x) + \\ &+ \Big[\int \Big(\partial_i\sigma(x)\Big)\Big(\partial_j\mu_+(dx)\Big) + (i \longleftrightarrow j)\Big] \end{aligned} \tag{7.3.40}$$

where the first two terms obviously vanish if $\underline{G} = \underline{0}$. The first because if $\underline{G} = \underline{0}$ the distribution μ_+ is invariant by time reversal (and coincides with the Liouville distribution μ_0 because $\sigma = 0$) and σ is odd by time reversal; the second because $\sigma = 0$ if $\underline{G} = \underline{0}$. Hence

$$\partial_i\partial_j\langle\sigma\rangle_+|_{G=0} = \Big(\partial_j\langle J_i^0\rangle_+ + \partial_i\langle J_j^0\rangle_+\Big)\Big|_{G=0} \tag{7.3.41}$$

Let us remark now that if $\underline{G} = \underline{0}$ it is $\partial_i \langle J_j \rangle_+ = \partial_i \langle J^0_j \rangle_+$, because J^0 and J are odd with respect to i and differ by infinitesimals $O(G)$, by (7.3.21).

Hence since $L_{ij} \stackrel{def}{=} \partial_i \langle J_j \rangle_+$, *c.f.r.* (7.3.22), one finds (by equating the coefficients of second order in $\underline{G}$ of the two sides of the (7.3.37) and, to simplify the result, using the fact that $\langle J_i \rangle_+ |_{\underline{G}=\underline{0}} = 0$)

$$\frac{1}{2}(L_{ij} + L_{ji}) = \frac{1}{2} \int_{-\infty}^{\infty} \langle J_i(S_t \cdot) J_j(\cdot) \rangle_+ |_{G=0} dt \tag{7.3.42}$$

which, setting $i = j$, shows us that the fluctuation theorem (7.3.8) reduces, in the limit in which $\underline{G} \to \underline{0}$ to the fluctuation dissipation theorem *for a single current* (*i.e.* to the simple Green–Kubo formula).

To see that the fluctuation theorem implies more generally the Onsager's reciprocity relations and the general fluctuation dissipation theorem, always in the limit $\underline{G} = 0$, it is necessary to make use of its more general formulation in (7.3.17).

One chooses, fixed j, as observable $\kappa_1(x) \equiv \kappa(x)$ the magnitude $\kappa(x) = G_j \partial_j \sigma(x)$ that is "odd" in the sense discussed in connection with (7.3.18), and one defines $q_1 \equiv q$ as

$$\int_{-\tau/2}^{\tau/2-1} \kappa(S_t x)\, dt = \tau G_j \langle \partial_j \sigma \rangle_+ q = \tau \langle \kappa \rangle_+ q \tag{7.3.43}$$

and proceeding as in the already seen case, one computes $\zeta(p, q)$ by computing first the Laplace transform

$$e^{\tau \lambda(\beta_1, \beta_2)} = \int e^{\beta_1 (p-1) \langle \sigma \rangle_+ + (q-1) \langle \kappa \rangle_+ - \tau \zeta(p,q)} dp dq \tag{7.3.44}$$

always with the *method of the cumulants* and neglecting the third order in $\underline{G}$. The $\zeta(p, q)$ is then computed by means of a Legendre transform (as after the (7.3.33)) on two variables β_1, β_2.

Comparing the result with the fluctuation theorem one obtains, after elementary computations, analogous to those already described, the relation

$$\langle G_j \partial_j \sigma \rangle_+ = \frac{1}{2} C_{12} + O(G^3) \tag{7.3.45}$$

analogous to (7.3.38). And this relation, *now asymmetric because j plays a special role* having been fixed *a priori*, combined with (7.3.38) is translated into $L_{ij} = L_{ji}$, essentially by repeating the observations that led to the (7.3.42), *c.f.r.* [Ga96a], and, at the same time, into the Green–Kubo relation

$$L_{ij} = \frac{1}{2} \int_{-\infty}^{\infty} \langle J_i(S_t \cdot) J_j(\cdot) \rangle_+ |_{G=0} dt \tag{7.3.46}$$

which is a stronger *nonsymmetric* version of the simpler (7.3.42).

The reversibility assumption used to link the Onsager relations and the fluctuation theorem is supposed for all $\underline{G}$ near $\underline{0}$, However Onsager reciprocity holds more generally under the only assumption of reversibility at $\underline{G} = \underline{0}$ A derivation of the reciprocity solely based upon the latter assumption and on the chaotic hypothesis is possible as well, see [GR97].

Bibliography: [An82],[GC95a],[GC95b][BGG97],[Ga95a],[Ga97], [Ga99a], [Gr97]. The connection between fluctuation theorem and Green–Kubo formulae has been observed empirically in the experiments in [BGG97] where it was correctly interpreted by one of the authors (P.G.) and from it the theory of Onsager's relations of this section started.

§7.4 The structure of the attractor for the Navier–Stokes equations. Dissipative Euler Equations. Barometric formula.

To conclude this work we turn to an attempt at a further understanding of Kolmogorov's theory and to the description of further properties of what we shall call the *Navier–Stokes attractor*, meaning with this elocution the statistical properties of the invariant distribution associated with the NS evolution under constant forcing and giving its statistics.

(A) Reversible and irreversible equations for a real fluid.

We shall consider the following four equations

$$\begin{aligned}
\dot{\underline{u}} + \underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}}\, \underline{u} &= -\frac{1}{\rho}\underline{\partial} p + \underline{g} + \nu\Delta\underline{u}, \qquad \underline{\partial}\cdot\underline{u} = 0 \qquad \text{NS}\\
\dot{\underline{u}} + \underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}}\, \underline{u} &= -\frac{1}{\rho}\underline{\partial} p + \underline{g} + \beta\Delta\underline{u}, \qquad \underline{\partial}\cdot\underline{u} = 0 \qquad \text{GNS} \qquad (7.4.1)\\
\dot{\underline{u}} + \underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}}\, \underline{u} &= -\frac{1}{\rho}\underline{\partial} p + \underline{g} - \chi\underline{u}, \qquad \underline{\partial}\cdot\underline{u} = 0 \qquad \text{ED}\\
\dot{\underline{u}} + \underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}}\, \underline{u} &= -\frac{1}{\rho}\underline{\partial} p + \underline{g} - \alpha\underline{u}, \qquad \underline{\partial}\cdot\underline{u} = 0 \qquad \text{GED}
\end{aligned}$$

that describe an incompressible fluid in a region Ω that will be a tridimensional torus, possibly deprived of some circular regions (*obstacles*). For simplicity it will be convenient to suppose that the obstacles, if present, are such that by repeating them periodically in space they would "occult" infinity (*i.e.* there is no straight line that can be drawn in space without intersecting the lattice formed by the obstacles and their copies).

On the boundary of the obstacles we shall put *slip boundary conditions* *i.e.* $\underline{u}\cdot\underline{n} = 0$ if $\underline{n}$ is the normal to the obstacles.

The first equation is the NS equation with viscosity ν. The second equation is the Gaussian Navier–Stokes equation, or GNS equation, introduced in §7.1.[1] As seen in §7.1 this means that β is, (7.1.4)

$$\beta(\underline{u}) = \frac{\int_\Omega \big(\underline{\partial}\wedge\underline{g}\cdot\underline{\omega} + \underline{\omega}\cdot\,(\underset{\sim}{\underline{\omega}}\cdot\underset{\sim}{\underline{\partial}}\underline{u})\big)\,d\underline{x}}{\int_\Omega(\underline{\partial}\wedge\underline{\omega})^2\,d\underline{x}} \qquad (7.4.2)$$

[1] The symbol for the multiplier necessary to fix the total vorticity $\eta L^3 = \rho\int\underline{\omega}^2\,dx$, with $\underline{\omega} = \underline{\partial}\wedge\underline{u}$, is here changed into β, *c.f.r.* (7.1.3).

The third equation, *c.f.r.* §6.2 (D), will be called *dissipative Euler equation*, or ED: and it represents a nonviscous ideal fluid that flows on a "sticky bottom": think to the case $d = 2$ in which the fluid flows on a real surface (*i.e.* a "rough" surface). The constant χ can be called *sticky viscosity*, *c.f.r.* §6.2. In the case $d = 3$, that here interests us, this equation does not seems to be a good model for a real fluid and it will be considered initially only for the purpose of comparison with the Navier–Stokes equation. But it will result, from the discussion, that the connection between the four equations is in reality very strict and they are in a certain sense *equivalent*.

The fourth equation will be called *Gaussian–dissipative Euler equation*, or GED, and here α is a multiplier defined so that the total kinetic energy $\varepsilon L^3 = \frac{\varrho}{2}\int \underline{u}^2\,dx$ is a constant of motion *in spite of* the action of the force $\underline{g}$; this means that α is given by

$$\alpha(\underline{u}) = \frac{\int_\Omega \underline{g}\cdot\underline{u}\,d\underline{x}}{\int_\Omega \underline{u}^2\,d\underline{x}} \tag{7.4.3}$$

A similar equation but with a different constraint has been considered in [SJ93]: the constraint considered there is that the energy content per unit volume and in every shell of momentum, in the sense of the §6.2, is *pre-fixed and equal to value predicted from the theory K41* (*i.e.* $\int_{k_n}^{k_{n+1}} K(k)dk \propto (\nu\eta)^{2/3}k_n^{-2/3}$, *c.f.r.* (6.2.12), if $k_n = 2^n k_0 = 2^n\,2\pi/L$). A reversible equation with variable reversible friction appeared earlier in [Ge86], see the review [MK00].

Both the equations GED and the GNS have a symmetry in $\underline{u}$, that makes them *reversible* in the sense that if S_t is the flow that solves the equations (so that $t \to S_t\underline{u} = \underline{u}(t)$ is the solution with initial datum $\underline{u}$), then the map $i : \underline{u} \to -\underline{u}$ *anticommutes* with the time evolution

$$i\,S_t \;=\; S_{-t}\,i \tag{7.4.4}$$

In absence of results on existence and uniqueness for the equations (7.4.1) we shall consider only the truncated equations with momentum cut–off K, *c.f.r.* §2.2, §3.2 and §6.2, so large that it will be possible to suppose heuristically that the solutions of the truncated equations can be a good model for the motion.

The truncation will be performed on a convenient orthonormal base in the space of the zero divergence fields $\underline{u}$: we shall consider natural, on account of the simple boundary conditions chosen, to use the base generated by the *minimax principle*, *c.f.r.* problems of the §2.2, applied to the Dirichlet quadratic form $\int_\Omega(\underset{\sim}{\partial}\,\underline{u})^2 d\underline{x}$ defined on the space of the divergenceless fields $\underline{u} \in C^\infty(\Omega)$ and tangent to the boundaries of the obstacles: $\underline{u}\cdot\underline{n} = 0$ on $\partial\Omega$ and $\underline{\partial}\cdot\underline{u} = 0$ in Ω.

The fields of the base will then verify, *c.f.r.* §2.2, $\Delta\underline{u}_j = -E_j\underline{u}_j + \underline{\partial}_j\mu$, with $\underline{u}_j, \mu_j \in C^\infty$, if μ_j is a suitable multiplier and E_j are eigenvalues.

For example *in the case of a container with no obstacles* let $\underline{u} = \sum_{\underline{k}\ne\underline{0}} \underline{\gamma}_{\underline{k}} e^{i\underline{k}\cdot\underline{x}}$ be the representation of $\underline{v}$ as Fourier series, with $\underline{\gamma}_{\underline{k}} = \overline{\underline{\gamma}}_{-\underline{k}}$ and $\underline{k}\cdot\underline{\gamma}_{\underline{k}} = 0$; here the "momentum" $\underline{k}$ has components that are integer multiples of the "lowest" momentum $k_0 = 2\pi/L$. Then we consider the equation

$$\dot{\underline{\gamma}}_{\underline{k}} = -\vartheta(\underline{k})\underline{\gamma}_{\underline{k}} - i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} (\underline{\gamma}_{\underline{k}_1}\cdot\underline{k}_2)\,\Pi_{\underline{k}}\underline{\gamma}_{\underline{k}_2} + \underline{g}_{\underline{k}} \tag{7.4.5}$$

in which the $\underline{k}$ takes only values $0 < |\underline{k}| < K$ for some suitably large *cut–off momentum* $K > 0$ and $\Pi_{\underline{k}}$ is the projection on the plane orthogonal to $\underline{k}$. This is an equation that defines a "truncation at scale K" of the equations (7.4.1) if

$$\begin{cases}\vartheta(\underline{k}) = -\nu\underline{k}^2 & \text{NS case}\\ \vartheta(\underline{k}) = -\beta\underline{k}^2 & \text{GNS case}\end{cases}\qquad \begin{cases}\vartheta(\underline{k}) = -\chi & \text{ED case}\\ \vartheta(\underline{k}) = -\alpha & \text{GED case}\end{cases} \tag{7.4.6}$$

We shall suppose, in this case with no obstacles, that the mode $\underline{k} = \underline{0}$ is absent, *i.e.* $\underline{\gamma}_{\underline{0}} = \underline{0}$: this is possible if, as we shall suppose, the external force $\underline{g}$ does not have a component on the Fourier mode $\underline{0}$, (*i.e.* it has average zero).

In the no obstacles case it is also easy to express the coefficients α, β for the truncated equations

$$\begin{aligned}\alpha &= \frac{\sum_{0<|\underline{k}|<K} \overline{\underline{g}}_{\underline{k}}\cdot\underline{\gamma}_{\underline{k}}}{\sum_{0<|\underline{k}|<K} \underline{\gamma}_{\underline{k}}^2}\\ \beta &= \beta_i + \beta_e, \qquad \beta_e = \frac{\sum_{\underline{k}\ne\underline{0}} \underline{k}^2 \underline{g}_{\underline{k}}\cdot\overline{\underline{\gamma}}_{\underline{k}}}{\sum_{\underline{k}} \underline{k}^4 |\underline{\gamma}_{\underline{k}}|^2}\\ \beta_i &= \frac{-i\sum_{\underline{k}_1+\underline{k}_2+\underline{k}_3=\underline{0}} \underline{k}_3^2\,(\underline{\gamma}_{\underline{k}_1}\cdot\underline{k}_2)\,(\underline{\gamma}_{\underline{k}_2}\cdot\underline{\gamma}_{\underline{k}_3})}{\sum_{\underline{k}} \underline{k}^4|\underline{\gamma}_{\underline{k}}|^2}\end{aligned} \tag{7.4.7}$$

where the $\underline{k}$ takes only the values $0 < |\underline{k}| < K$, with a cut–off at momentum $K > 0$.

The cases in which the region Ω contains obstacles is very similar, even though we cannot write simple expressions for the fields of the base nor for the truncated equations, that are formally very close to the (7.4.5), *c.f.r.* §2.2, so much that for brevity we shall always refer to (7.4.5)÷(7.4.7) *even* in the cases in which we shall consider other boundary conditions (obviously in such cases we shall have to think that in reality the equations are somewhat different, for example the $|\underline{k}|$ will be in reality $\sqrt{E_j}$ *etc*, but the differences will never be important except when explicitly mentioned).

Let us denote with $S_t^{\nu,ns}\underline{u}, S_t^{\eta,gns}\underline{u}, S_t^{\chi,ed}\underline{u}, S_t^{\varepsilon,ged}\underline{u}$ the solutions of the equations (7.4.5), or of the corresponding ones in the cases with obstacles, corresponding to a given initial datum $\underline{u}$. Or in general

$$S_t^\xi \underline{u}, \quad \xi = (\nu, ns), \, (\eta, gns), \, (\chi, ed), \, (\varepsilon, ged) \tag{7.4.8}$$

The label ξ specifies the model that we consider among the four models $\xi = ns, gns, ed, ged$; η is the total vorticity $\int_\Omega (\underline{\partial}\wedge\underline{u})^2 d\underline{x}$, constant in the GNS evolution, and ε is the total energy $\int_\Omega \underline{u}^2 d\underline{x}$, constant in the ED evolution. Note that, in general, the "phase space" is not the same for the various models, because in some of them (GNS and GED) the velocity fields are subjected to constraints.

Keeping the nonconservative force $\underline{g}$ constant, we shall suppose that for every equation, *i.e.* for every choice of the label ξ, (7.4.8), there is only one stationary distribution μ_ξ that describes the statistics of a given initial $\underline{u}$, chosen with a distribution μ_0 endowed with density on phase space: note that, being $K < \infty$, phase space has finite dimension.

The value of K will be fixed in the case NS setting $K = k_\nu$, *c.f.r.* eq. (6.2.9) and (D) in §6.2; in the case GNS one shall choose $K = K^1$ so large that the average value $\langle\beta\rangle_{\eta,gns} \stackrel{def}{=} \tilde\nu$ becomes independent of K (we suppose that this is possible) and then we shall choose $K = \max(k_{\tilde\nu}, K^1)$.[2] In the cases ED and GED we shall make analogous choices of K, always under the hypothesis that at $\chi, \underline{g}$ fixed there exists a value of K such that the time averages of the observable α become "effectively" K–independent.

What follows however *does not* depend on the choice made on K: hence one could also avoid presupposing this "ultraviolet" stability hypothesis with respect to the values of K (that could easily appear unreasonable) and it will be enough that one only supposed that K is "large".

We assume existence of the statistics: which means that, given an observable F on phase space $\mathcal{F}$ (of the velocity fields with cut–off to momentum K), for some probability distribution μ_ξ it is

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(S_t^\xi \underline{\gamma}) dt = \int_{\mathcal{F}} F(\underline{\gamma}') \mu_\xi(d\underline{\gamma}') \stackrel{def}{=} \langle F\rangle_\xi \tag{7.4.9}$$

for all choices of $\underline{\gamma}$ except for a set of volume zero (with respect to the volume measure on phase space). The latter will be called the *SRB distribution* for the equations (7.4.5),(7.4.6).

A particular role will be plaid, as we can imagine from the analysis of the §7.1, §7.2, §7.3, by the $\langle\eta\rangle_\xi, \langle\varepsilon\rangle_\xi$ and by the averages $\langle\alpha\rangle_\xi, \langle\beta\rangle_\xi$. Together, obviously, with the rate of entropy production $\sigma(\underline{\gamma})$ *defined*, in agreement with what was said at §7.3, by the *divergence* of the r.h.s. of the truncated equations and of its average $\langle\sigma\rangle_\xi$ (*c.f.r.* §7.1 where this quantity has been denoted $\delta(x)$).

Considering explicitly the case with no obstacles let

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[2] The discussion that follows hints that $K^1 \equiv k_{\tilde\nu}$.

• D_K the number of modes $\underline{k}$ with $0 < |\underline{k}| < K$: so that the number of (independent) components $\{\gamma_{\underline{k}}\}$ is $2D_K$ (note that $\gamma_{\underline{k}}$ has two independent complex components for each $\underline{k}$ but $\gamma_{\underline{k}} = \overline{\gamma}_{-\underline{k}}$).
• $2\,\overline{D}_K = \sum_{|\underline{k}|<K} 2\underline{k}^2$ (that in the case with obstacles would become the quantity $2\,\overline{D}_K = \sum_{\sqrt{E_j}<K} E_j$)).

One finds that the phase space contraction per unit time is σ given by

$$\begin{aligned}
\sigma =&2\,\overline{D}_K\nu && \xi = (\nu, ns)\\
\sigma =&2\,\overline{D}_K\beta - \overline{\beta}_e - \overline{\beta}_i && \xi = (\eta, gns)\\
\sigma =&2D_K\chi && \xi = (\chi, ed)\\
\sigma =&2D_K\alpha - \alpha && \xi = (\varepsilon, ged)
\end{aligned} \tag{7.4.10}$$

where $\overline{\beta}_i$, $\overline{\beta}_e$ are suitably defined. For example in the case without obstacles:

$$\overline{\beta}_e = \frac{\sum_{\underline{k}} \underline{k}^2\,\overline{\underline{g}}_{\underline{k}}\cdot\gamma_{\underline{k}}}{\sum_{\underline{k}} \underline{k}^4 |\gamma_{\underline{k}}|^2} - 2\frac{\big(\sum_{\underline{k}} \underline{k}^2\,\overline{\underline{g}}_{\underline{k}}\cdot\gamma_{\underline{k}}\big)\big(\sum_{\underline{k}} \underline{k}^4 \gamma_{\underline{k}}^2\big)}{\big(\sum_{\underline{k}} \underline{k}^4 |\gamma_{\underline{k}}|^2\big)^2} \tag{7.4.11}$$

hence $\sigma \simeq 2\,\overline{D}_K\beta$ for $\xi = (\eta, gns)$ and $\sigma \simeq 2D_K\alpha$ for $\xi = (\varepsilon, ged)$.
With these definitions (*c.f.r.* §7.1) the following conjecture has been proposed that I will call the *conjecture of statistical equivalence* of the GNS and the NS dynamics

Conjecture *(equivalence betwee NS and GNS): The statistics $\mu_{\nu,ns}$ and $\mu_{\eta,gns}$ of the NS equation and, respectively, of the GNS equation are* equivalent *in the limit in which the Reynolds number R tends to infinity* provided *the parameters η and ν are related so that that $\langle\sigma\rangle_{\nu,ns} = \langle\sigma\rangle_{\eta,gns}$ (or, equivalently, $\nu = \langle\beta\rangle_{\eta,gns}$).*

Equivalent means that the ratios between average values of the same "local observables" with respect to the two distributions tend to 1 as $R \to \infty$. By *local observable* we mean an observable that depends on the field $\underline{u}$ only through the components of scale contained between two fixed values in the inertial domain: *i.e.* only through the components of the field $\gamma_{\underline{k}}$ with "*scale*" $|\underline{k}|$ such that $k_1 < |\underline{k}| < k_2$ with $k_1 \gg k_0$ and $k_2 < \infty$: the "locality" is therefore understood in the "space of momenta"; the Reynolds number is defined here as $R = \eta^{1/3}L^{4/3}\nu^{-1}$, *c.f.r.* comments to the (6.2.9). In the case of nonperiodic boundary conditions the *scale* of a component γ_j of the field will be determined by the value $\sqrt{E_j}$ of the correspondent eigenvector $\underline{u}_j$.

And an analogous equivalence conjecture between statistics can be formulated for the equations ED and GED

Conjecture *(equivalence between ED and GED): The statistics $\mu_{\chi,ed}, \mu_{\varepsilon,ged}$ of the equations ED and of the equations GED are equivalent in the limit of large Reynolds number provided the parameters ε and χ are related so that $\langle\sigma\rangle_{\chi,ed} = \langle\sigma\rangle_{\varepsilon,ged}$ (or $\chi = \langle\alpha\rangle_{\varepsilon,ged}$).*

Anyone who has some familiarity with statistical mechanics will recognize in the just stated conjectures a strong analogy with the corresponding statements on the equivalence between statistical ensembles, *c.f.r.* [Ga95c]: with the limit $R \to \infty$ that plays the role of the thermodynamic limit.

The idea of the possibility of describing in terms of *statistical ensembles* the statistical properties of systems outside equilibrium has gradually developed in the recent literature and the idea of the possibility of equivalent descriptions in terms of different statistical ensembles emerged at the same time, *c.f.r.* [Ge86], [ES93], [SJ93], [Ga95b],[Ga96], [MK00].

On a heuristic basis the conjectures would be justified *if the rate of entropy creation would reach its average value on a time scale which is rapid with respect to the time scales characteristic of hydrodynamics.* The coefficients $\alpha \simeq (2D_K)^{-1}\sigma$, and $\beta \simeq (2\,\overline{D}_K)^{-1}\sigma$, (7.4.10), could then be identified with their average values $\langle\alpha\rangle_{\varepsilon,gne}$ or $\langle\beta\rangle_{\eta,gns}$ and *hence identified with the viscosity constant* ν or χ.

In this way the GNS and NS equations would be equivalent and both would be macroscopic manifestations of two equivalent microscopic mechanisms of dissipation. One esplicitly specified by the Gaussian constraint of constant total vorticity (and hence total dissipation, by the proportionality between the two quantities), the other one with *a priori* fluctuating dissipation but that can be phenomenologically modeled by means of a constant viscosity.

The same can be said of the relation between the equations ED and GED.

We now look at the problem of understanding how to extract from the conjectures just discussed some consequence observable in experiments. This will be possible by combining what said above with the chaotic hypothesis of the previous sections.

(B) Axiom C and the pairing rule.

Unfortunately it will be still necessary to propose assumptions that it will not be possible to justfy other than by possibly checking some of their consequences. Nevertheless since this is the conclusive section of a “foundations of fluid mechanics” I feel that it will be permitted to discuss them, also because a discussion somewhat out of balance in a really heuristic direction can be stimulating.

What follows has, therefore, to be seen as a collection of ideas that developed naturally while meditating on the many works consulted to accomplish the task that I undertook several years ago, *i.e.* of presenting to the students of my course of fluid mechanics a guided introduction to a very vast field of research.

The main difficulty for the application of the fluctuation theorem to the GNS or GED equations, in spite of their reversibility, is that they are systems *far out of equilibrium* hence we cannot imagine that the attractive set be the whole phase space: typically we expect indeed that the attractor has *finite Hausdorff dimension* (and there exist several arguments in favor of this idea), while the full phase space has *a priori* infinite dimension, [FP67], [Ru82], [Li84]), [Ru84].

Then the chaotic hypothesis tells us that "things proceed as if" the attractor was a smooth, finite dimensional, surface on which dynamics is well modeled by an evolution S that is hyperbolic.

If furthermore we suppose that the dynamics verifies the axiom C, the reversibility of the GED or GNS equations will imply the existence, *c.f.r.* §7.2, of a time reversal map i^* *that leaves invariant the attracting set.*

Hence the fluctuation theorem will hold, *c.f.r.* §7.3. But obviously the contraction of phase space that enters in its formulation *not will be* σ because the latter is the contraction $\sigma(x)$ of the *total* volume and not the contraction $\sigma_0(x)$ of the volume element on the surface of the attracting set.

Of course *one cannot hope to characterize* in a simple way the attracting set for the purpose of computing its element of surface. A theory that made reference to the "equations of the attracting set" would risk strongly to remain totally inapplicable (see, nevertheless, the case of the GOY model of§6.3 in which equations of the attracting set can be proposed and used, *c.f.r.* (6.3.22)).

Help comes from a property whose validity has been slowly changing from a "curiosity" to "interesting but exceptional" to "interesting and often verified". It is a remarkable property of the Lyapunov exponents of chaotic systems related in some way to Hamiltonian systems afflicted by dissipative phenomena.

It has been noted, starting with the work of Dressler, [Dr88], that in certain Hamiltonian systems with ℓ degrees of freedom and subject to particular forms of friction the Lyapunov exponents, arranged as

$$\lambda_\ell^+ \ge \lambda_{\ell-1}^+ \ge \ldots \ge \lambda_1^+ \ge \lambda_1^- \ge \lambda_2^- \ge \ldots \ge \ldots \lambda_\ell^-$$

in decreasing order and adding a superscript $\pm$ to distinguish the ones with the same subscript, are such that

$$\frac{1}{2}(\lambda_j^+ + \lambda_j^-) = constant \qquad \text{for each } j \tag{7.4.12}$$

In fact this property is even true, at least in the first examples in which it was found and if the metric that is used is suitably chosen, for the eigenvalues of the matrices of *local expansion and contraction*, *i.e.* also without considering the limit that appears in the definition of Lyapunov exponents; in such case the constant depends on the phase space point where the expansions and contractions are computed. To be precise we define here what we mean by *local Lyapunov exponents* over a time τ: they are the eigenvalues of the matrix $(J_\tau^T(x)J_\tau(x))^{1/2\tau}$ with $J_\tau(x)$ being the Jacobian matrix of the map S^τ as a map between $S^{-\tau/2}x$ and $S^{\tau/2}x$ (τ even).

The value of the constant in (7.4.12) is (obviously) the average value of the phase space contraction. For an illustration of a numerical check of the rule (in a case it can be proved to be rigorously valid see Fig. (7.4.1)).

The breakthrough on the above "*pairing rule*" has been due to an experimental discovery, [ECM90]: an actual mathematical proof came only later: this is just one example of a property that is mathematically important and yet relatively "easy" to prove but which has been missed by mathematicians. For an illustration of a numerical check of the rule (in a case in which it can be proved to be rigorously valid) see Fig. (7.4.1).

Recently the *pairing rule* (7.4.12), [DM97], has been shown valid, *in the local version*, for rather wide classes of systems (called *isokinetic*) subject to Gaussian dissipative constraints in which forcing takes place through the action of locally conservative external forces (but not globally conservative, like an electromotive electric field). A typical example being a system of particles subject to the Gaussian constraint of keeping total kinetic energy constant and to a force that tends to establish a current of matter in a (not simply connected) container. An important extension is in [WL98].

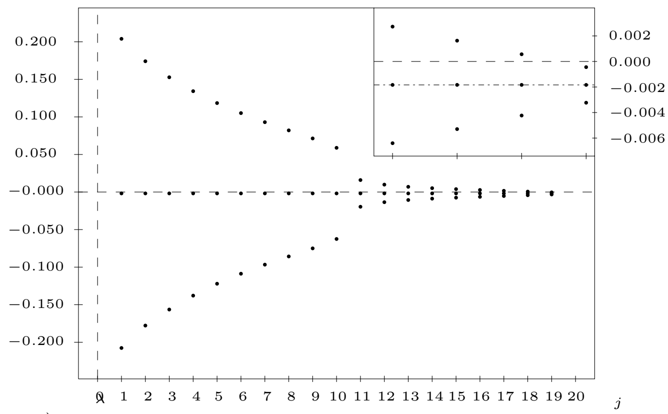

Fig. (7.4.1) *The* 38 *Lyapunov exponents for a model of electrical conduction, in a very large electromotive field; the model has* 19 *degrees of freedom and is governed by isokinetic equations. The small figure is an enlargement of the tail of large one and it shows the pairing rule and the fact that at such field the* 19*–th exponent is slightly negative and hence the attracting set has dimension* lower than that of the phase space. From [BGG97].

The interest of a dissipative reversible system for which the pairing rule holds in a local sense lies in the fact that it seems possible to establish for it in a natural way a relation between the contraction, $\sigma_0(x)$, around x of the surface element of the attracting set and that, $\sigma(x)$, of the volume element with full dimension.

Indeed, following the analysis in [BGG97], if $2N$ is the dimension of the phase space and $2(N-M)$ that of the attracting set (assumed smooth) it will be that the contraction of the phase space around points located on the surface of the attracting set A is $\sigma(x)=\sigma_0(x)+\sigma_\perp(x)$ where σ_0 is the rate of contraction "on" the attracting set and $\sigma_\perp$ that on the part of the stable manifold of the point x which is a manifold that partly "sticks out of A", *c.f.r.* footnote [5] in 7.2.

This can be interpreted by thinking that the tangent space in x consists of $2(N-M)$ directions, $N-M$ of which expansive and $N-M$ contracting, all tangent to A and in $2M$ directions *all contracting* that instead concern the part of stable manifold that sticks out of the attracting set; *furthermore*, it appears natural that the *pairs* of exponents are divided into $N-M$ *pairs* relative to the $2(N-M)$ directions tangent to A and in the $2M$ remaining ones.

The above is not the only possibility, but it is certainly the simplest. And if it is verified then we deduce immediately, assuming the local pairing rule (*i.e.* (7.4.12) with an x–dependent constant) that

$$\sigma_0(x)=\frac{N-M}{N}\sigma(x) \tag{7.4.13}$$

i.e. there is *proportionality between the the total contraction $\sigma(x)$ of phase space and that, $\sigma_0(x)$, of the element of surface on the attracting set.*

Since $\sigma(x)$ is directly accessible, or at least more directly accessible, than the individual Lyapunov exponents we realize the great potential of the pairing rule. For example combined with time reversibility, axiom C and the fluctuation theorem of the §7.3 it tells us that, *c.f.r.* (7.3.8):

$$\zeta(p)-\zeta(-p)=(1-\frac{M}{N})\langle\sigma\rangle_+ p \tag{7.4.14}$$

where $\zeta(p)$ is defined by (7.3.7) in terms of the *total* contraction (given by σ) of phase space.

Eq. (7.4.14) gives the result, perhaps surprising at first sight, that the slope of the $\zeta(p)-\zeta(-p)$ as a function of p *diminishes* by $1-\frac{M}{N}$ if the dimension of the attractor diminishes, *i.e.* if the viscous phenomena increase. This result, although still not accurately checked by any experiment, seems at least consistent with the results of the experiments in §6 of [BGG97] (that have inspired it) and with a few others that followed.

What has all this to do with (7.4.1)? First of all if the forcing field $\underline{g}$ is locally conservative (possible only if the container Ω has holes!)[3] one must

[3] Because it is not possible to define on a torus a vector field $\underline{g}$ with zero average locally conservative and not globally such (hence that cannot be trivially absorbed in the pressure term of the equation): in order that such a field exists it is necessary that there be holes, *i.e.* that the regions that the fluid can occupy be not simply connected (*e.g.* periodic).

remark that, *c.f.r.* [Ga96], the systems ED and GED are *naturally* among the cases in which the pairing rule has been proved. The first is a particular case of the theorem in [Dr88] while the second is a particular case of the theorem in [DM97].

The key observation is that the Euler equations can be thought of as *half* of the equations that describe the motion of the fluid: the other half of the equations describes the field of the displacements $\underline{\delta}(x)$ of the points of the incompressible fluid with respect to the positions that they have in a reference configuration

$$\dot{\underline{\delta}}(x) = \underline{u}(\underline{\delta}(x)), \qquad \dot{\underline{u}}(x) + \underset{\sim}{u} \cdot \underset{\sim}{\partial} \underline{u} = -\frac{1}{\rho} \underline{\partial}\, p(x) \tag{7.4.15}$$

as it was discussed in detail to §1.7, point (E), (1.7.29).

Having made this observation it is easy to realize that, at least formally, the ED equations are just equations obtained by imposing an external locally conservative force and a friction proportional to the velocity. While the GED case, correspond to imposing the same external force and an isokinetic constraint, *i.e.* the constraint $\int \underline{u}^2 dx = constant$, by means of the Gauss' principle.

Hence we get the formal validity of the pairing rule *for the equations (7.4.15) in a phase space with a double number of dimensions*: in which the displacements $\underline{\delta}(x)$ with respect to a reference configuration are also described, see [Ga96] for the details. Naturally the Lyapunov exponents of the GED equation *will not* verify the pairing rule because several of the pairs will consist in pairs of exponents of which one is relative to the GED and the other is relative to the additional degrees of freedom "of displacement".

But an attentive examination of the pairing rules proofs, see [DM97], [WL98], in the cases in which they are known to hold, induce to think that, in spite of the many examples, the pairing rule *is not general* and conclusive numerical evidence has been provided in [BCP98]. For example it *does not seems reasonable* that it holds *in the case of the GNS equation*. Hence to be able to obtain informations, relevant for real fluids, from the fluctuation theorem it is necessary, besides assuming the validity of the axiom C, some extension of the pairing rule that allows us *to establish a relation between the contraction rate σ of the whole phase space and that σ_0 of the surface of the attracting set*.

The discussion in [Ga96] proposes indeed *to define* in the case of the GNS equation the numbers c_j as

$$c_j = \frac{\lambda_j^+ + \lambda_j^-}{\langle\beta\rangle_+} \tag{7.4.16}$$

where $\langle\beta\rangle_+$ is the average of β, see (7.4.1), with respect to the distribution $\mu_{\eta,gns}$. Here the exponents $\lambda^{\pm})j$ are the ones of the GNS equation *coupled* with the displacement equation $\dot{\underline{\delta}} = \underline{u}(\underline{\delta}(\underline{x}))$.

In the case NS one must instead define c_j by means of the (7.4.16) with ν that replaces $\langle\beta\rangle_+$. And the paper [Ga97] proposes, then, the following generalization of the pairing rule

The local Lyapunov exponents verify (7.4.16) in the average and the average value is reached very rapidly for Reynolds number R large. So that one can consider (7.4.16) as locally true: hence one can repeat the argument that links the contraction of phase space to the contraction of the surface of the attracting set.

The volume contraction in the phase space of the equations in question is the same whether one considers the equations just as equations only for the velocity fields or also for the fields of velocity $\underline{u}$ and of displacement $\underline{\delta}$. This is due to the "triangular" structure of the Jacobian matrix due to the fact that the displacements equation $|\dot{V}\delta(\underline{x}) = \underline{u}(\underline{\delta}(\underline{x}))$ decouples from the equation of the velocity field, *c.f.r.* (7.4.15). We deduce that the fluctuations of the total contraction $\sigma(x)$ of phase space verifies a fluctuation theorem and the quantity $\zeta(p) - \zeta(-p)$ is linear in p.

The coefficient of proportionality is measurable from the statistics of the solutions of the equations for the field $\underline{u}$ and has value $P\langle\alpha\rangle_+$ in the cases of the GED equations and $\overline{P}\langle\beta\rangle_+$ in the case of the GNS equations where P is the number of pairs of Lyapunov exponents with an element > 0 and one < 0 divided by the total number of pairs, while $\overline{P}$ is $\sum_j^* c_j / \sum_j c_j$ where $\sum_j^*$ runs over the only j which corresponds to a pair of Lyapunov exponents of opposite sign.

On the basis of the equivalence conjectures we could hope to translate some predictions on the equations GNS or GED, immediately, into predictions for the equations NS and ED, respectively. Defining α for ED via the equations (7.4.7) and σ via the corresponding fourth of (7.4.10) we could expect that the fluctuations of σ verify a fluctuation relation with slope $P\,\chi$.

Likewise in the NS case we would expect that, defining β via (7.4.7) and σ via the second of (7.4.10), then σ verifies a fluctuation theorem with slope $\overline{P}\,\nu$. *This, of course, when the conditions of equivalence of the conjectures at point (A) are verified.*

What said may however be doubted because of the fact that the fluctuation theorem deals with the quantity σ, divergence of the equations of the motion, that is a "nonlocal" quantity in the space of the momenta (in the sense above indicated): rather it is global. In statistical mechanics the analogous observables are nonlocal observables that are, often, observables that have different distributions *even in statistically equivalent ensembles* (think to the total energy in the canonical ensemble and in the microcanonical ensemble).

Perhaps the mentioned relation is reasonably valid only if applied to the quantity σ_Δ defined as σ but replacing the integrals on Ω in (7.4.2) and (7.4.3) with integrals on a small volume Δ internal to the fluid: unfortunately a satisfactory analysis of this idea, *c.f.r.* [Ga96], [Ga00], is lacking although it is desirable, since it could lead to "very stringent" predictions that could even conceivably be experimentally checkable, [Ga00].

We address now the problem of showing that there exists a strict relation between the four equations (7.4.1). Developing an idea already in [Ga97].

(C) Relation between the NS and ED equations: the barometric formula.

Meditating on the ED or GED equations and on the NS or GNS equations one is led to think that the relation between them may be similar to the relation that one finds in statistical mechanics between the equilibrium distribution of a gas at different heights when the gas is in the field of gravity.

Locally a gas in a field appears simply as a homogeneous gas in equilibrium, but globally (on a length scale H on which the external potential changes substantially: *i.e.* $\beta mgH \sim 1$, if β is the inverse temperature and m the mass of the particles) one shall see that the pressure and the density are not constant. To describe their variations one arrives at the so called *barometric formula*, *c.f.r.* [MP72].

Likewise we can expect that the stationary states of the ED (or equivalently of the GED) are *also* "locally (in momentum space)" equivalent to stationary states for the NS or GNS: in the sense that if we consider observables that depend on the velocity field components $\underline{u}_{\underline{k}}$ with modes $\underline{k}$ on a certain scale $|\underline{k}| \sim \kappa$ then we should essentially see no difference at all provided the amount of energy present in this shell (that depends on some parameters related to the initial data that generate the stationary states for the two equations) is arranged to be the same.

The exact relation that determines κ will be called *barometric formula*: and it should not be difficult to determine the barometric formula on the basis of considerations of dimensional nature. Note that here "*locality*" has to be understood, as in (A),(B) *in the space of the momenta rather that in that of the coordinates.*

The determination of the barometric formula consists, essentially, in the development of a theory analogous to that of Kolmogorov K41 for the equations ED, *c.f.r.* [Ga96].

One can try to develop such theory in the case of a container without obstacles and on the basis of some hypothesis that at the moment seems reasonable to me. By changing the hypothesis the result could change in its analytic form but the fundamental idea on which the derivation that follows is based is independent of the details of the theory proposed as analogous to the theory K41.

We shall suppose, just to be concrete, as "reasonable" that in the case of the ED equations the stationary distribution equipartitions energy between the modes, *i.e.* $\langle|\underline{\gamma}_{\underline{k}}|^2\rangle = \gamma^2$ for all the $\underline{k}$ in the "inertial domain" $L^{-1} \ll |\underline{k}| \ll k_\chi$ where k_χ is the scale where the ultraviolet cut–off, necessary for giving a mathematical meaning to the equations, is performed *c.f.r.* §6.2 (D). Hence $\gamma^2(k_\chi L)^3 = \varepsilon$ will be the total energy.

For purpose of comparison with (D) in §6.2 we note that the quantity there called ε corresponds to $\eta\nu$ here.

In this case the distribution of energy (*i.e.* the amount $K(k)dk$ of energy per unit volume and between k and $k+dk$) is $K(k)=\frac{3\varepsilon}{4\pi}\frac{k^2}{k_\chi^3}$, for $k<k_\chi$: very different from the law $k^{-5/3}$ of Kolmogorov. It is rather analogous to the law of Rayleigh–Jeans of the black body, *c.f.r.* [Ga92].

In the theory K41 a key role is plaid by the quantity $v_\kappa^3\kappa$ that is identical to $\eta\nu$ for all values $k_0\ll\kappa\ll k_\nu$. Therefore let us compute the value of $v_\kappa^3\kappa$ in our case. We find

$$\frac{v_\kappa^3\kappa}{\varepsilon\chi}=\frac{((\kappa L)^3\gamma^2)^{3/2}\kappa}{\varepsilon\chi}=\frac{\big((k_\chi L)^3\gamma^2\big)^{3/2}k_\chi}{\varepsilon\chi}\big(\frac{\kappa}{k_\chi}\big)^{11/2}=\frac{\varepsilon^{3/2}k_\chi}{\varepsilon\chi}\big(\frac{\kappa}{k_\chi}\big)^{11/2} \tag{7.4.17}$$

and we see that the quantity $v_\kappa^3\kappa$ depends on κ in the ED case.

Given κ the SRB statistics for the ED equations in a stationary state with total energy ε attributes to this quantity the same value that it has in the SRB statistics for the NS equations in a stationary state with total vorticity η if

$$\frac{\varepsilon\chi}{\eta\nu}=\frac{\chi}{\sqrt{\varepsilon}k_\chi}\big(\frac{\kappa}{k_\chi}\big)^{-11/2} \tag{7.4.18}$$

provided (naturally) κ is smaller than the "Kolmogorov scales" k_ν, k_χ.

The "barometric formula" is then the statement of equivalence between NS and ED on the scale κ, i.e. if we only look to the properties of the velocity field that depend on $\gamma_{\underline{k}}$ for $\frac12\kappa<|\underline{k}|<\kappa$, if (7.4.18) holds and $\kappa\ll k_\nu,k_\chi$.

If we look on a different scale $\kappa'=2^n\kappa$ for some n (large) then we can expect equivalence between ED or (GED) and NS (or GNS), *but the pairs* ε,η will have now to be such that the equation (7.4.18) holds on the new scale.

The analogy with the usual barometric formula for the distribution of Boltzmann–Gibbs in the field of gravity justifies the name given to the (7.4.18). We see that $\eta\,\nu$ play the role of the gravity, $\varepsilon\,\chi$ that of the chemical potential while κ/k_χ that of the height.

The barometric formula is only an example of the consequences that we can draw from the equivalence conjectures between the stationary states for various equations that describe the dynamics of a given system. *It is interesting also because it gives us the possibility of obtaining the statistics of the NS equation on a given scale by simulating a different simpler equation* (as it is an equation in which no second derivatives appear).

The above analysis seems well in agreement with the spirit of the analysis in [SJ93] that first proposed, in a different context and with different perspectives, a vision that has strong similarity with that discussed here.

Nevertheless in order that the above be accepted as a correct, although phenomenological, analysis it is necessary to determine also the constant k_χ:

as is explained in §6.2 (D), we cannot determine it without a much more detailed theory of the equations ED; the multiplicative constant (*i.e.* k_χ) in (7.4.18) cannot be determined and "only" the exponent 11/2 in the barometric formula is heuristically established.

Problems.

[7.4.1] (*Gauss' principle applied to fluids*) Equations derived on the basis of the Gauss principle depend on the *effort function* $\mathcal{E}(\underline{a})$ of the accelerations that one wants to minimize. The choice $\mathcal{E}(\underline{a}) = ((\underline{f} - m\underline{a})/m)^2$ that was used in the problems of §7.1 was just an example. In systems with infinitely many degrees of freedom this ambiguity becomes even more obvious. For instance we could minimize conditioning to a *fixed total energy* (and to incompressibility) the "effort" $\mathcal{E}_1(\underline{a}) \stackrel{def}{=} \big((\underline{a} + \underline{\partial}p - \underline{f}), (\underline{a} + \underline{\partial}p - \underline{f})\big)$ or, subject to the same constraint, $\mathcal{E}_2(\underline{a}, s) \stackrel{def}{=} \big((\underline{a} + \underline{\partial}p - \underline{f}), (-\Delta)^{-1}(\underline{a} + \underline{\partial}p - \underline{f})\big)$. Check that, in toroidal geometry (*i.e.* periodic boundary conditions), imposing $\mathcal{E}_1(\underline{a})$ on divergenceless fields $\underline{u}$ with the constraint $\varphi \stackrel{def}{=} \int (\underline{u})^2\, d\underline{x}$, leads to the GED equations while imposing $\mathcal{E}_2$ leads to equations that look like the incompressible NS equations. In the latter case the equations obtained are not the GNS equations of §7.1, or (7.4.1), (7.4.2): they coincide with the second of (7.4.1) but the multiplier β is different. Compute β in the latter case. (*Idea:* β has to be such that energy rather than dissipation stays exactly constant in time.)

[7.4.2] Check that, in toroidal geometry, the GNS equations in (7.4.1), (7.4.2) can be obtained by applying the Gauss principle with effort $\mathcal{E}_1(\underline{a}) \stackrel{def}{=} \big((\underline{a}+\underline{\partial}p-\underline{f}), (\underline{a}+\underline{\partial}p-\underline{f})\big)$ and constraint $\varphi \stackrel{def}{=} \int (\underline{\partial}\underline{u})^2 d\underline{x} = const$ on the divergenceless fields $\underline{u}$. (*Idea:* Note that $\varphi = \int \underline{u} \cdot \Delta\underline{u}\, d\underline{x}$.)

Bibliography: [Ga96], [SJ93], [Ga95b],[Ga95]. The importance of the pairing rule in the isokinetic cases for the purpose of the application of the fluctuation theorem has been noted in the course of the work of interpretation of the experimental results in [BGG97] by one of the authors (F.B.). The fluid equation with constrained constant energy with effort function $\mathcal{E}_1(\underline{a})$ has been considered in [BPV98] with the purpose of checking the conjecture of equivalence of §7.4. The numerical experiment is performed on the GOY model, rather than on the NS equation, and the results seems to indicate that the equation with constrained energy and with effort given by the analogue of $\mathcal{E}_1$ in problem [7.4.1] gives results in agreement with the conjecture. This seems to be *not so* for the equation with constraint of constant dissipation (*i.e.* the analogue of the GNS equation for the GOY model); it is in this work that the the effort function $\mathcal{E}_1$ was introduced thus extending the conjectures. Experiments on 2–dimensional NS with all the above constraints, and more, have been performed in [RS99] with results that seem always compatible with the conjectures: however the latter experiments are performed with rather severe truncations of the NS equations so that they may not be testing the conjectures at really large Reynolds numbers.

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