CHAPTER IV

Incipient turbulence and chaos

Lack of periodicity is very common in natural systems, and is one of the distinguishing features of turbulent flow. Because instantaneous flow patterns are so irregular, attention is often confined to the statistics of turbulence, which, in contrast to the details of turbulence, often behave in a regular well–organized manner. The short–range weather forecaster, however, is forced willy–nilly to predict the details of the large scale turbulent eddies –the cyclones and anticyclones– which continually arrange themselves into new patterns. Thus there are occasions when more than the statistics of irregular flow are of very real concern. (E.N. Lorenz, 1962, [Lo63] p. 131.)

§4.1 Fluids theory in absence of existence and uniqueness theorems for the basic fluidodynamics equations. Truncated NS equations. The Rayleigh's and Lorenz' models.

Analysing the fundamental problems of the NS equation has, in particular, brought up clearly the lack of an adequate algorithm, *i.e.* convergent and constructive, for its solution. Furthermore even if we knew that the fluids equations had unique and regular solutions, for regular initial data, (for the NS equation (this is true if $d = 2$ and likely if $d = 3$, false if $d \geq 4$) this would not help much to the understanding of the physical properties of such solutions at large times.

An analogous situation is met in kinetic theory of gases. Assuming that the interaction between atoms is bounded below, one easily finds that Newton's equations admit global solutions. However this gives us no help in the derivation of the properties of gases (equation of state, fluctuations, transport coefficients, *etc*). And as soon as N, number of particles, is of the order of few decades it becomes impossible to make use of the methods for the construction of solutions, not because they are no longer valid (they are) but because not even the largest conceivable computers (not to speak of the existing ones) will ever be able to apply the solution algorithms to perform, accurately, and in a reasonable time the necessary calculations (*i.e.* in a time comparable to that of human life or even to the age of the Universe).

This did not hinder the development of a deep and (at least in various respects) satisfying theory of gases and materials.

Wishing to provide a theoretical frame for the study of the asymptotic behavior in time of 2 and 3–dimensional fluid motions we are *therefore* forced to change attitude.

The question that we shall attempt to answer is whether it is possible to set up a theory without really entering into the mathematical details of the properties of the solutions of the NS or Rayleigh equations (for instance). And we should add the adjective "possible" to the word "solutions" because, as we saw, no known methods exist to construct not even one of the weak solutions of the NS equation (or for that matter of the Rayleigh equations), aside from the trivial cases in which one can construct explicitly a global solution *c.f.r.* problems in §3.3. We shall mostly refer to the incompressible NS equations but identical comments can be made for other fluidodynamical equations.

The first remark is that the NS equation, in spite of the privileged role that it plaid so far in our analysis, *is not a fundamental equation.* We must keep in mind that it is a phenomenological equation, obtained under various assumptions, *c.f.r.* (B) in §1.2. It is possible to modify a little the mathematical interpretation of the assumptions made in its derivation to obtain equations different from the NS equations but which should be physically equivalent to them for what concerns predictions about fluid motions.

For instance considering, always for the sake of simplicity, a fluid in a cubic container with periodic boundary conditions, the hypothesis that the velocity gradient "be small", made in justifying the constitutive equation

$$T_{ij} = -p\delta_{ij} + \eta(\partial_i u_j + \partial_j u_i) \tag{4.1.1}$$

could be interpreted by saying that not only $\partial_i u_j$ must be small, but that there should exist a minimal length scale λ below which there are no variations of $\underline{u}$, *i.e.* by saying that the Fourier transform $\hat{\underline{u}}_{\underline{k}}$ of $\underline{u}$ does not have components with wavelength shorter than λ, *i.e.* with $|\underline{k}| > 2\pi\lambda^{-1}$.

Hence the equations that would be obtained would be the NS equations *truncated* at $|\underline{k}| < 2\pi\lambda^{-1}$: for which we have seen that there are global existence and uniqueness theorems and constructive approximation algorithms. If however the velocity develops, in the course of time, Fourier components which have non negligible amplitudes at $|\underline{k}| \simeq 2\pi\lambda^{-1}$ it will be necessary to give up using the NS equation to describe the motion: and pass to more elaborated equations that could be non fluidodynamic equations, possibly even involving the atomic nature of the fluid.

It is, for instance, clear that if $\lambda \simeq$ mean free path in the fluid and if the harmonics of $\underline{u}$ with wavelength $\simeq \lambda$ have importance in describing the motion, then (4.1.1), and hence the NS equation itself, is inadequate for the representation of the motion.

We can therefore take the point of view that a theory of fluids can be developed by using the NS equations truncated with an ultraviolet cut–off at a wave vector $|\underline{k}| \simeq K \stackrel{def}{=} W/\nu$ where W is a velocity variation characteristic of the initial datum and ν is the dynamical viscosity so that W/ν is a typical quantity with the dimension of inverse length: at least if initially the Fourier modes with non negligible amplitude are those with $|\underline{k}| \ll W\nu^{-1}$. Note that $W/\nu = L^{-1}R$ if R is the Reynolds number.

We shall worry about the validity of the model only if, in the course of time, the motion under investigation will develop Fourier harmonics with $|\underline{k}|$ of the order of $\sim K$. In such case it will be possible to continue using the same equations but with a larger ultraviolet cut–off, and so on until we are forced to use a cut–off so large that the continuum model for the fluid becomes unreasonable (*e.g.* when the cut–off reaches the atomic scale).

It is an empirical fact that by letting smooth initial data (with "few" harmonics) evolve they do not develop, as time goes, harmonics with an ever shorter wave length but the motion evolves asymptotically, if the external force is time idependent, by "confining" the relevant harmonics and their amplitudes. In §6.2, *c.f.r.* equation (6.2.9), we shall see that $K = L^{-1}R$ is usually a "generous" ultraviolet cut–off if one considers the evolution of a velocity field obtained after the system has evolved a long enough time to have reached a stationary state. Indeed one expects an effective cut–off to act at wave numbers smaller than $K' \stackrel{def}{=} L^{-1}R^{3/4}$, in the sense that the relevant harmonics will be related to modes $\le K'$ substantially smaller than the cut–off K, while higher harmonics, even if initially present, will rapidly decay so that the equations with a cut off at $K' = L^{-1}R^{3/4}$ should already faithfully represent the fluid motion.

This attitude allows us to attempt at developing an empirical theory of incompressible fluid motions by describing them with differential equations that are, or that are believed to be, equivalent to the NS equations, unless we realize that the theory itself implies phenomena that force us to change the model to obtain a more correct description.

In this way we can set up a procedure to build mathematical models for fluids that often reveal themselves self consistent (in the sense that they do not evolve interesting initial data towards situations in which the approximations and assumptions made in deriving the model fail).

The interest of this viewpoint is that one is led to models that do not present mathematical difficulties, for what concerns existence, uniqueness and constructive approximability of the solutions, and nevertheless are models that can be used to illustrate physical phenomena.

A classic example, and important as well, is provided us by the *Lorenz' model* for convection. It is a model derived via a hyper simplification of the Rayleigh's convection model, §1.5. The Lorenz' treatment of the model (including its derivation) has a special historical importance because it establishes in a clear way a method of analysis that has been followed in most successive theoretical studies of the onset of turbulence, [Lo63].

To avoid considering the Lorenz' model as a mathematical curiosity it is useful to keep in mind the analysis of §1.5, where we devoted attention to the physical foundations of Rayleigh's model.

We shall also illustrate some truncations of the bidimensional NS equation with periodic boundary conditions. Both the Lorenz' model and the truncations of the NS equation that we shall discuss, will have to be considered of mathematical or of exemplificatory interest, except perhaps in cases of very low Reynolds numbers: they will serve to illustrate in the coming sections, concrete examples in which certain mechanisms become manifest that we believe are at the origin of the development of turbulence and of chaotic motions. "Developed" or "strong" turbulence is a phenomenon important only at very large Reynolds' numbers and it will the object of study after that of the initial turbulence that appears at (relatively) small Reynolds' number, see Ch. VI, VII.

(A) The 2*–dimensional Saltzman's equations.*

We recall that in the theory of convection, §1.5, we denoted $\underline{u}$ the velocity field and ϑ the variation of the temperature field with respect to the linear temperature profile (*i.e.* the trivial "*thermostatic solution*") that one would have if the fluid remained motionless, just conducting heat from the top (colder) plate of the container to the bottom (warmer) one: see (1.5.17).

The 2–dimensional version of Rayleigh's model is obtained by assuming that $\underline{u},\vartheta$ in (1.5.17) are y–independent. In this case there is a velocity potential $\psi=\psi(x,z)$ such that:

$$\underline{u}=(-\partial_z\psi,0,\partial_x\psi) \tag{4.1.2}$$

and (1.5.17) can be rewritten in terms of ψ rather than of $\underline{u}$. With the notation

$$\underset{\sim}{u}\cdot\underset{\sim}{\partial}\, f\equiv(-\partial_z\psi\,\partial_x f+\partial_x\psi\,\partial_z f)\equiv\frac{\partial(\psi,f)}{\partial(x,z)} \tag{4.1.3}$$

The Rayleigh's equations (1.5.17) take a form called "*Saltzman's equation*"

$$\begin{aligned}\partial_t\vartheta+R\frac{\partial(\psi,\vartheta)}{\partial(x,z)}=&\sigma^{-1}\Delta\vartheta+R\partial_x\psi,\qquad \psi|_{z=0}=\psi|_{z=1}=0\\ \partial_t\Delta\psi+R\frac{\partial(\psi,\Delta\psi)}{\partial(x,z)}=&\Delta^2\psi+R\partial_x\vartheta,\qquad \vartheta|_{z=0}=\vartheta|_{z=1}=0\end{aligned} \tag{4.1.4}$$

which differ from the Saltzman's equation in [Lo63] only because we write them in a dimensionless form; R is the Reynolds number σ is the "Prandtl's number", (*c.f.r.* §1.5).

The equation $\underline{\partial}\cdot\underline{u}=0$ is now an identity, if $\underline{u}$ is given by (4.1.2), while the condition $\psi(x,0)\equiv\psi(x',1)$ translates the condition of vanishing total horizontal momentum and at the same time the condition of 0 horizontal velocity at the boundaries $z=0,1$ ("bottom" and "top" of the fluid).[1] The

[1] If $\psi(x,0)$ and $\psi(x,1)$ are constants it is: $u_z=\partial_x\psi=0$ if $z=0,1$. Furthermore, the two constant values of ψ for $z=0$ and $z=1$ are related because we require that the total horizontal momentum per unit horizontal length vanishes: in fact if the two constant values are equal it is: $\int u_x dxdz=-\int\partial_z\psi dxdz=-\int(\psi(x,1)-\psi(x,0))dx=0$.

fact that the constant value of ψ, common to $z=0$ and $z=1$, is set equal to 0 simply reflects the property of ψ of being defined up to an arbitrary additive constant.

(B) A priori estimates.

Consider first a continuously differentiable solution of the general 3–dimensional Rayleigh's equations, (1.5.17)

If $\delta T<0$, *i.e.* if the temperature $T_0-\delta T$ of the upper surface is warmer than that, T_0, of the lower surface, then by multiplying the first of (1.5.18) by $\underline{u}$ and the second by ϑ, one finds

$$\frac{1}{2}\frac{d}{dt}E(t)=-S(t)\qquad\text{with}\tag{4.1.5}$$
$$E(t)=\int\underline{u}^2\,d\underline{x}+\int\vartheta^2\,d\underline{x},\qquad S(t)=\int(\underset{\sim}{\partial}\underline{u})^2\,d\underline{x}+\sigma^{-1}\int(\underline{\partial}\vartheta)^2\,d\underline{x}$$

hence $E(t)\le E(0)$.

If, instead, $\delta T>0$ (*i.e.* the upper surface is colder than the lower) we find from (1.5.17):

$$\frac{1}{2}\frac{d}{dt}E(t)=-S(t)+2R\int\vartheta u_z\,d\underline{x}\le -S(t)+RE(t)\tag{4.1.6}$$

(having exploited $2ab\le a^2+b^2$), hence

$$E(t)\le E(0)e^{2Rt}\tag{4.1.7}$$

We can obain more interesting bounds if we consider the problem with periodic horizontal boundary conditions, *i.e.* we demand that $\underline{u},\vartheta$ be periodic functions in x,y at fixed z, with period l^{-1}, where $l>0$ is a fixed parameter (we are using dimensionless units, so that l is a dimensionless quantity). Then

$$S(t)\ge((2\pi l)^2+\pi^2)\,\min(1,\sigma^{-1})\,E(t)\stackrel{def}{=}\pi^2\,\mu\,E(t)\tag{4.1.8}$$

because the expansion of the fields $\underline{u},\vartheta$ in Fourier series in x and in a sine series in z has minimum momentum $2\pi l$ in the x–direction and π in the z–direction (in our units, the vertical dimension is 1). Hence, in the *uninteresting* case $\delta T<0$ (*c.f.r.* (1.5.18)), (4.1.5) implies

$$E(t)\le E(0)e^{-2\pi^2\,\mu\,t}\xrightarrow[t\to+\infty]{}0\tag{4.1.9}$$

and the "no–motion" state $\vartheta\equiv 0$, $\underline{u}=\underline{0}$ is always a *stable equilibrium.*

While if $\delta T>0$ the no–motion state is stable (by (4.1.6), (4.1.8)) if $2R<\pi^2\mu$, *i.e.* if R is small enough; and, for general data, we can only say

$$E(t)\le E(0)e^{-(2\pi^2\mu-2R)t}\tag{4.1.10}$$

although, even for R large, the estimate (4.1.10) could be excessively pessimistic (at least in some cases one can check that it can be improved to $E(t) \le const$).

In the 2–dimensional case one gets from (4.1.4) an estimate of $(\underset{\sim}{\underline{\partial}} \underline{u})^2$ and $(\Delta\psi)^2$, similar to the (4.1.10) but with $\underline{u}^2 = (\underline{\partial}\psi)^2$ replaced by $(\Delta\psi)^2$:

$$\begin{aligned}&\tfrac{1}{2}\dot{E}_1 \equiv \tfrac{1}{2}\tfrac{d}{dt}\int(\vartheta^2 + (\Delta\psi)^2)\,d\underline{x} \le \\ &\le -\int\big((\Delta\underline{\partial}\psi)^2 + \sigma^{-1}(\underline{\partial}\vartheta)^2\big)\,d\underline{x} + R\int(\vartheta\partial_x\psi + \Delta\psi\,\partial_x\vartheta)\,d\underline{x}\end{aligned} \tag{4.1.11}$$

This implies that the quantity $E_1 \equiv \int(\vartheta^2 + (\Delta\psi)^2)\,d\underline{x}$ is bounded by

$$E_1(t) \le E_1(0)e^{2Ct} \tag{4.1.12}$$

and C can be estimated in terms of R, σ, l, *c.f.r.* problems.

The *a priori* bound in (4.1.12) suffices, with (4.1.8), to develop a theory of existence of weak solutions. And, *in the 2-dimensional case*, the (4.1.12) permits us to derive a proof of an existence, uniqueness and regularity theorem (when the initial data are C^∞) analogous to the one discussed in §3.2 for the NS equations.

From the bounds on E_1 (hence on $\int(\underset{\sim}{\underline{\partial}}\underline{u})^2 d\underline{x}, \int\vartheta^2 d\underline{x}$), and thinking the equations written as in (1.5.17), we can repeat the auto–regularization theory of §3.2 in the present case. The only substantial change is that now the boundary conditions are different, and the velocity and temperature fields must be developed over a basis different from the Fourier basis, *c.f.r.* (4.1.13). One finds that if $|\vartheta_{\underline{k}}| < C_\beta|\underline{k}|^{-\beta}$ and $|\underline{u}_{\underline{k}}| < C_\alpha|\underline{k}|^{-\alpha}$ with $\beta \ge 0$ and $\alpha \ge 2$, and $C_a, C_\beta < \infty$, then $\vartheta, \underline{u}$ are C^∞.

(C) The Lorenz' model.

The model is obtained by considering the (4.1.4) with periodic conditions in the variable $x \in [0, 2a^{-1}]$, where a is a dimensionless parameter (we are using here the dimensionless equations: in the original coordinates the length scale associated with $2a^{-1}$ would be: $2a^{-1}H$).

Supposing that ϑ and ψ are of class C^∞ and choosing a suitable basis adapted to periodic conditions in x and to vanishing conditions at $z = 0, 1$, we can write

$$\begin{aligned}\vartheta(x,z) &= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=1}^{\infty}\vartheta_{k_1k_2}e^{i\pi k_1 lx}\sin\pi k_2 z\\ \psi(x,z) &= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=1}^{\infty}\psi_{k_1k_2}e^{i\pi k_1 lx}\sin\pi k_2 z\end{aligned} \tag{4.1.13}$$

with $\vartheta_{k_1k_2} = \overline{\vartheta}_{-k_1k_2}$ and $\psi_{k_1k_2} = \overline{\psi}_{-k_1k_2}$. We can then study special solutions that admit a development like (4.1.13) with $\vartheta_{\underline{k}}$ and $\psi_{\underline{k}}$ real: in fact, for symmetry reasons, if the initial data can be written as in (4.1.13) with $\vartheta_{\underline{k}}, \psi_{\underline{k}}$ real then this representation remains valid for all times $t > 0$, for every regular solution of the (4.1.4). The equations (4.1.4) become (after some trigonometry)

$$\begin{aligned}\dot\vartheta_{r_1r_2} =& -\sigma^{-1}\pi^2(l^2r_1^2+r_2^2)\vartheta_{r_1r_2}+\pi l r_1 R\psi_{r_1r_2}-\frac{\pi^2 l}{4}R\cdot\\ &\cdot\sum_{\substack{\varepsilon=\pm1\\ \eta=\pm1}}\ \sum_{\substack{|k_1+\eta h_1|=r_1\\ |k_2+\varepsilon h_2|=r_2}}\big(k_1h_2-\eta k_2 h_1\big)\psi_{k_1k_2}\vartheta_{h_1h_2}\sigma_2\\ \dot\psi_{r_1r_2} =& -\pi^2(l^2r_1^2+r_2^2)\psi_{r_1r_2}+\frac{\pi l r_1}{\pi^2(r_1^2l^2+r_2^2)}R\,\vartheta_{r_1r_2}-\frac{\pi^2 l}{4}R\cdot\\ &\cdot\sum_{\substack{\varepsilon=\pm1\\ \eta=\pm1}}\ \sum_{\substack{|k_1+\eta h_1|=r_1\\ |k_2+\varepsilon h_2|=r_2}}\frac{\pi^2(l^2h_1^2+h_2^2)}{\pi^2(l^2r_1^2+r_2^2)}\big(\eta k_1h_2-\varepsilon k_2h_1\big)\psi_{k_1k_2}\psi_{h_1h_2}\sigma_1\sigma_2\end{aligned}\tag{4.1.14}$$

where σ_1, σ_2 are the signs of $k_1+\eta h_1$ and $k_2+\varepsilon h_2$, respectively. And $\underline{r}, \underline{k}, \underline{h}$ are integer, non negative, components vectors with one component at least always positive (*c.f.r.* (4.1.13)).

The (4.1.14) are the "spectral form", *c.f.r.* §2.2, of the (4.1.4) for solutions with initial data that can be written as (4.1.13).

It is believed, and it has been shown by (numerical) experiments of Saltzman's, [Sa62], [Lo63], that at least as R and σ vary in suitable intervals there exist initial data which, following (4.1.14), evolve as $t \to +\infty$ towards a state in which only three spectral components $\vartheta_{\underline{k}}, \psi_{\underline{k}}$ are not negligible; and precisely the components

$$\psi_{11} \stackrel{def}{=} \sqrt{2}X, \quad \vartheta_{11} \stackrel{def}{=} \sqrt{2}Y, \quad \vartheta_{02} \stackrel{def}{=} -Z \tag{4.1.15}$$

The equations (4.1.14) truncated *over the latter* 3 *components* become

$$\begin{aligned}\dot X =& -\pi^2(1+a^2)X+\frac{\pi a R}{\pi^2(1+a^2)}Y\\ \dot Y =& -\sigma^{-1}\pi^2(1+a^2)Y-\pi^2 aRXZ+\pi aRX\\ \dot Z =& -4\sigma^{-1}\pi^2 Z+\pi^2 aRXY\end{aligned}\tag{4.1.16}$$

To simplify (4.1.16) we rescale the variables with suitable scaling parameters λ, μ, ν, ξ (which are symbols that are not to be confused with physical quantities that we occasionally denoted with the same letters)

$$x \equiv \lambda X, \qquad y = \mu Y, \qquad z \equiv \nu Z, \qquad \tau \equiv \xi t \tag{4.1.17}$$

so that the equations can be rewritten as

$$\begin{aligned}\dot x =& \xi^{-1}(-\pi^2(1+a^2)x+\frac{\pi a\lambda R}{\pi^2(1+a^2)\mu}y)\\ \dot y =& \xi^{-1}(-\sigma^{-1}\pi^2(1+a^2)y-\frac{\pi^2 aR\mu}{\lambda\nu}xz+\frac{\pi aR\mu}{\lambda}x)\\ \dot z =& \xi^{-1}(-4\sigma^{-1}\pi^2 z+\frac{\pi^2 aR\nu}{\lambda\mu}xy)\end{aligned}\tag{4.1.18}$$

Following Lorenz' normalizations we shall choose

$$\begin{aligned}&\xi^{-1}\sigma^{-1}\pi^2(1+a^2)=1,\quad \xi^{-1}\frac{\pi^2 aR\nu}{\lambda\mu}=1,\quad \xi^{-1}\frac{\pi^2 aR\mu}{\lambda\nu}=1,\\ &r\stackrel{def}{=}\xi^{-1}\frac{\pi aR\mu}{\lambda},\quad \xi^{-1}\frac{\pi aR\lambda}{\pi^2(1+a^2)\mu}\stackrel{def}{=}\sigma,\qquad b\equiv\frac{4}{(1+a^2)}\end{aligned}\tag{4.1.19}$$

obtaining the "*Lorenz' model*" equations

$$\begin{aligned}\dot x&=-\sigma x+\sigma y\\ \dot y&=-y-xz+rx\\ \dot z&=-bz+xy\end{aligned}\qquad\text{with}\qquad \begin{aligned}b&=4(1+a^2)^{-1}\\ \sigma&=R_{Pr}\\ r&=\frac{(\pi a)^2R^2\sigma}{(\pi^2(1+a^2))^3}\equiv\frac{R_{Ray}}{R_a}\end{aligned}\tag{4.1.20}$$

where $R_a\equiv\pi^4(1+a^2)^3a^{-2}$ and R_{Pr},R_{Ray} are the Prandtl and Rayleigh numbers, defined in §1.5, *c.f.r.* for instance (1.5.15).

Remark: (symmetry) It is important to note that the equations (4.1.20) are invariant under a (simple) symmetry group; namely under the 2–elements group consisting of the transformations:

$$x\to\varepsilon\,x,\ y\to\varepsilon\,y,\ z\to z\qquad \varepsilon=\pm1\tag{4.1.21}$$

which we call the "symmetry group" of (4.1.20).

The (4.1.20) are such that the thermostatic solution $\vartheta=\psi=0$ loses stability for $R_{Ray}>R_a$, *i.e.* for $r>1$, (as shown by Rayleigh in the more general context of the 3–dimensional Rayleigh's equation, and as it can be checked immediately by studying the linearized equation near a thermostatic solution). The value of a that yields the minimum of R_a is $a^2=1/2$, which shows that the convective instability arises by generating convective motions that are spatially periodic with a period of length $2a^{-1}H\equiv2\sqrt2H$ (in dimensional units).

The equations were studied by Lorenz, [Lo63], choosing

$$a^2=\frac12,\qquad b=\frac83,\qquad \sigma=10.\tag{4.1.22}$$

A choice simply due to the fact that $a^2=1/2$ (which implies the value $b=8/3$, *c.f.r.* (4.1.20)), is the value of a for which the static solution $\vartheta=\psi=0$ becomes unstable at the smallest value of the number R_{Ray}. The value of σ is a value of the order of magnitude of the Prandtl's number of water in normal conditions, *c.f.r.* table in §1.5.

(D) Truncated NS Models.

Likewise we can study systems of ordinary equations obtained by truncation from the NS equations. It is interesting to study them together with the Lorenz' model in order to illustrate other phenomena that, although very frequent, do not show up in its phenomenology.

We shall only consider a few truncations of the NS equations in 2–dimensions with periodic boundary conditions. The latter equations, written in spectral form, see (3.2.6), (3.2.26), are

$$\dot{\gamma}_{\underline{k}} = -\underline{k}^2\gamma_{\underline{k}} - i\sum_{\Delta(\underline{k})}\gamma_{\underline{k}_2}\gamma_{\underline{k}_3}\frac{(\underline{k}_2^{\perp}\cdot\underline{k}_3)(\underline{k}_3^2-\underline{k}_2^2)}{|\underline{k}_1|\,|\underline{k}_2|\,|\underline{k}_3|} + g_{\underline{k}} \tag{4.1.23}$$

where the sum runs over the set, denoted $\Delta(\underline{k})$, of all unordered pairs of coordinates, $\underline{k}_2,\underline{k}_3$ such that $\underline{k}_2+\underline{k}_3=\underline{k}$. It is the sum over the "triads of interacting modes" considered in §2.2; the velocity and external force fields are, respectively,

$$\underline{u}(\underline{x}) = \sum_{\underline{k}\neq\underline{0}}\gamma_{\underline{k}}\frac{\underline{k}^{\perp}}{|\underline{k}|}e^{i\underline{k}\cdot\underline{x}},\qquad \underline{g}(\underline{x}) = \sum_{\underline{k}\neq\underline{0}}g_{\underline{k}}\frac{\underline{k}^{\perp}}{|\underline{k}|}e^{i\underline{k}\cdot\underline{x}}, \tag{4.1.24}$$

and the viscosity ν is set equal to 1, for simplicity; *c.f.r.* also (2.2.10), (2.2.25), (3.2.26).

The truncations will have $\underline{k}$ and $\Delta(\underline{k})$ chosen among the following seven "modes"

$$\begin{array}{llll}\underline{k}_1=(1,1), & \underline{k}_2=(3,0), & \underline{k}_3=(2,-1), & \underline{k}_4=(1,2),\\ \underline{k}_5=(0,1), & \underline{k}_6=(1,0), & \underline{k}_7=(1,-2) & \end{array} \tag{4.1.25}$$

and their opposites.

One checks that the equations truncated on the modes in (4.1.25) are (setting $\gamma_i\equiv\gamma_{\underline{k}_i}$):

$$\begin{aligned}
\dot{\gamma}_1 &= -2\gamma_1 + \frac{4i}{\sqrt{10}}\gamma_2\,\overline{\gamma}_3 - \frac{4i}{\sqrt{10}}\gamma_4\,\overline{\gamma}_5, \qquad \dot{\gamma}_2 = -9\gamma_2 - \frac{3i}{\sqrt{10}}\gamma_1\gamma_3\\
\dot{\gamma}_3 &= -5\gamma_3 - \frac{7i}{\sqrt{10}}\,\overline{\gamma}_1\gamma_2 - \frac{9i}{5\sqrt{2}}\gamma_1\gamma_7 + r, \qquad \dot{\gamma}_4 = -5\gamma_4 - \frac{i}{\sqrt{10}}\gamma_1\gamma_5\\
\dot{\gamma}_5 &= -\gamma_5 + \frac{3i}{\sqrt{10}}\,\overline{\gamma}_1\gamma_4 + \frac{i}{\sqrt{2}}\gamma_1\,\overline{\gamma}_6, \qquad \dot{\gamma}_6 = -\gamma_6 - \frac{i}{\sqrt{2}}\gamma_1\,\overline{\gamma}_5\\
\dot{\gamma}_7 &= -5\gamma_7 - \frac{9i}{5\sqrt{2}}\gamma_3\,\overline{\gamma}_1
\end{aligned} \tag{4.1.26}$$

where the external force has been supposed to have only one component: with intensity r on the mode $\pm\underline{k}_3$. The (4.1.26) admit special solutions in which

$$\begin{aligned}\gamma_j &= x_j = \text{real}, & j &= 1,3,5\\ \gamma_j &= ix_j = \text{imaginary}, & j &= 2,4,6,7\end{aligned} \tag{4.1.27}$$

and such a solutions are generated by initial data of the form (4.1.27) (an easily checked symmetry property) provided r is real, as we shall suppose from now on.

The equations (1.4.26) for complex data become a set of seven real equations for solutions of the form (4.1.27)

$$\begin{aligned}
\dot x_1 &= -2x_1 - \frac{4}{\sqrt{10}}x_2x_3 + \frac{4}{\sqrt{10}}x_4x_5, \qquad \dot x_2 = -9x_2 - \frac{3}{\sqrt{10}}x_1x_3\\
\dot x_3 &= -5x_3 + \frac{7}{\sqrt{10}}x_1x_2 + \frac{9}{5\sqrt{2}}x_1x_7 + r, \qquad \dot x_4 = -5x_4 - \frac{1}{\sqrt{10}}x_1x_5\\
\dot x_5 &= -x_5 - \frac{3}{\sqrt{10}}x_1x_4 + \frac{1}{\sqrt{2}}x_1x_6, \qquad \dot x_6 = -x_6 - \frac{1}{\sqrt{2}}x_1x_5\\
\dot x_7 &= -5x_7 - \frac{9}{5\sqrt{2}}x_3x_1
\end{aligned} \tag{4.1.28}$$

which we shall call NS_7*–model* or seven modes truncated model of the 2–dimensional NS equations (studied in [FT79]).[2]

Remark: (symmetry) We note that the equations (4.1.28) are invariant under a (simple) symmetry group; namely under the 4–elements group consisting of the transformations:

$$\begin{aligned}
&\gamma_1 \to \varepsilon\,\eta\,\gamma_1, \qquad \gamma_2 \to \varepsilon\,\eta\,\gamma_2, \qquad \gamma_3 \to \gamma_3,\\
&\gamma_4 \to \eta\,\gamma_4, \qquad \gamma_5 \to \varepsilon\,\gamma_5, \qquad \gamma_6 \to \eta\,\gamma_6, \qquad \gamma_7 \to \varepsilon\,\eta\,\gamma_7
\end{aligned} \tag{4.1.29}$$

with $\varepsilon, \eta = \pm 1$, which we call the "symmetry group" of (4.1.28).

A simpler model, (considered in [BF79], [FT79])[3] is the following NS_5–model obtained by considering the special truncation

$$\begin{aligned}
\dot x_1 &= -2x_1 - \frac{4}{\sqrt{10}}x_2x_3 + \frac{4}{\sqrt{10}}x_4x_5, \qquad \dot x_2 = -9x_2 - \frac{3}{\sqrt{10}}x_1x_3\\
\dot x_3 &= -5x_3 + \frac{7}{\sqrt{10}}x_1x_2 + r, \qquad \dot x_4 = -5x_4 - \frac{1}{\sqrt{10}}x_1x_5\\
\dot x_5 &= -x_5 - \frac{3}{\sqrt{10}}x_1x_4
\end{aligned} \tag{4.1.30}$$

which is obtained by setting $x_6 = x_7 \equiv 0$ in (4.1.28). The (4.1.30) are invariant under the same 4–elements *symmetry group* in (4.1.29):

$$\begin{aligned}
&\gamma_1 \to \varepsilon\,\eta\,\gamma_1, \qquad \gamma_2 \to \varepsilon\,\eta\,\gamma_2, \qquad \gamma_3 \to \gamma_3,\\
&\gamma_4 \to \eta\,\gamma_4, \qquad \gamma_5 \to \varepsilon\,\gamma_5,
\end{aligned} \tag{4.1.31}$$

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[2] : In the reference the equations are written for variables ξ_i related to those called here x_i by $x_i \equiv \sqrt{50}\xi_i$, x_2 is denoted $-x_2$ and $r = R\sqrt{50}$ if R is the quantity defined in [FT79]; see also [Ga83], Cap. V, §8, (5.8.14).

[3] In the references the equations are written for the variables ξ_i related to the x_i by $x_i \equiv \sqrt{10}\xi_i$ and x_2 is denoted $-x_2$, see also [Ga83], Cap. V, §8, (5.8.9).

with $\varepsilon, \eta = \pm 1$, which we call the "symmetry group" of (4.1.30).

As we see, the equations (4.1.28), (4.1.30) are quite similar to those of the Lorenz' model. All such equations can be interpreted as equations describing systems of coupled gyroscopes, subject to an external torque, to friction and to some dynamical (*i.e.* anholonomic) constraints that force equality of various components of the angular velocities. This is a particular case of the general remark discussed in (E) of §2.2.
Likewise one could consider truncations involving more modes or also truncations on the corresponding equations in $d = 3$ dimensions. But the Lorenz' and NS models with five or seven modes, that we denoted respectively, NS_5 and NS_7, will be sufficient to illustrate many phenomena of the qualitative theory of the onset of chaotic motions and turbulence.

Problems.

[4.1.1]: Show that if $\underline{u}$ and ϑ verify the boundary conditions in (4.1.4) and are periodic in x with period l^{-1} then there exists a constant $C_1 > 0$ such that: $\int (\partial_x \psi)^2 dxdz \le C_1 l^2 \int (\Delta\psi)^2 dx\, dz$, where the integral in x is over the period. Show that the right hand side of (4.1.11) can be bounded above by $C(R, \sigma, l) \int (\vartheta^2 + (\Delta\psi)^2)\, dx\, dz$ and compute an explicit estimate for C_1 and the function $C(R, \sigma, l)$. (*Idea:* Use (4.1.3); and, for the calculation of C use $ab \le \varepsilon^{-1} a^2 + \varepsilon b^2$ with $a = \vartheta$ and $b = \partial_x \psi$, or $a = \Delta\psi$ and $b = \partial_x \vartheta$, fixing suitably ε in the two cases, to compensate the "undesired term" with the first integral on the right hand side of (4.1.11).)

[4.1.2]: Show that if $r < 1$ the origin is a global attracting set for the solutions of (4.1.20): all initial data tend to zero exponentially. (*Idea:* Multiply (4.1.20) by $(\alpha\, x, y, z)$, with $\alpha > 0$ getting, if $E = \alpha\, x^2 + y^2 + z^2$: $\frac{1}{2}\dot E = -\alpha\,\sigma x^2 - y^2 - bz^2 + (\alpha\,\sigma + r) xy$ and note that the matrix (often called the "stability matrix") $\begin{pmatrix} \alpha\sigma & -\frac{1}{2}(r + \alpha\,\sigma) \\ -\frac{1}{2}(r + \alpha\,\sigma) & 1 \end{pmatrix}$ has determinant $\alpha\,\sigma - \frac{1}{4}(\alpha\,\sigma + r)^2$. The latter has a maximum value at $\alpha\,\sigma = 2 - r$ where it has value $1 - r$. As long as the determinant is > 0 the matrix is positive definite, hence $\ge \lambda(r) > 0$. Setting $\delta = \min(\lambda(r), b)$ one realizes that $\frac{1}{2}\dot E \le -\delta E$ with $\delta > 0$ if $r < 1$.)

[4.1.3]: Show that solutions of (4.1.20) evolve so that (x, y, z) enters into any ball of radius $> br^{-1}(\sigma + r)$ after a finite time (dependingon the initial condition). (*Idea:* Given [4.1.2] it suffices to chek the statement for $r \ge 1$: change variables setting $z = \zeta - z_0$ and rewrite the equations for x, y, ζ; multiply them, scalarly, by (x, y, ζ) obataining, with the choice $z_0 = -(\sigma + r)$: $\frac{1}{2}\dot E = -\sigma x^2 - y^2 - bz^2 - b(\sigma + r)\zeta \le -\delta' E + b(\sigma + r)\sqrt{E}$, where $E = x^2 + y^2 + \zeta^2$, $\delta' = \min(\sigma, b, 1)$ hence $\dot E < 0$ if $E > b^2(\sigma + r)^2$.)

[4.1.4]: Show that if $r = 0$ every initial datum for the Lorenz' model, (4.1.20),(4.1.22), evolves approaching zero esponentially for $t \to \infty$, *i.e.* as e^{-ct} for some time constant c, and estimate the time constant . (*Idea:* Multiply (4.1.20) by x, y, z respectively and use that $\sigma > b > 1$ to deduce that the time constant is ≥ 1, *c.f.r.* [4.1.2] above).

[4.1.5] Show that if $r = 0$ the solutions of (4.1.26) and (4.1.28) approach zero esponentially and show that the time constant (see [4.1.4]) is ≥ 1 in both cases. Check that there are solutions, in both cases, that tend to zero exactly as e^{-t}. (*Idea:* The special solutions correspond to initial data with all components vanishing but a suitable one).

[4.1.6] Show that, if $r \ne 0$, the equations (4.1.26), (4.1.28) have, for small r, a time independent solution $\underline{x}_r$, often also called a "*fixed point*", that attracts exponentially initial data close enough to $\underline{x}_0$. And if $r \ne 0$ is small also the equations (4.1.20), (4.1.22) enjoy the same property. (*Idea:* By continuity the linear stability matrix, *i.e.* the matrix of the coefficients obtained by linearizing the equations near the time independent solution, keeps eigenvalues with real part $\lesssim 0$ by continuity, because for $r = 0$ they are ≤ -1 by the time constant estimates in [4.1.4], [4.1.5]).

[4.1.7]: Show that the first stability loss of (4.1.20), (4.1.22) happens at $r = 1$ with a single eigenvalue λ of the stability matrix becoming zero. Check that this bifurcation generates two new time independent solutions no longer invariant with respect to the symmetry, implicit in (4.1.20), $(x, y) \longleftrightarrow (-x, -y)$. (*Idea:* Compute explicitly the time independent solutions).

[4.1.8]: Identify the symmetries of (4.1.20), (4.1.28) and check that as r increases they are successively "broken" generating, by bifurcation, time independent motions with less and less symmetries. (*Idea:* Just solve explicitly for the time independent solutions).

[4.1.9]: Compute the maximum value r_c of r for which the time independent solutions of (4.1.20), (4.1.22) without symmetry are stable and show that the stability loss takes place because two nonreal conjugate eigenvalues reach the imaginary axis for $r = r_c$. (*Idea:* Compute the nontrivial time independent solutions, *i.e.* $z = r - 1$, $x = y = \pm\sqrt{r-1}$, and linearize the equation around one of them thus obtaining an expression for the stability matrix. Compute its characteristic equation, which will have the form: $\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0$ with c_j simply expressed in terms of the parameters, r, b, σ. The critical value of r in correspondence of which such time independent solutions lose stability is found by requiring that the above characteristic equation has two purely imaginary solutions $\lambda = \pm i\mu$ (one easily realizes that a third solution is always < 0); hence it is the value of r for which there is μ such that $-\mu^3 + c_1\mu = 0$ *and* $-c_2\mu^2 + c_0 = 0$; *i.e.* it is the value of r for which $c_1 = c_0/c_2$: this turns out to be a *linear equation* for r with solution $r_c = 470/19 = 24.736842$.)

[4.1.10]: Show that (4.1.26) for $r > \bar r_c$, with $\bar r_c$ suitably chosen, does not admit new stable or unstable time independent solutions (*i.e.* the number of time independent solutions is constant for r large enough), and there exists $\bar r_{c,0} > \bar r_c$ such that for $r > \bar r_{c,0}$ no time independent solution is stable. (*Idea:* Solve for all the time independent solutions and compute exactly the time independent solutions and their respective stability matrices; and discuss the sign of the real part of the eigenvalues. Show that that the fourth order equation for the eigenvalues of the time independent solutions with the least symmetry admit a purely imaginary solution $i\mu$ for $r = r_{c,0}$, (which implies a discussion of an equation of second degree only)).

[4.1.11]: The loss of stability, at $r = r_c$, of the time independent solutions of the Lorenz' model is *inverse*: in the sense that no stable periodic motions are generated . Check this empirically via a qualitative computer analysis. In the next sections we shall introduce the notion of vague attractivity which will allow an analytic check for r near r_c (this is a cumbersome, but instructive, check that the loss of stability takes place via the imaginary axis crossing by two conjugate *nonreal* eigenvalues of the linear stability matrix and with a vague attractivity index which is positive, see §4.2, §4.3).

[4.1.12]: Consider a 3–mode truncation of the $d = 2$ NS equation and suppose that the *shortest modes* $\underline{k}_0 = (0, \pm 1)$ are the sole modes with a novanishing component of the force $\underline{g}_{\underline{k}}$: $\underline{g}_{\underline{k}} = r\delta_{\underline{k},\underline{k}_0}\underline{k}_0^\perp$. Check that *laminar motion* $\underline{\gamma}_{\underline{k}} = \delta_{\underline{k},\underline{k}_0}\, r\underline{k}_0^\perp/\nu\underline{k}_0^2$ is linearly stable for *all* values of r. (*Idea:* Diagonalize the stability matrix).

[4.1.13]: As in [4.1.12] but with a 5 modes truncation. Is this true for any finite truncation? (*Idea:* Yes (*Marchioro's theorem*). Write (as suggested by *Falkoff*) the NS equations in dimension 2 and check that if $\Delta \stackrel{def}{=} \sum_{|\underline{k}|>k_0} (\frac{\underline{k}^2}{\underline{k}_0^2} - 1)|\gamma_{\underline{k}}|^2$ then $\frac{1}{2}\dot\Delta = \sum_{|\underline{k}|>k_0} \underline{k}^2\,(\frac{\underline{k}^2}{\underline{k}_0^2} - 1)|\gamma_{\underline{k}}|^2$, hence $\frac12\dot\Delta \le -\nu k_0^2\Delta$ and $\Delta(t) \le e^{-2\nu k_0^2 t}\Delta(0)$, *etc.*; see also proposition 4 in Sec. 3.2)

Bibliography: The analysis of subsections (A)÷(C) is taken from [Lo63] and that in (D) from [FT79]; the problems are taken from the same papers, see also [Ga83], Cap. V, §9. The last two problems admit a remarkable generalization: the $d = 2$ NS equation, *whether or not* truncated, forced on the *shortest mode* (*i.e.* on the *largest length scale*) is such that the laminar solution is stable for all values of the force intensity (*Marchioro's theorem*), *c.f.r.* [Ma86]. This means that in the $d = 2$ NS equation forced on the shortest mode turbulence is not possible. This result admits some generalizations, see [Ma86],[Ma87].

§4.2: Onset of chaos. Elements of bifurcation theory.

In particular regimes of motion, fluids can be described by *simple* equations with few degrees of freedom: this happens when the velocity, temperature, density fields can be expanded on suitable bases of functions adapted to the boundary conditions of the specific problem and have few appreciably nonzero components. This is the very definition of "simple" or "low excitation" motion.

Examples can be imagined on the basis of the analysis in §4.1. The Lorenz' and truncated NS models have been derived by developing the fields on the Fourier basis, or on similar bases adapted to the considered boundary conditions (*c.f.r.* (4.1.13) for an example): sometimes they can, in spite of their simplicity, be considered as good models for understanding some of the fluids properties. The Lorenz' model was indeed originally suggested by the results of certain numerical experiments;[1] however the examples of §4.1 are already rather complex and it is convenient to look at the matter starting somewhat more systematically.

For the purposes of a more technical discussion it is convenient to confine ourselves, for simplicity, to studying motions of a fluid subject to constant external forces. In general, because of the presence of viscous forces, motions will asymptotically (as $t \to \infty$) dwell in a small part of the space of possible states which we shall call *phase space* (*i.e.* velocity, pressure, density, temperature ... fields): the phase space volume contracts, in a sense to be better understood in the following, and even large sets of initial data rapidly reduce themselves "in size", essentially no matter how we decide to measure their size.[2]

Sometimes, after a short transient time, large regions of phase space are "squeezed" on an "attracting set" consisting of a simple fixed point (*i.e.* of a time independent state of motion of the fluid), converging asymptotically

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[1] The experiments (by Saltzman, *c.f.r.* §4.1 and [Lo63]) which indicated that sometimes the motion was well represented in terms of temperature and velocity fields with only three components on a natural basis.

[2] The size can, for instance, be measured by the volume occupied in phase space or in terms of its Hausdorff dimension, see §3.4: two possible notions with a very different meaning, as we shall see in Sect. 5.5.3 and in Ch. VII.

to it. Sometimes it will develop asymptotically on a periodic motion (hence converging to a closed curve in phase space), or to a quasi periodic motion (hence converging to a torus in phase space) or to an irregular motion (converging to a "*strange attracting set*").[3]

To visualize the above remarks think of the velocity, temperature, density, *etc*, by developing them over a suitable basis (for instance in the case of the NS equations with periodic boundary conditions the Fourier basis will be convenient). In this way every state of the fluid is represented by the sequence $x = \{x_i\}$ of the components of the coordinates on the chosen basis. A motion of the system will be a sequence $x(t) = \{x_i(t)\}$ whose entries are functions of time.

We shall say that the system has *apparently* "n degrees of freedom" if we shall be able to describe its state completely, at least for the purposes of our observations and of the precision with which we are planning to perform them, by n components of the sequence x. We shall also say that the system has "n degrees of freedom" or "n apparent modes". Imagining to increase the force we must think that, usually, the motions become more complicated and this can become mathematically manifest via the necessity of increasing the number of nonzero components, *i.e.* of nontrivial coordinates, of the vector $x = \{x_i\}$ describing the state of the system.

The attribute of "*apparent*" is necessary because, evidently, other coordinates could exist in terms of which it is still possible to describe the motions in a satisfactory way and which are *less in number*.

For instance a periodic motion of a fluid is described, in any coordinate system, by a periodic function $\underline{U}_{\underline{x}}(\varphi)$ of period 2π in φ and such that $\underline{u}(\underline{x}, t) = \underline{U}_{\underline{x}}(\omega t)$, if $T = 2\pi/\omega$ is its period. It is clear that $\underline{u}(\underline{x}, t)$ can also have many Fourier harmonics in $\underline{x}$, say n, of comparable amplitudes $\hat{\underline{u}}_{\underline{k}}(t)$; but, although the number of apparent degrees of freedom of the motion is at least n, the "true number" of degrees of freedom is, in this case, 1 because the "best" coordinate is precisely φ.

A first rough idea of the fluid motion and of its complexity can be obtained by just counting the number of "essential" coordinates necessary to describe it. As just noted this number might depend on the choice of the basis used to define the coordinates: but we should expect that one cannot describe a given motion with less than a certain minimum number of coordinates. We shall see that the attempt to define precisely the *minimum number* of coordinates naturally leads to the notion, or better to various notions, of fractal dimension of the region in phase space visited after a transient time. Such minimum number can also be called the "*dimension of the motion*" or the "*effective number of degrees of freedom*".

Adopting the above terminology (admittedly of phenomenological–empiri-

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[3] Where the adjective "strange" refers more to the complex nature of the asymptotic motion than to a strange geometric structure of the attracting set: we shall see that very complex motions are possible even when the attracting set is a very smooth geometric surface.

cal nature that we shall attempt to make more and more precise) we can say

(1) A time independent motion will be described by coordinates x which are constant, or "fixed", in time: *i.e.* by a 'fixed point". An asymptotically time independent motion will appear as a motion in which the coordinates tend to assume a time independent value as $t \to \infty$.
(2) An asymptotically periodic motion will appear in phase space as attracted by a closed curve run periodically, *i.e.* by a curve described by a point $x(t)$ whose coordinates are all asymptotically periodic. For t large the motion will then be, with a good approximation, $t \to \{x_i(t)\} = \{\xi_i(\omega t)\}$ with $\xi_i(\varphi)$ periodic with period 2π, and the curve $\varphi \to \{\xi_i(\varphi)\}$, $\varphi \in [0, 2\pi]$, is its "orbit" or "trajectory".
(3) A quasi periodic motion with q periods $2\pi\omega_1^{-1}, \ldots, 2\pi\omega_q^{-1}$ will appear as a function $t \to x(t) = \{\xi_i(\omega_1 t, \ldots, \omega_q t)\}$ with $\xi_i(\varphi_1, \ldots, \varphi_q)$ periodic of period 2π in their arguments. And the geometric set $\underline{\varphi} \to \{\xi_i(\underline{\varphi})\}$, $\underline{\varphi} \in T^q$, is the *invariant torus* described by the quasi periodic motion. Evidently it will not be reasonably possible to use less than one coordinate to describe a periodic motion, nor less than q coordinates to describe a quasi periodic motion with q rationally independent periods.[4]
(4) A motion that is not of any of the above types, *i.e.* neither time independent, nor periodic, nor quasi periodic, is called "*strange*".

Until recently it was, indeed, believed that the phenomenon of the onset of turbulence was extremely simple, see [LL71], Chap. 3. In the just introduced language it would consist in successive appearances of new independent frequencies;[5] their number growing as the driving force increased, with a consequent generation of new quasi periodic motions. Until the number of "excited" modes and of frequencies (q in the above notations), had become so large to make it difficult identifying them so that a relatively simple quasi periodic motion could look, instead, as erratic and unpredictable.

In the above "Aristotelic" vision according to which an arbitrary motion can conceptually be considered as composed by (possibly very many) circular uniform motions a "different one" which does not take place on a torus and that cannot be regarded as composed by circular motions is therefore "strange".

To an experimentalist who assumes the quasi periodic viewpoint it can be not too interesting to study the onset of turbulence, because it would appear as consisting in a progressive "excitation" of new modes of motion: a dull phenomenon, differing in different systems only by insignificant details.

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4 Rational independence of $(\omega_1, \ldots, \omega_q)$ means that no linear combination $\sum_i n_i\omega_i$ with n_i integers vanishes unless $n_i = 0$ for all $i = 1, \ldots, q$.

5 This means an increasing number of "modes" periodically moving with frequencies among which no linear relation with integer coefficients exists.

It is remarkable that the force of the above aristotelian–ptolemaic conception of motion, [LL71], Chap. 3,[6] has been such that no one had, until the 1970's, the idea of checking experimentally whether it was in agreement with the natural phenomena observable in fluids. This is most remarkable after it had become clear, starting with Copernicus and Kepler, how fertile was to subject to critique the astronomical conceptions on quasi periodicity: a critique which prompted the birth of modern Newtonian science, although at the beginning it did in fact temporarily reinforce the ancient astronomical and kinematical conceptions which attained the maximum splendor with the work of Laplace in which all celestial motions were still quasi periodic, *c.f.r.* [Ga99b]. And it is surprising that the quasi periodic theory of turbulence was abandoned only in the 1970's, 100 years after a definitive critique of the fundamental nature of quasi periodic motions had become available with the work of Poincaré.

Following the analysis of Ruelle–Takens, [RT71], [Ru89], [ER81], formulated at the end of the 1960's and preceded by the independent, and somewhat different in spirit, critique of Lorenz (*c.f.r.* comments at the end of (C) in §4.2, and [Lo63]), experimentalists were induced to perform experiments considered of little interest until then. They could in this way observe the ubiquity of "*strange motions*", governed by attracting sets in phase space that are neither fixed points, nor periodic motions, nor quasi periodic compositions of independent periodic motions.

The observations established that at the onset of turbulence motions have few degrees of freedom: observed after an initial transient time they can be described by few coordinates whose time evolution can be modeled by simple equations. But they also established that *nevertheless* such equations produce soon (as one increases the strength of the forcing) "strange" motions, which are not quasi periodic.

Furthermore from the works [Lo63], [RT71], and others, it emerged that motions can increase in complexity, still staying with few degrees of freedom, by following *few* different "*routes*" or "*scenarios*" which however can show up in rather arbitrary order in different systems giving thus rise to quite rich phenomenologies; much as the complexity of matter grows up from the "simplicity" of atoms.

This relies on an important mathematical fact: namely differential equations with few degrees of freedom often present few types of really different asymptotic motions and all that a given model does is to organize the appearance of the various asymptotic regimes according to a certain order.

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[6] The reader might have trouble finding even traces of this once widely held point of view. In fact the Ch. 3 of [LL71] has been rewritten in the subsequent editions (postumous to one of the authors). While the new version is certainly closer to the modern views on turbulence it is clear that the complete elimination of the original theory will eventualy produce great problems to historians of science. A more sensible choice would have been, perhaps, to keep the "ancient" viewpoint in small print at the end of the new chapter.

An order which, unfortunately, remains to date *essentially inscrutable*: in the sense that, given a few degrees of freedom system, it is usually impossible to predict the order and the many quantitative aspects of its asymptotic dynamical properties without actual experimentation.

Nevertheless the small number of *a priori* possibilities allows us to use the existing theory also for a certain non negligible number of quantitative predictions and, very important, also for the conception and design of the experiments.

A basic mathematical instrument to analyze the scenarios of the onset of turbulence is *bifurcation theory.* Here we give a general idea of the most well known results: their complete (and general) mathematical analysis is beyond the scopes of this volume: see problems for a few more details.

Consider a system described by a mathematical model consisting in a differential equation on a phase space R^n with n dimensions

$$\dot{\underline{x}} = \underline{f}(\underline{x}; r) \tag{4.2.1}$$

depending on a parameter r, to be thought of as a measure of the strength of the driving force; the vector field $\underline{f}(\underline{x}; r)$ is a C^∞ function, to fix ideas, and such that *a priori* it is $|\underline{x}(t)| \le \overline{C}(\underline{x}(0))$ with $C(\cdot)$ a suitable function. We shall denote $\underline{f}(\underline{x}; r)$ also as $\underline{f}_r(\underline{x})$ for ease of notation.

With the above assumptions, the last of which is an *a priori* estimate which usually translates a consequence of energy conservation, there are no regularity or existence problems for the solutions. These are problems that we have already given up considering in general (see Ch. III and §4.1): accordingly it is clear that (4.2.1) admits regular global solutions, unique for given initial data.

Typically imagine that $\underline{x} = (x_1, \ldots, x_n)$ are the components of a development of the velocity field on a basis $\underline{u}_k(\underline{\xi})$ and of the temperature fields on another suitable basis: let the bases be chosen to represent correctly the boundary conditions, hence likely to require a small (if not minimal) number of "non negligible" components. For instance if the fluid is entirely described by the velocity field (as in the case of incompressible NS fluids) it will then be

$$\underline{u}(\underline{\xi}) = \sum_{k=1}^{n} x_k \underline{u}_k(\underline{\xi}) \tag{4.2.2}$$

where n depends on the complexity of the motion that one studies. In some cases, n can be very small: as an example consider equation (4.1.20) regarded as a description in terms of 3 "not negligible" coordinates of the large time behavior of certain solutions of Rayleigh's equations, (4.1.14).

Remarks.

(1) It is convenient to repeat that we have always in mind studying systems with *a priori* infinitely many degrees of freedom (like a fluid): and, nevertheless, such systems may admit asymptotic motions that can be described

with few coordinates whose *minimum number n* we call the *"effective number of degrees freedom"*. This reduction of the number of degrees of freedom is due to the dissipative nature of the motions that we study.

(2) Of course since the equations that we consider depend on a parameter r the number n will vary with the parameter. But if the parameter is kept in a prefixed (finite) range of values we can imagine n to be large enough to describe all asymptotic motions that develop when r is within the given range. This n might be larger than needed for certain values of r's.

(3) As the driving force increases we expect that the velocity field acquires a structure with more complexity so that the number of coordinates ("components" or "modes") necessary to describe it increases; with the consequent change of the asymptotic states of the system from what we called "laminar" to more interesting ones.

(4) The number of coordinates should not, nevertheless, increase too rapidly with the driving strength r (although, naturally, the number n could increase without limit as $r \to \infty$). In any event one always imagines that n can be taken so large to provide enough coordinates to describe the asymptotic motions in every prefixed interval in which one is interested to let r vary.

(5) At first sight this viewpoint is perhaps not very intuitive and, at the same time, it is very rich: we are in fact saying in a more precise form what stated previously, *i.e.* that the motion of the fluid can show a quite high complexity in spite of being describable in terms of few parameters. Or, in other words, that it is not necessary that the system be described by many parameters to show "strange" behavior.

Obviously it will be more common to observe "strange" motions in systems with many effective degrees of freedom: but motions of too high complexity are difficult to study both from the theoretical viewpoint and the experimental one while those that arise in systems with few (effective) degrees of freedom are easier to study and to classify, hence they provide us with a good basis for the theory and it is interesting that remarkable phenomena of turbulence are observable when n is still small. *A new viewpoint on the study of turbulence is thus born, in which the phenomenology of the onset of turbulence acquires an important theoretical interest.*

We can audaciously hope that in the future the phenomena that appear at the onset of turbulence can become the blocks with which to construct the theory of "strong" turbulence, with many degrees of freedom, in a way similar (perhaps) to the construction of a theory of macroscopic systems from the theory of elementary atomic interactions, *i.e.* in a way similar to Statistical Mechanics.

The possibility of this viewpoint emerged after the works of Ruelle–Takens and it seems supported by a large amount of experimental and theoretical checks.

(A) Laminar motion.

Coming back to (4.2.1) imagine n fixed and suppose that for $r = 0$ ("no driving") the system has $\underline{x} = \underline{0}$ as the unique time independent solution. We shall suppose that this solution is globally attracting in the sense that if $\underline{x}_0 \neq \underline{0}$ is any initial state then the solution of the equations of motion with $\underline{x}_0$ as initial datum is such that

$$\underline{x}(t) \xrightarrow[t\to\infty]{} \underline{0} \tag{4.2.3}$$

This approach to $\underline{0}$ will usually take place exponentially fast (*i.e.* $|\underline{x}(t)| < c_1\, e^{-c_2 t}, c_1, c_2 > 0$ for $t \to \infty$) and with a time constant equal to the largest real part of the eigenvalues of the "stability matrix" $J_{ij} = \partial_{x_i} f_j(\underline{0}; 0)$, also called the "*Jacobian matrix*" of the fixed point.

In fact for $\underline{x}_0 \simeq \underline{0}$ the equation appears to be well approximated by the linear equation $\dot{\underline{x}} = J\underline{x}$: then $|\underline{x}(t)| \simeq |e^{Jt}\underline{x}_0| \leq \mathit{const}\, e^{\mathrm{Re}\,\lambda\, t}|\underline{x}_0|$, if λ is the eigenvalue of J with largest real part.[7]

This property is, for instance, easily verified in the models of §4.1, *c.f.r.* problems [4.1.4],[4.1.5],[4.1.13].

This situation persists for small r, at least for what concerns the solutions that begin close enough to the laminar motion which for $r > 0$ is defined as the phase space point $\underline{x}_r$ such that $\underline{f}(\underline{x}_r; r) = \underline{0}$, where $\underline{x}_r$ is a regular function of r tending to $\underline{0}$ for $r \to 0$ (*c.f.r.* problems [4.1.6], [4.1.7]).

(B) Loss of stability of the laminar motion.

As r increases the picture changes and the "laminar" solution may lose stability when r reaches a certain critical value r_c. This happens when (and *if*) an eigenvalue λ_r, of the stability matrix $J_{ij}(r) = \partial_{x_i} f_j(\underline{x}_r; r)$, with largest real part reaches (as r increases) the imaginary axis of the complex plane on which we imagine drawing the curves followed by the eigenvalues of $J(r)$ as functions of r.

This can happen in two ways: either because $\lambda_{r_c} = 0$ or because $\mathrm{Re}\,\lambda_{r_c} = 0$ but $\mathrm{Im}\,\lambda_{r_c} = \omega_0 \neq 0$. In the second case there will be two eigenvalues ($\lambda_{r_c} = \pm i\omega_0$) reaching the imaginary axis (because the matrix J is real, hence its eigenvalues are in conjugate pairs).

In the first case *we should not expect*, in general, that for $r > r_c$ the time independent solution $\underline{x}_r$, that we are following as r increases, continues to exist. This in fact would mean that the equation $\underline{f}(\underline{x}_{r_c} + \underline{\delta}; r) = \underline{0}$ is soluble with $\underline{\delta}$ tending to zero for $r \to r_c^{\pm}$: an event that should appear "unreasonable" (to reasonable people) for the arguments that follow.

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[7] In this respect the Grobman–Hartman theorem is relevant, *c.f.r.* problems [4.3.8], [4.3.9]: it, however, rather shows that identifying the nonlinear equation, near the fixed point, with its linearized version is not "always valid" and that it can be understood literally in the sense of its "generic" validity, *c.f.r.* (C) for a definition and an analysis of the notion of genericity).

Let $\underline{v}$ be the eigenvector of $J(r_c)$ with zero eigenvalue; then we can consider the graph of the first component f^1_r of the vector $\underline{f}_r$ in a coordinate system in which the n eigenvectors of $J(r_c)$ as coordinate axes, assuming for simplicity that $J(r_c)$ is diagonalizable and that $\underline{v}$ is the eigenvector with label 1; and the fixed point $\underline{x}_{r_c}$ is on the coordinate axis parallel to $\underline{v}$.

Then the graph, *c.f.r.* Fig. (4.2.1), of f^1_r as function of the component x_1 of $\underline{x}$ along $\underline{v}$, keeping the other components fixed to the value that they have in the point $\underline{x}_{r_c}$ taken as origin, will appear as a curve with a local maximum (or a minimum) near the origin: indeed "*in general*" the second derivative of f^1_r in the $\underline{v}$ direction will be nonzero near $r = r_c$ and one can therefore exclude the possibility of an inflection point (as a "rare" and "not general" event).

As r approaches r_c the maximum (or minimum) point on this curve gets closer to the origin and for $r = r_c$ coincides, by the described construction, with it and the curve appears tangent to the abscissae axis (*i.e.* to $\underline{v}$) and lies, locally, on the lower half plane (or upper if it is a minimum).

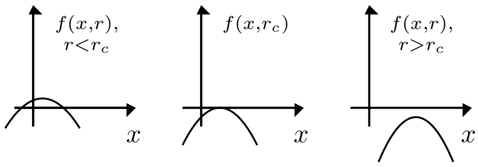

Fig. (4.2.1): *Loss of stability by fixed point through a stability matrix eigenvalue going through* 0*. There a collision between the stable fixed point (right) with an unstable one (left) and the two "annihilate" each other.*

Letting r increase it will happen, in general, that the graph of the curve will continue its "descent" (or "ascent") motion and the maximum (respectively the minimum), *c.f.r.* Fig. (4.2.1), will go below (above) the axis $\underline{v}$ hence it will no longer be possible to solve the equation $f^1(\underline{x}; r) = 0$, at least not with $\underline{x}_r \xrightarrow[r\to r_c]{} \underline{x}_{r_c}$. There will not exist, therefore, ("in general") solutions to $\underline{f}(\underline{x}; r) = \underline{0}$ near $\underline{x}_{r_c}$, for $r > r_c$.[8]

Naturally *there are also other possibilities*: for instance the "velocity of descent" of the maximum of the curve as a function of r could become zero for $r = r_c$ and change sign (in this case the fixed point solution would continue to exist). Or for $r = r_c$ the point $\underline{x}_r$ could become a horizontal inflection.

However the latter alternatives are not generically possible in the sense that, in order for them to happen, special relationships must hold between the coordinates of $\underline{x}_r$ and r for $r = r_c$; for instance the first possibility means

[8] Note that the above argument works in any dimension although it has a 1–dimensional flavour (and it is straighforward in 1 dimension). The possibility of reducing it to $d = 1$ is due to the hypothesis that J is diagonalizable so that the basis which is used exists. In general what this is saying is that the existence of a fixed point for $r > 0$, given that $\underline{f}(\underline{x}_0, 0) = \underline{0}$ cannot be inferred (and often does not exist) unless $\det J|_{r=0} \neq 0$, as well known from the implicit functions theorem.

that, if $\underline{x}_{r,max}$ is the point of a maximum for $f_1(\underline{x})$ as a function of x_1, then it must be $\partial_r f_1(\underline{x}_{r,max}) = 0$ for $r = r_c$ which in general has no reason to be true: *i.e.* there is no reason why *simultaneously* the derivatives of f^1_r with respect to *both* x_1 and r should vanish at $r = r_c, \underline{x} = \underline{r}_{r_c}$.
It is therefore worth pausing to clarify formally the notion of "*genericity*" that, so far, we have been using in an intuitive sense only.

(C) The notion of genericity.

We should first stress that we are considering various equations that are of interest to us because they are models of some physical phenomenon and we have already given up, *a priori*, to describe it *ab initio* by going back to the atomic hypothesis. Hence it should be possible to modify the equations of our models while still producing the same results within reasonable approximations.
For instance if $\underline{f}_r$ depends on certain empirical parameters it should not be possible to obtain qualitatively different results (at least if suitably interpreted) by changing the value of the parameters within the experimental errors. We want, in other terms, that the predictions of the model be insensitive to "reasonable" changes of the model itself.
Certainly, however, some properties of the models cannot be modified. In fact if, *a priori* , we know that fundamental principles imply that $\underline{f}$ should enjoy a certain property then modifications of the model that violate the property should be excluded.
Hence if $\underline{\partial} \cdot I\,\underline{f} = 0$ has, for a suitable matrix I, the meaning of energy conservation (as it is the case for Hamilton's equations) we shall not permit changes in the equation, *i.e.* in $\underline{f}$, that do not keep this property. If some conservation law requires that $\underline{f}(\underline{x})$ be odd in $\underline{x}$, then we shall not permit modifications of $\underline{f}$ violating such property (think for instance to the third law of dynamics that imposes that the force between two particles be an odd function of their relative position).
In every model of a physical phenomenon a few very special properties are imposed on $\underline{f}$: which usually translate physical laws, regarded as fundamental and whose validity should, therefore, not be discussed. After imposing such properties the function $\underline{f}$, *i.e.* the equations of motion, will have still (many) free parameters: and, then, the qualitative and quantitative aspects of the theory should not be sensitive to their small variations.

Suppose that a given problem is modeled by a differential equation $\dot{x} = f(x)$ with f in a space $\mathcal{F}$ of functions verifying all properties that *a priori* the system must verify (like *conservation laws* or *symmetries*). *If* one believes that *all a priori necessary properties are satisfied* one can then take the attitude, which appears the only reasonable one, that the significant physical properties do not change by changing a little f within the space $\mathcal{F}$.
"Little" must be understood in the sense of some metric measuring distances between functions in $\mathcal{F}$ and which should translate into a quantita-

tive form which functions, on the basis of physical considerations, we are willing to consider a "small change". For instance

(1) If f must be a function of one variable of the form $f(x) = r + ax^2$ then $\mathcal{F}$ could be identified with the pairs (r, a) and a natural metric could be the ordinary distance in R^2.
(2) If f must be a function of the form $f(x) = r + \sum_{k=1}^{\infty} a_k x^{2k}$ with $0 \le a_k \le 1$ then $\mathcal{F}$ is the space of such functions and it could be metrized in various ways (which, however, give raise to quite different notions of distance); for instance one could define the distance between f^1, f^2 as $|r^1 - r^2| + \sum_k |a_k^1 - a_k^2| 2^{-k}$, or as $\max_k(|r^1 - r^2| + |a_k^1 - a_k^2|)$, *etc.*
(3) If f must be an analytic function, holomorphic and bounded for $|x| < 1$ and if $\{a_k\}$ are its Taylor coefficients (so that $f(x) = \sum_{k=0}^{\infty} a_k x^k$) one could define the distance between f^1 and f^2 as $d(f^1, f^2) = \sup_{|x|<1} |f^1(x) - f^2(x)|$ or (but not equivalently) $d(f^1, f^2) = \sum_k |a_k^1 - a_k^2| 3^{-k}$.
(4) If f is a function of class $C^1([0,1])$ the distance could correspond to the metric induced by the norm on C^1, defined by the maximum of the modulus of f and of its derivative, or (not equivalently) by the norm defined by the integral of the absolute value of f plus that of its derivative.

With the above remarks and if $\mathcal{F}$ is a separable metric space,[9] we define

Definition *(stability and genericity): A property $\mathcal{P}$ of the functions $f \in \mathcal{F}$ is "stable" if, given an element $f \in \mathcal{F}$ which enjoys the property $\mathcal{P}$, it holds also for the elements close enough to f. A property is "*stable at $f \in \mathcal{F}$*" if it holds for all the elements close enough to f. A property $\mathcal{P}$ is "*generic*" in $\mathcal{F}$ if it holds for an open dense set in $\mathcal{F}$.*

Remark: In mathematics one often uses a somewhat different definition, calling generic also a property that holds on a set that, although not necessarily open and dense, is nevertheless a countable intersection of dense open sets:[10] it is not useful to discuss it here as in any event one should *not* attribute excessive importance to the details of the notion of genericity.

Example 1: Consider the space of the functions of the real variable x of the form $f(x) = r - ax^2$ with $(a, r) \in R^2$, and consider the equation $f(x) = 0$. Existence of a solution is a stable property if $r/a > 0$: but it is not generic because the complementary set $r/a < 0$ is open.

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[9] Separability means that in the metric space $\mathcal{F}$ there is a denumerable dense set of points: it would be very inappropriate in our context to consider more general spaces, "never" met in applications.

[10] Which is still dense under very general assumptions on the space $\mathcal{F}$ in which the sets are contained, by "Baire's theorem", *c.f.r.* [DS60], chap. I: *e.g.* it is sufficient that $\mathcal{F}$ be a metric space which is separable and complete.

Example 2 (*nongenericity of the bifurcation through* $\lambda = 0$): As a more interesting example consider the functions $f(x;r)$ of class C^2 (*i.e.* with two continuous derivatives) such that
(a) for $r \le r_c$ there is a solution x_r of the equation $f(x;r) = 0$ continuously dependent on r, and
(b) $f'(x_r;r) < 0$ for $r < r_c$ and $f'(x_{r_c};r_c) = 0$ (here $f' \equiv \partial_x f$).
Considering the differential equation $\dot x = f(x,r)$ (a), (b) imply that for $r < r_c$ the time independent solution x_r is linearly stable for $r < r_c$ and becomes unstable (or, more accurately, marginally stable) at $r = r_c$.

Let $\mathcal{F}$ be the space of the above considered functions regarded as a metric space with some metric (*e.g.* one can define the distance between two functions as the supremum of $\sum_{j=0}^{2} |\partial_x^j (f-g)|$). Then generically in $\mathcal{F}$ it will be $\partial_x^2 f(x_{r_c}, r_c) \ne 0$ and $\partial_r f(x_{r_c}, r_c) \ne 0$.

Hence *generically* the graph of $f(x, r_c)$ in the vicinity of $x = x_{r_c}$ will have a local maximum (or minimum) either above or below the axis $f = 0$; and the local maximum (or minimum) $y(r)$ of f near x_r will have in $r = r_c$ a derivative $y'(r_c)$ *different from zero.* Hence the graph of f "evolves" with r as described by the graphs of the figure Fig. (4.2.1) above.

The Fig. (4.2.1) shows the nongenericity of the existence of a solution x_r of the equation $f(x;r) = 0$ continuous as a function of r for $r > r_c$, in the cases in which, for $r = r_c$, the derivative of f in x_{r_c} vanishes. This argument can be suitably extended to cases of equations in more dimensions.

Note that a solution of $f(x;r) = 0$ is a time independent solution $x = x_r$ of the equation of motion $\dot x = f(x;r)$; hence the stability loss when an eigenvalue of the stability matrix reaches 0 at $r = r_c$, *leads generically to the disappearance of the time independent solution from the vicinity of* x_{r_c}, as illustrated by the Fig. (4.2.1). We say that if a fixed point for a differential equation $\underline{\dot x} = \underline{f}(\underline{x};r)$ loses stability because an eigenvalue of the stability matrix reaches 0 then, generically, *there is no bifurcation*: *i.e.* the fixed point cannot be continued for $r > r_c$ and possibly coexist (unstable) with other stable nearby fixed points.

Different is the case in which f has properties that guarantee *a priori* the existence of the solution x_r for r near r_c: see the example 5 below.

Example 3 (genericity of the annihilation phenomenon): A more accurate analysis shows that instead, generically, the existence of a solution x_r continuous in r of $f(x,r) = 0$ for $r \le r_c$, with $f'(x_r, r) < 0$ and $f'(x_{r_c}, r_c) = 0$), is accompanied by the existence of another family, x'_r, of solutions of the same equation verifying however $f'(x'_r, r) > 0$ and such that $x_r - x'_r \xrightarrow[r\to r_c]{} 0$; this is made manifest, in the one–dimensional case, by the preceding Fig. (4.2.1). Generically the loss of stability due to the passage through 0 of a real eigenvalue of the stability matrix comes together with a "collision" (always as the parameter r varies playing the role of "time" in our description) between two solutions, one stable and one not, which reciprocally "annihilate" and disappear from the vicinity of the collision point for $r > r_c$.

Example 4 (symmetric bifurcation through $\lambda = 0$): However consider the space $\mathcal{F}_0$ of the C^2 functions $f(x,r)$ (in the sense of the preceding examples) *which are odd in* x and which for $r < r_c$ have a negative derivative at the origin and for $r = r_c$ have zero derivative. Then *it is no longer generically true* that the solution $x = 0$ disappears for $r > r_c$: it is always false, instead. This is not a contradiction because belonging to $\mathcal{F}_0$ *is not generic* in the space $\mathcal{F}$ of the functions of example 2. But it is, tautologically, *generic* in $\mathcal{F}_0$ being the very property that defines such space.

It is interesting to see that in the latter space the stability loss with an eigenvalue passing through zero is not only accompanied by the "survival" of the solution $x = 0$, generically becoming unstable, but also by the appearance of two stable solutions x_r^+, x_r^- approaching $x_{r_c} = 0$ for $r \to r_c$ and existing for $r < r_c$ *or* for $r > r_c$, depending on whether the sign of $f''(0; r_c)$ is positive or negative.

The proof can be obtained by contemplating the following drawing, Fig. (4.2.2), illustrating the one dimensional case and the result can be extended to more dimensions, *c.f.r.* problem [4.2.9].

This shows how delicate can be the discussion and the interpretation of "genericity"; the same property can be generic in a certain context (*i.e.* in a certain space $\mathcal{F}$) and not generic in another (*i.e.* in another space $\mathcal{F}_0$).

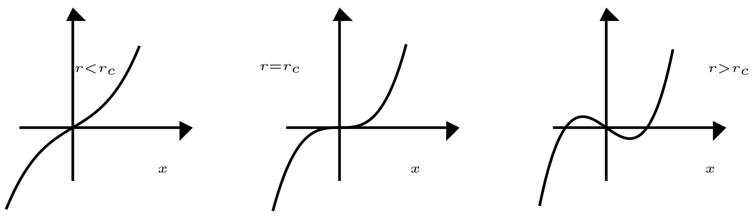

Fig. (4.2.2): *A bifurcation in presence of symmetry: the funcion f is supposed odd in x and the origin remains a fixed point even though it loses stability through* 0.

Interpreting the property $\underline{f}(\underline{x}, r) = -\underline{f}(-\underline{x}, r)$ as a "*symmetry*" of $\underline{f}$ we see that bifurcations "*in presence of a symmetry*" can be quite different and "non generic" when compared with the generic bifurcation behavior in absence of symmetry. This is a general fact.

Example 5: If S is an invertible C^k *map*, $k \ge 1$, of a compact C^k surface, *i.e.* a C^k "diffeomorphism", and if x is a periodic point for S with period τ, we say that x is *hyperbolic* if the stability matrix of x, regarded as fixed point of S^τ, does not have eigenvalues of modulus 1.
The δ–local *stable manifold* (or *unstable manifold*) of O is defined as the surface $W_O^{\delta,s}$ (or $W_O^{\delta,u}$) that
(a) is a graph defined on the ball of radius δ centered at O on the plane spanned by the eigenvectors of the stability matrix of S^τ in O corresponding to eigenvalues with modulus < 1 (or, respectively, with modulus > 1) and
(b) enjoy the property that $d(S^{n\tau}y, O) \xrightarrow[n\to+\infty]{} 0_{s,y}$ for $y \in W_O^{\delta,s}$ (or, respectively, $d(S^{-n\tau}y, O) \xrightarrow[n\to+\infty]{} 0$ for $y \in W_O^{\delta,u}$).

If S is of class C^k then one can prove that, if δ is small enough, $W_O^{\delta,s}$ and $W_O^{\delta,u}$ exist and are of class C^k and, furthermore, they depend regularly (in C^k) on parameters from which S possibly depends (assuming of course that S is of class C^k also with respect to these parameters), *c.f.r.* [Ru89b], p. 28.
An example of a classical genericity theorem is the following theorem (Kupka–Smale theorem, *c.f.r.* [Ru89b], p. 46.)

Theorem *(genericity of hyperbolicity and transversality of periodic orbits in C^∞ maps): Consider the set of all C^∞ diffeomorfisms of a C^∞–regular compact manifold M. Then*
(a) for all $T_0 > 0$ the set $\mathcal{I}_{T_0}$ of diffeomorphisms such that all periodic points with period $\le T_0$ are hyperbolic is open and dense.[11] *Each $f \in \mathcal{I}_{T_0}$ contains finitely many such periodic orbits: hence it is possible to define a $\delta = d_{f,T_0} > 0$ so that each periodic point for f with period $\le T_0$ admits a δ–local stable and unstable manifolds.*
(b) for all $T_1 > 0$ consider the set $\mathcal{I}_{T_0,T_1} \subseteq \mathcal{I}_{T_0}$ of diffeomorphisms such that if O, O' are points of two periodic orbits of period $\le T_0$ then $S^{T_1} W_O^{\delta,u}$ and $S^{-T_1} W_{O'}^{\delta,s}$ have intersections (if any) which are linearly independent, i.e. their tangent planes form an angle different from 0 or π. Then $\mathcal{I}_{T_0,T_1}$ is open and dense,
Hence the above hyperbolicity and transversality properties are two generic properties for each T_0, T_1.

We deduce that the set of C^∞–maps such that all their periodic points are hyperbolic and with global stable and unstable manifolds intersecting transversally is a countable intersection of open dense sets. In particular, by general results in the theory of sets they form a dense set (it is Baire's theorem, *c.f.r.* footnote[10]).

Example 6: (Hamiltonian equations) A Hamiltonian differential equation in R^{2n} (*i.e.* a vector field in R^{2n}) is not generic in the space of differential equations for motions in R^{2n}. One expects, therefore, in such systems to observe as generic phenomena various phenomena which are not generic in the space of all differential equations.

We conclude these remarks by noting another delicate aspect of the notion of genericity: at times, in real or numerical experiments, it may happen that one "forgets" to note an important property of a system, *e.g.* a symmetry property; in this case one might therefore be "surprised" to see occurrence of a nongeneric phenomenon that, being nongeneric, we do not expect to

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[11] If f, g are two C^∞ functions defined on a ball in R^n and $\delta(\xi) \stackrel{def}{=} |\xi|/(1+|\xi|)$ then the distance in C^∞ between f and g can be, for instance, defined by $d(f,g) \stackrel{def}{=} \sum_{k=0}^\infty 2^{-k}\delta(|\partial^k f - \partial^k g|)$ having set $|\partial^k f - \partial^k g| = \max_{x\in M} |\partial^k f(x) - \partial^k g(x)|$. By making use of an atlas of maps for the manifold M this naturally leads to a metric on the diffeomorphisms of M.

see. When this happens it is usually a very interesting event, because it may lead to the discovery of a missed conservation law or of a forgotten symmetry.

The idea that only generic properties may have relevance led, see [RT71], *to the modern viewpoint on turbulence theory and to the reduction and classification (partial but on the whole rather satisfactory) of the a priori possible phenomena occurring in incipient turbulence.*

The generality of the latter idea, its key role and its concrete use combined with the theory of dynamical systems to yield a general theory of turbulence, distinguish the ideas of Ruelle–Takens from those, which were formulated several years earlier, of Lorenz.

The latter did not, perhaps, fully stress the importance of the chaotic phenomena he discovered and he "confined" himself to stress the actual existence of chaotic motions[12] among solutions of simple equations describing for instance atmospheric evolution, and pointing out that in the theory of fluids equations one should expect instability in the form of nonperiodicity and a consequent impossibility of predictions of details of the asymptotic motion (as time t tends to ∞).

Instead Ruelle–Takens's viewpoint was more general (as they tried to relate *and prove* instability by means of the genericity notion) and was therefore better communicated and understood and immediately broadly tested. It is however also important to recall that while the observations of Lorenz were made at the beginning of the 1960's those of Ruelle–Takens were made at the end of the same decade when the electronic computers had become simple and common enough to be used to perform immediate large scale checks of the new ideas, which is indeed what happened, with important feedback on the further development of the very same ideas.

(D) Generic routes to the loss of stability of a laminar motion. Spontaneously broken symmetry.

Going back to the stability of laminar motion, regarded as a solution of the equation $\underline{f}(\underline{x}_r; r) = \underline{0}$, suppose that the stability loss is due to an eigenvalue of the stability matrix $J(r)$ of the fixed point $\underline{x}_r$ which reaches $\lambda_r = 0$ at $r = r_c$. In this case for $r > r_c$,, generically (see examples 2,4 in (C)), the laminar motion disappears and no motions remain confined near $\underline{x}_{r_c}$: even if the initial datum is close to $\underline{x}_{r_c}$ the asymptotic behavior will then be regulated by properties of the equations of motion in some *other region* of phase space, away from the point $\underline{x}_{r_c}$ (in whose vicinity, for $r < r_c$, wander the motions with initial data close to $\underline{x}_r$).

Hence we completely lose control of what happens because, in our generality, we do not know how the function $\underline{f}_r$ behaves far away from the fixed

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[12] Simply and acutely called "nonperiodic", putting on the same level periodic and quasi periodic motions by distinguishing them from their "opposites", *i.e.* from the nonperiodic motions.

point that has "ceased to exist" at $r = r_c$.

It is now necessary to say that often systems have some symmetry and the laminar motion that exists for $r = 0$, and then is followed by continuity as r grows has, usually, the same symmetry. *This is the case in all examples in* §4.1, *c.f.r.* (4.1.21), (4.1.29), (4.1.31). Therefore it can happen that the losses of stability are "nongeneric": in the same way as in the example 4 of (C) above. Through them the initial fixed point may be replaced in its role of attracting set, by other fixed points endowed with *less symmetry* which, in turn, can lose further symmetries as r increases. This is a sequene of events that we can call *spontaneous symmetry breakings*,

An interesting class of examples of sequences of bifurcations "non generic" because of symmetry breaking is provided by the models in §4.1; see problems [4.1.7], [4.1.8] and in the successive §4.4).

The bifurcations can continue until, when r attains a certain value, *all symmetries are broken.* The fixed points that remained stable up to this value of r lose their stability, this time in a "generic" way. If it happens "through $\lambda = 0$" one cannot say much, *in general*, as already noted. However if the bifurcation takes place because a pair of conjugate eigenvalues cross the imaginary axis, then we can still examine a few "generic" possibilities, some of which fairly simple.

In fact a stability loss due to a pair of conjugate eigenvalues of the stability matrix $J(r)$ passing through the imaginary axis at $r = r_c$ does not destroy the existence of the fixed point. This is trivially so because in this case the stability matrix $J(r_c)$ *does not have zero determinant* so that, by the implicit functions theorem, one can still find a fixed point that for $r > r_c$ "continues" the fixed point $\underline{x}_r$, that lost stability at $r = r_c$, into a generically unstable one.

The eigenvalues of $J(r)$, for $r > r_c$, will contain, generically, two conjugate eigenvalues $\lambda_\pm(r)$ with positive real part unless $\partial_r \mathrm{Re}\,\lambda_+(r_c) = 0$, a non-generic possibility that we exclude. Possibly changing the definition of r we can suppose that $\mathrm{Re}\,\lambda_\pm = r - r_c$

$$\lambda_\pm(r) = (r - r_c) \pm i\omega(r) \tag{4.2.4}$$

where $\pm\omega(r)$ is the imaginary part, with $\omega(r_c) \equiv \omega_0 > 0$:

When $r > r_c$ the fixed point $\underline{x}_r$, being unstable, "repels" essentially all motions that develop close enough to it. If the *nonlinear terms* of the Taylor series of $\underline{f}$ in $\underline{x}_{r_c}$ force, when $r = r_c$, initial data close to $\underline{x}_{r_c}$ to approach as $t \to \infty$ the fixed point then for $r > r_c$ the linear repulsion of the fixed point is compensated at a suitable distance from the fixed point $\underline{x}_r$, by the nonlinear terms. In this case we say that the vector field $\underline{f}$ is *vaguely attracting* at $\underline{x}_{r_c}$.

In the simple 2–dimensional case (see problem [4.2.4]) one checks the existence of a "*suitable*" system of coordinates $\underline{x} = (x_1, x_2)$ in which

(a) the nonlinear terms $\underline{w}$ of $\underline{f}$ are, for $r = r_c$, at least of third order in $\rho = |\underline{x} - \underline{x}_{r_c}|$ and, at the same time,

(b) if $\underline{\rho} = (\rho_1, \rho_2)$ and $\underline{\rho}^\perp = (\rho_2, -\rho_1)$ the third order in $\rho = |\underline{x} - \underline{x}_{r_c}|$ has the form $\gamma\,\rho^2\underline{\rho} - \gamma'\,\rho^2\underline{\rho}^\perp$, up to higher orders in ρ, for some $\gamma(r), \gamma'(r)$ computable in terms of the first three derivatives of $\underline{f}$ at $(\underline{x}_{r_c}, r_c)$; hence

$$\underline{f}(\underline{x}) = (r - r_c)\underline{\rho} - \omega'(r)\underline{\rho}^\perp + \gamma\,\rho^2\,\underline{\rho} + O(\rho^4) \tag{4.2.5}$$

where $\omega' = \omega'(\rho, r) = \omega(r) + \gamma'(r)\,\rho^2$. Finding such a system of coordinates is always possible (and easy, if tedious, to do explicitly) as discussed in problem [4.2.4].

Hence the notion of vague attractivity is simply formulated *in the above coordinates* as a property of the sign of the third order terms in the Taylor expansion of $\underline{f}_r(\underline{x})$ in $\underline{x} - \underline{x}_{r_c}$ and $(r - r_c)$: if $\gamma < 0$ then for r small the fixed point $\underline{x}_{r_c}$ is *vaguely attractive* and the vectors $\underline{w}$ "point towards the origin", while if $\gamma > 0$ they "point away from the origin" which is then *vaguely repulsive* (*c.f.r.* problem [4.2.4] and [Ga83] Chap. 5).

In the cases of "*vague repulsivity*" (*i.e.* $\gamma > 0$) not much can be said for $r > r_c$ since the motion will abandon the vicinity of the fixed point that loses stability and we shall be in a situation similar to the one corresponding to the passage of an eigenvalue through 0. If regions far away from O evolve again towards O (arriving close to it along an attracting direction) one says that there is an "*intermittency phenomenon*": its analysis is very close to the case (H1) in §4.3 and, to avoid quasi verbatim repetitons we do not discuss it here.

However in the case of *vague attractivity* (*i.e.* $\gamma < 0$) and for $r > r_c$, but $r - r_c$ small, the distance ρ (in the special system of coordinates introduced above) is such that the first term in (4.2.5) $(r - r_c)\rho$ can be "*balanced*" by the cubic term $\gamma\rho^3$ which, by assumption, would tend to recall the motion to the fixed point: then a periodic orbit is generated, evidently stable.

In the chosen coordinates it is a circular orbit (up to orders higher than the third in the distance to the unstable fixed point) and with radius $O((-\gamma)^{-1/2}\sqrt{r - r_c})$. Indeed, in the special coordinates (x_1, x_2) that we consider, the transformation can be written, setting $z = x_1 + ix_2$, simply

$$\dot{z} = \big(r - r_c + i\omega'(|z|, r)\big)\,z - \gamma\,|z|^2\,z \tag{4.2.6}$$

with an approximation of order $O(|z|^4)$: showing the validity of the above statement on the radius and showing also that the period will be $\sim 2\pi/\omega_0$ if $\omega_0 = \omega(r_c)$, see [4.2.4], because if $\rho = \sqrt{-\gamma^{-1}(r - r_c)}$ the function $z(t) = \rho e^{i\omega'(\rho,r)t}$ is an exact solution of the approximate equation.

In the vague attractivity case the asymptotic motion with initial data close enough to the unstable fixed point $\underline{x}_r$, will therefore be of somewhat higher complexity compared to the one taking place for $r < r_c$: no longer evolving towards a fixed point but towards a periodic stable motion. And the system, that for $r \le r_c$ had asymptotic motions without any time scale (*i.e.* it had just fixed points) will now acquire a "*time scale*" equal to the period $\sim 2\pi\omega_0^{-1}$ of this periodic orbit.

What we said so far in the 2–dimensional case can be extended, without essential changes (see the problems), to higher dimensions and it constitutes the "*Hopf's bifurcation theory*". Basically this is so because the motion in the directions transversal to that of the plane of the two conjugate eigenvectors with small real part eigenvalues is a motion which exponentially fast contracts to 0 on a time scale regulated by the maximum of the real part of the remaining eigenvalues (which is negative) so that the motion is "effectively two–dimensional", see problems.

When the evolution (with r) leads, as r increases, to a periodic stable orbit we can proceed by analyzing the stability properties of the latter. Again several possibilities arise and we shall discuss them in §4.3.

Problems. Hopf bifurcation.

[4.2.1]: Let $\dot{\underline{x}} = \underline{f}_r(\underline{x})$, $\underline{x} \in R^n$, be a differential equation (of class C^∞), that for $r < r_c = 0$ has a fixed point that loses stability because two conjugate eigenvalues $\lambda(r) = r + i\omega(r)$ e $\bar\lambda(r) = r - i\omega(r)$, with respective eigenvectors $v(r), \bar v(r)$, as r increases, pass across the real axis with an imaginary part $\omega_0 = \omega(0) > 0$. Note that the hypothesis $\mathrm{Re}\,\lambda(r) = r$ is not very restrictive and it can be replaced by the more general $\frac{d}{dr}\mathrm{Re}\,\lambda(r_c) > 0$. (*Idea:* If the derivative dos not vanish one can change variable setting $r' = \mathrm{Re}\,\lambda(r)$ for r close to $r_c = 0$).

[4.2.2]: In the context of [4.2.1] suppose $n = 2$ and, if $x = (x_1, x_2) \in R^2$ show that the linearized equations can be written, setting $z = x_1 + ix_2$, as $\dot z = \lambda(r)z$. Furthermore the equations can be written to third order (*i.e.* by neglecting fourth order terms) as

$$\dot z = \lambda(r)z + az^2 + b\bar z^2 + cz\bar z + a_1 z^3 + a_2 z^2\bar z + a_3 z\bar z^2 + a_4\bar z^3$$

with a, b, c, a_i suitable complex numbers and $\bar z$ denotes the complex conjugate of z..

[4.2.3]: In the context of [4.2.2] show the existence of a coordinate change $z = \zeta + \alpha\zeta^2 + \beta\zeta\bar\zeta + \gamma\bar\zeta^2$ that allows us to write, near the origin and near $r = 0$, the equation truncated to third order in a form in which no second order terms appear. (*Idea:* This is just a "brute force" check. Check that the coefficients α, β, γ can be determined as wanted).

[4.2.4]: *(normal form vague attractivity theorem)* In the context of [4.2.1],[4.2.2],[4.2.3] assume that the equation truncated to third order does not contain quadratic terms (not restrictive by [4.2.3]).Show that it is possible a further change of coordinates $z = \zeta + \alpha_1\zeta^3 + \alpha_2\zeta^2\bar\zeta + \alpha_3\zeta\bar\zeta^2 + \alpha_4\bar\zeta^3$ such that the equation assumes the *normal form*

$$\dot\zeta = \lambda(r)\zeta + \vartheta|\zeta|^2\zeta + O(|\zeta|^4)$$

with $\vartheta = \vartheta(r)$ a suitable complex function of r. Check also that the value of $\mathrm{Re}\,\vartheta$ can be computed as a function only of the derivatives of order ≤ 3, with respect to $\underline{x}$ at $\underline{x} = \underline{0}$, of the function $\underline{f}_r(\underline{x})$ defining the differential equation. The time independent solution is said "vaguely attractive" if $\gamma \equiv \mathrm{Re}\,\vartheta < 0$ for $r = r_c$. (*Idea:* Same as in the preceding problem, *c.f.r.* [Ga83] §5.6, §5.7).

[4.2.5]: *(Hopf's bifurcation in 2 dimensions)* In the context of problem [4.2.4], and supposing $\gamma = \mathrm{Re}\,\vartheta < 0$ at $r = r_c$, show that the equation $\dot{\underline{x}} = \underline{f}_r(\underline{x})$ admits a periodic attractive solution which, in the coordinates defined in the preceding problem, runs on a curve with equation $|\zeta| = \sqrt{(r-r_c)(-\gamma)^{-1}} + O((r-r_c))$, for $r > r_c$ and $r - r_c$ small enough. The period of the motion is $\sim 2\pi/\omega_0$. (*Idea:* Note that this is obvious for the third order truncated equation. The non truncated case reduces to the truncated one via a suitable application of the implicit functions theorem (*c.f.r.* problem [4.2.10]), as the higher order terms are "negligible").

[4.2.6]: *(Inverse Hopf's bifurcation in* 2 *dimensions)* In the context of problem [4.2.4], and supposing $\gamma = \mathrm{Re}\,\vartheta > 0$, show that the equation $\underline{\dot{x}} = \underline{f}_r(\underline{x})$ admits a periodic repulsive solution which, in the coordinates defined in the preceding problem, runs on a curve with equation $|\zeta| = \sqrt{(r-r_c)\gamma^{-1}} + O((r-r_c))$, for $r < r_c$ and $r - r_c$ small enough. The period of the motion is $\sim 2\pi/\omega_0$. (*Idea:* as in the preceding problem).

[4.2.7]: *(Center manifold theorem)* Under the assumptions of [4.2.1], with $n > 2$, introduce the system of coordinates $\underline{x} = (x, y, \underline{z})$ with x–axis parallel to $\mathrm{Re}\,\underline{v}(0)$, y–axis parallel to $\mathrm{Im}\,\underline{v}(0)$ and $\underline{z} \in R^{n-2}$ in the plane orthogonal to that of $\underline{v}$, $\overline{\underline{v}}$. Suppose that for some $r_0, \nu > 0$ the remaining eigenvalues of the stability matrix of $\underline{x} = \underline{0}$ stay with real part not larger than $-\nu < 0$, for $r \in (-r_0, r_0)$.
Then for all $k \geq 2$ and a suitable $\delta_k > 0$ it is possible to find a 2–dimensional surface $\Sigma_k(r)$ of class C^k and equations

$$\underline{z} = \underline{\zeta}_r(x, y), \qquad |x|, |y| < \delta_k,\ |r| \leq \frac{r_0}{2}$$

such that

(i) if $|x_0|, |y_0| \leq 2\delta_k$, $|z_k| \leq \delta_k$ the solutions $t \to S_t(\underline{x}_0)$ (with initial datum $\underline{x}_0 = (x_0, y_0, \zeta_r(x_0, y_0))$) of the equation remain on Σ_k if initially $\underline{x} \in \Sigma_k$ (*i.e.* Σ_k is S_t-invariant) and
(ii) there is D such that if $|S_\tau(\underline{x})| < \delta_k$ for $0 \leq \tau \leq t$ then $d(S_t(\underline{x}), \Sigma_k) \leq De^{-\nu t/2}$: *i.e.* the surface Σ_k is attractive for motions that are near the origin, as long as they stay near it; *i.e.* the motion gets close to Σ_k and it can go away from the origin only by "gliding" along Σ_k and
(iii) $|\underline{\zeta}_r(x, y)| \leq C(x^2 + y^2)$, with C a suitable constant.

This is a version of the *center manifold theorem*, *c.f.r.* [Ga83], §5.6, for instance.

[4.2.8] Show that the above theorem, combined with the results of the previous problem allows us to prove the following

Theorem *(Hopf bifurcation theorem) Suppose that for $r = r_c$ the differential equation in problem [4.2.1] has the origin as a fixed point which loses stability as r grows through r_c because of the crossing of the imaginary axis by two eigenvalues of the stability matrix and in the way described in [4.2.1]. Suppose that a suitable polynomial formed with the derivatives of $\underline{f}_r$ of order ≤ 3, evaluated at the time independent point and for $r = r_c$, is negative. Then for $r > r_c$ and $r - r_c$ small enough there is a periodic attractive solution located within a distance $O(\sqrt{r - r_c})$ from the origin; the period of this orbit is $\sim 2\pi/\omega_0$ and it tends to $2\pi/\omega_0$ as $r - r_c \to 0^+$.* (*Idea:* The center manifold theorem with k large (*e.g.* $k = 4$ is certainly enough) reduces the proof to the $n = 2$ case, precedently treated).

[4.2.9] (*Loss of stability through $\lambda = 0$ in higher dimension*) Consider a C^2 function $\underline{f}(\underline{x}, r)$ which for $r \leq r_c$ vanishes at a point $\underline{x}_r$ continuously dependent on r. Suppose that the stability matrix $\partial_{x_j} f_i = J_{ji}$ at the point $\underline{x}_r$ has only one real eigenvalue λ_r which, among the other eigenvalues, also has largest real part. Call the corresponding eigenvector $\underline{v}_r$ and, furthermore, suppose $\lambda_{r_c} = 0$. Using the center manifold theorem in [4.2.7] extend to dimension ≥ 2 the genericity and non genericity properties in examples 2,4 above.

[4.2.10] (*Implicit function to neglect higher orders in the Hopf bifurcation in* [4.2.5]) For simplicity suppose that $\omega(r) = \omega, \vartheta(r) = \gamma < 0$ are real constants. The parametric equation for the invariant curve of the equation in problem [4.2.4] can be written as

$$z(\varphi) = \rho_0\,(1+\delta(\varphi))\,e^{i(\varphi+h(\varphi))}, \qquad \rho_0 \stackrel{def}{=} (-\gamma^{-1}\,r)^{1/2}$$

where $\varphi \in T^1 = [0, 2\pi]$ and δ, h are functions in $C^1(T^1)$, and suppose that the solution to the equation in problem [4.2.4] is, without neglecting the higher order terms, equivalent to $\dot\varphi = \omega + \varepsilon$ for some constant ε. Check that the condition for this to happen has the form $(\delta', h', \varepsilon') = (\delta, h, \varepsilon)$ where $(\delta', h', \varepsilon') = H(\delta, h, \varepsilon)$ with H an operator on the space $C^1(T^1) \times C^1(T^1) \times R$ defined for $||\delta||_1, ||h||_1, |\varepsilon| < 2^{-1}$ by

$$\delta' = r^{3/2}\,\big(\partial_\varphi + 2r/(\omega+\varepsilon)\big)^{-1}\big(-r^{-1/2}\delta^2(1+\delta)(\omega+\varepsilon)^{-1} + R_1(\varphi;\delta,h,\varepsilon)\big)$$

$$h' = r^{3/2}\,(\omega+\varepsilon)^{-1}\,\partial_\varphi^{-1}\Big(R_2(\varphi;\delta,h,\varepsilon) - \int_0^{2\pi} R_2(\psi;\delta,h,\varepsilon)\,d\psi/2\pi\Big)$$

$$\varepsilon' = r^{3/2}\int_0^{2\pi} R_2(\psi;\delta,h,\varepsilon)\,d\psi/2\pi$$

with R_j, as well as the following A_j, B_j, D_j, C_j, a regular function in $r, \delta, h, \varepsilon, e^{i\varphi}$:

$$R_j(\varphi;\delta,h,\varepsilon) = A_j(\varphi) + B_j(\varphi;\delta,h,\varepsilon)\,\delta\, + C_j(\varphi;\delta,h,\varepsilon)\,h\, + D_j(\varphi;\delta,h,\varepsilon)\,\varepsilon$$

and the operators ∂_φ^{-1} and $(\partial_\varphi + 2r/(\omega+\varepsilon))^{-1}$ are defined as the multiplication by $(in)^{-1}$, $(in + 2r/(\omega+\varepsilon))^{-1}$ of the Fourier transforms of the operands. Note that the action of ∂_φ^{-1} is well defined because the operator acts on a 0–average function. Check that there is $c > 0$ such that H is a contraction in $C^1 \times C^1 \times R$ if $||\delta||_{C^1} < c$, $||h||_{C^1} < c$, $|\varepsilon| < c$ if r is small enough: hence H has a fixed point. For an alternative simpler approach see [Ga83]. (*Idea:* Note that ∂_φ^{-1} and $(\partial_\varphi + 2r/(\omega+\varepsilon))^{-1}$ can be written as integrals over φ once they make sense.)

Bibliography: [RT71],[Ga83]. A complete exposition of the bifurcation theory related to the above problems can be found in [Ru89b].

§4.3 Bifurcation theory. End of the onset of turbulence.

We shall study the stability of the periodic motions continuing the analysis of §4.2.

(E) Stability loss of a periodic motion. Hopf bifurcation.

Stability of periodic motions of the general equation (4.2.1) in R^n can be studied via a *Poincaré's map* defined as follows.

Consider a periodic motion Γ with period T_r as a closed curve in phase space and intersect it in one of its points O with a flat surface Σ transversal to Γ (*i.e. not* tangent to Γ) of dimension $n-1$, if n is the dimension of phase space. Imagine fixing on Σ coordinates $\underline{\eta} = (\eta_1, \ldots, \eta_{n-1})$ with origin O. As the intensity r of the driving force, *c.f.r.* (4.2.1), varies Γ, O, Σ change and Γ may even cease to exist; but if we study the system for r near some value r_c then Σ can be taken as fixed.

If $\underline{\xi}$ is a point of Σ in a vicinity U of the origin O, we imagine taking it as initial datum of a solution of the equation of motion $\dot{\underline{x}} = \underline{f}(\underline{x}; r)$, *c.f.r.* (4.2.1). It will follow closely the periodic motion on a trajectory that will come back to intersect Σ in a point $\underline{\xi}'$ after a time approximately equal to the period T_r of O if, as we shall suppose, U is chosen small enough.

1 Definition *(Poincaré map): We shall denote $S_\Sigma\underline{\xi}$ the point $\underline{\xi}'$ on Σ reached in this way starting from $\underline{\xi} \in \Sigma$. The map S_Σ, of the section Σ into itself, has O as a fixed point and is called a "Poincaré map". The map can be described in a system of cartesian coordinates $\underline{\eta} = (\eta_1, \ldots, \eta_{n-1})$ on Σ in which O has coordinates that we shall call $\underline{x}_r$. The matrix M of the derivatives in O of S_Σ with respect to the coordinates $\underline{\eta}$, i.e. $M_{ij} = \partial_{\eta_i} S_\Sigma(\underline{\eta})_j\, |_{\underline{\eta}=\underline{x}_r}$ will be called stability matrix of the periodic motion Γ.*

The stability matrix M (which is a $(n-1)\times(n-1)$ square matrix) besides depending on Γ depends also on the choice of O on Γ, on the choice of the section Σ through O, and finally on the system of coordinates defined on Σ. However it is not difficult to realize that the spectrum of the eigenevalues of M *does not depend on any of such choices*, *c.f.r.* problem [4.3.1].

The stability of the orbit is discussed naturally in terms of the map S_Σ. Indeed the orbit will be stable and exponentially attracting nearby points if the stability matrix M of Γ will have all eigenvalues with absolute value < 1. As the parameter r (which parameterizes the original equation and hence S_Σ itself) increases a loss of stability is revealed by one of the eigenvalues of M reaching the unit circle. We assume that this happens at $r = r_c$.

There are three *generic* possibilities

(1) an eigenvalue with maximum modulus of the matrix $M(r)$ reaches the unit circle in a nonreal point $e^{i\delta_c}$, $\delta_c \neq 0$, together with a conjugate eigenvalue (because $M(r)$ is real), while the others stay away of the real axis.
(2) the eigenvalue with maximum modulus is simple and reaches the unit circle in -1; or
(3) the eigenvalue with maximum modulus is simple and reaches the unit circle in 1.

Other possibilities (like more than two eigenvalues reaching simultaneously the unit circle) are nongeneric and will not be considered.

In the first two cases the determinant $\det(M(r_c) - 1)$ does not vanish and, by an implicit functions theorem, one finds that the fixed point $\underline{x}_r$ of the map S_Σ can be *"continued"* for $r > r_c$ as a, generically unstable, fixed point of S_Σ: which corresponds to the existence of an unstable periodic motion. This continuation, in the third case, is generically impossible as in the analogous case, examined in §4.2, of a fixed point for a differential equation that loses stability because an eigenvalue of the stability matrix reaches 0, see §4.2 and problem [4.3.2].

Consider the first case: it can happen that for $r = r_c$, *i.e.* at the moment of stability loss by the periodic motion Γ, the nonlinear terms of the Poincaré's map S_Σ are attractive (analogously to what we saw in §4.2 in discussing the vague attrattivity of fixed points). Then for fixed r slightly larger thn r_c

these terms will balance the linear repulsivity acquired by O and, at a certain distance from O, an invariant curve $\gamma = S_\Sigma\gamma$ is created on Σ: the action of S_Σ, "rotates" γ over itself with a *rotation number* approximately equal to $\delta_c/2\pi$ where δ_c is the argument of the complex eigenvalue $\lambda_{r_c} = e^{i\delta_c}$.[1] In this case one says that the periodic motion is vaguely attractive for $r = r_c$.

This means that motions with initial data on γ generate, in the n dimensional phase space, a surface $\mathcal{T}$ that is stable and is, topologically, an *invariant bidimensional torus* intersecting Σ on the closed curve γ. Furthermore a second time scale is generated, whose ratio to the period of the periodic orbit, still existing but now unstable, is approximately $2\pi/\delta_c$ and the motion on the invariant torus is quasi periodic or periodic.

We must expect, however, that as r increases beyond r_c quasi periodicity and periodicity alternate because the rotation number will continuously change from rational to irrational (in fact the map S_Σ, hence γ, change continuously and the rotation number depends continuously on the map, see problems [5.1.28], [5.1.29]). Below we shall see that rational values will be generically taken over small intervals of r so that the graph of the function $\delta(r)$ will look like a "smoothed staircase".

While the motion "turns" on $\mathcal{T}$ following essentially (at least for small $r - r_c$) the nearby periodic orbit Γ (by now unstable) the trajectory wraps around on the torus $\mathcal{T}$ to "reappear" right on the section Σ on the curve γ but in a point rotated *in the average* by an arc of length δ in units in which the curve Γ has length 2π. Thus the orbit closes approximately every $2\pi/\delta$ "revolutions" and this gives also the physical meaning of the rotations number.

Vague attractivity, as in the case of the fixed points of §4.2 can be discussed more precisely. For instance we consider the case in which the dimension of the surface Σ is 2 and the point O is chosen as origin of the coordinates x_1, x_2 in the plane Σ.

The coordinates can be combined to form a complex number $z = x_1 + ix_2$ and they can be chosen in analogy with the choice described in problem [4.2.4]. *This time, however, a further condition becomes necessary: namely that for $r = r_c$ none of the eigenvalues λ_i of the stability matrix shall verify*

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[1] Since γ is invariant the Poincaré map S_Σ can be represented on γ as a map of a circle into itself. One simply identifies the point of γ with curvilinear abscissa (counted from an origin fixed on γ) equal to s with the point of the unit circle with angle $2\pi\, s/\ell$, where ℓ is the length of γ; and the map S_Σ becomes a regular map $s \to g(s)$ of the circle into itself. As we shall see, *c.f.r.* (4.3.1), the curve γ is in suitable coordinates very close to a circle and the map $g(s)$ is very close to a rotation, hence $g(s)$ is monotonically increasing and the map is invertible as a map of the circle into itself. Imagining of "developing" the circle into a straight line the map S_Σ becomes a function g defined on $(-\infty, +\infty)$ such that $g(s + 2\pi) = g(s) + 2\pi$. Then, a theorem of Poincaré, *c.f.r.* problem [4.3.3], gives the existence of the limit $\rho = \lim_{n\to\infty}(2\pi\, n)^{-1} g^n(s)$ and its independence on s. It is natural to call this number ρ the "*rotation number*" of the map g: it is indeed the fraction of 2π that in the average the iterates of g impose to the points on the circle at every action. A perfect rotation $s \to s + \delta$ will have rotation numer $\delta/2\pi$.

$\lambda_i^3 = 1$ *or* $\lambda_i^4 = 1$.

Assuming in what follows the latter (generically true) condition a change of coordinates can be devised which will be at least of class C^1 and such that the map S_Σ assumes, in the new coordinates, the form

$$S_\Sigma z = \lambda(r)\, z\, e^{c(r)\,|z|^2 + O(z^4)} \tag{4.3.1}$$

where $\lambda(r_c) = e^{i\delta_c}$, $c(r)$ is a complex number, and $\lambda(r), c(r), O(z^4)$ are functions of z, r of class C^1 at least.

If Re $c(r_c) = \overline{c} < 0$ one has *vague attractivity* (the case in which third order terms neither attract nor repel for $r = r_c$, *i.e.* $\overline{c} = 0$, is not generic). The analysis is entirely analogous, *mutatis mutandis*, to the corresponding one in §4.2.

We suppose that the r–derivative of the absolute value part of $\lambda(r)$ at $r = r_c$ does not vanish so that it is not restrictive to suppose that also

$$\lambda(r) = e^{r - r_c + i\delta(r)} \tag{4.3.1a}$$

Neglecting $O(z^4)$ the circle $r - r_c - \mathrm{Re}\, c(r)\, |z|^2 = 0$ is exactly invariant, by (4.3.1), and the motion on it is a rotation by an angle ϑ about equal to $\delta(r_c)$ (precisely equal to $\delta(r) + \mathrm{Im}\, c(r)\, |z|^2$). This implies that there is an invariant curve which is, approximately, a circle of radius ρ which the map S_Σ, approximately, rotates by an angle ϑ

$$\rho = \big(-\, \overline{c}^{-1}\,(r - r_c) + O((r - r_c)^2)^{1/2}, \qquad \vartheta = \delta(r_c) + O(r - r_c) \tag{4.3.2}$$

because the terms of $O(z^4)$ cannot change too much the picture, as it can be seen by an application of some implicit function theorem, see problem [4.3.10].

Generically the above condition on the powers 3 and 4 of the eigenvalues λ_i will be verified; hence, in the vague attractivity case, an invariant torus is generated which is run by motions with two time scales (*for* $r - r_c$ *small positive and up to infinitesimal corrections in* $r - r_c$) respectively equal to $T_{r_c} = 2\pi\omega_0^{-1}$ (period of the periodic motion that lost stability), and $T_1 = 2\pi T_{r_c} \delta^{-1}$. This is again called a *Hopf bifurcation c.f.r.* problems of §4.2. As r decreases towards r_c the torus becomes confused, one says "collides", with the orbit Γ from which it "inherited" stability for $r > r_c$.

If, instead, the nonlinear terms of the map S_Σ tend, for $r = r_c$, to move trajectories away from O (*i.e.* if in the special coordinates mentioned above we have $\overline{c} > 0$) we say that there is *vague repulsivity* and in this case the nonlinear terms (generically of order of $r - r_c$) cannot balance the linear ones and the motion gets away from the vicinity of the unstable periodic orbit.

We are in a situation similar to the one met in §4.2 when a generic loss of stability took place because the eigenvalue with largest real part reached 0.

As in that case, if $\bar{c} > 0$, one still finds a curve invariant for S_Σ, *unstable* and existing fo r $r < r_c$: this therefore implies that in phase space there is an invariant unstable torus which for $r \to r_c$ merges (one says it "collides") with the periodic orbit which inherits its instability for $r > r_c$.

The nature of the bifurcation in the non generic cases in which $\lambda_i^3 = 1$ or $\lambda_i^4 = 1$ is quite involved and not yet completely understood.

At this point *one could be led in temptation* and think that the motion on the invariant torus, born via a vaguely attractive Hopf bifurcation from a periodic orbit, is run (possibly only generically) by *quasi periodic* motions with two angular velocities approximately equal to $\omega(r), \delta(r)$ and which vary with continuity. Naively we could think that in general the rotation number of the motion on the invariant curve γ will be $\delta^*(r)/2\pi$ for some $\delta^*(r)$ tending, see (4.3.2), to δ_{r_c} as $r \to r_c$, *in a strictly monotonic way*, at least if $\partial_r \mathrm{Re}\,\lambda(r_c) > 0$.

A more attentive exam of this thought would, however, reveal it a rushed conclusion. If it was so, indeed, the rotation number would evolve with continuity assuming, as $r > r_c$ varies near r_c, values at times rational and at times irrational. If for a certain r we have a rational rotation number this implies that the motion on γ is, in reality, asymptotic to a periodic motion. Then we imagine the map S_Σ to describe, as r varies, a curve in the space of the smooth maps of the circle: such maps have the generic property that when their rotation number is rational the motions that they generate are asymptotic to a finite number of periodic motions which are either stable or unstable with a stability matrix which is not equal to 1.[2]

Therefore we *cannot* expect that the torus is covered by periodic orbits as it could be when the rotation number varying continuously and strictly monotonic becomes rational: by doing so we take too fast for granted the same picture as in the cases in which the rotation on the torus is *exactly* linear.[3]

Indeed if we think that as r changes the curve described by S_Σ enters and exits, in the space of circle maps, open regions where the motions are regulated by a few attracting periodic orbits (with a stability eigenvalue which is < 1) then their asymptotic motions will be periodic and will keep a "*fixed*" period as r varies in (possibly very tiny) intervals: because the period, being an integer, cannot vary by a small amount. The torus will be covered, possibly, by periodic orbits only at the extreme values of such intervals of r.

We should expect that as r increases the rotation number will remain fixed

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[2] This is a very special case, intuitively appealing, of a more general theorem, *Peixoto's theorem*, that says that generically a differential equation on a 2–*dimensional* torus has a finite number of periodic solutions or fixed points, some of which attractive and some repulsive.

[3] Note If the rotation number is rational and the rotation is linear the torus is covered by a continuum of distinct periodic orbits. If the rotation number is irrational the torus is, instead, run densely by the motion on it which, therefore, will not have any non trivial attracting sets.

over segments with positive length > 0 on the axis r, and on such segments it will keep a rational value.

When this happens one says that in such intervals of r a *phase locking phenomenon* or *resonance* takes place between the two rotations (the one along the periodic orbit and the one on the section Σ).

This will not, however, forbid that the rotation number varies with continuity with r (because *natura non facit saltus*, as is well known). But the graph of the rotation number as a function of r will have a characteristic aspect, appearing piecewise *flat* and *hence* not strictly monotonic. Such a graph is often called a *Devil's staircase* (an obscure etymology because it is unclear why the latter Being could have spared time to dedicate himself to building such stairs, see problem [4.3.6]).

The case in which at stability loss of the fixed point is vaguely repulsive implies that the motion wanders away from the unstable orbit and not much can be said in general. If regions far away from O evolve again towards O (arriving close to it along an attracting direction) one says that there is an "*intermittency phenomenon*": its analysis is very close to the case (H1) below and, to avoid quasi verbatim repetitons we do not discuss it here.

(F) Loss of stability of a periodic motion. Period doubling bifurcation.

This is the case when the stability loss of the periodic orbit is due to the crossing through -1, for $r = r_c$, by an eigenvalue of the stability matrix $M(r)$. Again we must distinguish the *vaguely repulsive* case, (in which the nonlinear terms repel away from the fixed point for $r = r_c$), from the opposite case, *vaguely attractive*.

The case in which the third order terms do neither action will not be considered because it is nongeneric.

In the first case, again, for $r > r_c$ the motion simply drifts away from the periodic motion which has become unstable and nothing more can be said in general: an intermittency phenomenon can happen also in this case if regions far from O are mapped back close to O and its features can be discussed as in the case (H1) below. In the second case instead one easily sees that a new stable *periodic point* for S_Σ is generated, with period 2.

Indeed one can consider a point P near O and displaced by a small δ in the direction of the eigenvector $\underline{v}$ corresponding to the eigenvalue of M which is approximately -1. Then P returns close to O roughly on the *opposite side relative to* O (with respect to $\underline{v}$) and at a distance $\delta' > \delta$ a little further away than δ from the origin.

Iterating the point cannot get too far because, by the assumed vague attractivity, the cubic terms will eventually balance the tendency to get away from the linearly unstable O:[4] the motion therefore settles into an "equilibrium state" visiting alternatively at each turn, *i.e.* at every application of

[4] A proof is similar, but much simpler, to the one suggested in (E) in the analogous case of the Hopf bifurcation, *c.f.r.* problem [4.3.10] below.

the map S_Σ, two points located where the linear repulsion and the nonlinear attraction compensate each other. This is a periodic orbit for the map S_Σ with period 2 which is the intersection between the plane Σ and a periodic trajectory in phase space with a period of about twice[5] that $2\pi/\omega_0$ of the orbit that lost stability.

Geometrically this periodic orbit will be very close to the unstable orbit and almost twice as long: this explains the name of "*period doubling bifurcation*" that is given to this phenomenon.

(G) Stability loss of a periodic motion. Bifurcation through $\lambda = 1$.

This case is analogous to the already encountered one of fixed points of the differential equation (4.2.1) losing stability for $\lambda = 0$: the fixed point O_r for S_Σ generically ceases to exist at $r = r_c$; and for $r > r_c$ the motion migrates away from the vicinity of the periodic point that we followed up to $r = r_c$ and, in general, nothing can be said. Obviously in models with symmetries we can repeat what already said for the fixed points of (4.2.1) and it is possible that before the "*jump into the dark of the unknown*" (*i.e.* the migration of the motions towards more stable attracting sets, far away from the site in phase space where the orbit Γ_c that lost stability was located), a few nongeneric bifurcations may develop, accompanied by symmetry breakings. Or it is possible that regions far from Γ_{r_c} are again mapped close to Γ_{r_c} and an intermittency phenomenon arises: we describe it in the very similar case (H1) below.

(H) And then? Chaos! and its "scenarios".

When motions get far away from motions that have lost stability various possibilities arise: the simplest is when motions repelled by fixed points or periodic motions that have become unstable get near attracting sets, possibly far in phase space, that are still simple, *i.e.* of the same type of those considered so far (fixed points or periodic motions). In this case, as r changes further, all what has been said can be proposed again *in the same terms*. And the "real" problem is "postponed" to larger values of r. But things may be different, possibly after one ore more repetitions of the above analysis, and the examples that follow provide an interesting illustration.

(H1) Intermittency scenario.

Consider a periodic orbit Γ_r that loses stability at $r = r_c$ because of the crossing through 1 or -1 of an eigenvalue of the stability matrix. Generically the periodic orbit will disappear if the crossing is through 1 while if the crossing is through -1 it will persist unstable, varying continuously with r.

The Poincaré map S_Σ continues to contract approximately in all *but one* direction for $r - r_c$ small. The direction will be approximately that of the eigenvector $\underline{v}_c$ of the stability matrix $M(r_c)$ of S_Σ at the fixed point O on

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[5] Approximately because $r \neq r_c$.

which the map S_Σ is defined for $r \le r_c$. We call O_c the position of O at $r = r_c$.

And motion in that direction while not having small amplitude might still be essentially one dimensional and S_Σ can sometimes be considered as a map of a segment into itself. This will happen if the nonlinear terms of S_Σ at $r = r_c$ tend to recall, again, near O_c the motions that go far enough from $O_c \in \Sigma$.

We then see that the motion appears as having an approximately periodic component (developing near the orbit Γ_{r_c} that has lost stability) and a component transversal to it.

One can also say that it looks as if the orbit Γ_{r_c} lied (approximately if $r > r_c$) on a 2–dimensional invariant surface intersecting Σ along a curve τ approximately tangent to the direction $\underline{v}_c$ of the eigenvector corresponding to the real eigenvalue that has reached the unit circle.

At "each turn" the motion returns essentially to this curve τ, because the map S_Σ does not cause the motion to go far away from it, being unstable only in the direction $\underline{v}_c$ and *having assumed* that S_Σ brings back to near O_c points of Σ that are far from it (at least for r only slightly beyond r_c).

On the curve τ (that, near O_c, is very close to the straight line parallel to the eigenvector $\underline{v}_c$) we define the abscissa s and use it to describe the points of τ; let $s^* = s(r_c)$ be the abscissa of the point O_c of τ (we can imagine that $s^* = 0$).

In this case it can, sometimes, be a good *approximation* to consider the projection of the motion on the line parallel to the direction $\underline{v}_c$. Denoting always with s the abscissa on this line it will be possible to describe the motion by a map $s \to g_r(s)$. We are interested in the cases in which initial data close to the fixed point that has lost stability do not go too far away along this line, but remain indefinitely confined to a segment, say to $[-s_0, s_0]$, moving back and forth on it.

A picture illustrating the phenomenon is the following; represent the curve τ as a segment in the direction of $\underline{v}_c$ and a family of maps of the segment into itself which leads to a loss of stability at $r = r_c$ with eigenvalue 1, remaining attractive away from the point $s^* = 0$.

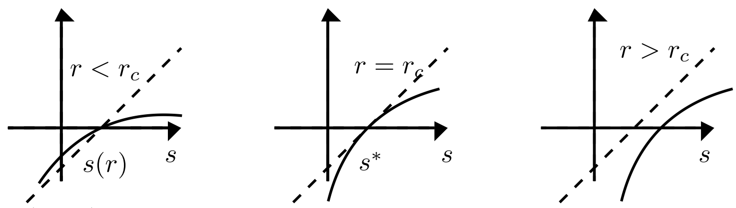

Fig. (4.3.1) *Case with eigenvalue* $+1$. *The* s *axis represents* τ*, the point* $s(r)$ *the fixed point; the map tends, for* $r > r_c$*, to send points away from* s^**; therefore if the non linear terms eventually prevail the map should look quite different further away from* s^**. Below in Fig. (4.3.2) a more global picture is presented. The case with eigenvalue* -1 *is simpler and is illustrated in Fig. (4.3.3).*

Reducibility of the motion generated by S_Σ on the section Σ can, however, *only be an approximation* because the map of $[-s_0, s_0]$ in itself will have necessarily (see below) to consist in an expansion of $[-s_0, s_0]$ followed by a *folding* of the interval into itself, *c.f.r.* Fig. (4.3.1). Therefore there will exist points $s = g_r(s_1) = g_r(s_2)$ that are images of distinct points s_1 and s_2. Hence if the motion did really develop on an invariant surface intersecting Σ on the curve τ then the uniqueness of solutions for the differential equation, that regulates the motions that we consider, would be violated. In Fig. (4.3.1) the "folding" is not illustrated, *i.e.* one should imagine that it happens at a distance from s^* not shown in the Fig. (4.3.1): see Fig. (4.3.2) for an example.

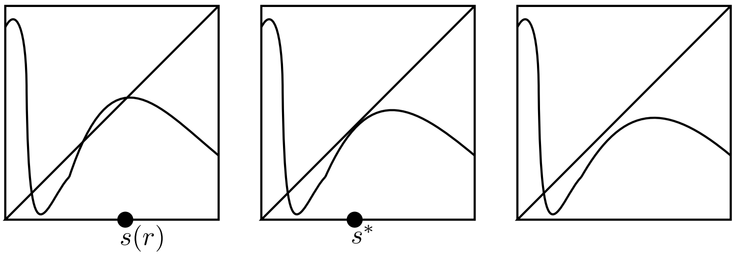

Fig. (4.3.2) *An example of an intermittency phenomenon: the picture represents the graph of $g(s)$ over $[-s_0, s_0]$ abd the square containing the graph is usually drawn as a visual aid to see "by inspection" that the interval $[-s_0, s_0]$ is invariant. The diagonal helps visualizing the fixed points and the iterates of the map.*

Hence the "one dimensional" description cannot be but approximated and in reality one must think that the segment $[-s_0, s_0]$ consists of several (infinitely many) segments which are essentially parallel and extremely close, so that they are not distinguishable by our observations. Nevertheless in this approximation we have the possibility of thinking the motion as 1–dimensional, at least in some cases in which the loss of stability is due to a single eigenvalue crossing in 1 or -1 the unit circle.

In the case of *crossing through* 1 we can imagine that g_r has a graph of the above type: respectively before the stability loss, at the stability loss and afterwards.

Stability loss will then be accompanied by an *intermittency phenomenon*, in fact for $r > r_c$ the map g_r that for $r = r_c$ was tangent to the diagonal will not have any longer points in common with the diagonal near the last tangency point and therefore motions *will spend a long time* near the point $s^*(r_c)$ of "last existence" of the fixed point (*no longer existent* for $r > r_c$). Then they will go away and, possibly, return after having visited other regions of phase space (that under the assumptions used here reduces to an interval).[6]

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[6] Unless, of course, the picture differs from that in Fig. (4.3.2) and the points are attracted by *other* stable fixed points or periodic orbits: in such cases, however, we can repeat the whole analysis.

In this situation the motion does not appear periodic nor quasi periodic: there are time intervals, even very long if $r \sim r_c$, in which it looks essentially periodic. This is so because the motion returns close to what for $r = r_c$ was still a fixed point of S_Σ and which will still be such in an approximate sense if $r - r_c > 0$ is small.

Such long time intervals are followed, because of the phenomenon illustrated by the third of Fig. (4.3.2), by erratic motions in which the point gets away from the periodic orbit that has lost stability until it is again "captured" by it because the map is, far from $s^*(r_c) = 0$, still pushing towards s^*. One checks, indeed, that for small $r - r_c > 0$ a number $O((r - r_c)^{-1/2})$ of iterations of S_Σ are necessary to get appreciably far away from the point with abscissa $s^*_{r_c}$ (which for $r > r_c$ no longer is a fixed point).

In this way "chaotic" or "turbulent" motions arise following what is called the "*Pomeau–Manneville scenario*" or "*intermittency scenario*", [PM80], [Ec81].

(H2) Period doubling scenario.

A different scenario develops when the stability loss is due to the crossing by an eigenvalue of $M(r)$ of the unit circle through -1. In this case the one dimensional representation described in (H1) will still be possible, but the graph of g_r, having to cut at 90^o the bisectrix (of the second and third quadrants of the plane (s, g_r)) at $r = r_c$ (so that $g'_{r_c}(0) = -1$), cannot become tangent to the graph of S_Σ, for $r > r_c$ and $r - r_c$ small.

This time the unstable fixed point O with abscissa $s(r)$ on the curve τ continues to exist even for $r - r_c > 0$ (at least for a while) because the stability matrix of O does not have an eigenvalue 1 for $r = r_c$ (hence the fixed point continues to exist as guaranteed by an implicit functions theorem). We imagine that the absolute value of the derivative of g_r at the unstable fixed point increases: and we can then apply a general theory of bifurcations associated with interval maps whose graph has the form in Fig. (4.3.3)

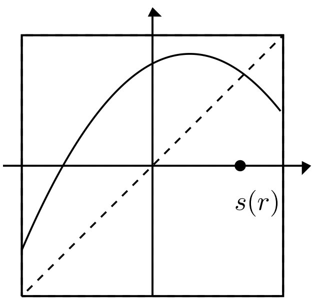

Fig. (4.3.3): *This illustrates a map g with a fixed point $s(r)$ with a stability eigenvalue about -1, and smaller than 1 in modulus. As r varies one should imagine that the slope at $s(r)$ reaches for $r = r_c$ the value -1 and then grows further (in modulus) for $r > r_c$ giving rise to a period doubling bifurcation.*

which is one of the simplest forms that allow us to define a map that stretches and then folds on itself (another has been illustrated in Fig. (4.3.2)).

This theory, due to Feigenbaum, predicts, not "generically" but "quite generally", that the first period doubling bifurcation is followed by an *infinite chain*, appearing within a short interval of r, *of period doubling bifurcations*, in which the initial orbit successively doubles in shape and period. The intervals of r at the end of which the successive bifurcations take place contract exponentially in size with a ratio asymptotically equal to $1/4.68..$ which is a universal constant (*i.e.* independent of the system considered within a vast class of possible systems) and is called the *Feigenbaum constant*.

Chaotic, nonperiodic, motions appear (or can appear) at the end of the chain of period doublings: and this transition to chaos is called the *Feigenbaum scenario* or *period doubling scenario*, *c.f.r.* [Fe78],[Fe80], [Ec81],[CE80].

Of course if the loss of stability occurs with O being vaguely repulsive *an intermittency phenomenon* is possible if the non linear terms can still be such to map regions far away from O back to the vicinity of O. Its discussion is completely analogous to the one in (H1).

(H3) The Ruelle–Takens scenario.

It remains to consider what happens when some Hopf's bifurcation generates a torus and, as r increases this torus loses stability. This torus will appear in a Poincaré's section as a closed curve γ (intersection between the torus and the section surface Σ).

Attempting to generalize the notion of Poincaré's map to analyze stability of the quasi periodic motion is doomed to failure because the map should transform a curve γ_0 (*i.e.* a small deformation of the invariant curve γ, representing the intersection between the surface Σ and the torus of which we study the stability) into a new curve γ_0', image of the former "after one turn". The failure is not so much to ascribe to the fact that such a map would act on an infinite dimensional space (acting upon curves in Σ rather than on point on Σ) but, rather, because in general *it will not be well defined* since new motions can develop on the torus.

For instance small periodic orbits or invariant sets appearing on the surface of the torus or nearby make the extension of Poincaré's map, that we would like to define, to depend in an important way on the choice of Σ and can make it ill defined (while one notes that in the preceding case of the stability analysis for a periodic orbit all sections are equivalent). For instance if there is an invariant set E on the torus which has a small diameter and the surface Σ does not intersect it we cannot derive from the behavior of the Poincaré's map, any instability phenomenon due to motions that stay inside E, simply because their trajectories never visit Σ.

Often one finds that the two time scales associated with the motion on the torus become commensurate (as r varies), *i.e.* one has *resonance* or *phase locking*, *c.f.r.* (F), and motion is asymptotically periodic and not quasi periodic.

Or it may happen that the loss of stability generates attracting sets that are contained in a surface of higher dimension which is not a torus or even if it is a torus the motion on it, generically, will *neither be periodic nor quasi periodic* making it difficult to build a general theory of the motions.

The problem is real because, as shown in [RT71], the stability loss of a bidimensional torus $\mathcal{T}_2$, *even admitting that it generates an invariant three dimensional torus* $\mathcal{T}_3$, is followed, generically, by a motion on $\mathcal{T}_3$ that is not a quasi periodic motion with three frequencies (as in the classic *aristotelic scenario*) but rather by a motion regulated by a *strange attracting set*, *i.e.* by an attracting set which is neither a time independent point nor a periodic orbit nor a quasi periodic motion. Hence the stability of an invariant torus provides us with a third scenario, called the *scenario of Ruelle–Takens*, because they pointed out the above stated genericity of strange motions generated by the loss of stability of a quasi periodic motion with two frequencies, [RT71],[Ec81].

To be precise in [RT71] it is shown that if for some reason an invariant torus of dimension *four*, only later reduced to three, is generated at a certain value of the parameter r, then generically motion on it will *not* be quasi periodic. And the point of the work was to criticize the idea that it could be common that a 2–dimensional torus run quasi periodically bifurcates into a 3–dimensional torus also run quasi periodically which in turn bifurcates into a 4 dimensional torus run quasi periodically and so on.

(I) Conclusions:

The three scenarios just discussed do not exhaust all possibilities but they cover fairly well the instabilities that are generically possible when a periodic motion or a quasi periodic one (with two frequencies) become unstable.

This is so at least in the cases in which the stability loss is due to the passage through the unit circle of either one real eigenvalue or of two conjugate ones of the stability matrix of the Poincaré map, while the others remain well inside the unit circle. Hence it is reasonable to think that the unstable motion generated on the surface on which the Poincaré map is defined is, within a good approximation, one dimensional and developing on a segment in the first case or on a closed curve in the second.

There are also *other possibilities*, like the eventual moving far away from the fixed point (or periodic motion) that has become unstable as it happens in cases of vague repulsivity: this case either give rise to a "*palingenesis*" because the phenomenology reappears anew because the motions near the unstable ones migrate elsewhere in phase space towards a stable fixed point or periodic orbit or invariant 2-dimensional torus, or it can be considered as a *fourth scenario* that we can call the *scenario of direct transition to chaos*, *i.e.* when the motions departing from the vicinity of the ones that have become unstable do not return intermittently close to them and are asymptotically governed by a strange attracting set.

It can also happen that a large number of eigenvalues of the stability matrix dwell near the unit circle (or the imaginary axis); a case in which the motion may "appear" chaotic because of the many time scales that regulate its development. One can hope, and this seems indeed the case, that this rarely happens because of the nongenericity of the crossing of the unit circle by more than one real or two conjugate eigenvalues (or of the imaginary axis by more than one real or two conjugate eigenvalues in the case of stability loss by a fixed point). In any event one can think that this is part of a *fifth scenario* or *scenario of the remaining cases*.

Among the "other cases" there is also the possibility of successive formation of quasi periodic motions with an increasing number of frequencies, *i.e.* there is the possibility of a scenario sometimes called the *scenario of Landau* (or of *Hopf*, who also proposed it, or *quasi periodicity scenario*). They are certainly mathematically possible and examples can be exhibited which, however, seem to have little relevance for fluidodynamics where this scenario has *never* been observed when involving quasi periodic motions with more than three frequencies; and the cases with three frequencies, or interpretable as such, observed are only two: one in a real experiment, *c.f.r.* [LFL83], and one in numerical simulations in a 7 modes truncation of the three dimensional Navier–Stokes equation, *c.f.r.* [GZ93].

Note also that, as discussed in (E) apropos of the Devil's staircases, already what happens in an ordinary Hopf bifurcation is in a way in "conflict" with the quasi periodicity scenario with increasing number of frequencies: in fact phase locking phenomena (with the consequent formation of stable periodic orbits) are generated rather than quasi periodic motions with monotonically varying rotation number.[7]

At this point it must also be said, as suggested by Arnold in support of Landau's scenario, that although it is true that quasi periodic motions cannot generically follow losses of stability by two frequencies motions on 2–dimensional tori, it remains that there are generic sets with *extremely small volume*[8] while there are nongeneric sets with *extremely large volume*.[9]

Examples of the latter phenomenon are, indeed, quasi periodic motions that arise in the theory of conservative systems: they are motions that are important, for instance, for the applications to celestial mechanics. They take place on sets with large positive volume in the space of the control parameters, and are therefore even easy to observe. Not more difficult than

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7 It is unreasonable to think that Landau ignored the phase locking phenomenon in this context: probably he thought that it had negligible consequences in the development of turbulence and that "things" wernt as if it was unexistent. The phase locking intervals would become shorter and shorter as the forcing strength increased. This is a very fruitful way of thinking common in Physics, think for instance (see [Ga99b]) to the geocentric hypothesis of ancient astronomy or to the ergodic hypothesis, see Ch. VII for other examples.

8 Think of the union of intervals of size $\varepsilon\, 2^{-n}$ centered around the n–th rational point.

9 Think to the set of points ω in $[0,1]$ such that $|\omega q - p| > C q^{-2}$ for a suitable C and for all pairs (p,q) with $q>0$: these are the "*diophantine*" numbers with exponent 2, which form a set of measure 1 but with a dense complement, *c.f.r.* problems in §5.1.

the phenomena that take place in correspondence of the complementary values of the control parameters (which may be even stable, *i.e.* take place in open sets).

Hence there is the possibility that by simply changing the notion of small or large, understanding for "large" what has a large volume (rather than what is "generic"), that the quasi periodicity scenario appears as acceptable. Thus, mathematically speaking, the critique that one can really move to the scenario of Landau is that we do not observe it or that it is *extremely rare*.

As r increases further all the possibilities seen above will "overlap" as they can appear in various regions of phase space and thus coexist, making it very difficult even a phenomenology of developed turbulence.

But for what concerns the onset of turbulence the first four scenarios described above provide a rather general list of possibilities and rarely transitions to chaos have been observed that significantly deviate from one of them (I only know of the two quoted above, [LFL83], [GZ93].

Finally one should note that although the growth of r usually favors development of instabilities, it is not necessary that bifurcations should always happen in the direction in which we have discussed them: sometimes as r increases the same phenomena are observed but in reverse order. For instance it is possible, and it is observed, that periodic orbits, instead of doubling, halve in a "reverse" Feigenbaum sequence reaching a final state consisting in a stable periodic orbit which, as r increases further, might become again unstable following any of the possible scenarios. Furthermore what happens in certain phase space regions is, or can be, largely independent of what happens in others.

When for the same value of r of the driving there are several attracting sets (with different attracting basins, of course) one says that the system presents a phenomenon of *hysteresis* also called of *non–uniqueness*, see [Co65] p. 416 and [FSG79] p. 107.

Several direct "transitions to chaos", (fourth scenario), *can be instead regarded as normal inverse evolutions according to one of the first three scenarios.* Indeed what at times happens is that in a "direct transition" to chaos motions get away towards an attracting set which, at the considered value of r, is chaotic but which is the result of an independent evolution (following one of the first three scenarios) of another attractive set located elsewhere in phase space. In conclusion we can say that the scenarios alternative to the four described ones have never been observed with certainty in models of physical interest (although they are mathematically possible, as simple examples show).

Therefore we shall call "normal" an evolution towards chaotic motions that follows one of the four scenarios described above: "anomalous" will be other possibilities including the *aristotelic scenario* in which quasi periodic motions with more and more frequencies appear without ever giving rise to a truly strange motion, the scenario often attributed to Landau who in this case is meant by some to play the unpleasant role of modern Ptolemy, (which to Landau should not sound bad as the critiques to Ptolemy are often not well founded either and in any event hardly diminish his scientific stature, [Ga99b].

(J) The experiments.

I shall not present here how the analysis of the preceding sections really looks from an experimental point of view: *i.e.* which are the actual observations and the related techniques and methodolgy. The reader will find a simple and informative expository introduction in [SG78].

Problems.

[4.3.1]: Consider the spectrum of the stability matrix M of the Poincaré map associated with a periodic orbit Γ. Check that it does not depend on the point O nor on the section plane Σ chosen to define it. (*Idea:* Check that changes of O, or of Σ, change M by a similarity transformation; *i.e.* there is a matrix J such that the new stability matrix M' is $M' = JMJ^{-1}$.)

[4.3.2]: (*Hopf bifucation for maps*) Proceeding in analogy to the analysis of §4.2, formulate the Hopf bifurcation theory for periodic orbits. (*Idea:* The role of the imaginary axis is now taken by the unit circle. The theory is essentially identical with the novelty of the possible stability loss through $\lambda = -1$, which gives rise to the period doubling bifurcations. The normal forms approach, *c.f.r.* problem [4.2.4] is also very similar. If one formulates and accepts a theorem analogous to the center manifold theorem, *c.f.r.* problem [4.2.7], the analogue of the Hopf bifurcation theorem is discussed in the same way taking into account, when necessity arises, the further condition mentioned above (*i.e.* $\lambda_i^a \neq 1$ for $a = 3, 4$) that there should not be eigenvalues of the stability matrix at $r = r_c$ which are third or fourth roots of unity, *c.f.r.* [Ga83]).

[4.3.3]: (*existence of the rotation number for circle maps*) Let $\alpha \to g(\alpha)$ be a C^∞ increasing function defined for $\alpha \in (-\infty, \infty)$ and such that:

$$g(\alpha + 2\pi) = g(\alpha) + 2\pi$$

Note that the map of the circle $[0, 2\pi]$ defined by $\alpha \to g(\alpha) \bmod 2\pi$ is also of class C^∞: show that the limit $\lim_n \frac{1}{n} g^n(\alpha)$, if existing, is independent of α. Show then that the limit exists (*Poincaré's theorem*).
(*Idea:* Monotony of g implies that $g^n(0) \leq g^n(\alpha) \leq g^n(2\pi) = g^{n-1}(g(2\pi)) = g^{n-1}(g(0) + 2\pi) = \ldots = g^n(0) + 2\pi$: this gives α–independence. Likewise $g^n(2\pi\, k) = g^n(0) + 2\pi\, k$: next note that

$$g^{n+m}(0) \leq g^n(0) + g^m(0) + 4\pi$$

because setting $g^m(0) = 2\pi k_m + \delta_m$, $0 \leq \delta_m < 2\pi$ it is $g^{n+m}(0) \leq g^n(2\pi k_m + \delta_m) \leq g^n(2\pi(k_m + 1)) = g^n(0) + 2\pi(k_m + 1) \leq g^n(0) + g^m(0) + 4\pi$. Hence calling $L = \liminf_{m\to\infty} m^{-1} g^m(0)$ and if m_0 is such that $m_0^{-1} g^{m_0}(0) < L + \varepsilon$ with ε positive, we shall write $n = k\, m_0 + r_0$ with $0 \leq r_0 < m_0$. Then $g^n(0) \leq k g^{m_0}(0) + 4\pi(k + 2)$ and dividing by n both sides and taking the limits as $n \to \infty$ we get $\limsup n^{-1} g^n(0) \leq m_0^{-1} g^{m_0}(0) + 8\pi/m_0 \leq L + \varepsilon + 8\pi/m_0$ and by the arbitrariness of ε and the possibility of choosing m_0 as large as wanted we get $\limsup n^{-1} g^n(0) \leq \liminf m^{-1} g^m(0)$, *i.e.* the limit esists. See §5.4 for an alternative proof. See problem [5.1.28], [5.1.29], below for a constructive proof and an algorithm to construct the rotation number and verify its continuity as a function of the map g.)

[4.3.4]: By using only a computer analyze the bifurcations of (4.1.30) and, proceeding by experimentation, check the existence of period doubling bifurcations. Try to find an interval of r, where pairs of stable and unstable periodic motions annihilate as r decreases, via a bifurcation through 0, *c.f.r.* [FT79]. (The results are described also in [Ga83], Chap. V, §8).

[4.3.5]: Study empirically, with a computer, the structure of the bifurcations of the time independent solutions and of the (possible) periodic orbits of (4.1.28), *c.f.r.* [Fr83]. (The results are described also in [Ga83], Chap. V, §8).

[4.3.6]: Build an example of a function on $[0,1]$ whose graph is a "devil's staircase", *i.e.* which is a nondecreasing function such that every rational value is taken in an interval in which the function is constant and, furthermore, which is continuous and strictly increasing in all points in which it has an irrational value. (*Idea:* Consider the n–th rational point x_n, $n = 1, 2, \ldots$ (enumerating rationals arbitrarily). Let $f(x) = x_1$ for x in the "first triadic interval" I_1 (*i.e.* in the open interval containing the numbers whose first digit in base 3 is not 1: $(\frac{1}{3}, \frac{2}{3})$).[10] Consider x_2 and if $x_2 < x_1$ let $f(x) = x_2$ in the first triadic interval to the left of I_1, *i.e.* $(1/9, 2/9)$, or if $x_2 > x_1$ in the first triadic interval to the right of I_1, *i.e.* $(7/9, 8/9)$. Proceed iteratively and let $f(x)$ be the function that we reach in this way: extend it by continuity to the other points (that form the Cantor set) obtaining a function that has as graph the stair of the wicked Being.)

[4.3.7]: (*a resonance*) Conjecture that every C^∞ map, S, of R^n in itself that has the origin as a hyperbolic fixed point (*i.e.* with a stability matrix without eigenvalues λ_i of modulo 1) can be transformed by a coordinate change of class C^2 at least into a map S' which is *exactly linear* in the vicinity of the origin. Show that the conjecture is false by showing that the map

$$x_1' = \lambda^2 x_1 + N_{123} x_2 x_3, \quad x_2' = \lambda^{-1} x_2, \quad x_3' = \lambda^3 x_3$$

with $\lambda > 1$ gives a counterexample because it cannot be transformed into $\xi_1' = \lambda^2 \xi_1$, $\xi_2' = \lambda^{-1}\xi_2$, $\xi_3' = \lambda^3 \xi_3$ with a C^2 change of coordinates. (*Idea:* Show that $\xi_j' = x_j + \sum_{i,k} m_{j,ik} x_i x_k + o(x^2)$ gives rise to incompatible conditions for the coefficients $m_{j,ik}$ if $N_{123} \neq 0$: because they should verify $(\lambda_j - \lambda_i\lambda_k) m_{ijk} = N_{ijk}$.)

[4.3.8]: The result in problem [4.3.7] is not incompatible with the *Grobman–Hartman theorem* which, instead, states that if S is of class C^1 at least then it can be transformed in the vicinity of the origin into a linear map via a coordinate change locally invertible and *continuous*, *but not necessarily differentiable*. One can show that if S is of class C^∞ then there exists, near a fixed point (the origin to fix the ideas), a C^1 coordinate change that linearizes locally the map if $\lambda_j \neq \lambda_i\lambda_k$ for all values of i, j, k. One says in this case that S verifies a "nonresonance condition" of order 2 at the fixed point. If, furthermore, $\lambda_i \neq \lambda_1^{k_1} \ldots \lambda_n^{k_n}$ for all n–ples of nonnegative integers $k_1, \ldots, k_n$ with sum ≥ 2, one says that S verifies a "nonresonance condition" to all orders at the fixed point: then the coordinate change can be chosen to be of class C^k, for all $k > 0$, close to the fixed point (how close may depend on the value prefixed for k), *c.f.r.* [Ru89b], moreover it depends continuously, in class C^1, from any parameters on which S possibly depends, provided S is C^∞ in these parameters). Analogous results hold for the exact linearization of differential equations near a fixed point where the stability matrix has eigenvalues λ_j with $\lambda_j \neq \sum_i k_i \lambda_i$ for $\underline{k}$ as above, *c.f.r.* [Ru89b], p. 25.

[4.3.9]: (*stable and unstable manifolds: normal form in absence of resonances*) Assume valid the *nonresonance* property to arbitrary order and prove that the results quoted in problem [4.3.8] can be used to show existence, regularity and regular dependence (on the (possible) parameters on which S depends) of two manifolds $W_O^{\delta,s}$ e $W_O^{\delta,u}$ contained in a vicinity of O, of small enough radius δ, such that $S^n x \xrightarrow[n\pm\infty]{} O$ if $x \in W_O^{\delta,s}$ or, respectively, $x \in W_O^{\delta,u}$. (*Idea:* This is obvious if S is linear, and it remains true also in the nonresonant nonlinear case, by using the coordinate change quoted in problem [4.3.8]. This is a little involved proof of the existence of the stable and unstable manifolds of an hyperbolic fixed point. Existence can be proved with more elementary means and under more general conditions, in which the equation is *not even linearizable*: the manifolds

[10] In general given an interval I it can be divided in three thirds, I_0, I_1, I_2: called here the 0-th, the 1-st and the 2–d (*i.e.* we use the C-language conventions).

always exist for any hyperbolic fixed point and have *the same regularity* of S provided the latter is at least of class C^1, *c.f.r.* [Ru89b], p.28.)

[4.3.10]: Applying an implicit function theorem prove that even if the terms $O(z^4)$ in eq. (4.3.1) are not neglected there is a curve γ invariant for S_Σ and close to the circle in eq. (4.3.2). (*Idea:* Let $\varepsilon = -2\,\mathrm{Re}\,c(r)$ and let the parametric equations of the curve γ be $z = \rho(1+\rho\,\xi(\alpha))e^{i\alpha}$ with $\alpha \in [0, 2\pi]$ with ρ as in eq. (4.3.2). Then the equation for γ becomes $K\xi(\alpha) = \xi(\alpha)$ with

$$\begin{aligned} K\xi(\alpha') =&(1-\varepsilon)\xi(\alpha) + \rho\xi(\alpha)^2\,\overline{\Lambda} + \rho^2\Lambda' \\ \alpha' =&\alpha + \delta + \rho^2\,\mathrm{Im}\,c(r) + \rho^2\xi(\alpha)\tilde{\Lambda} + \rho^4\Lambda \end{aligned}$$

where $\Lambda, \Lambda', \overline{\Lambda}, \tilde{\Lambda}$ are smooth functions of $\xi\alpha$. Then for $r - r_c > 0$ small (*i.e.* for ρ small) one finds, if $||\xi||_{C^1} \le 1$,

$$||K\xi_1 - K\xi_2||_{C^1} \le (1-\varepsilon/2)||\xi_1 - \xi_2||_{C^1}$$

so that the solution can be found simply as $\lim_{n\to\infty} K^n\xi_0$ with $\xi_0 = 0$, for instance.)

Bibliography: [RT71], [Fe78], [Fe80], [Ec81], [CE80], [ER81], [Ru89], [GZ93], [LL71], [SG78] and principally [Ru89b]. The scenario of Landau in its original formulation can be found in the first editions of the treatise [LL71]: in the more recent ones the text of the latter book has been modified and updated to take into account the novelties generated by [RT71].

§4.4: Dynamical tables.

In this section we analyze properties of the states that are reached at large time starting from randomly chosen initial data. Such states depend on the value of a parameter, r in the examples of §4.1, that we conventionally call the *Reynolds number* and which measures the intensity of the driving force applied to the system to keep it in motion.

This parameter will be considered, in order to offer a more cogent visual image, as a "time": so that we shall say that "as r increases" a bifurcation takes place from a certain stationary state (not necessarily time independent, nor quasi periodic but just statistically stationary) which leads to the "creation" of new stable stationary states. Or that "as r increases", somewhere in phase space happens a "creation" of a pair of fixed points or of periodic orbits one stable and one unstable, or an *"annihilation"* of such a pair happens, *etc.*

The following view of the onset of turbulence emerges from the discussion of §4.2, §4.3. The initial laminar motions (which in phase space is a fixed point, *i.e.* a "time independent motion") loses stability through successive bifurcations from which new laminar motions are born; they are less symmetric, at least if the system possesses some symmetries, as the examples in §4.1, see (4.1.21), (4.1.29), (4.1.31). The new motions attract, in turn, random initial data (by random we mean chosen with a probability distribution that has a density with respect to the volume in phase space).

This may go on until (and if) the laminar motion disappears because of a "collision" with a time independent unstable motion giving rise directly or intermittently to a chaotic motion. Or it becomes unstable via a Hopf bifurcation generating a stable periodic motion. In the latter case the periodic stable motion may lose stability as r increases becoming in turn unstable. This can happen in various ways

(1) by a "collision" with an unstable periodic orbit with a possible change of the asymptotic motion to a chaotic one following an intermittent or a direct scenario, or
(2) giving raise to a quasi periodic motion on a 2–dimensional torus following the Ruelle–Takens scenario, or
(3) giving rise to an asymptotic motion which is chaotic after several period doubling bifurcations following the Feigenbaum scenario.

In case (2) the torus will evolve possibly losing stability and generating strange attractors; likewise in case (3) the periodic motions generated by the period doubling bifurcations will continue to double in an ever faster succession (as r grows) eventually generating a chaotic attracting set.

It is also possible that, at certain values of r, stable periodic orbits or stable fixed points (or other types of attracting sets) appear somewhere else in phase space in regions which seemed to have no other distinguishing property and which dispute the role of attracting set to attracting sets existent elsewhere in phase space: this is the *hysteresis* phenomenon, *c.f.r.* (I) in §4.3.

The "dispute" takes place because the attraction basins of the attracting sets will in general have common boundaries which often show rather high geometrical complexity; so that it can be difficult to decide to which basin of attraction a particular initial datum belongs, and which will be the long time behavior of a motion following the given initial datum.

For instance a pair of fixed points or of periodic motions, one stable and one unstable, can be born at a certain value of r, in some region of phase space and evolve in a way similar to the evolution of the "principal series" of attracting sets, which we define loosely as

Definition *(principal series): we shall call "series" a family of invariant sets in phase space parameterized by the forcing strength r and varying continuously with r (several invariant sets may correspond to each r because of possible bifurcations). The family of fixed points, periodic orbits, invariant tori, other invariant sets that can be continuosly traced back to an attracting set that exists at $r = 0$ will be called a "principal series" of bifurcations, the others (when existent) "secondary series".*

Remark: Usually there will be only one such family because for $r = 0$ there will be a single globally attracting (time independent) point, that we called *laminar motion.*

As soon as there exist two or more distinct attracting sets, phase space divides into regions attracted by different attracting sets; and naturally new phenomena are possible if one looks at data starting on the boundary that separates the two basins of attraction. Such phenomena are "difficult to see" because it is impossible to get initial data that are exactly on the boundary of the basins, unless the system is endowed with some special symmetry that imposes a simple and precise form for the boundary between two basins.

The order in which different types of attracting sets appear in a bifurcation series, principal or secondary, is not necessarily an ordered sequence of fixed points first, then periodic orbits, then invariant tori *etc*: some steps may be missing or the order can be changed, even inverted.

(A) Bifurcations and their graphical representation.

It is important to find a method to represent the above phenomenology to understand by inspection the situation in a given particular case.

A convenient method consists in using the following "*dynamical tables*".

Consider the family of the stable fixed points and attracting sets that can be continuously followed as r grows starting from the fundamental laminar motion of the system, identified with a laminar motion existing for $r = 0$, that we suppose for simplicity existent unique and globally attractive for $r = 0$ and for r very small, *i.e.* we consider the "principal series" of bifurcations according to the definition above.

We shall represent in a plane the axis r as abscissae axis: a point on a line parallel to the axis r will symbolize the laminar motion corresponding to the parameter r.

The stability loss at $r = r_c$ will be marked by a point I, *c.f.r.* Fig. (4.4.1). The segment OI will be denoted F_0 (with F standing for "fixed").

If in r_c there is a bifurcation with symmetry loss we draw two new lines departing from I, *i.e.* as many as the newly created fixed points, drawing as dashed the lines representing the unstable fixed points and as continuous lines the stable ones. The vertical axis does not have *meaning* other than a symbolic one and the plane is only used to have space to draw a picture

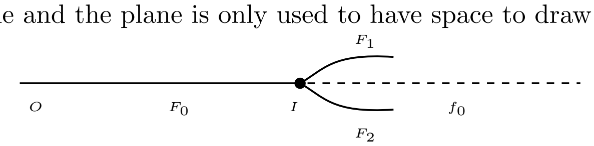

Fig. (4.4.1) *The segment OI represents a family of fixed points which bifurcate at I with a symmetry breaking bifurcation into two steble fixed points and an unstable one.*

Hence from a figure in which all continuous families of fixed points have been represented as lines parallel to the x axis we see immediately how many fixed points coexist in correspondence of a given r and from which "history" they arise as r grows: it is enough to draw a line perpendicular to the axis and count the intersections with the graph.

To continue in a systematic way it is convenient to establish more general rules for the drawings that we shall make. We shall use the following symbols

Fig. (4.4.2) *The drawings represent a family (F) of fixed points continuously evolving as r varies in* (r_1, r_2)*, or a family of periodic orbits (O), or a family of stable invariant tori (T), or of a strange attracting sets (S), respectively.*

to denote that in correspondence of the interval (r_1, r_2) there is a stable fixed point continuously depending on r in the case F, or a stable periodic orbit in the case O, or an invariant torus run quasi periodically in the case T, or of strange attracting set in the case S.

The same symbols, in *dashed form*, will be used to denote the corresponding unstable entities. For brevity we shall call "*eigenvalues*" of a fixed point or of a periodic orbit the eigenvalues of its stability matrix.

A bifurcation will be denoted by a black disk, that will carry a label to distinguish its *type*: for instance

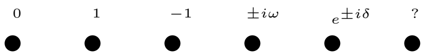

Fig. (4.4.3) *Symbols for types of bifurcations.*

will denote respectively the following cases

(1) Case denoted 0: a stability loss of a fixed point due to the passage through 0 of a real eigenvalue (with loss of some symmetry and persistence of the fixed point as an unstable point, or with a creation of a pair of fixed points (or annihilation: with ensuing direct or intermittent chaos).

(2) Case denoted 1: a stability loss by a periodic orbit due to the passage through 1 of a real eigenvalue with consequent breaking of some symmetry or a creation (annihilation) of a pair of periodic orbits (one stable and one unstable). Hence on the black disk of Fig. (4.4.1) we should put the label 0 over the label I.

(3) Case denoted -1: a stability loss by a periodic orbit due to the passage through -1 of a real eigenvalue. In this case a consequent appearance of a stable periodic orbit of almost double period and persistence of the preceding orbit as an unstable one is possible; but sometimes the orbit is not vaguely attractive and direct or intermittent chaotic transitions appear.

(4) Case denoted $\pm i\omega$: a stability loss by a fixed point due to the crossing of the imaginary axis by a pair of conjugate eigenvalues with either the consequent appearance of a periodic stable orbit and permanence of the fixed point as an unstable one, or with the disappearance of a stable periodic orbit and appearance of chaotic motion via a direct or intermittent transition.

(5) Case denoted $e^{\pm i\delta}$: a stability loss by a periodic orbit due to the passage through the unit circle of a pair of conjugate eigenvalues; also in this case the loss of stabiloity could result in a (direct or intermittent) chaotic transition or in the appearance of an invariant torus (stable or unstable) and persistence of the periodic orbit as an unstable orbit.

(6) There are, sometimes, bifurcations whose type is difficult to analyze or to understand or other: this case will be denoted by affixing on the bullet a label ? .

An *arrow pointer* will indicate that the history of an orbit or of an attracting set proceeds, as the parameter r varies, in the direction marked by the arrow and that it can possibly undergo further bifurcations *that are not indicated*.

(B) An example: the dynamical table for the model NS_5.

The discussion of the sections §4.2, §4.3 combined with the graphic conventions just established allows us to summarize the theoretical results on the equation (4.1.30), which are described in the problems of §4.1, §4.3 and the experimental ones obtained mainly in [FT79], via a *dynamical table* that provides us immediately with a "global view" of the phenomenology.

We recall that the NS_5, *i.e.* (4.1.30), possess a symmetry: they are invariant under a simple symmetry group with four elements, see §4.1.

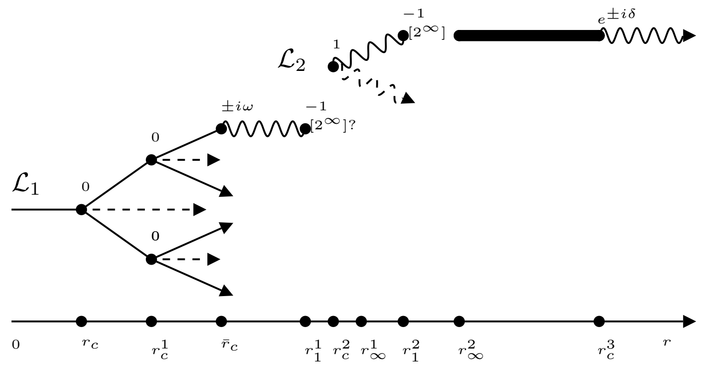

Fig. (4.4.4) *A rather detailed dynamical table representing the phenomenology observed in the model NS_5. The r_c, r_c^1, $\bar r_c$ are symmetry breaking bifurcations. Note the hysteresis in a small interval to the right of $r_c^2 < r_\infty^1$.*

The above Fig. (4.4.4) illustrates, in fact, the real dynamical table of the equation (4.1.30), computed in [FT79] (*c.f.r.* also the problems of §4.1, §4.3). The points r_c, r_c^1 marked on the r axis are the threshold values of the successive symmetry breaking bifurcations. The first marks the breaking of the symmetry (4.1.31) in the sense that the two new stable fixed points generated by the bifurcation are mapped into each other by the transformations with $\varepsilon = \pm 1$ but are still invariant under the subgroup with $\eta = \pm 1$ and $\varepsilon = 1$; at r_c^1 the symmetry is completely broken and the 4 resulting stable fixed points are distinct and mapped into each other by the group of symmetry.

In $\bar r_c$ *each* of the four stable fixed points generated by the preceding bifurcations loses stability undergoing a Hopf bifurcation, and each generates a stable periodic orbit (hence four symmetric orbits are generated): one of them is indicated by the wavy line (the others are related to it by the symmetries of the model and are hinted by the arrow pointers).

This orbit loses stability by period doubling in r_1^1 and a sequence of infinitely many period doubling bifurcations follows, indicated by the symbol $[2^\infty]$. One observes, in fact, *only three* such period doubling bifurcations because at the value $r_c^2 < r_\infty^1 = \{$ value where the sequence of period doubling bifurcations "should" accumulate (according to Feigenbaum's theory) $\}$ a hysteresis phenomenon develops (*c.f.r.* (I) in §4.3).

Namely, in *another region* of phase space (not far, but clearly distinct, from the one where the orbits, that we followed so far, underwent period doubling bifurcations), a pair of periodic orbits (one stable and one unstable) are "created".

The attraction basin of the new stable orbit rapidly extends, as r grows within a very small interval. While r is in this interval a *hysteresis phenomenon occurs* and some initial data are attracted by the new stable orbit and others are attracted by the doubling periodic orbits. Soon the attraction basin of the new orbit appears to swallow the region where the doubling periodic orbits should be located (hence they either disappeared or possess too small a basin of attraction to be observable in the simulations). In this way the "main series" of bifurcations denoted $\mathcal{L}_1$, terminates (without glory) its history.

Note that a system, like the one being considered, which has various symmetries necessarily presents hysteresis phenomena when the attracting sets do not have full symmetry (*i.e.* when symmetry breaking bifurcations have occurred). This is therefore the case for $r > \overline{r}_c$: the dynamic table in reality consists, for $r > \overline{r}_c$, of four equal tables, only one of which is described in Fig. (4.4.4) (the others would emerge, by symmetry,from the other "arrow pointers" associated with the other fixed points). Between these 4 *coexisting* attracting sets, images of each other under the symmetry group, hysteresis phenomena naturally take place. But the hysteresis phenomenon between the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ in Fig. (4.4.4) has a *different nature*, *i.e.* it is not trivially due to a coexistence of symmetric attracting sets, because no element of the first series is a symmetric image of one of the second.

In Fig. (4.4.4) we have not marked the unstable periodic orbits that remain, as a "wake", after each series of period doubling bifurcations: such unstable orbits can, in fact, be followed in the numerical simulations at least for a while (as r increases). Drawing them would make the figure too involved; but we can imagine that the symbol $[2^\infty]$ includes them. The interrogation mark that follows $[2^\infty]$ means that in reality one observes only few period doubling bifurcations because, as discussed above, at a certain point between r_c^2 and the value r_∞^1 (extrapolated from the three or four really observed values of the period doubling bifurcations, on the basis of Feigenbaum's universality theory) a "collision" seems[1] to take place between the stable orbits produced by the period doubling bifurcations and the attraction basin of the stable periodic orbits of the *new series* $\mathcal{L}_2$, with the consequence that the family $\mathcal{L}_1$ seems to end.

[1] Keep in mind that we are discussing experimental result

The new series $\mathcal{L}_2$, as described, is born (as r increases) as a stable periodic orbit which at birth is marginally stable with a stability matrix with a single eigenvalue 1 corresponding to a creation of a pair of periodic orbits, one stable and one unstable: following the stable orbit one observes (experimentally) that it loses stability for some $r = r_1^2$, by a period doubling to which an infinite sequence of period doubling bifurcations follow, in the sense that one is able to observe many of them and the only obstacle to observe more seems to be the numerical precision needed.

At the end of the evolution of this family which is at a a value r_∞^2 where the successive thresholds r_j^2 accumulate as $j \to \infty$ (at a fast rate numerically compatible with Feigenbaum's constant, 4.68...), one observes the birth of a strange attracting set, which is however followed at larger r by a periodic stable orbit, see Fig. (4.4.4).

The latter, if followed *backwards decreasing* r, loses stability at $r = r_c^3$ because of the passage of two complex eigenvalues though the unit circle *without* being followed by the birth of an invariant torus of dimension 2 (because one observes instead a chaotic motion): this means that the bifurcation (at decreasing r) is an "inverse" bifurcation. And, indeed, at $r = r_c^3$ the stable periodic orbit that exists for $r > r_c^3$ has, in fact, been observed to collide with another unstable periodic orbit which also exists for $r > r_c^3$ (at least for $r - r_c^3$ small) and the two annihilate leaving (for $r < r_c^3$) a strange attracting set with chaotic motion on it. The unstable periodic orbit has not beeen represented in the picture.

The set of lines connected with the one representing the laminar motion at small Reynolds number is the *main series* of bifurcations and the others are the *secondary series* (only one of them in the case of Fig. (4.4.4), if one does not count the other 3 existing because of the symmetry (4.1.29)).

The just described phenomenology is, however, as it often happens with experiments (real or numerical) valid only in a first approximation. Refining the measurements one can indeed expect to be able to see further details: for instance the interval $[r_\infty^2, r_c^3]$, which appears at first to correspond to motions regulated by "just" 4 symmetric strange attracting sets as shown in Fig. (4.4.4), reveals itself endowed with much higher complexity.

For instance one can "resolve", via experimentally more delicate and difficult observations and measurements, *c.f.r.* [Fr83], some features of the motions that take place for r in this interval as shown in Fig. (4.4.5) where the "*window*" (describing the birth of the periodic orbit and the successive Feigenbaum bifurcations) for $r > r_F$ has a really small width of the order of 10^{-2}! (not on scale in Fig. (4.4.5)) and terminates in the point r_F^∞ where the period doubling bifurcations accumulate: to it again a chaotic motion follows.

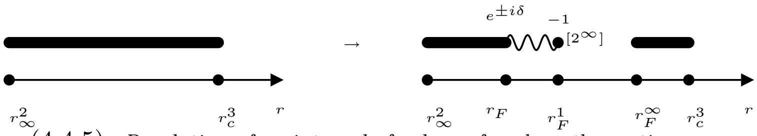

Fig. (4.4.5): *Resolution of an interval of values of r where the motion appears, at first, always chaotic. At a more accurate analysis the interval $[\overline{r}^2_\infty, r^3_c]$ splits into regions where it is still chaotic and into tiny regions (not on scale in the drawing) where stable periodic orbits exist and evolve through a rapid history of period doubling bifurcations.*

(C) The Lorenz model table.

The dynamical table for the Lorenz equation, built on the basis of the results discussed in the problems of §4.1, is simpler and shows how the laminar motion evolves, as r increases, by first losing symmetry at $r = 1$ (see (4.1.21)) into two new stable fixed points mapped into each other by the 2–elements symmetry group. Each of them loses stability, for some $r = r_c$, with two conjugate eigenvalues crossing the imaginary axis. However the bifurcation is not "direct", *i.e.* there will *not* be generation of a stable periodic orbit from each of the two fixed points, (*c.f.r.* Fig. (4.4.6)) at $r = r_c$; rather the points will lose stability because they "collide" with an unstable periodic orbit (represented by the dashed wavy line in the figure). After the stability loss the fixed points remain unstable and, as Fig. (4.4.6) indicates, no periodic stable motion is generated but we have directly a chaotic motion (this is a direct transition to chaos, according to the fourth scenario in §4.3).

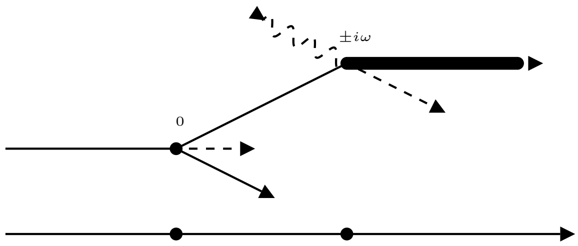

Fig. (4.4.6) *Dynamical table for the Lorenz model representing a symmetry breaking followed by a collision, between the resulting stable time independent point and an unstable periodic orbit, which engenders chaos.*

A more attentive analysis shows that also in this model there are intervals of r in (r_c, ∞) in which there is an attracting set consisting of a periodic (stable) motion which as r varies evolves with a sequence of period doubling bifurcations as in the case illustrated in Fig. (4.4.5), *c.f.r.* [Fr80].

(D) Remarks.

From bifurcation theory we have seen that certain bifurcations are not generic: hence usually we must imagine, when trying to build a dynamical table for a particular system, only tables in which generic bifurcations appear. This is for instance the case of the previously examined realistic dynamic tables.

In this way we can only build a relatively small number of combinations at each bifurcation. Examples of nongeneric (hence forbidden in the tables) bifurcations can be taken from the analysis of §4.2 and, obviously, there are many more, among which the bifurcation in the Fig. (4.4.1) in *absence* of an accompanying breaking of some symmetry. Some examples are provided in Fig. (4.4.7).

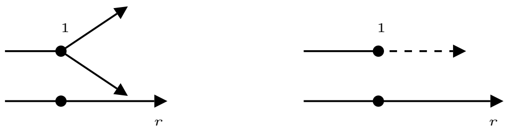

Fig. (4.4.7) *Forbidden non generic bifurcations.*

In principle we should expect that any dynamical table which only contains generic bifurcations could be realized by the phenomenology of some model: and in fact their variety is very large. One can simply look at [Fr83], [Fr80], [FT79] to be convinced. Examining the dynamical tables presented in such papers one sees by inspection the involved phenomenology of the sequences of bifurcations, with various hysteresis phenomena (*i.e.* coexistence of various attracting sets in correspondence to a fixed value of r, as in the case of Fig (4.4.4) in the interval between r_c^2 and r_∞^1).

In several cases *resolutions* of "apparently simple" attracting sets have been observed which, as in the example of Fig. (4.4.5), reveal a far more complex structure than that of Fig (4.4.4). But the greater complexity manifests itself over much smaller intervals of r and it can be easily "missed" if the observations are not very precise or not systematic enough.

Dynamical tables provide us with a method to rapidly visualize and organize *without boring and monotonous descriptions* the general aspects of the phenomenology of a given experiment.[2]

One should refrain from thinking that compiling dynamical tables is simple *routine*: they usually require and describe accurate and patient analysis. Particularly when hysteresis phenomena are relevant and make it difficult to follow a given attracting set as the Reynolds number varies; or when one wishes to follow the history of an attracting set that after a bifurcation survives but becomes unstable. The latter feat, understandably quite difficult, is at times necessary because the same attractive set may lose stability in correspondence of a certain Reynolds number to reacquire it again in correspondence of another value (typically "colliding" with another attracting set and "ceding" to it the instability while "taking" the stability, if usage of a pictorial language is allowed here).

[2] We shall no longer distinguish, unless necessary, between numerical experiments and experiments performed on real fluids: because we take for granted that both are "real" and with equal "dignity".

(E) Another example: the dynamic table of the NS_7 model.

The NS_7 model of §4.1 has been studied in detail in [FT79] where measurements sufficient to establish the following dynamical table have been performed.

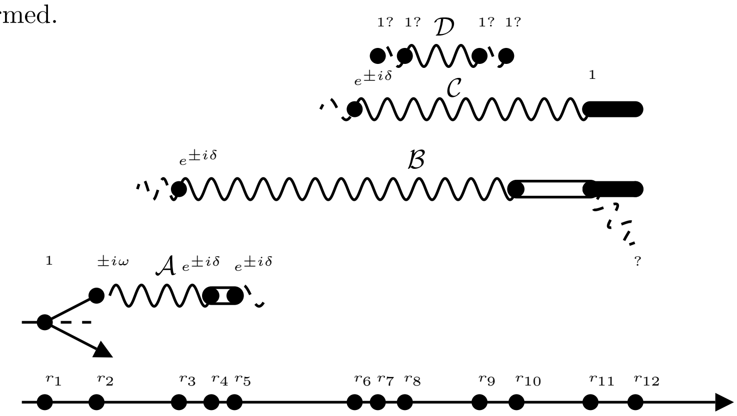

Fig. (4.4.8) *A NS_7 equations dynamical table. For $r < r_1$, "last bifurcation breaking the symmetry" in (4.1.29), the series is the same as that preceding r_c^1 in Fig. (4.4.4). Note the hysteresis phenomena between $\mathcal{B}, \mathcal{C}, \mathcal{D}$ or $\mathcal{A}, \mathcal{B}$ or $\mathcal{B}, \mathcal{C}$.*

The strange attracting sets that are born at large r are not different in the sense that *apparently* the bifurcation series $\mathcal{C}$ and $\mathcal{B}$ both end in the same attracting set for $r > r_{11}$ (even though in the table it appears drawn as different) modulo the symmetries of the model.

We see in this table several hysteresis phenomena. In this case too we did not represent the trivial hysteresis: one should always take into account that if a differential equation has certain symmetries then the attracting sets either are symmetric or the sets that are obtained by applying the symmetry transformations to one of them generate others (that are "copies") with the same properties. This gives rise to hysteresis phenomena that are "trivial" (in the sense of the comments to Fig. (4.4.4)) and are not marked because we have not drawn in the table the attracting or repelling sets that can be obtained from the drawn ones by applying the symmetries of the model, *c.f.r.* (4.1.29) (with the exception of the case of the bifurcation at r_1).

Not all bifurcations in the table of Fig. (4.4.8) are clearly understood: in particular those of the line $\mathcal{D}$. The bifurcations on which hangs the type label "?" have not been studied (because there remains always something to do, usually for good reasons).

Since the details that can be seen in dynamic tables depend on the precision with which measurements are performed (or, better, can be performed) it is expected that by performing more accurate measurements one can find some change of the tables in the sense that more details may appear, as already seen in the case of the NS_5 model.

(F) Table of the 2–dimensional Navier–Stokes equation at small Reynolds number. Navier–Stokes)

Finally we conclude by considering the dynamic table of the 2–dimensional Navier–Stokes equation with a large regularization parameter, *c.f.r.* Fig. (4.4.9). This delicate study indicates that the phenomenology that occurs at small values of the Reynolds number *appears to stabilize* as a function of the number of Fourier modes used in the regularization.

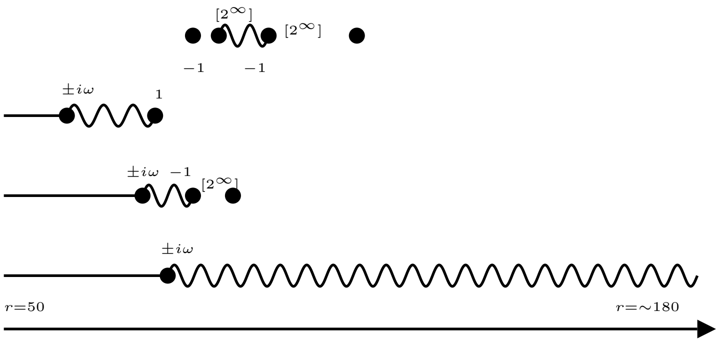

Fig. (4.4.9) *The stabilized phenomenology of the NS equations up to Reynolds number* ~ 200 *and with truncations at order* $M = 26$, 37, 50, 64 *(results, however, do not depend on* M*). Note the hysteresis phenomena.*

Up to Reynolds numbers of the order of 200 it seems that at least the laminar motions become independent from the value of the ultraviolet cut–off M that defines the regularization by selecting the Fourier modes such that $|\underline{k}| = \max(|k_1|, |k_2|) \le M$, *c.f.r.* §3.1, §3.2.

We consider the dynamical tables of these regularizations of the equations, that we shall denote as $NS^{(M)}$, with $M = 26, 37, 50$, 64, taken from [FGN88], drawn for r between $r = 50$ to about $r = 180$. The force acts only on one mode (and its opposite): $\underline{k} = (2, -1)$ and it is real. We only consider solutions in which the various Fourier components $\gamma_{\underline{k}}$ are either real or purely imaginary, extending the remark seen in §4.1, *c.f.r.* (4.1.27).

Note that there is a very good reason for the choice of the mode on which the forcing acts: the simplest choice, *i.e.* the mode with minimum length $\underline{k}_0 = (0, \pm 1), (\pm 1, 0)$, is not sufficient to create interesting phenomenology: see problem [4.1.13]. However any other choice of the mode seems sufficient to generate interesting phenomenologies. How much the results depend on which mode is actually chosen for the forcing has not been studied: however it seems that the choice does not affect "substantially" the phenomenology and it would be interesting to clarify this point. Also the restriction that the components $\gamma_{\underline{k}}$ are real or imaginary (rather than complex) is an important issue partially studied in the literature and deserving further investigation.

The tables coincide from $r = 0$ up to about $r = 180$, hence we draw only the table relative to $M = 64$ (which is a model with 5200 equations or so). The above Fig. (4.4.9) illustrates an interval of r going from 50 to ~ 180.

For larger values important variations of the form of the table as function of the cut–off M are still observed, particularly for $r > 200$. In all cases the motions eventually generate a quasi periodic bidimensional motion; however this happens at *cut–off dependent* values of r, between ~ 250 and ~ 600.

By the discussion in §4.3, (F), the "quasi periodic" motion should appear in a sequence of phase locked periodic motions: in the experiments the motion always appears quasi periodic which means that the steps of the "devil staircase" formed by the graph of the rotation number as a function of r are so short that they are not observable within the precision of the experiments.

This shows how delicate can be the process of (believed) "convergence" as $M \to \infty$ of the attracting sets, and of the corresponding phenomenologies, of the equations $NS^{(M)}$ to those of the full Navier–Stokes equation. And strictly speaking we only know that such convergence really takes place through (scanty) empirical evidence: the theory developed in Chap. 3, of the 2–dimensional Navier–Stokes equations (with periodic boundary conditions), although rather complete from the viewpoints of existence, uniqueness and regularity, *does not allow us to obtain asymptotic results as* $t \to \infty$ and it is therefore of little use for the questions which interested us here.

Problems.

[4.4.1]: On the basis of the phenomenology described in [FZ85], [FZ92] draw (from their original data) the dynamic tables describing the asymptotic motions of the systems considered.

[4.4.2]: Draw the dynamic tables describing the asymptotic motions studied in [Ri82] (from his original data).

Bibliography: The experimental results of simulations, exposed above, are taken mainly from the papers: [FT79], [FGN88], [Fr80], [Fr83], [Ri82], [Ga83]. The dynamical tables have been introduced and widely used in these references. For experiments on real fluids see [FSG79], [SG78].