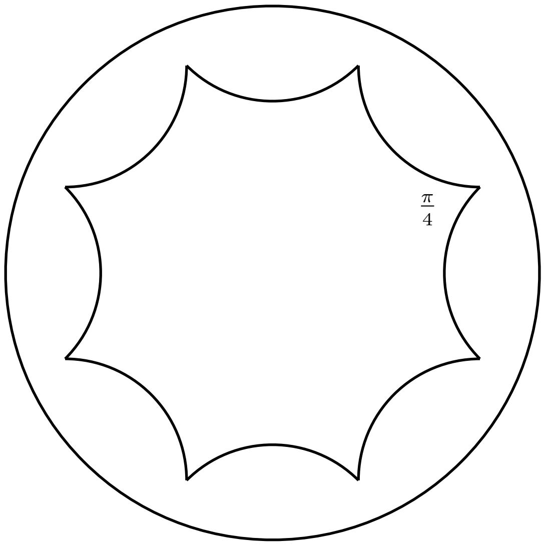
Foundations of Fluid Mechanics

Giovanni Gallavotti



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Cover: **Order and Chaos**, (free Author's reinterpretation of a well known geometrical object).

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Preface

The imagination is stricken by the substantial conceptual identity between the problems met in the theoretical study of physical phenomena. It is absolutely unexpected and surprising, whether one studies equilibrium statistical mechanics, or quantum field theory, or solid state physics, or celestial mechanics, harmonic analysis, elasticity, general relativity or fluid mechanics and chaos in turbulence.

So that when in 1988 I was made chair of Fluid Mechanics at the Università *La Sapienza*, not to recognize work I did on the subject (there was none) but, rather, to avoid my teaching mechanics, from which I could have a strong cultural influence on mathematical physics in Roma, I was not excessively worried, although I was clearly in the wrong place. The subject is wide, hence in the last decade I could do nothing else but go through books and libraries looking for something that was within the range of the methods and experiences of my past work.

The first great surprise was to realize that the mathematical theory of fluids is in a state even more primitive than I was conscious of. Nevertheless it still seems to me that a detailed analysis of the mathematical problems is essential for any one who wishes research into fluids. Therefore I dedicated (Chap.3) all the space necessary to a complete exposition of the theories of Leray, of Scheffer and of Caffarelli, Kohn and Nirenberg, taken directly from the original works.

The analysis is preceded by a long discussion of the phenomenological aspects concerning the fluid equations and their properties, with particular attention to the meaning of the various approximations. One should not forget that the fluid equations *do not have fundamental nature*, *i.e.* they ultimately are phenomenological equations and for this reason one "cannot ask from them too much". In order to pose appropriate questions it is necessary to dominate the heuristic and phenomenological aspects of the theory. I could not do better than follow the Landau–Lifshitz volume, selecting from it a small, coherent set of properties without (obviously) being able nor wishing to reproduce it (which, in any event, would have been useless), leaving aside most of the themes covered by that rich, agile and modern treatise, which the reader will not set aside in his introductory studies.

In the introductory material (Chaps.1,2) I inserted several modern remarks taken from works that I have come to know either from colleagues or from participating in conferences (or reading the literature). Here and there, rarely, there are a few original comments and ideas (in the sense that I did not find them in the accessed literature).

The second part of the book is dedicated to the qualitative and phenomenological theory of the incompressible Navier–Stokes equation: the lack of existence and uniqueness theorems (in three space dimensions) did not have practical consequences on research, or most of it. Fearless engineers write gigantic codes that are supposed to produce solutions to the equations: they do not care the least (when they are conscious of the problem, which unfortunately seems to be seldom the case) that what they study are *not* the Navier–Stokes equations, but just the informatic code they produced. *No one* is, to date, capable of writing an algorithm that, in an *a priori* known time and within a prefixed approximation, will produce the calculation of any property of the equations solution following an initial datum and forces which are not "very small" or "very special". Statements to the contrary are not rare, and they may appear even on the news: but they are wrong.

It should *not* be concluded from this that engineers or physicists that work out impressive amounts of papers (or build airliners or reentry vehicles) on the "solutions" of the Navier–Stokes equations are dedicating themselves to a useless, or risible, job. On the contrary their work is necessary, difficult and highly qualified. It is, however, important try and understand in which sense their work can be situated in the Galilean vision that wishes that the book of Nature be written in geometrical and mathematical characters. To this question I have dedicated a substantial part of the book(Chap. 4,5): where I expose *descriptive* or *kinematical* methods that are employed in the current research (or, better, in that part of the current research that I manage to have some familiarity with). These are ideas born in the seminal works of Lorenz and Ruelle–Takens, and in part based on stability and bifurcation theory and aim at a much broader and ambitious scope.

Chaotic phenomena are "very fashionable": a lot of ink flowed about them (and many computer chips burnt out) because they attract the attention even of those who like scientific divulgation and philosophy. But their perception is distorted because to make the text interesting for the nonspecialized public, often statements are made which are strong and ambiguous. Like "determinism is over", which is a statement that, if it has some basis of truth, certainly does not underline that nothing changes for those who cherish a deterministic conception of physical reality (a category to which all my colleagues and myself belong) or for those who did cherish it (like Laplace) when the "theory of chaos" was not, yet.

Hence in discussing chaotic properties of the simplest fluid motions I do not investigate at all philosophical themes, nor the semantic interpretation of the words illustrating objective properties. This is so in spite of the "light and non technical" appearance of this part of the book, which is in fact not light at all and it is *very* technical and collects a long sequence of steps, each of which is so simple not to require technical details.

I find it important that anyone is interested in science–related philosophical matters (in Greek times this encompassed all of philosophy but things have changed since; *c.f.r.* [BS98], [Me97]) should necessarily dedicate the time needed for a full understanding of the technical instruments (such as geometry, infinitesimal calculus and Newtonian physics) as already indicated by Galileo. It would be illusory to think one cpuld appreciate modern science without such instruments (*i.e.* "science", which is situated out of the elapsing time and which is called “modern” referring only to some of its “accidents”); divulgence is often terribly close to mystification.

The analysis is set with the aim of studying the initial development of turbulence, following the ideas of Ruelle–Takens, and, mainly, for the introduction and discussion of *Ruelle's principle.* This is a principle that, in my view, has not been appreciated as much as it could, perhaps for its “abstruse nature” or, as I prefer to think, for its originality. I became aware of it at a talk by Ruelle in 1973; I still recall how I was struck by the audacity and novelty of the idea. Since then I started to meditate on how it could lead to concrete applications; a difficult task. In the conclusive Chap. 7 I expose a few recent proposals of applications of the principle.

Section §6.1 is dedicated to the problem of the construction of invariant (*i.e.* stationary) distributions for the Navier–Stokes, equations: collecting from the literature heuristic ideas which seem to me quite interesting, even when far from physical or mathematical applications (or from the solution of the problem).

Kolmogorov “K41” theory cannot be absent in a modern text, no matter how introductory, and it is succinctly discussed in §6.2; while in §6.3 I describe some recent simulations which, in my view, have brought new ideas into the theory of fluids (multifractality): a selection of whose partiality I am aware and which is only partly due to space needs. Partly it is, however, a choice made because it concerns research done in he area of Roma and therefore is more familiar to me.

The last chapter contains several ideas developed precisely while I was teaching the fluid mechanics courses. Often I deal with very recent works which might have no interest at all in a few years from now. Nevertheless I am confident that the reader will pardon my temerity and consider it as a justifiable weakness at the end of a work in which I have limited myself only to classical and well–established results.

I tried to keep the book self-contained, not to avoid references to the literature (that is always present, apart from unavoidable involuntary omissions) but rather to present a unitary and complete viewpoint. Therefore I have inserted, in the form of problems with detailed hints for their solution, a notable amount of results that make the problems perhaps even more interesting than the text itself. I tried use problems with a guided solution to present results that could well have been part of the main text: they are taken from other works or summarize their contents. Students who will consider using the book as an introductory textbook on fluid mechanics should try to solve all the problems in detail, without having recourse to the quoted literature; I think that this is essential in order to dominate a subject that is only apparently easy.[1]

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[1] Among the problems one shall find a few classical results (like elementary tides theory) but also (1) phenomenology of nonhomogeneous chemically active continua, (2) Stokes's formula, (3) waves at a free boundary, (4) elliptic equations in regular domains (and the Stokes equation theory), (5) smoke–ring motions, (6) Wolibner–Kato theory for the 2–dimensional Euler equation theory, (7) potential theory needed for Leray's theory,

I wish to thank colleagues and students for the help they provided in correcting my notes. In particular I wish to thank Dr. Federico Bonetto and Dr. Guido Gentile. Some ideas that emerged during endless discussions have influenced the text particularly in the last few sections, and sometimes have avoided errors or unprecise statements.

Roma, 2001.

(8) Sobolev inequalities needed for the CKN theory, (9) several questions on numerical simulations, (10) some details on bifurcation theory, (11) a few comments on continued fractions and on the geodesic flows on surfaces of constant negative curvature, (12) the ergodic theorems of Birkhoff and Oseledec, (13) Lyapunov exponents for hyperbolic dynamical systems, (14) some information theory questions (for entropy). I think that until the last chapter, dedicated to more advanced themes, the only theorem used but not proved (not even with a hint to a proof) is the center manifold theorem (because I did not succeed developing a reasonably short self–contained proof, in spite of its rather elementary nature). Several theorems are hinted at by using a heuristic approach. This is because I find often missing in the literature the heuristic illustration of the ideas, which is generally very simple at least in the simplest nontrivial cases in which they usually were generated.

Indice

CHAPTER I

Continua and generalities about their equations

§1.1 Continua.

A *homogeneous continuum*, chemically inert, in d dimensions is described by

(a) A region Ω in ambient space ($\Omega \subset R^d$), which is the occupied volume.
(b) A function $P \to \rho(P) > 0$, defined on Ω, giving the *mass density*.
(c) A function $P \to T(P)$ defining the *temperature*.
(d) A function $P \to s(P)$ defining the *entropy density* (per unit mass).
(e) A function $P \to \underline{\delta}(P)$ defining the *displacement* with respect to a reference configuration.
(f) A function $P \to \underline{u}(P)$ defining the *velocity field*.
(g) An *equation of state* relating $T(P), s(P), \rho(P)$.
(h) A *stress tensor* $\underset{\approx}{\tau}$, also denoted (τ_{ij}), giving the force per unit surface that the part of the continuum in contact with an ideal surface element $d\sigma$, with normal vector $\underline{n}$, on the side of $\underline{n}$ exercises on the part of continuum in contact with $d\sigma$ on the side opposite to $\underline{n}$, via the formula

$$d\underline{f} = \underset{\approx}{\tau}\,\underline{n}\,d\sigma \qquad (\underset{\approx}{\tau}\,\underline{n})_i = \sum_{j=1}^{d} \tau_{ij} n_j \tag{1.1.1}$$

(i) A *thermal conductivity tensor* $\underset{\approx}{\kappa}$, giving the quantity of heat traversing the surface element $d\sigma$ in the direction of $\underline{n}$ per unit time via the formula

$$dQ = -\underset{\approx}{\kappa}\,\underset{\sim}{n}\,\underline{\partial} T\,d\sigma \tag{1.1.2}$$

(l) A *volume force density* $P \to \underline{g}(P)$.
(m) A relation expressing the stress and conductivity tensors as functions of the observables $\underline{\delta}, \underline{u}, \rho, T, s$.

Relations in (g), (m) are called the continuum *constitutive relations*: in a microscopic theory of continua they must be deducible, in principle, from the atomic model. However in the context in which we shall usually be the constitutive relations have a purely macroscopic character, hence they are phenomenological relations and they must be thought of as essential parts of the considered model of the continuum.

More generally one can consider non homogeneous continua, with more than one chemical components among which chemical reactions may occur: here I shall not deal with such systems, but the foundations of their theory are discussed in some detail in the problems at the end of §1 (*c.f.r.* problems [1.1.7]–[1.1.17]).

We can distinguish between solid and liquid (or fluid) continua. Liquids have a constitutive relation that allows us to express τ in terms of the thermodynamic observables and, furthermore, of the velocity field $\underline{u}$: in other words τ does not depend on the displacement field $\underline{\delta}$.

We always suppose the *validity of the principles of dynamics and thermodynamics*: *i.e.* we assume the validity of a certain number of relations among the observables (listed above) which describe a continuum.

A notation widely used below will be $\underset{\approx}{\tau}$ to denote a tensor τ_{ij}, $i, j = 1, \ldots d$; and $\underset{\approx}{\tau}\,\underline{u}$ to denote the result of the action of the tensor $\underset{\approx}{\tau}$ on the vector $\underline{u}$, *i.e.* the vector whose i-th component is $\sum_j \tau_{ij} u_j$. We shall often adopt the *summation convention over repeated indices*: this means that. for instance, $\sum_{j=1}^{3} \tau_{ij} n_j$ will be denoted (unless ambiguous) simply $\tau_{ij} n_j$.

In this way the relations imposed upon the observables describing the continuum by the laws of thermodynamics and mechanics are the following.

(I) Mass conservation.

If Δ is a volume element which in time t evolves into Δ_t it must be

$$\int_{\Delta} \rho(P,0)dP \equiv \int_{\Delta_t} \rho(P,t)dP \tag{1.1.3}$$

Choosing t infinitesimal one sees that the region Δ_t consists of the points that can be expressed as

$$P' = P + \underline{u}(P)t, \qquad P \in \Delta \tag{1.1.4}$$

and this relation can be thought of as a coordinate transformation $P \to P'$ with Jacobian determinant

$$\det \frac{\partial P'_i}{\partial P_j} = \det\big(1 + \frac{\partial u_i}{\partial P_j} t\big) = 1 + t \sum_{i=1}^{3} \frac{\partial u_i}{\partial P_i} + 0(t^2) \tag{1.1.5}$$

so that, neglecting $O(t^2)$:

$$\begin{aligned}\int_{\Delta_t}\rho(P',t)\,dP' &= \int_\Delta \rho(P+\underline{u}\,t,t)(1+t\,\underline{\partial}\cdot\underline{u})dP = \\ &= \int_\Delta \rho(P)dP + t\int_\Delta(\underline{\partial}\rho\cdot\underline{u}+\partial_t\rho+\rho\underline{\partial}\cdot\underline{u})dP\end{aligned} \tag{1.1.6}$$

hence we find, from (1.1.3):

$$\partial_t\rho+\underline{\partial}\cdot(\rho\,\underline{u})=0 \tag{1.1.7}$$

which is the *continuity equation*.

(II) Momentum conservation (I cardinal equation)

$$\frac{d}{dt}\int_\Delta \rho\,\underline{u}\,dP = \int_\Delta \rho\,\underline{g}\,dP + \int_{\partial\Delta}\underset{\approx}{\tau}\,\underline{n}\,d\sigma \tag{1.1.8}$$

To evaluate the derivative one remarks that at time ϑ

$$\begin{aligned}&\int_{\Delta_\vartheta}\underline{u}(P',\vartheta)\,\rho(P',\vartheta)\,dP' = \\ &= \int_\Delta \rho(P+\underline{u}(P)\vartheta,\vartheta)\,(1+\vartheta\underline{\partial}\cdot\underline{u})\,\underline{u}(P+\underline{u}(P)\vartheta,\vartheta)\,dP \\ &\int_{\partial\Delta}(\underset{\approx}{\tau}\,\underline{n})_i\,d\sigma = \int_\Delta\sum_j(\partial_j\tau_{ij})\,dP\end{aligned} \tag{1.1.9}$$

and, therefore (1.1.8) becomes

$$\partial_t(\rho u_i)+\sum_j\partial_j\big(u_j(\rho u_i)\big)=\rho g_i+\sum_{j=1}^3\partial_j\tau_{ij} \tag{1.1.10}$$

i.e. , by (1.1.7) and the summation convention, we find

$$\partial_t u_j+\underline{u}\cdot\underline{\partial}\,u_j=g_j+\frac{1}{\rho}\partial_k\tau_{jk} \tag{1.1.11}$$

(III) Angular momentum conservation.

This is a property that is automatically satisfied, as a consequence of the definition of stress tensor, (1.1.1): if one allowed a more general stress law $\underline{\tau}_i(\underline{n})d\sigma$, rather than $\tau_{ij}n_jd\sigma$ with a symmetric $\underset{\approx}{\tau}$, one would derive that it imposes that $\tau(\underline{n})_i d\sigma$ must have the form $\tau_{ij}n_jd\sigma$, and that $\tau_{ij}=\tau_{ji}$.

Let, indeed, Δ be a set with the form of a tetrahedron with three sides on the coordinate axes and a face with normal vector $\underline{n}$.

Let $\underline{\tau}_1, \underline{\tau}_2, \underline{\tau}_3$ and $\underline{\tau}_n$ be the stresses that act on the four faces, with normal vectors the unit vectors $\underline{i}, \underline{j}, \underline{k}$ of the coordinate axes and $\underline{n}$, respectively.

The angular momentum of Δ with respect to a point $P_0 \in \Delta$ is $\underline{K} = \int_\Delta (P - P_0) \wedge \rho\, \underline{u}\, dP \le 0(\ell^4)$, if ℓ is the diameter of Δ; also the momentum of the volume forces has size $0(\ell^4)$. On the other hand the momentum of the stresses is *a priori* of size $0(\ell^3)$ unless the total force due to the stresses vanishes:[1] hence in order that it be of size of order $0(\ell^4)$ (as it must by consistence to avoid infinite angular acceleration of Δ) it is necessary that a suitable relation between $\underline{\tau}_1, \underline{\tau}_2, \underline{\tau}_3$ and $\underline{\tau}_n$ be verified. To find it note that if the total stress force did not vanish to leading order as $\ell \to 0$, *i.e.* if

$$\underline{0} \ne \underline{\tau}(\underline{n}) d\sigma - \big(\underline{\tau}_1 d\sigma_1 + \underline{\tau}_2 d\sigma_2 + \underline{\tau}_3 d\sigma_3\big) \equiv \big(\underline{\tau}(\underline{n}) - \underline{\tau}_1 n_1 - \underline{\tau}_2 n_2 - \underline{\tau}_3 n_3\big) d\sigma \tag{1.1.12}$$

then it would follow that the total force would be $\underline{c}\, d\sigma, \underline{c} \ne 0$, hence the total angular momentum would be of size $0(\ell^3)$ with respect to some point P_0 of Δ. Therefore

$$\tau(\underline{n})_j = \tau_{ji} n_i \tag{1.1.13}$$

(with the summation convention).[2]

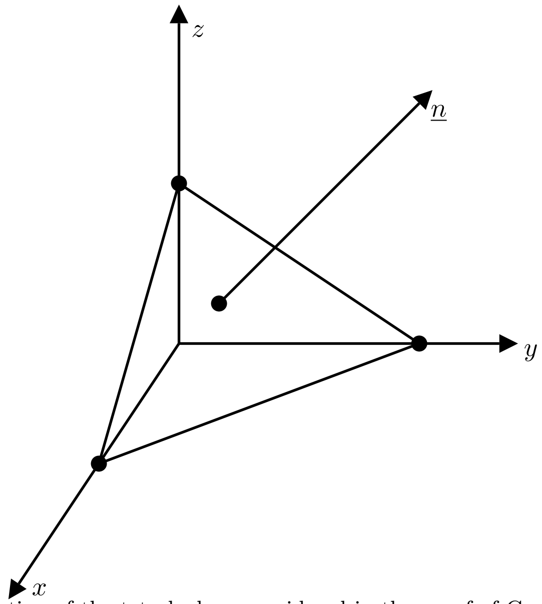

Fig. (1.1.1): Illustration of the tetrahedron considered in the proof of Cauchy's theorem. The unit vectors, $\underline{i}, \underline{j}, \underline{k}$, are not drawn.

Furthermore it is $\tau_{ji} = \tau_{ij}$ as one sees by noting that, at leading order in ℓ, the angular momentum of the stresses on the faces of an *infinitesimal cube*

[1] If the total force does not vanish it has size of order ℓ^2, *i.e.* proportional to the surface area of Δ and then by changing P_0 inside Δ one can find (many) points P_0 with respect to which the momentum of the stresses is of $P(\ell^3)$.

[2] One can also say that this relation follows from the first cardinal equation: indeed if $\underline{c}\, d\sigma \ne \underline{0}$ the total force would have size $0(\ell^2)$ while being equal to the time derivative of the linear momentum which has size $0(\ell^3)$.

with side ℓ, with respect to the cube center, should be of $O(\ell^4)$, as noted above, but it is

$$\ell^3\big[\big(\underline{j}\wedge(\underset{\approx}{\tau}\,\underset{\sim}{j})\big)+\underline{i}\wedge(\underset{\approx}{\tau}\,\underset{\sim}{i})+\underline{k}\wedge(\underset{\approx}{\tau}\,\underset{\sim}{k})\big] \tag{1.1.14}$$

Note that $(\underset{\approx}{\tau}\,\underline{j})_i=(\tau_2)_i,(\underset{\approx}{\tau}\,\underline{i})_i=\tau_{1i},(\underset{\approx}{\tau}\,\underline{u})_i=\tau_{3i}$ are the components of the stresses on the faces having as normals the coordinate unit vectors $\underline{j},\underline{i},\underline{k}$ respectively, and if one imposes that (1.1.14) vanishes, one gets:

$$\begin{aligned}0=&\det\begin{pmatrix}\underline{i}&\underline{j}&\underline{k}\\0&1&0\\\tau_{21}&\tau_{22}&\tau_{23}\end{pmatrix}+\det\begin{pmatrix}\underline{i}&\underline{j}&\underline{k}\\1&0&0\\\tau_{11}&\tau_{12}&\tau_{13}\end{pmatrix}+\det\begin{pmatrix}\underline{i}&\underline{j}&\underline{k}\\0&0&1\\\tau_{31}&\tau_{32}&\tau_{33}\end{pmatrix}\\=&\,\underline{i}\big(\tau_{23}-\tau_{32}\big)+\underline{j}\big(\tau_{13}-\tau_{31}\big)+\underline{k}\big(\tau_{21}-\tau_{12}\big)\end{aligned} \tag{1.1.15}$$

so that $\tau_{ij}=\tau_{ji}$ often called *Cauchy's theorem.*

(IV) Energy conservation.

This is a more delicate conservation law as it involves also the thermodynamic properties of the continuum.

We imagine that every infinitesimal fluid element Δ is a (ideally infinite) system in thermodynamic equilibrium and, hence, with a well defined value of the observables like internal energy, entropy and temperature.., *etc.*

The equation of state, characteristic of the continuum considered, will be a relation expressing the internal energy ε per unit mass in terms of the mass density ρ and of the entropy per unit mass s: $(\rho,s)\to\varepsilon(\rho,s)$.

Then the energy balance in a volume element Δ will be obtained by expressing the variation (per unit time) of its energy (kinetic plus internal)

$$\frac{d}{dt}\int_\Delta(\rho\,\frac{\underline{u}^2}{2}+\rho\,\varepsilon)\,dP \tag{1.1.16}$$

as the work performed by the volume forces, plus the work of the stresses on the boundary of the volume element and, also, plus the heat that penetrates by conduction from the boundary of Δ. This is the sum of the following addends

$$\begin{aligned}&\int_\Delta\rho\,\underline{g}\cdot\underline{u}\,dP+\int_{\partial\Delta}\underset{\approx}{\tau}\,\underline{n}\underset{\sim}{u}\,d\sigma+\int_{\partial\Delta}\kappa_{ij}(\partial_iT)\cdot n_j\,d\sigma= \qquad (1.1.17)\\&=\int_\Delta\rho\,\underline{g}\,\cdot\underline{u}\,dP+\int_\Delta\partial_i\left(\tau'_{ij}u_j\right)dP-\int_\Delta\underline{\partial}\cdot(p\,\underline{u})\,dP+\int_\Delta\partial_i(\kappa_{ij}\partial_jT)\,dP\end{aligned}$$

where we wrote (defining $\underset{\approx}{\tau}{}'$) $\tau_{ij}=-p\delta_{ij}+\tau'_{ij}$ with p being the pressure and where we assumed the validity of Fourier's law (1.1.2) for the heat transmission.

Equating (1.1.16),(1.1.17) and using (1.1.11) (multiplied by $\underline{u}$ and integrated over Δ) to eliminate the term with kinetic energy one gets

$$\frac{d}{dt}\int_\Delta \rho\varepsilon dP = \int_\Delta \left[\underset{\approx}{\tau}\,' \cdot \underset{\sim}{\underline{\partial u}} + \underline{\partial}\cdot(\underset{\approx}{\kappa}\,\underset{\sim}{\partial} T) - p\,\underline{\partial}\cdot\underline{u}\right] dP \tag{1.1.18}$$

and in this relation we recognize that the last term in the r.h.s. is $-p\,d|\Delta|/dt$, *i.e.* it is the work done per unit time by the pressure forces, while the term before the last yields the quantity of heat that enters by conduction in the volume element. The l.h.s. is the variation per unit time of the internal energy. Therefore from the first principle of thermodynamics $dE = dQ - pdV$ we see that the first term in the r.h.s. *must* represent an amount of heat entering the volume element. It is naturally interpreted as the quantity of heat generated by friction forces described by the tensor $\underset{\approx}{\tau}\,'$.

Note that $\underset{\approx}{\tau}\,'$ not only contributes to the energy balance through the heat generated per unit volume by friction, *i.e.* $\underset{\approx}{\tau}\,'\underset{\sim}{\underline{\partial u}}$, but also through the mechanical work $(\underline{\partial}\,\underset{\approx}{\tau}\,')\underset{\sim}{u}$ per unit volume: such contributions appear, in fact, summed together in (1.1.17) (in the form $\underline{\partial}(\underset{\approx}{\tau}\,'\underset{\sim}{u}) = \underset{\approx}{\tau}\,'\underset{\sim}{\underline{\partial u}} + \underset{\sim}{u}\,\underline{\partial}\underset{\approx}{\tau}\,'$).

This leads, therefore, to interpret $\underset{\approx}{\tau}\,'$ as an observable associated with the friction forces inside the fluid as well as with the non normal internal stresses.

In order that this interpretation be possible it is necessary, of course, that τ'_{ij} depends solely on the local thermodynamic quantities (s, T) and on the gradient (and, possibly, on the higher order derivatives) of the velocity field and, furthermore, it should vanish if the derivatives of the velocity vanish. Hence in what follows $\tau'_{ij} = 0$ if $\underline{\partial}\,\underset{\sim}{u} = \underset{\approx}{\underline{0}}$.

The differential form of the (1.1.18) is

$$\partial_t(\rho\varepsilon) + \underline{\partial}\cdot(\rho\varepsilon\underline{u}) \equiv \rho(\partial_t\varepsilon + \underline{u}\cdot\underline{\partial}\varepsilon) = \underset{\approx}{\tau}\,'\cdot\underline{\partial}\,\underset{\sim}{u} + \underline{\partial}\cdot(\underset{\approx}{\kappa}\,\underset{\sim}{\partial}T) - p\,\underline{\partial}\cdot\underline{u} \tag{1.1.19}$$

having used (in the first identity), the continuity equation.

(V) II^o law of thermodynamics and entropy balance

Eq. (1.1.19) can be combined with the second principle of thermodynamics $T\,dS = dE + p\,d|\Delta|$, and $S = \rho|\Delta|s, E = \rho|\Delta|\varepsilon$, $d|\Delta|/dt = \underline{\partial}\cdot\underline{u}\,|\Delta|$ which gives $T\,d\int_\Delta \rho\,s\,dP = d\int_\Delta \rho\,\varepsilon\,dP - \int_\Delta dP$. One obtains (using also the continuity equation)

$$T\,\rho\left(\partial_t s + \underline{u}\cdot\underline{\partial}\,s\right) = \underset{\approx}{\tau}\,'\cdot\underline{\partial}\,\underset{\sim}{u} + \underline{\partial}(\underset{\approx}{\kappa}\,\underset{\sim}{\partial}T) \tag{1.1.20}$$

which is the form in which, in applications, energy conservation is often used.

Introducing the *heat current* $\underline{J}_q \stackrel{def}{=} -\underset{\approx}{\kappa}\,\underset{\sim}{\partial} T$ then (1.1.20) can be put in the form

$$\rho\frac{ds}{dt} = -\underline{\partial}\cdot\frac{\underline{J}_q}{T} - \underline{J}_q\cdot\frac{\underline{\partial} T}{T^2} + \underset{\approx}{\tau}{}'\cdot\frac{1}{T}\underline{\partial}\,\underset{\sim}{u} = -\underline{\partial}\cdot\frac{\underline{J}_q}{T} + \sigma \tag{1.1.21}$$

where σ is interpreted as *entropy density generated per unit volume and unit time.* The σ is also written as

$$\sigma = \sum_j J_j X_j \tag{1.1.22}$$

where X_j is a vector with twelve components consisting in the $-T^{-2}\partial_i T$, $T^{-1}\partial_i u_j$ and J_j consists in the $J_{q\,j}, \underset{\approx}{\tau}{}'_{ij}$.

Remark: In general thermodynamic forces X_j are identified with parameters measuring how far from thermodynamic macroscopic equilibrium the local state of the fluid is (*e.g.* with a temperature or velocity gradient); and the corresponding currents are identified with the coefficients J_j that allow us to express the entropy creation rate as a linear combination of the forces X_j, *c.f.r.* (1.1.22). It is clear that the identification ("duality") of the forces and the thermodynamic currents is not free of ambiguities because in each problem, given the entropy creation rate σ, one can in general represent σ in several ways in the form (1.1.22). This is an ambiguity analogous to that present in the identification, in mechanics, of the canonical coordinates from a given Lagrangian: one gets different coordinates depending on which are the variables from which one wants to think that the Lagrangian depends.
An unavoidable further problem lies in the fact that a really precise and purely macroscopic definition of entropy creation rate is *not*, to date, well established in systems out of equilibrium unless the systems are *very* close to equilibrium. But this is not the place to enter into a discussion of the foundations of nonequilibrium thermodynamics, see §7.1÷§7.4.

A basic assumption often made in the dynamics of continua is that the relationship between the *thermodynamic forces* X_j and the *currents* or *fluxes* is *linear*, at fixed values of the state observables ρ, p, T

$$J_j = \sum_k L_{jk} X_k \tag{1.1.23}$$

at least if the "thermodynamic forces" X_k are "small". This relation can be combined with the invariance properties for the Galilean transformations, with other possible symmetries (present in fluids with more components and/or chemically active) and with two principles of the thermodynamics of irreversible processes, namely the *Onsager reciprocity* and the *Curie principle.* One obtains, in this way, important restrictions on the tensors $\underset{\approx}{\kappa}, \underset{\approx}{\tau}{}'$.

Onsager's relations imply that the matrix L is symmetric

$$L_{jk} = L_{kj} \tag{1.1.24}$$

while the Curie's principle says that some among the coefficients L_{jk} vanish. If $L_{jk} = 0$ one says that the current J_j "*does not directly depend on*" (or "*does not directly couple with*") the thermodynamic force X_k. Curie's principle states precisely that the currents J_j that have a vectorial character, *i.e.* are the components of an observable that transforms, under Galilean transformations, as a vector (such as the $\underline{J}_q$), do not couple nor depend directly on thermodynamic forces with different transformation (or "covariance") properties (such as the derivatives $\partial_i u_j$, that have a tensorial character). More generally there is no coupling between thermodynamic forces and currents with different transformation properties with respect to the symmetry groups of the continuum considered.

For instance the matrix κ_{ij} must be symmetric and τ'_{ij} must be (as it already follows from the II^d cardinal equation of dynamics, see above) symmetric and expressible in terms of the derivatives $\partial_i u_j$ via a linear combination of $\partial_i u_j + \partial_j u_i$ e $\delta_{ij}\,\partial_k u_k$ because these are the only tensors that one can form with a linear dependence on the derivatives $\partial_i u_j$

$$\kappa_{ij} = \kappa_{ji}, \qquad \tau'_{ij} = \eta\,(\partial_i u_j + \partial_j u_i) + \eta'\,\partial_k u_k\,\delta_{ij} \tag{1.1.25}$$

see (1.2.6), (1.5.2).

Onsager's relations are a macroscopic consequence of the microscopic reversibility of dynamics, *c.f.r.* [DGM84] and the Curie principle also is rooted on microscopic symmetries, [DGM84].

The second law of thermodynamics is imposed (not without some conceptual difficulties, see problem [1.1.17] below) by requiring that $\sigma \ge 0$: which is obtained by demanding that the tensor $\underset{\approx}{\kappa}$ be positive definite and that $\underset{\approx}{\tau}' \cdot \underset{\sim}{\partial\, u} \ge 0$, *i.e.* in the case (1.1.25) $\eta, \eta' + 2\eta \ge 0$.

Problems.

[1.1.1]: Let $X(\underline{x},t)$ be a generic observable and define a *current line* as a solution $t \to \underline{x}(t)$ of the equation

$$\dot{\underline{x}} = \underline{u}(\underline{x},t)$$

where $\underline{u}(\underline{x},t)$ is a given velocity field. The *substantial derivative* $\frac{dX}{dt}$, of X, is then defined by the t–derivative of $X(\underline{x}(t),t)$ and it is written as

$$\frac{d}{dt}X(\underline{x}(t),t) = \partial_t X + \underline{u}\cdot\underline{\partial}\,X$$

Show that

$$\rho\frac{d}{dt}X = \partial_t(\rho X) + \underline{\partial}\cdot(\rho X\underline{u})$$

(*Idea:* Use the continuity equation for ρ).

[1.1.2]: Check that the continuity equation can be read as: "the substantial derivative of ρ is $-\rho\underline{\partial}\cdot\underline{u}$".

[1.1.3]: (*A kinetic theory problem*) Consider a monoatomic rarefied gas, whose atoms have mass m and radius σ and occupy the semi space $z>0$. Imagine that the fluid is undergoing with isothermal stratified motion with a small shearing velocity field of size $v(z)=z\,v'$ parallel to the x–direction. Let ρ_n be the numerical density (number of atoms per unit volume), λ be the mean free path and $\bar v=(3k_BT/m)^{1/2}$ be the average thermal agitation velocity (k_B is Boltzmann's constant and T is the absolute temperature). Find a heuristic justification, neglecting the horizontal velocity components, for the statement that the number of particles crossing an ideal surface at height $z_0\gg\lambda$ coming from quotae $z>z_0$ and without suffering collisions is, approximately,

$$\int_{z_0}^{z_0+\lambda}dz\int_{-w_z\tau>\lambda}\rho_n\,dxdy\,f(\underline{w})d\underline{w}$$

if $f(\underline{w})=e^{\frac{-mw^2}{2k_BT}}(\frac{m}{2\pi k_BT})^{3/2}$ is Maxwell's distribution (with k_B representing the Boltzmann's constant.

[1.1.4]: (*kinetic theory for viscosity and heat conductivity)* In the context of problem [1.1.3] deduce that the variations of momentum and thermal kinetic energy (*i.e.* average of $\frac12$ the square of the velocity minus the average velocity) contained in the gas layer at height $z\le z_0$, per unit time and surface, are respectively (we denote by v' the derivative $\frac{dv(z)}{dz}$, by $\overline{v}$ the mean velocity $\overline{v}=(3k_BT/m)^{1/2}$ and the free flight time by $\tau=\lambda/\overline{v}$)

$$\frac{1}{\tau}\int_0^\lambda dh\int_{\lambda/\tau}^\infty\rho_n dw\frac{e^{\frac{-mw^2}{2k_BT}}}{(2\pi k_BT/m)^{1/2}}(2mhv'),$$
$$\frac{1}{\tau}\int_0^\lambda dh\int_{\lambda/\tau}^\infty\rho_n dw\frac{e^{\frac{-mw^2}{2k_BT}}}{(2\pi k_BT/m)^{1/2}}(2\frac{3}{2}k_Bh\frac{dT}{dz})$$

Deduce from this that the force per unit surface exerted by the fluid above the height z_0 on the part of the fluid below z_0 is $F=\eta v'$ with:

$$\eta=m\bar v\rho_n\lambda\gamma,\qquad \gamma=\int_{\sqrt3}^\infty e^{-p^2/2}\frac{dp}{\sqrt{2\pi}}$$

Deduce also that the amount of heat crossing per unit time and unit surface the height z_0 is $Q=\kappa\frac{dT}{dz}$ with

$$\kappa=\frac{3}{2}k_B\bar v\rho_n\lambda\gamma$$

so that, if the collision gross section is denoted σ, it is $\eta=m\frac{\gamma}{\pi\sigma^2}\sqrt{\frac{3kT}{m}}$. (*Idea:* Use the formula (in fact a definition) $\lambda\pi\sigma^2\rho_n=1$ for the mean free path in terms of the atomic diameter σ). See table at the end of the section.

[1.1.5]: (*Clausius–Maxwell relation between specific heat, viscosity and thermal conductivity*) In the context of problems [1.1.3],[1.1.4] assume that the stress tensor of the gas is $\tau'_{ij}=\eta(\partial_iu_j+\partial_ju_i)$ with η constant: compute the force per unit surface that the part of the gas above height z_0 exerts on the part of gas below it and deduce that η is the quantity studied in problem [1.1.4]; and that, therefore, the relation of *Clausius–Maxwell* holds between viscosity, heat conductivity and specific heat at constant volume $c_v\equiv\frac32RM_A^{-1}$ (if R is the gas constants and M_A is the atomic mass, *e.g.* $4g$ for helium)

$$\kappa=c_v\eta$$

and derive the independence of the viscosity η and of heat conductivity from the density and their proportionality to $\sqrt{T}$. Check that, by refining the calculations, (*e.g.* not neglecting the horizontal components of the velocities) the results only change by numerical factors of $O(1)$ independent on the physical quantities m, ρ, T: in particular the Clausius–Maxwell relation does not change (in rarefied gases).

[1.1.6]: (*energy flux in a perfect fluid*) Show that in a "perfect fluid" (*i.e.* with $\tau', \kappa, g = 0$, *c.f.r.* (1.1.18)) it is

$$\partial_t \int_V \rho\Big(\frac{v^2}{2}+\varepsilon\Big)\,dP = -\int_V \underline{\partial}\cdot\rho\underline{v}\,\Big(\frac{\underline{v}^2}{2}+w\Big)\,dP = \int_{\partial V}\rho\Big(\frac{\underline{v}^2}{2}+w\Big)\underline{v}\cdot\underline{n}d\sigma$$

where $w=\varepsilon+p/\rho$. Hence $\rho\Big(\frac{\underline{v}^2}{2}+w\Big)\underline{v}$ can be interpreted as *energy flux.* (*Idea:* See (1.1.18). Alternatively: if $d\sigma$ is a surface element with external normal $\underline{n}$ the amount of energy crossing $d\sigma$ in the direction $\underline{n}$ is $-\rho(\underline{v}^2/2+\varepsilon)\underline{v}\cdot\underline{n}d\sigma - p\underline{v}\cdot\underline{n}d\sigma$ because the first is the quantity of energy that "exits" through $d\sigma$ per unit time and the second is the work performed through $d\sigma$ by the part of fluid adjacent (but external) to it, per unit time).

[1.1.7] (*mass conservation in mixtures*) Suppose that a fluid consists of a mixture of n different fluids. Let $\rho_1,\ldots,\rho_n$ be the densities and $\underline{u}_1,\ldots,\underline{u}_n$ the respective velocity fields. Then $\rho=\sum_j\rho_j$ will be called the "total density" and $\underline{u}=\rho^{-1}\sum_j\rho_j\underline{u}_j$ the "velocity field" of the fluid. Show that the continuity equations can be written in the form

$$\partial_t\,\rho_k = -\underline{\partial}\cdot(\rho_k\underline{u}_k),\qquad \partial_t\rho=-\underline{\partial}\cdot(\rho\underline{u})$$

We shall set $\underline{J}=\rho\,\underline{u}$ and $\underline{J}_k=\rho_k\,(\underline{u}_k-\underline{u})$: check that if $\frac{d}{dt}\overset{def}{=}\partial_t+\underline{u}\cdot\underline{\partial}$ then

$$\frac{d\rho_k}{dt} = -\rho_k\,\underline{\partial}\cdot\underline{u}-\underline{\partial}\cdot\underline{J}_k$$

[1.1.8] (*mass conservation in chemically active mixtures*) In the context of problem [1.1.7] suppose that r chemical reactions are possible between the n species of fluid and that, otherwise, the particles interactions are modeled by hard cores so that the internal energy is entirely kinetic.
If the chemical equation for the j-th reaction is $\sum_{k=1}^n n_{jk}\,[k]=0$, where n_{jk} are stechiometric integers (*e.g.* $2[H_2]+[O_2]-2[H_2O]=0$ involves three species H_2, O_2, and H_2O of molecular mass $2,16,18$ respectively), one defines the *stechiometric coefficients* of the j-th reaction the quantities $\nu_{jk}=m_kn_{jk}$ where m_k is the molecular mass of the k-th species. Then: $\sum_{k=1}^n\nu_{jk}=0$, by mass conservation(*Lavoisier law*), $\sum_k\nu_{jk}\underline{u}_k=\underline{0}$ by momentum conservation and $\sum_k\nu_{jk}\frac12\underline{u}_k^2=\eta_j$ by energy conservation if η_j is the energy yield in the j–th reaction.
Let R_j be the number of chemical reactions of the j-th type that take place per unit volume and unit time (a number which can have either sign: $R_j>0$ means that the reaction proceeds in the direction of transforming molecules with negative stechiometric coefficients into molecules with positive coefficients and viceversa for $R_j<0$). Show that the equations of continuity are modified as

$$\partial_t\,\rho_k=-\underline{\partial}\cdot(\rho_k\underline{u}_k)+\sum_{j=1}^r R_j\,\nu_{jk},\qquad \partial_t\rho=-\underline{\partial}\cdot(\rho\underline{u})$$

Furthermore with the notations of [1.1.7]

$$\frac{d\rho_k}{dt}=-\rho_k\,\underline{\partial}\cdot\underline{u}-\underline{\partial}\cdot\underline{J}_k+\sum_{j=1}^r R_j\,\nu_{jk},\qquad \frac{d\rho}{dt}=-\rho\,\underline{\partial}\cdot\underline{u}$$

Finally setting $c_k = \rho_k/\rho$ it is

$$\rho\frac{d\,c_k}{dt} = -\underline{\partial}\cdot\underline{J}_k + \sum_{j=1}^{r} R_j\,\nu_{jk}$$

[1.1.9] (*momentum conservation in chemically active mixtures*) Check that in fluids with several chemically active components the equation corresponding to the I-th cardinal equation (*i.e.* to momentum conservation) is

$$\rho\frac{d\underline{u}}{dt} = -\underline{\partial}\,p + \underset{\sim}{\partial}\,\underset{\approx}{\tau}\,' + \sum_{k=1}^{n}\rho_k\,\underline{g}_k$$

where p is the sum of the partial pressures p_k of each species, $\underset{\approx}{\tau}\,'$ is the sum of the stresses $\underset{\approx}{\tau}\,'_k$ on each species *plus* the tensor $\sum_k \underline{u}_k\,\underset{\sim}{J}_k$, and $\underline{g}_k = -\underline{\partial}V_k$ is the force, with potential energy function V_k, per unit mass acting on the k-th species (which might be species dependent: think, for instance, to a ionized solution in an electric field) *provided* the total potential energy does not change in the chemical reactions (*i.e.* $\sum_k \nu_{jk}\,V_k = 0$). (*Idea:* Write the I-th cardinal equation for each species k:

$$\partial_t(\rho_k\underline{u}_k) + \underset{\sim}{\partial}\cdot(\rho_k\underline{u}_k\underset{\sim}{u}_k) = -\underline{\partial}p_k + \underset{\sim}{\partial}\underset{\approx}{\tau}\,'_k + \rho_k\underline{g}_k + \sum_j R_j\nu_{jk}\underline{u}_k$$

and sum over k taking into account the momentum conservation in [1.1.8].)

[1.1.10] (*energy conservation in chemically active mixtures*)In the context of [1.1.8], [1.1.9] call $\underline{J}_k = (\underline{u}_k - \underline{u})\,\rho_k$ the *diffusion current* of the k-th species, see [1.1.7], and check that the energy conservation equation is

$$\rho\frac{d\varepsilon}{dt} = -p\,\underline{\partial}\cdot\underline{u} + \underset{\approx}{\tau}\,'\underset{\sim}{\partial}\underline{u} + \sum_k \underline{g}_k\cdot\underline{J}_k - \underline{\partial}\cdot\underline{J}_q$$

where $\rho\varepsilon \stackrel{def}{=} \sum_k \rho_k\,(\varepsilon_k + \frac12(\underline{u}_k - \underline{u})^2)$ and $\underline{J}_q$ is suitably defined. (*Idea:* The energy in a volume element Δ due to the k–th species is $\int_\Delta \rho_k\,(\varepsilon_k + \frac12\underline{u}_k^2)\,d\underline{x}$ if ε_k is the internal energy per particle. We suppose that no interaction takes place between the species other than that giving rise to chemical reactions and other than the hard core pair interaction between the molecules. Then, setting $\vartheta_k \stackrel{def}{=} (\varepsilon_k + \frac12\underline{u}_k^2)$, the energy balance for the k–th species yields

$$\begin{aligned}\partial_t(\rho_k\vartheta_k) + \underline{\partial}\cdot(\rho_k\vartheta_k\underline{u}_k) = -\underline{\partial}\cdot(p_k\underline{u}_k) + \underset{\sim}{\partial}(\underset{\approx}{\tau}\,'_k\underline{u}_k) +\\ +\rho_k\underline{g}_k\cdot\underline{u}_k + \underline{\partial}\cdot(\underset{\approx}{\kappa}\,\underline{u}_k\cdot\underset{\sim}{\partial}T) + \delta_k\end{aligned}$$

where $\underset{\approx}{\kappa}\,\underline{u}_k\cdot\underset{\sim}{\partial}T$ is the heat flux into the species k and δ_k is the total energy variation of the species k due to the chemical reactions, $\delta_k = \sum_j \nu_{jk}(\eta_{jk} + \frac12\underline{u}_k^2)$, with η_{jk} being the dissociation energy of the k–th species into the components involved in the j–th reaction so that $\sum_k \delta_k = 0$. Adding and subtracting $\underline{u}$ where appropriate and summing over k one finds

$$\partial_t(\rho\varepsilon+\rho\frac{1}{2}\underline{u}^2)+\underline{\partial}\cdot(\sum_k\rho_k(\varepsilon_k+\frac{1}{2}\underline{u}_k^2)\,(\underline{u}_k-\underline{u})+\rho\varepsilon+\rho\frac{1}{2}\underline{u}^2)=$$
$$=-\underline{\partial}\cdot(p\,\underline{u})-\underline{\partial}\cdot(\sum_k p_k\,(\underline{u}_k-\underline{u}))+\underline{\partial}\cdot(\sum_k \underset{\approx}{\tau}{}'_k\,(\underline{u}_k-\underline{u}))+\underline{\partial}(\underset{\approx}{\tau}{}'\underset{\sim}{u})+$$
$$+\sum_k\rho_k\underline{g}_k\cdot(\underline{u}_k-\underline{u})+(\sum_k\rho_k\underline{g}_k)\cdot\underline{u}+\underline{\partial}\cdot(\underset{\approx}{\kappa}\cdot\underline{\partial}T)$$

hence, from [1.1.9], [1.1.8], one gets the above result with

$$\underline{J}_q\stackrel{def}{=}\sum_k\Big((\varepsilon_k+\frac{1}{2}\underline{u}_k^2)\,\underline{J}_k+\underset{\approx}{\tau}{}'_k(\underset{\sim}{u}_k-\underset{\sim}{u})-p_k\,(\underline{u}_k-\underline{u})\Big)-\underset{\approx}{\kappa}\underset{\sim}{\partial}T$$

see also [DGM84].)

[1.1.11] (*heat transport in chemically active mixtures*)The second law of thermodynamics, in the case of chemically active systems, takes the form $TdS=dU+pdV-\sum_k\mu_k\,d(\rho_kV)$ where μ_k is the chemical potential per unit mass of the k-th species. Proceeding as in *(V)* show that [1.1.8],[1.1.9],[1.1.10] imply

$$\rho\,(\partial_t s+\underline{u}\cdot\underline{\partial}s)=-\frac{1}{T}\underline{\partial}\cdot\underline{J}_q+\frac{1}{T}\underset{\approx}{\tau}{}'\cdot\underline{\partial}\underset{\sim}{u}+$$
$$+\frac{1}{T}\sum_k\Big(\mu_k\underline{\partial}\cdot\underline{J}_k-\sum_j\nu_{jk}R_j\,\mu_k+\underline{g}_k\cdot\underline{J}_k\Big)$$

(*Idea:* Combine [1.1.10] with the second of [1.1.8] or, using the invariance of ρV (mass conservation) with the third of [1.1.8]).

[1.1.12] (*entropy flow in chemically active mixtures*) Set, see also [1.1.10]

$$A_j=\sum_k\mu_k\nu_{jk},\qquad \underline{J}_s=\frac{1}{T}\,(\underline{J}_q-\sum_{k=1}^n\mu_k\underline{J}_k)$$
$$\sigma=-\,\underline{J}_q\cdot\frac{\underline{\partial}T}{T^2}-\sum_{k=1}^n\underline{J}_k\cdot(\underline{\partial}\frac{\mu_k}{T}-\frac{1}{T}\underline{g}_k)+\underset{\approx}{\tau}{}'\cdot\frac{1}{T}\underline{\partial}\underset{\sim}{u}-\sum_{j=1}^r R_j\frac{A_j}{T}$$

and check that the entropy balance equation in [1.1.11], generalizing (1.1.21), can be written

$$\rho\frac{ds}{dt}=-\underline{\partial}\cdot\underline{J}_s+\sigma$$

and (therefore, brushing aside conceptual problems on the identification of the various terms in the balance equation) the quantity $\underline{J}_s$ can be interpreted as the "*entropy current*" transported by the velocity fields, while σ can be interpreted as the quantity of entropy generated per unit volume (by the irreversible processes that develop during the fluids motions). If this interpretation is accepted then the second law of irreversible thermodynamics requires that $\sigma\le 0$. If this looks too strict an interpretation one should at least have that $\Gamma=\int\sigma\,dP\le 0$.

[1.1.13] (*entropy creation and thermodynamic forces and fluxes*) Check that, defining the "*thermodynamic forces*" $\underline{X}$ and the "*thermodynamic currents*" or "*fluxes*" as

$$\underline{X}=(-\frac{\partial_iT}{T^2},-\partial_i\frac{\mu_k}{T}+g_{ki},\frac{\partial_iu_j}{T},-\frac{A_j}{T}),\quad \underline{J}=(J_{qi},J_{ki},\tau'_{ij},R_j)$$

the entropy generated per unit volume σ, defined in [1.1.12], can be written

$$\sigma = \sum_j J_j X_j$$

thus extending to chemically active multicomponent fluids the results of the theory of homogeneous fluids. Formulate the Curie principle and Onsager reciprocity relations for such fluids. (*Idea:* They are "the same").

[1.1.14] (*entropy creation and constant transport coefficients)* Suppose that in a n components fluid the relation between thermodynamic forces and currents is linear: $J_i = \sum_k L_{ik} X_k$, and that L_{ik} are *constants* and satisfy Onsager relations. Then the *entropy production* per unit time $\Gamma \stackrel{def}{=} \int_\Omega \sigma \, dP$ has time derivative

$$\dot{\Gamma} = 2 \int_\Omega \sum_j J_j \cdot \partial_t X_j \, dP = 2 \int_\Omega \sum_j \partial_t J_j \cdot X_j \, dP$$

(*Idea:* $\Gamma = \int \sum_{jk} L_{jk} X_j X_k \, dP$ and differentiate).

[1.1.15] (*completeness of the equations for mixtures*) Consider the system in [1.1.8] and check that the number of equations equals the number of unknowns, listing the variables and the equations chosen. (*Idea:* For instance we can describe the system by the densities ρ_k, the velocity fields $\underline{u}_k$, the internal energies ε_k; then the first of [1.1.8] are equations for ρ_k, the [1.1.9] gives an equation for the $\underline{u}_k$ and [1.1.10] gives the equation for the ε_k. The equations of state $s_k = s_k(\varepsilon_k, \rho_k)$ and $\mu_k = \mu_k(\varepsilon_k, \rho_k)$ (which are not independent) give the entropy and the chemical potentials, hence the temperature $T = \frac{\partial s_k}{\partial \varepsilon_k}$ (which we have assumed to be the same for all species) and the partial pressures; one also needs the constitutive equations expressing the stresses and, more generally, the fluxes in terms of the forces (*i.e.* the matrix L) so that for instance $R_j = \sum_{j'} L_{jj'} \sum_k \nu_{j'k} \mu_k$ ("*law of mass action*").)

[1.1.16] (*Prigogine's principle*) Consider n fluids in mechanical equilibrium ($p = cost$ and $\underline{u} = \underline{0}$), with boundary conditions in which T is constant in time (at every point of the boundary) and the diffusive current of the k-th species vanishes ($\underline{J}_k = \underline{0}$ at every boundary point). Assume (a strong assumption) that L_{jk} are constants and that there are no volume forces ($\underline{g}_k = \underline{0}$). Taking into account that $\sum_k \underline{J}_k = \underline{0}$, check that the entropy produced per unit time and volume is

$$\sigma = \underline{J}_q \cdot \underline{\partial} \frac{1}{T} - \sum_{j=1}^{r} R_j \frac{A_j}{T} - \sum_{k=1}^{n-1} \underline{J}_k \cdot \underline{\partial} \frac{\mu_k - \mu_n}{T} .$$

Check then that the states that make the entropy production Γ, cf. problem [1.1.13], stationary (with respect to the variations of the forces $\underline{X}$) are time independent states (Prigogine). (*Idea:* Lagrange equations for the minimum are $L\underline{X} = \underline{0}$, *i.e.* $\underline{J} = \underline{0}$: hence $R_j = 0, \underline{J}_q = \underline{0}, \underline{J}_k = \underline{0}$ and, therefore, $\frac{d\rho_k}{dt} = 0$, $\frac{d\varepsilon}{dt} = 0$, $\frac{ds}{dt} = 0$ by [1.1.8],[1.1.9],[1.1.10].)

[1.1.17] (*Prigogine's minimal entropy production*) Show that the time independent states of the n fluids in [1.1.16] which minimize (in a strict sense) the entropy production and that are states of mechanical and thermal equilibrium (*i.e.* with p, T constants as a function of time, with $\underline{u} = \underline{0}$ and with a Gibbs function per unit mass $g = \varepsilon - Ts + p\rho^{-1} \equiv \sum_k \mu_k c_k$ which is a strict minimum at every fluid point) are states in stable equilibrium among the thermal and mechanical equilibrium states if the fluids can be regarded as perfect gases (Prigogine). (*Idea:* Imagine perturbing the state by slightly varying T and c_k keeping $\underline{\partial} p, \underline{u}=\underline{0}$; then the system evolves and one has, by [1.1.14]

$$\begin{aligned}\dot\Gamma =&2\int_\Omega \big(\underline{J}_q\cdot\partial_t\underline{\partial}\frac1T-\sum_{k=1}^{n-1}\underline{J}_k\cdot\partial_t\underline{\partial}\cdot\frac{\mu_k-\mu_n}{T}-\sum_{j=1}^r R_j\partial_t\frac{A_j}{T}\big)\,dP=\\ =&2\int_\Omega\big(-\underline{\partial}\cdot\underline{J}_q\partial_t\frac1T+\sum_{k=1}^{n-1}\big(\underline{\partial}\cdot\underline{J}_k-\sum_{j=1}^r R_j\nu_{jk}\big)\,\partial_t\frac{\mu_k-\mu_n}{T}\big)\,dP\end{aligned}$$

And recalling that $\sum_k\nu_{jk}=0$ and the continuity equations for the concentrations c_k (in [1.1.8] and in the hint to [1.1.11])

$$\begin{aligned}\dot\Gamma =&2\int_\Omega\big(-\underline{\partial}\cdot\underline{J}_q\partial_t\frac1T+\sum_{k=1}^{n-1}\Big(\underline{\partial}\cdot\underline{J}_k-\sum_{j=1}^r\nu_{jk}R_j\Big)\partial_t\frac{\mu_k-\mu_n}{T}\big)\,dP=\\ =&2\int_\Omega(-\underline{\partial}\cdot\underline{J}_q\partial_t\frac1T-\sum_{k=1}^{n-1}\rho\,\frac{dc_k}{dt}\,\partial_t\frac{\mu_k-\mu_n}{T}\big)\,dP=2\int_\Omega\big(\big(-\underline{\partial}\cdot\underline{J}_q-\\ &-\sum_{k=1}^{n-1}\rho\,\frac{dc_k}{dt}\,(\mu_k-\mu_n)\big)\,\partial_t\frac1T-\frac{\rho}{T}\sum_{h,k=1}^{n-1}\partial_{c_h}(\mu_k-\mu_n)\,\frac{dc_h}{dt}\frac{dc_k}{dt}\big)\,dP\end{aligned}$$

and by our time independence assumption it is $\frac{dc_h}{dt}=\partial_t c_h$; note that $T^{-1}(\mu_k-\mu_n)$ depends only on c_k,c_n by the perfect gas assumption (in fact in a perfect gas of mass m_k and temperature T the chemical potential is $\mu_k=k_BT\,(\log\rho_k-\frac32\log(k_BT)^{-1}-\frac32\log m_k))$.
However in thermodynamics it is $TdS=dU+pdV-\sum_k\mu_kd(c_k\rho V)$ (note that it is convenient to introduce c_k because $\rho|\Delta|$ is constant in time, while $\rho_k|\Delta|$ is not such, by the second equation in [1.1.8]).
Or: $Tds=d\varepsilon+pd\rho^{-1}-\sum_k\mu_kdc_k$, so that (recalling that $\underline{u}=\underline{0}$) one gets that $(-\underline{\partial}\cdot\underline{J}_q-\sum_{k=1}^{n-1}\rho\,\partial_tc_k\,(\mu_k-\mu_n))\partial_t\frac1T$ is equal, by the equation in [1.1.10], which in the present case becomes $\rho\partial_t\varepsilon=-\underline{\partial}\cdot\underline{J}_q$, to $\rho\partial_t\varepsilon-\rho\sum_k(\mu_k-\mu_n)\partial_tc_k\equiv\rho C\partial_tT$, to the heat generated per unit volume and C is the heat capacity during the transformation.
The thermal equilibrium condition is that in a volume Δ the function $G=U+p|\Delta|-TS=\sum_k\mu_k\rho_kV$ (hence, see [1.1.11], $dG=-SdT+|\Delta|dp+\sum_k\mu_kd\rho_k|\Delta|$) be a minimum at fixed T,p. Therefore in this case $dg=\sum_{k=1}^{n-1}(\mu_k-\mu_n)dc_k$ (where $g=\varepsilon-Ts+p\rho^{-1}$), and one sees that the quadratic form $(M\delta c,\delta c)=\sum_{h,k=1}^{n-1}\partial_{c_h}(\mu_k-\mu_n)\,\delta c_h\delta c_k$ must be positive definite. Hence

$$\dot\Gamma=2\int\big(-\frac{\rho C}{T^2}(\partial_tT)^2-\frac{\rho}{T}(M\partial_tc,\partial_tc)\big)\,dP\le0$$

in the states that are obtained by a small perturbation of an equilibrium state with minimal Γ.)

[1.1.18] The results on the minimality properties of Γ extend to cases more general than those treated in [1.1.16] but one cannot avoid the condition that the coefficients L_{ik} are constants, and therefore one cannot, on the basis of the above discussion, formulate a universal principle stating that the time independent states in a multicomponent fluid are obtained by *minimizing the entropy production* compatibly with the boundary condition and with the acting forces (because the constancy of L_{jk} is a rather restrictive assumption which is often not satisfied, not even approximately, *c.f.r.* [1.1.5]). One can then infer that there must be some conceptual problem in the interpretation of the thermodynamics of fluids? for instance the definition of entropy produced per unit mass in [1.1.12] and (1.1.21) is not completely free of ambiguities and it has a phenomenological nature. Hence a possible refoundation of nonequilibrium thermodynamics will have to be based on the principles of mechanics rather than on an extension, in some sense arbitrary, of the macroscopic equilibrium thermodynamics. It is conceivable that a generalization of classical thermodynamics to nonequilibrium phenomena (even if statistically stationary) may simply not be possible, at least not without a deep revision of the basic concepts. Nothing, in fact, allows us to believe that a so simple and deep theory, such as equilibrium thermodynamics, is really susceptible of extensions to other nonequilibrium phenomena with the exception of a few cases which are very special (even though very important).

Bibliography: [LL71], [DGM84]. Problems [1.1.7] ÷ [1.1.17] provide a concise exposition of the first 82 pages of [DGM84].

From [LL71]	*kinematic viscosity*	*thermal conductivity*	*Prandtl number*
	$\nu\ cm^2/sec$	$\chi\ cm^2/sec$	ν/χ
Air	$1.50\,10^{-1}$	$2.05\,10^{-1}$	$7.33\,10^{-1}$
Water	$1.00\,10^{-2}$	$1.48\,10^{-3}$	6.75
Alcohol	$2.20\,10^{-2}$	$1.33\,10^{-3}$	$1.66\,10^{+1}$
Glycerine	6.8	$9.38\,10^{-4}$	$7.25\,10^{+3}$
Mercury	$1.20\,10^{-3}$	$2.73\,10^{-2}$	$4.40\,10^{-2}$

§1.2 Equations of motion of a fluid in general. Ideal and incompressible cases. Incompressible Euler, Navier–Stokes and Navier–Stokes-Fourier equations.

At an *internal point* P in the region Ω the equations of motion of a fluid described by the fields $\underline{u}$, T, p are therefore (see §1.2)

$$\begin{aligned}
&(1)\qquad \partial_t\rho+\underline{\partial}\cdot(\rho\underline{u})=0\\
&(2)\qquad \rho\big(\partial_t\underline{u}+\underset{\sim}{\underline{u}}\cdot\underset{\sim}{\underline{\partial}}\underline{u}\big)=-\underline{\partial}p+\rho\underline{g}+\underset{\sim}{\underline{\partial}}\underset{\approx}{\underline{\tau}}\,'\\
&(3)\qquad T\rho\big(\partial_t s+\underline{u}\cdot\underline{\partial}s\big)=\underset{\approx}{\underline{\tau}}\,'\underset{\sim}{\underline{\partial}}\underline{u}+\underline{\partial}(\underset{\approx}{\underline{\kappa}}\,\underset{\sim}{\underline{\partial}}T)\\
&(4)\qquad s=s(\rho,\varepsilon),\qquad T^{-1}=\partial_\varepsilon s(\rho,\varepsilon),\qquad p=-T\rho^2\partial_\rho s(\rho,\varepsilon)\\
&(5)\qquad \tau'_{ij}=\theta_{ij}(\underset{\sim}{\underline{\partial}}\underline{u},\rho,T),\qquad \theta_{ij}(0,\rho,T)\equiv 0\\
&(6)\qquad \kappa_{ij}=\xi_{ij}(\rho,T)
\end{aligned}\tag{1.2.1}$$

and the equationa (4) (equation of state), (5), (6) (constitutive equation) imply that (1), (2), (3) are a system of five equations for the five unknowns $\underline{u}, T, \rho$.

Equations (5), (6) could be more general, if one allowed the stress tensor to depend also on the higher order derivatives of the velocity field $\underline{u}$, or if one allowed the thermal conductivity tensor, too, to depend on the derivatives of $\underline{u}$. As a rule we shall not consider so general models (see, however, §7.4).

One should note that changing frame of reference the equations (1.2.1) remain invariant: in fact if $t \to \rho(\underline{x},t)$, $T(\underline{x},t)$, $\underline{u}(\underline{x},t)$ is a solution of (1), (2), (3) then $t \to \rho'(\underline{x}',t)$, $T'(\underline{x}',t)$, $\underline{u}'(\underline{x}',t)$, with

$$\rho'(\underline{x}',t) = \rho(\underline{x}'+\underline{v}t,t), \quad T'(\underline{x}',t) = T(\underline{x}'+\underline{v}t,t), \quad \underline{u}'(\underline{x}',t) = \underline{u}(\underline{x}'+\underline{v}t,t)-\underline{v} \tag{1.2.2}$$

gives the motion as seen from an inertial reference frame moving with velocity $\underline{v}$ and coinciding with the preceding frame at time $t = 0$. One checks immediately that (1.2.2) solves (1.2.1): the main point is, naturally, that velocity appears only via its derivatives in the constitutive equations.

This invariance property could not possibly hold if the dependence of the constitutive equations on velocity did not manifest itself through the derivatives of $\underline{u}$: it is (also) for this reason that one does not (usually) consider constitutive equations in which there is an explicit dependence on $\underline{u}$ (and not just on its derivatives).

The functions $s(\rho,\varepsilon)$ are not arbitrary but they must satisfy conditions imposed by the laws of statistical thermodynamics and of statistical mechanics: for instance $\rho^{-1}s(\rho,\varepsilon)$ must be a convex function of its arguments, monotonically increasing in ε and decreasing in ρ, see [Ga99a].

Let us examine the class of particular cases of the (1.2.1) in which ρ is constrained to stay constant: these are the incompressible fluids.

(A) Incompressible non viscous fluid (Euler equations).

The simplest such fluids are the non viscous ($\tau' = 0$) non conducting ($\kappa = 0$) ones:

$$\tau'_{ij} = 0 \;, \qquad \kappa_{ij} = 0 \;. \tag{1.2.3}$$

In these cases, since ρ is everywhere constant, the (1.2.1) become:

$$\begin{aligned}
&(1) \qquad \underline{\partial}\cdot\underline{u} = 0\\
&(2) \qquad \partial_t\underline{u} + \underset{\sim}{\underline{u}}\cdot\underline{\partial}\,\underline{u} = -\frac{1}{\rho}\underline{\partial}p + \underline{g}\\
&(3) \qquad \partial_t s + \underline{u}\cdot\underline{\partial}s = 0\\
&(4) \qquad s = \sigma(T)
\end{aligned} \tag{1.2.4}$$

where we chose to think the entropy as a function of T since σ is not singular when $(\partial p/\partial\rho) = \infty$ (a relation expressing incompressibility).

Property (3) gives $(ds/dt) = 0$: *i.e.* s is constant along the lines of current of an incompressible fluid. Hence the case of a fluid which at the initial time is "*isoentropic*", *i.e.* the case $s(\underline{x},0) = s_0 =$ constant, is particularly interesting. This is in fact a property that remains true as time evolves and the temperature will be given at every point by a constant $T = f(s_0)$ and, therefore, it disappears from the equations of motion

$$\begin{aligned}
&\underline{\partial}\cdot\underline{u} = 0\\
&\partial_t\underline{u} + \underset{\sim}{\underline{u}}\cdot\underset{\sim}{\underline{\partial}}\,\underline{u} = -\frac{1}{\rho}\underline{\partial}p + \underline{g}
\end{aligned} \tag{1.2.5}$$

and we get four equations for the four unknowns $\underline{u}, p$, that are called *Euler equations.*

One should, however, ask the question of how could eq. (1.2.4) be possibly related with a real fluid. Density, in such a fluid, would be fixed and entropy would be a function of the temperature alone: but the equation of state would then determine the pressure and, therefore, (1.2.4) would be over determined. For instance if at the initial time s, hence T, were constant over the whole volume, they they would remain constant, hence p would be constant as well (being a function of them) so that $\underline{\partial}\, p = \underline{0}$ and we would have four equations for the three unknowns $\underline{u}$.

This means that (in the incompressible case) the quantity p that appears in (1.2.4) cannot be naively identified with the pressure in the physical sense of the word and, consequently, the interpretation of (1.2.4) is more delicate than it looks at first, *c.f.r.* remarks following (1.3.10).

(B) Incompressible, non heat conducting, viscous fluid (Navier–Stokes equations).

The next simplest case is that of a viscous incompressible non conducting fluid: in this case $\underset{\approx}{\kappa} = \underset{\approx}{0}$ but $\tau'_{ij} \neq 0$. Since τ'_{ij} must vanish for $\partial_i u_j = 0$ the simplest model is the one corresponding to the constitutive equation:

$$\tau'_{ij} = \eta\,(\partial_i u_j + \partial_j u_i) + \eta'\,\underline{\partial}\cdot\,\underline{u}\,\delta_{ij} \qquad\qquad \kappa = 0 \qquad\qquad (1.2.6)$$

with scalar η, η' depending only on ρ and s (and not on the derivatives of $\underline{u}$), which can be intended as a first order term of a series of τ'_{ij} in powers of $\underset{\sim}{\underline{\partial}\,\underline{u}}$ in which the higher order terms are neglected. The coefficient η is a function of ρ and s which is usually called *dynamic viscosity* while one calls $\nu = \eta/\rho$ *kinematic viscosity.* Incompressibility is expressed by $\underline{\partial}\cdot\underline{u} = 0$ and hence, in incompressible cases, the second term can be omitted.

An incompressible fluid with constitutive equation given by (1.2.6) and ν= constant is called an *incompressible Navier–Stokes fluid*, or "*NS-fluid*", and it is described by the equations

$$\begin{aligned}
&(1) \qquad \underline{\partial}\cdot\underline{u} = 0\\
&(2) \qquad \partial_t\underline{u} + \underset{\sim}{\underline{u}}\cdot\underset{\sim}{\underline{\partial}}\,\underline{u} = -\frac{1}{\rho}\underline{\partial}p + \nu\Delta\underline{u} + \underline{g}\\
&(3) \qquad \rho T(\partial_t s + \underline{u}\cdot\underline{\partial}s) = \frac{\eta}{2}\sum_{ij}(\partial_i u_j + \partial_j u_i)^2\\
&(4) \qquad s = \sigma(T)
\end{aligned} \qquad (1.2.7)$$

The first two equations should determine p and $\underline{u}$ while the fourth establishes a suitable relation σ between s and T which allows us to compute, via (3) and by integration along the current lines, the entropy density s starting form its initial value. The problem decouples and the "real" equations are the first two, called the *Navier–Stokes equations* or *NS–equations.*

The interpretation problem mentioned immediately after (1.2.5) evidently remains in the present case.

However one should remark that if the fluid is enclosed in a container Ω the above equations might lead to results that might be physically unacceptable: for onstance if the NS equations are subject to a boundary condition $\underline{u} = \underline{0}$ then the third equation shows that it will not be in general possible to fix the temperature at the boundary because the equation would imply that $\partial_t s > 0$ on the boundary and s (hence T) will increase with time and could not be held fixed. Hence in presence of thermoconduction the above equations will be an acceptable model only in special cases; see Sec. 5 for a treatment of the problem in presence of boundaries. The incompressible NS equations will therefore be acceptable only if the internal generation of heat can be completely neglected either because "small" or because it is assumed to be removed by some mechanism which is not described by the equations themselves or which is implemented through special boundary conditions, see Sec. 5.

(C) Incompressible thermoconducting viscous fluid

More difficult is the description of an incompressible fluid which, besides being viscous is also thermoconductor. If the fluid density did not depend from the temperature, then the equations for $\underline{u}$ would be identical to the Navier–Stokes equations with constant density (incompressibility means that the constant density is also pressure independent). In this case entropy should depend only on the temperature, $\sigma(T) = \int c\, dT/T$ for some function $c = c(T)$, and (1.2.1) should become the equations:

$$\begin{aligned} &(1) \qquad \underline{\partial} \cdot \underline{u} = 0 \\ &(2) \qquad \partial_t \underline{u} + \underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}} \underline{u} = -\frac{1}{\rho} \underline{\partial} p + \nu \Delta \underline{u} + \underline{g} \\ &(3) \qquad \rho\, c\, (\partial_t T + \underline{u} \cdot \underline{\partial} T) = \frac{\eta}{2} \sum_{ij} (\partial_i u_j + \partial_j u_i)^2 + \kappa \Delta T \end{aligned} \tag{1.2.8}$$

with $\rho = constant$. Hence the problem *seems* again to decouple into the temperature independent one of solving (1) and (2) and then into the one of solving (3) which is the Fourier equation in presence of transport of matter. The interpretation problems mentioned for the Euler and NS equations are still present.

In conclusion physical conditions under which one can assume with good approximation a constant density with a varying temperature are quite rare in applications to fluids.

For instance in convection problems variability of density as a function of temperature is essential: see the analysis in the following §1.3, §1.5.

The (1.2.8), which have therefore a rather limited interest, will be called "Navier–Stokes–Fourier equations".

(D) On the physical meaning of an incompressibility condition.

Since in real fluids it is $(\partial p/\partial\rho)_s < \infty$ we must ask in which cases a real fluid can be considered as incompressible.

To evaluate qualitatively the meaning of an incompressibility hypothesis and its possible validity one can have recourse to *dimensional considerations*

The idea behind the analysis is: imagine that the fluid had a motion which is regular and which is characterized by a "typical velocity variation" δv in the sense that velocity varies of the order of magnitude δv with respect to its average over space and time. Likewise imagine that δT is a "typical temperature variation" with respect to the average temperature and δp is a "typical variation of pressure", *etc.* Furthermore imagine that the above variations show up on a length scale of size l or on a time scale τ.

In this situation quantities like $\underset{\sim}{\underline{\partial}}\underline{u}$, $\partial_t\underline{u}$, $\Delta\underline{u}$, $\underline{\partial}T$, $\partial_t T$, ΔT, $\underline{\partial}p$ can be estimated to have "typically" size of order of magnitude

$$|\underset{\sim}{\underline{\partial}\underline{u}}| \sim \frac{\delta v}{l}\,,\ |\partial_t\underline{u}| \sim \frac{\delta v}{\tau}\,,\ |\Delta\underset{\sim}{\underline{u}}| \sim \frac{\delta v}{l^2}\,,\ |\underline{\partial}T| \sim \frac{\delta T}{l}\,,\ |\Delta T| \sim \frac{\delta T}{l^2},\ |\partial p| \sim \frac{\delta p}{l} \tag{1.2.9}$$

which are interpreted as a maximal order of magnitude for such quantities.

Since we suppose that $\underline{u}$, ρ, T are related via the equations of motion it follows that certain relations must hold among the various quantities in (1.2.9). More precisely there must exist instants in which a given term of the equations has the same order of magnitude of any other (otherwise if, for instance, a term was always (much) smaller than another *we could neglect it* and the equation would become simpler).

Hence, for instance, since $\partial_t\underline{u}$ + other terms $= -\rho^{-1}\underline{\partial}p$ (*c.f.r.* (1.2.1), eq. (2)), the remark is that *in some instant and in some point it must be* $\tau^{-1}\delta v \sim \rho^{-1}\delta p/l$; *and since* $\ldots + \underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}}\underline{u} + \ldots = -\rho^{-1}\underline{\partial}p$ *there will be an instant and a point where* $(\delta v)^2/l \sim \delta p/\rho l$.

In the isoentropic case, *i.e.* when $(\partial\rho/\partial s)_p = -\rho^2(\partial T/\partial p)_s \equiv \rho^2\chi_s$ can be neglected in evaluating density variations,[1], one finds

$$\frac{\Delta\rho}{\rho} \cong \left(\frac{\partial\rho}{\partial p}\right)_s \frac{\delta p}{\rho} \tag{1.2.10}$$

and we realize that the condition of validity of the incompresssibility assumption, *i.e.* $\Delta\rho/\rho \ll 1$, can be obtained by estimating the largest values that $\delta p/\rho$ can take.

(E) The case of incompressible Euler equations

In the case of Euler equations (1.2.5), with $\underline{g} = 0$ for simplicity, the above remark tells us that $\delta p/l\rho$ can reach the following two sizes

$$\frac{\delta p}{l\rho} \sim \frac{\delta v}{\tau} \quad \text{or} \quad \frac{\delta p}{l\rho} \sim \frac{(\delta v)^2}{l} \tag{1.2.11}$$

[1] Where χ_s is the coefficient of heating in adiabatic compressions and the identity is derived from $\delta w = T\,\delta s + \rho^{-1}\delta p$ if w is the *enthalpy* per unit mass.

hence the condition $\Delta\rho/\rho \ll 1$ becomes $\left(\frac{\delta v}{v_s}\right)\frac{l}{\tau v_s} \ll 1$ and $\left(\frac{\delta v}{v_s}\right)^2 \ll 1$ where $v_s^{-2} = (\partial\rho/\partial p)_s$ and v_s has, as it is well known from elasticity theory, the meaning of speed of sound $v_s \equiv v_{sound}$ in the fluid. Hence the condition (1.2.10) becomes

$$\frac{\delta v}{v_{sound}} \ll 1, \qquad \frac{l}{\tau v_{sound}} \ll 1 \tag{1.2.12}$$

This relation is interpreted as follows; the Euler fluid can be considered "incompressible" *if the velocity variations are small with respect to the sound speed* and, furthermore, *if the variations manifest themselves over a length scale small with respect to the length run at sound speed over a the time scale over which velocity variations are appreciable.*

(F) The case of the Navier–Stokes equations.

In the case of the NS equations, still with $\underline{g} = \underline{0}$, the new term $\nu\,\Delta\underline{u}$ in the second of (1.2.7) adds to (1.2.11) a new comparison term $\nu\,\delta v\,l^{-2}$: hence the (1.2.10) adds $(\delta v)\nu/v_s^2 l \ll 1$ to the conditions (1.2.12) the $(\delta v)\nu/v_s^2 l \ll 1$; or

$$\frac{\nu}{v_{sound} l} \ll 1 \tag{1.2.13}$$

Since the second of the (1.2.8) coincides with the second of the (1.2.7) and it is the only equation containing p, we see that also (1.2.12), (1.2.13) are all the conditions of validity of the incompressibility assumption, under the hypothesis that the coefficient of heating by compression is neglegible.

If, finally, one supposes that in the considered equations it is also $\underline{g} \neq \underline{0}$ we add a new term in the (2) and we see that δp can also become such that $\delta p/\rho l \sim g$ thus leading to the further condition

$$\frac{lg}{v_{sound}^2} \ll 1 \tag{1.2.14}$$

Which means that, in presence of gravity, *the speed acquired in a free fall from a height equal to the characteristic length over which velocity changes must be small compared to the sound speed in order that the fluid could be considered as incompressible*; hence (1.2.12), (1.2.13) and (1.2.14) express incompressibility conditions in the various cases envisaged, always *if* the heating coefficient for an adiabatic compression can be considered zero.

(G) The case in which heating in adiabatic compressions is not negligible.

If $\left(\frac{\partial\rho}{\partial s}\right)_p \equiv \rho^2\chi_s$, *c.f.r.* (1.2.10), is not zero, *i.e.* if the coefficient of heating under adiabatic compression is not zero, we must add to (1.2.10) the term

$$\left(\frac{\partial\rho}{\partial s}\right)_p \frac{\delta s}{\rho} \equiv \rho\chi_s\delta s \tag{1.2.15}$$

and δs is evaluated, in the Navier–Stokes case, via the third of (1.2.7) yielding the two estimates

$$T\frac{\delta s}{\tau}\simeq\nu\frac{(\delta v)^2}{l^2}\quad\text{and}\quad T\delta s\frac{\delta v}{l}\simeq\nu\frac{(\delta v)^2}{l^2}\tag{1.2.16}$$

And we thus see that incompressibility is justified, in the NS equations case, from the validity of (1.2.12), (1.2.13), (1.2.14) *and in addition*

$$\rho\chi_s\nu\frac{\delta v^2}{l^2}\frac{\tau}{T}\ll1\quad\text{and}\quad\rho\chi_s\nu\frac{(\delta v)}{lT}\ll1\tag{1.2.17}$$

Finally, in the case of the (1.2.8) we must add to the last of (1.2.16) the $\rho T\frac{\delta s}{\tau}\approx\kappa\delta T l^{-2}$ e $\rho T\frac{\delta v}{l}\delta s\approx\kappa\frac{\delta T}{l^2}$; and therefore (via (1.2.15)) one finds the further conditions:

$$\kappa\frac{\delta T}{T}\frac{\tau\rho\chi_s}{l^2}\ll1\quad\text{e}\quad\kappa\frac{\delta T}{T}\frac{\rho\chi_s}{l\delta v}\ll1\tag{1.2.18}$$

which complete the list of the incompressibility conditions, apart from the problems with boundary conditions, that we have only mentioned in Sect. 1.2.2 and 1.2.3, and which will be studied in some detail in a simple case in Sect. 1.5.

Note to §1.2: dimensional arguments.

One can ask whether the notion of "dimensional argument" can be rendered more precise from a mathematical viewpoint.

It is useful to recall, for this purpose, that analytic functions enjoy a remarkable property: namely if $x\to f(x)$ is a function of the variable x defined for $x\in D$, one says that f is analytic if for every $x_0\in D$ the Taylor series

$$f(x)=\sum_{k=0}^{\infty}f^{(k)}(x_0)(x-x_0)^k/k!\tag{1.2.19}$$

converges for $|x-x_0|$ small enough. Or, equivalently, f is analytic if it is the sum of its own Taylor series around every point.

Then it follows that if f is analytic on D and we suppose that D is the closure of a bounded open set, then it is possible to find a value $\rho>0$ such that the Taylor series of f around *any* $x_0\in D$ has convergence radius at least $\rho>0$. Hence we shall be able to define $f(x)$ for complex values of x: if $|z-x_0|<\rho$ one sets

$$f(z)=\sum_{k=0}^{\infty}f^{(k)}(x_0)(z-x_0)^k/k!\tag{1.2.20}$$

and if the same point z is closer than ρ to two points x_0 and x_0' the two formulae for $f(z)$ obtained by choosing in (1.2.20) once x_0 and another time x_0' must coincide, because the two functions of z so defined must agree for the real z's common to the two intervals of radius ρ and centers x_0,x_0' (hence for infinitely many points and, hence, their identity follows from the identity principle for holomorphic functions).

Thus to say that a function $x\to f(x)$ is analytic on a closed and bounded real domain D is equivalent to saying that it is holomorphic in a complex domain $D_\rho\equiv\{z|\exists x\in D,\ |z-x|\le\rho\}\equiv\{|z-D|<\rho\}$ for some $\rho>0$.

We shall then say that a function f defined on a closed and bounded domain D is "regular on scale ρ" if it is analytic and the convergence radius of its Taylor series around any point is at least ρ, or equivalently if it is holomorphic in D_ρ.

The above notion of regularity is particularly relevant for *dimensional estimates*: indeed if f is *regular on scale* ρ *in* D we shall say that it has a *typical size* $|f|_\rho = \max_{z\in D_\rho} |f(z)|$ and we shall be able to estimate its derivatives as

$$|\partial_x^n f(x)| \le n!|f|_\rho \rho^{-n} \qquad\qquad \forall\, x \in D \tag{1.2.21}$$

i.e. the n-th derivative is simply estimated by dividing the *size of* f by the typical length ρ raised to the n-th power. Just as in the dimensional estimates that are introduced in various arguments in theoretical physics.

Hence the *regularity* on scale ρ and the typical size of a physical quantity that depends on a parameter x have a clear meaning when f is analytic and holomorphically extendible over a distance ρ in the complex and, in the extended domain, it is bounded by a constant $|f|_\rho$ which is identified as the "*typical size*" of f.

In the previous analysis a regular velocity field $\underline{u}(\underline{x}, t)$ must be interpreted as an analytic function in each of the variables x_j and t continuable in the complex, in each variable, by l in the x_j and by τ in the t, remaining bounded therein by δv, and likewise the $s = s(\underline{x}, t)$, $p = p(\underline{x}, t)$ must be analytic and continuable by l and τ, in x_j and t, respectively, staying bounded by δs and δp.

Thus we see that "accepting dimensional estimates" corresponds mathematically to admitting precise regularity properties on the functions under investigation.

Whenever such properties do not hold it becomes necessary to reexamine the dimensional argument: and sometimes it can turn out to be grossly incorrect. This happens when in the problem appear "several scales" very different from each other.

For instance sometimes the function $f(x)$ can be written as a sum of infinitely many functions $f_1(x) + f_2(x) + \ldots$ with f_i regular on scale ρ_i and of order of magnitude δ_i and, furthermore, $\rho_i \xrightarrow[i\to\infty]{} 0$. It is clear that in such cases one shall be very cautious in formulating dimensional arguments. For an explicit example consider a sequence $f_i(x) \equiv c_i f(x/\rho_i)$ with $\rho_i = 2^{-i}$, $c_i = 2^{-i^2}$ or $c_i = 2^{-ki}$ with k integer and fix $f(x)$ to be a rapidly decreasing function (*e.g.* if $D = [0, +\infty)$ we can take $f(x) = e^{-x}$).

Finally we mention that (1.2.21) is a simple consequence of Cauchy's theorem

$$f^{(n)}(x) = \frac{n!}{(2\pi i)} \oint \frac{f(z)}{(z-x)^{n+1}}\, dz \tag{1.2.22}$$

where the contour can be chosen as a circle around x contained in D_ρ: by choosing exactly the radius of the cirdle to be ρ and bounding above the right hand side by the absolute values one immediately gets (1.2.21).

The problems in which there are many length or time scales are called *multiscale problems*: dimensional arguments are in such cases called *scaling arguments*. In recent times new methods for their analysis have been developed, like the "*renormalization group method*". But since a long time they attract the interests of physicists and mathematicians and many beautiful phenomena in mathematics and physics have been understood, I just mention here the almost everywhere convergence of Fourier series for square integrable functions in mathematics and the ultraviolet stability of some quantum fields in three space time dimensions in relativistic physics and, as we shall see in the last sections of this book, in the developed turbulence theory in fluid mechanics, see [Ca66],[Fe72],[BG95].

Problems: *Stokes formula.*

[1.2.1]: Consider a viscous fluid occupying the entire space outside a sphere of radius R and moving with a velocity $\underline{v}_0$ at ∞. Suppose the motion time independent and the velocity so small that one can neglect the transport term $\underline{u}\cdot\underset{\sim}{\underline{\partial}}\underline{u}$ in the NS equation $\underline{0}=-\rho^{-1}\underline{\partial}p+\eta\rho^{-1}\Delta\underline{u}$ and $\underline{\partial}\cdot\underline{u}=0$. This is the *Stokes equation* which can be written

$$\Delta\,\mathrm{rot}\,\underline{u}=\underline{0},\qquad \underline{\partial}\cdot\underline{u}=0,\qquad \underline{u}=\underline{0}\quad \text{if } |\underline{x}|=R$$

Show that there is at most one smooth solution $\underline{u}$ tending to $\underline{v}_0$ as $r\to\infty$ and such that $r^2|\underset{\sim}{\underline{\partial}}\underline{u}|$ is bounded as $r\to\infty$.

The solution, if existent, must have the form

$$\underline{u}=\underline{v}_0+f_1(r)\underline{v}_0+f_2(r)\underline{n}\cdot\underline{v}_0\,\underline{n}+f_3(r)\underline{n}\wedge\underline{v}_0$$

with $r\equiv|\underline{x}|,\underline{n}\equiv\underline{x}/r$ and $f_j\xrightarrow[r\to\infty]{}0$. Furthermore $f_3\equiv0$ by parity symmetry. (*Idea:* Uniqueness follows because the difference $\underline{\delta}$ between two solutions must be such that $\Delta\underline{\delta}=\underline{\partial}\pi$ for some π; hence multiplying both sides by $\underline{\delta}$ and integrating by parts one gets that $\underset{\sim}{\underline{\partial}}\underline{\delta}=0$. The equation is linear and the only vectors linearly depending on $\underline{v}_0$ which can be made with $\underline{v}_0$ and $\underline{x}$ are $\underline{v}_0,\underline{x},\underline{x}\wedge\underline{v}_0$. Uniqueness implies that the solution must be parity invariant).

[1.2.2]: Chose $\underline{v}_0$ along the x–axis and check that if $\underline{u}$ has the form in [1.2.1] then

$$\begin{aligned}\underline{\partial}\cdot\underline{u}&=v_0\,x\,(\frac{f_1'}{r}+4(\frac{f_2}{r^2})+r(\frac{f_2}{r^2})')\\ \mathrm{rot}\,\underline{u}&=v_0\,\big(f_1'-\frac{1}{r}f_2\big)\,(0,\frac{z}{r},-\frac{y}{r})\end{aligned}$$

Check also that the choices $f_1=r^{-1},f_2=r^{-1}$ and $f_1=-r^{-3},f_2=3r^{-3}$ generate fields $\underline{u}$

$$\frac{1}{r}\big(\underline{v}_0+\underline{n}\,\underline{v}_0\cdot\underline{n}\big),\ \text{ or }\ \frac{1}{r^3}\big(-\underline{v}_0+3\underline{n}\,\underline{v}_0\cdot\underline{n}\big)$$

which have 0 divergence and a harmonic rotation (*i.e.* a rotation with 0 laplacian).

[1.2.3]: Hence one can look for a solution like

$$\underline{u}=\underline{v}_0-\frac{a}{r}\big(\underline{v}_0+\underline{n}\,\underline{v}_0\cdot\underline{n}\big)+\frac{b}{r^3}\big(-\underline{v}_0+3\underline{n}\,\underline{v}_0\cdot\underline{n}\big)$$

Show that the coefficients a and b are uniquely determined by the condition $\underline{u}=\underline{0}$ for $|\underline{x}|=R$ and have the value

$$a=\frac{3}{4}R,\qquad b=\frac{1}{4}R^3$$

(*Idea:* Note, for instance by [1.2.2], that the combinations in front of a and b have 0 divergence and rotation with 0 laplacian).

[1.2.4]: Compute the pressure field associated with the velocity field determined in [1.2.2] showing that $p(\underline{x})=-3\,\eta\,R\,\underline{v}_0\cdot\underline{n}/2r^2$. (*Idea:* $-\underline{\partial}p+\eta\Delta\underline{u}=\underline{0}$.)

[1.2.5]: The force exerted on the sphere has, if we choose the z axis parallel to the force, a z–component

$$F=R^2\int\Big(-p\cos\vartheta+(\tau'_{zz}\cos\vartheta+\tau'_{zx}\sin\vartheta\cos\varphi+\tau'_{zy}\sin\vartheta\sin\varphi)\Big)\sin\vartheta\,d\vartheta d\varphi$$

where $\tau'_{zj}=\eta(\partial_j u_x+\partial_z u_j)$, and show that this implies that $F=\eta Rv_0S$ where S is a constant. Compute S ($S=6\pi$, *Stokes formula*).

[1.2.6]: (*meaning of approximations*) Discuss under which assumptions the approximation in [1.2.1] can be acceptable. Show that the conditions imposed are realizable around the sphere because they are: $v_0^2/R \ll \nu v_0/R^2$, *i.e.* $v_0\nu^{-1}R \ll 1$ which is read, see the coming sections, by saying that the "Reynolds' number" is small. However such conditions are *not valid* far away from the sphere because there they become $v_0^2\,R/r^2 \ll \nu v_0\,R/r^3$, *i.e.* $v_0\nu^{-1}r \ll 1$. Hence at large distances the velocity field determined via the Stokes' approximations [1.2.1], [1.2.3], cannot be taken as correct. (*Idea:* Compare the size of the transport term $\underset{\sim}{u} \cdot \underset{\sim}{\partial}\, \underline{u}$ to the size of the viscous term $\nu\,\Delta\underline{u}$ by using the solution in [1.2.3].)

Bibliography: The discussion reported in (D,E,F) follows the ideas in [LL71].

§1.3 The rescaling method and estimates of the approximations.

The procedure illustrated in §1.2 to evaluate the orders of magnitude involved in the incompressibility approximations is simple but, in a way, not very systematic.

In fact the claim that (adimensional) quantities $\varepsilon \ll 1$ can be neglected is a satisfactory statement *only* if one is able to evaluate the error made and to show that corrections really have size ε with respect to the terms that are not neglected, as implicitly supposed in the analysis.

This can only be an asymptotic statement and what one really means, or what one should mean, is that it is possible to write the solution of the equations, that we want to approximate, as a series in the parameter ε. But from the discussion we see that ε appears in various forms and it is by no means clear what it does really mean that "we neglect terms of the order ε", in particular, when ε appears both as an order of magnitude of certain quantities and as an argument of relevant functions (as it happens when we say that a function varies on the scale l and $l/\tau v_{sound} = \varepsilon$ is small (*c.f.r.* (1.2.12)).

To make more precise the above intuitive idea we shall translate into a more mathematical form some of the arguments discussed in §1.2, trying to construct, at least in principle, an algorithm that allows us to write the equations necessary for the evaluation of the error as a series in a parameter (on in several parameters) $\ll 1$.

(1) Incompressible Euler equation.

For instance consider the case (A), §1.2, of the incompressible Euler equation, with $\underline{g} = \underline{0}$ for simplicity. Assume that the system is a *perfect gas* with constitutive equations.

$$s = c_V \,\log T - c\,\log\rho, \qquad \tau'_{ij} = 0, \qquad \kappa_{ij} = 0 \tag{1.3.1}$$

which, via the thermodynamic relation $p = -T\rho^2\,(\frac{\partial s}{d\rho})_T$, implies $p = c\,\rho T$.

Furthermore let $v^2_{sound} = (\frac{\partial p}{\partial \rho})_s = cT\,(1 + c/c_V)$ be the velocity sound.[1]

Let $\underline{\bar{u}}, \bar\rho, \bar s$ be an initial datum with the property of satisfying the first of the (1.2.12). This state can be assigned in terms of three functions $\underline{\bar w}(\underline{\xi}), \bar r(\underline{\xi}), \bar\sigma(\underline{\xi})$ very regular in their arguments $\underline{\xi} \in R^3$ as

$$\underline{\bar u}(\underline{x}) = \varepsilon\, v_{sound}\, \underline{\bar w}(\frac{\underline{x}}{l}), \quad \bar\rho(\underline{x}) = \bar r(\frac{\underline{x}}{l}), \quad \bar s(\underline{x}) = \bar\sigma(\frac{\underline{x}}{l}) \tag{1.3.2}$$

where ε is a very small parameter, so that the initial data in (1.3.2) satisfy *a priori* the condition that the initial velocity $\underline{\bar u}$ be small compared to the

[1] In the case of a perfect monoatomic gas $c_V = \frac{3}{2} R M_0^{-1}, c = R M_0^{-1}$ with R the gases constant, and M_0 the atomic mass.

sound speed; and they vary on a length scale l, which is a parameter with dimension of a length. The velocity v_{sound} depends on T (which depends on $\underline{x}$) and here we define it as equal to the value corresponding to the average value of T (computed from the equation of state in the initial configuration). We shall imagine the system in infinite space and that the functions in (1.3.2) are constants outside a bounded set and that the initial $\underline{\bar{w}}$ vanishes outside this set.

We now ask if there exists a solution to (1.2.1), (1.3.1) satisfying (1.2.12) also at positive times, and if this solution is *well approximated* by the solution of (1.2.4) with the same initial data, and *the better the smaller* ε is.

We shall limit ourselves to the analysis of the case $\bar{r}$ = *constant*, $\bar{\sigma}$ = *constant*, even though it will be instructive to write a few more general equations.

To pose correctly the question we ask whether (1.2.1), (1.3.1) with the initial datum (1.3.2), has a solution depending *regularly* on t through the "rescaled time" $\vartheta = \varepsilon t l^{-1} v_{sound}$: so that the second'of the (1.2.12) is automatically satisfied (because the time scale τ will be such that $\varepsilon\,\tau\,v_{sound} l^{-1} \simeq 1$ and, hence, $l/(\tau\,v_{sound}) \simeq \varepsilon \ll 1$).

More formally we ask the question whether a solution of (1.2.1) with equation of state (1.3.1) exists such that

$$\begin{aligned}\underline{u}(\underline{x},t) &= \varepsilon\, v_{sound}\, \underline{w}(\underline{x} l^{-1}, \varepsilon\, t\, v_{sound} l^{-1})\\ \rho(\underline{x},t) &= r(\underline{x} l^{-1}, \varepsilon\, t\, v_{sound} l^{-1})\\ s(\underline{x},t) &= \sigma(\underline{x} l^{-1}, \varepsilon\, t\, v_{sound} l^{-1})\end{aligned} \tag{1.3.3}$$

with $\underline{w}(\underline{\xi},\vartheta), r(\underline{\xi},\vartheta), \sigma(\underline{\xi},\vartheta)$ *regular functions of their arguments* and depending on ε so that they can be written as

$$\begin{aligned}\underline{w} &= \underline{w}_0 + \varepsilon \underline{w}_1 + \varepsilon^2 \underline{w}_2 + \dots\\ r &= r_0 + \varepsilon r_1 + \varepsilon^2 r_2 + \dots\\ s &= \sigma_0 + \varepsilon \sigma_1 + \varepsilon^2 \sigma_2 + \dots\end{aligned} \tag{1.3.4}$$

The regularity of $\underline{w}, r, \sigma$ implies that in the case in (1.3.3) the conditions (1.2.12) will continue to be satisfied at positive times and therefore *we expect*, if the discussion in §1.2 is correct, that the (1.3.3) verify the Euler incompressible equation, at a first approximation.

The latter property has now a precise mathematical meaning. In fact inserting the (1.3.4),(1.3.3) in (1.2.1), (1.3.1) and imposing that equations (1.2.1),(1.3.1) are verified at all orders in ε, we obtain equations for the $\underline{w}_j, r_j, \sigma_j$ which, solved with the natural initial data

$$\begin{aligned}&\underline{w}_0\big|_{\vartheta=0} = \underline{\bar{w}}, \quad r_0\big|_{\vartheta=0} = \bar{r} = constant, \quad \sigma_0\big|_{\vartheta=0} = \bar{\sigma}\\ &w_j, r_j, \sigma_j\big|_{\vartheta=0} = 0\end{aligned} \tag{1.3.5}$$

privide us with "solution to lowest order in ε", given by (1.3.3) with $\underline{w}, r, \sigma$ replaced by $\underline{w}_0, r_0, \sigma_0$ and the higher order corrections.

Then the question that we asked above is whether the functions

$$\varepsilon v_{sound}\, \underline{w}_0(\underline{x}l^{-1}, \varepsilon t v_{sound} l^{-1}), \quad r_0 = constant, \quad \sigma_0(\underline{x}l^{-1}, \varepsilon\, t v_{sound} l^{-1}) \tag{1.3.6}$$

verify incompressible Euler equations (1.2.4). Or

$$\begin{aligned} & r_0 = constant, \qquad \underline{\partial}_{\underline{\xi}} \cdot \underline{w}_0 = 0 \\ & \partial_\vartheta \underline{w}_0 + \underset{\sim}{w}_0 \cdot \underline{\partial}_{\underline{\xi}} \underline{w}_0 = -\underline{\partial}_{\underline{\xi}} p', \qquad \partial_\vartheta \sigma_0 + \underset{\sim}{w}_0 \cdot \underline{\partial}_{\underline{\xi}} \sigma_0 = 0 \end{aligned} \tag{1.3.7}$$

if $p'(\underline{\xi}, \vartheta)$ is a suitable function.

The equations for the successive orders should determine recursively $\underline{w}_j, r_j, \sigma_j$ and therefore all the corrections, *systematically.*

We would verify in this way, in a precise sense, that the slow velocity motions of the perfect gas under analysis is well approximated by the incompressible Euler equations. And, if we could devise an algorithm to compute the corrections w_j, r_j, σ_j, $j \ge 1$, it would make sense also to estimate the errors of the approximation

$$\begin{aligned} \underline{u}(\underline{x}, t) =& \varepsilon v_{sound}\, \underline{w}_0(\frac{\underline{x}}{l}, \varepsilon \frac{v_{sound} t}{l}), \quad \rho(\underline{x}, t) = r_0(\frac{\underline{x}}{l}, \varepsilon \frac{v_{sound} t}{l}), \\ s(\underline{x}, t) =& \sigma_0(\frac{\underline{x}}{l}, \varepsilon \frac{v_{sound} t}{l}) \end{aligned} \tag{1.3.8}$$

of the solutions of (1.2.1),(1.3.1) via the solutions of the incompressible Euler equation (1.2.4), (*i.e.* (1.3.6) and (1.3.7)).

It is useful to underline, again, that in our situation, the second of (1.2.12) follows from the first because from (1.3.8) we see that the scale of time evolution is $l v_{sound}^{-1} \varepsilon^{-1}$ (hence the second equation of (1.2.12) becomes $\varepsilon^2 \ll 1$ which coincides with the first); and if the property of approximation of (1.3.3) via the (1.3.4),(1.3.5) holds then the validity of (1.2.12) at the initial time (guaranteed for $\varepsilon \ll 1$ from (1.3.2)) implies its validity at the successive instants, at least up to the time $t = \tau_0 l \varepsilon^{-1} v_{sound}^{-1}$ if τ_0 is the instant until which the Euler equation (1.3.6), in the adimensional variables $\underline{\xi}, \vartheta$, with initial data (1.3.5) admits a solution that stays regular in $\underline{\xi}, \vartheta$.

Therefore we shall proceed to checking that the assumption that (1.2.1) admit a solution that can be developed in powers of ε is a *consistent* assumption and that it really leads to (1.3.7) at the lowest order in ε at least. Once we shall have succeeded, at least formally, we shall have obtained a precise qualitative check of the incompressibility assumptions. A quantitative check will require, then, in principle also an analysis of the series (1.3.4) or at least the analysis of the terms neglected and the possible proof that they tend to 0 for $\varepsilon \to 0$ in a way that can be estimated explicitly.

Note that the assumption $\bar r$ constant and $\bar\sigma=$ constant for $\vartheta=0$ (*i.e.* the case to which we confine our attention) implies that s is constant for all times because the right hand side of the third of the (1.2.1) vanishes. Then it follows that the pressure is a function $p=p(\rho)$ of the density (and $p(\rho)$ is the "adiabatics equation": $p(\rho)=C\rho^{1+c/c_V}$, with C suitable and independent of $\underline{x},t$, determined from the initial conditions).

The (1.3.3),(1.3.4),(1.3.5), can be inserted into (1.2.1); taking into account the assumptions on the constitutive equations made when considering (1.3.1) and supposing $\underline{g}=\underline{0}$, one finds (writing only the lowest orders in ε)

$$
\begin{aligned}
&\frac{\varepsilon v_{sound}}{l}\big(\partial_\vartheta r_0+\underline{\partial}_{\underline{\xi}}\cdot(r_0\underline{w}_0)\big)=0 \qquad (1.3.9)\\
&r_0\frac{\varepsilon^2 v^2_{sound}}{l}\big(\partial_\vartheta\underline{w}_0+\underset{\sim}{w}_0\cdot\underset{\sim}{\partial}_{\underline{\xi}}\underline{w}_0\big)+\ldots=-\frac{v^2(r)}{l}\big(\underline{\partial}r_0+\varepsilon\underline{\partial}r_1+\varepsilon^2\underline{\partial}r_2\ldots\big)\\
&\frac{\varepsilon v_{sound}}{l}\big(\partial_\vartheta\sigma_0+\underset{\sim}{w}_0\cdot\underset{\sim}{\partial}_{\underline{\xi}}\sigma_0\big)=0
\end{aligned}
$$

where $v^2(r)=\frac{\partial p(\rho)}{\partial\rho}\Big|_{\rho=r}$ is essentially still the square of the sound speed (at density r so that we denote it differently from the average quantity v^2_{sound}).

We realize that, in order that the second equations be consistent, it must be $\underline{\partial}r_0=\underline{0}$, *i.e.* the assumptions are consistent only if r_0 is *constant* as a function of $\underline{\xi}$. And the first of the (1.3.9) will say that $\underline{\partial}\cdot\underline{w}_0$ is constant in $\underline{\xi}$ (being equal to $\partial_\vartheta r_0$ with r_0 constant in $\underline{\xi}$) and, hence, vanishing if we suppose that $\underline{w}_0$ tends to zero at infinity for all times ϑ; likewise r_1 must be constant in $\underline{\xi}$. Hence also $\partial_\vartheta r_0=0$ and $r_0=\bar r$ stays constant and $\underline{\partial}\cdot\underline{w}_0=0$. In such case the (1.3.9) become the dimensionless equations:

$$
r_0=\bar r,\qquad \underline{\partial}\cdot\underline{w}_0=0,\qquad \sigma_0=\bar\sigma,\qquad \partial_\vartheta\underline{w}_0+\underset{\sim}{w}_0\cdot\underset{\sim}{\partial}_{\underline{\xi}}\,\underline{w}_0=-\frac{1}{r_0}\underline{\partial}r_2 \qquad (1.3.10)
$$

and, by the equation of state, $T=$ *constant*.

Thus we have obtained in the rescaled variables, (1.3.7), and in the adiabatic case the incompressible Euler equations.

We see another interesting property: namely what we call "pressure" in the incompressible Euler equations really is, up to a constant, the deviation from the average density to second order in ε.

In principle we should derive (infinitely many) other differential equations which should allow us to evaluate the corrections at the various orders in ε. But such equations would certainly be involved (if possible at all) and of little interest since we shall not have a grip on a theory for them (because we are unable, to this date, to really build a satisfactory theory for the lowest order, *i.e.* for the incompressible Euler equations, as we shall realize in the coming sections). Hence there is a serious risk that what said so far will remain for a long time at a formal level.

The above remarks help understanding the importance of the following theorem that considers the (1.2.1), (1.3.1) with initial data having the form

(1.3.5) with $\underline{\overline{w}} \in C^\infty$, and with $\underline{\overline{w}}_0$ vanishing outside a bounded set. And it allows us to say that the solution of the (1.2.1) tends, as $\varepsilon \to 0$, to the solution of the incompressible Euler equation in the following sense. We consider the solution of the equations (1.3.10), $w_0(\vartheta,\underline{x}),\sigma_0(\vartheta,\underline{x}),r_0$, then the following theorem holds, [Eb77], [EM94]:

1 Theorem *(incompressible Euler limit): The Euler equation (1.3.10) with the initial data (at $\vartheta = 0$) $\underline{w}_0(\underline{x},0) = \underline{\overline{w}}(\underline{x})$, $r_0 = \overline{r}$, $\sigma_0 = \overline{\sigma}$ admits a solution of class C^∞, that rapidly vanishes at infinity together with its derivatives, for times $\vartheta \le \tau_0$, if τ_0 is small enough (but depending on the initial data).*
The existence time τ_0, dimensionless by definition, can be chosen so that, for the times $t < \tau_0\, l/\varepsilon v_{sound}$, also Eq. (1.2.1) and Eq. (1.3.1) with initial data of the form (1.3.3) with ε a positive parameter, admit a solution of class C^∞, $\underline{u}_\varepsilon(\underline{x},t),\rho_\varepsilon(\underline{x},t),s_\varepsilon(\underline{x},t) \equiv \bar\sigma$. And one has

$$\begin{aligned} &\varepsilon^{-1}|\underline{u}_\varepsilon(\underline{x},t) - \varepsilon v_{sound}\,\underline{w}_0(\underline{x}l^{-1},\varepsilon\, v_{sound}\, t\, l^{-1})| \xrightarrow[\varepsilon\to 0]{} 0 \\ &|\rho_\varepsilon(\underline{x},t) - \bar r| \xrightarrow[\varepsilon\to 0]{} 0 \end{aligned} \tag{1.3.11}$$

uniformly for $t < \tau_0\, l/\varepsilon v_{sound}$.

Remarks:

(1) Note that the theorem is formulated without involving at all the higher order terms of the series (1.3.9). Hence, independently of their existence, it is rigorously established that, at least for a small time t, $t < \tau_0 l/\varepsilon\, v_{sound}$ (but of the order of ε^{-1}, *i.e.* "*the smaller ε is the better the incompressibility property is satisfied*"), the incompressible Euler equation provides an effective approximation to the solution of the (1.2.1),(1.3.1).
(2) The role of the previous statement (1) is to insure that the theorem be not empty it is obviously necessary to show that the incompressible Euler equation, with the initial data considered in the theorem, admits a solution up to a time $\tau_0 > 0$, which maintains the necessary properties of regularity. Such a theorem is indeed possible and it will be discussed in §3.1.
(3) It would be interesting to show that the time τ_0 of the theorem is of the order of the maximum time for which the incompressible Euler equation admits a regular solution (with the initial datum considered). This would be particularly interesting in the case of a fluid in a space with dimension $d = 2$: in this case, as we shall see, the incompressible Euler equation admits a global solution (*i.e.* a solution for all times) without losing regularity (*i.e.* data initially of class C^∞ remain such). Unfortunately the proof of the theorem *does not* allow us to conclude this much and the time τ_0 is an estimate that turns out to be shorter than the maximum time for which one can show, see §3.1, existence of regular solutions for the Euler equations.

(2) The incompressible Navier–Stokes equation.

In this case one must add in the right hand side of the second of (1.3.9) the term

$$\varepsilon \frac{\rho \nu v_{sound}}{l^2} \Delta_{\underline{\xi}}\, \underline{w}_0 \tag{1.3.12}$$

plus the corresponding higher orders. In §1.2, we saw that, to derive the conditions of validity of the approximations, the (1.2.13) had to be added to the (1.2.12) if also $\underline{g} = \underline{0}$ is assumed, for simplicity. Proceeding exactly in parallel to the preceding case of the Euler equations, and using the notations of (1.3.9), we see that incompressibility with initial data $\bar{r}$ and $\bar{s}$ constant (see (1.3.2)) is consistent if

$$\frac{\nu}{\varepsilon l v_{sound}} \equiv \nu_0 \ll 1 \qquad \text{independently of } \varepsilon \tag{1.3.13}$$

which is again (1.2.13). This means that (1.2.13) now demands that the length scale over which the fields change be of the order of magnitude of $l = \nu \varepsilon^{-1} v_{sound}^{-1} \nu_0^{-1}$. As in the Euler case one can prove the following theorem. Consider the adiabatic Navier–Stokes equations:

$$\begin{aligned} & r_0 = constant, \qquad s_0 = constant \\ & \partial_\vartheta \underline{w}_0 + \underset{\sim}{w}_0 \cdot \underset{\sim}{\partial}_{\underline{\xi}}\, \underline{w}_0 = -\frac{1}{r_0} \underline{\partial} p_0 + \nu_0 \Delta \underline{w}_0 \end{aligned} \tag{1.3.14}$$

with ν_0 a positive constant, initial data $\underline{w}_0 \in C^\infty$ and vanishing outside a bounded region, $r_0 = \bar{r} = constant, \sigma_0 = \bar{\sigma} = constant$; let $\underline{w}_0(\vartheta, \underline{\xi})$ be a solution of class C^∞ (in $\vartheta, \underline{\xi}$). Then, [KM81], [EM94]:

2 Theorem *(incompressibility; the NS case):*
(i) The Navier–Stokes equation (1.3.14) admits a C^∞–solution for times $\vartheta < \tau_0$ for some $\tau_0 > 0$,
(2) Let $\varepsilon > 0$ be a positive parameter. Assume that the constitutive equations are $\kappa_{ij} = 0$, (perfect non heat conducting gas) and $\tau'_{ij} = \rho\,\nu(\partial_j u_i + \partial_i u_j)$ with ν verifying (1.3.13) (NS stress). Given $l_0 > 0$ the time $\tau_0 > 0$ can be chosen so that (1.2.1) with equation of state of a perfect gas (see (1.3.1)) and initial data

$$\underline{u}(\underline{x}) = \varepsilon v_{sound} \underline{\bar{w}}_0(\frac{\underline{x}}{l}), \quad \rho(\underline{x}) = \bar{r}, \quad s(\underline{x}) = \bar{\sigma}, \qquad \text{with } l = \frac{l_0}{\varepsilon} \tag{1.3.15}$$

admits a C^∞–solution, which we shall denote $\underline{u}_\varepsilon(\underline{x}, t), \rho_\varepsilon(\underline{x}, t)$, and $s_\varepsilon(\underline{x}, t)$, defined for times $t < \tau_0\, l_0/\varepsilon^2 v_{sound}$. Furthermore

$$\begin{aligned} & \varepsilon^{-1} |\underline{u}_\varepsilon(\underline{x}, t) - \varepsilon\, v_{sound}\, \underline{w}_0(\varepsilon\, \underline{x} l_0^{-1}, \varepsilon^2\, v_{sound}\, t l_0^{-1})| \xrightarrow[\varepsilon \to 0]{} 0 \\ & |\rho_\varepsilon(\underline{x}, t) - \bar{r}| \xrightarrow[\varepsilon \to 0]{} 0, \qquad |s_\varepsilon(\underline{x}, t) - \bar{\sigma}| \xrightarrow[\varepsilon \to 0]{} 0 \end{aligned} \tag{1.3.16}$$

Remarks:

(1) One can make comments identical to the ones that follow theorem 1 above. One sees that in the limit $\varepsilon \to 0$ entropy is conserved: which is no longer obvious since the right hand side of the third of the (1.2.1) no longer vanishes. Nevertheless friction influences the equation for the velocity. This is, at first, looks strange but it is understood if one takes into account that $\underline{w}_0$ is a rescaled velocity and a variation of $O(1)$ due to friction (*i.e.* due to the term $\nu_0\Delta\underline{w}_0$ in (1.3.14)) generates a variation of energy of the order $O(\varepsilon^2)$ and, therefore, a quantity of heat and an increase of entropy (and temperature) $O(\varepsilon^2)$ which is not contradictory to the third of the (1.3.16).

(2) Interpreting theorems 1,2 above one can say: *on time and space scales $O(\varepsilon^{-1})$ the system follows the incompressible Euler equation*; while *on time scales $O(\varepsilon^{-2})$ and space scales $O(\varepsilon^{-1})$ the system follows the Navier–Stokes equations.*

(3) What can one then say if the initial datum is given without any free parameter ε? *i.e.* can the just stated theorems be concretely applicable as approximation theorems when ε is fixed? The risk being that they are just conceptual theorems illustrating the asymptotic nature of the incompressibility assumption.

(4) A proposal is the following. Wishing to apply such theorems in a given particular case one should check that the initial datum can be written in the form (1.3.4). Then if ε is small one shall be able to say that the incompressible Euler equation holds (for times up to $O(\varepsilon^{-1})$), and one should also be able to give the approximation error by using the estimates of the differences in (1.3.11),(1.3.16): indeed such estimates are constructive, *i.e.* computable, in the proofs (not described here) of the theorem. As the time increases, beyond $O(\varepsilon^{-1})$, we expect that the velocity field becomes more uniform in space and that it will, after a time very long with respect to $O(\varepsilon^{-1})$, be described by a regular function of $\varepsilon^{-1}\underline{x}l_0^{-1}$ for some l_0 which should depend on the initial data.

(5) In this situation we shall be in the assumptions of theorem 2 and the fluid will now evolve following the incompressible Navier–Stokes equation, with an approximation controlled up to times of order $O(\varepsilon^{-2})$, and it will proceed towards equilibrium (which is simply the state in which the velocity field vanishes, because we are supposing that there are no external forces) keeping a variability on scales of length of order $\varepsilon^{-1}l_0$ and of time of order $\varepsilon^{-2}\,\nu\,l_0^{-1}\,v_{sound}$.

(6) The above is a scheme of interpretation of an incompressible evolution: however it is just a "proposal" because there are *no other known theorems* that support such proposal and it is not so clear, in the above proposal, the cross–over between the two regimes can be described and how. From what said above not only l_0 is *not* calculated but there is no hint nor any idea on how to calculate it, nor there is any idea on which physical properties a calculation of l_0 could be based.

Bibliography: This section is based on the ideas and results of the paper [EM94]; the original theorems 1, 2 are in [Eb77],[Eb82], [KM81], [KM82]. I have preferred the approach in [EM94] because it is closer in spirit to the analysis by Landau and Lifshitz reported in (D,E,F) of §1.2.

§1.4 Elements of hydrostatics.

Hydrostatics deals with solutions of the Euler, Navier–Stokes or more general continua, with vanishing velocity fields and with time independent thermodynamic functions.

These solutions are very rare, as we shall see by considering a few model cases.

(1) *Hydrostatics in absence of thermoconduction.*

(A) Isoentropic case

Equations (1.2.1) become simply

$$-\frac{1}{\rho}\underline{\partial}\, p + \underline{g} = \underline{0}, \qquad \varepsilon = \varepsilon(\rho, s) \tag{1.4.1}$$

As an example we shall treat the case of a perfect monoatomic gas

$$\varepsilon = \varepsilon_0 \left(\frac{\rho}{\rho_0}\right)^{2/3} e^{(s-s_0)/c_v} \Rightarrow \varepsilon = c_v T, \quad p = \frac{2}{3}\rho c_v T = \rho\frac{RT}{M_A} = \frac{nRT}{v} \tag{1.4.2}$$

where ε_0, ρ_0, s_0 are values of ε, ρ, s in a reference thermodynamic state; $n = M/M_A$ with M the total fluid mass and M_A is the atomic mass (n is called "molar number"); $v = M/\rho$ is the specific volume of the fluid; R is the gas constant $R = 8.31\ 10^7$ erg $^\circ$K^{-1}; c_v is the specific heat at constant volume (per unit mass), *i.e.* $c_v = \frac{3}{2}R/M_A$. If the gas was diatomic the factor 3/2 would become, everywhere, 5/2.

Suppose that the force density $\underline{g}$ is conservative, $\underline{g} = -\underline{\partial} G$. In the isoentropic case the relation between ρ and p is simply the adiabatic equation of state $\rho = R(p)$: $\rho = p^{1/\gamma}\, const$ with $\gamma = 5/3$ for a monoatomic perfect gas. Therefore it is convenient to define the "*pressure potential*"

$$\Phi(p) = \int^p \frac{dp'}{R(p')} \tag{1.4.3}$$

so that (1.4.1) is solved by

$$\Phi(p(\underline{x})) + G(\underline{x}) = \text{constant}, \qquad \rho = R(p(\underline{x})) \tag{1.4.4}$$

that permits us to determine $p(\underline{x})$ in terms of G and consequently to determine $\rho(\underline{x}), \varepsilon(\underline{x})$ *etc* (note that $\Phi(p)$ is strictly increasing in p and, hence, invertible). This holds at least if the values of $G(\underline{x})$ are among those of Φ; *i.e.* they are in $\Phi([0,+\infty))$, up to an additive constant.[1]

One should also remark that if, in the isoentropic case, $\underline{g}$ was not conservative no hydrostatic solution could exist: a non conservative force will necessarily set the fluid in motion. This is physically obvious and (as we shall see) *remains essentially true also in the case of non isoentropic fluids.*

(B) Non isoentropic case.

The non isoentropic hydrostatics is analogous. Now $s = s_0(\underline{x})$ hence $\rho = r(p, s_0)$ so that the equation in the cases when $\underline{g}$ is conservative:

$$-\underline{\partial}\, p = r(p, s_0(\underline{x}))\, \underline{\partial} G \tag{1.4.5}$$

implies that $r(p, s_0(\underline{x}))$ must be a function of the form $R(G(\underline{x}))$, hence $p(\underline{x})$ must also be a function of the form $\pi(G(\underline{x}))$: then $s_0(\underline{x})$ must have the form $\Sigma(G(\underline{x}))$. The interesting consequence is, therefore, that in this case *hydrostatic solutions in which entropy cannot be expressed as a function of the potential of the volume forces is not possible.*

For general $\underline{g}$ one finds that *hydrostatic solutions can exist only if the volume force is proportional to a conservative force.* See problems.

We shall discuss here the latter question in a specific case in which there are no problems on the possible existence of solutions. Since (1.4.1) is very restrictive there are not many such cases and only the particularly symmetric ones are easy to treat.

Consider for instance a fluid occupying the half space $z > 0$ and subject to a gravity force with potential $G = gz$, and look for *stratified hydrostatic solutions*, *i.e.* solutions in which the thermodynamic functions depend only on z. We shall denote them $s = s_0(z)$, $T = T(z)$, $p = p(z)$, $\varepsilon = \varepsilon(z)$ and $\rho = \rho(z)$, $(\rho = r(p, s))$. Then (1.4.5) simplifies and we find

$$-\frac{1}{\rho(z)}\frac{dp}{dz} = \frac{dG}{dz}, \qquad \rho(z) = r(p(z), s_0(z)) \tag{1.4.6}$$

which is an ordinary differential equation for $p(z)$ determining it once the data $p(0) = p_0$ and the function $s = s_0(z)$ are known.

More specifically consider a perfect gas in a gravity field; the (1.4.6) become, if one imagines that $T = T_0(z)$ is *a priori* assigned (instead of the entropy)

$$-\frac{1}{\rho(z)}\frac{dp}{dz} = g, \quad T = T(z), \quad p = \frac{2}{3}\rho c_v T \tag{1.4.7}$$

so that if we take $T = (1 + \gamma z)\, T_0$, $\gamma \ge 0$ we find

$$\frac{2c_v T}{3p}\frac{dp}{dz} = -g \Rightarrow \frac{dp}{p} = -\frac{3}{2}\frac{1}{c_v T_0}\frac{g dz}{1+\gamma z} \tag{1.4.8}$$

[1] Or, in other words if $G(\underline{x})$ is bounded below, otherwise the equation does not admit hydrostatic solutions.

whose solution is

$$p = (1 + \gamma z)^{-3g/(2c_v T_0 \gamma)}\, p_0 \tag{1.4.9}$$

In the isothermal case, $\gamma = 0$, this becomes the well known

$$p = p_0 e^{-3gz/(2c_v T_0)} \tag{1.4.10}$$

while in the incompressible case (1.4.7) has the equally well known solution

$$p = p_0 - \rho g z \tag{1.4.11}$$

Hence it is possible that a gas in which temperature is not constant stays in a "stratified equilibrium".

(C) Stability of equilibria

Temperature and density gradients can generate instabilities of the equilibrium of a fluid because it could become energetically convenient to displace a volume element of the fluid by exchanging its position with another and by taking advantage of the external field or of the density differences due to temperature differences.

We consider the two following questions about the stratified equilibria in (B) above: (1) under which conditions are they stable, (2) under which conditions it is possible to suppose $\rho =$ constant and therefore use (1.4.11).

The result will be a remarkable stability criterion about the development of convective motions; *in the case of an adiabatic perfect gas*, *i.e.* if heat conduction is negligible, and in a gravity field g there will be stability in a gravity field g if

$$\frac{\partial T}{\partial z} \geq -\frac{g}{c_p} \tag{1.4.12}$$

This means that temperature can decrease with height, but not too much. If the variation ΔT between two horizontal planes at distance h is such that $\Delta T > gh/c_p$, and if the higher plane is colder, convection phenomena start "spontaneously", *i.e.* they are generated by the smallest perturbations.

To obtain the criterion (1.4.12) let $z \to T(z), s(z), p(z),\ \rho(z)$ be the thermodynamic functions expressed in terms of the height z.

Let Δ be an infinitesimal cube at height z containing gas with specific volume $v = v(p, z)$. Imagine to displace the mass in Δ and to transfer it in a volume Δ' at height $z' = z + \delta z > z$ adiabatically (because we suppose $\underset{\approx}{\kappa} = 0$ and no heat exchange is possible by conduction).

The new volume occupied by the mass $M = \Delta v(p, s)^{-1}$ will be of size $\Delta' = \Delta\, v(p', s)/v(p, s)$ because the gas will have to acquire pressure p' keeping entropy s, (as in absence of heat conduction the transformations are adiabatic).

At the same time the mass originally in Δ' will have to be moved in Δ. Since this mass is $M' = \Delta'/v(p',s')$ it will occupy, at the new pressure p, a volume

$$\begin{cases} \Delta'' = M'v(p,s') = \Delta' \ \ v(p,s')/v(p',s') = \\ = \Delta \ \ v(p',s)v(p,s')/v(p,s)v(p',s') = (1+O(\delta z^2))\,\Delta \end{cases} \tag{1.4.13}$$

as one sees by a Taylor expansion of $\log v(p,s)$ using that $s-s'$ and $p-p'$ have order of magnitude δz.

We interpret this by saying that the mass to be moved away from Δ' to make space for the one coming from Δ "does indeed fit" in the volume Δ left free (up to a negligible higher order correction). Therefore the proposed transformation will be energetically favored (in a gravity field) if $M = \Delta/v(p,s) < M' = \Delta' \ /v(p',s') = \Delta \ v(p',s)/v(p,s)v(p',s')$, *i.e.* if

$$\frac{v(p',s)}{v(p',s')} > 1 \tag{1.4.14}$$

If (1.4.14) holds then the equilibrium is unstable and small perturbation will induce the permutation of the two volumes of gas generating a nonzero velocity field $\underline{u} \neq 0$ and raise "convective currents"

To see the "usual" meaning of (1.4.14), *i.e.* of

$$-\left(\frac{\partial v}{\partial s}\right)_p \frac{ds}{dz} > 0 \tag{1.4.15}$$

one can use the relation $(\partial v/\partial s)_p \equiv T/c_p(\partial v/\partial T)_p$. Since, in most substances, it is $(\partial v/\partial T)_p > 0$ the (1.4.15) becomes $-ds/dz > 0$, so that:[2]

$$\begin{aligned} \frac{\delta s}{\delta z} &= \left(\frac{\partial s}{\partial T}\right)_p \frac{\delta T}{\delta z} + \left(\frac{\partial s}{\partial p}\right)_T \frac{\delta p}{\delta z} = \frac{c_p}{T}\frac{\delta T}{\delta z} - \left(\frac{\partial v}{\partial T}\right)_p \frac{\delta p}{\delta z} = \\ &= \frac{c_p}{T}\frac{\delta T}{\delta z} + \left(\frac{\partial v}{\partial T}\right)_p \frac{g}{v} < 0 \Leftrightarrow \frac{\delta T}{\delta z} < -\frac{T}{c_p}\frac{g}{v}\left(\frac{\partial v}{\partial T}\right)_p \end{aligned} \tag{1.4.16}$$

is the general instability condition. In the perfect gas case on has stability if (1.4.12) holds.

Finally, on the basis of the analysis in §1.2, we see that the assumption $\rho =$ constant is licit if we limit our interest to a portion of fluid spanning a height H such that $gH \ll v_{sound}^2$, and a variation of temperature δT such that $(\partial\rho/\partial s)_p\,\delta s/\rho \approx \rho\chi_s\, c_p\delta T/T \approx \,(\rho\chi_s v_{sound}^2/T)\,(\delta T/T) \ll 1$.

(2) *Hydrostatics in presence of thermoconduction.*

In this case too one finds that hydrostatic solutions are rare and special. For the purposes of an example, and to avoid repetitions, we pose a slightly

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[2] From $G = U + PV - TS \Rightarrow dG = -SdT + VdP$.

different problem compared to the ones already discussed and we treat only a simple example.

We ask whether a fluid in a container Ω *and in a conservative force field* $\underline{g}$, $\underline{g} = -\underline{\partial} G$, *can, at least in particular circumstances, conduct heat without developing motion* (*i.e.* if it can "look" like a solid conductor). We shall assume therefore that the temperature on the walls is a preassigned function $\underline{\xi} \to \vartheta(\underline{\xi})$.

For a fluid verifying (1.2.8) one has

$$\begin{aligned} &\partial_t \rho = 0 \\ &\underline{\partial} p = -\rho \underline{\partial} G, \qquad T = \tau(v,p) \\ &T \rho \partial_t s = \kappa \Delta T \\ &T(\underline{\xi}) = \vartheta(\underline{\xi}), \quad \underline{\xi} \in \partial\Omega \ . \end{aligned} \tag{1.4.17}$$

The first equation says that $\rho = \rho(\underline{\xi})$ and the second that ρ must be a function of $\underline{\xi}$ through G so that p can be expressed easily in terms of ρ, namely

$$\text{if} \qquad \rho(\underline{\xi}) = V(G(\underline{\xi}))^{-1}, \quad \text{then} \quad p(\underline{\xi}) = \pi(t) + W(G(\underline{\xi})) \tag{1.4.18}$$

where $V(G)$ is a suitable function and $W(G) = -\int_{G_0}^{G} V(G')^{-1}\, dG'$. Since we must have $T = \tau(v,p)$ it is

$$T(\underline{\xi}) = \tau\big(V(G(\underline{\xi})), \pi(t) + W(G(\underline{\xi}))\big) \tag{1.4.19}$$

for each $\xi \in \Omega$, and, hence, also for $\underline{\xi} \in \partial\Omega$. Then it will also be $\pi(t) = \pi_0$ = constant. Thus hydrostatic solutions are possible only if the temperature assigned on the boundary depends on $\underline{\xi}$ via $G(\underline{\xi})$. In this case also $T(\underline{\xi})$ is a function of $G(\underline{\xi})$ and therefore $s(\underline{\xi})$ has the same property.

Furthermore assuming that $\vartheta(\underline{\xi})$ depends on $\underline{\xi}$ via $G(\underline{\xi})$ it is not clear that there is a solution of

$$\partial_t \rho = 0, \quad p = \pi_0 + W(G), \quad \Delta T = 0, \quad T(\underline{\xi}) = \vartheta(\underline{\xi}) \ \text{on} \ \partial\Omega \tag{1.4.20}$$

In fact the last two conditions on T determine T uniquely (as the solution of a "Dirichlet problem" $\Delta T = 0$ in Ω, $T = \vartheta$ on $\partial\Omega$); and it is not necessarily true that T will be a function of $\underline{\xi}$ via $G(\underline{\xi})$: the latter is a very restrictive condition.

To understand how strong the latter restriction is consider the case of a gravity field

$$G(\underline{\xi}) = gz, \quad \vartheta(\underline{\xi}) = T_0(z), \quad \underline{\xi} \in \partial\Omega \tag{1.4.21}$$

In this case we see that T, ρ, s must be functions of z alone and therefore the equation $\Delta T = 0$ becomes $d^2T/dz^2 = 0$, *i.e.* for a suitable γ

$$T(z) \equiv \vartheta(z) \equiv (1 + \gamma g z)\, T_0. \tag{1.4.22}$$

We thus see that to have hydrostatic solutions not only $\vartheta(\underline{\xi})$ must be a function of z alone but it must be a linear function.

Finally, if $G = gz$ and $T = (1+\gamma gz)\,T_0$ we see that the (1.4.20) can be satisfied if $W(G)$ is chosen as solution of the equation obtained by imposing the equation of state $T = \tau(v,p)$:

$$(1+\gamma G)\,T_0 = \tau\left(\frac{1}{W'(G)}, p_0 + W(G)\right) \tag{1.4.23}$$

which is a differential equation for W which, once solved, gives W, V and therefore ρ and p in terms of $G = gz$.

Obviously the conclusion is that convective motions are necessarily generated inside a fluid in a conservative force field and not in thermal equilibrium, apart from very special cases.

The only case in which, under rather general assumptions, one can have static thermoconduction is an incompressible fluid, see (1.2.8); in this case the equations are

$$\begin{aligned}&\rho = \rho_0\,, \qquad \underline{\partial} p = -\rho_0 \underline{\partial} G, \quad \rho_0 c_p \partial_t T = \kappa \Delta T\\ &s = \sigma(T) \equiv \int^T c_p(T)\frac{dT'}{T'}\end{aligned} \tag{1.4.24}$$

where now s depends only on T and $dS/dT = c_p$ because

$$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p = 0 \tag{1.4.25}$$

while p has to be thought of as no longer related to s or T because

$$\left(\frac{\partial p}{\partial T}\right)_v = -\left(\frac{\partial s}{\partial v}\right)_T = 0 \tag{1.4.26}$$

One can ask how to reconcile the possibility of a solution of (1.4.24), in which T depends on time, with the impossibility of such a solution that we have just shown in the case of a compressible fluid. In fact the incompressible fluid is in a suitable sense a limit case of the compressible fluid.

In reality a compressible fluid close to an incompressible one (in the sense discussed in §1.3) cannot be, for the above discussion, a static thermoconductor and it will start “flowing”. However the motion will be the slower the closer we are to a situation in which the fluid can be regarded as incompressible.

Therefore the question of the connection between (1.4.24) and (1.4.20) implies a study of a nonstatic problem and it will be analyzed later (*c.f.r.* §1.5).

(3) *Current lines and the Bernoulli theorem.*

A fluid motion is called *static* if the velocity and thermodynamic fields describing it are time independent.[2]

For such motions it makes sense to define the "*current lines*" as geometric, time independent, curves; they are just the solutions of the differential equations

$$\dot{\underline{\xi}} = \underline{u}(\underline{\xi}) \ . \tag{1.4.27}$$

Current lines play an important role particularly in the case of isoentropic Euler flows. A simple but important property associated with them is "*Bernoulli's theorem*".

Let $\rho = \rho(p)$ be the adiabatic equation of state of the fluid; then we define, as above, the pressure potential $\Phi(p) = \int^p dp'/\rho(p')$ and therefore the Euler equations are

$$\underline{\partial} \cdot (\rho \underline{u}) = 0 \qquad \underset{\sim}{u} \cdot \underset{\sim}{\partial}\, \underline{u} = -\rho^{-1} \underline{\partial}\, p - \underline{\partial}\, G \tag{1.4.28}$$

Multiplying the second equation by $\underline{u}$ we recognize that it becomes

$$\underset{\sim}{u} \cdot \underset{\sim}{\partial} \left[\frac{\underline{u}^2}{2} + \Phi(p) + G \right] = 0 \tag{1.4.29}$$

If $t \to \underline{\xi}(t)$ is a point that moves on a current line according to (1.4.27) and if $X(\underline{\xi}, t)$ is a function then $\partial_t X + \underline{u} \cdot \underline{\partial}\, X$ is the t–derivative dX/dt of $X(\underline{\xi}(t), t)$ evaluated in $(\underline{\xi}(t), t)$. Hence we see that,setting $X(\underline{\xi}, t) = \underline{u}^2(\underline{\xi})/2 + \Phi(p(\underline{\xi})) + G(\underline{\xi})$, the (1.4.29) says that X is constant along the current lines of the fluid:

$$\frac{\underline{u}^2}{2} + \Phi(p) + G = \text{ constant} \tag{1.4.30}$$

This is an equation expressing the *vis vivae* theorem, as the following classical alternative derivation shows.

Let S' be a surface element through $\underline{\xi}'$ with normal $\underline{n}'$ parallel to the fluid velocity $\underline{u}'$ in $\underline{\xi}'$: draw the current line through every point of S', forming in this way a "current tube" which we shall cut, at a point $\underline{\xi}''$, with an element of surface S'' orthogonal to the velocity $\underline{u}''$ in $\underline{\xi}''$.

Consider the fluid enclosed in the current tube at time $t = 0$. At time $t + \delta t$ the surface S' will be displaced forward by $\underline{u}'\delta t$ while the other surface S'' will be displaced forward by $\underline{u}''\delta t$.

The kinetic energy variation of the considered part of the fluid will be, by the static state assumption, simply

$$\frac{1}{2}\underline{u}''^2 \rho'' \underline{u}'' \cdot \underline{n}'' \delta t S'' - \frac{1}{2}\, \underline{u}'^2 \rho' \underline{u}' \cdot \underline{n}' \delta t S' \tag{1.4.31}$$

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[2] Often one calls such flows "stationary": here this appellation is avoided because we shall reserve the name "stationary" for states of the fluid that have well defined statistical properties: see Chaps. 5,6,7.

which must equal the work of the applied forces. The external forces perform a work given by the variation of the potential energy (changed in sign)

$$G'\rho' S' \underline{u}' \cdot \underline{n}' \delta t - G''\rho'' S'' \underline{u}'' \cdot \underline{n}'' \delta t \tag{1.4.32}$$

while the calculation of the pressure forces is more delicate because we must take into account that such forces not only work on the external faces and on the bases of the tube but also inside it. To compute the work done by the pressure forces we divide the tube into sections $S' = S_1, S_2, \ldots, S_n = S''$ normal to the velocity and spaced so that the center of S_{i+1} follows the center of S_i by an amount much smaller than the quantity $\underline{n}_i \cdot \underline{u}_i \delta t$, if $\underline{u}_i$ and $\underline{n}_i$ are velocity and, respectively, normal vector to S_i.

Under such conditions the fluid element can be regarded as rigid and subject to a force equal to the difference between the pressures on its two bases times their area. Then the work can be computed as

$$\sum_{i=1}^{n-1} (p_i - p_{i+1})\, S_i\, \underline{u}_i \cdot \underline{n}_i\, \delta t \tag{1.4.33}$$

because the pressure forces do not perform work on the lateral face of the current tube (since they are orthogonal to it: recall that the stress tensor is $-p\,\delta_{ij}$).

Mass conservation imposes that $\rho_i S_i \underline{u}_i \cdot \underline{n}_i\, \delta t = Q$ for all i. Hence (1.4.33) becomes

$$Q \sum_{i=1}^{n-1} \frac{p_i - p_{i+1}}{\rho_i} = -Q \int_{p'}^{p"} \frac{dp}{\rho(p)} = -(\Phi(p'') - \Phi(p'))Q \ . \tag{1.4.34}$$

And summing (1.4.31), (1.4.32), (1.4.34) we find

$$\frac{\underline{u}'^2}{2} + \Phi(p') + G' = \frac{\underline{u}''^2}{2} + \Phi(p'') + G'' \tag{1.4.35}$$

In the case of incompressible motions (1.4.35) becomes simpler becauseqq

$$\Phi(p) = \frac{p}{\rho} \quad \text{(incompressible case)} \tag{1.4.36}$$

where ρ is the (constant) fluid density.

From (1.4.35), (1.4.36) we read that increasing the velocity implies that the pressure diminishes (in the incompressible case) or (in the more general isoentropic case) the potential of pressure diminishes. In the incompressible case to a shrinking of the tube section corresponds an increase of the velocity and therefore a decrease of the pressure. It is a property on which several pumps rely.

Problems

[1.4.1] (*integrability of a vector field*) The (1.4.1) shows that only force fields for which there is an integrating factor $\mu(\underline{x})$, *i.e.* such that $\underline{g} = \mu(\underline{x})\underline{\partial} G$ for some G, can generate hydrostatic solutions; show that, in such solutions, the pressure depends on $\underline{x}$ via $G(\underline{x})$ and that also the product $\rho\mu$ is a function of G. Show also that in the 2–dimensional cases every force field admits, at least locally in the vicinity of a point where it does not vanish, an integrating factor (but in general this is only a local property). (*Idea:* Let $\rho = r(p, s)$ be the equation of state and let $\underline{g} = \mu(\underline{x})\underline{\partial} G$; since two scalar functions with proportional gradients have the same level surfaces the (1.4.1) implies that: p is a function of $\underline{x}$ via G: $p(\underline{x}) = \pi(G(\underline{x}))$ and, again by (1.4.1), $\mu\,\rho = r(\pi(G(\underline{x})), s_0(\underline{x}))\mu(\underline{x}) = F(G(\underline{x}))$ for a suitable F).

[1.4.2] In the context of [1.4.1] show that if the entropy density $s_0(\underline{x})$ is known then one can compute the pressure. Note that, however, in general one needs to check compatibility relations between $s_0(\underline{x}), \mu(\underline{x})$ and the equation of state $\rho = r(p, s)$ in order that the equation be soluble. (*Idea:* The pressure must be a function $\pi(G)$. Then $\partial p/\partial G \equiv r(\pi(G(\underline{x})), s_0(\underline{x}))\,\mu(\underline{x}) = \pi'(G(\underline{x}))$ and from this differential equation one deduces π by fixing its value at a point $\underline{x}_0$ and by integrating the equation along a curve which leads from $\underline{x}_0$ to $\underline{x}$, after having expressed $s_0(\underline{x})$ and $\underline{x}$ in terms of G along the curve. The procedure depends upon the curve and therefore compatibility conditions are necessary.)

[1.4.3] In the context of the above two problems assume that the volume force $\underline{g}$ is conservative with potential G, and assume that the entropy $s_0(\underline{x})$ is given and it is a function of the potential, $s_0(\underline{x}) = S(G(\underline{x}))$, show that the compatibility conditions in [1.4.2] are satisfied and that a hydrostatic solution of the second of the (1.2.1) is possible. (*Idea:* The ρ can be expressed in terms of the equation of state $\rho = r(p, s)$ and of the solution $p = \pi(G)$ of the differential equation

$$\frac{\partial}{\partial G}\pi = r(\pi, S(G)) \quad \pi(G_0) = P_0$$

and the hydrostatic solution will then be $p(\underline{x}) = \pi(G(\underline{x}))$.

[1.4.4] (*temperature and in hydrostatic states*)Check that the hydrostatic solutions in [1.4.3] will, in general, correspond to states of the fluid in which temperature changes from point to point and they will, therefore, be really possible only for very special temperature distributions because in general the temperature will be incompatible with the hydrostatic solution of the third of the (1.2.1). (*Idea:* Note that the equation of state allows us to express T as a function of s, p and it will not be, in general, true that the third of the (1.2.1), will hold, *c.f.r.* [1.4.3], unless $\underset{\sim}{\partial}\underset{\approx}{\kappa}\,\underline{\partial} T = 0$ of course).

[1.4.5] (*calm air condition*) Imagine air as a perfect diatomic gas with molecular mass $m_A = 28.8\,m_H$, $m_H =$ hydrogen mass $= 1.67 \cdot 10^{-24}$ g and take $k_B = 1.38 \cdot 10^{-16}$ erg $^\circ K^{-1}$, $g = 9.8 \cdot 10^2$ cm/sec^2, $c_p = \frac{7}{2}\frac{k_B}{m_A} = \frac{7}{2}\frac{R}{M_A}$, $R = k_B N_0$ and $N_0 =$ Avogadro number, $R = 8.31 \cdot 10^7$ erg/$^\circ K$, $N_0 = 6.022\,10^{23}$). Compute, if the ground temperature is $\overline{T} = 20^\circ$C, which is the value of T_0 such that if the temperature at height $z = 10^3$ m is $T \ge T_0$ then convective currents will not develop. (*Idea:* $T_0(z) = \overline{T} - gzc_p^{-1}$ is the limit case as given by (1.4.12); thus one finds $T_0 > \overline{T} - 9.6^\circ K$ (*i.e.* a gradient of $0.96 \cdot 10^{-2}\,^\circ K/m$). If $T < T_0$ air cannot be observed in a hydrostatic stratified equilibrium.)

[1.4.6] In "real" and calm atmosphere in equilibrium the temperature gradient that is observed is $\sim 0.6 \cdot 10^{-2}$ °K/m and therefore the calm atmosphere in normal conditions is in stratified equilibrium. Check this statement by finding and consulting some geophysical data.

[1.4.7] (*incompressibility estimate for air*) Express the condition under which a perfect gas in mechanical equilibrium in the gravity field and at constant temperature can be considered as incompressible. (*Idea:* from the discussion in §1.2 one sees that density variations on the scale l over which sensible variations of pressure occur are such that: $\frac{\Delta\rho}{\rho} \simeq \frac{gl}{c^2}$, where c is the sound velocity. Take, in the case of air, $c \simeq 10^3$ km/h. Check that the characteristic scale over which density variation take place is $\simeq c^2/g$, *i.e.* $\simeq 10^4$m. Hence one can consider that in normal conditions air is incompressible (for what concerns the hydrostatic state) over length scales of the order of a kilometer er and therefore one can use (1.4.11) to evaluate the height from a measurement of pressure. For larger heights ρ cannot any more be considered as constant and to compute the height z in terms of p it becomes necessary to know also how temperature changes with height. At least for quota differences not too large it is possible to evaluate the height from pressure measurements, independently of the temperature distribution: it is the principle on which altimeters work. Using a "naive" altimeter, based on the formula $p = -\rho gz + \gamma_0$ (*i.e.* on an empirical gauge performed under ideal atmospheric conditions) can lead to important errors if the atmospheric conditions are not "ideal".) (*Idea:* It is

$$\frac{\Delta\rho}{\rho}\frac{1}{l} = \frac{9.8}{(10^6 \cdot 10^{-3}/3 \cdot 6)^2}\mathrm{m}^{-1} = 1.27 \cdot 10^{-4}\mathrm{m}^{-1}.)$$

[1.4.8] (*gravity and calm planetary athmospheres*) Consider a perfect gas in equilibrium in a gravitational field generated by a sphere of given mass and radius and, defining "stratified equilibria" states in which the thermodynamic quantities depend only upon the distance from the center of the sphere, repeat the analysis performed in this section in the case of the half space. Apply the results to the Earth's atmosphere and to that of some other planet (*e.g.* Mars and Venus), computing which could be the maximum temperature gradient compatible with a stratified equilibrium. Compare the results with the average gradients at the surface of the planets as deduced from known astrophysical data. (*Idea:* Part of the problem is to look for, and find, the necessary astrophysical data.)

[1.4.9] (*a case of impossibility of hydrostatic states*) Consider a perfect gas with equation of state (1.3.1) (*i.e.* $s = c_v \log T - c \log\rho$ and therefore $p = \rho Tc$ and $\varepsilon = \frac{3}{2}cT$, where $c = R/M_0, c_v = \frac{3}{2}c$ if R is the gas constant and M_0 is the mass of a mole). Suppose that viscosity and thermal conductivity are given by the Clausius–Maxwell relations ($\eta = c_1T^{1/2}$, $\kappa = c_2T^{1/2}$, with suitable c_1, c_2: *c.f.r.* [1.1.5]). Suppose that the stress tensor is expressed in terms of the viscosity as $\tau_{ij} = \eta(\partial_i u_j + \partial_j u_i)$. Assume also that the gas is enclosed in a cubic container Ω with walls temperature fixed $T_0(P)$, $P \in \partial\Omega$. Show that in general the gas cannot stay in equilibrium (*i.e.* keep $\underline{u} = \underline{0}$, and $T, p =$ constant) and find a distribution of temperature on the walls T_0 which does not permit configurations of (mechanical and thermal) equilibrium in presence of a gravity force. (*Idea:* Show that the equations are

$$\begin{aligned}
&\partial_t\rho + \underline{\partial}(\rho\underline{u}) = 0\\
&\rho\, c_v\,(\partial_t T + \underline{u}\cdot\underline{\partial}T) = -p\underline{\partial}\cdot\underline{u} + \underline{\partial}(\kappa\underline{\partial}T)\\
&\partial_t\underline{u} + \underset{\sim}{\underline{u}}\cdot\underset{\sim}{\underline{\partial}}\underline{u} = -\frac{1}{\rho}\underline{\partial}p + +\underset{\approx}{\underline{\tau}}{}'\cdot\underset{\sim}{\underline{\partial}}\underline{u} + \frac{\eta}{\rho}\Delta\underline{u} + \underline{g}\\
&p = \rho Tc, \qquad \varepsilon = \frac{3}{2}cT
\end{aligned}$$

and check that $\underline{u} \equiv \underline{0}$ (mechanical equilibrium) implies that $\Delta T + \frac{1}{2}(\underline{\partial}T)^2/T = 0$ (*i.e.* $\Delta T^{3/2} = 0$) and $\underline{\partial}\log p = \underline{g}/(cT)$), hence that $\underline{g}$ must be parallel to $\underline{\partial}T$ (considering the rotation of the last expression and using that $\underline{g}$ is conservative): this is in general false. For instance if $T_0(x,y,z) = \vartheta x^{2/3}$ for (x,y,z) on $\partial\Omega$ then $T(x,y,z) = \vartheta x^{2/3}$ is solution of the equation for T but its rotation is not parallel to $\underline{g}$.)

[1.4.10] (*elementary tide theory*) Consider a homogeneous spherical planet T of radius R coated by an ocean of depth $h > 0$, large enough and of density negligible with respect to that of the planet. Let L be its small, lonely, satellite (also spherical and homogeneous). Denote by M_T and M_L the respective masses and assume that the motion of the two heavenly bodies about their center of mass be circular uniform. Let ρ be the distance TL of the two heavenly bodies: $\rho \gg R \gg h$. Assuming, for simplicity, the satellite on the equator plane and the planet rotation axis orthogonal to it, compute the equilibrium configuration of the fluid surface and evince Newton's formula according to which the *tidal excursion* (*i.e.* the maximal height variation between successive high and low tide) is $\mu = \frac{3}{2}R(\frac{R}{\rho})^3\frac{M_L}{M_T}$. (*Idea:* If G is the center of mass, its distance from the center T is $\rho_B = \frac{M_L}{M_T+M_L}\rho$ and the angular velocity of revolution of the two heavenly bodies is ω, such that $\omega^2\rho = k(M_L + M_T)\rho^{-2}$, if k is the gravitational constant. Let $\underline{n}$ be a unit vector out of T and note that, imagining the observer standing on the frame of reference rotating around G with angular velocity ω (so that the axis TL has a fixed unit vector $\underline{i}$), the potential energy (gravitational plus centrifugal) in the point $r\underline{n}$ has density proportional to

$$-k\frac{M_T}{r} - k\frac{M_L}{(\rho^2 + r^2 - 2\alpha\rho r)^{1/2}} - \frac{1}{2}\omega^2\,(\rho_B^2 + r^2 - 2\rho_B r\alpha)$$

if $\alpha \equiv \underline{i}\cdot\underline{n} \equiv \cos\vartheta$. Develop this in powers of r/ρ to find

$$\begin{aligned}&-k\frac{M_T}{r} - k\frac{M_L}{\rho}\frac{3}{2}\alpha^2(\frac{r}{\rho})^2 - \frac{1}{2}k\frac{M_T}{\rho}(\frac{r}{\rho})^2 + const \equiv\\ &\equiv -k\frac{M_T}{\rho}\big(\frac{\rho}{r} + (\frac{r}{\rho})^2\frac{M_L}{M_T}(\frac{3}{2}\alpha^2 + \frac{1}{2})\big) + cost\end{aligned}$$

because the linear terms cancel in virtue of Kepler's law ($\omega^2\rho^3 = k(M_T + M_L)$); and therefore the equation of the equipotential surface is

$$\frac{\rho}{r} + (\frac{r}{\rho})^2\big(\frac{M_L}{M_T}\frac{3}{2}\cos^2\vartheta + \frac{1}{2}\big) = cost$$

Hence, setting $r = (1+\varepsilon)R$, we find: $\varepsilon \simeq (\frac{R}{\rho})^3\frac{M_L}{M_T}\frac{3}{2}\cos^2\vartheta + cost$; the constant is determined by imposing that the solid of equation $r = (1+\varepsilon(\vartheta))R$ has the same volume as the ball $r = R$ and, of course, h has to be large compared to μ (otherwise ...).)

[1.4.11] (*tides and Moon mass*) Knowing that on the open Atlantic (*e.g.* St. Helena island) the tide excursion is of about 90 cm, [EH69], and supposing that this would be the tidal excursion on a Earth uniformly covered by a layer of water in a time independent state and subject to the only action of the Moon, estimate the ratio between the mass of the Moon and that of the Earth. Suppose $R = 6378\,Km$, $\rho = 363.3\,10^3\,Km$ equal to the minimum distance Earth–Moon.

[1.4.12] (*ratio of Moon and Sun tides*) Estimate the ratio between the Moon tide and the Sun tide. (*Idea:* $\frac{\varepsilon_L}{\varepsilon_S} = \frac{M_L}{M_S}(\frac{\rho_S}{\rho_L})^3 \simeq 2$, supposing that the Sun mass is $M_S = 10^6 M_T$, and that the Moon mass is (approximately) the one deduced from the problem [1.4.11]).

[1.4.13] Taking into account the result of [1.4.12] compute again the Moon mass and the ratio between the Sun tide and the Moon tide. (*Idea:* The Moon tide will then be of about 50cm rather than the 80 cm of [1.4.11].)

[1.4.14] (*tidal slowing of a planet rotation*) Let ω_D the daily rotation velocity of the planet T above, and suppose that the daily rotation takes place on the same plane of the satellite L orbit. Assume that the planet is uniformly coated by a viscous fluid which adheres to the bottom of the ocean while at the surface it is in equilibrium with the satellite (*i.e.* the tide is in phase with the satellite and therefore rotates with an angular velocity $\omega_D - \omega$ with respect to the planet surface). Let $\omega_D \gg \omega$ and let the depth of the ocean be h; suppose as well that the friction force be η times the gradient of velocity: then the momentum of the friction forces with respect to the rotation axis will be

$$A = \int_0^\pi [\eta \frac{(\omega_D - \omega) R \sin\vartheta}{h}] \cdot [R \sin\vartheta] \cdot [2\pi R^2 \sin\vartheta \, d\vartheta] = \frac{8\pi}{3} \frac{\eta R^4 (\omega_D - \omega)}{h}$$

Estimate the daily and annual deceleration of the planet assuming that the annual revolution velocity is $\omega_D/365$, and that $\eta = 0.10\, gs^{-1}\, cm^{-1}$, $R = 6.3\, 10^3\, Km$, $h = 1.Km$, $\omega = 2\pi d^{-1}$, $M_T = 5.98 \times 10^{27}$ g, estimating the number of years necessary in order that the planet rotation velocity (around its axis) be reduced by a factor e^{-1}. (*Idea:* The inertia moment of the planet is $I = \frac{2}{5} M_T R^2$ and therefore $\dot\omega_D = -A/I$, *i.e.* $\omega_D(t) = \omega_D(0) e^{-t/T_0}$, $T_0 = 3M_T h/20\eta R^2$. The result is $T_0 = 1.5 \cdot 10^7$ years which means that (in this friction model) the day would, at the moment, be longer by about .55 sec every century, *i.e.* by about $0.65 \cdot 10^{-5}$% a century, obviously showing the inadequacy of the model, see [MD00] for a theory of tides and despinning.)

[1.4.15] (*tides on Mars*) Had Mars an ocean uniformly covering it, how wide would the Sun tide be there? (*Idea:* $\sim$ 5.23 cm because the radius of Mars is $3.394 \cdot 10^8$ cm, its distance to the Sun is $227.94 \cdot 10^{11}$ cm and its mass is $0.64 \cdot 10^{27}$ g.)

[1.4.16] (*tides on Europe and Moon*) Assuming that Europe (satellite of Jupiter) had a deep enough uniform ocean estimate the height of the tide generated by Jupiter. Same for the pair Moon–Earth (*Idea:* write a small computer program to solve the general problem of the static tide generated on a satellite by its planet to solve all problems of this kind and play with various cases like Titan–Saturn *etc.*)

Bibliography: [LL71], [BKM74].

§**1.5 The convection problem. Rayleigh's equations.**

We now investigate in more detail the (1.2.1), and the (1.2.8), to find a "simple model" (*i.e.* simpler than (1.2.1) themselves) for some incompressible motions in which nontrivial thermal phenomena take place. Essentially we search for some concrete case in which (1.2.8) is derived as a "consequence exact in some asymptotic sense" of (1.2.1). We shall find a physically interesting situation, known as the "*Rayleigh regime*", describing a simple incompressible heat conducting and viscous fluid flowing between two surfaces at constant temperature.

(A) General considerations on convection.

The problem we shall address here is to deduce equations, simpler than the general ones in (1.2.1), valid under physically significant situations and that can still describe at least a few of the phenomena of interest, *i.e.* motions generated by density differences due to temperature differences. We look for a system of equations that could play the role plaid by the incompressible NS equations in the study of purely mechanical fluid motions (*i.e.* motions in which temperature variations and the heat and matter transport generated by them can be neglected).

Incompressibility in the simple form of the assumption that $\rho = const$ is obviously not interesting as, by definition, one has convection when the density variations due to temperature variations *are not* negligible.

Therefore we ask whether a physically compressible fluid, a perfect gas to be specific, admits motions that preserve the volume, *i.e.* such that $\underline{\partial}\cdot\underline{u} = 0$ with a good approximation, without having constant density.

In general, however, the divergence $\underline{\partial}\cdot\underline{u}$ *is not* a constant of motion and, therefore, one can doubt that the above question is a well posed one. And in fact we shall find that solutions with $\underline{\partial}\cdot\underline{u} = 0$ can only exist in an approximate sense, giving up the requirement that the equation of state be exactly verified and replacing it, in some sense, by the $\underline{\partial}\cdot\underline{u} = 0$. More precisely we shall find approximations that transform the general equations (1.2.1) into equations that are approximate but which include among them the $\underline{\partial}\cdot\underline{u} = 0$, assuming that it is verified at the initial time, even though it is not necessarily $\rho = const$.

(B) The physical assumptions of the Rayleigh's convection model.

Consider, for definiteness, a perfect gas

$$s = c_V \log e - c\log\rho \Rightarrow s = c_V \log T - c\log\rho + \,cost \tag{1.5.1}$$

i.e. $p = c\rho T$, $e = c_V T$, where c_V is the specific heat at constant volume, $c = R/M_{mol}$ is the ratio between the gas constant R and the molar mass, and the internal energy is denoted e to avoid confusion with the adimensional small quantity ε introduced in the following. However the assumption of perfect gas is not necessary and the only change in considering a general fluid is that some quantities will have values that cannot be computed unless one specifies the substance under study.

We suppose the fluid to be enclosed between two horizontal planes at height $z = 0$ and $z = H$, subject to a gravity force $\underline{g} = (0, 0, -g)$.

Boundary conditions are fixed by assigning the "ground temperature and the "temperature in quota", $T = T_0$, if $z = 0$, and $T = T_0 - \delta T$, if $z = H$. Furthermore we shall assume that the velocity field is tangent to the planes $z = 0$ and $z = H$ and that the total horizontal momentum $\int u_j\, d\underline{x}$, $j = 1, 2$, vanishes (in the following we shall take the notations $\underline{u} = (u_x, u_y, u_z)$ and $\underline{u} = (u_1, u_2, u_3)$ as equivalent).

We shall study motions in which the pressure is close to the static barometric pressure, $p = p_0 - \rho_0 g z$, and in which density is close to a given ρ_0 and p_0 is large with respect to $\rho_0 g H$.

Supposing that the viscosity coefficients η, η', *c.f.r.* (1.2.6), are constant (the problems in §1.1 show that this is an assumption that can be reasonable if the temperature variations are sufficiently small, see also below) it follows that the "exact" equations are, if $\nu = \eta/\rho, \nu' = \eta'/\rho$, *c.f.r.* §1.1 and (1.2.1):

$$\begin{aligned}&\rho T\big(\partial_t s + \underset{\sim}{u}\cdot\underset{\sim}{\partial} s\big) = \kappa\Delta T + \frac{\eta}{2}(\underline{\partial}\underset{\sim}{u} + \underset{\sim}{\partial}\underline{u})^2 + (\eta+\eta')(\underline{\partial}\cdot\underline{u})^2\\ &\partial_t\underline{u} + \underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{u} = \nu\Delta\underline{u} + (\nu+\nu')\underline{\partial}(\underset{\sim}{\partial}\cdot\underset{\sim}{u}) - \frac{1}{\rho}\underline{\partial}p + \underline{g}\\ &\partial_t\rho + \underline{\partial}\cdot(\rho\underline{u}) = 0\end{aligned} \tag{1.5.2}$$

to which one adds the equation of state $s = s(e,\rho)$ in (1.5.1), or (equivalently) the two relations $s = c_V\log T - c\log\rho$, $p = c\rho T$, and also the mentioned boundary conditions. We shall suppose κ,ν constant, for simplicity.

It is interesting to make the side–remark that the condition that friction generates entropy at a positive rate is expressed by $\eta(\underline{\partial}\underset{\sim}{u} + \underset{\sim}{\partial}\underline{u})^2 + \eta'(\underline{\partial}\cdot\underline{u})^2/2 \ge 0$, *i.e.* $\eta' \ge -2\eta$. The case $\eta'+\eta = 0$ is therefore possible, from this point of view, hence it is theoretically relevant.

Note that, given $s_0, \underline{u}_0, \rho_0$ at time $t=0$, we can compute T_0, p_0 at the same instant via the equation of state and, hence, via (1.5.2), $\dot{\underline{u}}, \dot{s}, \dot{\rho}$ (at $t=0$): so we can compute $s, \underline{u}, \rho$ at time $dt > 0$. Motion is therefore formally determined by the equations (1.5.1),(1.5.2), aside from problems that might be generated by boundary conditions..

If, furthermore, at $t=0$ one has $\underline{\partial}\cdot\underline{u} = 0$ it is not necessarily $\underline{\partial}\cdot\dot{\underline{u}} = 0$: hence it is not necessarily $\underline{\partial}\cdot\underline{u} = 0$ at time $dt > 0$.

It follows that the condition $\underline{\partial}\cdot\underline{u} = 0$ can be added only provided we eliminate one of the scalar relations, *e.g.* the continuity equation. And this can only be consistent if the incompressibility conditions seen in §1.2 are realized and if, also, the temperature variations do not cause important density variations.

If α is the thermal expansion coefficient at constant pressure ($\alpha \sim T_0^{-1}$ if δT is small, in our perfect gas case), the latter condition simply means that $\varepsilon \equiv \alpha\delta T \ll 1$. And the incompressibility condition seen in §1.2 is formulated in the same way by requiring that a typical variation v of the velocity has to be small with respect to the sound velocity v_{sound}.

An estimate of v can be obtained by remarking that motions that develop starting from a state close to rest are essentially due to the density variations due to temperature variations, which naturally generate a small archimedean force with acceleration $\alpha\,\delta T\,g$.

Thus a typical velocity in a motion close to rest, at least initially, is the one acquired by a weight that falls from a height H with acceleration $g\alpha\delta T$:

$$v = \sqrt{Hg\alpha\delta T} \tag{1.5.3}$$

and the time scale of such motions will be the time of fall, of the order $\tau_c = H/v$. We shall make some simplifying assumptions, namely

(h1) *We shall only consider motions in which the space scale and the time scale over which the velocity varies are of the order of H and, respectively, of $\nu^{-1}H^2$ or $\tau_c = H/v$ assuming that the latter two times have the same order of magnitude.*

Furthermore

(h2) *We shall suppose that all velocities have the same order of magnitude, otherwise (c.f.r. §1.2) the discussion on incompressibility would be more involved; in this way there will be only one "small" parameter ε:*

$$\varepsilon \sim \alpha\delta T \sim \frac{v}{v_{sound}} \sim \frac{\nu H^{-1}}{v_{sound}} \sim \frac{gH}{v_{sound}^2} \tag{1.5.4}$$

Here $\sim$ means that the ratio of the various quantities stays fixed as $\varepsilon \to 0$: in other words the ratios of the various quantities should be regarded as further parameters; the notations are $\alpha = -\rho^{-1}\big(\frac{\partial\rho}{\partial T}\big)_p \approx T^{-1}$ and $v_{sound}^{-2} = \big(\frac{\partial\rho}{\partial p}\big)_T \approx (cT)^{-1}$.[1] Note that the convective instability condition (1.4.12) ($\frac{dT}{dz} > -\frac{g}{c_p}$) becomes, since $\frac{dT}{dz} \approx \frac{\delta T}{H}$, $\frac{\alpha\delta T}{gH} v_{sound}^2 > 1$ and hence the conditions (1.5.4) correspond to unstable situations (although "marginally" so because this parameter is $O(1)$) with respect to the birth of convective motions, at least in absence of thermoconduction, *c.f.r.* (C) in §1.4.

Convective motions in turn can be more or less stable with respect to perturbations: the latter is a different, more delicate, matter that we shall analyze later. Their instability will thus be possible (and even be strong) depending on other characteristic parameters: we shall see that, for instance, convective motions arise even though the adiabatic stability condition (1.5.4) holds but the quantity $R_{Pr} = \nu\rho c_p/\kappa$ is large.

Writing $\rho^{-1}\delta\rho = \rho^{-1}\big(\partial\rho/\partial T\big)_p\delta T + \rho^{-1}\big(\partial\rho/\partial p\big)_T\delta p$ we see that the fluid can be considered incompressible if, estimating $\rho^{-1}\delta p$ as $|\dot{\underline{u}}| \sim v^2 H^{-1} \sim |\underset{\sim}{\underline{u}} \cdot \underset{\sim}{\partial}\, \underline{u}|$ and using (1.5.3) (see 1.2)

$$\alpha\delta T \ll 1, \qquad \frac{Hg\alpha\delta T}{v_{sound}^2} \ll 1 \tag{1.5.5}$$

Hence under the assumptions (1.5.4) the incompressibility condition (1.5.5) will simply be $\varepsilon \ll 1$.

Under the hypothesis (1.5.5), and supposing that the velocity and temperature variations in the motions that we consider take place over typical scales of length of the order H and of time of the order $\nu^{-1}H^2$, the equations of

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[1] In fact the sound velocity is defined as $(\partial\rho/\partial p)_s^{-1/2} = \sqrt{c(1 + c_V/c)T} = \sqrt{c_p T}$, rather than by $\sqrt{cT}$, because usually one considers adiabatic motions; but the two definitions give the same order of magnitude in simple ideal gases because $c_V = 3c/2$.

motion will be written, setting $T = T_0 + \vartheta - \delta T/H z$, as

$$\begin{aligned} &\underline{\partial} \cdot \underline{u} = 0 \\ &\dot{\underline{u}} + \underset{\sim}{u} \cdot \underline{\partial}\underline{u} = \nu\Delta\underline{u} + \underline{g} - \frac{1}{\rho}\underline{\partial}p \\ &\dot{\vartheta} + \underset{\sim}{u} \cdot \underset{\sim}{\partial}\vartheta - \frac{\delta T}{H}u_z = \chi\Delta\vartheta + \frac{\nu}{2c_p}(\underset{\sim}{\partial}\underline{u} + \underline{\partial}\underset{\sim}{u})^2 \end{aligned} \tag{1.5.6}$$

where we set $\chi = \kappa\rho^{-1}c_p^{-1}$; the continuity equation has been eliminated (and its violations will be "small" if (1.5.5) holds)[2] and the last equation is obtained from the first of (1.5.2) by noting that, within our approximations, the thermodynamic transformation undergone by the generic volume element must be thought as a transformation at constant pressure, so that[3] $Tds = c_p dT$.

We shall suppose that ν, χ, c_p in (1.5.6) are constants (again for simplicity). And we shall always imagine, without mention, that the boundary conditions are the ones specified before (1.5.2).

(C) The Rayleigh model.

The (1.5.6), valid under the hypotheses (1.5.5),(1.5.4), are still very involved and it is worth noting that the conditions (1.5.4) allow us to perform further simplifications because there are regimes in which the equations can contain terms of different orders of magnitude.

For instance we can consider the case in which the (1.5.4) hold and one supposes that the external force g tends to 0 and the height H tends to ∞,

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[2] Indeed the terms $\partial_t\rho$ e $\underline{u} \cdot \underline{\partial}\rho$ of the continuity equation have (both by (1.5.4)) order of magnitude $O(\rho\alpha\delta T\,\nu/H^2)$, while the third term $\rho\underline{\partial} \cdot \underline{u}$ has order $O(\rho v/H)$ and, by (1.5.4), $vH \sim \nu$ so that the ratio of the orders of magnitude is $O(\varepsilon)$. To lowest order the continuity equation is thus $\underline{\partial} \cdot \underline{u} = 0$.

[3] A more formal discussion is the following. Imagine s as a function of p, T (in a perfect gas it would be $s = c_p \log T + c \log p$); then

$$Tds = c_p dT + \Big(\frac{\partial s}{\partial p}\Big)_T dT \tag{1.5.7}$$

and we can estimate the ratio $\dot{p}/\dot{T}$ by estimating $\dot{T}$ as $\delta T H^2/\nu$ and $\dot{p}$ by remarking that $O(|\underline{\partial}p|) \sim O(\rho\dot{\underline{u}}) \sim \rho v\nu/H^2$ and hence the variations δp of p have size $O(\delta p) \sim O(\rho v\nu/H)$ and, therefore, $O(\dot{p}) = O(\rho v\nu^2/H^3)$. Hence if we compare $\dot{p}/p$ to $\dot{T}/T$ we get $\rho v\nu T/p\delta T H$ which has size $O(\varepsilon)$ by (1.5.4): for instance in the free gas case this is $(v/v_{sound}) \cdot (\nu/Hv_{sound}) \cdot (1/\alpha\delta T) = O(\varepsilon)$ (we consider a fixed fluid so that the parameters v_{sound}, ρ, ν are regarded as constants). Hence in the variation of entropy we can suppose $p = constant$ to leading order, so that $Tds = c_p dT$ (and in the free gas $c_p = c_v + c$). Note that the perfect gas assumption is not necessary for the above argument: it is only made to perform an explicit computation of $\big(\partial s/\partial p\big)_T$ which is a constant in this argument as it is a property of the fluid.

Note also, as it will be used in the following, that if $\chi \simeq \nu$ (*c.f.r.* [1.1.5]), the term $\eta(\underset{\sim}{\partial}\underline{u} + \underline{\partial}\underset{\sim}{u})^2/2$ has size $O(\eta v^2 H^{-2}) = O(\rho\nu v^2 H^{-2}) = O((v/v_{sound})^2(\nu/Hv_{sound})^2) = O(\varepsilon^4)$ so that it can be eliminated from (1.5.6). See [EM94].

as $\varepsilon \to 0$. This facilitates the estimate of the various orders of magnitude in terms of $\varepsilon = \alpha\,\delta T$. It will be possible, in fact, to fix $g = g_0\varepsilon^2$ and $H = h_0\varepsilon^{-1}$ (keeping fixed ν, v_{sound}, p_0, T_0) which, for small ε, is a regime that we shall call the *Rayleigh regime.* In this regime the typical velocity will be $v = \sqrt{\alpha\delta T g H} = O(\varepsilon)$.

In this situation we can see further simplifications, as $\varepsilon \to 0$, because several terms in the last two equations (1.5.6) have order of magnitude in ε which is $O(\varepsilon^3)$; hence all terms of order $O(\varepsilon^4)$ (or smaller) can be neglected in the limit in which $\varepsilon \to 0$. Indeed

(I) the term $\nu c_p^{-1}(\underset{\sim}{\partial}\underline{u})^2$ in of order $O(\varepsilon^4)$; *i.e.* one can neglect the heat generation, by friction, inside the fluid.

(II) $\underline{g} - \rho^{-1}\underline{\partial}p$ differs from $-\alpha\vartheta\underline{g} + \underline{\partial}p'$, for some suitable p', by $O(e^4)$.

Before discussing the validity of the above (I) and (II) note that the typical velocity variations, *c.f.r.* (1.5.3),(1.5.4), will have order $v = \sqrt{gH\alpha\,\delta T} = O(\varepsilon)v_{sound}$ while the typical deviations of temperature and pressure from the hydrostatic equilibrium values will have order $\alpha\,\delta T = \varepsilon$ or v/v_{sound} (*i.e.* again $\le O(\varepsilon)$) respectively, if measured in adimensional form).

Hence neglecting terms of order $(\alpha\,\delta T)^4$ allows us to keep in a significant way the nonlinear terms in (1.5.6), which have order $O(\varepsilon^3)$.

We first discuss (II); we remark that by the definition of ϑ, see (1.5.6), it is

$$\begin{aligned}\text{rot}\,(-\frac{1}{\rho}\underline{\partial}p) =&\frac{1}{\rho^2}(\underline{\partial}\rho\wedge\underline{\partial}p) = \frac{1}{\rho^2}\big(\frac{\partial\rho}{\partial T}\big)_p(\underline{\partial}\vartheta + \frac{\delta T}{H}\frac{\underline{g}}{g})\wedge\underline{\partial}p = \\ =&\frac{\alpha\rho c}{\rho^2}\underline{\partial}\vartheta\wedge\rho\underline{g} + O(\varepsilon^4) \equiv -\alpha\,c\,\underline{\partial}\wedge(\vartheta\underline{g}) + O(\varepsilon^4)\end{aligned} \tag{1.5.8}$$

because if, as we are assuming, we think that $H = h_0\varepsilon^{-1}, g = g_0\varepsilon^2$, with $\varepsilon = \alpha\,\delta T$ then

(a) in the first line, noting that $dT = d\vartheta - \frac{\delta T}{H}dz$, $\underline{\partial}z = -\underline{g}/g$, we use $(\frac{\partial\rho}{\partial T})_p = \rho/T = \alpha\rho$ and the part of $\underline{\partial}\rho$ proportional to $(\frac{\partial\rho}{\partial p})_T\underline{\partial}p$ does not contribute; furthermore,
(b) $\underline{\partial}p - \rho\underline{g}$ has order $O(\varepsilon^3)$ (*i.e.* the order of $\dot{\underline{u}}$ and hence of the product of νH^{-2} times the typical velocity $v = O(\varepsilon)$, see above). Thus we can replace $\underline{\partial}p$ in the last term in the first line of (1.5.8) with $\rho\underline{g}$ up to $O(\varepsilon^4)$ and

Therefore (II) implies $\text{rot}\,(-\rho^{-1}\underline{\partial}p + \alpha\underline{g}\vartheta) = O(\varepsilon^4)$, *i.e.* for some p' it is

$$-\frac{1}{\rho}\underline{\partial}p = -\alpha\vartheta\underline{g} + \underline{\partial}p' + O((\alpha\,\delta T)^4) \tag{1.5.9}$$

or, since $\underline{g}$ is conservative, also $\underline{g} - \rho^{-1}\underline{\partial}p = -\alpha\vartheta\underline{g} + \underline{\partial}p' + O(\varepsilon^4)$.

We now turn to (I), analyzing with the method of §1.2 its physical significance. In the Rayleigh regime the term that we want to neglect has order of magnitude

$$\frac{\nu}{c_p}(\underset{\sim}{\partial}\,\underline{u})^2 \sim \frac{\nu}{c_p}\frac{Hg\alpha\,\delta T}{H^2} = O(\varepsilon^4) \tag{1.5.10}$$

and it has, therefore, to have order of magnitude small compared to the order of magnitude of the other terms of the equation, *i.e.* $\dot\vartheta, \underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\vartheta$, $\frac{\delta T}{H}u_z, \chi\Delta\vartheta$. And one has

$$\begin{aligned}\dot\vartheta \sim & O(\frac{\nu\delta T}{H^2}), \qquad \frac{\delta T}{H}u_z \sim O(\frac{\delta T}{H}\sqrt{g\alpha\,\delta T\,H})\\ \underset{\sim}{u}\cdot\underset{\sim}{\partial}\vartheta \sim & O(\sqrt{gH\alpha\,\delta T}\frac{1}{H}\delta T), \qquad \chi\Delta\vartheta \sim O(\chi\frac{\delta T}{H^2})\end{aligned} \tag{1.5.11}$$

and comparing (1.5.10) with (1.5.11) one finds that the incompressibility conditions (1.5.5) and the conditions of validity of (I) and (II) can be summarized into

$$\begin{aligned}\sqrt{gH\alpha\,\delta T} \ll v_{sound}, \qquad & \varepsilon \equiv \alpha\,\delta T \ll 1,\\ c_p\,\delta T \gg Hg\alpha\,\delta T, \qquad & \frac{\nu}{c_p}\frac{\sqrt{gH\alpha\,\delta T}}{H\delta T} \ll 1\end{aligned} \tag{1.5.12}$$

supposing, as said above, that motions take place over length and time scales given by H and by $H^2\nu^{-1}$ respectively and the conditions can be simultaneously satisfied by choosing

$$\varepsilon = \alpha\,\delta T, \quad g = g_0\varepsilon^2, \quad H = h_0\varepsilon^{-1}, \quad \frac{g_0h_0}{c_pT_0} \approx 1, \quad \frac{\nu}{\sqrt{c_pT_0}h_0} \approx 1 \tag{1.5.13}$$

where g_0, h_0, T_0 are fixed and we used $\alpha \approx T^{-1}$, $v_{sound}^2 \sim c_pT$, provided it is $\varepsilon \ll 1$, (note that these relations are just the (1.5.4)).

In such conditions the (1.5.11) have all size $O(\varepsilon^3)$ and the equations, including the boundary conditions specified before (1.5.2), become

$$\begin{aligned}&\underline{\partial}\cdot\underline{u} = 0\\ &\underline{\dot u} + \underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u} = \nu\Delta\underline{u} - \alpha\vartheta\underline{g} - \underline{\partial}p'\\ &\dot\vartheta + \underset{\sim}{u}\cdot\underset{\sim}{\partial}\vartheta = \chi\Delta\vartheta + \frac{\delta T}{H}u_z\\ &\vartheta(0) = 0 = \vartheta(H), \quad u_z(0) = 0 = u_z(H), \quad \int u_x d\underline{x} = \int u_y d\underline{x} = 0\end{aligned} \tag{1.5.14}$$

and we do not write the equation of state nor the continuity equation because s and ρ no longer appear in (1.5.14) (and, in any event, the equation of state will not hold other than up to a quantity of order $O((\alpha\,\delta T)^2)$. The function p' is related, *but not equal*, to the pressure p: within the approximations it is $p = p_0 - \rho_0 g z + p'$.

We must expect, for consistency, that $\underline{u} = O((\alpha\,\delta T)) = O(\varepsilon)$ and $\vartheta = O(\varepsilon)$, and that the equations make sense up to $O((\alpha\,\delta T)^4)$, *as they now consist entirely of terms of order* $O(\varepsilon^3)$.

In fluidodynamics one defines various *numbers* by forming dimensionless quantities with the parameters that one considers relevant for the stability of the flows studied. In the present case the nonlinear terms in (1.5.14) make sense, and one can define a number measuring the strength of the flow, namely $R = v/v_c$ with $v_c = \nu/H$ (*i.e.* $R = \sqrt{\alpha\,\delta T\, gH}/(\nu/H)$): instabilities can arise for large R, *i.e.* for large velocity variations.

One should stress that, in the considered regime, the Reynolds number is a "free" parameter, ε–independent (by (1.5.13)) in the sense that it is possible to keep R constant while $\varepsilon \to 0$.

In a general flow the "*Reynolds number*" R of a velocity field is defined as the ratio between a typical velocity and the "geometric speed", *i.e.* a velocity formed by the viscosity and a typical length scale. Sometimes there are several numbers that one can imagine to define because there are various different length or time scales. In the present situation the number R, or better R^2, formed by using the "geometric speed" scale ν/H is called *Grashof number*, see [LL71].

In fact there is a second 'natural "geometric speed scale": namely $H^{-1}\chi$. Often $\chi \sim \nu$ (as, in perfect gases, the Clausius–Maxwell relation implies, *c.f.r.* problem [1.1.5]: $\chi = \nu\, c_p/c_V$): but there are materials for which $\nu\chi^{-1} \equiv R_{Pr}$, called the *Prandtl number*, is very large and, therefore, the speed $H^{-1}\chi$ is very different from $H^{-1}\nu$ and instability phenomena can arise at lower velocity gradients. The following table gives an idea of the orders of magnitude (*c.f.r.* [LL71], p. 254) of the experimental values of R_{Pr}

Mercury	0.044
Air	0.733
Water	6.75
Alcohol	16.8
Glycerine	7250.

The convective instability problem, *i.e.* the determination of the values of the parameters R and R_{Pr} in correspondence of which the trivial solution $\underline{u} = \underline{0}$, $\vartheta = 0$ of (1.5.14) loses stability (in the sense of linear stability), was investigated by Rayleigh who did show, as it will be seen in §4.1, that the convective instability is controlled by the size of the product $R^2 R_{Pr}$, sometimes called the *Rayleigh number*:

$$R_{Ray} = \frac{g\alpha\,\delta T\, H^3}{\chi\nu} = R^2\, R_{Pr} \tag{1.5.15}$$

or, sometimes, the *Péclet number*, [LL71].

Remarks:
(1) Note that if the (1.5.4) hold then $R = O(1)$, *i.e.* it stays fixed as $\varepsilon \to 0$. Hence, physically, R_{Ray} large is in general related to large R_{Pr} as well as to large R. However in perfect gases $R_{Pr} = 1$, *c.f.r.* [1.1.5].
(2) Obviously, since the problem allows us to define two independent dimensionless numbers (except in the free gas case, as remarked) we must expect that there is a two–parameters family of phenomena described by (1.5.14) and one should not be surprised that for each of them one could define a characteristic number having the form $R^a R^b_{Pr}$: considering the "large" quantity of possible pairs of real numbers (a, b) one realizes that there is the possibility to make famous not only one's own name, but also that of friends (and enemies), by associating it to a "convective number".

To organize rationally the convective numbers it is useful to define the following adimensional quantities

$$\begin{aligned}\tau =& t\nu H^{-2},\ \xi = xH^{-1},\ \eta = yH^{-1},\ \zeta = zH^{-1},\\ \vartheta^0 =& \frac{\alpha\vartheta}{\alpha\,\delta T},\ \underline{u}^0 = (\sqrt{gH\alpha\,\delta T})^{-1}\,\underline{u}\end{aligned} \tag{1.5.16}$$

where the functions $\underline{u}^0, \vartheta^0$ are regarded as functions of the arguments (τ, ξ, η, ζ).
One checks easily that the Rayleigh equations in the new variables take the form

$$\begin{aligned}&\dot{\underline{u}} + R\underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}}\,\underline{u} = \Delta\underline{u} - R\vartheta\underline{e} - \underline{\partial}p, \qquad R^2 = \frac{gH^3\alpha\,\delta T}{\nu^2}\\ &\dot{\vartheta} + R\underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}}\,\vartheta = R_{Pr}^{-1}\Delta\vartheta + Ru_z, \qquad R_{Pr} = \frac{\nu}{\kappa}\\ &\underline{\partial} \cdot \underline{u} = 0\\ &u_z(0) = u_z(1) = 0, \qquad \vartheta(0) = \vartheta(1) = 0, \qquad \int u_x d\underline{x} = \int u_y d\underline{x} = 0\end{aligned} \tag{1.5.17}$$

where after the change of variables we eliminated the labels 0 and recalled t, x, y, z the adimensional coordinates τ, ξ, η, ζ in (1.5.16) redefining p suitably; furthermore we have set $\underline{e} = (0, 0, -1)$.

The equations (1.5.17) hold under the hypothesis that (1.5.13) hold: note again that in such case $R = g_0 h_0^3 \nu^{-2}$ and $R_{Pr} = \nu\kappa^{-1}$ can be fixed independently of ε. This is important because it shows that various regimes depending on two parameters (the parameters R, R_{Pr}) exist in which the equations are admissible, if ε is small.

Remark: one can also note that if δT was < 0, *i.e.* if the temperature increased with height, the equations (1.5.17) and (1.5.14) would "only" change because of the sign of ϑ in the first of the (1.5.17) or of $\vartheta\underline{g}$ in the second of (1.5.14). This can be seen by looking back at the derivation, or by remarking that changing the sign of δT is equivalent to exchange the role of $z = 0$ and $z = H$. The heat transport equations between two horizontal planes with the one above warmer that the lower one (where "up" and "down" are defined by the direction of gravity) are therefore, if $\delta T > 0$

$$\begin{aligned}
&\dot{\underline{u}} + R\underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}}\, \underline{u} = \Delta \underline{u} + R\vartheta \underline{e} - \underline{\partial} p, \qquad R^2 = \frac{gH^3\alpha\,\delta T}{\nu^2}\\
&\dot{\vartheta} + R\underset{\sim}{\underline{u}} \cdot \underset{\sim}{\underline{\partial}}\, \vartheta = R_{Pr}^{-1}\Delta\vartheta + Ru_z, \qquad R_{Pr} = \frac{\nu}{\chi}\\
&\underline{\partial} \cdot \underline{u} = 0\\
&u_z(0) = u_z(1) = 0, \qquad \vartheta(0) = \vartheta(1) = 0, \qquad \int u_x d\underline{x} = \int u_y d\underline{x} = 0
\end{aligned} \tag{1.5.18}$$

which are, however, less interesting because they do not imply any (linear) instability of the thermostatic solution, see §4.1.

(D) Rescalings: a systematic analysis.

One can ask for a more systematic way to derive (1.5.17). One can again use the method, actually very general, employed in §1.3.

Suppose that $\varepsilon \equiv \alpha\,\delta T$, $\alpha = T^{-1}$, $v_{sound}^2 = c_p T$ and

$$\begin{aligned}
\varepsilon =&\alpha\delta T, \qquad \varepsilon' = \frac{gH}{cT} \approx \varepsilon, \qquad \varepsilon'' = \frac{\nu}{Hv_{sound}} \approx \varepsilon, \qquad \varepsilon \to 0\\
R =&\frac{gH^3\alpha\,\delta T}{\nu^2}, \qquad \text{and} \quad R_{Pr} = \frac{\nu}{\kappa} \qquad \text{fixed}
\end{aligned} \tag{1.5.19}$$

which is a "regime" that we shall (as above) call the *Rayleigh convective regime* with parameter ε. *We are, automatically, in this regime if the parameters are chosen as in* (1.5.13). We now look for a solution of (1.5.2) which can be written as

$$\begin{aligned}
\underline{u}(\underline{x},t) =&(gH\alpha\,\delta T)^{1/2}\, \underline{u}^0(\underline{x}H^{-1}, t\nu H^{-2})\\
T(\underline{x},t) =&T_0 - \frac{\delta T}{H} z + \delta T \vartheta^0(\underline{x}H^{-1}, t\nu H^{-2})\\
\rho(\underline{x},t) =&\rho_0(zH^{-1}) + \varepsilon r^0(\underline{x}H^{-1}, t\nu H^{-2})\\
p =&\varepsilon p_0(zH^{-1}) + \varepsilon^2 p^0(\underline{x}H^{-1}, t\nu H^{-2}), \qquad g = \varepsilon^2 g_0
\end{aligned} \tag{1.5.20}$$

where $\underline{u}^0(\underline{\xi},\tau), \vartheta^0(\underline{\xi},\tau), r^0(\underline{\xi},\tau)$ *can be thought of as power series* in ε with coefficients *regular* in $\underline{\xi},\tau$: note that under the assumptions (1.5.19) the three parameters $\varepsilon, \varepsilon', \varepsilon''$ are estimated by ε, which therefore is our only small parameter. Furthermore the functions $T_0 - \delta T z H^{-1}$ and $\rho_0(zH^{-1}), p_0(zH^{-1})$ are solutions of the "time independent problem", *i.e.* of (1.4.6), with boundary conditions $\rho_0(0) = \overline{\rho}$ and T_0; namely

$$\rho_0(\zeta) = \overline{\rho}, \qquad p_0(\zeta) = p_0(0) + \overline{\rho} g_0 h_0 \zeta \tag{1.5.21}$$

obtained from (1.4.10) by applying the equation of state to express ρ_0 in term of p (given by (1.4.10)) and of the temperature $T_0 - \delta T\zeta$.

One can now check, by direct substitution of (1.5.20) into (1.5.2), with $\underline{u}^0 = \underline{u}^1 + \varepsilon\underline{u}^2 + \ldots$, $\rho = \rho_0 + \varepsilon r^1 + \varepsilon r^2 + \ldots$, $p^0 = p^2 + \varepsilon p^3 + \ldots$ and $\vartheta^0 = \vartheta^1 + \varepsilon\vartheta^2 + \ldots$, that the lowest orders $\underline{u}^1, \vartheta^1$, $\overline{\rho}, p_0$ verify (1.5.17).

As in footnote [3] one has to note that $\dot{p}$ turns out to be of $O(\varepsilon^4)$ in an equation in which all terms have the same size of order $O(\varepsilon^3)$ so that $\dot{p} = 0$ up to order $O(\varepsilon^4)$ and (since $p = c\rho T$) we can replace $T\dot{\rho}$ by $-\rho\dot{T}$ in the entropy equation. Also here the assumption of perfect gas is not essential.

Hence the lowest order of $\underline{u}^0, \vartheta^0, \rho$ describes, at least heuristically, the asymptotic regime in which the (1.5.17) are exact: we shall say that the *convective Rayleigh equation (1.5.17) is expected to be is exact in the limit "$\varepsilon \to 0$ with the relations (1.5.13) fixed"*.

This is a mathematically transparent method, apt to clarify the meaning of the approximations and it is the version for the (1.5.2) (*i.e.* for the problem of the gas between two planes at given temperatures) of the analysis of the incompressibility assumption of §1.3. Hence we can hope in the validity of theorems of the type of the ones in §1.3: however such theorems have not (yet) been proved in the present case.

This viewpoint is clearly more systematic because it allows us, in principle, also to find the higher order (in ε) corrections which, in a similar way, should verify suitable equations.

To determine that, in a suitable limit, a certain regime (*i.e.* certain simplified equations) are "asymptotically exact" it is usually necessary to proceed empirically as above (or as in §1.2) and only *a posteriori*, once the structure of the equations has been understood and the relevant adimensional parameters have been identified, it becomes possible to "guess" the right rescaling and the limit in which the equations "become exact".

It is convenient to note here that it is not impossible that for the same equation one can find several distinct "regimes" in which the solutions are described by "rescaled" equations (simpler than the original ones, but usually different and depending on the regime). Although we do not attempt to discuss an example for the model (1.5.2) considered here, there are many other examples: we already met in §1.3 an instance in which a fluid can be found in a "Euler regime" or in the (different) "Navier–Stokes regime".

Problems

[1.5.1]: Examine some consequence of a violation of the (1.5.4).

Bibliography: [LL71]: §50,§53,§56; and [EM93]: from the latter work, in particular, I have drawn many of the basic ideas and the methods of the present section.

§1.6 Kinematics: incompressible fields, vector potentials, decom-

positions of a general field.

It is important to keep in mind various representations of velocity fields in terms of other vector fields with special properties. Much as in electromagnetism it can be important to represent electric or magnetic fields in terms of potentials (like the Coulomb potential or the vector potential). Indeed, sometimes, the basic equations expressed in terms of such auxiliary fields take more transparent or simpler forms. In this section and in the problems at the end of it we shall discuss some among the simplest representation theorems of vector fields of relevance in fluidodynamics (and electromagnetism).

(A) Incompressible fields in the whole space as rotations of a vector potential.

Consider incompressible fluids: the continuity equation will require that

$$\underline{\partial} \cdot \underline{u} = 0 \quad \text{in } \Omega \tag{1.6.1}$$

Suppose $\underline{u}$ of class C^∞ on $\Omega = R^3$ and rapidly decreasing at infinity: *i.e.* for each $p, q \geq 0$ let $|\xi|^q \underset{\sim}{\underline{\partial}}^p \underline{u}(\xi) \xrightarrow[|\xi|\to\infty]{} 0$.[1] Then there is a vector field $\underline{A}$ such that

$$\text{rot } \underline{A} = \underline{u}, \qquad \underline{\partial} \cdot \underline{A} = 0 \tag{1.6.2}$$

Constructing $\underline{A}$ is elementary if one starts from the Fourier transform representation of $\underline{u}$

$$\underline{u}(\underline{\xi}) = \int \hat{\underline{u}}(\underline{k}) e^{i\underline{k}\cdot\underline{\xi}} \, d\underline{k} \tag{1.6.3}$$

Indeed (1.6.1) means $\underline{k} \cdot \hat{\underline{u}}(\underline{k}) = 0$ *i.e.* $\hat{\underline{u}}(\underline{k}) = i\underline{k} \wedge \underline{a}(\underline{k})$ where $\underline{a}(\underline{k})$ is a suitable vector orthogonal to $\underline{k}$ and unique for $\underline{k} \neq \underline{0}$. Hence

$$\underline{u}(\underline{\xi}) = \int i\underline{k} \wedge \underline{a}(\underline{k}) e^{i\underline{k}\cdot\underline{\xi}} d\underline{k} = \text{rot} \int \underline{a}(\underline{k}) e^{i\underline{k}\cdot\underline{\xi}} d\underline{k} = \text{rot } \underline{A}(\xi) \tag{1.6.4}$$

However the vector field $\underline{A}$ in general, although being of class C^∞, will not have a rapid decrease at infinity.

This can be evinced by remarking that for $\underline{k} \to \underline{0}$ the expression for $\underline{a}(\underline{k})$ in terms of $\hat{\underline{u}}(\underline{k})$ will not, in general, be differentiable in $\underline{k}$. In fact from $\underline{\partial} \cdot \underline{u} = 0$ it follows that $\hat{\underline{u}}(\underline{0}) = \underline{0}$.[2] This implies only that $\underline{u}(\underline{k})$ has order $\underline{k}$ for $\underline{k} \to \underline{0}$. However to have that $\underline{a}(\underline{k}) = i\underline{k} \wedge \hat{\underline{u}}(\underline{k})/\underline{k}^2$ be regular in $\underline{k} = \underline{0}$ one should have that $\underline{k} \wedge \hat{\underline{u}}(\underline{k})$ has the form of a product of $\underline{k}^2$ times a regular function of $\underline{k}$: but the vanishing of $\hat{\underline{u}}(\underline{0})$ implies only that $\underline{a}(\underline{k})$ is bounded as

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[1] Here and in the following we shall denote with symbols like $\underline{\partial}^p$ a generic derivative of order p with respect to the coordinates $\underline{x}$.

[2] Because $\hat{\underline{u}}(\underline{k})$ is of class C^∞ in $\underline{k}$ by the decrease of $\underline{u}$ and of its derivatives as $\xi \to \infty$ hence $\hat{\underline{u}}(\underline{k}) = \hat{\underline{u}}(\underline{0}) + O(\underline{k})$ and $0 \equiv \underline{k} \cdot \hat{\underline{u}}(\underline{k}) = \underline{k} \cdot \hat{\underline{u}}(\underline{0}) + O(\underline{k}^2)$: since $\underline{k}$ is arbitrary it must be $\hat{\underline{u}}(\underline{0}) = \underline{0}$.

$\underline{k} \to 0$ with, in general, a limit depending on the direction along which one lets $\underline{k}$ tend to $\underline{0}$ (and therefore it is not differentiable in $\underline{k}$ at $\underline{0}$). Hence $\underline{a}(\underline{k})$ is bounded and approaches rapidly zero as $\underline{k} \to \infty$, but it is not everywhere differentiable in $\underline{k}$ and, as a consequence, its Fourier transform $\underline{A}(\xi)$ will be C^∞ but it will not tend to zero rapidly for $\xi \to \infty$.

The above can also be derived directly from the formula, which does not involve Fourier transforms,

$$\underline{A}(\xi) = \frac{1}{4\pi} \int_{R^3} \frac{d\eta}{|\xi - \eta|} \operatorname{rot} \underline{u}(\eta) \tag{1.6.5}$$

i.e. $\underline{A} = -\Delta^{-1} \operatorname{rot} \underline{u}$, well known in electromagnetism ($\underline{A}$ can be interpreted as the magnetic field generated by a current $\underline{u}$ according to the "*Biot–Savart law*").

(B) An incompressible field in a finite convex volume Ω as a rotation: some sufficient conditions.

Often one needs to consider divergenceless vector fields representing the velocity $\underline{u}$ of a fluid in a finite container Ω which we always suppose with a very regular boundary of class C^∞. We ask whether given $\underline{u} \in C^\infty(\Omega)$ it is possible to find $\underline{A} \in C^\infty(\Omega)$ so that

$$\underline{u} = \operatorname{rot} \underline{A} \qquad \operatorname{div} \underline{A} = 0 \tag{1.6.6}$$

A field $\underline{A}$ verifying (1.6.6) and in $C^\infty(\Omega)$ will be called a *vector potential* for $\underline{u}$.

Evidently to show that a field $\underline{A}$ exists it will suffice to show, see Sec. 1.6.1, that $\underline{u}$ can be extended outside Ω to a function $C^\infty(R^3)$ vanishing outside a bounded domain $\tilde\Omega \supset \Omega$ and, everywhere, with vanishing divergence.

Referring to the problem in (A) the difficulty is that *the existence of an extension is not evident*. And if Ω contains "holes" it is not true in general (see the problem [1.6.12] for an example). In what follows we shall exhibit one such field by considering convex domains Ω for which the geometric construction is possible. The theorem can be extended to far more general domains (regular convex domains whose boundary points can be connected to ∞ by a curve that has no other points in Ω, *i.e.* domains "with no holes" see problems [1.6.11]%[1.6.14]).

To show the existence of an extension of $\underline{u}$ outside Ω we need to understand better the structure of the divergence free vector fields. We consider the case in which Ω is strictly convex and with analytic boundary (see problem [1.6.8]) for the general case).

Consider the components u_2 and u_3 of $\underline{u}$: we shall extend them to functions, that we still denote with the same name, defined on the whole R^3 and there of class C^∞ and vanishing outside a sphere $\tilde\Omega$ of large enough radius containing Ω in its interior. For this purpose we consider the cylinder of all the lines parallel to the axis 1 that cut Ω: the ones tangent to $\partial\Omega$ will define a closed line λ on $\partial\Omega$ and smooth.

Let Σ be a surface of class C^∞ containing λ and intersecting transversally (*i.e.* with a non zero angle) every line parallel to the axis 1. We consider the function u_1 on $\Sigma\cap\Omega$ and extend it to a function defined on the whole Σ, of class C^∞ there, and vanishing outside the sphere $\tilde\Omega$.

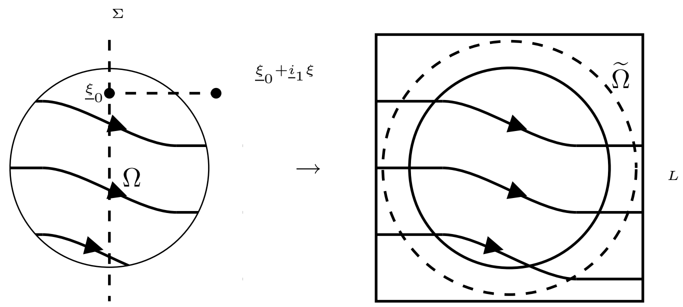

Fig. (1.6.1): The directed lines reprent the field $\underline{u}$, the circle represents Ω and the vertical dotted line the surface Σ while the horizontal dotted line represents a segment parallel to the axis 1. The dotted circle represents $\tilde\Omega$.

Every point of R^3 can be represented as $\underline{\xi}_0+\underline{i}_1\xi$ with $\underline{\xi}_0\in\Sigma$ and $\xi\in R$ and we shall denote it $(\underline{\xi}_0,\xi)$, see Fig. (1.6.1). We then define

$$u_1(\underline{\xi}_0+\underline{i}_1\xi)=u_1(\underline{\xi}_0)+\int_0^\xi -\sum_{j=2}^3\partial_j u_j(\underline{\xi}_0+\underline{i}_1\xi')\,d\xi' \tag{1.6.7}$$

and we thus obtain a velocity field $\underline{u}$ with zero divergence defined in R^3 and extending the field given in Ω. By construction this field is identically zero outside a cylinder parallel to the axis $\underline{i}_1$ containing Ω. Furthermore if one moves along the axis $\underline{i}_1$ and by a distance large enough away from Ω it becomes constant and parallel to the axis 1 itself

$$\underline{u}(\underline{\xi}_0+\xi\underline{i}_1)=\begin{cases}V_+(\underline{\xi}_0)\,\underline{i}_1 & \xi>0 \text{ large}\\ V_-(\underline{\xi}_0)\,\underline{i}_1 & -\xi>0 \text{ large}\end{cases} \tag{1.6.8}$$

where $V_\pm(\xi_0)=u_1(\xi_0)+\int_0^{\pm\infty}-\sum_{j=2}^3\partial_j u_j\,d\xi'$, *c.f.r.* (1.6.7). Here “large” means $|\xi|>L_0$ for some L_0: let $L>L_0$. This is illustrated in the left square in Fig. (1.6.2).

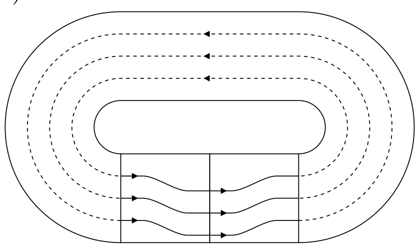

Fig. (1.6.2): The left square contains the extension of the field current lines until they become parallel to the 1–axis. The right square is the mirror image of the left square and the curved dotted lines match the lines exiting from the right square with the corresponding ones of the left square.

We now set

$$\begin{aligned} &u'_j(\underline{\xi}_0 + (L+\xi)\underline{i}_1) = -u_j(\underline{\xi}_0 + (L-\xi)\underline{i}_1) \qquad && j = 2,3 \\ &u'_1(\underline{\xi}_0 + (L+\xi)\underline{i}_1) = +u_1(\underline{\xi}_0 + (L-\xi)\underline{i}_1) \qquad && 0 < \xi < 2L \end{aligned} \tag{1.6.9}$$

It is clear that $\underline{u}'$ extends $\underline{u}$ from $\tilde{\Omega}$ to R^3 and that for $\xi \to \pm\infty$ it has for every "transversal coordinate" $\underline{\xi}_0 \in \Sigma$, the same limit $V_-(\underline{\xi}_0)\underline{i}_1$; furthermore $\underline{\partial} \cdot \underline{u}' = 0$.

Consider now a cylinder Γ parallel to $\underline{i}_1$, such that $\tilde{\Omega} \subset \Gamma$, with a bounded circular base orthogonal to $\underline{i}_1$. Continue the lines of the cylinder which are parallel to the axis into a smooth bundle of curves parallel to the axis $\underline{i}_1$ so that each closes onto itself as symbolically drawn in Fig. (1.6.2).

In the closed tube $\tilde{\Gamma}$ consider a vector field $\underline{\tilde{u}}$ defined so that it remains tangent to the just constructed curves (which define $\tilde{\Gamma}$). The field $\underline{\tilde{u}}$ will be equal to $V_-(\underline{\xi}_0)\underline{i}_1$ at the point where $\tilde{\Gamma}/\Gamma$ joins the curve that has transversal coordinate $\underline{\xi}_0$.

One then continues the function $\underline{\tilde{u}}$ in $\tilde{\Gamma}$, outside Γ, so that the flow of $\underline{\tilde{u}}$ through equal surface elements normal to each curve, among the ones considered, remains constant.

The field $\underline{u}$ is of class C^∞, vanishes out of $\tilde{\Gamma}$ and everywhere it as zero divergence.

We can then think of $\underline{\tilde{u}}$ as a field on R^3 vanishing outside the tube $\tilde{\Gamma}$ and write it as

$$\underline{\tilde{u}} = \text{ rot } \underline{A} \;, \quad \underline{\partial} \cdot \underline{A} = 0 \;. \tag{1.6.10}$$

where $\underline{A}$ is a suitable vector field of class C^∞ (*c.f.r. (A)*). This shows that the restriction of $\underline{A}$ to Ω has the properties that we wish for a vector potential.

(C) Ambiguities for vector potentials of incompressible fields.

It is clear that, once one has found a field $\underline{A}$ that is a vector potential for an incompressible field $\underline{u}$, one can find infinitely many others: it suffices to alter $\underline{A}$ by a gradient field. Then we ask the question: given $\underline{u} \in C^\infty(\Omega)$, with $\underline{\partial} \cdot \underline{u} = 0$ and given a vector potential $\underline{A}_0$ for $\underline{u}$ which ambiguity is left for $\underline{A}$? We still suppose that the domain Ω is simply connected.

If $\underline{A}$ and $\underline{A}_0$ are two vector potentials for $\underline{u}$ then $\underline{A} - \underline{A}_0$ is such that $\underline{\partial} \cdot (\underline{A} - \underline{A}_0) = 0$ and rot $(\underline{A} - \underline{A}_0) = \underline{0}$: *i.e.* there is $\varphi \in C^\infty(\Omega)$ such that

$$\underline{A} = \underline{A}_0 + \underline{\partial}\varphi \;, \quad \Delta\varphi = 0 \;. \tag{1.6.11}$$

Therefore we see that $\underline{A}$ *is determined up to the gradient of a harmonic function* φ.

If we did also require that $\underline{A}\cdot\underline{n}=0$ on $\partial\Omega$ then given a vector potential $\underline{A}_0$ for $\underline{u}$ and calling φ a solution of the *Neumann problem*

$$\Delta\varphi=0,\qquad \partial_{\underline{n}}\varphi=-\underline{A}_0\cdot\underline{n} \tag{1.6.12}$$

we realize that there exists a unique vector potential $\underline{A}$ such that

$$\underline{u}=\ \mathrm{rot}\ \underline{A}\ ,\quad \underline{\partial}\cdot\underline{A}=0\qquad \text{in }\Omega,\quad \underline{A}\cdot\underline{n}=0\qquad \text{su }\partial\Omega \tag{1.6.13}$$

and more generally one can imagine other properties (like boundary conditions or other) apt to single out a vector potential for an incompressible field among the many possible ones.

(D) A regular vector field in Ω can be represented as the sum of a rotation field and a gradient field.

Let $\underline{\xi}\to\underline{w}(\underline{\xi})$ be a vector field defined on R^3 and there of class C^∞ and rapidly decreasing at ∞. If $\hat{\underline{w}}(\underline{k})$ is its Fourier transform, it will be possible, uniquely, to write, for $\underline{k}\neq 0$,

$$\hat{\underline{w}}(\underline{k})=i\underline{k}\wedge\underline{a}(\underline{k})+\underline{k}\,f(\underline{k})\quad\text{with}\quad \underline{a}(\underline{k})\cdot\underline{k}=0 \tag{1.6.14}$$

with $\underline{a}(\underline{k})$, $f(\underline{k})$ bounded by $|\underline{k}|^{-1}$ near $\underline{k}=\underline{0}$, rapidly decreasing and of class C^∞ for $\underline{k}\neq\underline{0}$. Hence

$$\underline{w}=\ \mathrm{rot}\ \underline{A}+\underline{\partial}\varphi \tag{1.6.15}$$

Or *every vector field can be represented as a sum of a solenoidal field and a gradient field*, because the fields that have zero divergence are also called solenoidal. Note that, as in the case *(A)*, the potentials will have in general a slow decay (normally proportional to $\hat{\underline{w}}(\underline{0})$ and decaying almost as $O(|\underline{\xi}|^{-2})$ as $\underline{\xi}\to\infty$.

The same result also holds, therefore, to represent a field $\underline{w}\in C^\infty(\Omega)$ when Ω is a finite region. It suffices to extend $\underline{w}$ to $C^\infty(R^3)$ and apply (1.6.14), (1.6.15).

(E) The space $X_{\rm rot}(\Omega)$ and its complement in $L_2(\Omega)$.

The space $L_2(\Omega)$ admits a remarkable decomposition into a direct sum of two orthogonal subspaces, "*the rotations and the gradients*". This is not an extension of the decompositions discussed so far: indeed we shall see that also a purely solenoidal field $\underline{u}=\mathrm{rot}\,\underline{X}$ will admit a non trivial decomposition (*i.e.* a representation with a nonzero gradient component), *unless it is also tangent to the boundary of* Ω. In the last case the decomposition that we are about to describe will coincide with the previous ones.

Consider the space $X_{\rm rot}\subset L_2(\Omega)$ defined as

$$X_{\rm rot}=\{\text{closure in }L_2(\Omega)\text{ of the fields }\underline{u}\text{ with zero divergence, }C^\infty(\Omega)\ \text{and zero in the vicinity of the boundary}\}\ \equiv\ \overline{X^0_{\rm rot}} \tag{1.6.16}$$

where $X^0_{\rm rot}$ is the set of the C^∞ fields, with zero divergence and *vanishing* in the vicinity of the boundary $\partial\Omega$.

The space $X_{\rm rot}$ should be thought of, in a sense to be made precise below, as the set of vector fields $\underline{u}$ with zero divergence and with some component vanishing on the boundary $\partial\Omega$. One difficulty, for instance, is the fact that if $\underline{u}\in X_{\rm rot}$ then $\underline{u}$ is in $L_2(\Omega)$ but it is not necessarily differentiable[3]so that the divergence of $\underline{u}$, $\underline{\partial}\cdot\underline{u}$, can only vanish (in general) in a "weak sense"; precisely

$$\int_\Omega \underline{u}\cdot\underline{\partial}f\,d\xi\equiv 0,\qquad \forall\, f\in C^\infty(\Omega) \tag{1.6.17}$$

(as it can be seen by approximating $\underline{u}$ with $\underline{u}_n\in X^0_{\rm rot}$ in the norm of $L_2(\Omega)$, so that $\int_\Omega \underline{u}\cdot\underline{\partial}f\,d\xi=\lim_n\int_\Omega \underline{u}_n\cdot\underline{\partial}f\,d\xi=-\lim_n\int_\Omega\underline{\partial}\cdot\underline{u}_n f\,d\xi=0$, because $\underline{u}_n\cdot\underline{n}\equiv 0$ on $\partial\Omega$). And (1.6.17) shows that if $\underline{u}\in X_{\rm rot}\cap C^\infty(\Omega)$ then

$$\begin{aligned}&-\int_\Omega \underline{\partial}\cdot\underline{u}\,f d\underline{x}+\int_{\partial\Omega}\underline{u}\cdot\underline{n}\,f\,d\sigma=0\Rightarrow\\ &\Rightarrow\underline{\partial}\cdot\underline{u}=0 \text{ in } \Omega,\qquad \underline{u}\cdot\underline{n}=0\ \text{ in } \partial\Omega\end{aligned} \tag{1.6.18}$$

by the arbitrariness of$f$.

It is now convenient to introduce, to simplify notations, the following notion

Definition *(generalized derivatives and distributions): given a function $f\in L_2(\Omega)$ we say that f has a "generalized derivative" F_j, in $L_2(\Omega)$, with respect to x_j, or a derivative $\partial_j f=F_j$ "in the sense of the theory of distributions", if there is a function $F_j\in L_2(\Omega)$ for which we can write*

$$\int_\Omega f(\underline{x})\,\partial_j\varphi(\underline{x})\,d\underline{x}=-\int_\Omega F_j(\underline{x})\,\varphi(\underline{x})\,d\underline{x},\qquad \text{for all } \varphi\in C^\infty_0(\Omega) \tag{1.6.19}$$

where $C^\infty_0(\Omega)$ is the space of the functions φ of class C^∞ and which vanish in the vicinity of the boundary $\partial\Omega$ of Ω.

Analogously we can define the *divergence* in "the sense of distributions" of a fixed $\underline{u}\in L_2(\Omega)$, its *rotation*, and the higher order derivatives.

Thus the regular functions in $X_{\rm rot}$ vanish on $\partial\Omega$ only in the sense that $\underline{u}\cdot\underline{n}=0$. The others verify the (1.6.17) and therefore have $\underline{\partial}\cdot\underline{u}=0$ in the sense of distributions and $\underline{u}\cdot\underline{n}=0$ on $\partial\Omega$ in a weak sense (exactly expressed by the first of (1.6.17)).

The space of the fields $\underline{f}\in L_2(\Omega)$ in the orthogonal complement of $X_{\rm rot}$ also admits an interesting description; if $\underline{f}\in X^\perp_{\rm rot}$, it will be for every C^∞ field $\underline{A}$ vanishing near $\partial\Omega$

$$0\equiv\int\underline{f}\cdot{\rm rot}\,\underline{A}\,d\xi\ \Rightarrow\ {\rm rot}\ \underline{f}=\underline{0}\quad \text{in the sense of distributions} \tag{1.6.20}$$

[3] Because the operation of closure in L_2 can generate functions that are not regular.

and it is possible to show also (*c.f.r.* problems [1.6.16]%[1.6.19]) that $\underline{f}$ can be written as $\underline{f} = \underline{\partial}\varphi$ with φ function in $L_2(\Omega)$ and gradient (in the sense of distributions) in $L_2(\Omega)$.

Concluding: *the most general vector field in* $L_2(\Omega)$ *can be written as*

$$\begin{aligned} &\underline{w} = \underline{u} + \underline{\partial}\varphi \qquad \textit{with} \\ &\underline{u} \in X_{\rm rot}\ , \ \varphi \in L_2(\Omega) \ \textit{and with gradient in} \ L_2(\Omega) \end{aligned} \tag{1.6.21}$$

and such decomposition is unique (because $X_{\rm rot}$ and $X_{\rm rot}^{\perp}$ are a decomposition of a Hilbert space in two complementary orthogonal spaces). *Furthermore the functions in* $X_{\rm rot}$ *must be thought of as vector fields with zero divergence (in the sense of distributions) and tangent to* $\partial\Omega$ *(in the sense of (1.6.17)). If* $\underline{u}$ *is continuous in* Ω *up to and including the boundary then* $\underline{u}$ *is really tangent on* $\partial\Omega$.

One usually says: *A vector field in* $L_2(\Omega)$ *can be uniquely written as sum of a solenoidal vector field tangent to the boundary and of a gradient vector field.*

(F) The "gradient–solenoid" decomposition of a regular vector field.

We note that a vector field $\underline{w} \in C^\infty(\Omega)$ can be naturally written in the form

$$\underline{w} = \underline{u} + \underline{\partial}\varphi \quad \text{with} \quad \underline{u}\cdot\underline{n} = 0 \quad \text{on} \quad \partial\Omega \ \ \textit{and} \qquad \underline{\partial}\cdot\underline{u} = 0 \quad \text{on} \quad \Omega \tag{1.6.22}$$

and the decomposition is "regular", *i.e.* one can find $\underline{u} \in C^\infty(\Omega)$ and $\varphi \in C^\infty(\Omega)$, unique in this regularity class (up to an arbitrary additive constant in φ). In fact we shall have that φ satisfies

$$\begin{aligned} \partial_n\varphi &= \underline{n}\cdot\underline{w} \qquad \text{su } \partial\Omega \\ \Delta\varphi &= \underline{\partial}\cdot\underline{w} \qquad \text{su } \Omega \end{aligned} \tag{1.6.23}$$

and this equation determines $\varphi \in C^\infty(\Omega)$ up to an additive constant. Note that (1.6.23) is a *Neumann problem* hence, in order that it be soluble, it is necessary that the compatibility condition expressed by electrostatics Gauss' theorem be satisfied: $\int_{\partial\Omega}\underline{w}\cdot\underline{n}\,d\sigma = \int_\Omega \underline{\partial}\cdot\underline{w}\,d\underline{x}$. This condition is automatically verified because of the integration theorem of Stokes. The regularity in class $C^\infty(\Omega)$ of the solution of the Neumann problem with boundary data of class $C^\infty(\partial\Omega)$ is one of the fundamental properties of the theory of the elliptic equations and one finds it discussed in many treatises (*c.f.r.* for instance [So63]) and it is summarized in the problems of §2.2.

Clearly $\underline{u} = \underline{w} - \underline{\partial}\varphi \in C^\infty(\Omega)$ satisfies (1.6.22). Hence the decomposition (1.6.22) exists and is unique (up to an additive constant in φ), hence the sense in which $\underline{u}\cdot\underline{n} = 0$ is, in such cases, literal.

Problems and complements.

[1.6.1]: Show the impossibility, in general, of a solution of (1.6.22) with $\underline{u} = 0$ on $\partial\Omega$. (*Idea:* consider a field $\underline{w}$ like

$$\begin{aligned} &w_1(x,y,z) = -\partial_y f(x,y)\chi(z) \\ &w_2(x,y,z) = \partial_x f(x,y)\chi(z), \qquad\qquad x,y \in R^2 \\ &w_3(x,y,z) = g(x,y)\chi(z), \qquad\qquad z \in [0,+\infty) \end{aligned}$$

where $\chi(z) \equiv 1$ for $|z| \le z_0$ and $\chi(z) \equiv 0$ for $|z| > z_1 > z_0$, and $\chi \in C^\infty(R^3)$. Then the equations (1.6.22) become

$$\begin{cases} \Delta\varphi = g(xy)\chi'(z), & \Omega = R^2 \times [0,+\infty) \\ \partial_z\varphi = g(xy), & z = 0 \end{cases}$$

$$\begin{cases} \partial_x\varphi = -\partial_y f, & z = 0 \\ \partial_y\varphi = \partial_x f \end{cases}$$

but he first pair of equations determines φ up to a constant and, therefore, the other two equations will be false for suitable choices of the *arbitrary* function f.)

[1.6.2]: Show the incompatibility of (1.6.22), with $\underline{u} = \underline{0}$ on $\partial\Omega$ in an example with bounded Ω. (*Idea:* Consider Ω = sphere of radius R, use polar coordinates and imitate the example in [1.6.1] by letting ρ play the role of z (with $\rho \ge 0$, $\rho \le R$) and to (θ,φ) the role of (x,y).)

[1.6.3]: (*extension of a function, brutal method*) If $\vartheta(x)$ is a C^∞ function which is 1 for $x \in (-\infty,+1)$ and 0 for $x \in [+2,+\infty)$, consider the function

$$x \Rightarrow f(x) = \vartheta(x) \sum_{k=0}^{\infty} \frac{c_k}{k!} x^k e^{-xc_k^2} \tag{$*$}$$

and show that it is of class C^∞ in $x \in [0,+\infty)$ *for any sequence* c_k. Furthermore

$$f^{(0)}(0) = c_0, \qquad f^{(p)}(0) = c_p + \text{ polynomial in } c_0,\dots,c_{p-1}, \quad \forall p > 0$$

Show also that $e^{-xc_h^2}$ could be replaced by $e^{-x|c_h|^\alpha}$ with $\alpha > 0$ arbitrary.

[1.6.4]: Using the result in [1.6.3] show that, given an arbitrary sequence $f^{(j)}$, there is a function $f \in C^\infty([0,+\infty)]$ such that $f^{(j)}(0) = f^{(j)}$, $\forall j$. Furthermore, setting $F_j = \sum_{k=0}^{j} |f^{(k)}|$ there is a polynomial $L_j(x)$, explicitly computable, for which

$$||f||_{C^{(j)}([0,+\infty))} \le L_j(F_j) \tag{$**$}$$

where $||\cdot||_{C^{(j)}(a,b)} = \sup_{\substack{x\in[a,b]\\0\le k\le j}} |\frac{d^k f(x)}{dx^k}|$ is the "C^j–metric" on the functions on $[a,b]$.

[1.6.5]: Check that (*) and (**) imply existence of an extension f^a of f from $[-\infty,0]$ to $[-\infty,a]$, with $a > 0$, which is continuous in the "C^∞–sense (*i.e.* such that if f_n is a sequence for which $||f_n||_{C^j(-\infty,0])} \xrightarrow[n\to\infty]{} 0$ *for every* j then also $||f_n^a||_{C^j(-\infty,a])} \xrightarrow[n\to\infty]{} 0$ *for every* j).

[1.6.6] (*extensions in more dimensions*) Show that the preceding construction can be generalized to prove the extendibility of a C^∞–function, rapidly decreasing in the semi space $z \le 0$, to a C^∞–function defined on the whole space and vanishing for $z > \varepsilon$ with a prefixed ε. The extension can be made so that it is continuous in the sense C^∞ (*c.f.r.* [1.6.5]). (*Idea:* It suffices to consider the case in which $f(\underline{x},z)$ is periodic in $\underline{x}$ with arbitrary period L. In this case f can be written as: $f(\underline{x},z) = \sum_{\underline{\omega}} e^{i\underline{\omega}\cdot\underline{x}} f_{\underline{\omega}}(z)$ where

$\underline{\omega} = 2\pi L^{-1}(n_1, n_2)$ with $\underline{n}$ a bidimensional integer components vector; and we can use the theory of the one–dimensional extensions to extend *each* Fourier coefficient as

$$f_{\underline{\omega}}(z) = \sum_{k=0}^{\infty} \frac{c_k(\underline{\omega})}{k!} z^k\, e^{-zc_k(\underline{\omega})^2 e^{|\underline{\omega}|}}, \qquad \text{per} \quad z \ge 0$$

where one should note the insertion of $e^{|\underline{\omega}|}$ in the exponent, for the purpose of controlling the $\underline{x}$–derivatives, which "only" introduce powers of components of $\underline{\omega}$ in the series.)

[1.6.7]: Show that there is a family of points $\underline{x}_j$ and of C^∞–functions $\chi_j^r(\underline{x})$ vanishing outside a sphere of radius r centered at $\underline{x}_j$ such that: $\sum_j \chi_j^r(\underline{x}) \equiv 1$ and, furthermore, such functions can be chosen to be identical up to translations, *i.e.* to be such that $\chi_j^r(\underline{x}) = \chi^r(\underline{x} - \underline{x}_j)$ for a suitable χ^r. (*Idea:* Consider a pavement of R^3 by semi open cubes with side $r/4$ and centers $\underline{x}_j$; let $f_j(\underline{x})$ be the characteristic function of the cube centered at $\underline{x}_j$. Clearly $\sum_j f_j(\underline{x}) \equiv 1$; let $\gamma(\underline{x}) \ge 0$ be a C^∞–function vanishing outside a small sphere of radius $r/4$ and with integral $\int \gamma(\underline{x})d\underline{x} = 1$. Let: $\chi_j^r(\underline{x}) = \int \gamma(\underline{x}-\underline{y}) f_j(\underline{y})d\underline{y}$ and check that such functions have all the properties that we ask for.)

[1.6.8]: (*continuity of extensions of smooth functions*) Use the preceding problems to show that a C^∞–function f in a bounded domain Ω with a C^∞–boundary can be extended to a C^∞–function in the whole space, vanishing after a distance $\varepsilon > 0$ from Ω, with arbitrarily prefixed ε, and so that the extension is continuous in the metric of C^p (for all $p \ge 0$). (*Idea:* Using the partition of the identity of the preceding problem reduce the problem to that of the extension of a function on Ω vanishing outside a small neighborhood of a boundary point; then reduce the problem of extending the latter function to that of extending a C^∞–function defined on a semi space, discussed above.)

[1.6.9] Show that there are sequences $\{a_k\}, \{b_k\}$, $k = 0, 1, \ldots$, such that for each $n = 0, 1, \ldots$:

$$\sum_{k=0}^{\infty} |a_k||b_k|^n < \infty, \qquad \sum_{k=0}^{\infty} a_k b_k^n = 1, \qquad -b_k \xrightarrow[k\to\infty]{} \infty$$

(taken from [Se64]) (*Idea:* fix $b_k = -2^k$ and determine the numbers X_k^N by imposing

$$\sum_{k=0}^{N-1} X_k^N b_k^n = 1 \qquad n = 0, 1, \ldots, N-1$$

which can be solved via the Cramer rule and the well known properties of the Vandermonde matrices (and determinants) M with $M_{nk} = b_k^n$. In this way one finds $X_k^N = A_k B_{k,N}$ where

$$A_k = \prod_{j=0}^{k-1} \frac{2^j + 1}{2^j - 2^k}, \qquad B_{k,N} = \prod_{j=k+1}^{N-1} \frac{2^j + 1}{2^j - 2^k}$$

where $A_0 = 1$, $B_{N-1,N} = 1$; and then check that: $|A_k| \le 2^{-(k^2-2k)/2}$, $B_{k,N} \le e^4$. Furthermore $B_{k,N}$ increases and thus there is the limit $\lim_{N\to\infty} B_{k,N} = B_k$, $1 \le B_k \le e^4$. And setting $a_k = B_k A_k$ we see that the sequences a, b so built cheerfully enjoy the wanted properties, [Se64].)

[1.6.10]: (*linear and continuous extension of a smooth function*) Let f be a C^∞–function defined in a semi space $x \in R^d \times [-\infty, 0]$; show that if a_k, b_k denote the sequences of problem [1.6.9], setting

$$f^0(x,t) = \sum_{k=0}^{\infty} a_k \Phi(b_k t) f(x, b_k t) \qquad t > 0 \quad \text{e} \quad f^0(x,t) = f(x,t) \qquad t \le 0$$

with $\Phi \in C^\infty(R)$, $\Phi(t) \equiv 1$ for $|t| \le 1$, and $\Phi(t) \equiv 0$ for $|t| \ge 2$, then f^0 extends f to the whole R^{d+1} *linearly and continuously* in the C^p–metric (for all $p \ge 0$).

[1.6.11] Consider a regular domain $\Omega \subset R^3$ and assume that on a sphere S containing Ω it is possible to define a C^∞ vector field $\underline{w} \ne \underline{0}$ such that: (1) $\underline{w}$ is orthogonal to ∂S, (2) every flux line of $\underline{w}$ crosses Ω in a connected piece (possibly reduced to a point or to the empty set), (3) there is a regular surface Σ such that the flux lines of $\underline{w}$ are transversal to $\Sigma \cap \Omega$ and every point of Ω is on a line that intersects $\Sigma \cap \Omega$. We shall, briefly, say that Ω *has the section property.* Show that if $\underline{u} \in C^\infty(\Omega)$ and $\underline{\partial} \cdot \underline{u} = 0$ then there is $\underline{A} \in C^\infty(\Omega)$ such that $\underline{u} = \mathrm{rot}\,\underline{A}$. (*Idea:* The section property is what suffices to repeat the argument in *(B)*.)

[1.6.12] Show that if $\Omega \subset R^3$ is not *simply connected by surfaces* in he sense that not every regular closed surface contained in Ω can be continuously deformed shrinking it to a point, still staying inside Ω, then *it is not in general true that a solenoidal field* $\underline{u} \in C^\infty(\Omega)$ *is the rotation of a field* $\underline{A} \in C^\infty(\Omega)$. (*Idea:* Let Ω = a convex region between two concentric spheres of radii $0 < R_1 < R_2$ and consider the field $\underline{u} = \frac{\underline{r}}{r^3}$ generated by the Coulomb potential. If it could be $\underline{u} = \mathrm{rot}\,\underline{A}$ for some $\underline{A} \in C^\infty(\Omega)$ one could extend $\underline{A}$ to the whole space in class C^∞ and, in particular, it would be extended to the interior of the smaller sphere. Then it would ensue (by Gauss' theorem) $0 \equiv \int_{|\underline{r}|<R_1} \underline{\partial} \cdot \underline{u} \equiv \int_{|\underline{r}|=R_1} \underline{u} \cdot \underline{n} = 4\pi$! Note the generality of the above example.)

[1.6.13]: (*local extension of an incompressible field*) Let $\Omega \subset R^3$ be regular and connected. Consider the layer of width ε around $\partial\Omega$. It is possible to set up, in a regular set $\Omega_\varepsilon \supset \Omega$ with boundary at distance $> \varepsilon/2$ from that of Ω, an atlas of orthogonal coordinates (x_1, x_2, x_3) such that $\partial\Omega$ has equation $x_3 = 0$ in each chart and such that the direction 3 coincides everywhere on $\partial\Omega$, with the external normal to $\partial\Omega$ and coincides in every point of $\partial\Omega_\varepsilon$, with the external normal to $\partial\Omega_\varepsilon$. Using this coordinate system show that a $C^\infty(\Omega)$ solenoidal field can be extended to a solenoidal field in $C^\infty(\Omega_\varepsilon)$ parallel, in a neighborhood of $\partial\Omega_\varepsilon$, to the direction 3 of the coordinate system in a (arbitrary) chart. The extension can be obtained by extending u_1, u_2 arbitrarily to functions of $C^\infty(\Omega_\varepsilon)$ vanishing near $\partial\Omega_\varepsilon$ and setting

$$u_3(x_1, x_2, x_3) = u_3(x_1, x_2, 0) + \frac{1}{h_1 h_2} \int_0^{x_3} d\xi \Big(-\partial_1(h_2 h_3 u_1) - \partial_2(h_1 h_3 u_2) \Big)$$

if $h_1 dx_1^2 + h_2 dx_2^2 + h_3 dx_3^2$ is the metric of the considered system of coordinates and if the integrand functions are evaluated at the point (x_1, x_2, ξ), while those outside are evaluated at (x_1, x_2, x_3). (*Idea:* Note that the divergence of a vector field with components u_1, u_2, u_3 in an orthogonal system of coordinates is expressible as

$$\frac{1}{h_1 h_2 h_3} \Big(\partial_1(h_2 h_3 u_1) + \partial_2(h_1 h_3 u_2) + \partial_3(h_1 h_2 u_3) \Big)$$

in terms of the orthogonal metric.)

[1.6.14]: A regular simply connected domain $\Omega \subset R^3$ contained in a sphere S has the "property of the normal" if it is possible to define a $C^\infty(\overline{S/\Omega})$–field of unit vectors that extends the external normal vectors to $\partial\Omega$ to the entire domain $\overline{S/\Omega}$ ending, on the boundary ∂S, in the external normal vectors to ∂S. Note that a necessary and sufficient condition so that Ω has this property is that each point can be connected to ∞ by a continuous curve that has no other points in Ω. And note that a necessary and sufficient property for the latter property is that Ω be simply connected by surfaces, see [1.6.12].

[1.6.15] If the regular connected domain $\Omega \subset R^3$ has the property of the normal, see [1.6.14], (or is simply connected by surfaces) then every $C^\infty(\Omega)$ and solenoidal field $\underline{u}$ is the rotation of a field $\underline{A} \in C^\infty(\Omega)$. (*Idea:* The problem [1.6.13] allows us to reduce to the case in which $\underline{u}$ is orthogonal to $\partial\Omega$ and it is parallel to the normal to $\partial\Omega$ in the vicinity of $\partial\Omega$; on the other hand [1.6.14] allows us to extend $\underline{u}$ from Ω to S: it suffices to define $\underline{u}$ as parallel to the vector field connecting the external normals to $\partial\Omega$ to those of ∂S and fix the size of $\underline{u}$ such as to conserve the volume (*c.f.r.* the analogous extension discussed in *(B)* above). Hence the problem is reduced to the "standard" case in which $\underline{u}$ is in $C^\infty(S)$ and ends on ∂S normally to ∂S. The latter is a particular case of the one treated in *(B)*.)

[1.6.16]: (*representing square integrable functions as weak derivatives*) Let $\Omega \subset R^3$ be a simply connected regular region in the interior of a cube Q_1. Let $Q = \cup_{j=1}^8 Q_j$ be the cube with side L twice the previous one and union of 8 copies of Q_1; suppose that the center of Q is the origin. Let Ω_j, $j = 1, \ldots, 8$, be the "copies" of Ω in the eight cubes ($\Omega_1 \equiv \Omega$). We associate with each Q_j a sign $\sigma_j = \pm 1$ so that the sign of Q_1 is $+$ and the sign of every cube is opposite to the one of the adjacent cubes (*i.e.* with a common face). If $f \in L_2(\Omega)$ consider the extension f^e of f to $L_2(Q)$ defined by setting $f = 0$ in Q_1/Ω and then $f^e(x') = \sigma_j\, f(x)$ if $x' \in Q_j$ is a copy of $x \in Q_1$ (*i.e.* it has the same position relative to the sides of Q_j as x relative to the sides of Q_1). Consider Q as a torus (*i.e.* we identify its opposite sides) so that it makes sense to define the Fourier transform of a function in $L_2(Q)$. Show that the Fourier transform of f^e is a function $\hat f^e(\underline{k})$ (with $\underline{k} = \frac{2\pi}{L}\underline{n}$ for $\underline{n}$ an integer components vector) such that $\hat f^e(\underline{k}) = 0$ if $\underline{k} = (k_1, k_2, k_3)$ has a zero component (*i.e.* if $k_1k_2k_3 = 0$). Deduce that given $p \geq 0$ and $j = 1, 2, 3$ there is a $F^{(p,j)} \in L_2(\Omega)$ such that

$$f = \partial_{x_j}^p\, F^{(p,j)}, \qquad ||F^{(p,j)}||_{L_2(\Omega)} \leq 8\Big(\frac{L}{2\pi}\Big)^p ||f||_{L_2(\Omega)} \tag{!}$$

where the derivatives are intended in the sense of distributions in $L_2(\Omega)$. If $f \in C_0^\infty(\Omega)$ then $F^{(p,j)}$ can be chosen as the restriction of a $C^\infty(Q)$–function. (*Idea:* Define $\bar F^{(p,j)} \in L_2(Q)$ via the Fourier transform of f^e: $\hat{\bar F}^{(p,j)}(\underline{k}) = (ik_j)^{-p}\, \hat f^e(\underline{k})$ if $k_1k_2k_3 \neq 0$ and $\hat{\bar F}^{(p,j)}(\underline{k}) = 0$ otherwise. Then restrict $\bar F^{(p,j)}(\underline{x})$ to Ω and check the wished properties.)

[1.6.17]: (*Approximating an irrotational field with a gradient*)
Let $\gamma_n(\underline{x}) = e^{(\underline{x}^2 - n^{-2})^{-1}} c_n$ for $|\underline{x}| < 1/n$ and let $\gamma_n = 0$ otherwise. The constant c_n is such that $\int \gamma_n \equiv 1$, ($c_n \propto n^3$). Let $\underline{f} \in L_2(\Omega)$ and $\operatorname{rot} \underline{f} = \underline{0}$ in the sense that for each $\underline{A} \in C_0^\infty(\Omega)$ it is $\int_\Omega \underline{f} \cdot \operatorname{rot} \underline{A} = \underline{0}$. Show, (with the notations of the problem [1.6.16] and if $*$ denotes the convolution product in the domain Q considered as a torus):

(i) $\int_Q \underline{f}^e \cdot \operatorname{rot} \underline{A} = \underline{0}$ for $\underline{A} \in C_0^\infty(\cup_{k=1}^8 \Omega_k)$
(ii) if n is large enough: $\int_Q \gamma_n * \underline{f}^e \cdot \operatorname{rot} \underline{A}^e \equiv 8 \int_\Omega \gamma_n * \underline{f}^e \cdot \operatorname{rot} \underline{A}^e \equiv \underline{0}$
(large enough means that $\frac{1}{n}$ is smaller than the distance between $\partial\Omega$ and the support of $\underline{A}$);
(iii) $\operatorname{rot}(\gamma_n * \underline{f}^e) = \underline{0}$, in $\Omega_k^{(n)} = \big\{\underline{x} | \underline{x} \in \Omega_k,\, d(\underline{x}, \partial\Omega_k) > \frac{1}{n}\big\}$
(iv) $\lim_{n\to\infty} \gamma_n * \underline{f} = \underline{f}$, in $L_2(\Omega)$.

[1.6.18]: Within the context of problems [1.6.16],[1.6.17] suppose Ω convex and let $\varphi_n(\underline{x})$ be defined for $\underline{x} \in \Omega_k^n$ in the interior of $\cup_{k=1}^8 \Omega_k^{(n)}$ by

$$\varphi_n(\underline{x}) = \int_{\Omega_k^{(n)}} \frac{d\underline{y}}{|\Omega_k^{(n)}|} \int_0^1 ds\, (\underline{x} - \underline{y}) \cdot (\gamma_n * \underline{f}^e)(\underline{y} + s(\underline{x} - \underline{y}))$$

Check that for $\underline{x} \in \cup_{k=1}^8 \Omega_k^{(n)}$

$$\partial_j \varphi_n(\underline{x}) = \gamma_n * f_j^e(\underline{x}), \qquad \int_{\Omega_k^{(n)}} \frac{d\underline{y}}{|\Omega_k^{(n)}|} |\varphi_n(\underline{x})| \leq C\, ||f||_{L_2(\Omega)}$$

(*Idea:* The first relation is simply due to $\mathrm{rot}\,\gamma_n * f^e = \underline{0}$ in $\cup_{k=1}^8 \Omega_k^{(n)}$, *c.f.r.* problem [1.6.17]. The integral in the second can be bounded by

$$L\sqrt{3}\int_0^1 ds \int_{\Omega_k^{(n)}} \frac{d\underline{x}d\underline{y}}{|\Omega_k^{(n)}|} |(\gamma_n * \underline{f}^e)(\underline{y} + s(\underline{x} - \underline{y}))| \le$$
$$\le \frac{L\sqrt{3}}{|\Omega_k^{(n)}|}|\int_0^1 \frac{ds}{(1-s)^3} \int_{|\underline{x}-\underline{z}|<(1-s)L\sqrt{3}} d\underline{x}d\underline{z}|(\gamma_n * \underline{f}^e)(\underline{z})| \le$$
$$\le C_1 L \int_{\Omega_k^{(n)}} d\underline{z}|(\gamma_n * \underline{f}^e)(\underline{z})| \le$$
$$\le C_2 L^{5/2} ||\gamma_n * \underline{f}^e||_{L_2(Q)} \le C L^{5/2} ||\underline{f}||_{L_2(\Omega)}$$

(after changing variables $\underline{y} \to \underline{z} = \underline{y} + (\underline{x} - \underline{y})s$) for suitable constants C_1, C_2, C.)

[1.6.19] (*Representing an irrotational field defined in a convex domain as a gradient*) Let $F_j^{(n)} \in L_2(Q)$ and $\varphi_n \in \cup_{k=1}^8 \Omega_k^{(n)}$ be chosen, in the context of problems [1.3.17], [1.3.18], so that

$$\partial_j F_j^{(n)} = \gamma_n * f_j^e \text{ in } L_2(Q), \qquad ||\varphi_n||_{L_2(\Omega_k^{(n)})} \le C||f||_{L_2(Q)}$$

Check first that the bounds obtained in problem [1.6.18] imply that in $\Omega_k^{(n)}$ it is $F_j^{(n)}(\underline{x}) = \partial_j \varphi_n(\underline{x}) + c_k^n$ with the constant c_k^n bounded by $C||f||_{L_2(\Omega)}$.
Then extend φ_n to Q by setting it equal to 0 outside $\cup_{k=1}^8 \Omega_k^{(n)}$ and let $\varphi = \lim \varphi_{n_q}$ be a weak limit in $L_2(\Omega)$ of φ_n on a subsequence $n_q \to \infty$. Show that φ admits generalized first derivatives and $\underline{\partial}\varphi \equiv \underline{f}$. (*Idea:* One remarks that if $b \in C_0^\infty(\Omega)$ it is $\int_\Omega \partial_j b \varphi = \lim_{n\to\infty} \int_\Omega \partial_j b\, \varphi_n = -\int_\Omega b \partial_j \varphi_n = -\int_\Omega b f_j$ so that $|\int \varphi \partial_j b| \le ||f||_{L_2(\Omega)}\, ||b||_{L_2(\Omega)}$.)

[1.6.20]: (*a confined solenoidal field with non closed flux lines*) Consider the vector field defined in the cylinder $0 \le x \le L$, $y^2 + z^2 \le R^2$ by

$$u_1 = v, \qquad u_2 = -\omega z, \qquad u_3 = \omega y$$

fixing L such that $\omega v^{-1} L/(2\pi)$ is irrational. Extend the field $\underline{u}$ to $y^2 + z^2 \le 4R^2$ making v, ω C^∞–functions of $y^2 + z^2 \equiv r^2$ vanishing in the vicinity of the external boundary of the cylinder (where $y^2 + z^2 = 4R^2$). Show that the field has zero divergence in the cylinder. Continue the field in such a way that its flux lines exit the right face of the cylinder (where $x = L$ and the other two coordinates are y, z, say) and reenter from the left side (where $x = 0$) at the point corresponding to the same coordinates y, z: this is done by defining $\underline{u}$ outside the cylinder so that the flux thus generated is incompressible (using the continuation technique discussed in *(B)*). Show that the flux lines of $\underline{u}$ exiting in L at the points of coordinates y, z such that $y^2 + z^2 < R^2$ "at every turn around" have new coordinates y', z' still inside the disk of radius R but they *never* close on themselves. Therefore *the flux lines of a confined solenoidal field are not necessarily closed.*

Bibliography: [So63], Sect. 5,6; [CF88a], Sec. 1; [Se64].

§1.7 Vorticity conservation in Euler equation. Clebsch potentials and Hamiltonian form of Euler equations. Bidimensional fluids.

(A) Thomson theorem:

A very important property for Euler fluids is the vorticity conservation law. This law is basic for the understanding (very imperfect to date) of the evolution of structures like "*smoke rings*".

Consider a not necessarily incompressible isoentropic Euler fluid; hence with a relation between density ρ and pressure p given by $\rho = R(p)$, see Sec. 1.4.3, so that the pressure potential $\Phi(p)$ is defined by $\underline{\partial}\Phi = \rho^{-1}\underline{\partial}p$. Suppose also that the external force $\underline{g}$ be conservative: $\underline{g} = -\underline{\partial}G$.

Let γ be a contour that we follow in time, $t \to \gamma(t)$, $\gamma(0) = \gamma$, where $\gamma(t)$ is the contour into which γ evolves if its points follow the current that passes through them at the initial time.

In formulae the contour $\gamma(t)$ has parametric equations $s \to \underline{\ell}(s,t)$ that can be expressed in terms of the equations $s \to \underline{\ell}(s)$ of $\gamma = \gamma(0)$ by saying that $\underline{\ell}(s,t)$ is the value of the solution at time t of

$$\begin{cases} \dot{\xi} = \underline{u}(\xi, t) \\ \xi(0) = \underline{\ell}(s) \end{cases} \tag{1.7.1}$$

Then the *Thomson theorem* holds

$$\frac{d}{dt}\int_{\gamma(t)} \underline{u}(\underline{\ell}, t) \cdot d\underline{\ell} = 0 \tag{1.7.2}$$

i.e. "vorticity" of a contour is conserved along flow lines. One says also that "*vorticity is advected by the fluid flow*".

Checking (1.7.2) is simple; the equation for $\gamma(t)$ is, up to $O(t^2)$,

$$s \to \underline{\ell}(s) + \underline{u}(\underline{\ell}(s), 0)\, t = \underline{\ell}' \tag{1.7.3}$$

so that, up to $O(t^2)$

$$d\underline{\ell}' = d\underline{\ell} + t\underset{\sim}{\underline{\partial}}\,\underline{u}(\underline{\ell}, 0)\underset{\sim}{d\underline{\ell}} + O(t^2) \tag{1.7.4}$$

and, up to $O(t^2)$

$$\begin{aligned} &\int_{\gamma(t)} \underline{u}(\underline{\ell}', t) \cdot d\underline{\ell}' = \int_\gamma (\underline{u}(\underline{\ell}, 0) + t\,\underset{\sim}{\underline{u}}\,(\underline{\ell}, 0) \cdot \underset{\sim}{\underline{\partial}}\,\underline{u}(\underline{\ell}, 0) + t\,\partial_t\underline{u}(\underline{\ell}, 0))\cdot \\ &\cdot (d\underline{\ell} + t\,\underset{\sim}{\underline{\partial}}\underline{u}(\underline{\ell}, 0))\,\cdot \underset{\sim}{d\underline{\ell}} = \int_\gamma \underline{u}\cdot d\underline{\ell} + t\int_\gamma (\underset{\sim}{\underline{u}}\cdot \underset{\sim}{\underline{\partial}}\,\underline{u}\,d\underline{\ell} + \partial_t\underline{u}d\underline{\ell} + \underline{u}\cdot\underset{\sim}{\underline{\partial}}\,\underline{u}\,\underset{\sim}{d\underline{\ell}}) = \\ &= \int_\gamma \underline{u}\cdot d\underline{\ell} + t\int_\gamma (-\underset{\sim}{\underline{\partial}}(\Phi + G) + \underset{\sim}{\underline{\partial}}\frac{\underline{u}^2}{2})\,\underset{\sim}{d\underline{\ell}} = \int_\gamma \underline{u}\cdot d\underline{\ell} \end{aligned} \tag{1.7.5}$$

which gives (1.7.2). If $\underline{g}$ is not conservative the right hand side of (1.7.2) becomes $\int_\gamma \underline{g}\cdot d\underline{\ell}$.

(B) Irrotational isoentropic flows:

An immediate consequence is that, if for $t=0$ rot $\underline{u}=0$ and if the external force is conservative, then rot $\underline{u}$ must remain zero at all successive times: indeed if γ is an infinitesimal contour which is the boundary of a surface $d\sigma$ with normal $\underline{n}$ it will be

$$\int_\gamma \underline{u}\cdot d\,\underline{\ell} \equiv \text{ rot } \underline{u}\cdot\underline{n}\,d\sigma \tag{1.7.6}$$

and then the arbitrariness of $d\sigma$ and $\underline{n}$ and the invariance of vorticity just say that rot $\underline{u}\equiv 0$ at all times $t>0$.

For this reason, in the case of flows which are isoentropic, or more generally such that density is a function of the pressure, "irrotational" or "potential" flows can exist and they are in fact very interesting for their simplicity that allows us to describe them (in simply connected domains) in terms of a scalar function $\varphi(\xi,t)$, called "*velocity potential*", as $\underline{u}=\underline{\partial}\varphi$.

A simple property of such flows is that, possibly changing the potential function φ by a suitable function of time alone, *it must be*

$$\frac{\partial\varphi}{\partial t}+\frac{\underline{u}^2}{2}+\Phi(p)+G=0 \tag{1.7.7}$$

where G is the potential energy of the field of the volume forces (assumed conservative) and $\Phi(p)$ is the pressure potential. For instance in the case of an incompressible fluid in a gravity field it will be $\Phi(p)+G=\frac{1}{\rho}p+gz$.

In fact from the momentum equation (*i.e.* (1.1.11)) we see that

$$\underline{\partial}\partial_t\varphi+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{u}+\underline{\partial}(\Phi+G)=\underline{0} \tag{1.7.8}$$

But $\underset{\sim}{u}\cdot\underset{\sim}{\partial}\underline{u}\equiv\underline{\partial}u^2/2-\underline{u}\wedge(\text{rot }\underline{u})$ is in this case $\underline{\partial}\underline{u}^2/2$ hence (1.7.8) expresses that the gradient of (1.7.7) vanishes, *i.e.* the left hand side can only be a function of time which, by changing φ by an additive function of time alone, can be set equal to zero.

Note the relation between (1.7.8) and Bernoulli's theorem, *c.f.r.* (1.4.30): if the motion is "*static* (*i.e.* $\underline{u}$ is time independent) then φ can be chosen time independent and (1.7.8) gives

$$\frac{\underline{u}^2}{2}+\Phi(p)+G=\text{ constant} \tag{1.7.9}$$

Hence the quantity on the left hand side of (1.7.9) which in static flows is *constant along the current lines* is, in the static irrotational cases, *constant in the whole fluid.*

It is, however, important to keep in mind that *regular* irrotational flows (*i.e.* of class C^∞) are not possible in ideal incompressible fluids in finite, even if simply connected, containers. Simply because in such cases the velocity field would be a gradient field $\underline{u} = \underline{\partial}\varphi$, with φ a suitable scalar *harmonic* function. At the boundary of the container the velocity will have to be tangent and, therefore, the normal derivative of φ will vanish. From the uniqueness of the solutions of the Neumann problem we see that the only harmonic function φ whose gradient enjoys this property is the constant function corresponding to the velocity field $\underline{u} = \underline{0}$.

Hence irrotational isoentropic (or incompressible) motions can exist, possibly, in unbounded domains or they will be singular in some point. They can be also possible in fluids with a free surface, *i.e.* which occupy a region depending on time where the boundary condition is *not* a Neumann boundary condition, *e.g.* in the case of water waves in an open container, see (F).

(C) Eulerian vorticity equation in general and in bidimensional flows:

What said until now is, strictly speaking, valid only if $d = 3$: if $d = 2$ the rotation $\mathrm{rot}\,\underline{u}$ has to be redefined.

One can, however, always imagine that a bidimensional fluid is a tridimensional fluid observed on the plane $z = 0$. It will suffice to define the velocity $\underline{u}(x, y, z) \equiv (\underline{u}(x, y), 0)$ for each z and think of the thermodynamic fields simply as defined everywhere and independent of the z coordinate. We can call this fluid the "3–dimensional extension" of the bidimensional fluid or a 3–dimensional "stratified flow", see also §1.4.

One checks easily that the three dimensional extensions of solutions of the bidimensional equations of Euler and of NS satisfy the corresponding three dimensional equations. Hence the above considerations about the vorticity hold for them.

The rotation of the bidimensional field $\underline{u} = (u_1, u_2)$ regarded as extended to three dimensions is the rotation of $(u_1, u_2, 0)$ hence it is the vector

$$(0, 0, \partial_1 u_2 - \partial_2 u_1) \tag{1.7.10}$$

Therefore we often say that *in two dimensional fluids the rotation of the velocity field $\underline{u} = (u_1, u_2)$ is a scalar ζ*:

$$\zeta = \partial_1 u_2 - \partial_2 u_1 \tag{1.7.11}$$

Note that a bidimensional field $\underline{u}$ with zero divergence, that we represent in $d = 3$ as $\underline{u} = \mathrm{rot}\,\underline{A}$, in $d = 2$ is represented instead as

$$\underline{u} = \underline{\partial}^\perp A, \quad A \text{ scalar} \;, \quad \underline{\partial}^\perp = (\partial_2, -\partial_1) \tag{1.7.12}$$

This can be seen, for instance, by going through the arguments used in the preceding section §1.6 on the representations of velocity fields.

A more analytic form of the just discussed properties can be obtained by remarking that vorticity evolves according to an equation obtained by applying the rotation operator to the momentum equation: $\partial_t \underline{u} + \underset{\sim}{u} \cdot \underset{\sim}{\partial} \underline{u} = -\underline{\partial}(\Phi + G)$. One finds, by using $\frac{1}{2}\underset{\sim}{\partial}\underset{\sim}{u}^2 = \underline{u} \wedge \text{ rot } \underline{u} + \underset{\sim}{u} \cdot \underset{\sim}{\partial} \underline{u}$ and by setting $\underline{\omega} = \text{rot}\, \underline{u}$, the *Eulerian vorticity equation* for isoentropic fluids:

$$\begin{aligned} &\partial_t \text{ rot } \underline{u} - \text{ rot } (\underline{u} \wedge \text{ rot } \underline{u}) = \underline{0}, \qquad \text{or :} \\ &\partial_t \underline{\omega} + \underset{\sim}{u} \cdot \underset{\sim}{\partial}\, \underline{\omega} = \underset{\sim}{\omega} \cdot \underset{\sim}{\partial}\, \underline{u} \end{aligned} \tag{1.7.13}$$

where we used the identity $\text{rot}\,(\underline{a} \wedge \underline{b}) = \underset{\sim}{b} \cdot \underset{\sim}{\partial}\, \underline{a} - \underset{\sim}{a} \cdot \underset{\sim}{\partial}\, \underline{b}$ if $\underline{\partial} \cdot \underline{a} = \underline{\partial} \cdot \underline{b} = 0$. The (1.7.13) shows again how $\text{rot}\, \underline{u} = \underline{0}$ is possible, if true at time $t = 0$, *i.e.* shows the possibility of irrotational flows.

In the $d = 2$ incompressible case (1.7.13) simplifies considerably, the vorticity being now a scalar, and it becomes, *c.f.r.* (1.7.10), (1.7.11)

$$\partial_t \zeta + \underline{u} \cdot \underline{\partial}\, \zeta = 0 \tag{1.7.14}$$

which says that the *vorticity is conserved along the current lines.*
This gives rise, in the case $d = 2$, and for an isoentropic Euler fluid in a time independent container Ω to the existence of a first integral

$$E(\zeta) = \int_\Omega \zeta(\xi)^2 d\xi \tag{1.7.15}$$

that is called *enstrophy.* And any function f generates via

$$F(\zeta) = \int_\Omega f(\zeta(\xi))\, d\xi \tag{1.7.16}$$

a first integral for the isoentropic Euler equations.

Rather than isoentropic equations one can, of course, consider equations in which the relation between p and ρ is fixed: the only property really used has been that $\rho^{-1}\underline{\partial}\, p = \underline{\partial}\Phi$ for some Φ.
In particular we can apply the above considerations to incompressible fluids.

(D): Hamiltonian form of the incompressible Euler equations and Clebsch potentials. Normal velocity fields.

To exhibit the Hamiltonian structure of the Euler equations one can introduce the "*Clebsch potentials*", denoted (p, q), *c.f.r.* [La32], §167, p. 248, as $C^\infty(R^3)$–functions. The potentials can be used to construct the incompressible velocity field

$$\underline{u} \overset{def}{=} q\, \underline{\partial}\, p - \underline{\partial}\, \gamma, \qquad \gamma \overset{def}{=} \Delta^{-1} \underset{\sim}{\partial} \cdot (q\, \underset{\sim}{\partial}\, p) \tag{1.7.17}$$

Then we define

$$H(p,q)=\frac{1}{2}\int \underline{u}^2\,d\underline{x}\equiv\frac{1}{2}\int\big(q\underline{\partial}\,p-\underline{\partial}\Delta^{-1}\underset{\sim}{\partial}(q\underset{\sim}{\partial}\,p)\big)^2\,d\underline{x} \tag{1.7.18}$$

and, denoting by $\delta F/\partial f(x)$ the *functional derivative* of a generic functional F of f, compute the Hamiltonian equations of motion

$$\dot p=-\frac{\delta H}{\delta q(x)}=-\underline{u}\cdot\underline{\partial}p,\qquad \dot q=\frac{\delta H}{\delta p(x)}=-\underline{u}\cdot\underline{\partial}q \tag{1.7.19}$$

which are immediately checked if, in the evaluation of the functional derivatives, one uses that $\underline{u}$ has zero divergence and hence γ "does not contribute at all" to the functional equation.[1] We now compute $\dot{\underline{u}}$:

$$\begin{aligned}\dot{\underline{u}}=&\dot q\,\underline{\partial}p+q\,\underline{\partial}\,\dot p-\underline{\partial}\,\dot\gamma=\\ =&-\underset{\sim}{u}\cdot(\underset{\sim}{\partial}\,q)(\underline{\partial}p)-q\underline{\partial}(\underset{\sim}{u}\cdot\underset{\sim}{\partial}p)-\underline{\partial}\dot\gamma= \qquad (1.7.20)\\ =&-\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,(q\,\underline{\partial}\,p)+\underset{\sim}{u}\cdot(q\underset{\sim}{\partial}\,\underline{\partial}\,p)-q(\underline{\partial}\,\underset{\sim}{u})\cdot\underset{\sim}{\partial}\,p-q\underset{\sim}{u}\cdot\underline{\partial}\,\underset{\sim}{\partial}\,p-\underline{\partial}\dot\gamma\\ =&-\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\big(q\,\underline{\partial}p-\underline{\partial}\,\gamma\big)-\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{\partial}\,\gamma-\frac{1}{2}\underline{\partial}\underset{\sim}{u}^2-(\underline{\partial}\,\underset{\sim}{u})\cdot(\underset{\sim}{\partial}\,\gamma)-\partial\,\dot\gamma=\\ =&-\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}-\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{\partial}\,\gamma-\frac{1}{2}\underline{\partial}\underset{\sim}{u}^2-\underline{\partial}(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\gamma)+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{\partial}\gamma-\underline{\partial}\,\dot\gamma\\ =&-\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}-\underline{\partial}\big(\frac{1}{2}\underset{\sim}{u}^2+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\gamma+\dot\gamma\big)\end{aligned}$$

Hence we see that $\underline{u}$, defined in]equ(7.17), satisfies the incompressible Euler equation and the pressure π is written

$$\begin{aligned}\frac{1}{\rho}\pi=&\frac{1}{2}\underset{\sim}{u}^2+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\gamma+\dot\gamma,\qquad\text{with}\quad\gamma=\Delta^{-1}\underset{\sim}{\partial}\cdot(q\underset{\sim}{\partial}p)\quad\text{e}\\ \dot\gamma=&\Delta^{-1}\underset{\sim}{\partial}\big(-(\underline{u}\cdot\underline{\partial}q)\underset{\sim}{\partial}p-q\underset{\sim}{\partial}\big(\underline{u}\cdot\underline{\partial}p)\end{aligned} \tag{1.7.21}$$

Conclusion: *the incompressible Euler equations can be represented in a Hamiltonian form in which the Clebsch potentials are canonically conjugated coordinates.*

We now want to see which is the generality of the representation of an incompressible velocity field as in (1.7.17). The key remark is that, if $\underline{u}$ is given by (1.7.17) then $\underline{\omega}=\mathrm{rot}\,\underline{u}$ is

$$\underline{\omega}\equiv\mathrm{rot}\,\underline{u}=\underline{\partial}q\wedge\underline{\partial}p \tag{1.7.22}$$

[1] Indeed $\dot p(\underline{x})=-\int\underline{u}(\underline{y})\cdot(\delta(\underline{x}-\underline{y})\underline{\partial}_{\underline{y}}p(\underline{y})-\underline{\partial}_{\underline{y}}\frac{\delta\gamma}{\delta q(\underline{x})})\,d\underline{y}\equiv-\underline{u}(\underline{x})\cdot\underline{\partial}p(\underline{x})$ and, likewise, $\dot q(\underline{x})=\int\underline{u}(\underline{y})\cdot q(\underline{y})\underline{\partial}_{\underline{y}}(\delta(\underline{x}-\underline{y})-\frac{\delta\gamma}{\delta p(\underline{x})})\equiv-\underline{u}(\underline{x})\cdot\underline{\partial}q(\underline{x})$.

which shows that the flux lines of $\underline{\omega}$ are orthogonal to both $\underline{\partial}p$ and $\underline{\partial}q$ so that they are the intersections between the level surfaces of the functions q and p. This suggests the following proposition

Proposition: *A smooth incompressible velocity field with vorticity $\underline{\omega}$ never vanishing in a simply connected region Ω admits a representation in terms of Clebsch potentials if and only if there are two regular functions α, β such that the flux lines of the vorticity, inside Ω, can be written in the form: $\alpha(x, y, z) = a$, $\beta(x, y, z) = b$ as the constants a e b vary in some domain D.*

Remarks:
(1) a "domain" is here a set in which every point is an accumulation point of interior points.
2) the proposition holds for any Ω: in particular for very small regions Ω in which $\underline{\omega} \neq \underline{0}$: in such regions the vorticity lines are essentially parallel and, therefore, they satisfy the assumptions of the proposition. It follows that *locally* it is always possible to represent a *smooth* velocity field via Clebsch potentials p, q, γ (and the representation is far from unique, as we shall see).

proof: the remark in (1.7.22) shows the necessity. Viceversa if (α, β) are two functions whose level surfaces intersect along flux lines of $\underline{\omega}$ it will be

$$\underline{\omega} = \lambda\, \underline{\partial}\alpha \wedge \underline{\partial}\beta \tag{1.7.23}$$

for a suitable λ, *function of* α, β. The (1.7.23), in fact, simply expresses that the lines with tangent $\underline{\omega}$ are parallel to the intersections of the level surfaces; furthermore λ is *constant* along the curve determined by fixed values of α and β because $\underline{\omega}$ has zero divergence and hence $\underline{\partial} \cdot \underline{\omega} = \underline{\partial}\lambda \cdot \underline{\partial}\alpha \wedge \underline{\partial}\beta = 0$ *i.e.* λ has zero derivative in the direction of $\underline{\omega}$.[2]
Hence if (q, p) are defined as functions of α, β so that

$$\frac{\partial(q,p)}{\partial(\alpha,\beta)} \equiv \partial_\alpha q \partial_\beta p - \partial_\beta q \partial_\alpha p = \lambda(\alpha, \beta) \tag{1.7.24}$$

we get two potentials (q, p) which, thought of as functions of (x, y, z) (via the dependence of (α, β) from (x, y, z)), are by definition such that $\mathrm{rot}\,(q\,\underline{\partial}p - \underline{u}) \equiv \mathrm{rot}\,(q\,\underline{\partial}\,p) - \underline{\omega} = \underline{0}$: *i.e.* there is a γ such that $\underline{u} = q\,\underline{\partial}\,p - \underline{\partial}\,\gamma$, because Ω is simply connected.
The (1.7.24) is always soluble, and in several ways; for instance one can fix $p = \beta$ and deduce

$$p = \beta, \qquad q = \int_{\alpha_0(\beta)}^{\alpha} \lambda(\alpha', \beta) d\alpha' \tag{1.7.25}$$

[2] This can also be seen by noting that the flux of $\underline{\omega}$ through the base of the cylinder of flux lines of $\underline{\omega}$ between surfaces over which α varies by $d\alpha$ and β by $d\beta$ of is $\lambda d\alpha d\beta$ hence λ must be constant in this cylinder because $\underline{\omega}$ has zero divergence.

where $\alpha_0(\beta)$ is an arbitrary function (but obviously such that there is at least one point $(a,b) \in D$ with $a = \alpha_0(b)$): and in this way (1.7.25) give us p, q as functions of α, β and, through them, as functions of (x, y, z).

It is not difficult to realize that there are velocity fields whose vorticity *cannot be* globally described in terms of potentials q, p. A simple example is provided by the field constructed in the problem [1.6.20], regarded as a vorticity field of a suitable velocity field. Since the flux lines fill *densely* a bidimensional surface (the cylindrical surface $0 \le x \le L$ and $y^2 + z^2 = r^2$, if $r < R$), it is clear that distinct flux lines on this surface cannot be intersections of two regular surfaces. Furthermore the example of [1.6.20] can be modified so that the flux lines fill even a three dimensional region (*c.f.r.* problems below).
We then set the following definition

Definition *(normal solenoidal field): A normal solenoidal field in* Ω *is an incompressible* C^∞*–field in* Ω *that admits a Clebsch representation, (1.7.17), in terms of two potentials* q, p*, with the function* γ *determined from* p, q *by solving a Neumann problem in* Ω*.*

Problem [1.6.20] shows that it is "reasonable" that the normal velocity fields are dense among all solenoidal fields, where dense has to be given a suitable meaning.
One should think, intuitively, the class of normal fields as that of the fields whose vorticity lines are either closed or begin and end at the boundary of Ω (*i.e.* at ∞ when Ω is the whole space), or that are always closed (as it is possible in the case of a parallelepiped with periodic boundary conditions).
The interest of the notion of normal field is not so much related to their density (*i.e.* to the approximability of any solenoidal field by one with flux lines closed or ending on the boundary) rather it is related on the invariance property of such fields. If the initial field is normal then it stays normal if it evolves under the Euler equations; furthermore *the evolution equations for such fields have a Hamiltonian form* provided, of course, the solution of the equation (1.7.19) is regular (of class C^∞ or, somewhat enlarging the notions above, at least of class C^2 so that all the differentiation operations considered on $\underline{\omega} = \text{rot}\,\underline{u}$ be meaningful) and $\underline{\omega}$ does not vanish: this is the content of the above informal analysis (due to Clebsch and Stuart, [La32]).

Obviously it will be interesting to try writing in Hamiltonian form the evolution equations for a velocity field more general than a normal one. This is possible: however it has the "defect" that it poses the equations into Hamiltonian form in a phase space much more general than that the incompressible fields. The incompressible Eulerian motions appear immersed into a phase space in which a rather strange Hamiltonian evolution takes place *but which reduces to the incompressible Euler equations* only on the small subspace representing incompressible velocity fields.
The interest of the Clebsch-Stuart Hamiltonian representation lies in the

fact that, although describing only motions of a "dense" class of velocity fields, it describes them with independent *globally defined coordinates* p, q. The alternative Hamiltonian representations of the general incompressible motion the canonical coordinates are not independent from each other: if one wants to describe an incompressible Eulerian field some constraints among them must be imposed (just for the purpose of imposing the possibility of interpreting them as coordinates describing an incompressible velocity field). See the problems for an analysis of this representation.

(E): Lagrangian form of the incompressible Euler equations, Hamiltonian form on the group of diffeomorphisms.

Clebsch's potentials are interesting because they show, for instance, that the Euler equations in 3–dimensions also admit infinitely many constants of motion: they are functions of the form (1.7.16) with ζ replaced by p or q by the (1.7.19). They involve, however, quantities p, q with a mechanical meaning that is not crystal clear and which makes their use quite difficult.

Are there more readily interpretable representations? In fact a fluid is naturally described in terms of the variables $\underline{\delta}(\underline{x})$ giving the "displacement" of the fluid points with respect to a reference position $\underline{x}$ that they occupy in a fluid configuration with constant density ρ_0.

It becomes natural to define a perfectly *incoherent* fluid as the mechanical system with Lagrangian:

$$\mathcal{L}(\dot{\underline{\delta}}, \underline{\delta}) = \int \frac{\rho_0}{2}\, \dot{\underline{\delta}}^2(\underline{x})\, d^3\underline{x} \tag{1.7.26}$$

with equations of motion determined by the action principle, *i.e.* $\rho_0 \frac{d}{dt}\dot{\underline{\delta}} = \underline{0}$, which are the equations of motion of a totally incoherent fluid.

The ideal incompressible fluid should be defined, formally and *alternatively* to the viewpoint of §1.1 as an incoherent fluid on which an ideal incompressibility constraint is imposed.

Denoting as above by $\delta F/\delta f(x)$ the functional derivative of a generic functional F of f, we note that the determinant of the Jacobian matrix $J = \partial\underline{\delta}/\partial\underline{x}$ of the transformation $\underline{x} \to \underline{\delta}(\underline{x})$ is $\det J = \underline{\partial}\delta_1 \wedge \underline{\partial}\delta_2 \cdot \underline{\partial}\delta_3$. Then the action principle applied to the Lagrangian (1.7.26) with the constraint that $\det J = 1$ leads immediately to

$$\rho_0 \frac{d}{dt}\dot{\delta}_k = \frac{\delta}{\delta\,\delta_k(\underline{x})} \int Q(\underline{x}')\, \det J(\underline{\delta}(\underline{x}'))\, d\underline{x}' \equiv -\varepsilon_{kij}\left(\underline{\partial}\delta_i \wedge \underline{\partial}\delta_j\right) \cdot \underline{\partial} Q \tag{1.7.27}$$

where Q is the Lagrange multiplier necessary to impose the constraint: it has to be determined by requiring that $\det J = 1$.

By algebra the (1.7.27) becomes

$$\rho_0 \frac{d}{dt}\dot{\underline{\delta}} = -(\det J)\, J^{-1} \underline{\partial} Q \equiv -\det J\, \frac{\partial \underset{\sim}{x}}{\partial \underline{\delta}}\, \underset{\sim}{\partial} Q \tag{1.7.28}$$

If $\underline{\delta} = \underline{\delta}(\underline{x})$, setting $\underline{u}(\underline{\delta}) = \dot{\underline{\delta}}(\underline{x})$, $\rho = \rho_0 / \det J$ and $p(\underline{\delta}) = Q(\underline{x})$ and changing variables from $\underline{x}$ to $\underline{\delta}$ in (1.7.28) we get

$$\frac{d}{dt}\underline{u} = -\frac{1}{\rho}\underline{\partial} p \tag{1.7.29}$$

where now the independent spatial variable is $\underline{\delta}$ that, being dummy, can be renamed $\underline{x}$ thus leading, really, to the incompressible Euler equations because p will have to be determined by imposing incompressibility which in the new variables is $\underline{\partial} \cdot \underline{u} = 0$ (provided the initial datum $\underline{u}$ has zero divergence $\underline{\partial} \cdot \underline{u} = 0$). Hence $p = -\Delta^{-1}\underline{\partial} \cdot (\underset{\sim}{u} \cdot \underset{\sim}{\partial} \underline{u})$.

Naturally $\dot{\underline{\delta}}(\underline{x}) = \underline{u}(\underline{\delta}(\underline{x}))$ is the evolution equation for $\underline{\delta}$ which, however, *decouples* from (1.7.29) in the sense that in (1.7.29) $\underline{\delta}$ no longer appears because it is a dummy variable which can equally well be called $\underline{x}$.

From the Lagrangian form of the equations of motion one can readily derive a Hamiltonian form. It will suffice to consider the Lagrangian

$$\mathcal{L}_i(\dot{\underline{\delta}}, \underline{\delta}) = \int \frac{\rho_0}{2}\, \dot{\underline{\delta}}(\underline{x})^2\, d^3\underline{x} - \int [\Delta^{-1}\underline{\partial}(\underset{\sim}{u} \cdot \underset{\sim}{\partial}\underline{u})]\,(\det J(\underline{\delta}) - 1)\, d\underline{x} \tag{1.7.30}$$

where $J(\underline{\delta}) = \partial\underline{\delta}/\partial\underline{x}$ and, furthermore, the expression in square brackets has to be computed *by thinking of* $\underline{u}$ *as a function of* $\underline{\delta}$ *(i.e.* $\underline{u}(\underline{\delta}) = \dot{\underline{\delta}}(\underline{x}(\underline{\delta}))$ with $\underline{x}(\underline{\delta})$ inverse of $\underline{\delta}(\underline{x})$*) and imagining that the differentiation operators and* Δ *act on the variables* $\underline{\delta}$ *as independent variables which eventually have to be set equal to* $\underline{\delta}(\underline{x})$.

In other words we recognize in the expression in square brackets the value of the multiplier introduced in (1.7.27), which turns the just made statements into an immediate formal consequence of the above Lagrangian form of the Euler equation.

The resulting equations for $\underline{p}(\underline{x}) = \delta\mathcal{L}_i/\partial\dot{\underline{\delta}}(\underline{x})$ e $\underline{q}(\underline{x}) = \underline{\delta}(\underline{x})$ are Hamiltonian equations for which, as a consequence of the preceding analysis, "incompressible" initial data, *i.e.* initial data with a zero divergence $\underline{u}$ and $\det J = 1$, evolve remaining incompressible. Hence the Euler equations describe a particular class of motions of a Hamiltonian system, which generates an evolution on the family of maps $\underline{x} \to \underline{\delta}(\underline{x})$ of R^3 into itself: precisely the class of the ideal incompressible motions.

Equation (1.7.30) implies that *for "incompressible data"* we shall have $\underline{p}(\underline{x}) = \rho_0\dot{\underline{\delta}}(\underline{x}) = \rho_0\underline{u}(\underline{\delta}(\underline{x}))$ and, obviously, $\underline{q}(\underline{x}) = \underline{\delta}(\underline{x})$, while if the datum $\underline{u}$ is not incompressible the evolution generated by (1.7.30) *is not* the eulerian evolution: *i.e.* the equations of motion of (1.7.30) are correct (namely are equivalent to Euler's) only for the incompressible motions.

Proceeding formally (along a path that it is an open problem to make mathematically rigorous) one can finally check that the "surface" $\mathcal{S}$ of the incompressible transformations of R^3 is such that the restriction to the phase space $\mathcal{S}\times\mathcal{T}(\mathcal{S})$ (*i.e.* the space of the points in $\mathcal{S}$ and the cotangent vectors $\dot{\underline{\delta}}$ to it) of such Hamiltonian motions is itself a family of Hamiltonian motions on $\mathcal{S}\times\mathcal{T}(\mathcal{S})$. From (1.7.30), we see that the Hamiltonian will be quadratic in $\underline{p}$ (because the (1.7.30) is quadratic in $\dot{\underline{\delta}}$ and, *on the surface* $\mathcal{S}$, $\underline{p}=\rho_0\dot{\underline{\delta}}(\underline{\delta})$) *i.e.* $H(\underline{p},\underline{q})=\frac12(\underline{p},G(\underline{q})\underline{p})$ with G a suitable operator on the space of the incompressible transformations.

The surface $\mathcal{S}$ can, finally, be seen as a *group of transformations* (the group of diffeomorphisms of R^3 into itself which preserve the volume) and Arnold has shown that G can then be interpreted as a metric on the group $\mathcal{S}$ which is (left)–invariant and, therefore, the Eulerian motions can be thought of as *geodesic motions* on $\mathcal{S}$, [Ar79].

(F): Hamiltonian form of the Eulerian potential flows; example in 2*–dimensions.*

It is difficult to find concrete applications of the Clebsch potentials or of the Lagrangian and Hamiltonian interpretations of Euler equations given in (E). Nevertheless in special cases the Hamiltonian nature of the Euler equations emerges naturally in other forms and it can become quite useful for the applications.

Furthermore there is an important and vast class of problems that evidently cannot be studied via Clebsch potentials: namely the irrotational incompressible flows in "free boundary" domains representing motions of a fluid in contact with a gas under a gravity field.

As an example we describe a very remarkable application to 2*–dimensional fluid motions. It is the theory of vorticity-less waves, i.e. of potential flows, in* $d=2$.

Consider the plane x,z and on it a fluid whose free surface has equation $z=\zeta(x)$. This means also that we consider only "*non breaking waves*". Let $\underline{u}$ be a velocity field describing the fluid, which we suppose ideal and incompressible. The system evolves via Euler equations and in a (conservative) force field of intensity g oriented towards $z<0$: *gravity.*

The first relation that we shall look for is the *boundary condition* on the free surface of the fluid. It is clear that if $\underline{u}(x,\zeta(x,t),t)=\underline{u}$ is the velocity of a point that at time t is in $(x,\zeta(x,t))=(x,\zeta)$ then at the instant $t+dt$ it will be in $(x+u_xdt,\zeta+u_zdt)$ hence in order that the fluid remains connected (*i.e.* no holes develop in it) it is necessary that this point coincides with $\zeta(x+u_xdt,t+dt)$, up to $O(dt^2)$.

Hence, denoting $\dot\zeta=\partial_t\zeta,\zeta'=\partial_x\zeta$, we obtain the boundary condition $-\partial_x\zeta\,u_x+u_z=\partial_t\zeta$ in $(x,\zeta(x,t))$; the latter relation has to be coupled with the condition that the pressure be constant, *e.g.* $p=0$, at the free boundary. If $\underline{n}(x)\overset{def}{=}(-\zeta'(x),1)/(1+\zeta'(x)^2)^{1/2}$ is the external normal to the set of the

(x, z) with $z < \zeta(x)$, the two conditions can be written as

$$\begin{aligned} &(\underline{n} \cdot \underline{\partial}\, \varphi)(x, \zeta(x)) = \frac{\partial_t \zeta(x)}{(1 + \zeta'(x)^2)^{1/2}} \\ &\partial_t \varphi + g\zeta + \frac{1}{2}(\underline{\partial}\varphi)^2 = 0 \end{aligned} \tag{1.7.31}$$

respectively, *c.f.r.* (1.7.7). The relations show that the two functions $(\zeta(x), \dot\zeta(x))$ uniquely determine the velocity field as well as their values at time $t + dt$.

In fact given $\zeta, \dot\zeta$ we can compute the velocity field $\underline{u} = \underline{\partial}\varphi$ by determining φ as the harmonic function satisfying $\Delta\varphi = 0$ in $D = \{(x, z)|\, x \in [0, L], z \in (-\infty, \zeta(x))\}$ with boundary condition $\partial_n \varphi$ given by the first of (1.7.31): a Neumann problem; furthermore from the second relation we can compute $\partial_t \varphi(x, z)$ by solving a Dirichlet problem in D with boundary condition $(\partial_t \varphi)(x, \zeta(x))$. The latter field can be used to compute $\ddot\zeta(x)$ by differentiating the boundary condition rewritten in the form $\partial_t \zeta = -\partial_x \zeta\, u_x + u_z$.

We can now put the equations into Hamiltonian form. We expect that, the motion being "ideal", the equations of motion must follow from a naive aplication of the action principle. We shall restrict the analysis to motions which are horizontally periodic over a length L, for simplicity. Since we have seen that the motions are naturally described in the coordinates $\zeta(x), \dot\zeta(x)$ we consider the action functional of $\zeta, \dot\zeta$ as the difference betwen the kinetic energy of the fluid and its potential energy in the gravity field.

Hence the domain D in which we shall consider the fluid is $D = \{(x, z)|\, x \in [0, L], z \in (-\infty, \zeta(x))\}$. In D the velocity field will be $\underline{u} = \underline{\partial}\varphi$ and its time derivative will be $\underline{\partial}\psi$ with φ harmonic, solutions of

$$\Delta\varphi = 0 \qquad \text{in } D, \qquad \underline{n} \cdot \underline{\partial}\varphi = \frac{\dot\zeta}{(1 + \zeta'(x))^{1/2}} \qquad \text{in } \partial D \tag{1.7.32}$$

The potential energy will obviously be infinite, being $\int_0^L \int_{-\infty}^{\zeta(x)} \rho\, g\, z\, dx dz$. However what counts are the energy variations which are formally the same as those of $\rho \int_m^L dx \frac{1}{2} g\zeta(x)^2$ where m is any quantity less than the minimum of $\zeta(x)$.

The kinetic energy will be $\frac{\rho}{2} \int_0^L dx \int_{-\infty}^{\zeta(x)} dz\, |\underline{\partial}\varphi(x, z)|^2$. Thus the action of the system, *i.e.* the difference between kinetic and potential energy is, if $ds \stackrel{def}{=} (1 + \zeta'(x)^2)^{1/2} dx$ and if we introduce the auxiliary function $\psi(x) \stackrel{def}{=} \varphi(x, \zeta(x),$

$$\begin{aligned} \mathcal{L} &= \frac{\rho}{2} \int_0^L \int_{-\infty}^{\zeta(x)} (\underline{\partial}\varphi)^2\, dx dz - \frac{\rho}{2} \int_0^L g\zeta(x)^2 dx = \\ &= \frac{\rho}{2} \int_0^L (\psi(x) \frac{\dot\zeta(x)}{(1 + \zeta'(x))^{1/2}}\, ds - g\zeta(x)^2\, dx) \end{aligned} \tag{1.7.33}$$

after integrating by parts the first integral, defining if $\psi(x) \stackrel{def}{=} \varphi(x, \zeta(x)$ and making use of the first relation (1.7.31), (1.7.32).

The linearity of (1.7.32) shows that ψ is linear in $\dot\zeta(x)/(1+\zeta'(x))^{1/2}$: *i.e.* $\psi = M\dot\zeta/(1+(\zeta')^2)^{1/2}$ for some linear operator M (which depends on $\zeta(x)$, *i.e.* on the shape of D).

In general if φ_1,φ_2 are two harmonic functions in D with respective "Dirichlet value" φ_1,φ_2 on the boundary, ∂D, and respective "Neumann values" on the same boundary $\sigma_1 = \underline{n}\cdot\underline{\partial}\varphi_1, \sigma_2 = \underline{n}\cdot\underline{\partial}\varphi_2$ then, (on the boundary) by definition, $\varphi_i = M\sigma_i$ for a suitable linear operator M and $\int_{\partial D} ds\,\big((M\sigma_1)\,\sigma_2 - \sigma_1\,(M\sigma_2)\big) = \int_D(\varphi_1\Delta\varphi_2 - \varphi_2\Delta\varphi_1)\,dxdz = 0$ so that the operator M is *symmetric* on $L_2(ds,[0,L])$. Note that the operator M transforms, by definition and by (1.7.34), a Neumann boundary datum for a harmonic function into the corresponding Dirichlet datum. Hence $\mathcal{L}$ becomes

$$\mathcal{L} = \frac{\rho}{2}\int_0^L \big(M(\frac{\dot\zeta}{(1+\zeta')^{1/2}})\frac{\dot\zeta}{(1+\zeta')^{1/2}}\,ds - g\zeta^2\,dx\big) \tag{1.7.34}$$

Therefore we can write the equations of motion in terms of the Hamiltonian corresponding to the Lagrangian $\mathcal{L}$. Note that $\delta\mathcal{L}/\delta\dot\zeta(x) = M(\dot\zeta(x)/(1+(\zeta')^2)^{1/2})$, where we denote by $\delta/\delta\varphi(x)$ a generic functional derivative and we recall that $(1+\zeta'(x)^2)^{-1/2}ds \equiv dx$: therefore we can define the variable $\pi(x)$ conjugate to $\zeta(x)$ as $\pi(x) \stackrel{def}{=} \rho M\dot\zeta/(1+(\zeta')^2)^{1/2} = \rho\psi(x)$ and the Hamiltonian in the conjugate variables π,ζ is

$$\mathcal{H} = \frac{1}{2}\int_0^L \Big(\frac{1}{\rho}\pi(x)G\pi(x) + \rho g\zeta(x)^2\Big)\,dx \tag{1.7.35}$$

where if Γ^D is the operator that solves the Dirichlet problem in D we set $G(\zeta)\pi(x) = \rho\,\big(-\partial_x\zeta(x)\partial_x\big(\Gamma^D\pi\big)(x,z) + \partial_z\big(\Gamma^D\pi\big)(x,z)\big)_{z=\zeta(x)}\big)$.

The check that the equations of motion generated by (1.7.35) are correct, *i.e.* coincide with (1.7.31) proceeds as follows. The first Hamilton equation is just $\dot\zeta = \frac{1}{\rho}G\pi$ which coincides with the first of (1.7.31). More interesting is the computation of the functional derivative $-\delta\mathcal{H}/\delta\zeta(x)$. It is

$$\begin{aligned}
&-g\rho\zeta(x) - \rho\frac{\delta}{\delta\zeta(x)}\int_0^L dy\int_{-\infty}^{\zeta(y)} dz\,\frac{1}{2}(\underline{\partial}\varphi)^2(y,z) =\\
&= -g\rho\zeta(x) - \frac{\rho}{2}(\underline{\partial}\varphi)^2(x,\zeta(x)) -\\
&-\int_D \rho\underline{n}\cdot\underline{\partial}\varphi(y,\zeta(y))\frac{\delta}{\delta\zeta(x)}(\Gamma^D\frac{\pi}{\rho})(y,z)\big|_{z=\zeta(y)}ds
\end{aligned} \tag{1.7.36}$$

where $ds = (1+\zeta'(y)^2)^{1/2}\,dy$. The last functional derivative can be studied by remarking that by its definition $(\Gamma_D\pi)(x,z)|_{z=\zeta(x)} \equiv \pi(x)$ hence near $z=0$ we get

$$\frac{1}{\rho}\Gamma^D\pi(y,z) = \frac{1}{\rho}\,\varphi(y,\zeta(y)) + (z-\zeta(y))\,\frac{1}{\rho}\,\partial_z(\Gamma^D\pi)(y,\zeta(x)) +$$

$$+ O((z-\zeta(x))^2) = \frac{1}{\rho}\pi(y) + (z-\zeta(y))u_z(y) + O((z-\zeta(x))^2) \Rightarrow$$

$$\Rightarrow \frac{1}{\rho}\frac{\delta(\Gamma^D\pi)(y)}{\delta\zeta(x)} = -\delta(x-y)\,u_z(y) \tag{1.7.37}$$

Since $\rho\,\underline{n}\cdot\underline{\partial}\varphi(y,\zeta(y)ds = \rho\dot\zeta(y)dy$ this means that

$$-\frac{\delta\mathcal{H}}{\delta\zeta(x)} = -\rho g\zeta(x) - \frac{\rho}{2}(\underline{\partial}\varphi)^2(x) + \rho\dot\zeta(x)u_z(x) \tag{1.7.38}$$

Furthermore $\dot\pi = \rho\,\frac{d}{dt}(\varphi(x,\zeta(x)) = \rho\,(\partial_t\varphi(x,\zeta(x)) + u_z(x)\dot\zeta(x))$ (because $u_z = \partial_z\varphi$). The latter relation combined with (1.7.38) for $-\delta\mathcal{H}/\delta\zeta(x)$ gives the pressure condition on the free surface.

The above analysis is taken from [DZ94],[DLZ95],[CW95].

(G) Small waves.

The Hamilton equations in (F) considerably simplify in the case of "small waves". Indeed non linear terms will be neglected and the domain D occupied by the fluid will be considered to be the half–space $z<0$. In this approximation M,G take the value that they have for $\zeta\equiv 0$ and admit a simple and well known exact expression. See the problems for a more quantitative analysis of this approximation.

It will then be possible to develop π,ζ in Fourier modes

$$\begin{aligned}\pi(x) =& \frac{1}{\sqrt{2L}}\sum_k p_k e^{ikx}, \qquad \zeta(x) = \frac{1}{\sqrt{2L}}\sum_k q_k e^{ikx}\\ \varphi(x,z) =& \frac{1}{\rho}\frac{1}{\sqrt{2L}}\sum_k p_k e^{ikx}e^{|k|z}\end{aligned} \tag{1.7.39}$$

where k is an integer multiple of $2\pi/L$, and the Hamiltonian can be written in terms of the Fourier coefficients

$$H(p,q) = \frac{1}{2}\sum_k(\frac{|k|}{2\rho}|p_k|^2 + \frac{\rho g}{2}|q_k|^2) \tag{1.7.40}$$

In (1.7.40) there are redundant variables because $p_k = \overline{p_{-k}}$ and $q_k = \overline{q_{-k}}$. Therefore if we decompose p_k, q_k into real and imaginary parts, writing $p_k = p_{1k} + ip_{2k}$ and $q_k = q_{1k} + iq_{2k}$, we find

$$H(p,q) = \frac{\rho g}{4}q_0^2 + \sum_{k>0}\sum_{j=1,2}\frac{1}{2}(\frac{k}{\rho}p_{jk}^2 + \rho g q_{jk}^2) \tag{1.7.41}$$

Recalling that the Fourier transform is a canonical transformation this shows that the problem of the small waves, spatially periodic horizontally with period L in a deep bidimensional fluid is equivalent to studying infinitely many harmonic oscillators: *hence it is exactly soluble.*

The normal modes, *i.e.* the action–angle coordinates, of the problem are naturally given by the Fourier transform and the mode with $k = 0$ is immobile: *i.e.* q_0 = *constant* p_0 = *constant*. The second relation is evidently trivial because φ, hence also π, are defined up to an additive constant; the first, instead, has the simple interpretation that the average level (*i.e.* $q_0 = \int_0^L \zeta(x)\,dx$) remains constant: *i.e.* the constancy of q_0 expresses mass conservation.

It is interesting to study the velocity fields corresponding to the normal modes of oscillation that we have just derived: see the following problems that also illustrate a more classical approach to linear waves, [LL71].

Problems: *Sound and surface waves. Radiated energy.*

[1.7.0] Think of a 2–dimensional fluid as a stratified 3–dimensional one (see (C) above) and define vorticity as the field perpendicular to the fluid motion and with intensity $\omega = \partial_z u_x - \partial_x u_z$ (see (C) above). Find an expression for the velocity field in terms of Clebsch potentials. (*Idea:* Vorticity lines are straight lines perpendicular to the plane x, z. Hence proceeding as in (D) above we can choose as functions α, β, see (1.7.23), the functions $\alpha = z, \beta = x$, for instance. Then the function $\lambda(z, x)$ in (1.7.24) is precisely: $\lambda(z, x) = \omega(x, z)$ and, by (1.7.25)

$$p = x, \quad q(x, z) = \int_{-\infty}^{z} dz'\omega(x, z')$$

thus the function γ will be $\gamma = \Delta^{-1}(x\Delta q + \partial_x q)$, where Δ^{-1} is the inverse of the Laplace operator in the domain occupied by the fluid.)

[1.7.1]: (*sound waves in a Euler fluid*) Consider a compressible adiabatic Euler fluid, with zero velocity, pressure p_0 and temperature T_0. Imagine to perturb the initial state by a small irrotational perturbation $\underline{u}' = \underline{\partial}\varphi$, vanishing at ∞. Suppose that the motion is adiabatic and that $\underline{u}'$ as well as the variations p' of the pressure and ρ' of the density are small, neglecting their squares. Show that sound waves are generated in the fluid and compute their propagation speed. (*Idea:* Write $\rho = \rho_0 + \rho', p = p_0 + p'$ and check that

$$\partial_t \rho' + \rho_0 \underline{\partial} \cdot \underline{u}' = 0, \qquad \partial_t \underline{u}' + \rho_0^{-1} \underline{\partial} p' = 0, \qquad p' = (\frac{\partial p}{\partial \rho})_s \rho'$$

and, furthermore, $\underline{u}' = \underline{\partial}\varphi$ for a suitable φ; and the second relation implies $\underline{\partial}(\partial_t \varphi + \rho_0^{-1} p') = 0$. Hence $\partial_t \varphi = -\rho_0^{-1} p'$ because at ∞ one has $p' = 0$ and $\underline{u}' = \underline{0}$. So that

$$\partial_t^2 \varphi = -\frac{1}{\rho_0} \partial_t p' = -\frac{1}{\rho_0} \big(\frac{\partial p}{\partial \rho}\big)_s \partial_t \rho' = \big(\frac{\partial p}{\partial \rho}\big)_s \Delta\varphi$$

therefore φ and, by linearity, $\partial_t \varphi$ evolve according to the wave equation with speed $c = \big(\frac{\partial p}{\partial \rho}\big)_s^{1/2}$.)

[1.7.2]: (*free surface boundary condition*) Consider a non viscous incompressible bidimensional fluid, filling the region of the plane (x, z) defined by $z \le \zeta(x, t)$, where $\zeta(x, t)$ is the *free surface* at time t. Show that the boundary condition relating $\underline{u}(\underline{x}, t)$ to $\zeta(x, t)$ is

$$\Big(1 + (\partial_x \zeta)^2\Big)^{1/2} \underline{n} \cdot \underline{u} = \partial_t \zeta$$

if $\underline{n} = \Big(1 + (\partial_x \zeta)^2\Big)^{-1/2} (-\partial_x \zeta, 1)$ is the external normal to the free surface. Show also that the boundary condition has the meaning that the fluid elements that are on the surface *remain there* forever: a necessary consequence of the hypothesis that the displacement of the fluid elements due to the fluid motion is a regular transformation of the domain occupied by the fluid. (*Idea:* In a regular transformation the interior points remain interior.)

[1.7.3]: (*linearity conditions for motions of a free surface*) In the context of [1.7.2] suppose that the fluid evolves with a regular motion in which the surface waves have amplitude a which evolves sensibly over a time scale τ, and suppose that the variations of $\underline{u}$ at fixed time are sensible over a length scale $\lambda \equiv k^{-1}$. Show that the condition under which the transport term (also called inertial term) $\underset{\sim}{u} \cdot \underset{\sim}{\partial}\underline{u}$ in Euler equation can be neglected is that $a \ll \lambda$, *i.e.* $ka \ll 1$. (*Idea:* Proceed as in the analysis of §1.2. The order of magnitude of $\partial_t \underline{u}$ is $a\tau^{-2}$, that of $\underline{u}$ is $a\tau^{-1}$ and that of $\underset{\sim}{u} \cdot \underset{\sim}{\partial}\underline{u}$ is $a^2\tau^{-2}\lambda^{-1}$.)

[1.7.4]: (*linearized Euler equations for motions of a free surface*) In the context of problem [1.7.2] and assuming the inertial term, discussed in [1.7.3], to be negligible consider an irrotational motion: $\underline{u} = \underline{\partial}\varphi$ with pressure at the free surface $p_0 = 0$ constant. Show that Euler equations become

$$\partial_t\varphi + \rho^{-1}p + gz = 0, \quad \Delta\varphi = 0, \quad u_z(x, \zeta(x,t)) = \partial_t\zeta(x,t), \quad p(x,\zeta) = 0$$

(*Idea:* The first is (1.7.7), with $\frac{1}{2}\underline{u}^2$ neglected, *c.f.r.* [1.7.3]; the second is the zero divergence condition for $\underline{u} = \underline{\partial}\varphi$; the third is the boundary condition in [1.7.2] if one neglects the terms of order $O((a/\lambda)^2)$; the fourth expresses that the pressure at the free surface is constant.)

[1.7.5]: (*wave solutions for motions of a free surface*) In the context of [1.7.4] find a solution with $\zeta(x,t) = a\cos(kx - \omega t)$ showing that its existence follows immediately from the analysis in (G) assuming $\omega^2 = kg$; hence surface waves with phase velocity $\sqrt{gk}/k$ and group velocity $\frac{1}{2}\sqrt{g/k}$ are possible. (*Idea:* At $z = \zeta(x,t)$ one must have , *c.f.r.* [1.7.4], $\partial_t\varphi = -ag\cos(kx - \omega t)$; hence $\varphi(x,z) = \frac{ag}{\omega}e^{kz}\sin(kx - \omega t)$ solves the first two equations of [1.7.4], and the $u_z = \partial_t\zeta$ yields $\frac{gk}{\omega} = \omega$.)

[1.7.6]: (*current lines equations in free surface waves*) Write the equations of the current lines of the motion found in [1.7.5]. (*Idea:*

$$\dot{x} = agk\omega^{-1}e^{kz}\cos(kx - \omega t), \quad \dot{z} = agk\omega^{-1}e^{kz}\sin(kx - \omega t).)$$

[1.7.7]: (*circles and current lines in free surface waves*)From the equations deduced in [1.7.6] deduce that if $|z|$ is large then, *approximately*, the current lines with an average position (x_0, z_0) are circles of radius $agk\omega^{-2}e^{kz_0}$ around (x_0, z_0).

[1.7.8]: (*shallow "water" waves*) Show that if the fluid, rather than occupying the region $z \le \zeta(x,t)$, occupies a region $-h \le z \le \zeta(x,t)$ and if $h \gg \lambda \gg a$ then the equations derived in [1.7.4] are simply modified by adding the further boundary condition $\partial_z\varphi|_{z=-h} = 0$. Evaluate the speed of phase propagation analogous to the one in [1.7.5], showing that $\omega^2 = gk\tanh kh$. Show that also this problem can be put into a Hamiltonian form and leads to an integrable Hamiltonian system (of harmonic oscillators); calculate the normal modes.

[1.7.9]: (*velocity in shallow "water" waves*) Show, in the context of [1.7.8], that if $\lambda \gg h$ the phase and group velocity coincide and have value $\sqrt{gh}$.

[1.7.10]: What would change if in the preceding problems we considered the fluid 3–dimensional? (*Idea:* "nothing".)

[1.7.11]: Consider in the context of problem [1.7.1] an irrotational 2–dimensional infinite fluid and consider a plane wave with speed c. Show that the velocity v at a given point and at a given time is related to the density and pressure variations ρ' and p' by $p' = \rho v c$ e $p' = c^2\rho'$. (*Idea:* The second relation follows as explained in the hint for [1.7.2]; and if $\varphi = f(x/c - t)$ is the velocity potential of a plane wave in the direction x one has $v = \partial_x\varphi = c^{-1}f'(x/c-t)$, while $p' = -\rho\partial_t\varphi$, always from the hint for [1.7.1]: hence $p' = -\rho f'(x/c-t)$.)

[1.7.12]: (*energy flux through a surface orthogonal to a plane wave in a fluid*) Note that in a plane wave the average energy flux through a surface with normal $\underline{n}$ parallel to the direction of the wave velocity $\underline{v}$ is $\rho_0 c\,\overline{v^2}$, where the bar denotes time average. (*Idea:* The energy flux is, *c.f.r.* [1.1.6], $\rho\underline{v}(\varepsilon+\frac{v^2}{2}+\frac{p}{\rho})$; neglecting $\underline{v}v^2$ one has $\rho\underline{v}(\varepsilon+\frac{p}{\rho}) = w_0\rho\underline{v}+\rho w'\underline{v}$ where $w_0 = \varepsilon_0 + p_0/\rho_0$, and $w' = \left(\frac{\partial w}{\partial p}\right)_s p'$. But w is the enthalpy per unit mass, so that $\left(\frac{\partial w}{\partial p}\right)_s = 1/\rho_0$; hence $\overline{\rho\underline{v}(\varepsilon+p/\rho)} = w_0\,\overline{\rho\underline{v}} + \overline{p'\underline{v}}$. The first average vanishes because $\overline{\rho\underline{v}}$ is the average variation of fluid mass to the left of the surface, while [1.7.11] implies $p' = \rho v c$ hence $\overline{p'v} = \rho c\,\overline{v^2}$.)

[1.7.13]: (*energy emitted by small oscillations or deformations of a large body in an infinite fluid*) In the context [1.7.1] suppose that a body Γ of arbitrary form and linear dimension l is immersed in the fluid. Imagine that the body oscillates either by varying the volume (homotetically) or by displacing the center of mass along the z axis. Let $\omega/2\pi$ be the oscillations and define $\lambda = 2\pi c/\omega$, with c equal to the sound speed.
Supposing that $\lambda \ll l$ and also that such oscillations have a small enough amplitude compared to l we can assume that every surface element of the body emits a plane wave and that the velocity of the point $\underline{x} \in \partial\Gamma$, where the external normal is $\underline{n}$, is $\underline{u}$. Show that the energy emitted per unit time is $\rho_0 c\int_{\partial\Gamma}(\underline{u}\cdot\underline{n})^2\,d\sigma$. (*Idea:* If $\lambda \ll l$ every wave goes its own way and the result follows immediately from [1.7.12].)

[1.7.14]: (*energy emitted by small oscillations or deformations of a small body in an infinite fluid*) In the context of [1.7.13] suppose that $\lambda \gg l$ and that the volume $t \to V(t)$ of the body varies as a given function of time; show that the energy I emitted per unit time is

$$I = \frac{\rho}{4\pi c}\,\overline{(\partial_t^2 V)^2}$$

(*Idea:* In the region $l \ll r \ll \lambda$ one can suppose that $\Delta\varphi = 0$ (because $c^{-2}\partial_t^2\varphi \sim O(\omega^2 c^{-2}\varphi) \sim O(\lambda^{-2}\varphi)$ while the order of magnitude of $\Delta\varphi$ should be much larger being $O(r^{-2}\varphi)$); then for a suitable function a of t we have $\varphi = \frac{a}{r}+O(\frac{1}{r^2})$ and incompressibility demands that the flux through the surface of a (large compared to l) sphere is such that $\dot V = 4\pi a$. Hence $\varphi = -\frac{\dot V(t)}{4\pi r}$. At distance $r \gg \lambda$ it will be, instead, a solution of the wave equation *i.e.* (in the dominant spherical term) $\varphi = -\frac{f(t-r/c)}{4\pi r}$ (for some f because φ must be a solution of the wave equation with speed c): it follows that the behavior for r in both regions can be interpolated by the single expression $\underline{u} = \underline{\partial}\varphi = \frac{(\partial_t^2 v)(t-r/c)\underline{n}}{4\pi c r}$. And integrating by parts on a surface of radius r the result follows.)

[1.7.15]: (*energy emitted by a small rigid body vibrating in a fluid*) Suppose that the body of volume V is rigid, hence $V = const$, and that it moves along the z–axis with an oscillatory motion and pulsation ω. Then the velocity potential will not have ar^{-1} as leading term for r large (although small relative to λ, *i.e.* in the region where the potential can be considered harmonic) because (as in [1.7.14]) this would mean that $\dot V \neq 0$) and its leading behavior will rather be the next (dipole) term $\varphi = \underline{A}(t)\cdot\underline{\partial}\frac{1}{r}$, where $\underline{A}$ is (by symmetry) parallel to the axis z. Noting that $\underline{\partial}\left(\frac{\underline{A}(t-r/c)}{r}\right)$ solves the wave equation, the potential φ can be interpolated for all r ($r \gg l$) by $\varphi \simeq -\frac{\dot{\underline{A}}(t-r/c)\cdot\underline{n}}{cr}$. Show that the velocity at large distance and the irradiated energy per unit time are

$$\underline{v} \simeq \underline{n}\,\underline{n}\cdot\partial_t^2\underline{A}/c^2 r, \quad I = 4\pi\rho\,\overline{(\partial_t^2\underline{A})^2}/3c^3$$

[1.7.16]: (*energy emitted by a small rigid body slowly and harmonically vibrating in a fluid*) Suppose that the body is a sphere of radius R moving as $z(t) = L\cos\omega t$, $L \ll c/\omega$. Show that

$$I \simeq 2\pi\rho\omega^2 \left(\frac{R^3}{2}\right)^2 \frac{2}{3}\frac{(\partial_t^2 z)^2}{c^3}$$

(*Idea:* If a sphere moves at constant velocity $\underline{v}_0$ then the field φ that generates the potential flow is $\varphi = ar^{-2}\underline{v}_0 \cdot \underline{n} \equiv -a\underline{v}_0 \cdot \underline{\partial}\frac{1}{r}$ (because $\underline{v}_0 \cdot \underline{\partial}\frac{1}{r}$ is the only scalar field which is linear in $\underline{v}_0$, vanishing at ∞ and harmonic): and A is determined by imposing $\underline{n} \cdot \underline{\partial}\varphi|_{\underline{r}=R\underline{n}} = \underline{v}_0 \cdot \underline{n}$: *i.e.* $a = \frac{1}{2}R^3$. Hence $\underline{A} = \underline{v}_0\frac{R^3}{2}$ and $\dot{\underline{A}} = \frac{R^3}{2}\underline{k}\partial_t^2 z$, if the motion is slow, $\omega L \ll c$, *c.f.r.* [LL71], §68, 69.)

[1.7.17]: (*comparison between energy emitted by a small rigid body and the electro-magnetic energy emitted by a charge when both have a harmonic motion*) Compare the electro-magnetic energy emitted by a charge e oscillating as $z(t) = L\cos\omega t$, $L \ll c/\omega$, with $c =$ speed of light and the energy irradiated by a ball oscillating in the in the same way in a fluid. (*Idea:* The expressions coincide if one identifies the electro-magnetic e^2/c^3 with $2\pi\rho\omega^2\left(2^{-1}R^3\right)^2/c^3$: recall that the energy emitted per unit time by an oscillating charge is $\frac{2}{3}e^2c^{-3}\,\ddot{\underline{z}}$, *c.f.r.* [Be64], (11.19) or (13.5).)

[1.7.18]: (*non integrability of nonlinear surface waves in deep fluids*) Recall that a Hamiltonian system obtained by perturbing a system of harmonic oscillators with a perturbing function containing non quadratic terms in the canonical variables of the oscillators can be formally reduced to an integrable system provided suitable non resonance conditions are satisfied, *c.f.r.* [Ga83], §10. Then one can put the problem of trying to show that such conditions are verified in the theory of waves in a deep fluid studied in (F). Attempt an analysis of which kind of problems one has to tackle in order to check the non resonance conditions (and identify some of them). *Comment*: The problem can be exactly solved and one can show that the system can be (remarkably enough) written as an integrable one provided one neglects the terms of order above the third (included), see [DZ94],[DLZ95], or even above the fourth (included), see [CW95]. But if the fifth order terms are not neglected then the system *cannot* be integrated by quadratures, [CW95].

Bibliography: [LL71], §12,13,68,69, [La32] §167, p. 248; [Ar79] Appendix 2, D; [Be64] vol. II, §11, (11.19); [DZ94], [DLZ95], [CW95].