CHAPTER II

Empirical algorithms.

§2.1 Incompressible Euler and Navier–Stokes fluidodynamics. First empirical solutions algorithms. Auxiliary friction and heat equation comparison methods.

(A) Euler equation

Imagine an incompressible Euler fluid in a fixed volume Ω with a (C^∞)–regular boundary. The equations describing it are

$$\begin{array}{lll} (1) & \underline{\partial}\cdot\underline{u}=0, & \text{in } \Omega\\ (2) & \partial_t\underline{u}+\underset{\sim}{\underline{u}}\cdot\underset{\sim}{\underline{\partial}}\underline{u}=-\rho^{-1}\underline{\partial}p-\underline{g}, & \text{in } \Omega\\ (3) & \underline{u}\cdot\underline{n}=0, & \text{in } \partial\Omega\\ (4) & \underline{u}(\xi,0)\equiv\underline{u}_0(\xi), & t=0 \end{array} \qquad (2.1.1)$$

where $\underline{n}$ denotes the external normal to $\partial\Omega$ and the "*boundary condition*" (3) expresses the condition that the fluid "glides" (without friction) on the boundary of Ω.

Given a t–independent external field $\underline{g}(\xi)\in C^\infty(\Omega)$ the problem of fluidodynamics with fixed walls is:

(1) Given $\underline{u}_0\in C^\infty(\Omega)$ with $\underline{\partial}\cdot\underline{u}_0=0$, $\underline{u}_0\cdot\underline{n}=0$ on $\partial\Omega$, is there a solution $t\to\underline{u}(\xi,t)$, $p(\xi,t)$ of (2.1.1) valid for small enough times and with $\underline{u}$ and p of class C^∞ or, at least, with continuous derivatives?
(2) Are there solutions global in time?
(3) Are such solutions unique?
(4) Under which assumptions on $\underline{g}$ and $\underline{u}_0$ can one find uniform estimates as a function of time on the derivatives of $\underline{u}$ and p?

Obviously it would be desirable to have a positive answer to (2), (3) while (4) would have importance in view of a consistency check with the physics supposedly described by the equations, which we *stress that have been deduced under the hypothesis that velocity gradients stay small.*

In this section we shall look for a heuristic and constructive algorithm for the existence of the solutions. Given $\underline{u}_0(\xi)$ we can imagine computing $\underline{u}(\xi,t)$, for t very small, as

$$\underline{u}(\xi,t)=\underline{u}_0(\xi)+\underline{\dot u}(\xi,0)\,t \tag{2.1.2}$$

However to compute $\underline{\dot u}(\xi,0)$ we need to know p. The pressure p can be computed from (2) in (2.1.1); indeed the divergence of (2), and equation (3) in (2.1.1), give

$$\begin{aligned}&-\rho\,\underline{\partial}\cdot(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u})+\rho\,\underline{\partial}\cdot\underline{g}=\Delta p && \text{in } \Omega\\ &\partial_n p=-\rho\,[(\underset{\sim}{u}\cdot\underset{\sim}{\partial})\,\underline{u}]\cdot\underline{n}+\rho\,\underline{g}\cdot\underline{n} && \text{in } \partial\Omega\end{aligned} \tag{2.1.3}$$

which shows that p_0 is determined up to a constant; note that the inhomogeneous Neumann problem in (2.1.3) satisfies automatically the well known compatibility condition imposing that the integral on the boundary of $\partial_n p$ be equal to the volume integral of the datum of the problem (*i.e.* the l.h.s. of the first of (2.1.3)), because of the integration theorem of Stokes.

Inserting the function p_0 so computed into (2) in (2.1.1) at $t=0$, we see that we can compute $\underline{\dot u}(\xi,0)$ and that, by construction, $\underline{\partial}\cdot\underline{\dot u}(\xi,0)=0$. Therefore it makes sense to define (2.1.2) and, in fact, we just found an *approximation algorithm* which could even be of interest in numerical simulations. We see also which is the mechanism permitting us to eliminate the pressure by expressing it as a function of the velocity field.

We set, given $t_0>0$, for $k\ge 1$:

$$\begin{aligned}&\underline{u}(\xi,kt_0)=\underline{u}(\xi,(k-1)t_0)+t_0\,\underline{\dot u}(\xi,(k-1)t_0)=\underline{u}_k(\xi)\\ &\underline{u}(\xi,0)\equiv\underline{u}_0\end{aligned} \tag{2.1.4}$$

where

$$\begin{aligned}&\underline{\dot u}(\xi,(k-1)t_0)=-\frac{1}{\rho}\underline{\partial}p_{k-1}(\xi,(k-1)t_0)-\\ &-\underset{\sim}{u}(\xi,(k-1)t_0)\cdot\underset{\sim}{\partial}\,\underline{u}(\xi,(k-1)t_0)+\underline{g}\\ &p_{k-1}=\ \text{solution of}\ \begin{cases}\Delta p_{k-1}=-\rho\underline{\partial}\cdot(\underset{\sim}{u}_{k-1}\cdot\underset{\sim}{\partial}\underline{u}_{k-1})+\underline{\partial}\cdot\underline{g} & \Omega\\ \partial_n p_{k-1}=-\rho(\underset{\sim}{u}_{k-1}\cdot\underset{\sim}{\partial}\underline{u}_{k-1})\underline{n}+\underline{g}\cdot\underline{n} & \partial\Omega\end{cases}\end{aligned} \tag{2.1.5}$$

and the question is whether the limit

$$\underline{u}(\xi,t)=\lim_{\substack{k\to\infty\\ t_0\to0,\,kt_0=t}}\underline{u}_k(\xi) \tag{2.1.6}$$

exists and gives a solution of (2.1.1).

For the time being we shall consider the problem of the existence of the limits a "technical" one and, therefore, we can consider that the equation (2.1.1) is "formally solved".

The (2.1.6) with k finite and large gives, at least in principle (because the calculation of $\underline{u}_k$ is obviously very difficult also from the point of view of numerical solutions) an approximation algorithm for an incompressible motion that generates evolutions that, on one hand, can be interesting by themselves even not considering applications to fluids, and that, on the other hand, can be considered as models for a real fluid evolution.

The approximations should become better as the "discretization time" t_0 approaches 0.

Remark: the algorithm defined by (2.1.4),(2.1.5),(2.1.6) is, at times, called the *Euler algorithm*: more generally this algorithm provides solutions of a differential equation $\dot{\underline{x}} = \underline{f}(\underline{x})$, with $\underline{x}(0) = \underline{x}_0$ that, at time kt_0, is given by the recursive relation $\underline{x}_k = \underline{x}_{k-1} + t_0\underline{f}(\underline{x}_{k-1})$, $k \ge 1$.

(B) Navier–Stokes–Euler equation.

We look if, at least in a heuristic sense like the one analyzed in (A), a similar treatment of the incompressible Navier–Stokes equations is possible. The equations are

$$
\begin{array}{lll}
(1) & \underline{\partial} \cdot \underline{u} = 0 & \text{in } \Omega \\
(2) & \partial_t \underline{u} + \underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}} \underline{u} = -\frac{1}{\rho} \underline{\partial} p + \underline{g} + \nu \Delta \underline{u} & \text{in } \Omega \\
(3) & \underline{u} \cdot \underline{n} = 0, & \text{in } \partial\Omega \\
(4) & \underline{u}(\xi, 0) = \underline{u}_0(\xi) & t = 0
\end{array}
\qquad (2.1.7)
$$

provided one can assume that friction between fluid and boundaries is negligible and, therefore, *the fluid glides without friction along the walls* of the container.

In this case the equations are discussed exactly as in the Euler case (with obvious modifications) and we find, therefore, an approximation algorithm similar to (2.1.4), (2.1.5): in (2.1.5) one has to add on the r.h.s. of the first equation the term $\nu\,\Delta\underline{u}_{k-1}$ while the equation in (2.1.5) for $\Delta\,p_{k-1}$ is unchanged (because $\nu\,\Delta\underline{u}_{k-1}$ has zero divergence) and the equation for $\partial_n p_{k-1}$ is modified by adding $\nu\,\underline{n} \cdot \Delta\underline{u}_{k-1}$.

Therefore there are no really new problems.

(C) Navier–Stokes equations and algorithmic difficulties.

In the applications friction against the walls *is by no means negligible*, to the extent that the physically significant boundary condition is $\underline{u} = 0$ rather than $\underline{u} \cdot \underline{n} = 0$.

We then call the (2.1.7) NSE–equations for the fluid in Ω while we shall call NS–equations for a fluid in Ω the equation

$$\begin{array}{lll}(1) & \underline{\partial}\cdot\underline{u}=0 & \text{in } \Omega\\ (2) & \partial_t\,\underline{u}+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}=-\frac{1}{\rho}\underline{\partial}\,p+\underline{g}+\nu\,\Delta\underline{u} & \text{in } \Omega\\ (3) & \underline{u}=0 & \text{in } \partial\Omega\\ (4) & \underline{u}(\xi,0)=\underline{u}_0(\xi) & t=0\end{array}\qquad(2.1.8)$$

The analysis of (2.1.8) is, however, radically different: indeed if we attempt at determining p at time $t=0$ we find

$$\begin{array}{ll}\Delta p=-\rho\,\underline{\partial}\cdot(\underset{\sim}{u}_0\cdot\underset{\sim}{\partial}\underline{u}_0)+\rho\underline{\partial}\cdot\underline{g} & \text{in } \Omega\\ \underline{\partial}\,p=-\rho\underline{g}+\rho\nu\Delta\underline{u}_0 & \text{in } \partial\Omega\end{array}\qquad(2.1.9)$$

which in general *will not admit a solution* because it is not necessarily true that the tangential derivatives of p, which obviously are already determined by just the normal derivative (via the solution of the corresponding inhomogeneous Neumann problem), are compatible on $\partial\Omega$ with (2.1.9).

One could, for a moment, hope that the fact that $\underline{u}_0$ is not arbitrary, being with zero divergence, implies *ipso facto* compatibility: but it is easy to convince oneself that this is not the case, see (E) and problems [2.1.8], [2.1.9] below.

This is a serious difficulty showing that we must necessarily expect that on the boundary of Ω *interesting and difficult phenomena must take place.*

The first effect of the difficulty is that, on the basis of what said until now, it does not yet allow us to give a prescription for a numerical solution of the NS–equation with *viscous adherence* to the boundary.

And one can legitimately suspect that (2.1.8) is not a well posed problem: it will certainly be necessary to interpret suitably (2.1.8) since if we interpret it in a strict sense, in which all functions involved are $C^\infty(\Omega)$, it simply looks inconsistent because it does not allow us to compute $\dot{\underline{u}}$, not even at $t=0$ (because it is defined by an insoluble equation).

A way to proceed to develop an algorithm which, at least on a heuristic basis, permits us to compute a solution to (2.1.8), giving at the same time a suitable interpretation to it and bypassing the problem just met, is the following.

Along the normal direction to $\partial\Omega$, we imagine extending the volume Ω by a length ε and we denote Ω_ε this extended volume; we suppose that the fluid occupies it and there it verifies there the equation

$$\begin{array}{ll}\underline{\partial}\cdot\underline{u}=0 & \text{in } \Omega_\varepsilon\\ \partial_t\underline{u}+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{u}=-\frac{1}{\rho}\underline{\partial}p+\underline{g}-\nu\Delta\underline{u}-\sigma_\varepsilon(\xi)\underline{u} & \text{in } \Omega_\varepsilon\\ \underline{u}\cdot\underline{n}=0 & \text{in } \partial\Omega_\varepsilon\\ \underline{u}(\xi,0)=\underline{u}_0 & \text{in } \Omega_\varepsilon\end{array}\qquad(2.1.10)$$

where $\underline{u}_0$ is extended arbitrarily between Ω and Ω_ε, assuming that it is extended, together with a prefixed number p of derivatives ($p \ge 2$), continuously and assuming also that the extension vanishes near the new boundary $\partial\Omega_\varepsilon$. The function $\sigma_\varepsilon(\xi)$ vanishes inside Ω and increases rapidly from 0 to a value $\overline{\sigma}(\varepsilon) \stackrel{def}{=} \overline{\sigma}_\varepsilon$ with (large) average slope $\overline{\sigma}(\varepsilon)/\varepsilon$. This additive term has the interpretation of "*friction*" that slows down the fluid in the corridor between $\partial\Omega$ and $\partial\Omega_\varepsilon$.

The level lines of $\sigma_\varepsilon(\xi)$ will be, by their definition, parallel to $\partial\Omega$ and $\partial\Omega_\varepsilon$, the gradient of σ_ε, denoted $\sigma'_\varepsilon \underline{n}$ with σ'_ε such that $|\underline{n}| = 1$, is a vector field extending, to the layer between the two boundaries, the fields formed by their normals (draw a picture).

One can expect that in the limit $\varepsilon \to 0$ the solution to (2.1.10), for which we can apply the constructive algorithm of the preceding case (B) because on $\partial\Omega_\varepsilon$ the condition $\underline{u} \cdot \underline{n} = 0$ holds, will be such that the limit

$$\lim_{\varepsilon\to 0} \underline{u}_\varepsilon(\xi, t) = \underline{u}(\xi, t) \qquad \xi \in \Omega \tag{2.1.11}$$

exists and $\underline{u}(\xi, t)$ solves (2.1.8), *under the further condition that* $\overline{\sigma}(\varepsilon) \to \infty$*, possibly fast enough, for* $\varepsilon \to 0$.

Remark: If Ω has special forms, *e.g.* it is a cube, then it might be convenient to take Ω_ε to be a torus. We shall do so below in the case of a similar simpler, one dimensional, problem.

The algorithm (2.1.10), that we shall call "*auxiliary friction algorithm*" or "*friction method*", will encounter considerable numerical difficulties because of the term $\sigma_\varepsilon \underline{u}$ and of the divergence towards $+\infty$ (for $\varepsilon \to 0$) of $\bar{\sigma}_\varepsilon$; in fact, for $\varepsilon \to 0$, $\dot{u}_t$ computed from (2.1.10) would tend to ∞, unless one could guarantee *a priori* that $u_\varepsilon \to 0$ when σ_ε becomes large: and it does so quickly enough to control the product $\sigma_\varepsilon \underline{u}_\varepsilon$ (which appears quite difficult a task from a mathematical viewpoint).

(D) Heat equation.

To understand better what is happening let us examine a simpler model case. Consider the heat equation

$$\begin{aligned} \partial_t T &= c\,\partial_x^2 T, \qquad\qquad x \in [-\pi, \pi] \\ T(-\pi) &= T(\pi) = 0 \\ T(x, 0) &= T_0(x) \end{aligned} \tag{2.1.12}$$

If we attempted to find a solution to this equation with the preceding method we should set

$$T_k = T_{k-1} + t_0 \dot{T}_{k-1} \equiv T_{k-1} + t_0 c T''_{k-1} \quad k \ge 1 \tag{2.1.13}$$

but it is clear that, unless $T''(\pm\pi) = 0$, this will already be impossible because T_1 will not verify the boundary conditions.

And even if $T_0''(\pm\pi) = 0$ we see that $T_1''(\pm\pi) \neq 0$ unless $T_0''''(\pm\pi) = 0$. Hence if $T_0(x)$ does not have *all* even derivatives vanishing in the neighborhood of $\pm\pi$ the algorithm will not work.

The "*auxiliary friction method*" of (C) above, setting $c = 1$, would lead to equations

$$\begin{aligned} &\dot T = T'' - \sigma_\varepsilon T \\ &T(-\pi-\varepsilon) = T(\pi+\varepsilon),\, T'(-\pi-\varepsilon) = T'(\pi+\varepsilon) \\ &T(x,0) = T_0(x) \end{aligned} \tag{2.1.14}$$

having extended T_0 arbitrarily out of $[-\pi,\pi]$ to a periodic function (see the comment following (2.1.11)) in $[-\pi-\varepsilon,\pi+\varepsilon]$, and imagining to identify the points $\pi+\varepsilon$ and $-\pi-\varepsilon$, see Fig. (2.1.1) below.

If we choose σ_ε equal to a constant $\bar\sigma_\varepsilon$ for $x \in [\pi + \frac{2}{3}\varepsilon, \pi+\varepsilon]$ and to 0 for $x \in [\pi, \pi + \frac{\varepsilon}{3}]$ and $\sigma_\varepsilon(x) = \sigma_\varepsilon(-x)$, then we can show that the iterative method works, in principle, at least if the initial datum T_0 is regular enough (and if $\bar\sigma_\varepsilon \to \infty$ fast enough as $\varepsilon \to 0$).

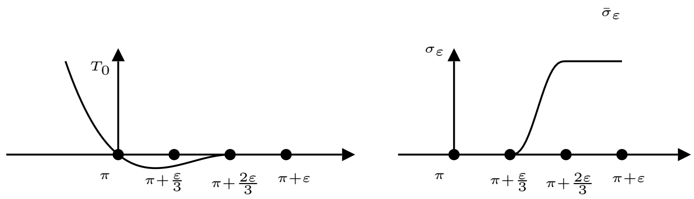

Fig. (2.1.1): Extension of the initial datum $T_0(x)$ to the right of π; and graph of the auxiliary function σ_ε between π and $\pi+\varepsilon$.

This time the problem is so simple that one can show that, *if* $T_0(x)$ *is* C^∞*, and analytic in the interior of* $[-\pi,\pi]$, then the algorithm (2.1.13) converges to the solution of the heat equation with the correct boundary conditions (2.1.12). But the convergence of the function $T_{\varepsilon,k}(\xi,t)$, provided by the algorithm at the k–th step, to a limit $T(\xi,t)$

$$T(\xi,t) = \lim_{\varepsilon\to 0} \lim_{\substack{k\to\infty \\ kt_0=t}} T_{\varepsilon,k}(\xi) \tag{2.1.15}$$

is delicate and, as a numerical algorithm, it is not very good because $\dot T$ risks to become very large as $\varepsilon \to 0$, because $\sigma_\varepsilon \to \infty$ badly affects the term $\sigma_\varepsilon T$ in (2.1.14).

Furthermore if T_0 is not analytic but, for instance, "only" C^∞, the algorithm may converge but, in general, it *will not converge* to the usual solution of the heat equation. See below and problems [2.1.1]%[2.1.7].

An apparently better algorithm consists in transforming the equation into

$$T_\varepsilon(\xi,t) = e^{-\sigma_\varepsilon t} T_0(\xi) + \int_0^t e^{-\sigma_\varepsilon(\xi)(t-\tau)} T_\varepsilon''(\xi,\tau)\, d\tau \tag{2.1.16}$$

which we can try to solve iteratively by setting

$$T_{\varepsilon k}(\xi)=e^{-\sigma_\varepsilon t_0 k}T_0(\xi)+t_0\sum_{h=0}^{k-1}e^{-\sigma_\varepsilon(\xi)(k-h)t_0}T''_{\varepsilon\,h}(\xi),\qquad k\ge 1 \tag{2.1.17}$$

This yields a numerical algorithm in which σ_ε only appears in $e^{-\sigma_\varepsilon t}$, which can be computed without involving large quantities (even when $\sigma_\varepsilon t$ is large). Provided obviously $T''_\varepsilon(\xi)$ does not become too large, which we do not expect, not at least if compared with the corresponding $e^{-\sigma_\varepsilon(\xi)t_0}$ (the reason is that in the well known solution of the heat equation T'' has, usually, a discontinuity at the boundary but it is not infinite). See problems.

(E) An empirical algorithm for NS:

The above discussion suggests an analogous approach for (2.1.8) and (2.1.9) and leads to the following algorithm for (2.1.10). One writes (2.1.10) as

$$\begin{aligned}&\underline{u}\cdot\underline{n}=0 \text{ on } \partial\Omega,\qquad \underline{\partial}\cdot\underline{u}=0,\ \text{in }\Omega,\qquad \underline{u}(\xi,t)= \qquad (2.1.18)\\ &=\underline{u}_0(\xi)e^{-\sigma_\varepsilon(\xi)t}+\int_0^t e^{-\sigma_\varepsilon(\xi)(t-\tau)}(-\underset{\sim}{u}_\tau\cdot\underset{\sim}{\partial}\underline{u}_\tau-\frac{1}{\rho}\underline{\partial}p_\tau-\underline{g}-\nu\Delta\underline{u}_r)\,d\tau\end{aligned}$$

where u_τ, p_τ denote $u(\xi,\tau), p(\xi.\tau)$ and one sets up the approximation algorithm

$$\begin{aligned}&\underline{u}_k(\xi)=\underline{u}_0(\xi)e^{-\sigma_\varepsilon(\xi)t}+\sum_{h=0}^{k-1}t_0e^{-\sigma_\varepsilon(\xi)(k-h)t_0}(-\underset{\sim}{u}_h\cdot\underset{\sim}{\partial}\underline{u}_h-\\ &-\frac{1}{\rho}\underline{\partial}p_h+\underline{g}+\nu\Delta\underline{u}_h)\quad k\ge 1 \qquad (2.1.19)\\ &\Delta p_k=-\rho\,\underline{\partial}(\underset{\sim}{u}_k\cdot\underset{\sim}{\partial}\underline{u}_k)+\rho\,\underline{\partial}\cdot\underline{g}-\underline{u}_k\cdot\underline{\partial}\sigma_\varepsilon \qquad \text{in }\Omega_\varepsilon\\ &\partial_n p_k=-\rho(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{u}_k)\cdot\underline{n}+\rho\,\underline{g}\cdot\underline{n}+\rho\,\nu\Delta\underline{u}_k\cdot\underline{n} \qquad \text{in }\partial\Omega_\varepsilon\end{aligned}$$

where $\underline{\partial}\sigma_\varepsilon=\sigma'_\varepsilon\underline{n}$ and σ'_ε has support near $\partial\Omega_\varepsilon$. This algorithm does not encounter obvious problems as $\sigma_\varepsilon\to\infty$, provided $\underline{u}_k\cdot\underline{n}$ is so small that $\sigma'_\varepsilon\underline{u}_k\cdot\underline{n}$ remains bounded as $\varepsilon\to 0$ (see the last term in the second equation). Since we expect that $\underline{u}_k\cdot\underline{n}$ be of order $O(\varepsilon)$, this seems reasonable and we should have

$$\underline{u}(\xi,t)=\lim_{\varepsilon\to 0}\lim_{\substack{k\to\infty\\ kt_0=t}}\underline{u}_{k,\varepsilon}(\xi) \tag{2.1.20}$$

However, unlike the heat equation case, the algorithm (2.1.19) does not eliminate completely the problem, certainly relevant for numerical calculations, that is due to the presence of quantities like $\underline{u}_k\cdot\underline{\partial}\,\sigma_\varepsilon$ which are products of large quantities ($\underline{\partial}\sigma_\varepsilon$) times quantities ($\underline{u}_k$) that we expect to be small (but which we do not know *a priori* that they are really such).

Hence this algorithm has only a theoretical character which, rather than solving a problem, is well suited to illustrate some of its difficulties. We succeed in giving a meaning to a heuristic method of construction of solutions, but the method remains only a matter of principle, conditioned to the solution of an elliptic equation which may lead to a numerical stability problems (at least).

We could ask where did go the conceptual compatibility problem, at the origin of our worries. Obviously it is still around: indeed one should expect that the "exact" solutions of (2.1.18) or of (2.1.10) for $t > 0$ small do not verify any more the boundary conditions $\underline{u} = \underline{0}$ on $\partial\Omega$ (that now is an internal surface in the domain Ω_ε where the approximating solution is studied). But we can hope that the violation of the property of vanishing on the boundary $\partial\Omega$ should rapidly become, as time increases, very small for $t > 0$ and tend to vanish for every prefixed value of $t > 0$ as ε tends to zero, at least if σ_ε becomes vertical enough near $\partial\Omega$ and large enough (for $\varepsilon \to 0$).

The impossibility of satisfying the boundary condition at $t = 0$ (on $\partial\Omega$) implies that $\underline{\dot{u}}_\varepsilon$ will be not zero, and not even small, at $t = 0$ in some point on the boundary $\partial\Omega$ but it will rapidly become very small, the earlier the closest to 0 will ε be: and in the limit $\varepsilon \to 0$ one should attain a limit $\underline{u}$ which at time $t = 0$ *does not* satisfy the NS–equation, *but that verifies it at all later times* $t > 0$.

In this way the friction model for the boundary condition clarifies how it could be that the equation cannot be solved at $t = 0$ but it is solved at all $t > 0$ and which is a physical mechanism that produces such a result.

Should one worry that the solution found (*if* found) had nothing to do with the initial datum $\underline{u}_0$ then it should be noted that, although the initial datum cannot be obtained as a limit from $t > 0$, in the sense that not all derivatives of $\underline{u}$, for $t > 0$ (time derivatives included) tend to the corresponding ones of the initial datum[1] nevertheless motion is generated from the initial datum hence we can expect that the $\underline{u}$ tends to $\underline{u}_0$ in some sense, *e.g.* in $L_2(\Omega)$ or in some other sense (for instance pointwise in the internal points of Ω).

The important point to retain is that the physical meaning of the singularity at $t = 0$ has been understood, at least as a proposal to be checked, through a concrete model together with a description of the mechanism of transition from $t = 0$, where the equation cannot be solved in general, to $t > 0$ where, instead, its solubility does not meet any *a priori* difficulty of principle.

The following questions become natural at this point

[1] We have already seen that this cannot happen for the time derivative which, for the initial datum, is not even defined.

Question (Q1): Do the friction equations (2.1.10), once exactly solved for $\varepsilon > 0$*, really have the property of converging for* $\varepsilon \to 0$ *to a solution of the (2.1.8) (boundary conditions included) at least for* $t > 0$*.*

Question (Q2): Does the algorithm (2.1.19) really converge for $k \to \infty$*,* $t_0 \to 0$*,* $kt_0 \to t$*, to a solution of (2.1.18).*

and we attempt at understanding them by referring to simpler problems in which they already occurr.

(F) Precision of the same algorithms applied to the heat equation, (Q1).

Questions (Q1) and (Q2) will not be analyzed for the NS–equation because of the difficulties that are involved (which, to date, are not yet solved in the case $d = 3$). We can, however, analyze them in the far easier case of the heat equation and the study is very instructive as an illustration of mechanisms and difficulties that one can expect to find again in the case of more ambitious theories like that of the NS–equation. Let us, therefore, look at the problem (Q1) posed in *(E)* in the case of the equations (2.1.12), (2.1.14).

We shall however suppose, for simplicity, that σ_ε in (2.1.14), (2.1.16) is given as a discontinuous function represented in Fig. (2.1.2) by the solid lines rather than by the dashed lines.

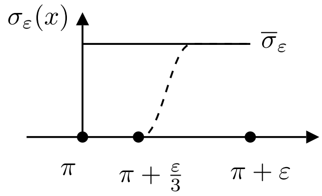

Fig. (2.1.2) The function $\sigma_\varepsilon(x)$ in $[\pi, \pi + \varepsilon]$, while in $[-\pi - \varepsilon, -\pi]$ it is defined as the mirror image: the graph is the solid line jumping from 0 at π to $\overline{\sigma}_\varepsilon$ at points $> \pi$. The dashed line refers to the smooth graph of the function previously used.

Furthermore it is convenient to use the fact that (2.1.14) satisfies periodic boundary conditions and to translate the "potential" σ_ε to the center of the interval (this is done via the change of coordinates $x' = x - \pi - \varepsilon$). In this new representation the boundary layer of width ε is now at the center of the interval $[-\pi - \varepsilon, \pi + \varepsilon]$ as in the Fig. (2.1.3): half of it, namely $[-\varepsilon, 0]$, corresponds to the previous left boundary layer and the other half, namely $[0, \varepsilon]$, corresponds to the previous right boundary layer.

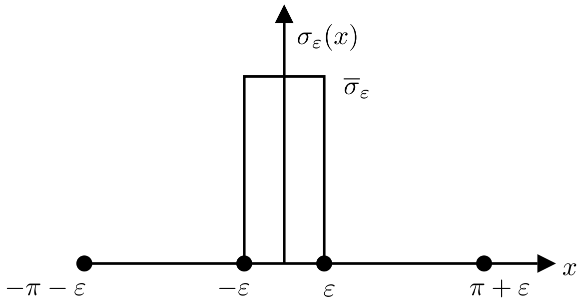

Fig. (2.1.3) The function σ_ε after the shift of th origin. The points $-\pi-\varepsilon$ and $\pi+\varepsilon$ are identified by the periodic boundary conditions.

We shall therefore consider the equation

(1) $\dot T = T'' - \sigma_\varepsilon T, \qquad$ in $[-\pi-\varepsilon, \pi+\varepsilon]/\{\pm\varepsilon\}$

(2) periodic boundary conditions and matching of the values and of the first derivatives of T in $\pm\,\varepsilon$ (2.1.21)

(3) $\sigma_\varepsilon = 0$ if $|x| > \varepsilon$ and $\sigma_\varepsilon = \overline{\sigma}_\varepsilon$ if $|x| < \varepsilon$

(4) initial datum, if T_0 is the datum in (2.1.14) : $\vartheta_0(x) = T_0(x-\pi-\varepsilon)$ if $x > \varepsilon$, and $T_0(x+\pi+\varepsilon)$ if $x < -\varepsilon$ and zero in $|x| < \varepsilon$

The exclusion of the points $\pm\varepsilon$ in the first of (2.1.21) is replaced by the matching condition which is now posed because σ_ε has been chosen discontinuous, see Fig. (2.1.3), and the equation (2.1.12) must, as a consequence, be somehow interpreted at the discontinuity points.[2]

The following proposition solves the question analogous to the (Q1), in the case of the heat equation

1 Proposition *(heat equation approximation algorithm): The heat equation, in the form (2.1.21), with ϑ_0 generated (c.f.r. (4) in (2.1.21)) by a smooth datum $T_0(x)$, say $T_0 \in C^\infty([-\pi,\pi])$, admits a solution $t \to T^\varepsilon(x,t)$ which for $\varepsilon \to 0$ and $\overline{\sigma}_\varepsilon \to +\infty$ fast enough converges for all $x \in [-\pi,\pi]$ to a smooth solution of the heat equation (2.1.12), (described in problem* [2.1.6]*).*

The equation can be solved exactly: the proof can therefore be reduced to a simple but instructive check, *c.f.r.* problems [2.1.1]%[2.1.7].

(G) Analysis of the precision of the algorithms in the heat equation case, (Q2).

We now study the question (Q2) following the (2.1.20), *i.e.* we study whether the algorithm (2.1.17) of solution of (2.1.16) does really gener-

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[2] The matching condition yields a possible interpretation: obviously it is only one of the possible interpretations and as such it has a physical significance, that however we shall not discuss here, given its auxiliary nature in the present context.

ate the wanted smooth solution, *i.e.* the solution guaranteed by the last proposition.

It is easy to convince oneself that *this is not the case*, in general, because this is already not true in the analogous case of the problem (2.1.12) in which the Dirichlet boundary condition is replaced by the even easier periodic boundary condition ($T(-\pi) = T(\pi),\ T'(\pi) = T'(-\pi)$). In such case there is even no need of introducing σ_ε nor of enlarging the domain: because the boundary conditions are respected by the algorithm (2.1.13) and the conceptual problems discussed in (D) (following (2.1.13)) do not even arise.

Note also that the algorithms (2.1.13) and (2.1.17) coincide in this periodic boundary conditions case.

Nevertheless the question whether the algorithm (2.1.13) does really produce the correct solution can be posed in this simpler problem (*i.e.* (2.1.12) with periodic boundary conditions and C^∞ periodic initial datum) and it is clearly a simpler question than the one in (Q2) about the algorithm (2.1.19) for the case of the NS equations with Dirichlet boundary conditions. In fact the following proposition holds

2 Proposition: *(anomaly of approximations convergence for the heat equation) Consider the equation (2.1.12) but with periodic boundary conditions (as above) and C^∞, periodic, initial datum. Let, for $k \ge 1$*

$$T_k = T_0 + ct_0 \sum_{h=1}^{k} T''_{k-h}, \qquad \longleftrightarrow \qquad T_k - T_{k-1} = c\, t_0\, T''_{k-1} \tag{2.1.22}$$

Let $\omega = 2\pi k$ with k integer and let $\hat T_0(\omega)$ be the Fourier transform of $T_0(x)$ then
(i) if, for $\tau, b, \eta > 0$, it is $|\hat T_0(\omega)| < \tau e^{-b|\omega|^{2+\eta}}$ (implying, among other things, that T_0 is analytic entire in x) the sequence (2.1.22) converges to the usual, well known, solution $T(x,t)$ of (2.1.12) for $t_0 \to 0$,
(ii) in general $\hat T_k(\omega)$ has a lmit $\hat T(\omega,t)$ as $k \to \infty$ with $kt_0 \to t$, for every ω
(iii) unless the condition in (i) on the Fourier transform holds it is not in general true that $T_k(x) \xrightarrow[kt_o \to t,\, t_0 \to 0]{} T(x,t)$.

proof: The Fourier transform of T_k can be computed as

$$\hat T_k(\omega) = \hat T_{k-1}(\omega)(1 - c\omega^2 t_0) = \hat T_0(\omega)(1 - c\omega^2 t_0)^k \tag{2.1.23}$$

where $\omega = 2\pi n$ with n integer > 0.
Hence (writing $t_0 = \frac{t}{k}$) we see that

$$\hat T_k(\omega) \xrightarrow[k\to\infty, t\to 0]{kt_0 = t} e^{-c\,\omega^2 t} \hat T_0(\omega) \tag{2.1.24}$$

But this does not mean that $T_k(\xi) \to T(\xi,t)$, where T is the "usual" solution of (2.1.12) with Fourier transform given by the r.h.s. of (2.1.24). One sees this immediately if T_0 has support $[-a,a]$ strictly contained in $[-\pi,\pi]$, $(a<\pi)$. In this case it is clear (from (2.1.22)) that T_k will remain identically zero in the set where it initially was zero with all its derivatives (*i.e.* outside of $[-a,a]$), hence it cannot be analytic for $t>0$ (while $T(x,t)$ is analytic).
Convergence is, however, guaranteed if

$$|(1-c\omega^2\frac{t}{k})^k\hat{T}_0(\omega)| \le \gamma_t(\omega) \quad \text{and} \qquad \sum_\omega |\gamma_t(\omega)| < \infty \tag{2.1.25}$$

with the series of the $|\gamma_t(\omega)|$ uniformly converging in t for t in an arbitrarily prefixed bounded interval. It is easy to see that this happens if, and essentially only if (see problems [2.1.10], [2.1.11]), $\hat{T}_0(\omega) \to 0$ as fast as $\tau e^{-b|\omega|^2}$, or faster, for $\tau, b>0$: at least if t is small enough and even for all times if the decay in ω is faster (*e.g.* as assumed in hypothesis in (i) of the proposition).

Therefore to make the method work *strong regularity properties must be imposed* on $T_0(x)$: it must be more regular than an entire analytic function of x (simple analyticity would "just" demand that the Fourier transform decays exponentially, *i.e.* as $\tau e^{-b|\omega|}$ for some $\tau, b>0$ but the above proof would not work).

(H) Comments:

(1) *A fortiori*, we must expect that very strong regularity conditions have to be imposed upon $\underline{u}_0$, besides imposing suitable properties on the values of $\underline{u}$ and of its derivatives on the boundary $\partial\Omega$ (*c.f.r.* the remark following (2.1.13)), so that the algorithm (2.1.19) could converge to the correct solution of the NS–equation (which, unlike the heat equation, has not yet been shown to even admit a solution).
(2) However the regularity conditions could, in the end, simply reduce to analyticity properties of $\underline{u}_0$ and of the boundary $\partial\Omega$ of Ω (whose regularity also influences that of p_0 hence of $\dot{\underline{u}}_0$ and $\underline{u}$, via the Neumann problem, (2.1.3)).
Unfortunately the problem is open, at least if $d=3$, and it is not even known whether under conditions of this type the NS equation admits a solution which is well defined and keeps, in general, a regularity comparable to that of the initial data for all times $t>0$.
(3) Note that the proposition is in apparent contradiction with the theory of the heat equation. Usually one says that the heat transport equation in a conducting rod, with fixed temperature at the extremes (equal to 0 or to each other in the above examples) and with regular initial datum "*admits a unique solution*". The solutions that we construct as limits of the approximations with time step t_0 starting from an initial datum which, for instance, is of class C^∞ and vanishes outside an interval $[-a,a]$ with $a<\pi$, and is analytic inside $(-a,a)$ are, in some sense, solutions of the

heat equation for a rod $[-a,a]$[3] Such solutions, regarded as functions on $[-\pi,\pi]$ vanish outside $[-a,a]$ hence they are not C^∞ in $[-\pi,\pi]$ (in general they will have discontinuous first derivatives at $\pm a$, to say the least).

(4) In other words the key to the uniqueness theorem lies in the requirement that T is really at least once differentiable with continuous derivatives on the *whole* interval $[-\pi,\pi]$ and for all times $t\ge 0$. Obviously one could take the alternative view of calling *solutions of the heat equation* the limits of sequences of discretized equations: in this case we could possibly have an existence and uniqueness theorem *once the discretized approximation is fixed* (possibly under additional conditions on the initial datum) but the solutions would be (in general) different from the "classical" solutions.[4] Furthermore the solution provided by the limit of a sequence of discretized equations may depend on which discretized equation is chosen (*i.e.* on which algorithm of approximation is used).

(5) One thus sees that rather deep interpretation problems arise here, which cannot be solved on a purely mathematical ground: to understand which is the correct meaning to give to a "solution" in physically interesting cases or applications it is necessary to go back to the physical properties that the equation translates into a mathematical model.
It is possible that the physical interpretation requires one or another solution depending on the physical origin of the problem. *It is clear that, if such considerations already apply to a simple equation like the* 1*–dimensional heat equation, with greater reason they will be relevant for equations like the Euler or Navier–Stokes equations* and lead to exposing major existence problems. In general uniqueness problems for partial differential equations are delicate both physically and mathematically: even in cases in which one usually says that there are "no problems" (as one says, for instance, for the heat equation). *This shows, once again, that dogmatic attitudes on notions like existence of solutions, or uniqueness, only lead to failure to see the existence of interesting problems.*

(6) The real importance of the above analysis, already for the heat equation, is shown by the fact that it makes at least less convincing the paradox that claims the incompatibility of the heat equation with special relativity: heat "waves" can apparently travel with infinite speed and "an initial datum with support in a finite region will evolve, by the heat equation, into a datum which does not vanish at an arbitrary distance".

This is, to say the least, a hasty conclusion because, if one defined "solution" what is obtained as limit of the above described Euler algorithm, one would instead find a solution that not only does not have infinite velocity but which in fact does not propagate at all (because as we have seen it will remain nonzero only where it was so initially). This is obviously only one more argument beyond the well known one that remarks that the heat

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[3] I do not know a reference for detailed analysis of this property.

[4] Which, when the initial data are at least C^1–functions, *by definition* are those of class C^1 in $[-\pi,\pi]$ and $t\ge 0$.

equation is a phenomenological macroscopic equation derived from assumptions that involve non relativistic ideas and notions (like the "radius of the molecules" which appears in the conductivity coefficient c (see [1.1.5]) and which, alone, would invalidate the "paradoxical" infinite speed propagation of heat "predicted" by the heat equation.

(7) Conclusion: the algorithms discussed in this section are interesting for the conceptual difficulties that they illustrate and for the multiplicity of aspects that they bring up on the theory of fluids. They have limited usefulness for applications, if not none at all.

Problems: *Well posedness of the heat equation and other remarks.*

[2.1.1]: Consider the eigenvalue problem associated with (2.1.14):

$$T'' - \sigma_\varepsilon T = -\lambda^2 T, \qquad \text{in } [-\pi-\varepsilon, \pi+\varepsilon] \qquad (*)$$

and show that for $|x| > \varepsilon$ the eigenvector T has the form $Ae^{i\lambda x} + Be^{-i\lambda x}$ while for $|x| < \varepsilon$ it has the form $\alpha \cosh\sqrt{\sigma_\varepsilon - \lambda^2}\; x + \beta \sinh\sqrt{\sigma_\varepsilon - \lambda^2}\; x$.

[2.1.2] Find the matching conditions determining λ for the eigenvalue problem in [2.1.1]. (*Idea:* If A_+, B_+ and A_-, B_- are the coefficients of the same solution $x > \varepsilon$ or $x < -\varepsilon$ then the periodicity condition imposes, for each η

$$A_+ e^{i\lambda(\pi+\varepsilon+\eta)} + B_+ e^{-i\lambda(\pi+\varepsilon+\eta)} = A_- e^{i\lambda(-(\pi+\varepsilon)+\eta)} + B_- e^{-i\lambda(-(\pi+\varepsilon)+\eta)}$$

Hence: $A_- = A_+ e^{2i\lambda(\pi+\varepsilon)},\ B_- = B_+ e^{-2i\lambda(\pi+\varepsilon)}$.
To simplify we study only eigenfunctions which are even in x (which suffices to study the (2.1.14) with an even initial datum: $T_0(x) = T_0(-x)$). The parity condition means: $A_+ e^{2i\lambda(\pi+\varepsilon)} e^{-i\lambda x} + B_+ e^{-2i\lambda(\pi+\varepsilon)} e^{i\lambda x} = A_+ e^{i\lambda x} + B_+ e^{-i\lambda x}$ *i.e.* $B_+ e^{-2i\lambda(\pi+\varepsilon)} = A_+$. Thus for $x \ge 0$ the even eigensolution with eigenvalue λ will have the form

$$\begin{aligned} T(x) &= A\,(e^{i\lambda x} + e^{2i\lambda(\pi+\varepsilon)} e^{-i\lambda x}) \qquad && x > \varepsilon \\ T(x) &= \alpha \cosh\sqrt{\sigma_\varepsilon - \lambda^2}\; x && |x| < \varepsilon \end{aligned} \qquad (**)$$

with suitable A, α. Hence the matching condition will be

$$\begin{aligned} A\,(e^{i\lambda\varepsilon} + e^{2i\lambda(\pi+\varepsilon) - i\lambda\varepsilon}) &= \alpha \cosh\sqrt{\sigma_\varepsilon - \lambda^2}\; \varepsilon \\ i\lambda A\,(e^{i\lambda\varepsilon} - e^{2i\lambda(\pi+\varepsilon) - i\lambda\varepsilon}) &= \alpha\sqrt{\sigma_\varepsilon - \lambda^2} \sinh\sqrt{\sigma_\varepsilon - \lambda^2}\; \varepsilon \end{aligned}$$

so that the eigenvalue λ is determined by

$$\frac{1 + e^{2\pi i\lambda}}{1 - e^{2\pi i\lambda}} \equiv i \operatorname{cotg} \pi\lambda = \frac{\lambda i}{\sqrt{\sigma_\varepsilon - \lambda^2}} \coth \varepsilon\sqrt{\sigma_\varepsilon - \lambda^2} \qquad (***)$$

The odd eigenfunctions are treated similarly.)

[2.1.3]: Show that if the eigenvalues associated with the even eigenfunctions of (*) are labeled, as λ increases, by $0, 1, 2, \ldots$ it is

$$\lambda_n = \left(n + \frac{1}{2}\right) + O(\varepsilon n), \qquad n \le \lambda_n \le n+1 \qquad \text{with } n \text{ integer}$$

if $\sigma_\varepsilon \varepsilon^2 \equiv E \xrightarrow[\varepsilon\to0]{} \infty$. (*Idea:* It suffices to draw the graph of both sides of (***), paying attention to distinguish the cases $\sigma_\varepsilon > \lambda^2$ and $\sigma_\varepsilon < \lambda^2$).

[2.1.4]: Show that the square of the L_2–norm of $T_n(x)$, $\int_{-\pi-\varepsilon}^{\pi+\varepsilon} |T_n(x)^2 dx$, is

$$|A|^2\, 4\pi\,(1+\frac{\sin 2\pi\lambda_n}{2\pi\lambda_n}) + \varepsilon|\alpha|^2(1+\frac{\sinh 2\sqrt{E-\varepsilon^2\lambda_n^2}}{2\sqrt{E-\varepsilon^2\lambda_n^2}})$$

so that if T_n is normalized to 1 in L_2 it is $|A^2| < 1/8\pi$ for $n \ge 1$ (noting that for $n \ge 1$ it is $\lambda_n \ge 1$). Show also that $|T_n(x)| \le C$ for some n–independent C. (*Idea:* The matching conditions in [2.1.3] imply

$$\Big|\frac{\alpha}{A}\Big| \le \begin{cases} \min\Big(2/|\cos\varepsilon\sqrt{\lambda_n^2-\sigma_\varepsilon}|, 2/\sqrt{\lambda_n^2-\sigma_\varepsilon}\,|\sin\varepsilon\sqrt{\lambda_n^2-\sigma_\varepsilon}|\Big) & \text{if } \sigma_\varepsilon < \lambda_n^2 \\ \min\Big(2/\cosh\varepsilon\sqrt{\sigma_\varepsilon-\lambda_n^2}, 2/\sqrt{\sigma_\varepsilon-\lambda_n^2}\,\sinh\varepsilon\sqrt{\sigma_\varepsilon-\lambda_n^2}\Big) & \text{if } \sigma_\varepsilon \ge \lambda_n^2 \end{cases}$$

so that for $n \ge 1$ (*i.e.* for $\lambda_n \ge 1$) we see that $|\alpha/A| \le 4$.)

[2.1.5]: Check that [2.1.3], [2.1.4] imply that $T_n(x)$, normalized in L_2, can be written as

$$T_n(x) = \frac{(e^{i\lambda_n x} - e^{-i\lambda_n x} + O(\varepsilon n))}{\text{normalization}} = \frac{1}{\sqrt{\pi}}\sin(n+\frac12)x + O(\varepsilon n), \qquad x > \varepsilon$$

Show also that the results in [2.1.4] agree with the corresponding results for the equation $T'' = -\lambda^2 T$ in $[0, 2\pi]$ with boundary conditions $T(0) = T(2\pi) = 0$, which yield eigenvalues $\lambda_n^2 = -\frac{n^2}{4}$ with eigenvectors $\pi^{-1/2}\sin\frac{nx}{2}$. (*Idea:* Note that the two problems are equivalent if one thinks of $[-\pi,\pi]$ and $[0,2\pi]$ as of two identical circles and if the point $x=\pi$ in the first case is identified with the point $x = 0 = 2\pi$ of the second case: with this change the odd order eigenfunctions for the problem in $[0, 2\pi]$ become the even ones for the problem in $[-\pi,\pi]$ (and *viceversa*)).

[2.1.6]: Show that the solution of (2.1.21) with initial data ϑ_0 even in x and as in (2.1.21) can be written, for $t > 0$, in terms of the eigenfunctions analyzed in problems [2.1.1]%[2.1.4] as

$$T_\varepsilon(\xi,t) = \sum_{k=1}^{\infty} e^{-\lambda_k^2 t}\langle T_k, \vartheta_0\rangle T_k(x)$$

where $\langle T, T'\rangle \equiv \int_{-\pi-\varepsilon}^{\pi+\varepsilon} T(x)T'(x)\,dx$: see (2.1.21) for the definition of ϑ_0.

[2.1.7]: Suppose that ϑ_0 is even in x. Show that the results of [2.1.4] imply $\langle T_k, \vartheta_0\rangle \xrightarrow[\varepsilon\to0]{} \langle \overline{T}_k, \tilde T_0\rangle$, where $\tilde T_0$ is the limit $\lim_{\varepsilon\to0}\vartheta_0(x)$ and $\overline{T}_k$ are the eigenfunctions of the problem

$$\begin{aligned} &T'' = -\lambda^2 T, \quad \text{in} \quad [-\pi,\pi]/\{0\} \quad \text{with periodicity } 2\pi \\ &T(0) = 0 \end{aligned} \qquad (a)$$

i.e. for $x > 0$

$$\overline{T}_k(x) = \frac{1}{\sqrt{\pi}}\sin\Big(k+\frac12\Big)x, \qquad \bar\lambda_k = k + \frac12 \qquad (b)$$

Furthermore check that: $\lambda_k \xrightarrow[\varepsilon\to 0]{} \overline{\lambda}_k$ and $\lambda_k^2 \ge C\,\overline{\lambda}_k^2$ for each $k, |\varepsilon| < 1$, if C is suitably chosen. Infer from this that

$$T_\varepsilon(\xi,t) \xrightarrow[\varepsilon\to 0]{} T(\xi,t) = \sum_{k=1}^{\infty} e^{-\overline{\lambda}_k^2 t} \langle \overline{T}_k, \tilde{T}_0\rangle\, \overline{T}_k(x) \tag{c}$$

and that the limit $T(\xi,t)$ solves the heat equation on $[-\pi,\pi]/\{0\}$ with boundary condition $T(0) = 0$ at $x = 0$. Check that this easily leads to a solution of the problem (2.1.12) because (2.1.12) can be interpreted as the above "translated" by π. (*Idea:* By [2.1.4] the terms of the series for $T_\varepsilon(\xi,t)$ can be estimated by a constant times $e^{-\lambda_n^2 t}$ and obviously $\langle T_k, T_0\rangle T_k(x)$ converges to the corresponding value in the expression for $T(\xi,t)$.)

[2.1.8]: Consider, in the case $d = 2$, a solenoidal nonconservative force $\underline{g} \in C^\infty(\Omega)$ and let $\gamma = \partial\Omega$ be such that $I = \int_\gamma \underline{g}\cdot d\underline{x} \ne 0$. Show that in this case (2.1.9) is not, in general, soluble if the initial datum is $\underline{u}_0 = \underline{0}$. Find an example. (*Idea:* Let Ω be a disk of radius r and let $\underline{g}(\underline{x}) \stackrel{def}{=} \underline{\omega}\wedge\underline{x}$ with $\underline{\omega} = \omega\underline{e}$ orthogonal to the disk. Then the normal component of $\underline{g}$ on γ is 0 so that the uniqueness of the solutions for the Neumann problem implies that $p = const$ hence $\underline{\partial} p = \underline{g} = \underline{0}$ but $I = 2\pi r^2\omega$).

[2.1.9]: Consider a velocity field in the half space $z \ge 0$ with components

$$u_1 = -y\,\chi(z)\,f(x^2+y^2), \qquad u_2 = x\,\chi(z)\,f(x^2+y^2), \qquad u_3 = 0$$

where $\chi(z) = z$ per $z \le h$, $h > 0$ prefixed, and $\chi(z) \equiv 0$ for $z > 2h$, while $f(r^2) = 0$ for $r > R$, with $R > 0$ prefixed. Show that $\underline{u}$ has zero divergence but (2.1.9) is no soluble, for χ, f generic. (*Idea:* Note that $\Delta\underline{u} = \chi(z)\,(8f' + 4r^2 f'')(-y,x,0)$, if $r^2 \equiv x^2+y^2$ and $z < h$. Hence $\underline{u}$ and $\Delta\underline{u}$ both vanish on the boundary $z = 0$. Furthermore

$$-\underline{\partial}\cdot(\underline{u}\cdot\underline{\partial}\,\underline{u}) \equiv \sigma(z^2,r^2) \equiv\ (f(r^2)^2 + 2f(r^2)f'(r^2)\,r^2)\chi(z)^2$$

Hence (assuming $\underline{g} = \underline{0}$) the gradient $\underline{\partial} p$ of p vanishes on $z = 0$ and the corresponding Neumann problem can be solved by the method of images. The potential p is then the electrostatic potential generated by a charge distribution σ with cylindrical symmetry around the z–axis and with center at the origin and with reflection symmetry across the plane $z = 0$.
The at large distance the electric field can be computed, to leading order, in R^{-1} and one sees that its component tangential to the plane $z = 0$ does not vanish and it has order R^{-4} if $\int r^2\sigma(r^2,z^2) r\,dr\,dz \ne 0$ (in electrostatic terms one can say that the electric field is dominated at large distance by the lowest nonzero dipole moment which is in the present case the quadrupole, yielding therefore a field proportional to R^{-4}). Check that the dipole moment is identically zero, for any f and χ, beginning with the remark that $\sigma \equiv \chi^2\,\frac{\partial r^2 f^2}{\partial\, r^2}$.)

[2.1.10]: Consider the heat equation in $[-\pi,\pi]$, $\dot{T} = T''$, with periodic boundary conditions and analytic initial datum $T_0(x)$ with Fourier transform $\hat{T}_0(\omega)$, $\omega = 0,\pm1,\ldots$; hence there are constants $\tau, b > 0$ such that $|\hat{T}_0(\omega)| \le \tau e^{-b|\omega|}$. Show that the L_2–norm of $(1-\omega^2 t/k)^k \hat{T}_0(\omega)$ may diverge as $k\to\infty$ although for each ω one has $(1-\omega^2 t/k)^k \hat{T}_0(\omega) \xrightarrow[k\to\infty]{} e^{-\omega^2 t}\hat{T}_0(\omega)$, *c.f.r.* (2.1.25). This implies that in general even if T_0 is analytic the method of approximation in (2.1.22) *does not* converge to the solution neither in the sense of L_2 nor pointwise and staying uniformly bounded. (*Idea:* Take $\hat{T}_0(k) \equiv \tau e^{-b|\omega|}$ and estimate the sum $\sum_\omega (1-\omega^2 t/k)^{2k}\tau^2 e^{-2b|\omega|}$ by the single term with $\omega^2 t = 3k$.)

[2.1.11]: In the context of [2.1.10] show that even if $\hat{T}_0(\omega) = \tau e^{-b\omega^2}$ still one cannot have L_2 convergence of $T_k(x)$ for all times $t > 0$. (*Idea:* Same as previous.)

Bibliography: [Bo79].

§2.2 Another class of empirical algorithms. Spectral method. Stokes problem. Gyroscopic analogy.

A method substantially different from the one discussed in §2.1 is the "*cut–off*" or "*spectral*" method. The name originates from the use of the representation of $\underline{u}$ on the basis generated by the Laplace operator on $X_{\rm rot}(\Omega)$, *c.f.r.* (1.6.16): it is, therefore, a method associated with the spectrum of this operator.

(A) Periodic boundary conditions: spectral algorithm and "reduction" to an ordinary differential equation.

We shall first examine a fluid occupying the d–dimensional torus T^d, *i.e.* an incompressible fluid enclosed in a cubic container with periodic boundary conditions ("opposite sides identified").
In this case the velocity and pressure fields, assumed regular, will admit a Fourier representation which can be regarded, obviously, as the expansion of the fields on the plane waves basis or, equivalently, on the basis generated by the eigenvectors[1] of the Laplace operator on T^d, *i.e.*

$$\underline{u}(\underline{\xi},t)=\sum_{\underline{k}}\hat{\underline{u}}_{\underline{k}}e^{i\underline{k}\cdot\underline{\xi}},\qquad p(\underline{\xi},t)=\sum_{\underline{k}}p_{\underline{k}}e^{i\underline{k}\cdot\underline{\xi}} \tag{2.2.1}$$

where $\underline{k}=2\pi L^{-1}\underline{n}$ with $L=$ side of the container and $\underline{n}$ is a vector with integer components. We adopt the following convention for the Fourier transforms

$$\begin{aligned}&\underline{u}(\underline{x})=\sum_{\underline{k}}e^{i\underline{k}\cdot\underline{x}}\hat{\underline{u}}_{\underline{k}},\qquad \hat{\underline{u}}(\underline{k})=L^{-d}\int_{T^d}e^{-i\underline{k}\cdot\underline{x}}\underline{u}(\underline{x})\,d\underline{x}\\&||\underline{u}||_2^2\equiv|\underline{u}|_2^2\equiv||\underline{u}||_{L_2(T^d)}=L^d\sum_{\underline{k}}|\hat{\underline{u}}_{\underline{k}}|^2=\int_{T^d}|\underline{u}(\underline{x})|^2\,d\underline{x}\end{aligned} \tag{2.2.2}$$

The incompressibility condition (*i.e.* zero divergence), in the case $d=3$, requires that for $\underline{k}\neq\underline{0}$

$$\hat{\underline{u}}_{\underline{k}}=\gamma^1_{\underline{k}}\,\underline{e}^1_{\underline{k}}+\gamma^2_{\underline{k}}\,\underline{e}^2_{\underline{k}}\equiv\underline{\gamma}_{\underline{k}} \tag{2.2.3}$$

where $\underline{e}^1_{\underline{k}}$, $\underline{e}^2_{\underline{k}}$ are two unit vectors orthogonal to $\underline{k}$. In the case $d=2$ it must be $\hat{\underline{u}}=\gamma_{\underline{k}}\underline{k}^\perp/|\underline{k}|$, if $\underline{k}=(k_1,k_2)$ and $\underline{k}^\perp=(k_2,-k_1)$.

[1] We consider in this section only real vector fields: nevertheless it is occasionally convenient to express them in terms of complex plane waves rather than using the sines and cosines waves. We shall not discuss further this matter of notation.

We consider only incompressible Euler and NS equations in which the applied external force $\underline{g}(\underline{\xi})$ has zero average $L^{-d}\int \underline{g}(\underline{\xi})\,d\underline{\xi} = \underline{0}$: this is to exclude that the center of mass of the fluid accelerates uniformly (note that with periodic boundary conditions the center of mass will move as a body of mass equal to that of the fluid subject to the sum of the volume forces); hence

$$\partial_t \int \rho\underline{u}\,d\underline{\xi} = \rho \int \underline{g}\,d\underline{\xi} = \underline{0} \tag{2.2.4}$$

and, possibly changing reference frame, it is not restrictive to suppose $\int \underline{u}\,d\underline{\xi} = \underline{0}$. Likewise we can fix the arbitrary additive constant in the pressure so that $\int p\,d\underline{\xi} \equiv 0$.

With such conventions and hypotheses we can rewrite the (2.2.1) as

$$\underline{u}(\underline{\xi},t) = \sum_{\underline{k}\neq\underline{0}} \underline{\gamma}_{\underline{k}}(t)e^{i\underline{k}\cdot\underline{\xi}}, \qquad p(\underline{\xi},t) = \sum_{\underline{k}\neq\underline{0}} p_{\underline{k}}e^{i\underline{k}\cdot\underline{\xi}} \tag{2.2.5}$$

and the Euler or NS equations become ordinary equations for the components $\underline{\gamma}_{\underline{k}}$ of the field $\underline{u}$. To write them explicitly remark that

$$\underset{\sim}{u}(\underline{\xi})\cdot\underset{\sim}{\partial}\,\underline{u}(\underline{\xi}) = \sum_{\underline{h},\underline{k}} e^{i(\underline{h}+\underline{k})\underline{\xi}}(\underset{\sim}{\gamma}_{\underline{h}}\cdot i\underset{\sim}{k})\,\underline{\gamma}_{\underline{k}} \tag{2.2.6}$$

Furthermore define, for $\underline{k}\neq\underline{0}$, the operator $\Pi_{\underline{k}}$ of orthogonal projection of R^3 on the plane orthogonal to $\underline{k}$ by

$$(\prod\nolimits_{\underline{k}}\underline{w})_i = w_i - \frac{\underline{w}\cdot\underline{k}}{\underline{k}^2}k_i \tag{2.2.7}$$

and note the following obvious identity

$$\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2\underline{\gamma}_{\underline{k}_2} \equiv (\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2)\prod\nolimits_{\underline{k}_1+\underline{k}_2}\underline{\gamma}_{\underline{k}_2} + (\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2)(1-\prod\nolimits_{\underline{k}_1+\underline{k}_2})\underline{\gamma}_{\underline{k}_2} \tag{2.2.8}$$

Consequently we see that the partial differential equations

$$\underline{\partial}\cdot\underline{u} = 0\,, \quad \dot{\underline{u}} + \underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{u} = -\rho^{-1}\underline{\partial}p + \underline{g} + \nu\Delta\underline{u} \tag{2.2.9}$$

can be written as the ordinary differential equations

$$\begin{aligned} \dot{\underline{\gamma}}_{\underline{k}} &= -\nu\underline{k}^2\gamma_{\underline{k}} - i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}(\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2)\prod\nolimits_{\underline{k}}\underline{\gamma}_{\underline{k}_2} + \prod\nolimits_{\underline{k}}\underline{\hat{g}}_{\underline{k}} \\ p_{\underline{k}} &= -\rho\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}\frac{1}{\underline{k}^2}(\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2)(\underline{k}\cdot\underline{\gamma}_{\underline{k}_2}) - \frac{i\rho}{\underline{k}^2}\underline{\hat{g}}_{\underline{k}}\cdot\underline{k} \end{aligned} \tag{2.2.10}$$

Therefore the pressure "disappears" and the equations for the "essential components" of the fields describing our system become

$$\dot{\underline{\gamma}}_{\underline{k}} = -\nu\underline{k}^2\gamma_{\underline{k}} - i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}(\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2)\prod\nolimits_{\underline{k}}\underline{\gamma}_{\underline{k}_2} + \underline{\hat{g}}_{\underline{k}}, \qquad \underline{\gamma}_{\underline{k}}(0) = \underline{\gamma}^0_{\underline{k}} \tag{2.2.11}$$

having assumed that $\Pi_{\underline{k}}\hat{g}_{\underline{k}} \equiv \hat{g}_{\underline{k}}$, since the gradient part of $\hat{g}_{\underline{k}}$, (*i.e.* the component of $\hat{g}_{\underline{k}}$ parallel to $\underline{k}$), can be included in the pressure, as we see from the second of the (2.2.10). To the (2.2.11) *we must always add the reality condition* for $\underline{u}$, *i.e.* $\underline{\gamma}_{\underline{k}} = \overline{\underline{\gamma}}_{-\underline{k}}$: we shall always assume such relation, *c.f.r.* footnote [1] above.

The Euler equations are simply obtained by setting $\nu = 0$.

If $\nu > 0$ the friction term gives rise to very large coefficients as $\underline{k}^2 \to \infty$ and therefore it will possibly generate problems in solution algorithms. It is therefore convenient to rewrite (2.2.11) as

$$\underline{\gamma}_{\underline{k}}(t) = e^{-\nu\underline{k}^2 t}\underline{\gamma}^0_{\underline{k}} + \int_0^t e^{-\nu\underline{k}^2(t-\tau)}\Big(\hat{g}_{\underline{k}} - i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}(\underline{\gamma}_{\underline{k}_1}(\tau)\cdot\underset{\sim}{k}_2)\prod_{\underline{k}}\underline{\gamma}_{\underline{k}_2}(\tau)\Big)d\tau \tag{2.2.12}$$

in which we see that the friction term is, in fact, a term that can help constructing solution algorithms, because it tends to “eliminate” the components with $|\underline{k}| \gg 1/\sqrt{\nu}$ *i.e.* the “*short wave*”components, also called the “*ultraviolet*” components, also called the “*short wave*” components of the velocity field.

Note that (2.2.12) suggests naturally a solution algorithm

$$\underline{\gamma}^{(n)}_{\underline{k}} = e^{-\nu\underline{k}^2 t}\underline{\gamma}^0_{\underline{k}} + t_0\sum_{m=0}^{n-1} e^{-\nu k^2 t_0(n-m)}\Big(\hat{\underline{g}}_{\underline{k}} - i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}\underline{\gamma}^{(m)}_{\underline{k}_1}\cdot\underset{\sim}{k}_2\,\Pi_{\underline{k}}\gamma^{(m)}_{\underline{k}_2}\Big) \tag{2.2.13}$$

where the $\gamma^{(m)}$ are computed in mt_0; and therefore

$$\lim_{\substack{n\to\infty\\ nt_0=t}} \underline{\gamma}^{(n)}_{\underline{k}} = \underline{\gamma}_{\underline{k}}(t) \tag{2.2.14}$$

should be a solution of the equation.

This proposal can be subject to criticism of the type analyzed in the preceding §2.1 and we can expect that it might be correct only under further regularity hypotheses on $\underline{u}^0$, *i.e.* on $\underline{\gamma}^0$.

The (2.2.13) requires summing infinitely many terms. For concrete applications it is therefore necessary to find an approximation involving only sums of finitely many terms. One of the most followed methods it to introduce a *ultraviolet cut–off*: this means introducing a parameter N (the *cut–off*) and to constrain $\underline{k}, \underline{k}_1, \underline{k}_2$ in (2.2.12), or in (2.2.11), to be $\leq N$.

Thus one obtains a system of finitely many ordinary equations and their solution $\underline{\gamma}^N_{\underline{k}}(t)$ *should* tend to a solution of the NS equations in the limit $N \to \infty$.

In the case of equation (2.2.13) with ultraviolet cut–off N we denote the approximate solution $\underline{\gamma}^{(n)N}_{\underline{k}}(t)$ and in the limit $N \to \infty, n \to \infty, nt_0 = t$ one shoud get a solution of the NS equations. The order in which the above limits have to be taken should have no consequences on the result, or it should be prescribed by the theory of the convergence: but it is not *a priori* clear that the limit really exists, nor that the solution to the equation is can be actually built in this way.

The simplicity of this algorithm, compared to those of §2.1 should be ascribed mainly to the boundary conditions that we are using. The algorithm name of *spectral method* will become more justified when we shall generalize it to the case of non periodic boundary conditions.

The algorithm has a great conceptual and practical advantage which makes it one of the most used algorithms in the numerical solutions of the Euler or NS equations. Unlike the method in §2.1 this algorithm makes manifest that viscosity appears explicitly as a damping factor on the velocity components with large wave number and rather than appearing as a "large" factor ($\sim \nu \underline{k}^2$) it appears as a "small" factor ($\sim e^{-\tau \underline{k}^2 \nu}$).

(B) Spectral method in a domain Ω with boundary and the boundary conditions problem.

We shall now build a cut–off algorithm also in the case of a bounded domain Ω with a (smooth) boundary.

We note that the real reason why we succeed at "exponentiating" terms containing viscosity is that the velocity field has been developed in eigenfunctions of the Laplace operator *which is the operator associated with the linear viscous terms of the NS–equation.*

The case of periodic boundary conditions has been very simple, because in absence of boundary it is possible to find a basis for the divergenceless fields $\underline{u}$ which is at the same time a basis of eigenvectors for the Laplace operator Δ appearing in the friction term. In presence of a boundary it will not in general be possible to find eigenvalues of the Laplace operator which have zero divergence and which *at the same time* also vanish on the boundary (*i.e.* there are no eigenvectors, in general, of the Laplace operator with Dirichlet boundary conditions and zero divergence).

However if we define the "divergenceless Laplace operator" as the operator on $X_{\rm rot}$ defined by the quadratic form on $X^0_{\rm rot}$ (*c.f.r.* §1.6, (1.6.16))

$$D(\underline{u}) = \int_\Omega (\underline{\partial}\, \underset{\sim}{u})^2\, dx \tag{2.2.15}$$

one can show the following theorem (*c.f.r.* the problems at the end of the section where the proof is described)

Theorem *(spectral theory of the Laplace operator for divergenceless fields): In the space $X_{\rm rot}$, (c.f.r. (1.6.16)), there is an orthonormal basis of vectors satisfying:*

(1) $\underline{u}_j \in C^\infty(\Omega)\,, \quad \int_\Omega \underline{u}_i \cdot \underline{u}_j = \delta_{ij}$

(2) $\underline{\partial}\cdot\underline{u}_j=0$ *in* Ω
(3) there is $\mu_j\in C^\infty(\Omega)$ *and* $\lambda_j>0$ *such that:* (2.2.16)

$$\Delta\underline{u}_j-\underline{\partial}\,\mu_j=-\lambda_j^2\underline{u}_j\quad\textit{in }\Omega$$

(4) $\underline{u}_j=0$ *in* $\partial\Omega$
(5) there are constants $\alpha,c,c',c_k>0$ *such that*

$$c\,j^{2/d}\le|\lambda_j|\le c'\,j^{2/d},\qquad|\partial^k\underline{u}_j(\underline{x})|\le c_k j^{\alpha+k/d} \tag{2.2.17}$$

for all $\underline{\underline{x}}\in\Omega$*, if* $d=2,3$ *is the space dimension.*

Then each divergenceless datum $\underline{u}\in X_{\rm rot}(\Omega)$ will be written as

$$\underline{u}(\underline{\xi},t)=\sum_{j=1}^\infty\gamma_j(t)\,\underline{u}_j(\underline{\xi}) \tag{2.2.18}$$

and therefore we can express in terms of the γ_j the results of the actions on $\underline{u}$ of the operators appearing in the Euler and Navier–Stokes equations.

If $\Pi_{\rm grad}$ and $\Pi_{\rm rot}=1-\Pi_{\rm grad}$ are the projection operators on the spaces $X_{\rm rot}^\perp=X_{\rm grad}$ and on $X_{\rm rot}$, *c.f.r.* §1.6, the actions of the Laplace operator and of the nonlinear *transport operator* are respectively

$$\begin{aligned}\Delta\underline{u}&=\sum_{j=1}^\infty-\lambda_j^2\,\gamma_j\underline{u}_j(\underline{\xi})-\underline{\partial}\,\Big(\sum_{j=1}^\infty\mu_j(\underline{\xi})\gamma_j\Big)\\ \underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}&=\sum_{j_1,j_2=1}^\infty\gamma_{j_1}\gamma_{j_2}\,\underset{\sim}{u}_{j_1}\cdot\underset{\sim}{\partial}\,\underline{u}_{j_2}=\\ &=\sum_{j_1,j_2=1}^\infty\gamma_{j_1}\gamma_{j_2}\Pi_{\rm rot}\,\big(\underset{\sim}{u}_{j_1}\cdot\underset{\sim}{\partial}\,\underline{u}_{j_2}\big)+\sum_{j_1,j_2=1}^\infty\gamma_{j_1}\gamma_{j_2}\Pi_{\rm grad}\big(\underset{\sim}{u}_{j_1}\cdot\underset{\sim}{\partial}\,\underline{u}_{j_2}\big)\end{aligned} \tag{2.2.19}$$

and the NS–equation becomes, if we set $\prod_{\rm grad}\big(\underset{\sim}{u}_{j_1}\cdot\underset{\sim}{\partial}\big)\,\underline{u}_{j_2}\equiv\underline{\partial}\,\pi_{j_1j_2}$,

$$\begin{aligned}\dot\gamma_j&=-\,\nu\lambda_j^2\gamma_j-\sum_{j_1,j_2=1}^\infty\gamma_{j_1}\gamma_{j_2}\big\langle(\underset{\sim}{u}_{j_1}\cdot\underset{\sim}{\partial}\,\underline{u}_{j_2}),\,\underline{u}_j\big\rangle+g_j\\ \rho^{-1}p&=-\,\nu\sum_{j=1}^\infty\mu_j(\underline{\xi})\,\gamma_j-\sum_{j_1j_2}\gamma_{j_1}\gamma_{j_2}\,\pi_{j_1j_2}(\underline{\xi})\end{aligned} \tag{2.2.20}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product $\langle f,g\rangle=\int_\Omega f(\underline{x})g(\underline{x})\,d\underline{x}$. The (2.2.20) can be written

$$\begin{aligned}\gamma_j(t)&=e^{-\nu\lambda_j^2t}\gamma_j^0+\int_0^te^{-\nu\lambda_j^2(t-\tau)}\Big(g_j-\sum_{j_1,j_2=1}^\infty\gamma_{j_1}(\tau)\gamma_{j_2}(\tau)C_j^{j_1j_2}\Big)\,d\tau\\ C_j^{j_1\,j_2}&=\int_\Omega(\underset{\sim}{u}_{j_1}\cdot\underset{\sim}{\partial}\underline{u}_{j_2})\cdot\underline{u}_j\,d\underline{\xi}\equiv\big\langle(\underset{\sim}{u}_{j_1}\cdot\underset{\sim}{\partial}\,\underline{u}_{j_2}),\underline{u}_j\big\rangle\end{aligned} \tag{2.2.21}$$

This shows that also in the case in which Ω has a boundary it is still possible to write the NS–equations so that viscosity appears as a "smallness" factor rather than as a "large" additive term.

One can also derive a discretization of the Euler equations by setting $\nu = 0$ in (2.2.21). However the boundary condition is now $\underline{u} \cdot \underline{n} = 0$ rather than $\underline{u} = 0$ and therefore we cannot expect that the series for $\underline{u} = \sum_{j=1}^{\infty} \gamma_j \underline{u}_j$ is "well" convergent because, otherwise, this would imply $\underline{u} = 0$ on the boundary, since $\underline{u}_j = 0$ on the boundary. *It would therefore not be a good idea to use the basis above to represent solutions of the Euler equations in the same domain.*

This is a general problem because the Euler and NS–equations are studied, usually, by imposing different boundary conditions. The limit $\nu \to 0$ in which, naively, the NS–equations should "reduce" to the Euler equations *must be* a singular limit and the convergence of the solutions of the NS–equations to solutions of the Euler equation when $\nu \to 0$ must be quite *improper* near the boundary $\partial\Omega$, where interesting surface phenomena will necessarily take place. Of course the case of periodic boundary conditions is *the* remarkable exception.

The (2.2.20),(2.2.21) can be treated as the analogous (2.2.10) and (2.2.12) and reduced, with an ultraviolet cut–off to a finite number of equations, generating a general spectral algorithm.

(C) The Stokes problem.

One calls "*Stokes problem*" the NS–equation linearized around $\underline{u} = \underline{0}$, *c.f.r.* the problems of §1.2

$$\begin{aligned} \dot{\underline{u}} &= -\rho^{-1}\underline{\partial} p + \nu\Delta\underline{u} + \underline{g}, \qquad \underline{\partial}\cdot\underline{u} = 0 \qquad &&\text{in } \Omega\\ \underline{u} &= \underline{0}, &&\text{in } \partial\Omega\\ \underline{u}|_{t=0} &= \underline{u}^0 \in X_{\rm rot}(\Omega) \end{aligned} \tag{2.2.22}$$

and we look for $C^\infty(\Omega\times(0,+\infty))$–solutions that for $t\to 0$ enjoy the property $\underline{u} \to \underline{u}^0$ at least in the sense of $L_2(\Omega)$ (see footnote [1] in (E) of §2.1), *i.e.* in the sense that the mean square deviation of $\underline{u}$ from $\underline{u}^0$, *i.e.* $\int |\underline{u} - \underline{u}^0|^2 \, d\underline{\underline{x}}$, tends to zero with t). We shall take, to simplify, $\underline{g} = \underline{0}$.

The theorem in (B) above allows us to obtain a complete solution of the problem. Indeed we develop $\underline{u}^0$ on the basis $\underline{u}_1, \underline{u}_2, \ldots$, *c.f.r.* (2.2.16)

$$\underline{u}^0(\underline{x}) = \sum_{j=1}^{\infty} \gamma_j^0 \, \underline{u}_j(\underline{x}) \tag{2.2.23}$$

and we immediately check that the solution is

$$\underline{u}(\underline{x},t) = \sum_{j=1}^{\infty} \gamma_j^0 e^{-\nu\lambda_j^2 t} \underline{u}_j(\underline{x}), \qquad p(\underline{x},t) = -\nu\rho \sum_{j=1}^{\infty} \gamma_j^0 e^{-\nu\lambda_j^2 t} \mu_j(\underline{x}) \tag{2.2.24}$$

From the properties (1)%(5) of the theorem in (B) above it follows that $\underline{u} \in C^\infty(\Omega \times (0, +\infty))$ and that $\underline{\partial} \cdot \underline{u} = 0$ for $t > 0$.

It is easy to see that this solution is unique. One also realizes the strict analogy with the heat equation of which the Stokes equation can be regarded as a "vectorial" version.

In particular it can happen that, if $\underline{u}^0 \in X_{\rm rot}(\Omega)$ but $\underline{u}^0$ does not have a series representation like (2.2.23) with coefficients γ_j^0 rapidly vanishing as $j \to \infty$ (for instance because $\underline{u}_0$ "does not match well" the boundary conditions so that all we can say is that $\sum_j |\gamma_j^0|^2 < \infty$), then $\nu \Delta \underline{u}^0$ could differ from a field vanishing on the boundary by more than we might expect, *i.e.* by an amount that is not "just" the gradient of a scalar field p: in fact the Neumann problem that should determine p is over determined and it can turn out to be impossible to solve it, as in the cases discussed in §1.6, §2.1 in connection with examples of the same pathology for the heat equation.

The pathology manifests itself *only* at $t = 0$ and it can be explained as in the heat equation case (via a physical model for the boundary condition as given, for instance, by the auxiliary friction method in (C) of §2.1). Obviously in (2.2.22) this problem shows up only at $t = 0$: if $t > 0$ in fact the (2.2.24) show that the boundary condition is strictly satisfied: and the over determined Neumann problem for p *becomes necessarily compatible* and has the second of the (2.2.24) as a solution, which at $t = 0$ might no longer make sense because of the possibly poor convergence of the series.

(D) Comments:

(1) Note that the spectral method for the NS–equations induces us into believing that at least for $t > 0$ the boundary condition is satisfied by the solutions (if existent): one could expect that friction implies that the coefficients $\gamma_j(t)$ tend to zero for $j \to \infty$ much faster than they do at $t = 0$, thanks to the coefficients $e^{-\lambda_j^2 \nu (t-\tau)}$. Hence the series (2.2.18) *should be* well convergent and, therefore, its sum should respect the boundary conditions which are automatically satisfied term by term in the series.

(2) *However* we shall see that the argument just given, which is essentially correct in the case of the heat equation (discussed in §2.1) and in the Stokes equation case, becomes now much more delicate and, mainly, if $d = 3$, it is no longer correct, basically because of the non linear terms in the transport equations: *c.f.r.* the analysis in §3.2.

(E) Gyroscopic analogy in $d = 2$.

The NS equations in $d = 2$–dimensions can be put in a form that closely reminds us of the rigid body equations of motion. The NS–equations, (2.2.10), with $\underline{g} = \underline{0}$ for simplicity, can be written in terms of the *scalar* observables $\gamma_{\underline{k}_1}, \gamma_{\underline{k}_2}, \gamma_{\underline{k}_3}$, related to the vector observable $\underline{\gamma}_{\underline{k}}$ via the $\underline{\gamma}_{\underline{k}} = \gamma_{\underline{k}}\, \underline{k}^\perp / |\underline{k}|$ because, if $d = 2$, the zero divergence property allows us to express $\underline{\gamma}_{\underline{k}}$ in terms of scalar quantities $\gamma_{\underline{k}}$:

$$\underline{\gamma}_{\underline{k}} = \gamma_{\underline{k}} \frac{\underline{k}^\perp}{|\underline{k}|}, \qquad \gamma_{\underline{k}} = -\overline{\gamma}_{-\underline{k}}, \qquad \text{if} \quad \underline{k}^\perp = (k_2, -k_1), \qquad \underline{k} = (k_1, k_2) \tag{2.2.25}$$

If $\underline{k}_1 + \underline{k}_2 + \underline{k}_3 = \underline{0}$ we then note that the equations (2.2.10) can be written

$$\begin{aligned}
\dot{\overline{\gamma}}_{\underline{k}_1} &= -\nu \underline{k}_1^2 \overline{\gamma}_{\underline{k}_1} - i\Big\{\gamma_{\underline{k}_2}\gamma_{\underline{k}_3} \frac{(\underline{k}_2^\perp \cdot \underline{k}_3)(\underline{k}_3^\perp \cdot \underline{k}_1^\perp)}{|k_1||k_2|k_3|} + (2 \leftrightarrow 3)\Big\} + ..\\
\dot{\overline{\gamma}}_{\underline{k}_2} &= -\nu \underline{k}_2^2 \overline{\gamma}_{\underline{k}_2} - i\Big\{\gamma_{\underline{k}_3}\gamma_{\underline{k}_1} \frac{(\underline{k}_3^\perp \cdot \underline{k}_1)(\underline{k}_1^\perp \cdot \underline{k}_2^\perp)}{|\underline{k}_1||\underline{k}_2||\underline{k}_3|} + (1 \leftrightarrow 3)\Big\} + ..\\
\dot{\overline{\gamma}}_{\underline{k}_3} &= -\nu \underline{k}_3^2 \overline{\gamma}_{\underline{k}_3} - i\Big\{\gamma_{\underline{k}_1}\gamma_{\underline{k}_2} \frac{(\underline{k}_1^\perp \cdot \underline{k}_2)(\underline{k}_2^\perp \cdot \underline{k}_3^\perp)}{|\underline{k}_1|\,|\underline{k}_2|\,|\underline{k}_3|} + (1 \leftrightarrow 2)\Big\} + ..
\end{aligned} \tag{2.2.26}$$

Note the symmetry properties

$$\underline{k}_1^\perp \cdot \underline{k}_2 = \underline{k}_2^\perp \cdot \underline{k}_3 = \underline{k}_3^\perp \cdot \underline{k}_1 \stackrel{def}{=} a(\underline{k}_1, \underline{k}_2, \underline{k}_3) \tag{2.2.27}$$

with $a(\underline{k}_1, \underline{k}_2, \underline{k}_3) \equiv -a$ where a is $\pm$ twice the area of the triangle formed by the vectors $\underline{k}_1, \underline{k}_2, \underline{k}_3$ (it is a symmetric function under permutations of $\underline{k}_1, \underline{k}_2, \underline{k}_3$). The sign is $+$ if the triangle $\underline{k}_1\underline{k}_2\underline{k}_3$ is circled clockwise and $-$ otherwise.

Keeping this symmetry into account together with the relations

$$\begin{aligned}
&\underline{k}_1^\perp \cdot \underline{k}_2 = -\underline{k}_2^\perp \cdot \underline{k}_1, \qquad \underline{k}_3 = -\underline{k}_1 - \underline{k}_2\\
&\underline{k}_1^\perp \cdot \underline{k}_2^\perp \equiv \underline{k}_1 \dot{\underline{k}}_2, \qquad \text{hence, for instance,}\\
&\underline{k}_2^\perp \cdot \underline{k}_3^\perp - \underline{k}_1^\perp \cdot \underline{k}_3^\perp = \underline{k}_1^2 - \underline{k}_2^2
\end{aligned} \tag{2.2.28}$$

one finds (patience is required)

$$\begin{aligned}
\dot{\overline{\gamma}}_{\underline{k}_1} &= -\nu \underline{k}_1^2 \overline{\gamma}_{\underline{k}_1} - i(\underline{k}_3^2 - \underline{k}_2^2)\, \tilde{a}\, \gamma_{\underline{k}_2}\gamma_{\underline{k}_3} + \ldots\\
\dot{\overline{\gamma}}_{\underline{k}_2} &= -\nu \underline{k}_2^2 \overline{\gamma}_{\underline{k}_2} - i(\underline{k}_1^2 - \underline{k}_3^2)\, \tilde{a}\, \gamma_{\underline{k}_1}\gamma_{\underline{k}_3} + \ldots\\
\dot{\overline{\gamma}}_{\underline{k}_3} &= -\nu \underline{k}_3^2 \overline{\gamma}_{\underline{k}_3} - i(\underline{k}_2^2 - \underline{k}_1^2)\, \tilde{a}\, \gamma_{\underline{k}_1}\gamma_{\underline{k}_2} + \ldots
\end{aligned} \tag{2.2.29}$$

where $\tilde{a} = a(\underline{k}_1, \underline{k}_2, \underline{k}_3)/|\underline{k}_1|\,|\underline{k}_2|\,|\underline{k}_3|$.

These equations are analogous to those for the angular velocity of a solid with a fixed point. The analogy becomes even more clear in the variables $\omega_{\underline{k}} = \gamma_{\underline{k}}/|\underline{k}|$ which obey the equations

$$\underline{k}_1^2\, \dot{\overline{\omega}}_{\underline{k}_1} = -\underline{k}_1^4 \nu\, \overline{\omega}_{\underline{k}_1} + (\underline{k}_2^2 - \underline{k}_3^2)\, a\, i\, \omega_{\underline{k}_2}\omega_{\underline{k}_3} + \ldots \text{ etc} \tag{2.2.30}$$

We see also an interesting property: namely every triple or "*triad*" $\underline{k}_1, \underline{k}_2, \underline{k}_3$ such that $\underline{k}_1 + \underline{k}_2 + \underline{k}_3 = \underline{0}$ contributes to the equations (2.2.30) in such a manner that, if the $\underline{\gamma}_{\underline{k}}$ relative to the other values of $\underline{k}$ (different from

$\pm\underline{k}_1, \pm\underline{k}_2, \pm\underline{k}_3$) were zero, the equations would describe the motion (*with friction*) of a "complex" (because the $\omega_{\underline{k}}$ are complex quantities) gyroscope.

Hence Euler equations can be interpreted as describing infinitely many coupled gyroscopes, each associated with a *triad* such that $\underline{k}_1+\underline{k}_2+\underline{k}_3=\underline{0}$: they are not independent and their coupling is described by the constraint that, if a vector $\underline{k}$ is common to two triads, then the $\omega_{\underline{k}}$'s, thought of as components of one or of the other gyroscope, *must be equal* (because $\omega_{\underline{k}}$ depends only upon $\underline{k}$ and not on which of the (infinitely many) triads the vector $\underline{k}$ it is regarded to belong to).

The motions of a complex gyroscope are not as simple as those of the ordinary gyroscopes, not even in absence of friction ($\nu\equiv 0$), and we understand also from this viewpoint the difficulty that we shall meet in the qualitative analysis of the properties of the solutions of the equations.

Note, finally, that the *single* complex gyroscope (*i.e.* described by the equations relative to a single triad) may admit motions that can be interpreted as motions of a system of "real" gyroscopes, even though writing $\omega_{\underline{k}}=\rho_{\underline{k}}e^{i\vartheta_{\underline{k}}}$ with $\rho_{\underline{k}}, \vartheta_{\underline{k}}$ real one finds that in general the phases $\vartheta_{\underline{k}}$ are not constant.

It is indeed easy to see that if the phases of the initial datum have special values then the phases remain constant and the $\rho_{\underline{k}}$ obey equations that are exactly like those obeyed by the three components of the three angular velocities of an ordinary gyroscope (hence in absence of friction they can be integrated by "*quadratures*"). For instance this happens if $\vartheta_j\equiv-3\pi/2$, see also (4.1.27) in §4.1.

(F) Gyroscopic analogy in $d=3$.

A gyroscopic analogy is possible, *c.f.r.* [Wa90], also in the $d=3$ case and it is based on the same identities introduced between (2.2.26) and (2.2.29) and on the new notion of *elicity*. We sketch it quickly here, leaving the details to the interested reader. In the case $d=3$, with $\underline{g}=\underline{0}$, given $\underline{k}$ we introduce, [Wa90], two *complex* mutually orthogonal unit vectors $\underline{h}_s(\underline{k})$, $s=\pm1$, also orthogonal to $\underline{k}$

$$\underline{h}_{s,\underline{k}}=\underline{v}_0(\underline{k})+is\,\underline{v}_1(\underline{k}),\qquad s=\pm1 \tag{2.2.31}$$

where $\underline{v}_0,\underline{v}_1$ are two mutually orthogonal *real* unit vectors orthogonal to $\underline{k}$ and, furthermore, such that $\underline{v}_0(-\underline{k})=\underline{v}_0(\underline{k})$ and $\underline{v}_1(-\underline{k})=-\underline{v}_1(\underline{k})$. In this way $\overline{\underline{h}}_{s,\underline{k}}=\underline{h}_{-s,\underline{k}}=\underline{h}_{s,-\underline{k}}$. Suppose, moreover, that the three vectors $\underline{v}_0,\underline{v}_1,\underline{k}$ form a counterclockwise triple.

The basis $\underline{h}_{-1,\underline{k}},\underline{h}_{+1,\underline{k}}$ in R^3 will be called the *elicity base* and we shall say that the vector $\underline{h}_{s,\underline{k}}$ has elicity s. Then the Fourier components $\underline{\gamma}_\kappa$ of an arbitrary divergenceless velocity field $\underline{v}$ can be written as

$$\underline{\gamma}_{\underline{k}}=\sum_{s=\pm1}\gamma_{\underline{k},s}\,\underline{h}_{s,\underline{k}} \tag{2.2.32}$$

where $\gamma_{\underline{k},s}$ are scalar quantities such that $\gamma_{\underline{k},s}=\overline{\gamma}_{-\underline{k},s}$.

The NS–equations (2.2.10) become

$$\begin{aligned}&\dot{\overline{\gamma}}_{\underline{k}_3,s_3}=-\nu\underline{k}_3^2\,\overline{\gamma}_{\underline{k}_3,s_3}-\\&-i\sum_{\underline{k}_1+\underline{k}_2+\underline{k}_3=\underline{0}}\gamma_{\underline{k}_1,s_1}\gamma_{\underline{k}_2,s_2}\,[\underline{h}_{s_1,\underline{k}_1}\cdot\underline{k}_2]\,[\underline{h}_{s_2,\underline{k}_2}\cdot\underline{h}_{s_3,\underline{k}_3}]\end{aligned}\tag{2.2.33}$$

The expression $[\underline{h}_{s_1,\underline{k}_1}\cdot\underline{k}_2]\,[\underline{h}_{s_2,\underline{k}_2}\cdot\underline{h}_{s_3,\underline{k}_3}]$ can be studied by noting that the vector $e^{-is\mu}\underline{h}_s$ can be obtained by rotating clockwise by μ the basis $v_0(\underline{k}), s\underline{v}_1(\underline{k})$: this remark allows us to reduce the calculation of this product to the same calculation in the planar case ($d=2$).

Given the triangle $\underline{k}_1,\underline{k}_2,\underline{k}_3$ and the elicities s_1,s_2,s_3 and having established a Cartesian reference system on its plane, so that the triangle $\underline{k}_1\underline{k}_2\underline{k}_3$ is circled clockwise, we can find three angles μ_1,μ_2,μ_3 such that the clockwise rotation by $s_j\mu_j$ of the basis $\underline{v}_0(\underline{k}_j), s_j\underline{v}_1(\underline{k}_j)$ brings it into a basis $\underline{\tilde v}_0(\underline{k}_j),\underline{\tilde v}_1(\underline{k}_j)$ with $\underline{\tilde v}_0(\underline{k}_j)$ directed as the axis $\underline{k}_j^\perp$ orthogonal to $\underline{k}_j$ and lying in the plane of the triangle and with components (on this plane) $(-k_{j2},k_{j1})$ if $\underline{k}_j=(k_{j1},k_{j2})$. Then

$$[\underline{h}_{s_1,\underline{k}_1}\cdot\underline{k}_2]\,[\underline{h}_{s_2,\underline{k}_2}\cdot\underline{h}_{s_3,\underline{k}_3}]=e^{-i\tilde\mu}\frac{\underline{k}_1^\perp\cdot\underline{k}_2}{|\underline{k}_1|}\,\big(\frac{\underline{k}_2^\perp\cdot\underline{k}_3^\perp}{|\underline{k}_2|\,|\underline{k}_3|}-s_2s_3\big)\tag{2.2.34}$$

where $\tilde\mu=s_1\mu_1+s_2\mu_2+s_3\mu_3$; and we see that we can use the expressions already obtained in the case $d=2$. If a is defined by setting $a=-\underline{k}_1^\perp\cdot\underline{k}_2$ (twice the area of the triangle formed by the vectors $\underline{k}_j$) and if $\mu=-\frac{\pi}{2}+s_1\mu_1+s_2\mu_2+s_3\mu_3$ and $\omega_{s,\underline{k}}=|\underline{k}|^{-1}\gamma_{\underline{k},s}$ then by the identities noted in (2.2.27) and (2.2.28) one gets

$$\begin{aligned}&\underline{k}_1^2\,\dot{\overline{\omega}}_{\underline{k}_1,s_1}=-\underline{k}_1^4\nu\,\overline{\omega}_{\underline{k}_1,s_1}-\\&-(\underline{k}_3^2-\underline{k}_2^2+(\underline{s}\wedge\underline{\kappa})_1\kappa_1\,\sigma)\,a\,e^{-i\mu}\,\omega_{\underline{k}_2,s_2}\omega_{\underline{k}_3,s_3}+\ldots\ \text{etc}\end{aligned}\tag{2.2.35}$$

where $\sigma=s_1s_2s_3$ and $\underline{\kappa},\underline{s}$ are defined by $\underline{\kappa}=(|\underline{k}_1|,|\underline{k}_2|,|\underline{k}_3|)$ and $\underline{s}=(s_1,s_2,s_3)$: which shows that, once more, the equations can be written in terms of *triads* as in the case $d=2$. It is therefore still possible to give a "gyroscopic" interpretation to the Euler and NS equations.

The 2–dimensional equations can be obtained from the (2.2.35) simply by eliminating the labels s_j from the ω and setting $\sigma=0$, because in this case the vectors $\underline{v}_1$ have to be replaced by $\underline{0}$.

We see, furthermore, that if Δ denotes the triad $(\underline{k}_1,s_1),(\underline{k}_2,s_2),(\underline{k}_3,s_3)$ then the quantities

$$E=\frac{1}{6}\sum_\Delta\sum_{\underline{k},s\in\Delta}|\underline{k}|^2|\omega_{\underline{k},s}|^2,\qquad\Omega=\frac{1}{3}\sum_\Delta\sum_{\underline{k},s\in\Delta}|\underline{k}|^4|\omega_{\underline{k},s}|^2\tag{2.2.36}$$

are constants of motion in the case $\nu=0,\underline{g}=\underline{0}$ and $d=2$; in this case the index s should not be present, but we have used the three dimensional; notation for homogeneity purposes.

The first quantity is proportional to the kinetic energy and the second is proportional to the *enstrophy*[2] (*i.e.* to the total vorticity). The quantities E, Ω are sums of positive quantities and we shall see how this property will make them particularly useful in obtaining *a priori* estimates on the solutions of the Euler and NS equations.

Remark the mechanism by which the 2–dimensional fluids conserve energy and enstrophy: it is the same by which a solid with a fixed point conserves energy and angular momentum: here $\underline{k}_1^2$, $\underline{k}_2^2$, $\underline{k}_3^2$ play the roles of principal inertia moments.

In the corresponding case $d=3$ the energy E is still conserved (because $\underline{s}\wedge\underline{\kappa}\cdot\underline{\kappa}=0$) while, since in general $\sum_i \kappa_i^2(\underline{s}\wedge\underline{\kappa})_i\kappa_i\neq 0$, the Ω is no longer a constant of motion.

Nevertheless in the case $d=3$, always with $\nu=0, \underline{g}=\underline{0}$, there is another constant of motion because the identity

$$\begin{aligned}&\big(s_3|\underline{k}_3|(\underline{k}_3^2-\underline{k}_1^2)+\underline{k}_3^2(s_2|\underline{k}_2|-s_1|\underline{k}_1|)\big)+\\ &+\big(s_2|\underline{k}_2|(\underline{k}_1^2-\underline{k}_3^2)+\underline{k}_2^2(s_1|\underline{k}_1|-s_3|\underline{k}_3|)\big)+\\ &+\big(s_1|\underline{k}_1|(\underline{k}_3^2-\underline{k}_2^2)+\underline{k}_1^2(s_3|\underline{k}_3|-s_2|\underline{k}_2|)\big)\equiv 0\end{aligned}\tag{2.2.37}$$

together with (2.2.35) implies that

$$\tilde\Omega=\frac{1}{3}\sum_{\Delta}\sum_{\underline{k},s} s\,|\underline{k}|\,|\gamma_{\underline{k},s}|^2\tag{2.2.38}$$

is a constant of motion (as it can also be directly seen from the Euler equations by remarking that such quantity is proportional to $\int\underline{u}\cdot\underline{\partial}\wedge\underline{u}\,dx$).

However $\tilde\Omega$ cannot be directly used in a priori estimates because it is the sum of quantities with sign not defined.

Remarks:

(1) Note that there can be velocity fields in which all the components have elicity $s=1$ (or all $s=-1$); it then follows from the (2.2.35) that, given $K>0$, there exist solutions of the Euler equations having the form $\underline{u}(\underline{x})=\sum_{\alpha,|\underline{k}_\alpha|=K} e^{i\underline{k}\cdot\underline{x}}c_{\underline{k}}\underline{h}_+(\underline{k})$ where k_α denotes here the component α, $\alpha=1,2,3,$ of the vector $\underline{k}$ (which, therefore, has all the components with modulus equal to K). Note in fact that in this case it is $\underline{s}\wedge\underline{\kappa}\equiv 0$ besides $\underline{k}_i^2=3K^2$ hence $\underline{k}_i^2-\underline{k}_j^2=0$.
(2) If we consider the NS equation (*i.e.* $\nu\neq 0$) in absence of external field the solutions in (1) are either identically zero or vanish exponentially.
(3) In presence of an external field which also has Fourier components $\underline{g}_{\underline{k}}$ which do not vanish only for $\underline{k}_\alpha\equiv K$ one can find an exact, non zero, time independent solution.

[2] From $\varepsilon\nu$ ("inside") and $\sigma\tau\rho\varepsilon\varphi\omega$ ("turn around").

Problems: interior and boundary regularity of solutions of elliptic equations and for Stokes equation.

Here we mainly present the theory of the Laplace operator on divergenceless field in a bounded convex region Ω with smooth boundary. The key idea that we follow is to reduce the problem to the case of in which Ω is instead a torus, where the problem is easy. This is an intuitive and alternative approach with respect to the classical ones, see for instance [Mi70], [LM72], [Ga82].

[2.2.1]: (*weak solutions*) Define a function $x, t \to T(x,t)$ to be a *weak solution* with periodic boundary conditions on $[a,b]$, of the heat equation, (2.1.12), "in the sense of the periodic $C^\infty(a,b)$–functions belonging to a set $\mathcal{P}$ of such functions dense in $L_2([a,b])$" if the function T is in $L_2([a,b])$ and if furthermore

$$\partial_t \int_a^b \varphi(x)\,T(x,t)\,dx - \int_a^b \varphi''(x)\,T(x,t)\,dx = 0, \qquad \text{for all } \varphi \in \mathcal{P}$$

We say that the initial datum of a weak solution is ϑ_0 if it is: $\int_a^b \varphi(x)\,\vartheta_0(x)\,dx = \lim_{t\to 0}\int_a^b \varphi(x)\,T(x,t)\,dx$ for each $\varphi \in \mathcal{P}$.
For instance $\mathcal{P}$ can be the set of *all* $C^\infty([a,b])$ and periodic functions: we say, in such case, that T is a solution "*in the sense of distributions*" in the variable x on the circle $[a,b]$. If $\mathcal{P}$ is the space of the trigonometric polynomials periodic in $[a,b]$ we say that T is the solution in the sense of trigonometric polynomials on $[a,b]$.
We say that a sequence $f_n \in L_2([a,b])$ tends *weakly* to $f \in L_2([a,b])$ in the sense $\mathcal{P}$ if $\lim_{n\to\infty}\int_a^b \varphi(x) f_n(x)dx = \int_a^b \varphi(x) f(x)dx$ for each $\varphi \in \mathcal{P}$.
Show that the heat equation on $[a,b]$ admits a unique weak solution in the sense of trigonometric polynomials (hence in the sense of distributions) for a given initial datum $\vartheta_0 \in C^\infty(a,b)$.
Extend the above notions (when possible) to the case of functions on a d–dimensional bounded domain Ω. (*Idea:* Choose $a = -\pi, b = \pi$ (for simplicity) and write the condition that T is a solution by choosing $\varphi(x) = e^{i\omega x}$, with ω integer.)

[2.2.2]: (*weak solutions and heat equation*) Let ϑ_0 be a C^∞–function with support in $[-a,a]$, $a < L/2$. Show that the algorithm (2.1.23) for the heat equation on a circle of length L (identified with the segment $[-L/2, L/2]$) produces a solution that converges weakly in the sense of trigonometric polynomials to the solution of the heat equation with periodic boundary conditions on $[-L/2, L/2]$. Compare this result with [2.1.10], [2.1.11]. (*Idea:* Let $\hat\vartheta_0(\omega)$, with ω integer multiple of $2\pi L^{-1}$, be the Fourier transform of the initial datum; and note that the Fourier transform of the approximation at time $t = kt_0$, with $t_0 > 0$ and k integer, is $\hat\vartheta_k(\omega) = \hat\vartheta_0(\omega)(1 - \frac{c\omega^2 t}{k})^k$, *c.f.r.* §2.1. The weak convergence becomes equivalent to the statement that $\hat\vartheta_k(\omega)$ tends, for each fixed ω and and for $k \to \infty, t_0 \to 0$ (with $kt_0 = t$), to $\hat\vartheta_0(\omega)e^{-c\omega^2 t}$.)

[2.2.3]: (*Weak solutions ambiguities*) In the context of [2.2.1] we see that the algorithm of [2.2.2], *c.f.r.* (2.1.23), produces a sequence of functions $\hat\vartheta_k(\omega)$ with Fourier transform $\vartheta_{t_0}(x,t)$, $t \equiv kt_0$, periodic on $[-L/2, L/2]$ which, thought of as an element of $L_2([-L/2, L/2])$ converges weakly in the sense of the trigonometric polynomials to a solution $T(x,t)$ of the heat equation on $[-L/2, L/2]$. Note that this happens *for any* $L > a$: *i.e.* in a sense the algorithm produces different weak solutions depending on which is the length of the periodic bar that we imagine to contain the initial heat. Convince oneself that this is not a contradiction, and that on the contrary it is a useful example to meditate on the caution that has to be used when considering the notion of weak solution

[2.2.4]: (*extension of a function to a periodic function with control of its L_2 norm*) Given $\underline{u} \in C_0^\infty(\Omega)$ we can extend it to a C^∞–periodic function on a cube T_Ω with side

$L > 2\,\mathrm{diam}\,\Omega$ containing Ω and, as well, a translate Ω' of Ω such that $\Omega' \cap \Omega = \emptyset$. The extension can be done so that the extension vanishes outside Ω and Ω' and, furthermore, on the points of Ω' has a value *opposite* to the one that it has in the corresponding points of Ω. Then the extension, which will be denoted by $\underline{\tilde u}$, has a vanishing integral on the whole T_Ω. Show that, (by the definition of D, *c.f.r.* (2.2.15))

$$\int_\Omega |\underline{u}(\underline{x})|^2\,d\underline{x} \equiv \frac{1}{2}|\underline{\tilde u}|_2^2 \le \frac{L^2}{8\pi^2}D(\underline{\tilde u}) = \frac{L^2}{4\pi^2}D(\underline{u}) \qquad \text{per} \quad \underline{u} \in X^0_{\rm rot}$$

(*Idea:* Write the "norm", *i.e.* the square root of the integral of the square, of the extension of $\underline{u}$ to $L_2(T_\Omega)$ and the value of $D(\underline{\tilde u})$ by using the Fourier transform and remarking that $2||\underline{u}||^2_{L_2(\Omega)} \equiv ||\underline{u}||^2_{L_2(T_\Omega)}$, and treat in a similar way $D(\underline{\tilde u})$.)

[2.2.5]: (*a lower bound for the Dirichlet quadratic form of a solenoidal vector field*) Given a convex region Ω with analytic boundary $\partial\Omega$ consider the space $X_{\rm rot} \equiv \overline{X^0_{\rm rot}}$ closure in $L_2(\Omega)$ of the space $X^0_{\rm rot}$ of the divergenceless fields vanishing in a neighborhood of the boundary. Consider the quadratic form $(\underline{u},\underline{v})_D$ (called the "*Dirichlet form*") associated with the Laplace operator:

$$(\underline{u},\underline{v})_D = \int_\Omega \underset{\sim}{\partial u} \cdot \underline{\partial} \cdot \underset{\sim}{v}\,d\underline{x}, \qquad \text{and set} \qquad D(\underline{u}) = \int_\Omega (\underset{\sim}{\partial u})^2\,d\underline{x}$$

defined on $X^0_{\rm rot}$. Show that the greatest lower bound of $D(\underline{u})/|\underline{u}|_2^2$ on $X^0_{\rm rot}$ is strictly positive. Show that it is in fact $\ge (2\pi L^{-1})^2$ if L is twice the side of the smallest square containing Ω. (*Idea:* Note that the infimum is greater or equal to the infimum of $\int_Q (\underline{\partial}\,\underline{u})^2 / \int_Q \underline{u}^2$ over all C^∞ periodic fields $\underline{u}$ defined and with zero average on a square domain Q containing Ω. Indeed every function in $C^\infty_0(\Omega)$ can be extended trivially, see [2.2.4], to a periodic function $\underline{\tilde u} \in C^\infty(Q)$ and if Q has side L one can obviously also request that it has zero average (by defining it as opposite to $\underline{u}$ in the points of the "copy" Ω' of Ω that we can imagine contained in Q and without intersection with Ω): write then $D(\underline{\tilde u})$ by using the Fourier transform.)

[2.2.6]: (*bounding in L_2 a solenoidal field with the L_2 norm of its rotation*) Show that in the context of [2.2.4] it is

$$|\underline{u}|_2^2 \le \frac{L^2}{4\pi^2}\int_\Omega ({\rm rot}\ \underline{u})^2\,d\underline{\xi}, \qquad \text{in } X^0_{\rm rot}$$

(*Idea:* Make use again of the Fourier transform as in [2.2.4], [2.2.5] and note that $\underline{\partial}\cdot\underline{\tilde u} = 0$ implies that $|\underline{k}|^2|\hat{\underline{\tilde u}}(\underline{k})|^2 = |\underline{k} \wedge \hat{\underline{\tilde u}}(\underline{k})|^2$, and furthermore $|\underline{u}|_2^2 = \frac{1}{2}|\underline{\tilde u}|_2^2 \le \frac{1}{2}\frac{L^2}{4\pi}D(\underline{\tilde u}) = \frac{L^2}{4\pi}D(\underline{u})$.)

[2.2.7]: (*"compactness" of Dirichlet forms*) Let $\underline{u}_n \in X^0_{\rm rot}$ be a sequence such that $|\underline{u}_n|_2 = 1$ and $D(\underline{u}_n) \le C^2$, for some $C > 0$, and show the existence of a subsequence of $\underline{u}_n$ converging in L_2 to a limit. (*Idea:* Imagine $\underline{u}_n$ continued to a function $\underline{\tilde u}$ defined on T_Ω and changed in sign in Ω', as in [2.2.4], [2.2.5], [2.2.6]. Then the hint of [2.2.6] implies $|\hat{\underline{\tilde u}}_n(\underline{k})| \le \frac{C}{|\underline{k}|}\sqrt{\frac{2}{L^d}}$, with the convention (2.2.2) on Fourier transform. Let $\{n_i\}$ be a subsequence such that $\hat{\underline{\tilde u}}_{n_i}(\underline{k}) \xrightarrow[i\to\infty]{} \hat{\underline{\tilde u}}_\infty(\underline{k})$, $\forall\,\underline{k}$ (which exists because $\underline{k}$ takes countably many values); we see that given an arbitrary $N > 0$:

$$\begin{aligned}
&4|\underline{u}_{n_i} - \underline{u}_{n_j}|_2^2 \equiv |\underline{\tilde u}_{n_i} - \underline{\tilde u}_{n_j}|^2_{L_2(T_\Omega)} = L^d \sum_{\underline{k}} |\hat{\underline{\tilde u}}_{n_i}(\underline{k}) - \hat{\underline{\tilde u}}_{n_j}(\underline{k})|^2 \le \\
&\le L^d \sum_{|\underline{k}|\le N} |\hat{\underline{\tilde u}}_{n_i}(\underline{k}) - \hat{\underline{\tilde u}}_{n_j}(\underline{k})|^2 + \frac{L^d}{N^2}\sum_{|\underline{k}|>N} |\underline{k}|^2 |\hat{\underline{\tilde u}}_{n_i}(\underline{k}) - \hat{\underline{\tilde u}}_{n_j}(\underline{k})|^2 \qquad (2.2.39)\\
&\xrightarrow[i,j\to\infty]{} \le \frac{2}{N^2}\sup D(\underline{\tilde u}_{n_i}) \le \frac{4\,C^2}{N^2}
\end{aligned}$$

hence the arbitrariness of N implies that $\underline{u}_{n_i}$ is a Cauchy sequence in $L_2(T_\Omega)$, hence in $L_2(\Omega)$, which therefore converges to a limit $\underline{u}_\infty \in X_{\rm rot}$.)

[2.2.8]: (*existence of a minimizer for the Dirichlet form on solenoidal vector fields*) Let $\underline{u}_n \in X^0_{\rm rot}$, $n \ge 1$, be a sequence such that there is a $\underline{u}_0$ for which

$$|\underline{u}_n|_2 \equiv 1\,,\quad D(\underline{u}_n) \to \inf_{|\underline{u}|_2=1,\ \underline{u}\in X^0_{\rm rot}} D(\underline{u}) \equiv \lambda_0^2, \qquad |\underline{u}_n - \underline{u}_0|_2 \to 0 \tag{2.2.40}$$

Show that $D(\underline{u}_n - \underline{u}_m) \xrightarrow[n,m\to\infty]{} 0$ hence, since $\sqrt{D(\underline{u})}$ is a metric, $\underline{u}_0$ is in the "*domain of the closure of the quadratic form*",[3] *i.e.* there is also the limit $D(\underline{u}_0) = \lim_{n\to\infty} D(\underline{u}_n)$ and $D(\underline{u}_0) = \lambda_0^2$. In other words the quadratic form (extended to the functions of its domain reaches its minimum value in $\underline{u}_0$).
Note that if the "surface" $D(\underline{u}) = 1$ is interpreted as an "ellipsoid" in $L_2(\Omega)$ then the search of the above infimum is equivalent to the search of the largest $\underline{u}$ (in the sense of the L_2–norm) on the surface such that $D(\underline{u}) = 1$: this is $\lambda_0^{-1}\,\underline{u}_0$ so that $\underline{u}_0$ has the interpretation of direction of the largest axis of the ellipsoid and λ_0^{-1} that of its length.
Finally note that strictly analogous results can be derived for the quadratic form $D_1(\underline{u}) = D(u) + \int_\Omega |\underline{u}|^2\, d\underline{x}$. (*Idea:* Note the remarkable *quadrangular equality*:

$$D\left(\frac{\underline{u}_n + \underline{u}_m}{2}\right) + D\left(\frac{\underline{u}_n - \underline{u}_m}{2}\right) = \frac{D(\underline{u}_n)}{2} + \frac{D(\underline{u}_m)}{2} \xrightarrow[n,m\to\infty]{} \lambda_0^2$$

and note that $\frac12|\underline{u}_n + \underline{u}_m|_2 \xrightarrow[m,n\to\infty]{} 1$ and deduce that $\liminf_{n,m\to\infty} D\left(\frac{\underline{u}_n+\underline{u}_m}{2}\right) \ge \lambda_0^2$ and, *hence*, $\frac12 D(\underline{u}_n - \underline{u}_m) \to 0$.)

[2.2.9]: (*recursive construction of the eigenvalues of the Dirichlet form on solenoidal fields*) In the context of [2.2.8], define $\lambda_1 > 0$ as

$$\lambda_1^2 = \inf_{\substack{|\underline{u}|_2=1\\ \underline{u}\in X^0_{rot},\ \text{and}\ \langle \underline{u},\underline{u}_0\rangle_{L_2}=0}} D(\underline{u})$$

and show that there is a vector field $\underline{u}_1 \in L_2(\Omega)$ such that $D(\underline{u}_1) = \lambda_1^2$, then define λ_2^2 *etc.* In the geometric interpretation of [2.2.8] $\underline{u}_1$ is the direction of the next largest axis of the ellipsoid and λ_1^{-1} is its length. Show also that $(\underline{u}_0, \underline{u}_1)_D = 0$. (*Idea:* Repeat the construction in [2.2.8]. Then remark that if $\underline{w} = x\underline{u}_0 + y\underline{u}_1$ then $D(\underline{w}) \equiv x^2\lambda_0^2 + y^2\lambda_1^2 + 2(\underline{u}_0, \underline{u}_1)_D xy = 1$ is an ellipse in the plane x, y with principal axes coinciding with the x and y axes.)

[2.2.10]: (*minimax principle for solenoidal fields*) Consider the nondecreasing sequence $\lambda_j, j = 0, 1, 2, \ldots$ constructed in [2.2.9] and show the validity of the following "*minimax principle*":

$$\lambda_j^2 = \min\max D(\underline{u})$$

where the maximum is taken over the normalized vectors that are in a subspace W_j, with dimension $j+1$, of the domain of the closure (see footnote [3]) of $D(\underline{u})$, while the minimum

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[3] The domain of a quadratic form defined on a linear subspace $\mathcal{D}$ of a (real) Hilbert space H consists in the vectors u for which one can find a sequence $u_n \in \mathcal{D}$ with $||u_n - u_m|| \xrightarrow[n,m\to\infty]{} 0$ and $D(u_n - u_m) \xrightarrow[n,m\to\infty]{} 0$: in such case the sequence $D(u_n)$ converges to a limit ℓ and we set $D(u) \stackrel{def}{=} \ell$, and the set of such vectors u is called the *domain of the closure of the form* or simply the "domain of the form". If u, v are in the domain of the form D one can also extend the "scalar product" $(u, v)_D = \big(D((u + v)) - D(u) - D(v)\big)/2$.

is over the choices of the subspaces W_j. Find the simple geometric interpretation of this principle in terms of the ellipsoid of [2.2.8], [2.2.9]. Note that in the minimax principle we can replace the $j+1$–dimensional subspaces of the domain of the form with the $j+1$–dimensional subspaces of X^0_{rot}, provided we replace the minimum with an infimum. (*Idea:* The principle is obvious by [2.2.8] in the case $j = 0$ and it is an interpretation of [2.2.9] in the other cases.)

[2.2.11]: (*minimax and bounds on eigenvalues of the solenoidal Dirichlet form*) Consider the sequence $\lambda_j(\Omega)$, $j = 0, 1, \ldots$ constructed in [2.2.9],[2.2.10] and consider the analogous sequence associated with the quadratic form $D(\underline{u})$ defined on the fields $\underline{u}$ of class $C^\infty(T_\Omega)$ periodic and with zero divergence on the cube T_Ω, defined in [2.2.4]. Denoting $\lambda_j(T_\Omega)$ the latter, show that $\lambda_j(\Omega) \ge \lambda_j(T_\Omega)$ and $\lim_{j\to\infty} \lambda_j(\Omega) = +\infty$. Likewise we can get upper bounds on the eigenvalues $\lambda^2_j(\Omega)$ by comparing the quadratic form D with the corresponding one on T'_Ω, a cube contained in the interior of Ω with periodic boundary conditions. (*Idea:* Make use of the minimax principle of [2.2.10] and note that every function in $X^0_{\rm rot}(\Omega)$ can be extended to a function on T_Ω. For the limit note that $\lambda_j(T_\Omega)$ are explicitly computable via a Fourier transform. To obtain the lower bound we extend linearly and continuously in the C^p topology for each p the periodic functions on T'_Ω to functions $X^0_{rot}(\Omega')$ defined on a slightly larger domain Ω' with smooth boundary containing the cube T'_Ω and contained in Ω keeping control of the L_2 norms of $\underset{\sim}{\underline{\partial u}}$: see problems [2.2.33], [2.2.34] and [2.2.35] for more details.)

[2.2.12]: Show that if $\mathcal{H}$ is the closed subspace spanned (in $L_2(\Omega)$) by the vectors $\underline{u}_j$ then in the domain of the closure of the form D orthogonal to $\mathcal{H}$ there cannot exist $\underline{w} \neq \underline{0}$. (*Idea:* If $\underline{w}$ belonged to the domain of the closure of the form one would find, proceeding as in [2.2.8], a vector $\underline{u}$ such that $D(\underline{u}) = \inf D(\underline{w}) = \overline{\lambda}^2 < \infty$, with the infimum taken over all vectors in the domain of the form and orthogonal to $\mathcal{H}$: this contradicts that $\overline{\lambda}_j \to \infty$ as $j \to \infty$; in other words we would have forgotten one element of the sequence $\underline{u}_j$.)

[2.2.13]: (*heuristic equation for the eigenfunctions of the solenoidal Dirichlet form*) Write the condition that $D(\underline{u})$ is minimal in the space of the divergenceless C^∞–fields which vanish on the boundary of Ω and are normalized to 1 in $L_2(\Omega)$, assuming that the vector field $\underline{u}$ which realizes the minimum exists and is a C^∞–function. Show that it verifies

$$\Delta\underline{u} = -\lambda^2\underline{u} - \underline{\partial}\mu$$

where μ is a suitable function. (*Idea:* One can use the Lagrange multipliers method to impose the constraint $\underline{\partial} \cdot \underline{u} = 0$).

[2.2.14]: Show that the function μ in [2.2.13] can be determined via the projection $P_{X^\perp_{\rm rot}}(\Delta\underline{u})$, where $P_{X^\perp_{\rm rot}}$ is the projection operator discussed in (F) of §1.6 (*i.e.* it is the function whose gradient is the gradient part of the gradient–solenoid decomposition of the field $\Delta\underline{u}$).

[2.2.15]: Show that [2.2.13],[2.2.14] imply that we should expect that the vectors of the basis $\underline{u}_n$ satisfy the equations

$$\begin{aligned}\Delta\underline{u}_n =& -\lambda^2_n\underline{u}_n - \underline{\partial}\mu_n, \qquad \underline{\partial}\cdot\underline{u} = 0 \qquad \text{in } \Omega\\ \underline{u}_n =& \underline{0}, \qquad \text{in } \partial\Omega\end{aligned}$$

for a suitable sequence of potentials μ_n.

[2.2.16]: Let $f \in C^\infty([0, H])$, show that

$$|f(0)|^2 \le 2(H^{-1}||f||^2_2 + H||f'||^2_2)$$

(*Idea:* Write $f(0) = f(x) + \int_x^0 f'(\underline{\xi})\,d\underline{\xi}$, and average the square of this relation over $x \in [0,H]$, then apply Schwartz' inequality).

[2.2.17]: Let $f \in C_0^\infty(\Omega \times [0,H])$ be the space of the $f(\underline{x},z)$ of class C^∞ and vanishing if $\underline{x}$ is close to the boundary (assumed regular) $\partial\Omega$ of Ω. Show that

$$\int_\Omega |f(\underline{x},0)|^2\,d\underline{x} \le 2(H\int_\Omega\int_0^H |\underline{\partial}_{\underline{x}} f|^2\,d\underline{x}\,dz + H^{-1}\int_\Omega\int_0^H |f|^2\,d\underline{x}\,dz)$$

(*Idea:* Apply the result of [2.2.16]).

[2.2.18]: (*a "boundary trace" theorem*) Let $f \in C^\infty(\Omega)$, show that there is $C > 0$ such that, if L is the side of the smallest cube containing Ω:

$$\int_{\partial\Omega} |f(x)|^2\,d\sigma \le C(L^{-1}||f||_2^2 + L||\underline{\partial} f||_2^2)$$

This is an interesting "*trace theorem*" on the boundary of Ω (a "Sobolev inequality"), [So63]. The constant C can also be chose so that it i invariant under homotety (in the sense that dilating the region Ω by a factor $\rho > 1$ the constant C does not change and, therefore, C depends only on the geometric form of Ω and not on its size). (*Idea:* Make use of the method of partition of the identity in [1.5.7],[1.5.8] to reduce the present problem to the previous ones).

[2.2.19] (*Traces*) Let Ω be a domain with a bounded smooth manifold as boundary $\partial\Omega$: *i.e.* such that $\partial\Omega$ is covered by a finite number N of small surface elements each of which can be regarded as a graph over a disk δ_i tangent to $\partial\Omega$ at its center $\xi_i \in \partial\Omega$, so that the parametric equations of σ_i can be written $z = z_i(\xi)$, $\xi \in \delta_i$ and the points $\underline{x}$ of Ω close enough to σ_i can be parameterized by $\underline{x} = (\xi, z_i(\xi) + z)$ with $\xi \in \delta_i, z \ge 0$. Note that if Ω has the above properties also the homothetic domains $\rho\Omega$ with $\rho \ge 1$ have the same properties and the σ_i can be so chosen that $diam(\sigma_i) = cL$ if L is the diameter of Ω and N, c are the same for all domains $\rho\Omega$ with $\rho \ge 1$.
Let $f \in \mathcal{C}^\infty(\Omega)$ and define $\partial^{\underline{\alpha}} \equiv \frac{\partial^{|\underline{\alpha}|}}{\partial_1^{\alpha_1}\ldots\partial_d^{\alpha_d}}$ and

$$||f||^2_{W^n(\Omega)} = \sum_{j=0}^n L^{2j-d}\sum_{|\underline{\alpha}|=j}\int_\Omega |\partial^{\underline{\alpha}} f(\underline{x})|^2\,d\underline{x}$$
$$||f||^2_{W^n(\sigma_i)} = ||f_{\delta_i}||_{W^n(\delta_i)}, \qquad ||f||^2_{W^n(\partial\Omega)} = \max_i ||f_{\delta_i}||_{W^n(\delta_i)}$$

where $f_{\delta_i}(\xi) = f\Big(\underline{x}(\xi, z_i(\xi))\Big)$. Check that problem [2.2.18] implies

$$||f||_{W^{n-1}(\partial(\Omega))} \le \Gamma\,||f||_{W^n(\Omega)}, \qquad n \ge 1$$

and the constant Γ can be taken to be the same for all domains of the form $\rho\Omega$, $\rho \ge 1$

[2.2.20] (*A "trace" theorem*) Given a scalar function $f \in L_2(\Omega)$ suppose that it admits generalized derivative up to the order n included, with n even. Hence $(-\Delta)^{n/2}$ exists in a generalized sense, because $|\langle f, (-\Delta)^{n/2} g\rangle| \le C||g||_2$ for $g \in C_0^\infty(\Omega)$ and for a suitable C, *c.f.r.* §1.6, (1.6.19) and problem [2.2.1] above. Suppose $n > d/2$.
Show that f is continuous in every point in the interior of Ω, together with its first j derivatives if $n - d/2 > j \ge 0$. Find an analogous property for n odd.
(*Idea:* Let Q_ε be a cube entirely contained in Ω and let $\chi \in C_0^\infty(Q_\varepsilon)$. The function $\chi f \equiv f_\chi$ thought of as an element of $L_2(Q_\varepsilon)$ admits generalized derivatives of order $\le n$ and $(-\Delta)^{n/2} f_\chi$ exists in a generalized sense (this is clear if $n/2$ is an integer because $\chi\partial^p g = \sum_{j=0}^p \partial^j(\chi^{(j)}\,g)$ where $\chi^{(j)}$ are suitable functions in $C_0^\infty(Q_\varepsilon)$).
Hence thinking of $g \in C_0^\infty(Q_\varepsilon)$ as a periodic function in Q_ε we see that there is a

constant C_{Q_ε} such that the relation $|\langle f_\chi,(-\Delta)^{n/2}g\rangle|\le C_{Q_\varepsilon}||g||_{L_2(Q_\varepsilon)}$ holds for each $g\in C_0^\infty(Q_\varepsilon)$, and therefore it must hold for each periodic $C^\infty(Q_\varepsilon)$–function g. Then, if $\hat f_\chi(\underline k)$ is the Fourier transform of f_χ as element of $L_2(Q_\varepsilon)$ (so that $\underline k=2\pi\varepsilon^{-1}\underline m$ with $\underline m$ an integer components vector), we get $\varepsilon^d\sum_{\underline k}|\hat f_\chi(\underline k)|^2|\underline k|^{2n}\le C^2_{Q_\varepsilon}$. Hence, setting $n=\frac d2+j+\eta$ with $1>\eta>0$, we see that the Fourier series of $\partial^j f_\chi$ is bounded above by the series

$$\sum_{\underline k}|\underline k|^j|\hat f_\chi(\underline k)|\equiv\sum_{\underline k}|\underline k|^{j+\eta+d/2}|\hat f_\chi(\underline k)||\underline k|^{-\eta-d/2}\le$$
$$\le\Big(\sum_{\underline k}|\underline k|^{2n}|\hat f_\chi(\underline k)|^2\Big)^{1/2}\Big(\sum_{\underline k}|\underline k|^{-2\eta-d}\Big)^{1/2}\le$$
$$\le C_{Q_\varepsilon}\varepsilon^{-d/2}(\frac{\varepsilon}{2\pi})^{\eta+d/2}(\sum_{\underline m}|\underline m|^{-d-2\eta}\Big)^{1/2}\equiv\Gamma_d\varepsilon^{n-j-d/2}C_{Q_\varepsilon}$$

hence f has j continuous derivatives in Q_ε. If n is odd one can say, *by definition* that $(-\Delta)^{n/2}f$ exists if it is $|<f,\partial^j g>|\le C||g||_2$ for each derivative of order $j\le n$ and for each $g\in C_0^\infty(\Omega)$; then the discussion is entirely parallel to the one in the n even case.)

[2.2.21] (*An auxiliary trace theorem*) Let $W^n(\Omega)$ be the space of the functions $f\in L_2(\Omega)$ with generalized derivatives of order $\le n$ and define.
Show that the method proposed for the solution of [2.2.19] implies that, if $n>j+d/2$ and $d(\underline x,\partial\Omega)$ denote the distance of $\underline x$ from $\partial\Omega$, then

$$L^j|\partial^{\underline\alpha}f(\underline x)|\le(\frac{d(\underline x,\partial\Omega)}{L})^{-j-d/2}\,\Gamma\,||f||_{W^n(\Omega)},\qquad|\underline\alpha|=j$$

and Γ can be chosen to be independent of Ω. (*Idea:* Let Q_1 be the unit cube. Let $\chi_1\in C_0^\infty(Q_1)$ be a function identically equal to 1 in the vicinity of the center of Q_1. Let $\chi_\varepsilon(\underline x)=\chi_1(\underline x\varepsilon^{-1})$ and show that the constant C_{Q_ε} considered in the estimates of problem [2.2.19] can be taken equal to $\gamma L^{d/2}\varepsilon^{-n}||f||_{W^n(\Omega)}$, with γ independent from Ω. Then choose $\varepsilon=d(\underline x,\partial\Omega)$.)

[2.2.22] (*Trace theorem*) Infer from problem [2.2.19], [2.2.21] that if $j<n-1-d/2$ there is a constant Γ such that $\partial^{\underline\alpha}f(\underline x)$, $|\underline\alpha|=j$ with $\underline x\in\partial\Omega$ is continuous and

$$L^j|\partial^{\underline\alpha}f(\underline x)|\le\,\Gamma\,||f||_{W^n(\Omega)},\qquad|\underline\alpha|=j$$

(*Idea:* a point $\underline x\in\partial\Omega$ will be in some σ_i, *c.f.r.* problem [2.2.19], and at a distance $O(1)$ from its boundary. Then apply problem [2.2.21].) With a little extra effort one can obtain the "same" result under the weaker condition $j<n-d/2$.

[2.2.27] (*a first regularity property of the eigenfunctions of the Dirichlet form: scalar case*) Let $f_0\in L_2(\Omega)$, $f_0=\lim f_n$, $||f_n||_2=1$ and $D(f_n)\to\lambda_0^2$, where λ_0 is associated with the first of the above minimax problems (*c.f.r.* [2.2.10]). Show that the function f_0 admits first generalized derivatives and also the generalized derivative Δ and $-\Delta f_0=\lambda_0^2 f_0$. (*Idea:* For each $g\in C_0^\infty(\Omega)$ we get, setting $(f,g)_D\equiv\int_\Omega\underline\partial f\cdot\underline\partial g\,d\underline\xi$, that

$$|(f_n,g)_D-\lambda_0^2\langle f_n,g\rangle|\xrightarrow[n\to\infty]{}0\Rightarrow|(f_0,(-\Delta-\lambda_0^2)g|)\,=\,0$$

Hence $|(f_0,-\Delta g)|\le\lambda_0^2||g||_2$ and f_0 has a Laplacian in a generalized sense, by definition of generalized derivative, and $-\Delta f_0=\lambda_0^2 f_0$.)

[2.2.28]: (*smoothess of the lowest eigenfunction of the Dirichlet form*) Check that under the hypotheses of [2.2.27] the f_0 is in $C^\infty(\Omega)$ and $f_0=0$ on $\partial\Omega$. (*Idea:* From $(-\Delta f_0)=$

$\lambda_0^2 f_0$ it follows that $(-\Delta)^{n/2} f_0 = \lambda_0^n f_0$, for each $n > 0$ hence, by the results of [2.2.24], [2.2.26], the f_0 verifies the properties wanted. The vanishing on the boundary follows from the trace theorem, [2.2.18], and of the fact that all the functions approximating f_0 have by assumption value zero on $\partial\Omega$).

[2.2.29]: (*pointwise estimates on the derivatives of the lowest eigenfunction of the Dirichlet form*) Show that [2.2.24] and the elliptic estimates in [2.2.26] allow us to estimate the derivatives of the eigenfunctions, normalized in L_2, f_0 as $||f_0||_{C^j(\Omega)} \le \Gamma\, L^{-d/2}\,(1 + (L\lambda_0)^{(j+d)})$ for all values of j.

[2.2.30]: (*pointwise estimates on the derivatives of other eigenfunctions of the Dirichlet form*) Show that, by the minimax principle, and by what has been seen in the above problems, what we have obtained for f_0 applies to the other eigenvectors generated, via the minimax principle, from the Dirichlet quadratic form $D(f)$. In particular: $||f_p||_{C^j(\Omega)} \le \Gamma\, L^{-d/2}\,(1 + (L\lambda_p)^{(j+d)})$ for all values of j, p.

[2.2.31]: (*pointwise upper estimates on the derivatives of other eigenfunctions of the solenoidal Dirichlet form*) Adapt the theory of the quadratic form $D(f)$ on the scalar fields in $f \in C_0^\infty(\Omega)$ to the theory of the form $D(\underline{u})$ in (2.2.15) on the space $X^0_{\rm rot}(\Omega)$ and deduce the theorem that leads to (2.2.16),(2.2.17). (*Idea:* Let $\underline{f}_0 \in L_2(\Omega)$, $\underline{f}_0 = \lim \underline{f}_n$, $\underline{f}_n \in X^0_{rot}(\Omega), ||\underline{f}_n||_2 = 1$ and $D(\underline{f}_n) \to \lambda_0^2$, where λ_0 is associated with the first of the above minimax problems (*c.f.r.* [2.2.10]). Show that the function $\underline{f}_0$ admits a generalized Laplacian Δ and $-\Delta\underline{f}_0 = \lambda_0^2\underline{f}_0 + \underline{\partial}\mu_0$ with $\underline{\partial}\mu_0 \in L_2(\Omega)$. (*Idea:* For each $\underline{g} \in C_0^\infty(\Omega), \underline{g} \equiv \underline{g}_{rot} + \underline{\partial}\gamma$ we get, setting $(f, g)_D \equiv \int_\Omega \underline{\partial} f \cdot \underline{\partial} g\, d\underline{\xi}$, that

$$|(\underline{f}_n, \underline{g})_D - \lambda_0^2 \langle \underline{f}_n, \underline{g}\rangle| \xrightarrow[n\to\infty]{} 0 \Rightarrow |(\underline{f}_0, (-\Delta - \lambda_0^2)\underline{g}_{rot})| = 0$$

hence $|(\underline{f}_0, -\Delta\underline{g})| = \lambda_0^2 |(\underline{f}_0, \underline{g}_{rot})| \le \lambda_0^2 ||\underline{g}||_2$ (because $||\underline{g}_{rot}||_2 \le ||\underline{g}||_2$ as the solenoid gradient decomposition of $L_2(\Omega)$ is orthogonal, *c.f.r.* Sec. 1.6.5). Hence f_0 has a generalized Laplacian and $D\underline{f}_0 - \lambda_0^2\underline{f}_0 = \underline{0}$ in the space $\underline{X}_{rot}$: *i.e.* there is μ_0 such that $\underline{\partial}\mu_0 \in X_0^\perp(\Omega)$ and $-\Delta\underline{f}_0 - \lambda_0^2\underline{f}_0 = \underline{\partial}\mu_0$. Analogously one finds that $\Delta^n f_0 = \lambda_0^n + \underline{\partial}\mu^n$ with $\underline{\partial}\mu^n$ in $X^\perp_{rot}(\Omega)$ for all $n > 0$, *etc.*)

[2.2.32]: (*completeness of the eigenfucnctions of the Dirichlet form*): Show that the sequence $\underline{u}_j$ constructed in [2.2.9] is an orthonormal basis in $X_{rot}(\Omega)$ which, from [2.2.31], is such that $\underline{u}_j$ are $C^\infty(\Omega)$ functions with zero divergence and vanishing on the boundary $\partial\Omega$. (*Idea:* If there existed $\underline{w} \ne \underline{0}$ in $X^0_{rot}(\Omega)$ (*c.f.r.* [2.2.9]) but out of the linear span $\mathcal{H}$ closed in $X_{rot}(\Omega)$ (which here plays the role of L_2 in the scalar problems treated in the preceding problems) one could suppose it orthogonal to $\mathcal{H}$: indeed $\lambda_j^k(\underline{w}, \underline{u}_j) = (\underline{w}, (-\Delta)^k\underline{u}_j) = ((-\Delta)^k\underline{w}, \underline{u}_j)$ hence

$$|(\underline{w}, \underline{u}_j)| \le ||(-\Delta)^k\underline{w}||\,\lambda_j^{-k}, \qquad |\partial^r\underline{u}_j(x)| \le \Gamma L^{-d/2}(1 + (L\lambda_j)^{r+d})$$

where the second inequality follows from [2.2.29], [2.2.30] (adapted to the non scalar case as in [2.2.30]). Furthermore from the first of (2.2.17) we see that $(\underline{w}, \underline{u}_j)$ tend to zero faster than any power in j and, also, that the series $\underline{w}^{||} = \sum_j (\underline{w}, \underline{u}_j)\underline{u}_j(x)$ converges very well, so that its sum is in the domain of the closure of D (one verifies immediately that the sum converges in the sense of the norm $\sqrt{D(\underline{u})}$, so that by the footnote [2] we see that $\underline{w}^{\|}$ is in the domain of D). This means that $\underline{w} - \underline{w}^{||} = \underline{w}^\perp$ is a vector in the domain of the form D which does not vanish and which is orthogonal to the space spanned by the vectors $\underline{u}_j$: which is impossible by the remark in [2.2.12].)

[2.2.33]: (*estimates on the large order eigenvalues and eigenfunctions of the scalar Dirichlet form*) Show that $\lambda_j^2 \ge Cj^{2/d}$ for some $C > 0$, *i.e.* find lower bound similar to the upper bound in [2.2.11] to the eigenvalues of the quadratic form D in the scalar case. (*Idea:* Let $Q' \subset \Omega$ be a cube of side size $3L$. It contains 3^d cubes of size L. Let

Q be the one among them which is at the center and suppose that Q contains Ω. Let $u \in C^\infty(Q)$ be *periodic* over Q and imagine it extended to the whole Q' by periodicity and to the whole Ω by setting it to 0 outside Q'. Let χ_Q be a $C^\infty(\Omega)$ function that has value 1 on and near Q and vanishes near the boundary of Q' and outside Q'. Let $w = \chi_Q u \in C^\infty(\Omega)$.
Since $\underline{\partial} w = \underline{\partial}\, \chi_Q\, u + \chi_Q\, \underline{\partial}\, u$ we see that

$$\int_\Omega (\underline{\partial} w)^2 dx \le 2 \int_\Omega \Big((\underline{\partial}\chi_Q)^2 |u|^2 + \chi_Q^2 (\underline{\partial} u)^2 \Big)\, dx \le \gamma^{-1} \int_Q (u^2 + (\underline{\partial} u)^2)\, dx$$

for γ a suitable constant. Hence if u varies in a $j+1$–dimensional subspace $W \subset C^\infty(T)$ it is

$$\max_{u\in W} \int_Q (u^2 + (\underline{\partial} u)^2)\, dx \ge \gamma \max_{u\in W} \int_\Omega (\underline{\partial} w)^2\, dx$$

Taking the minimum over all $j+1$–dimensional spaces $W \subset C^\infty(T)$ we get $\Lambda_j^2(Q) \ge \lambda_j^2(\Omega)$ by the minimax principle [2.2.10], if $\Lambda_j^2(Q)$ are the eigenvalues associated with the quadratic form $\int_Q (u^2 + (\underline{\partial} u)^2)\, dx$ to which the same arguments and results (including the minimax principle) obtained above for the quadratic form $\int_Q (\underline{\partial} u)^2\, dx$ apply with the obvious changes, see [2.2.8]. The latter eigenvalues have the form $(2\pi L^{-1} \underline{m})^2$ where $\underline{m}$ is an arbitrary integer component vector, with multiplicity 2 for each $\underline{m}$, so that we get the inequality $\lambda_j^2(\Omega) \le C j^{2/d}$ and the inequality analogous to the first of (2.2.17) in the present scalar case follows from [2.2.29]. [2.2.30] and from the latter inequality; the analogue of the second of (2.2.17) follows from the above inequality and from [2.2.30]. Lower bounds on $\lambda_j(\Omega)$ can be obtained by the minimax principle by enclosing Ω in a cube $Q' \supset Q \supset \Omega$.)

[2.2.34]: (*Estimates on the large order eigenvalues and eigenfunctions of the scalar Dirichlet form*) Let Q, Q', χ_Q be as in [2.2.33]. Let $\underline{u}$ be a C^∞ divergenceless field periodic on Q. Then $\underline{u}$ can be represented on Q as $\underline{u} = \underline{a} + \text{rot}\, \underline{A}$ where $\underline{a}$ is a constant vector and $\underline{A}$ has zero divergence, is $C^\infty(Q)$ and is periodic on Q. We extend $\underline{A}$ to a $C^\infty(Q')$ divergenceless field by periodicity and set it $\underline{0}$ outside Q'. Then $\underline{w} = \text{rot}\, (\chi_Q \frac{1}{2} \underline{x} \wedge \underline{a} + \chi_Q \underline{A})$ extends the field $\underline{u}$ to a field in $X^0_{rot}(\Omega)$. Check that

$$\int_\Omega (\underline{\partial}\, \underline{w})^2 \le \gamma^{-1} \int_Q (\underline{u}^2 + (\underline{\partial}\, \underline{u})^2)\, d\underline{x}$$

for some $\gamma > 0$. This implies the relations in (2.2.17), by the same argument in [2.2.33] and by the fact that the eigenvalues of the quadratic forms $\int_Q (\underline{u}^2 + (\underline{\partial}\, \underline{u})^2)\, d\underline{x}$ and can be explicitly computed and shown to have the form $1 + (2\pi L^{-1} \underline{m})^2$ where $\underline{m}$ is an arbitrary integer component vector, with multiplicity 4 for each $\underline{m}$. (*Idea:* The inequality and the determination of γ can be easily performed by writing the relations in Fourier transform over Q of $\underline{u}, \underline{A}$.)

[2.2.35]: (*bounds on the multipliers μ_j of the eigenfunctions of the solenoidal Dirichlet form*) Show that also the potentials μ_j in [2.2.15] can be bounded by a bound like the second of (2.2.17) $|\partial^k \mu_j| \le c_k j^{\alpha + k/d}$ and estimate α, c_k. (*Idea:* simply use $-\underline{\partial} \mu_j = -\Delta \underline{u}_j + \lambda_j^2 \underline{u}_j$ and then use (2.2.17).)

Bibliography: the gyroscopic analogy in $d = 3$ is taken from [Wa90]; the theory of the elliptic equations and Stokes problem, is based upon [So63],

[Mi70]. For a classical approach to the Dirichlet and Neumann problems see also [Ga82].

§2.3 Vorticity algorithms for incompressible Euler and Navier–Stokes fluids. The $d = 2$ case.

So far we tried to set up "*internal approximation algorithms*": by this we mean algorithms in which one avoids (or tries to avoid) approximating the wanted *smooth* solutions with velocity fields that are singular or have high gradients.

The interest of such methods lies in the fact that the fluid motions considered, real or approximate, are always motions in which make the hypotheses underlying the microscopic derivation of the equations can be considered valid.

However one can conceive "*external approximation algorithms*", in which one uses approximations that violate the regularity properties of the macroscopic fields, assumed in deriving the equations of motion: the regularity properties (*necessary for the phyisical consistency of the models*) of the solutions *should* (therefore) be recovered only in the limit in which the approximation converges to the solution.

Certainly such a program can leave us quite perplexed; but it is worth examining because, in spite of what one might fear, it has given positive results in quite a few cases and, in any event, it leads to interesting mathematical problems and to applications in other fields of Physics.

There is essentially only one method and it relies on Thomson's theorem. We shall examine it, as an example, in the case of a periodic container with side size L.

Consider first $d = 2$. The divergence condition is imposed by representing the velocity field $\underline{v}$ as

$$\underline{u} = \underline{\partial}^\perp A \qquad \underline{\partial}^\perp = (\partial_2, -\partial_1) \tag{2.3.1}$$

with A a scalar (smooth, see §1.6), and the vorticity is also a scalar

$$\zeta = \mathrm{rot}\, \underline{u} = -\Delta A \tag{2.3.2}$$

so that $\underline{u} = -\underline{\partial}^\perp \Delta^{-1} \zeta$, *c.f.r.* (C) in §1.7.

The Euler equations ($\nu = 0$) or the Navier–Stokes ($\nu > 0$) equations can be written in terms of ζ:

$$\begin{cases} \partial_t \zeta + \underline{u} \cdot \underline{\partial} \zeta = \nu \Delta \zeta + \gamma \\ \underline{u} = -\underline{\partial}^\perp \Delta^{-1} \zeta \end{cases} \tag{2.3.3}$$

where $\gamma = \mathrm{rot}\, \underline{g}$.

We now assume that the initial vorticity field is singular, and precisely it is a linear conbination of Dirac's delta functions

$$\zeta_0(\underline{\xi}) = \sum_{j=1}^{n} \omega_j \delta(\underline{\xi} - \underline{\xi}_j^0) \tag{2.3.4}$$

i.e. we suppose that the vorticity is concentrated in n points $\underline{\xi}_1^0, \ldots, \underline{\xi}_n^0$ where it is singular and proportional to ω_j: which we take to mean

$$\oint_{\underline{\xi}_j^0} \underline{u}_0(\underline{x}) \cdot d\underline{x} = \omega_j \tag{2.3.5}$$

if the contour turns around point $\underline{\xi}_j^0$ excluding the other $\underline{\xi}$'s.

To find the velocity field corresponding to (2.3.4) we need the inverse of the Laplace operator Δ with periodic boundary conditions. The Green function G, kernel of $-\underline{\Delta}^{-1}$, with periodic boundary conditions has the form

$$G(\underline{\xi}, \underline{\eta}) \equiv \Delta^{-1}_{\underline{\xi}\,\underline{\eta}} = -\frac{1}{2\pi} \log |\underline{\xi} - \underline{\eta}|_L + G_L(\underline{\xi}, \underline{h}) \equiv G_0(|\underline{\xi} - \underline{\eta}|_L) + \Gamma_L(\underline{\xi} - \underline{\eta}) \tag{2.3.6}$$

where $G_0(\underline{\xi} - \underline{\eta}) \equiv -\frac{1}{2\pi} \log |\underline{\xi} - \underline{\eta}|$ is the Green function for the Laplace operator Δ on the whole plane and $|\underline{\xi} - \underline{\eta}|_L$ is the metric on the torus of side L defined by $|\underline{\xi} - \underline{\eta}|_L^2 = \min_{\underline{n}} |\underline{\xi}_i - \underline{\eta}_i - \underline{n}L|^2$; and Γ_L is of class C^∞ for $|\underline{\eta}_i - \underline{\xi}_i| \neq L$ and such that $G(\underline{\xi}, \underline{\eta})$ is L–periodic and C^∞ for $\underline{\xi} \neq \underline{\eta}$. See problems following [2.3.11] for a proof of this interesting property.

The function $\underline{u}^0 = -\partial^\perp \Delta^{-1} \zeta^0$ has singular derivatives (for instance rot $\underline{u}^0 = \sum_i \omega_i \delta(\underline{\xi} - \underline{\xi}_i)$) and therefore not only we are not in the situation in which it makes physically sense to deduce that the evolution of $\underline{u}^0$ is governed by the Euler equations but, worse, we even have problems at interpreting the equations themselves.

Consider the Euler equation: $\nu = 0$, and suppose that the external force γ vanishes. In reality the interpretation ambiguity is quite trivial, in a sense, because if we suppose that $\zeta(\underline{\xi}, t)$ has the form

$$\zeta(\underline{\xi}, t) = \sum_{i=1}^{n} \omega_i \delta(\underline{\xi} - \underline{\xi}_j(t)) \tag{2.3.7}$$

where $t \to \underline{\xi}_j(t)$ are suitable functions, then we can find a meaningful equation that has to be verified by $\underline{\xi}_j(t)$ in order to interpret it as a solution of the Euler equations.

Note that (2.3.7) would be consequence of vorticity conservation because it says that the vorticity is transported (see (1.7.14)) by the flow that generates it, *provided the initial value ζ^0 of ζ is regular*: since, however, ζ_0 is not regular we can think that (2.3.7) is part of the definition of solution of the

(2.3.3) which, strictly speaking, does not make mathematical sense when ζ is not regular.

By substitution of (2.3.7) into the Euler equations, (2.3.3) with $\nu = 0$, we find

$$\sum_{i=1}^{n} \omega_i \underline{\partial}\delta(\underline{\xi} - \underline{\xi}_i(r)) \cdot \dot{\underline{\xi}}_i - \sum_{j=1}^{n} \omega_j \underline{\partial}^{\perp} G(\underline{\xi}, \underline{\xi}_j(t)) \cdot \sum_{p=1}^{n} \omega_p \underline{\partial}\delta(\underline{\xi} - \underline{\xi}_p(t)) = 0 \quad (2.3.8)$$

i.e. setting $\underline{\xi} = \underline{\xi}_i(t)$ we get

$$\dot{\underline{\xi}}_i = \underline{\partial}^{\perp}_{\underline{\xi}_i} \sum_{j=1}^{n} \omega_j G(\underline{\xi}_i, \underline{\xi}_j) \equiv \underline{\partial}^{\perp}_{\underline{\xi}_i} \sum_{h \neq i}^{n} \omega_h G(\underline{\xi}_i, \underline{\xi}_h) + \underline{\partial}^{\perp}_{\underline{\xi}} \omega_i G(\underline{\xi}, \underline{\xi}_i)|_{\underline{\xi} = \underline{\xi}_i} \quad (2.3.9)$$

which has no meaning because the "*autointeraction*" term

$$\underline{\partial}^{\perp}_{\underline{\xi}} G(\underline{\xi}, \underline{\xi}_i)\Big|_{\underline{\xi} = \underline{\xi}_i} = -\frac{1}{2\pi} \frac{(\underline{\xi} - \underline{\xi}_i)^{\perp}}{|\underline{\xi} - \underline{\xi}_i|^2}\Big|_{\underline{\xi} = \underline{\xi}_i} \quad (2.3.10)$$

has no meaning.

However one can think that the components of $\underline{\partial}^{\perp}_{\underline{\xi}} G(\underline{\xi}, \underline{\xi}_i)|_{\underline{\xi} = \underline{\xi}_i}$ *are numbers having the limit value for* $\underline{\xi} \to \underline{\xi}_i$ *of an odd function of* $\underline{\xi} - \underline{\xi}_i$ *and that they can, therefore, be interpreted as zero.* Obviously this remark can only have a heuristic value and it cannot change the sad fact that (2.3.9) does not have mathematical sense.

Hence we shall *define* a solution of (2.3.3), with $\nu = \gamma = 0$ and initial datum (2.3.4), the (2.3.7) with $\underline{\xi}_j(t)$ given by the solution of the equation

$$\dot{\underline{\xi}}_j = \underline{\partial}^{\perp}_{\underline{\xi}_j} \sum_{n \neq j} \omega_n G(\underline{\xi}_j, \underline{\xi}_n) \quad (2.3.11)$$

which coincides with (2.3.9) deprived of the meaningless term.

The entire procedure can, rightly, look arbitrary and it is convenient to examine through which mechanisms one can imagine to approximate regular solutions to Euler equation ($\nu = \gamma = 0$ in (2.3.3)) via "solutions" of (2.3.11).

The idea is quite simple. A continuous vorticity field can be thought of as a limit for $\varepsilon \to 0$ of

$$\zeta^0_\varepsilon(\underline{x}) = \sum_i \zeta^0_i \, |\Delta_i| \, \delta(\underline{x} - \underline{x}_i) \quad (2.3.12)$$

where $\zeta^0_i = \zeta^0(\underline{x}_i)$ and the sum is over small squares Δ_i, and with sides ε and centered at points $\underline{\xi}_i$, of a pavement of T_L.

This means that for the purpose of computing the integrals $\int \zeta^0(\underline{x}) f(\underline{x}) \, d\underline{x}$, at least, one can proceed by computing

$$\lim_{\varepsilon \to 0} \int \zeta^0_\varepsilon(\underline{x}) f(\underline{x}) \, d\underline{x}, \qquad \text{for all } f \in C^\infty(T_L) \quad (2.3.13)$$

i.e., as it is usual to say, "the functions ζ^0_ε approximates *weakly* ζ^0" as $\varepsilon \to 0$.

Then we can hope that, letting the field (2.3.12) evolve as prescribed by (2.3.11), and defining in this way a singular vorticity field $\zeta_\varepsilon(\underline{x}, t)$, one has that

$$\lim_{\varepsilon\to 0} \zeta_\varepsilon(\underline{x}, t) = \zeta(\underline{x}, t) \tag{2.3.14}$$

exists *in the weak sense*, *i.e.* $\lim_{\varepsilon\to 0} \int \zeta_\varepsilon(\underline{x}, t) f(\underline{x})\, d\underline{x} = \int \zeta(\underline{x}, t) f(\underline{x})\, d\underline{x}$, for all functions $f \in C^\infty(T_L)$, and it is a regular function, if such was ζ^0 to begin with, verifying the (2.3.3) with initial datum ζ_0.

It is therefore very interesting that the latter statement is actually true if $\zeta_0 \in C^\infty(T_L)$, [MP84],[MP92]. This theorem is of the utmost interest because it shows the very "possibility" of external approximations to the solutions of the Euler equation.

The method can be suitably extended to the theory of the Navier–Stokes equations, $\nu \neq 0$, and to the forced fluid case $\gamma \neq 0$. For the moment we supersede such extensions and, instead, we study more in detail the most elementary properties of the equation (2.3.11).

One has to remark first that (2.3.11) can be put in *Hamiltonian form*; set $\underline{x}_i = (x_i, y_i)$

$$\begin{aligned} &p_i = \sqrt{|\omega_i|}\; x_i, \quad q_i = \frac{\omega_i}{\sqrt{|\omega_i|}} y_i, \quad (\underline{p}, \underline{q}) \equiv (p_1, q_1, \ldots, p_n, q_n) \\ &H(\underline{p}, \underline{q}) = -\frac{1}{2} \sum_{h\neq k} \omega_h \omega_k G(\underline{\xi}_h, \underline{\xi}_k) \end{aligned} \tag{2.3.15}$$

where $\underline{\xi}_h, \underline{\xi}_k$ have to be thought as expressed in terms of the (p_h, q_h), (p_k, q_k). Then we can check that (2.3.11) becomes

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \qquad \dot{q}_j = \frac{\partial H}{\partial p_j} \tag{2.3.16}$$

i.e. the (2.3.11) are equivalent to the system of Hamiltonian equations (2.3.16). If $L = \infty$, *i.e.* for the equations in the whole space, H is

$$H(\underline{p}, \underline{q}) = \frac{1}{8\pi} \sum_{h\neq k} \omega_h \omega_k \log\Big(\Big| \frac{p_h}{\sqrt{|\omega_h|}} - \frac{p_k}{\sqrt{|\omega_k|}} \Big|^2 + \Big| \frac{q_h}{\sigma_h \sqrt{|\omega_h|}} - \frac{q_k}{\sigma_k \sqrt{|\omega_k|}} \Big|^2 \Big) \tag{2.3.17}$$

where σ_i is the sign of ω_i.

We shall consider the case $L = \infty$ in more detail: but even in the latter case the equations are difficult to solve, except in the trivial case of the "*two vortices problem*" when

$$H = \frac{1}{4\pi} \omega_1 \omega_2 \log \Big[\Big(\tfrac{p_1}{\delta_1} - \tfrac{p_2}{\delta_2} \Big)^2 + \Big(\tfrac{q_1}{\delta_1} \sigma_1 - \tfrac{q_2}{\delta_2} \sigma_2 \Big)^2 \Big] \tag{2.3.18}$$

with $\delta_j = \sqrt{|\omega_j|}$ and the Hamilton equations become, if Δ is the argument of the logarithm,

$$\dot p_1 = -\frac{\omega_1\omega_2}{2\pi}\frac{\frac{\sigma_1}{\delta_1}\Big(q_1\frac{\sigma_1}{\delta_1}-q_2\frac{\sigma_2}{\delta_2}\Big)}{\Delta} \qquad \dot p_2 = +\frac{\omega_1\omega_2}{2\pi}\frac{\frac{\sigma_2}{\delta_2}\Big(q_1\frac{\sigma_1}{\delta_1}-q_2\frac{\sigma_2}{\delta_2}\Big)}{\Delta}$$
$$\dot q_1 = \frac{\omega_1\omega_2}{2\pi}\frac{\frac{1}{\delta_1}\Big(\frac{p_1}{\delta_1}-\frac{p_2}{\delta_2}\Big)}{\Delta} \qquad \dot q_2 = -\frac{\omega_1\omega_2}{2\pi}\frac{\frac{1}{\delta_2}\Big(\frac{p_1}{\delta_1}-\frac{p_2}{\delta_2}\Big)}{\Delta}$$

so that we see that $\sigma_1\delta_1 p_1 + \sigma_2\delta_2 p_2 =$ const, $\delta_1 q_1 + \delta_2 q_2 =$ const, *i.e.* in terms of the original coordinates

$$\omega_1 x_1 + \omega_2 x_2 = \text{cost}, \qquad \omega_1 y_1 + \omega_2 y_2 = \text{cost} \tag{2.3.19}$$

and if $\omega_1 + \omega_2 \neq 0$ we can define the *center of vorticity* x as

$$x = \frac{\omega_1 x_1 + \omega_2 x_2}{\omega_1+\omega_2}, \qquad y = \frac{\omega_1 y_1 + \omega_2 y_2}{\omega_1+\omega_2} \tag{2.3.20}$$

If $\omega_1/\omega_2 > 0$ the vorticity center can be interpreted as the "center of mass" of two points with masses equal to $|\omega_i|$: the faster vortex is closer to the vorticity center. If $\omega_1/\omega_2 < 0$ then the center of vorticity leaves both vortices located at points P_1 and P_2 "on the same side". The equations are solved by the motion in which the line P_1P_2 joining the two vortices rotates with angular velocity $(\omega_1+\omega_2)/(2\pi\Delta)$ counterclockwise, if $\Delta = (x_1-x_2)^2 + (y_1-y_2)^2$, around the vorticity center (*c.f.r.* problems). The distance $\sqrt{\Delta}$ has to be > 0 as we shall exclude initial data in which a pair of vortices occupy the same point.

If instead $\omega_1 + \omega_2 = 0$ and $\omega \stackrel{def}{=} \omega_1$, the two vortices proceed along two parallel straight lines perpendicular to the line joining them and with velocity $\omega/(2\pi\sqrt{\Delta})$, going to the right of the vector that joins P_2 to P_1 if $\omega > 0$ (and to the left otherwise), (*c.f.r.* problems).

In general the "problem of n vortices" with intensities $\omega_1,\ldots,\omega_n$, and *vanishing total vorticity* $\omega = \sum_i \omega_i = 0$, admits, if $L = \infty$, *four* first integrals

$$\begin{aligned} I_1 =& \sum_i \sigma_i\sqrt{|\omega_i|}p_i, \qquad I_2 = \sum_i \sqrt{|\omega_i|}q_i,\\ I_3 =& \frac{1}{2}\sum_i \sigma_i(p_i^2+q_i^2), \qquad I_4 = H(\underline{p},\underline{q}) \end{aligned} \tag{2.3.21}$$

the I_1, I_2, I_3 can be simply written in the original coordinates as

$$I_1 = \sum_i \omega_i x_i, \quad I_2 = \sum_i \omega_i y_i, \quad I_3 = \frac{1}{2}\sum_i \omega_i|\underline{\xi}_i|^2 \tag{2.3.22}$$

while I_4 is given by (2.3.17). Their constancy in time follows directly from the equations of motion in the coordinates $\underline{x}_i$, (2.3.11) with G given in

(2.3.6), by multiplying them with ω_i and summing over i, or multiplying them by $\omega_i \underline{x}_i$ and summing over i. If L is finite only I_1, I_2, I_4 are first integrals (recall that if L is finite we only consider periodic boundary conditions).

In general, however, such integrals are not in involution, in the sense of analytical mechanics, with respect to the Poisson brackets (that we denote with curly brackets as usual). With the exception of a few notable cases.
For instance $\{I_4, I_j\} = 0$ simply expresses that I_1, I_2, I_3 are constants of motion; while $\{I_1, I_2\} = 0$ only if $\sum_i \omega_i \equiv \omega = 0$, (because I_1 only depends on $\underline{p}$ and I_2 only depends on the $\underline{q}$, hence the calculation of the parenthesis is easy and one sees that it yields, in fact, ω); furthermore $\{I_3, I_2\} = I_1$ and $\{I_3, I_1\} = -I_2$.

On the basis of general theorems on integrable systems we must expect that also the three vortices problem *with vanishing total vorticity* be integrable by quadratures. And in fact this is a generally true property. All "*confined motions*" (*i.e.* such that the coordinates of the points stay bounded as $t \to \infty$) will in general be quasi periodic and the others will be reducible to superpositions of uniform rectilinear motions and quasi periodic motions: *c.f.r.* problems. Here the word "superposition" has the meaning of the classical nonlinear superposition that one considers in mechanics in the theory of quadratures and of quasi periodic motions, *c.f.r.* [Ga99b].
The interest of the condition $\omega = 0$ of zero total vorticity is that this condition must automatically hold if one requires that the velocity field generated by the vortices tends to 0 at ∞ quickly (*i.e.* faster than the distance away from the origin): the circulation at ∞ has indeed the value ω.
In reality also the general three vortices problem with $\omega \neq 0$, representing vorticity fields slowly vanishing at ∞, is integrable in general by quadratures, *c.f.r.* problems.

Concerning the four or more vortices problems one can show, by following the same method used by Poincaré to show the non integrability by quadratures of the three body problem in celestial mechanics, that the problem is in general *not integrable* by quadratures: it does not admit enough other analytic constants of motion, [CF88b].

Finally if one considers the Euler equations in domains Ω different from the torus and from R^2 one obtains the (2.3.11) with the Green function $G(\underline{\xi}, \underline{\eta})$ of the Dirichlet problem in Ω. In fact the boundary condition $\underline{u} \cdot \underline{n} = 0$ imposes, by (2.3.1), that the potential A must have *tangential* derivative zero on the boundary of Ω and therefore it must be constant and the constant can be fixed to be 0. Therefore $A = \Delta^{-1}\zeta$ where Δ is the Laplace operator with vanishing boundary condition.
In these cases, in general, only I_4 is a constant of motion: the case in which Ω is a disk is exceptional: because also I_3 is, by symmetry, a constant of motion and, therefore, in this case the two vortices problem is still integrable

by quadratures.

Problems: *Few vortices Hamiltonian motions. Periodic Green function.*

[2.3.1]: (*two vortices problem on a plane*) Show that the equations of motion for two vortices of intensity ω_1, ω_2 located at (x_1, y_1) and (x_2, y_2) are respectively

$$\begin{aligned} \dot x_1 =& -\omega_2\,(y_1-y_2)/2\pi\Delta, \qquad & \dot x_2 =& \;\omega_1\,(y_1-y_2)/2\pi\Delta \\ \dot y_1 =& \;\omega_2\,(x_1-x_2)/2\pi\Delta, \qquad & \dot y_2 =& -\omega_1\,(x_1-x_2)/2\pi\Delta \end{aligned}$$

and deduce that setting $\zeta = (x_1 - x_2) + i(y_1 - y_2)$ it is $\dot\zeta = \frac{i(\omega_1+\omega_2)}{2\pi|\zeta|^2}\zeta$. Derive from this the properties of the motions for the two vortices problem discussed after (2.3.20).

[2.3.2]: Suppose $\omega_1 + \omega_2 \neq 0$, $\omega_1, \omega_2 \neq 0$ and set $c^{-1} = \sqrt{|\omega_1 + \omega_2|}$ and $d^{-1} = \sqrt{|\omega_1^{-1} + \omega_2^{-1}|}$. Let σ_1, σ_2 be, respectively, the signs of ω_1, ω_2 and let ϑ_1, ϑ_2 be the signs, respectively, of $\omega_1 + \omega_2$ and of $\omega_1^{-1} + \omega_2^{-1}$. Show that the transformation $(p_1, p_2, q_1, q_2) \longleftrightarrow (p, p', q, q')$:

$$\begin{aligned} p =& (\sigma_1|\omega_1|^{1/2}p_1 + \sigma_2|\omega_2|^{1/2}p_2)\,c \qquad & q =& (|\omega_1|^{1/2}q_1 + |\omega_2|^{1/2}q_2)\,c\,\vartheta_1 \\ p' =& (p_1|\omega_1|^{-1/2} - p_2|\omega_2|^{-1/2})\,d \qquad & q' =& (\sigma_1 q_1|\omega_1|^{-1/2} - \sigma_2 q_2|\omega_2|^{-1/2})\,d\,\vartheta_2 \end{aligned}$$

is a canonical map. (*Idea:* Poisson brackets between the p, p', q, q' are canonical: check.)

[2.3.3]: (*integrability by quadratures of two vortices planar motions*) Note that in the coordinates (p, p', q, q') the Hamiltonian of the two vortices problem depends only on the coordinates (p', q') and it is integrable by quadratures in the region $(p', q') \neq (0, 0)$, at (p, q) fixed. Show that the action–angle coordinates (A, α) can be identified with the polar coordinates on the plane (p', q'):

$$A = \frac{1}{2}\,(p'^2 + q'^2), \qquad \alpha = \arg\,(p', q'), \qquad \text{and} \qquad H(p', q') = \frac{\omega_1\omega_2}{4\pi}\log A + cost$$

(*Idea:* The map $(p', q') \longleftrightarrow (A, \alpha)$ is an area preserving map, hence a canonical map.)

[2.3.4]: Show that the results of [2.3.2] and [2.3.3] can be adapted to the case $\omega_1 + \omega_2 = 0$ and study it explicitly. (*Idea:* For instance the map

$$p' = p_1 - p_2, \quad q = (q_1 + q_2), \quad p = (p_1 + p_2)/2, \quad q' = (q_1 - q_2)/2$$

is canonical and transforms H into $\frac{1}{4\pi}\log(p'^2 + q'^2) + const$. However the motions do not have periodic components corresponding, in the preceding cases, to rotations of the vortices around the vorticity center, which is now located at ∞.)

[2.3.5]: Show that, *c.f.r.* (2.3.21), $\{I_3, I_1\} = -I_2$, $\{I_3, I_2\} = I_1$ and $\{I_1, I_2\} = \sum_i \omega_i$.

[2.3.6]: (*the planar three vortices problem*) Given three vortices with intensity ω_1, ω_2, ω_3, with $\omega_j > 0$, consider the transformation $(p_1, p_2, p_3, q_1, q_2, q_3) \longleftrightarrow (p, p', p_3, q, q', q_3)$ of problem [2.3.2] (it is a canonical transformation in which the third canonical coordinates are invariant) and compose it with the transformation of the same type $(p, p', p_3, q, q', q_3) \longleftrightarrow (P, p', p'', Q, q', q'')$ in which (p', q') are invariant while the transformation on (p, p_3, q, q_3) is still built as in [2.3.2] by imagining that in (p, q) there is a vortex with intensity $\omega_1 + \omega_2 = \omega_{12}$ and in (p_3, q_3) there is a vortex with intensity ω_3. Show that in the new coordinates (P, p', p'', Q, q', q'') the Hamiltonian is a function only of (p', p'', q', q'') while I_3 is the sum of a function of (P, Q) only and of a function of (p', p'', q', q'') only.

[2.3.7]: (*integrability by quadratures of the planar three vortices problem*) Show, in the context of [2.3.6], that the P, Q are constants of motion and the surfaces $H = \varepsilon, I_3 = \kappa$ are bounded surfaces in phase space. Hence they are regular 2–dimensional surfaces (*i.e.* they do not have singular points or degenerate into lower dimensional objects) then their connected components are 2–dimensional tori ("Arnold–Liouville theorem"), see [Ar79], and motions on these tori are quasi periodic with two frequences. The three vortices problem, *i.e.* the determination of the motions of the two degrees of freedom system obtained by fixing the values of P, Q, is therefore integrable by quadratures if $\omega_i > 0$. *This does not go on*: the planar four vortices problem is not integrable by quadratures, [CF88b].

[2.3.8]: Show that, by using the result of [2.3.4], the analysis about the three vortices problem and its integrability by quadratures remains true in the case in which vorticities do not have all the same sign, but are such that $\omega_1 + \omega_2 \neq 0$ and $\omega_1 + \omega_2 + \omega_3 \neq 0$. This time, however, the surfaces $I_3 = \kappa, H = \varepsilon$ will not in general be bounded and, therefore, the invariant surfaces will have the form, in suitable coordinates, of a product of a space $R^1 \times T^1$, or R^2, or T^2 and such coordinates can be chosen so that the evolution is linear and motion will be quasi periodic only in the third case.

[2.3.9]: (*integrability by quadratures of two vortices in a disk*) Consider the two vortices problem in a circular region. Show that this is also integrable by quadratures. (*Idea:* This time I_1, I_2 are not constants of motion, but H, I_3 still are (the second because of the circular symmetry of the problem); furthermore all motions are obviously confined.)

[2.3.10]: (*integrability by quadratures of two vortices in a torus*) As in [2.3.9] but assuming that the two vortices are confined on a torus (rather than moving on the plane). Show that the two vortices move rectlinearly because the total vorticity *must* be 0. (*Idea:* Note that on the torus the velocity field $\underline{u}$ must be periodic and of the form $\underline{u} = \underline{\partial}^{\perp} A$ with a suitable regular A, so that we can only consider vortices with total vorticity zero. This time I_3 is not a constant of motion but I_1, I_2 are, furthermore they are in involution, namely $\{I_1, I_2\} = 0$ if $\{\cdot, \cdot\}$ denotes the *Poisson bracket.*)

[2.3.11]: (*quadratures for three vortices in a torus*) Show that also the three vortex problem on the torus will be integrable outside the level surfaces of H, I_1, I_2 which are not compact. (*Idea:* I_1 and I_2 are in involution, because the total vorticity vanishes on the torus, *c.f.r.* [2.3.10], and they are in involution with H. Then apply Arnold–Liouville theorem, *c.f.r.* [2.3.7].)

[2.3.12]: (*Green's function for periodic boundary conditions*) Consider $G_N(\underline{x} - \underline{y}) = \sum_{|\underline{n}| \le N} G_0(\underline{x} - \underline{y} - \underline{n}L) - \sum_{0 < |\underline{n}| \le N} G_0(\underline{n}L)$, where the sum runs over the integer components vectors $\underline{n} = (n_1, n_2)$ and G_0 is defined after (2.3.6). Check the existence of the limit $\lim_{N\to\infty} G_N(\underline{x} - \underline{y}) = G(\underline{x} - \underline{y})$, which is a periodic function of $\underline{x} - \underline{y}$ with period L in each coordinate, and which differs from $G_0(\underline{x} - \underline{y})$ by a C^∞–function of $\underline{x}, \underline{y}$ for $\underline{x} - \underline{y}$ small with respect to L.
Check that the only singularity of $G(\underline{\xi} - \underline{\eta})$ occurs at $\underline{\xi} = \underline{\eta}$. (*Idea:* Note that $G_0(\underline{x}) = -\frac{1}{2\pi} \log |\underline{x}|$ and $|\underline{\xi} - \underline{n}\, L| = |\underline{n}|\, L\, (1 + (-2\underline{n} \cdot \underline{\xi}\, L + \underline{\xi}^2)/(\underline{n}\, L)^2)^{1/2}$; and setting $\varepsilon = (-2\underline{n} \cdot \underline{\xi}\, L + \underline{\xi}^2)/(\underline{n}\, L)^2$ one has $-G_0(\underline{\xi} - \underline{n}\, L) = \frac{1}{4\pi} \log(1 + \varepsilon) + \frac{1}{2\pi} \log |\underline{n}| L$.
Developing in powers of ε the latter expression becomes $\frac{1}{4\pi}(\varepsilon - \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)) + \frac{1}{2\pi} \log |\underline{n}| L$.

So that $-G_0(\underline{\xi} - \underline{n}L)$ becomes:

$$\frac{1}{4\pi}\Big(-2\frac{\underline{n}\cdot\underline{\xi}\,L}{|\underline{n}|^2L^2} + \frac{\underline{\xi}^2}{|\underline{n}|^2L^2} - 2\frac{(\underline{n}\cdot\underline{\xi})^2L^2}{(|\underline{n}|^2L^2)^2} + O(|\underline{n}|^{-3}) + \frac{1}{2\pi}\log|\underline{n}|L\Big)$$

Summing over $\underline{n}$ we note that the terms linear in $\underline{\xi}$ add up to 0. Furthermore the terms in $(\underline{n}\cdot\underline{\xi})^2$ have the form $\sum_{i,j} n_in_j\xi_i\xi_j$ and by symmetry we get the same result if the latter sum is replaced by $\sum_i \xi_i^2 n_i^2$ and by symmetry between the components of $\underline{n}$ we get again the same result if we replace this by $\sum_i \xi_i^2\underline{n}^2 = 2\underline{\xi}^2\underline{n}^2$. This means that when summing over $\underline{n}$ the contributions from the terms quadratic in $\underline{\xi}$ cancel exactly. Thus summing over $\underline{n}$ gives the same result as summing

$$\sum_{|\underline{n}|<N}\Big[G_0(\underline{\xi}-\underline{n}L) - G_0(\underline{n}L) - \underline{\partial}G_0(\underline{n}L)\cdot\underline{\xi} - \frac{1}{2}\underset{\approx}{\partial}\,G_0(\underline{n}L)\cdot\underline{\xi}\,\underline{\xi}\Big]$$

which is a sum of terms of size $O(|\underline{n}|^{-3})$, which converges because the dimension is $d = 2$. And in fact the derivatives of order $\alpha \ge 0$ with respect to $\underline{\xi}$ of the sum above are expressed as sums of quantities which have size in $\underline{n}$ of order $O(|\underline{n}|^{-3-\alpha})$ so that the limit is C^∞ in the sense stated.)

[2.3.13]: Show that if $|\underline{x}-\underline{y}|_L^2$ is defined as $\sum_{i=1}^2(|x_i - y_i| \bmod L)^2$, *i.e.* if $|\underline{x}-\underline{y}|$ is the natural metric on the torus of side L, then the (2.3.6) holds with Γ_L of class C^∞ for $|x_i - y_i| \ne L$ on the torus.

[2.3.14]: (*the images method*) Show that the function $G(\underline{x}-\underline{y})$ is such that $\Delta_{\underline{y}}G(\underline{x}-\underline{y}) = \delta(\underline{x}-\underline{y})$: for this reason the construction in [2.3.12] is called the "*images method*" to construct the Green function of the laplacian with periodic boundary conditions. (*Idea:* It suffices to show this for $\underline{x} = (\frac{L}{2}, \frac{L}{2})$, because of the translation invariance of G.)

Bibliography: The theorem of external approximation, following (2.3.14), is taken from [MP84]; for systems integrable by quadratures see [Ar79], [Ga83], [Ga86].

§2.4 Vorticity algorithms for incompressible Euler and Navier–Stokes fluids. The $d = 3$ case.

In the 3–dimensional case the analogue of the point vortex is a closed oriented curve γ, that we shall call *filament*, on which rot $\underline{u} = \underline{\omega}$ is *concentrated* and is *tangent* to it, so that γ is a flux line for $\underline{\omega}$.

(A) Regular filaments. Divergences and infinities.

To understand the evolution of a vorticity filament consider the Euler equation in the form (1.7.3)

$$\dot{\underline{\omega}} + \underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{\omega} - \underset{\sim}{\omega}\cdot\underset{\sim}{\partial}\underline{u} = 0, \qquad \frac{d\underline{\omega}}{dt} = \underset{\sim}{\omega}\cdot\underset{\sim}{\partial}\underline{u} \tag{2.4.1}$$

It is easy to find the meaning of (2.4.1) as an equation of evolution for a curve γ if we look at a point $\underline{\xi} \in \gamma$ and at an infinitesimal element, or *vorticity element* $\alpha\,\underline{\omega}$ of the filament, with α infinitesimal. The evolution by transport by the fluid of the element between $\underline{\xi}$ and $\underline{\xi}+\alpha\underline{\omega}$ is

$$\begin{aligned}\underline{\xi}\to\underline{\xi}'&=\underline{\xi}+\underline{u}(\underline{\xi})\,dt\\ \underline{\xi}+\alpha\underline{\omega}\to\tilde{\underline{\xi}}'&=\underline{\xi}+\alpha\underline{\omega}+\underline{u}(\underline{\xi}+\alpha\,\underline{\omega})\,dt\end{aligned}\tag{2.4.2}$$

which implies that the arc of γ between $\underline{\xi}$ and $\underline{\xi}+\alpha\,\underline{\omega}$ evolves into the arc between $\underline{\xi}'$ and $\tilde{\underline{\xi}}'$ with

$$\tilde{\underline{\xi}}'-\underline{\xi}'=\alpha\,(\underline{\omega}+\underset{\sim}{\omega}\cdot\underset{\sim}{\partial}\,\underline{u}\,dt)=\alpha\,\underline{\omega}',\qquad\text{if}\quad\underline{\omega}'\stackrel{def}{=}\underline{\omega}+\underset{\sim}{\omega}\cdot\underset{\sim}{\partial}\,\underline{u}\,dt\tag{2.4.3}$$

This shows, by the second relation in (2.4.1), that the line element $\alpha\,\underline{\omega}$ evolves into $\alpha\,\underline{\omega}'$ while the line is transported by the current: hence γ remains always tangent to $\underline{\omega}$ and if the length of a line element of γ is changed, in the evolution of γ, by a factor $(1+\lambda\,dt)$ then $\omega'=\omega(1+\lambda\,dt)$ describes also the corresponding evolution of the modulus $\omega=|\underline{\omega}|$ of $\underline{\omega}$.

Hence the filament shape evolves simply because it is transported and deformed by the fluid. The vorticity, instead, changes *proportionally* to the expansion of the line element corresponding to it: if the line gets longer the vorticity increases.

Since vorticity is a zero divergence field, its flux is constant along its flux lines, in particular along γ; hence if γ is a vorticity filament it must be

$$\underline{\omega}(\underline{\xi})=\Gamma\,\delta_\gamma(\underline{\xi})\,\underline{t}_\gamma(\underline{\xi})\tag{2.4.4}$$

where $\underline{t}_\gamma(\underline{\xi})$ is the unit vector tangent to γ in $\underline{\xi}\in\gamma$ and $\delta_\gamma(\underline{\xi})$ is a uniform distribution concentrated on γ, defined by

$$\int f(\underline{\xi})\delta_\gamma(\underline{\xi})\,d\underline{\xi}\stackrel{def}{=}\int_\gamma f(\underline{\xi})\,dl\tag{2.4.5}$$

for each $f\in C^\infty$, if $d\,l$ is the line element for γ.

To check that Γ *is time independent* imagine the distributions δ_γ realized as (limit of) a function different from 0 in a infinitesimal tubular neighborhood $\mathcal{T}$ with cross–section, in $\underline{\xi}\in\gamma$, given by $s(\underline{\xi})$. Then, denoting $\chi_{\mathcal{T}}(\underline{\xi})$ the characteristic function of $\mathcal{T}$, it must be

$$\underline{\omega}(\underline{\xi})=\Gamma\chi_{\mathcal{T}}(\underline{\xi})\frac{1}{s(\underline{\xi})}\underline{t}_\gamma(\underline{\xi})\tag{2.4.6}$$

if $\underline{t}_\gamma(\underline{\xi})$ is the unit tangent vector to γ in $\underline{\xi}$.

Of course as the time varies the tube $\mathcal{T}$ is transformed into $\mathcal{T}'$ and the section of the tube contracts by $(1+\lambda\,dt)$ while the line element expands by $(1+\lambda\,dt)$ because the tube $\mathcal{T}$ evolution is by an incompressible transport. At the same time we know that vorticity varies by the same factor and, therefore, if $\underline{\xi}$ and $\underline{\xi}'$ are (via the evolution) corresponding points on the curve γ and on its image γ', we note that

$$\underline{\omega}'(\underline{\xi}') = \Gamma\,\chi_{\mathcal{T}'}(\underline{\xi}')(1+\lambda\,dt)\frac{1}{s(\underline{\xi})}\,\underline{t}'_{\mathcal{T}'} \equiv \Gamma\,\chi_{\mathcal{T}'}(\underline{\xi}')\frac{1}{s'(\underline{\xi}')}\,\underline{t}'_{\mathcal{T}'} \tag{2.4.7}$$

which, comparing with (2.4.6), implies that $\Gamma' \equiv \Gamma$.

The velocity field $\underline{u}$ associated with a vorticity filament can be computed via a formula often called *Biot–Savart formula* because it says that the velocity field of the vorticity field is the "*magnetic field*" of an electrric current of intensity Γ circulating on the filament as computed from the Biot-Savart law (units aside, of course)

$$\underline{u}(\underline{x}) = \frac{\Gamma}{4\pi}\oint_\gamma \frac{d\underline{\rho}\wedge(\underline{x}-\underline{\rho})}{|\underline{x}-\underline{\rho}|^3} \tag{2.4.8}$$

where $d\underline{\rho}$ is the line element of γ: *it is in fact the solution of* $\underline{\partial}\wedge\underline{u} = \underline{\omega}$ hence it is the magnetic field generated by the intensity of current $\underline{\omega}$ in (2.4.6).

Then the evolution of a system of several vorticity filaments should (naively) be described by

$$\frac{d\underline{\rho}}{dt} = \sum_{j=1}^n \frac{\Gamma_j}{4\pi}\oint_{\gamma_j} \frac{d\underline{l}\wedge(\underline{\rho}-\underline{l})}{|\underline{\rho}-\underline{l}|^3} \qquad \text{if} \quad \underline{\rho}\in\cup_{j=1}^n\gamma_j \tag{2.4.9}$$

because they should be transported by the flow $\underline{u}$.

One can then try to see if a generic vorticity field $\underline{\omega}$ is approximable by a family of many filaments γ with a small vorticity circulation Γ which, as an approximation parameter ε varies, should become denser and denser approximating better and better $\underline{\omega}$ in the sense that, for every fixed $f \in C^\infty(R^3)$

$$\lim_{\varepsilon\to 0}\int \underline{f}(\underline{x})\cdot\underline{\omega}_\varepsilon(\underline{x})\,d\underline{x} = \int \underline{f}(\underline{x})\cdot\underline{\omega}(\underline{x})d\underline{x} \tag{2.4.10}$$

Two are the difficulties of this "conjecture", which is suggested by the success of the analogous result in dimension $d = 2$ in §2.3. The most evident is, perhaps, that in this $d = 3$ case it is no longer possible to neglect the autointeraction of the filament. It is already so in the simple case of a circular filament of vorticity $\underline{u}$. Indeed at a point $\xi\in\gamma$, if R is the radius of the circle γ, it will be

$$\dot{\underline{\rho}} = \frac{\Gamma}{4\pi}\oint_\gamma d\underline{\rho}'\wedge\frac{\underline{\rho}-\underline{\rho}'}{|\underline{\rho}-\underline{\rho}'|^3} \tag{2.4.11}$$

showing that $\dot{\underline{\rho}}$ is orthogonal to the plane of the filament and it has size v

$$v = \frac{\Gamma}{4\pi}\int_0^{2\pi} 2R^2\,d\alpha\frac{\sin^2\frac{\alpha}{2}}{|(2R\sin\alpha/2)^3|} = \frac{\Gamma}{2^4\pi R}\int_0^{2\pi}\frac{d\alpha}{|\sin\alpha/2|} = \infty \tag{2.4.12}$$

More generally $\underline{u}$ diverges near every point of γ where there is a curvature $R^{-1} > 0$. Hence it has no meaning to consider the evolution of the filament.

A further difficulty, independent of the previous one, is that a generic solenoidal velocity field *does not have* a corresponding vorticity field whose flux lines are closed. Indeed in general the flux lines of the field $\underline{\omega} = \mathrm{rot}\,\underline{u}$, although they cannot "terminate", they will wander around densely filling regions of R^3 without ever closing (*c.f.r.* [1.6.20]). Hence it is not very natural to think of an arbitrary divergenceless velocity field as "well" approximated by fields with closed flux lines.

The latter is an aspect in which the 3–dimensional fluid is deeply different from a 2–dimensional one, in which instead an arbitrary vorticity field is naturally thought of as a limiting case of a field in which vorticity is concentrated in points.

Among the two difficulties the second looks less serious: after all it is a difficulty that can certainly be circumvented by contenting ourselves with approximations of $\underline{\omega}$ with a system of closed vorticity filaments in a sense weak enough and, of course, we are quite free *a priori* to define the meaning of the "approximation" as we wish. The more so as it is an "external" approximation which, therefore, can only be justified *a posteriori.*

The first difficulty is, however, almost "uneliminable".

A way to eliminate it could be to consider filaments so *irregular* to have an undefined tangent and, in fact, such that $d\underline{\rho}' \wedge (\underline{\rho} - \underline{\rho}')$ oscillates so strongly in sign and size to produce a finite result for the integral, (2.4.12), defining the velocity field on the filament points.

Alternatively we could imagine filaments with flux Γ "vanishing" in a sense to define so that the velocity in (2.4.12) is finite.

(B) Thin filament. Smoke ring.

We shall examine the second possibility first, and proceed heuristically to derive the equations of motion of a vorticity filament with "*evanescent*" vorticity, or "*thin filament*".

Given a regular closed curve γ let γ_δ be a tiny tube with radius δ centered around it: imagine that in γ_δ a vorticity field is defined and directed as the tangent $\underline{t}(\underline{x})$ to the curve parallel to γ through $\underline{x} \in \gamma_\delta$. Here it is not important to specify in which sense the tiny tube is filled by curves "parallel" to γ because the result will not depend on such details.

The vorticity $\underline{\omega}$ will therefore be $\Gamma_\delta \sigma_\delta(\underline{x})\underline{t}(\underline{x})$ where $\sigma_\delta(\underline{x})$ is a function that, in the direction perpendicular to γ, decreases in a regular way to 0 near the surface of the tiny tube. Moreover the integral over a section orthogonal to the tiny tube of $\sigma_\delta(\underline{x})$ is fixed to equal 1, so that the tiny tube is a flux tube of the vorticity field with flux Γ_δ.

The velocity field corresponding to the vorticity field $\underline{\omega}$ will be given by the Biot–Savart formula

$$\underline{u}(\underline{x}) = \frac{\Gamma_\delta}{4\pi}\int_{\gamma_\delta} d^3\underline{y}\,\sigma_\delta(\underline{y})\frac{\underline{t}(\underline{y})\wedge(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^3} \tag{2.4.13}$$

where $\underline{t}$ denotes the unit tangent vector, and the calculation leading to (2.4.12) tells us that if $\Omega = \Gamma_\delta \log\delta$ is kept fixed while $\delta \to 0$, then

$$\lim_{\delta\to 0}\underline{u}_\delta(\underline{x}) = \frac{\Omega}{4\pi}\frac{1}{R(\underline{x})}\underline{b}(\underline{x}) \tag{2.4.14}$$

if $\underline{b}(\underline{x})$ is the unit vector *binormal* (we recall that this is the unit vector orthogonal to the plane of the tangent and the normal) to the curve γ and $R(\underline{x})$ is the radius of curvature of the curve γ in $\underline{x}$.

We then say that the velocity field of a *thin filament* γ with intensity Ω is

$$\underline{u}(\underline{x}) = \frac{\Omega}{4\pi}\frac{1}{R(\underline{x})}\underline{b}(\underline{x}) \qquad \underline{x}\in\gamma \tag{2.4.15}$$

which is sometimes called the "smoke rings equation", because it is a model for the motion of smoke rings, as long as they remain thin and well delimited. A more appropriate name is the equation for the *motion by curvature* of the curve γ.

The simplest case is when γ is a circle of radius R. In such case the (2.4.15) tells us that the circle moves by uniform rectilinear motion orthogonally to its own plane, with velocity $\Omega/4\pi R$ oriented to see the flux on the circle proceed counterclockwise.

The general case can also be "*exactly*" studied: *i.e.* the motion of thin filaments is integrable by quadratures! This is a *very remarkable* result of Hasimoto, *c.f.r.* [Ha72],[DS94].

The key remark is that (2.4.15) implies that the curve moves *without stretching*: the arc length of the curve is invariant. This can be seen form (2.4.3) which shows that the stretching of a vector oriented as the line element is proportional to $\underset{\sim}{\omega}\cdot\underset{\sim}{\partial}\underline{u}$, *i.e.* to $\underline{t}\cdot\partial_s(R^{-1}\underline{b}) = 0$: because the derivative of the binormal unit vector with respect to the curvilinear abscissa is proportional to the normal unit vector, by Frenet formulae, (2.4.16).

The inextensibility of the curve γ during its evolution by curvature allows us to label its points by their curvilinear abscissa with origin on a prefixed point of the curve. During the evolution the points of γ will keep the same abscissa on γ.

It is then important to recall the *Frenet's formulae* that express, on a curve γ, how the three unit vectors $\underline{T} = (\underline{t},\underline{n},\underline{b})$, *tangent, normal and binormal* to it, change with the curvilinear abscissa in terms of the *radius of curvature* R and of the *torsion* τ as

$$\partial_s\begin{pmatrix}\underline{t}\\ \underline{n}\\ \underline{b}\end{pmatrix} = \begin{pmatrix}0 & R^{-1} & 0\\ -R^{-1} & 0 & -\tau\\ 0 & \tau & 0\end{pmatrix}\begin{pmatrix}\underline{t}\\ \underline{n}\\ \underline{b}\end{pmatrix} \tag{2.4.16}$$

where torsion and curvature are computed at the point of γ with curvilinear ascissa s, see problem [3.4.1] below.

If we set

$$\psi(s,t) = \frac{1}{R(s,t)} e^{i\sigma(s,t)} \qquad \sigma(s,t) = \kappa(t) + \int_0^s \tau(s',t)\, ds' \qquad (2.4.17)$$

where $\kappa(t)$ is a suitable function of t, then *Hasimoto's theorem* can be formulated as

Theorem *(Integrability of the motion by curvature): The function $\psi(s,t)$ satisfies the nonlinear Schrödinger equation*

$$i\frac{4\pi}{\Omega}\partial_t \psi = \partial_s^2 \psi + \frac{1}{2}|\psi|^2 \psi \qquad (2.4.18)$$

which is an equation which is integrable by quadratures.

Therefore, this tells us that R, τ vary, in a sense, quasi–periodically, *c.f.r.* [CD82].

The derivation of (2.4.18) from Frenet's formulae and from the equation of motion by curvature, (2.4.15), is discussed in the problems [2.4.1]%[2.4.3].

Once (2.4.18) is solved the curvature and the torsion at a generic point are known as functions of time on the inextensible curve γ. Therefore the Frenet formulae allow us to compute, always via a quadrature, the unit vectors $\underline{t}(s), \underline{n}(s), \underline{b}(s)$ as functions of time. Hence (2.4.15), with a further quadrature, will also give the actual positions in space of the points of γ as functions of their initial position (which plays the role of a label for the points of γ).

Hence the problem is "completely soluble" by quadratures. However this is not the appropriate place to discuss the qualitative features of the smoke ring motions: it is clear that the motions of thin filaments is not strongly related with the problem of an external algorithm for soving the Euler equations in $d = 3$ which would require considering filaments which are *not thin*, as it already appears from the analysis in (A).

Nevertheless the problem just discussed has some relation with the Euler equation. With the notations introduced above imagine $\delta > 0$ and $\Omega = \Gamma_\delta \log \delta$ fixed and consider the solution of the Euler equation with initial vorticity $\underline{\omega}_\delta(\underline{x})$. Then one can ask whether the following property is valid

Assuming that such solution exists for all times $t > 0$ denote it $\underline{u}_\delta(\underline{x}, t)$. Is then the limit: $\lim_{\delta\to 0} \underline{u}_\delta(\underline{x}, t) = \underline{u}(\underline{x}, t)$ existing? And, if yes, is such limit the solution of the Hasimoto equation of the curvature motion with initial curve γ?

Answering is difficult and, in general, the problem is open. However suppose that the curve γ is a circle, and γ_δ is a tube obtained as the region swept by a disk of radius δ centered at a point of γ and orthogonal to γ by letting the center glide on γ. Suppose also that initial vorticity field

$\underline{\omega}_\delta$ is everywhere perpendicular to this disk. Then the answer to the just posed questions is *affirmative*, see [BCM00]. This is interesting because it clarifies the meaning and the importance of the heuristic considerations at the beginning of this section (B).

(C) Irregular filaments: Brownian filaments.

We now go back to the problem of devising external algorithms for the $d = 3$ Euler equations posed in (A) above and we shall consider the case of an irregular curve. It is then possible that even the velocity given by (2.4.8), suitably interpreted (because the contour integral cannot be, *a priori*, condidered defined over irregular curves) is finite.

The latter possibility can be illustrated through a simple example in which one can see that a very irregular ("fractal") curve can generate a velocity field that is, in some sense, finite on the curve itself.

Consider a curve with parametric equations in cylindrical–polar coordinates (polar on the x, y–plane) is

$$r(\alpha) = R\,(1 + \varepsilon(\alpha)), \qquad z = R\,\vartheta(\alpha) \tag{2.4.19}$$

with $\varepsilon(\alpha), \vartheta(\alpha)$, for the time being, arbitrary.

Setting $\varepsilon(0) = \varepsilon_0, \vartheta(0) = \vartheta_0$, $\varepsilon = \varepsilon(\alpha), \vartheta = \vartheta(\alpha)$ and calling ε', ϑ' the derivatives of ε, ϑ in α and $s \equiv \sin\alpha$, $c \equiv \cos\alpha$ we see that

$$\begin{aligned} d\underline{\rho}' &= (-R(1+\varepsilon)s + R\varepsilon' c\,,\, R(1+\varepsilon)c + R\varepsilon' s\,,\, R\vartheta')\,d\alpha \\ \underline{\rho}' - \underline{\rho} &= (R(1+\varepsilon)c - R(1+\varepsilon_0)\,,\, R(1+\varepsilon)s\,,\, R(\vartheta - \vartheta_0)) \end{aligned} \tag{2.4.20}$$

hence, setting also $\eta = \varepsilon - \varepsilon_0$, $\mu = \vartheta - \vartheta_0$:

$$|\underline{\rho}' - \underline{\rho}|^2 = (2(1-c)(1 + \varepsilon + \varepsilon_0 + \varepsilon\varepsilon_0) + \eta^2 + \mu^2)R^2 \tag{2.4.21}$$

while the components of $d\underline{\rho}' \wedge (\underline{\rho}' - \underline{\rho})$ are immediately computed from (2.4.20) and are

$$\begin{aligned} &\frac{d\underline{\rho}'}{d\alpha} \wedge (\rho' - \rho) = \\ &= \begin{cases} R^2[(1+\varepsilon)c\mu + \varepsilon' s\mu - \vartheta'(1+\varepsilon)s], \\ R^2[\vartheta'(1+\varepsilon)(c-1) + \vartheta'\eta + \mu(1+\varepsilon)s - \mu\varepsilon' c], \\ R^2[-(1+\varepsilon)\eta - (1+\varepsilon)(1+\varepsilon_0)(1-c) + (1+\varepsilon_0)\varepsilon' s] \end{cases} \end{aligned} \tag{2.4.22}$$

Suppose that the curve is chosen as a sample in an ensemble of curves randomly drawn with a probability distribution such that, as α varies near $\alpha_0 = 0$, the quantities ε, ϑ are mutually independent and each is a random function with independent increments. This means in particular that we assume, for $\alpha_1 < \alpha_2 < \alpha_3$, that the quantity $\varepsilon(\alpha_2) - \varepsilon(\alpha_1)$ is, as a random variable, "very little" (see below) dependent from $\varepsilon(\alpha_3) - \varepsilon(\alpha_2)$, and suppose the analogous property on ϑ. Suppose, furthermore, for simplicity, that the fluctuations of ε,η satisfy, for α_1,α_2 near 0, a continuity property like for instance

$$\langle(\varepsilon(\alpha_2)-\varepsilon(\alpha_1))^2\rangle\le(D|\alpha_2-\alpha_1|)^{2-2a} \tag{2.4.23}$$

with $D,a>0$ (the parameter a is a measure of lack of regularity of the random curves); finally suppose that the large fluctuations have small probability (for instance bounded by a Gaussian function). In reality under the above hypotheses one expects that $a\equiv 1/2$ and that the distribution of the increments of $\varepsilon(\alpha)$ and $\vartheta(\alpha)$, for the considered values of α is necessarily Gaussian. Hence it is a distribution of the kind that one encounters in the theory of Brownian motion.

Intuitively we imagine that the random functions ε,ϑ are continuous with probability 1 but have, for α close to $\alpha_0=0$, increments between α_1 and α_2, proportional to $(D|\alpha_2-\alpha_1|)^{1/2}$: hence they are *not differentiable*.

We pose the problem of whether, at least, the *average velocity* of the curve in the point $\underline{\rho}$ corresponding to $\alpha=0$ is finite. Velocity is given by the integral (2.4.11) which, considering the (2.4.22), in the point $\alpha_0=0$ becomes

$$\begin{aligned}\underline{u}=&\frac{\Gamma}{4\pi R}\int\frac{d\alpha}{(\alpha^2+\eta^2+\mu^2)^{3/2}}\cdot\\&\cdot(\mu+\alpha\varepsilon'\mu-\vartheta'\alpha,-\vartheta'\alpha^2/2+\vartheta'\eta+\alpha\mu-\mu\varepsilon',-\eta-\alpha^2/2+2\varepsilon'\alpha)\end{aligned} \tag{2.4.24}$$

where we set $1+\varepsilon\simeq 1,\cos\alpha-1=-\alpha^2/2,\sin\alpha=\alpha$ for simplicity, imagining that
(1) ε and ϑ are small perturbations (although random) and
(2) taking into account that the convergence problems in the above analysis are due to what happens for $\alpha\simeq 0$.

The quantities ε',ϑ' suffer from interpretation problems because, by assumption, such derivatives have no meaning: but (2.4.24) and the formal expression $\varepsilon'(\alpha)=(\varepsilon(\alpha+\delta)-\varepsilon(\alpha))/\delta$, in the limit as $\delta\to 0$, shows that by the independence of the increments of ε and ϑ, the terms containing ε' and ϑ' can be considered as contributing zero to the average of (2.4.22), (2.4.24).

Discarding the terms that contain ε',η' we see that the only component of $\underline{u}$ that has nonzero average is the third, and that such component has average

$$\begin{aligned}v=&\frac{\Gamma}{8\pi R}\langle\int d\,\alpha\frac{-\alpha^2}{(\alpha^2+\eta^2+\mu^2)^{3/2}}\rangle=\\=&\frac{-\Gamma}{8\pi R}\int d\alpha d\eta d\mu\,\frac{\alpha^2}{(\alpha^2+\eta^2+\mu^2)^{3/2}}f_\alpha(\eta)g_\alpha(\mu)\end{aligned} \tag{2.4.25}$$

where $f_\alpha(\eta),g_\alpha(\mu)$ are the probability distributions of $\eta=\varepsilon(\alpha)-\varepsilon(0),\mu=\vartheta(\alpha)-\vartheta(0)$.

To take advantage, in a simple way, of the assumptions on the distributions of η, ϑ it is convenient setting

$$\eta = (D|\alpha|)^{1/2}\overline{\eta}, \qquad \vartheta = (D|\alpha|)^{1/2}\overline{\vartheta} \tag{2.4.26}$$

and suppose that the variables $\overline{\eta}, \overline{\vartheta}$ have an *α–independent, Gaussian, distribution*: this simplifies some formal aspects of the calculations. We find

$$\begin{aligned}|v| \le&\text{cost} \int d\alpha d\,\overline{\eta} d\,\overline{\mu}\, e^{-\overline{\eta}^2-\overline{\mu}^2} \frac{\alpha^2}{(\alpha^2 + D|\alpha|(\overline{\eta}^2+\overline{\mu}^2))^{3/2}} \simeq\\ \simeq&\text{cost} \int d\alpha\, x dx\, e^{-x^2} \frac{\alpha^2}{(\alpha^2 + D|\alpha|\, x^2)^{3/2}} \simeq\\ \simeq&\text{cost} \int \frac{\alpha^2\, d\alpha}{(\alpha^2)} < \infty\end{aligned} \tag{2.4.27}$$

This remark on the finiteness of $\underline{u}$ admits the following generalization. Imagine a vorticity filament with equations

$$\alpha \to \underline{\rho}(\alpha) + \underline{\xi}(\alpha) \tag{2.4.28}$$

with a C^∞ function $\underline{\rho}(\alpha)$ and with $\underline{\xi}(\alpha)$ sample of a random trajectory that, locally near every one of its points, is "essentially" a Brownian motion. The analysis leading to (2.4.27) can be extended to classes of curves that are periodic and with increments that are "very little" mutually dependent. An example of classes of curves with these properties is illustrated in the problems.[1]

Then the evolution equation for the generic point on the curve, which we shall label by $\underline{\rho}_0 + \underline{\xi}_0$ with $\underline{\rho}_0 = \underline{\rho}(\alpha_0)$, $\underline{\xi}_0 = \underline{\xi}(\alpha_0)$,*c.f.r.* (2.4.28), is written

$$\begin{aligned}\frac{d(\underline{\rho}_0 + \underline{\xi}_0)}{dt} =& -\frac{\Gamma}{4\pi} \int \frac{(d\underline{\rho}' + d\underline{\xi}') \wedge (\underline{\rho}' + \underline{\xi}' - \underline{\rho}_0 - \underline{\xi}_0)}{|\underline{\rho}' - \underline{\rho}_0 + \underline{\xi}' - \underline{\xi}_0|^3} \stackrel{\text{def}}{=}\\ \stackrel{\text{def}}{=}& -\frac{\Gamma}{4\pi} \int \frac{d\underline{\rho}' \wedge (\underline{\rho}' + \underline{\xi}' - \underline{\rho}_0 - \underline{\xi}_0)}{|\underline{\rho}' - \underline{\rho}_0 + \underline{\xi}' - \underline{\xi}_0|^3}\end{aligned} \tag{2.4.29}$$

where having eliminated the terms $d\underline{\xi}'$, that correspond to the terms with the derivatives of ε, ϑ in (2.4.24), is in a sense a step analogous to having eliminated, in $d = 2$, the autointeraction terms of the vortices (see (2.3.9)), and it is hopefully justified by what we have seen in the above particular case in which these autointeraction terms between the vortices gave formally a zero contribution to the average velocity.

[1] Note that we cannot assume that $\underline{\xi}(\alpha)$ it to be exactly a Brownian path with α as time variable because the increments cannot be really independent because the curve must, in the end, be closed. The precise meaning that is given to the motion $\underline{\xi}(\alpha)$ is discussed in detail in the problems following [2.4.4].

The equation (2.4.29) generates then two equations, one for the *average* $\underline{\rho}$ and one for the *fluctuations* $\underline{\xi}$

$$\begin{aligned}\frac{d\underline{\rho}}{dt} &= -\langle\frac{\Gamma}{4\pi}\int\frac{d\underline{\rho}'\wedge(\underline{\rho}'+\underline{\xi}'-\underline{\rho}-\underline{\xi})}{|\underline{\rho}'-\underline{\rho}+\underline{\xi}'-\underline{\xi}|^3}\rangle\\ \frac{d\underline{\xi}}{dt} &= -\frac{\Gamma}{4\pi}\int\frac{d\underline{\rho}'\wedge(\underline{\rho}'+\underline{\xi}'-\underline{\rho}-\underline{\xi})}{|\underline{\rho}'-\underline{\rho}+\underline{\xi}'-\underline{\xi}|^3}-\frac{d\underline{\rho}}{dt}\end{aligned} \tag{2.4.30}$$

To show the correctness of (2.4.30) should mean something similar to what can be shown in $d = 2$. But before proceeding we must stress that now the problem is rather more involved and an answer is not known.

Suppose that $\underline{x} \to \underline{\omega}(\underline{x}) \in C^\infty$ and that the flux lines of the vorticity feld $\underline{\omega}$ are *all closed*.[2] Imagine to cut orthogonally the flux lines by a surface and to pave the suface with small squares of side $\approx \lambda$ and let $\underline{x}_j$ be the center of the j-th square and let Γ_j^λ be the flux of $\underline{\omega}$ through the j-th square. Call γ_j the flux curve of $\underline{\omega}$ passing through $\underline{x}_j$. Then the vorticity field

$$\underline{\omega}^\lambda(\underline{x}) = \sum_j \Gamma_j^\lambda \delta_{\gamma_j}(\underline{x})\, \underline{t}_{\gamma_j}(\underline{x}) \tag{2.4.31}$$

approximates *weakly* the field $\underline{\omega}$ in the limit in which the size λ of the squares tends to 0.

We can now imagine to evolve the curves $\gamma_j + \underline{\xi}_j$, where $\underline{\xi}_j$ is a sample of a random motion "similar" to a Brownian motion but with periodic sample paths and with a suitable mean square dispersion D_j^λ (see problem [2.4.4] and following ones for an example) and compute the vorticity field at time t by using the (2.4.30).

We ask the question whether it is possible to determine D_j^λ so that the vorticity field at time t has a weak limit as $\lambda \to 0$, converging to a *regular* vorticity field solving the Euler equation with initial datum $\underline{\omega}$. It is by no means clear that this or something similar to this could be true.

Heuristically we may expect that by choosing $D_j^\lambda \equiv D^\lambda$, j–independent, and setting $\underline{\omega}^D(\underline{x}, t) = \lim_{\lambda\to 0}\underline{\omega}^\lambda$, then the limit $\lim_{D\to 0}\underline{\omega}^D \equiv \underline{\omega}(\underline{x}, t)$ should satisfy the Euler equations. This should hold *even if* the fluctuations $\underline{\xi}$ are fixed as time independent, thus leading us to consider just the first of the (2.4.30) as a closed system of equations (because now $\underline{\xi}$ has, by assumption, the same distribution at all times).

Hence from the first of the (2.4.30):

$$\frac{d\underline{\rho}}{dt} = -\langle\frac{\Gamma}{4\pi}\int\frac{d\underline{\rho}'\wedge(\underline{\rho}'+\underline{\xi}'-\underline{\rho}-\underline{\xi})}{|\underline{\rho}'-\underline{\rho}+\underline{\xi}'-\underline{\xi}|^3}\rangle \tag{2.4.32}$$

---

[2] Which is general is not true, even when $\underline{\omega}$ vanishes outside a bounded region, but which constitutes an interesting class of cases.

with $\underline{\xi}(t)$ defined by a time independent distribution (rather than by the second of the (2.4.30)), is an evolution equation that is interesting in itself, even if it turns out to be only indirectly related to the evolution problem for a vorticity filament. In fact to give a meaning to the motion of a filament of vorticity it is necessary to consider both the equations in (2.4.30) and we see that the same "average filament" evolves in a different way depending on the distribution of the initial fluctuations $\underline{\xi}$, *i.e.* depending on the actual structure that is assigned to the filament.

The (2.4.32) and (2.4.30) give a method to give a meaning to the evolution of a filament (and to the notion of filament itself) alternative to the one, also natural, of considering the filament as a tiny vorticity tube initially with a constant section and to follow its evolution. From the viewpoint of numerical simulations the (2.4.32) and (2.4.30) are somewhat simpler than the equations arising from considering the tiny tube model for the vorticity field, because the objects that are described are, respectively one and two–dimensional while the tubes are 3–dimensional.

We see in this way the generation of the idea of making even more "external" the approximation algorithm by using as vorticity filaments, rather than regular closed curves, very irregular curves like the samples of an ensemble of random curves with a probability distribution that assigns essentially independent increments to the coordinates of their points.

But wishing to avoid such radically "external" algorithms of solution of the Euler equations, which present to us obvious conceptual and computational difficulties, it would be necessary to give up using vorticity based algorithms that work so well in 2–dimensional fluids. Therefore abandoning them should be only a "last resrt" because the computational difficulties do not seem overwhelming, as proved by the existence of empirical solution methods for the NS equation (which is an equation of similar complexity), *c.f.r.* [Ch82], [Ch88].

(D) Irregular filaments: quasi periodic filaments.

Another road to pursue for an alternative generalization of the 2-dimensional vorticity algorithms can be obtained by concentrating the vorticity, rather than on closed lines, on lines that are *not* closed and fill densely 2–dimensional or even 3–dimensional surfaces.

Such velocity fields can be observed in real experiments, think for instance to real smoke rings that move in air.

If the filament lines are distributed densely on the surface of a 2–dimensional torus, for instance, or fill its interior, the rotation of $\underline{u}$ for each of them must be infinitesimal and only their density will make sense.

Consider, as an example, the case of a filament filling densely the surface of a torus $\mathcal{T}$ in the simple case in which $\mathcal{T}$ is a 2-dimensional torus and on $\mathcal{T}$ the flux line of $\underline{\omega}$ is a dense quasi periodic trajectory.

We imagine that the torus $\mathcal{T}$ is tangent to the x_1, x_2 plane at the origin and that it has there an external normal parallel to the x_3 axis. The torus will be represented parametrically as

$$\begin{aligned} x_1 =& X_1(\xi_1,\xi_2),\ x_2 = X_2(\xi_1,\xi_2),\ x_3 = X_3(\xi_1,\xi_2) \qquad \text{with} \\ X_1(\xi_1,\xi_2) =& \xi_1 + O(\underline{\xi}^2),\ X_2(\xi_1,\xi_2) = \xi_2 + O(\underline{\xi}^2),\ X_3(\xi_1,\xi_2) = O(\underline{\xi}^2) \end{aligned} \tag{2.4.33}$$

where the $\underline{\xi}$ are angles on a standard torus $\underline{\xi} \in [0,2\pi]^2 \stackrel{def}{=} \mathcal{T}$.

On $\mathcal{T}$ we imagine the curve φ with equations $s \to \underline{\xi}(s) = (s, \eta s)$, $0 \le s \le q_n$, with $\eta =$ irrational and $\eta = \lim_{n\to\infty} p_n/q_n$ where p_n and q_n are the "convergents" of the continued fraction for η(cf. problem [5.1.7] below) p.97). Note that the curve φ will fill densely the torus and that, therefore, if n is large the closed curve φ_n with equations $s \to \underline{\xi}(s) = (s, p_n s/q_n)$ "essentially draws" the torus.

Let $\underline{\tau}$ be the unit vector tangent at the origin to $\mathcal{T}$ in the direction of the curve φ and let $\underline{\nu}$ be the unit vector orthogonal to $\underline{\tau}$ and tangent to the torus at the origin. Consider a surface element $d\sigma = dh\, d\ell$ around the origin where $d\ell$ is the size in the direction of $\underline{\tau}$ and dh is the size in the direction of $\underline{\nu}$.

The sum of the lengths of the segments of the curve φ_n that are contained in $d\sigma$ will be $N = q_n d\sigma/S$, asymptotically in n, where S is a geometric constant (by the "ergodicity" of quasi periodic motions).

We imagine that a vorticity field $\underline{\omega}_n$ is concentrated on the curve φ_n and is parallel to it: so that $\underline{\omega}_n = \gamma q_n^{-1} \underline{\tau}\, \delta_{\varphi_n}(\underline{\xi})$ where δ_{φ_n} is a Dirac's delta distribution uniformly distributed along the curve φ_n. If we now let $n \to \infty$ we see that, for any smooth function $f(\underline{\xi})$, it is

$$\int f(\underline{\xi})\, \underline{\omega}_n(\underline{\xi})\, d\underline{\xi} \xrightarrow[n\to\infty]{} \int \underline{\omega}(\underline{\xi}) f(\underline{\xi})\, d\underline{\xi}, \qquad \underline{\omega}(\underline{\xi}) = \underline{\tau}(\underline{\xi}) \delta_{\mathcal{T}}(\underline{\xi})\, d\underline{\xi} \tag{2.4.34}$$

where $\delta_{\mathcal{T}}$ is a Dirac distribution is concentrated on the surface $\mathcal{T}$ and there proportional to the surface area (the proportionality constant is a function on $\mathcal{T}$ that depends on the actual shape of $\mathcal{T}$ (*i.e.* on the parametric equations $\underline{X}(\underline{\xi})$ in (2.4.32)).

We interpret $\underline{\omega}(\underline{\xi})$ as a vorticity field concentrated on the quasi periodic filament φ on $\mathcal{T}$.

This vorticity distribution induces a velocity $\underline{u}(\xi)$ which is finite. Indeed we can compute it at the origin (to fix the ideas) supposing first that the torus is flat in the vicinity of the origin; setting $\underline{\rho} = (l, h, z)$, $\underline{\tau} = (0, 1, 0)$, $\underline{\nu}=$ (normal to the torus at the origin), the contribution to the velocity by a neighborhood of size of order ε around the origin is, by the Biot–Savart formula (2.4.8), proportional to the (improper) integral

$$\int_{|l|<\varepsilon} \gamma \frac{\underline{\tau} \wedge \underline{\rho}}{|\underline{\rho}|^3} \delta(z)\, dh\, dz\, dl = \int_{|l|<\varepsilon} \gamma \frac{l \underline{\nu}\, dh\, dl}{(h^2 + l^2)^{3/2}} \equiv 0 \tag{2.4.35}$$

for small ε, by parity. In the general case in which the torus has curvature at the origin, or near it, this means that the Biot–Savart integral (which would be logarithmically divergent by "power counting") defining $\underline{u}$ is in fact (improperly) convergent.

If we really concentrate the vorticity on φ_n, then we see that the above interpretation of the limit as $n \to \infty$ simply results from the limit as $n \to \infty$ of the velocity field $\underline{u}_n$ *deprived* of a part that is improperly defined as $n \to \infty$: this is (again) analogous to the prescription in the $d = 2$ case of point vortices in which the deleted part (last term in (2.3.9)) could be interpreted as a rotation of the vortex around itself and with infinite angular speed.

Similarly if $\mathcal{T}$ is a domain bounded by a torus and densely filled by flux lines and if the vorticity on its interior can be considered distributed with a density $\underline{\omega}$ then the integral expressing the value of the field $\underline{u}$ is convergent.

There is, therefore, also the possibility of studying the evolution of a family of flux tubes of dimension 2 or 3 densely filled by one more filament by letting the latter evolve to be transported by the current lines generated by itself. Alternatively one can study cases in which the vorticity is concentrated on surfaces ("vorticity sheets") or in volumes ("tubes") and such as to approximate some smooth vorticity field. In such cases there would be no problem in giving a meaning to the Biot–Savart integral, at least at the initial time.

In practice the algorithm seems simpler in the case of a vorticity filament concentrated on a line dense on a surface, if compared to the case of a vorticity "sheet" concentrated on a surface. But the convergence problems of all the above algorithms are very little studied and only on a empirical (numerical) basis, [Ch82].

Problems.

[2.4.1]: (*time derivative of the principal frame on a curve*) Consider a closed moving curve γ. Show that the three orthogonal vectors $\underline{T} = (\underline{t}, \underline{n}, \underline{b})$ evolve so that there exist three functions A, B, C such that

$$\partial_t \begin{pmatrix} \underline{t} \\ \underline{n} \\ \underline{b} \end{pmatrix} = \begin{pmatrix} 0 & A & B \\ -A & 0 & C \\ -B & -C & 0 \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{n} \\ \underline{b} \end{pmatrix} \stackrel{def}{=} M\underline{T} \tag{2.4.36}$$

(*Idea:* Since the three vectors are orthonormal they must evolve as $t \to O(t)\underline{T}(t)$ where $O(t)$ is a rotation matrix. Then the matrix $\dot{O}(t) = MO(t)$ with M an antisymmetric matrix).

[2.4.2]: (*principal frame motion and Frenet relations*) Consider the motion by curvature, *i.e.* according to (2.4.15), of a curve (necessarily inextensible) γ. Then the points $\underline{x} = \underline{r}(s)$ are labeled by their curvilinear abscissa s and, therefore, as time varies their positions can be expressed via a function $\underline{x} = \underline{\rho}(s, t)$. Writing $\partial_s \underline{T} = F\underline{T}$ and $\partial_t \underline{T} = M\underline{T}$, the Frenet formulae (2.4.17) and the relations in [2.4.1], show that the identities $\partial_t\partial_s\underline{T} = \partial_s\partial_t\underline{T}$ imply the relations

$$\begin{aligned} \partial_t R^{-1} =& \partial_s A + B\tau \\ 0 =& \partial_s B - R^{-1}C - \tau A \\ \partial_t \tau =& -\partial_s C - R^{-1}B \end{aligned}$$

and, setting $\Omega' = \Omega/4\pi$, the $\partial_t \underline{\rho} = \underline{u} = \Omega' R^{-1} \underline{b}$ imply $\partial_s(R^{-1}\underline{b}) = \partial_t(\partial_s \underline{r})$, which in turn imply

$$A = \Omega' \frac{\tau}{R}, \quad B = \Omega' \partial_s R^{-1}$$

hence A, B, C are uniquely determined by R, τ. (*Idea:* For the first relations simply differentiate the (2.4.36) with respect to t and (2.4.16) with respect to s using (2.4.16) and, respectively, (2.4.36) to express the ∂_t and, respectively, the ∂_s of the unit vectors: one gets six relations each of the above ones being obtained twice. Proceed similarly for the second relations, by taking also into account that $\partial_s \underline{\rho} = \underline{t}$.)

[2.4.3]: (*Hasimoto's theorem*) Starting from the expressions in [2.4.2] for A, B, C check that the equations for R, τ are

$$\partial_t \begin{pmatrix} R^{-1} \\ \tau \end{pmatrix} = \begin{pmatrix} \partial_s A + B\tau \\ -\partial_s C - R^{-1} B \end{pmatrix} \qquad \partial_s B = R^{-1} C + \tau A$$

and check that, setting $\psi = R^{-1} e^{i\sigma}$, $\sigma = \kappa(t) + \int_0^s \tau(s')ds'$, $\kappa(t) = \int_0^t \big(2^{-1} R(0,t')^{-2} - C(0,t')\big)\, dt'$, the "Hasimoto identity" holds, *i.e.* the ψ satisfies the nonlinear Schrödinger equation, (2.4.18).

[2.4.4]: (*A gaussian process*) Consider the periodic functions in $L_2([0, 2\pi])$, $\alpha \to \varepsilon_N(\alpha)$ with zero average and with only N harmonics. These are the functions that can be expressed as: $\varepsilon_N(\alpha) = \pi^{-1} \sum_{k=1}^N (c_k \cos k\alpha + s_k \sin k\alpha)$. Define a probability distribution on the set of functions of the type considered, by assigning to the coefficients c_k, s_k the Gaussian distribution

$$\prod_{k=1}^N \frac{e^{-\frac{1}{2}(c_k^2+s_k^2)k^2} dc_k\, ds_k}{\sqrt{2\pi k^{-2}}^2}$$

and show that $\langle(\varepsilon_N(\alpha) - \varepsilon_N(\beta))^2\rangle = 2\,\pi^{-1} \sum_{k=1}^N k^{-2}(1 - \cos k(\alpha - \beta)) \equiv C_N(\alpha - \beta) < |\alpha - \beta|_{2\pi}$ where $|\alpha - \beta|_{2\pi} = \min_n |\alpha - \beta - 2\pi n|$ and also that $C_N(x) \xrightarrow[N\to\infty]{} \pi(|\alpha - \beta|_{2\pi} + O((|\alpha - \beta|_{2\pi}^2)^2)$. *The Gaussian process defined in [2.4.4] and discussed in problems following [2.4.4] has periodic sample paths: it differs therefore from the usual brownian motion. However the difference is quite trivial, see [IM65] p.21, problem 3.* (*Idea:* Note that the series limit of $C_N(x)$ as $N \to \infty$ is the Fourier series for the function $|x|_{2\pi} - |x|_{2\pi}^2/2$ in the interval $[-\pi, \pi]$.)

[2.4.5]: In the context of [2.4.4] show that the probability that $|\varepsilon_N(\alpha) - \varepsilon_N(\beta)|$ is larger than $\sqrt{\gamma C_N(\alpha - \beta)}$ is $2\int_\gamma^\infty e^{-\gamma^2} d\gamma/2/\sqrt{2\pi}$. (*Idea:* Note that $\varepsilon_N(\alpha) - \varepsilon_N(\beta)$ must have a Gaussian distribution with dispersion (or "width", or "covariance") $C_N(\alpha - \beta)$,because it is a linear combination of Gaussian random variables (the c_k, s_k).)

[2.4.6]: Show that the probability that, given two "dyadic" points $\alpha, \beta < \pi$ of order p, $\alpha = 2\,\pi\, h 2^{-p}$ and $\beta = \alpha + 2\,\pi\, 2^{-p}$ *adjacent* it is $|\varepsilon_N(\alpha) - \varepsilon_N(\beta)| > \gamma p 2^{-p/2}$ is estimated above by $P_p = c\, 2^p\, \gamma\, p\, \exp -\gamma^2 p^2/2$ for some constant $c > 0$. (*Idea:* The probability of the simultaneous validity of any number of events is bounded by the sum of the respective probabilities: hence the result follows immediately from the problem [2.4.5] because in this case the number of events is 2^p).

[2.4.7]: (*Wiener's theorem for brownian paths*) Given two dyadic points $\alpha = 2\pi h 2^{-p}$ and $\beta = 2\pi k 2^{-q}$, $\alpha, \beta < \pi$, not necessarily of the same order, show that, for instance, if $\alpha < \beta$ and $q < p$ there exists a sequence of n points $\alpha = x_1 \le x_2 \le x_3 \ldots \le x_n = \beta$ such that $x_{i+1} - x_i = 2^{-(p-i)}\sigma_i$ with $\sigma_i = 0, 1$ suitable. This means that x_i and x_{i+1} are either the same or adjacent "on scale" $2^{-(p-i)}$.
Deduce that the probability that $|\varepsilon_N(\alpha) - \varepsilon_N(\beta)| < \gamma\,|\alpha - \beta|^{1/2} \log|\alpha - \beta|^{-1}$ for *any pair* of dyadic points $\alpha, \beta < \pi$ can be estimated by $1 - C\gamma e^{-\gamma^2/2}$, for some constant $C > 0$ (Wiener theorem).

(*Idea:* First note that $(\beta-\alpha)/2\pi = (k2^{p-q}-h)/2^p$ so that expanding in base 2 the numerator $k2^{p-q}-h = \sum_j^{<p} n_j 2^j$ with $n_j = 0,1$ we get the representation $(\beta-\alpha)/2\pi = \sum_j \sigma_j 2^{-j}$, with $\sigma_j = 0,1$ and trivially related to the n_j.
Write $\varepsilon_N(\alpha)-\varepsilon_N(\beta) = \sum(\varepsilon_N(x_{i+1})-\varepsilon_N(x_i))$ and note that the probability that, *for all* the i, it is $|\varepsilon_N(x_{i+1})-\varepsilon_N(x_i)| < \gamma(p-i)2^{-(p-i)/2}$ is estimated, by the result of [2.4.6], by $1-C\gamma e^{-\gamma^2/2}$.
Moreover $\sum 2^{-(p-i)}\sigma_i \equiv |\beta-\alpha|/2\pi$ hence we get the inequality

$$\sum 2^{-(p-i)/2}(p-i)\sigma_i \le 12\,(|\beta-\alpha|/2\pi)^{1/2}\log_2 4\pi|\beta-\alpha|^{-1}$$

in fact if $\overline{p}$ is the smallest p_j for which $\sigma_j = 1$ it is $2^{-\overline{p}} \le |\beta-\alpha|/2\pi < 2^{-\overline{p}+1}$ and $\sum_j p_j\sigma_j 2^{-p_j/2} \le \sum_{m=\overline{p}}^{\infty} m2^{-m/2} \le 12\,\overline{p}2^{-\overline{p}/2}$ from which the latter inequality follows.)

[2.4.8]: Hence the random functions (defined on the dyadics) $\varepsilon_N(\alpha)$ are *uniformly* Hölder continuous, with exponent $\sim 1/2$, and "modulus of continuity"

$$\gamma = \sup_{\alpha,\beta} \frac{|\varepsilon(\alpha)-\varepsilon(\beta)|}{(|\alpha-\beta|_{2\pi}\log|\alpha-\beta|_{2\pi}^{-1})^{1/2}}$$

that is finite with a probability tending to 1 for $\gamma\to\infty$. One can then consider the limit as $N\to\infty$ of the probability distribution on the space $C([0,2\pi])$ of the continuous functions generated by the Gaussian distribution P_N introduced in [2.4.4]. The measurable sets will be defined by the set of functions that in m prefixed angles $\alpha_1,\ldots,\alpha_m$ take values in prefixed intervals $I_1,\ldots,I_m$. Such sets are called "cylinders", for obvious reasons, and they play the role analogous to that of the intervals in the theory of integration of functions of one variables. Furthermore, (in analogy to the ordinary integration theory) all sets approximable will be measurable that can be approximated via a denumerable sequence of operations of union, intersection and complementation on a denumerable collection of cylindrical sets. Check that the probability of each cylindrical set converges to a limit as $N\to\infty$. One can check (using Wiener theorem) that the measure thus constructed is completely additive (*i.e.* if a cylinder can be represented as a countable union of other cylinders then its measure is the sum of the measures of the cylinders that add up to it); and that the set of Hölder continuous functions is measurable and has probability 1. One defines in this way a probability distribution ("periodic Brownian motion") on the space of the continuous functions which are Hölder continuous (even with exponent 1/2).

Bibliography: The tiny filaments theory is mainly taken from [DS94] which discusses some remarkable integrable extensions; see also [Ha72] and the important extension to the theory of motion by curvature in the case of discrete curves, [DS95]. Other pertinent references are [BCM00], [Ne64], [CD82].