

Renormalization group approach to the theory of the Fermi surface.G. Benfatto¹, G. Gallavotti²

Abstract: we describe an approach to the theory of the Fermi surface based on the renormalization group. A notion of quasi particle appears naturally in the theory. The beta function is finite to all orders of perturbation theory and we compute and discuss it to second order. The normality of the Fermi surface is linked to the repulsivity of the quasi particle potential: the remarkable cancellations that occur in the beta function lead to think that a normal Fermi surface can perhaps occur, but only when the interaction between the quasi particles is repulsive. The approach was first proposed in an earlier paper, [BG1], and we give here an improved version of it, referring to [BG1] only to make use of technical estimates derived there.

A natural approach to the theory of the Fermi surface in an interacting Fermi liquid is to slice momentum space into layers $I_1, I_0, I_{-1}, I_{-2} \dots$, with $I_1 = \{\text{set of } (k_0, \vec{k}) \text{ such that } k_0^2 + [(\vec{k}^2 - p_F^2)/2m]^2 \geq p_0^2\}$ and, if $n \leq 0$, $I_n = \{\text{set of } (k_0, \vec{k}) \text{ such that } (2^{n-1}p_0)^2 \leq k_0^2 + [(\vec{k}^2 - p_F^2)/2m]^2 \leq (2^n p_0)^2\}$; here p_0 is an arbitrary momentum scale, p_F is the Fermi momentum and m is the particle mass. Then the fermion propagator $g(t, \vec{x})$ appearing in the perturbation theory [LW] can be written:

$$g(t, \vec{x}) = \sum_{n=-\infty}^1 g^{(n)}(t, \vec{x}) \equiv \int_{I_n} \frac{dk_0 d^d \vec{k}}{(2\pi)^{d+1}} \frac{e^{-(ik_0 t + \vec{k} \cdot \vec{x})}}{-ik_0 + (\vec{k}^2 - p_F^2)/2m} \quad (1)$$

Hence $g^{(n)}$ is the part of the propagator coming from layers at distance $O(2^n p_0)$ from the Fermi sphere ($k_0 = 0, |\vec{k}| = p_F$ in the $(d+1)$ -dimensional momentum space). The momentum scale p_0 used to measure the distance from the Fermi surface will be conveniently chosen identical to the inverse of the range of the potential. The above decomposition of the propagator generates a representation $\psi^\pm = \sum_{n=-\infty}^1 \psi^{(n)\pm}$.

In order to clarify the basic difficulty, let us consider, for simplicity, the case $d = 3$; then the asymptotic form of $g^{(n)}$ is (if $\beta \equiv p_F/m$), for large n and $|\vec{x}| + |t|$:

$$g^{(n)}(t, \vec{x}) \sim \text{const } 2^{2n} \frac{p_F^2 p_0}{\beta} \left(t \frac{\sin(p_F |\vec{x}|)}{p_F |\vec{x}|} + \frac{\cos(p_F |\vec{x}|)}{\beta} \right) \gamma(2^n t p_0, 2^n \vec{x} p_0) \quad (2)$$

and γ is n -independent and fast decreasing (provided, as usual, the oscillations due to the sharpness of the boundary of I_n are eliminated by means of the device of introducing a

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smooth partition of unity mimicking the slicing). The (2) is not suited for a renormalization group analysis of the main functional integral $\int P_g(d\psi) \exp[-V^{(1)}(\psi)]$, where:

$$\begin{aligned}
V^{(1)}(\psi) = & \int d\vec{x}d\vec{y}dt \lambda_1(\vec{x} - \vec{y}) \psi_{t\vec{x}}^+ \psi_{t\vec{y}}^+ \psi_{t\vec{y}}^- \psi_{t\vec{x}}^- + \\
& + \alpha_1 \int d\vec{x}dt \psi_{t\vec{x}}^+ \frac{-\Delta + p_F^2}{2m} \psi_{t\vec{x}}^- + \nu_1 \int d\vec{x}dt \psi_{t\vec{x}}^+ \psi_{t\vec{x}}^-
\end{aligned} \tag{3}$$

and $P_g(d\psi)$ denotes that the functional integral of $\exp(-V^{(1)})$ has to be evaluated using Feynman rules with propagator g .

The unsuitability of the decomposition (2) stems from the fact that $g^{(n)}$ does not *scale*, *i.e.* it does not depend on (t, \vec{x}) via $(2^n t p_0, 2^n \vec{x} p_0)$, but it also oscillates on scale p_F^{-1} and, worse, it contains a scaleless singularity $|\vec{x}|^{-1}$. Hence, unlike the analogous properly scaling decomposition met in relativistic quantum field theory or in the theory of the critical point in Statistical Mechanics, we cannot use (2) even for assigning in an unambiguous way a dimension to the field.

For instance, if one disregards the above difficulties, one might be tempted to say that the field scales like $2^{2n/2} = 2^n$. This, as we shall see, would be quite misleading and, essentially, wrong. If, nevertheless, one insists in the above path and defines the effective potential on scale $2^{-h} p_0^{-1}$ as:

$$\exp[-V^{(h)}(\psi)] = \int P(d\psi^{(1)}) \dots P(d\psi^{(h+1)}) \exp[-V^{(1)}(\psi + \psi^{(1)} + \dots + \psi^{(h+1)})] \tag{4}$$

trying to identify relevant and irrelevant terms in some *reasonable* way, then one is led to think that $V^{(h)}$ becomes quite singular as $h \rightarrow -\infty$, approaching a potential very large ($\rightarrow \infty$), but with very small range ($\rightarrow 0$) and integral, in its two body part.

This result, which is not free of unpleasant ambiguities due to the lack of scaling of (2), is bad. In fact, on the one hand, it does not sound unreasonable physically since the δ -function potential is trivial (if $d > 1$); hence it could be interpreted as saying that the system tends to become non interacting on large scale, even though $V^{(h)}$ is not approaching zero. On the other hand the divergence of $V^{(h)}$ is a knell for a quantitative application, because the RG (renormalization group) approach is typically a (resummed) perturbation theory approach and does not accommodate for large potentials (*i.e.* there are no known techniques to deal with them).

We think that a way out of the impasse is to find a better decomposition of the propagator allowing us to see that the fields ψ on scale n are really propagating with a propagator $g^{(n)}$ smoothly depending on $(2^n t p_0, 2^n \vec{x} p_0)$, so that no ambiguity could possibly be met in the RG power counting arguments.

Our proposal is to introduce *quasi particles* fields $\psi_{x,\vec{\omega}}^\pm$ in terms of which the *particle* fields ψ_x^\pm can be written:

$$\psi_x^{\pm(n)} = \int_{|\vec{\omega}|=1} d\vec{\omega} e^{\pm i p_F \vec{\omega} \vec{x}} \psi_{x,\vec{\omega}}^{\pm(n)} \tag{5}$$

where $x = (t, \vec{x})$ and the integral is over the unit sphere so that $p_F \vec{\omega}$ is a Fermi sphere momentum. The quasi particles propagator $g^{(n)}(x, \vec{\omega}; x', \vec{\omega}')$ is chosen so that:

$$\begin{aligned} g^{(n)}(x, \vec{\omega}; x', \vec{\omega}') &= \delta(\vec{\omega} - \vec{\omega}') g^{(n)}(x - x', \vec{\omega}) \\ g^{(n)}(x) &= \int d\vec{\omega} e^{ip_F \vec{\omega} \vec{x}} g^{(n)}(x, \vec{\omega}) \end{aligned} \quad (6)$$

Although there are ∞ -many ways of representing $g^{(n)}(x)$ in the form (6), it is remarkable (and not totally obvious) that, at least if d is odd, there exists one way in which $g^{(n)}(x, \vec{\omega})$ has the property that, if $\xi_n = (2^n p_0 t, 2^n p_0 \vec{x}) \equiv (\tau_n, \vec{\xi}_n)$, for large n :

$$g^{(n)}(x, \vec{\omega}) \sim 2^n C_d p_F^{d-1} p_0 (\tau_n - i \vec{\omega} \vec{\xi}_n) \gamma(\xi_n) \quad (7)$$

with γ exponentially decreasing in $|\xi_n|$, see [BG1], and we choose from now on units in which the Fermi velocity $\beta = p_F/m$ is equal to 1.

The above relation tells us that the quasi particle fields scale, *in all dimensions*, like $2^{n/2}$, *i.e.* have dimension 1/2, and also allows us to set up the familiar description of the effective potential in terms of relevant and irrelevant contributions.

Of course we must regard the initial interaction as an interaction between quasi particle fields rather than between particle fields. All the effective potentials (and, consequently, all the Schwinger functions) will be described in terms of quasi particle fields and, eventually, one will have to perform the appropriate integrations over the quasi particle momenta $\vec{\omega} p_F$ to express the physical observables.

Taking the spin equal to zero, for simplicity, we can write the part of second and fourth degree in the ψ of a typical effective potential as:

$$\begin{aligned} & \int \prod_{j=1}^4 dx_j d\vec{\omega}_j e^{ip_F (\vec{\omega}_1 \vec{x}_1 + \vec{\omega}_2 \vec{x}_2 - \vec{\omega}_3 \vec{x}_3 - \vec{\omega}_4 \vec{x}_4)} \\ & V_1(x_1, \dots, x_4, \vec{\omega}_1, \dots, \vec{\omega}_4) \psi_{x_1 \vec{\omega}_1}^+ \psi_{x_2 \vec{\omega}_2}^+ \psi_{x_3 \vec{\omega}_3}^- \psi_{x_4 \vec{\omega}_4}^- + \\ & + \int \prod_{j=1}^2 dx_j d\vec{\omega}_j e^{ip_F (\vec{\omega}_1 \vec{x}_1 - \vec{\omega}_2 \vec{x}_2)} \\ & \psi_{x_1 \vec{\omega}_1}^+ [V_2(x_1, x_2, \vec{\omega}_1, \vec{\omega}_2) + \vec{V}_3(x_1, x_2, \vec{\omega}_1, \vec{\omega}_2) \vec{D}_{\vec{\omega}_2} + V_4(x_1, x_2, \vec{\omega}_1, \vec{\omega}_2) \partial_t] \psi_{x_2 \vec{\omega}_2}^- \end{aligned} \quad (8)$$

where $\vec{D}_{\vec{\omega}} \equiv \vec{\partial}_{\vec{x}} + i(\vec{\omega}/2p_F) \Delta_{\vec{x}}$.

The functions V_i , which in general will be distributions involving, at worst, derivatives of delta functions, are subject to a rather strong constraint. Namely (8) has to be such that it can be rewritten as a function of $\int d\vec{\omega} e^{ip_F \vec{\omega} \vec{x}} \psi_{x\vec{\omega}}^\pm$, *i.e.* in terms of the particle fields.

Our prescription to identify the relevant terms in (8) is to extract them by a localization operator \mathcal{L} . This is a linear operator whose definition is immediately suggested by the

theory of the scalar fields localization operators. The operator \mathcal{L} annihilates the terms of degree > 4 in the fields. On the parts of degree 2 and 4 it is defined, on the basis of the Wick monomials, by:

$$\begin{aligned} \mathcal{L} : \psi_{x_1\vec{\omega}_1}^+ \psi_{x_2\vec{\omega}_2}^- &::= \psi_{x_1\vec{\omega}_1}^+ (\psi_{x_1\vec{\omega}_2}^- + (x_2 - x_1) \mathcal{D}_{\vec{\omega}_2} \psi_{x_1\vec{\omega}_2}^-) : \\ \mathcal{L} : \psi_{x_1\vec{\omega}_1}^+ \psi_{x_2\vec{\omega}_2}^+ \psi_{x_3\vec{\omega}_3}^- \psi_{x_4\vec{\omega}_4}^- &::= 2^{-1} \sum_{j=1,2} : \psi_{x_j\vec{\omega}_1}^+ \psi_{x_j\vec{\omega}_2}^+ \psi_{x_j\vec{\omega}_3}^- \psi_{x_j\vec{\omega}_4}^- : \end{aligned} \quad (9)$$

where $\mathcal{D}_{\vec{\omega}} = (\partial_t, \vec{\mathcal{D}}_{\vec{\omega}})$.

The above definition (9) constitutes a simplification over the definition of \mathcal{L} in ref. [BG, §11]: it would be made possible by the remark that the sum of all the terms contributing to the beta function and represented by graphs of the type described in a) at the end of [BG, §10] do in fact add up to zero (hence the new definition of \mathcal{L} adopted in §11 of [BG1] to deal with such graphs would not really be necessary). We still do not have a general proof of this property: but, after checking it in a few cases, we decided to use it in this paper to present it as a conjecture. We continue using it as it considerably simplifies the exposition, but what follows would be essentially unchanged if we used the more involved definition of \mathcal{L} in [BG1].

Hence the relevant part of the interaction on scale h will take the form:

$$\begin{aligned} V_L^{(h)} = & \int dx d\vec{\omega}_1 d\vec{\omega}_2 e^{ip_F(\vec{\omega}_1 - \vec{\omega}_2)\vec{x}} : \psi_{x\vec{\omega}_1}^+ [2^h \nu_h + i\alpha_h \beta \vec{\omega}_2 \vec{\mathcal{D}}_{\vec{\omega}_2} + \zeta_h \partial_t] \psi_{x\vec{\omega}_2}^- : + \\ & - \int dx \prod_{i=1}^4 d\vec{\omega}_i e^{ip_F(\vec{\omega}_1 + \vec{\omega}_2 - \vec{\omega}_3 - \vec{\omega}_4)\vec{x}} \lambda_h(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) : \psi_{x\vec{\omega}_1}^+ \psi_{x\vec{\omega}_2}^+ \psi_{x\vec{\omega}_3}^- \psi_{x\vec{\omega}_4}^- : \end{aligned} \quad (10)$$

where the 2^h in front of ν_h is introduced for convenience, as it turns out that $\lambda_h, \alpha_h, \zeta_h$ are *marginal*, while ν_h is relevant (with dimension 1).

For instance, to first order, the coefficients in the relevant part of $V^{(0)}$ are, if $\hat{\lambda}_0$ is the Fourier transform of $\lambda_1(\vec{x})$, see (3):

$$\begin{aligned} \lambda_0(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) = & \frac{1}{4} [\hat{\lambda}_0(p_F(\vec{\omega}_1 - \vec{\omega}_3)) - \hat{\lambda}_0(p_F(\vec{\omega}_2 - \vec{\omega}_3)) - \\ & - \hat{\lambda}_0(p_F(\vec{\omega}_1 - \vec{\omega}_4)) + \hat{\lambda}_0(p_F(\vec{\omega}_2 - \vec{\omega}_4))] \\ \nu_0 = 2\nu_1 \quad \alpha_0 = \alpha_1 \quad \zeta_0 = \zeta_1 \end{aligned} \quad (11)$$

The reason why we start describing $V^{(0)}$ is that we can use (7) only for $n \leq 0$. For $n = 1$ the large \vec{k} 's are important and the problem is an ultraviolet one while (7) does not hold, even qualitatively. In other words it is interesting to introduce the quasi particle fields only to describe the large scale components of the particle fields. The effective potential $V^{(0)}$ has to be constructed by solving the ultraviolet problem of integrating $\exp[-V^{(1)}]$

over $\psi^{(1)}$. The latter problem should be treated by different methods and it can be made rather easy to study by modifying the dispersion relation from $(\vec{k}^2 - p_F^2)/2m$ to a dispersion relation differing from it only for large $|\vec{k}|$ and diverging very fast as $|\vec{k}| \rightarrow \infty$.

In writing (10) by applying the localization operator to (8), one could expect to find that ν_h, α_h, ζ_h are not constant, but rather depend on $\vec{\omega}_1, \vec{\omega}_2$: it is however important to remark that they are constant. In fact we know *a priori* that the initial effective potential, as well as all the effective potentials on lower scales, can be expressed in terms of only the particle fields. This implies not only that ν_h, α_h, ζ_h depend on the $\vec{\omega}$'s in a special way (namely as in (10)), but also that the λ_h depend on the $\vec{\omega}$'s in a special way. In fact one can check that this means that the function λ_h can be expressed in terms of a three argument's function $\Lambda_h(\vec{\omega}, \vec{\omega}', \vec{\omega}'')$ having rotation symmetry and such that $\Lambda_h(\vec{\omega}, \vec{\omega}', \vec{\omega}'') = -\Lambda_h(\vec{\omega}, \vec{\omega}'', \vec{\omega}')$, as:

$$\lambda_h(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) = \Lambda_h(\vec{\omega}_1, \vec{\omega}_3, \vec{\omega}_4) - \Lambda_h(\vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) \quad (12)$$

and $\Lambda_h(\vec{\omega}, \vec{\omega}', -\vec{\omega}') \equiv \lambda_h(\vec{\omega}, -\vec{\omega}, \vec{\omega}', -\vec{\omega}')/2$.

To check that, with the above definitions of relevant terms, the beta function is well defined, *i.e.* to check the consistency of a perturbation approach based on the RG, we simply apply, [BG1], the techniques summarized in [G] to study the beta function. The application is straightforward and one ends up with a recursion relation, for $d > 1$ and, for simplicity, spin 0:

$$\begin{aligned} \lambda_{h-1} &= \lambda_h + B_1^{(\geq 2)}(\alpha_h, \zeta_h, \nu_h, \lambda_h) + 2^{\varepsilon h} B_2^{(\geq 1; \geq 1)}(\alpha_h, \zeta_h, \nu_h; \lambda_h) \\ \nu_{h-1} &= 2\nu_h + 2^{\varepsilon h} B_3^{(\geq 2)}(\alpha_h, \zeta_h, \nu_h, \lambda_h) \\ \alpha_{h-1} &= \alpha_h + \beta'' \nu_h^2 + 2^{\varepsilon h} B_4^{(\geq 2)}(\alpha_h, \zeta_h, \nu_h, \lambda_h) \\ \zeta_{h-1} &= \zeta_h - \beta'' \nu_h^2 + 2^{\varepsilon h} B_5^{(\geq 2)}(\alpha_h, \zeta_h, \nu_h, \lambda_h) \end{aligned} \quad (13)$$

where $B_j^{(\geq p; \geq q)}(x; y)$ denotes a formal power series in x, y in which the x variables appear to order p or higher, and the y appear to order q or higher; $B_j^{\geq p}(x)$ denotes, likewise, a formal power series in which the x variables appear to order $\geq p$. Here $\varepsilon > 0$ can be taken any number $< 1/4$.

The above result is formally derived in [BG1] together with explicit bounds on the n -th order coefficients of the formal series: the bounds are h independent and grow like $n!$; similar ideas are used in [FT] to study the perturbation theory of the Schwinger functions without dealing with the theory of the beta function. The above form of the r.h.s. is far from the *a priori* most general: in fact the special nature of the r.h.s. reflects a large number of cancellations which in a sense are the main technical novelty of the problem (which is, otherwise, a rather easy modification of the analogous field theory problem reviewed in [G]). Such cancellations are essential also in the analysis of the Schwinger functions of the theory, which are directly related to the structure of the Fermi surface, [BG1].

There are two problems with (13). Probably the deepest is that, even though we are dealing with a Fermi system, we cannot eliminate the $n!$ in the estimates of the n -th order coefficients of the series for the B functions in (13). It follows, by the methods used in [G] to study the beta function in the scalar field theories, that the n -th order coefficients of the above series are bounded by $DC^{(n-1)}(n-1)!$ for some C, D . One could hope for better results. For instance, if $d = 1$ (hence $\vec{\omega}$ takes only the two values ± 1), the power series for the B_j probably do converge. This seems within the range of the presently known techniques, because one should notice that the problem is very close, if $d = 1$, to the problem of the theory of the beta function in the Gross-Neveu model (see [GK]: it seems to us that this paper implies that the beta function is convergent; unfortunately in the quoted reference the proof of the convergence of the beta function is not directly required so that a formal proof of the result that we are proposing is strictly speaking not yet available).

Coming back to $d > 1$, we have illustrated only one of the two problems that arise in this case. The second problem is simply that the recursive relation (13) is not easy to study even if one is willing to forget about the higher order terms and about the convergence question. It seems of interest, therefore, to analyze the problem of studying the structure of the recursion for λ_h obtained by neglecting the higher order terms as well as all the terms involving the other running constants.

The recursion relation has, for a suitable kernel B , the general form:

$$\lambda_{h-1}(\vec{\omega}_1, \dots) = \lambda_h(\vec{\omega}_1, \dots) + \int B(\vec{\omega}_1, \dots; \vec{\omega}'_1, \dots; \vec{\omega}''_1, \dots) \lambda_h(\vec{\omega}'_1, \dots) \lambda_h(\vec{\omega}''_1, \dots) \quad (14)$$

+ remainder consisting in higher order or mixed terms

We want to see if, at least to second order, one can find a solution which, starting from an initial λ_0 , evolves so that λ_h stays bounded as $h \rightarrow -\infty$: as we have discussed in [BG, §12], this is essential for the existence of a normal Fermi surface and of a normal Fermi liquid. If $d = 1$ and the spin is 0, the second order terms which come from the contributions of two Feynman graphs, the *direct* and the *exchange* graph:

$$\quad (15)$$

cancel exactly so that, to second order and neglecting mixed terms, $\lambda_{h-1} = \lambda_h$. A more detailed analysis of the one dimensional spinless model shows that, in fact, a similar result is valid at any order and one expects that $\lambda_h \rightarrow \lambda_\infty(\lambda_0)$ as $h \rightarrow -\infty$, [BG1].

But the $d > 1$ case is different. In this case λ_h is not a number and the two contributions do not cancel each other. Starting with λ_0 given by (11), we inquire whether it is possible to make a consistent expansion in which the running couplings evolve to zero, if one uses (14) truncated to second order.

If one assumes that λ_h stays bounded as $h \rightarrow -\infty$, then one can check that the parts B_d and B_e , which add up to B in (14) and come from the direct and the exchange graphs, behave differently. Namely, if the λ_h are uniformly bounded, $B_e(\lambda_h) \rightarrow 0$ as $h \rightarrow -\infty$ at fixed $\vec{\omega}$'s and $\sum_{h=-\infty}^0 |B_e \lambda_h| \leq C$ with an $\vec{\omega}$ -independent C : in some sense $B_e(\lambda_h)$ vanishes as $h \rightarrow -\infty$. Similarly the part containing mixed terms in λ_h and the other running couplings also approaches zero, at fixed $\vec{\omega}$, if $\alpha_h, \zeta_h, \nu_h, \lambda_h$ stay bounded.

Write $\lambda_h = P_C \lambda_h + (1 - P_C) \lambda_h$, where:

$$P_C \lambda(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) = \frac{1}{4} [\tilde{\lambda}(\vec{\omega}_1; \vec{\omega}_3) - \tilde{\lambda}(\vec{\omega}_1; \vec{\omega}_4) - \tilde{\lambda}(\vec{\omega}_2; \vec{\omega}_3) + \tilde{\lambda}(\vec{\omega}_2; \vec{\omega}_4)] \quad (16)$$

$$\tilde{\lambda}(\vec{\omega}_1; \vec{\omega}_3) = \lambda(\vec{\omega}_1, -\vec{\omega}_1, \vec{\omega}_3, -\vec{\omega}_3)$$

and suppose, in addition, that we try to argue that the relevant term selected in (9) actually is such that only the part expressed with the kernel $P_C \lambda_h$ is really relevant: while the part expressed with the kernel $(1 - P_C) \lambda_h$ can be considered irrelevant.

Then one has to work out again the beta function theory with the newly defined notion of relevant terms: one can check that the new beta function is, at second order, simply related to the previous one, and can be finite to all orders if $\lambda_h(\vec{\omega}_1; \vec{\omega}_3)$ becomes short ranged in the $\vec{\omega}$'s as $h \rightarrow -\infty$, for instance if:

$$\tilde{\lambda}_h(\vec{\omega}_1, \vec{\omega}_3) = f_h(2^{-h} |\vec{\omega}_1 - \vec{\omega}_3|) - f_h(2^{-h} |\vec{\omega}_1 + \vec{\omega}_3|) \quad (17)$$

with f_h smooth and constant at ∞ . This is a strong requirement and one would have to check that it is in turn a property of the λ_h evolving under the evolution defined by the beta function, or by a finite truncation of it, like (14) truncated to second order.

The above definition of relevant terms is very appealing as it would say that the relevant part of the effective potential is expressed by the function $\tilde{\lambda}_h(\vec{\omega}_1; \vec{\omega}_3)$, which can be regarded as the *interaction potential* between two *Cooper pairs* of quasi particles (the first with quasi momenta $\pm \vec{\omega}_1 p_F$ and the second with quasi momenta $\pm \vec{\omega}_3 p_F$), see also [A].

We have been able to perform only some simple consistency check, of the above hypothesis, that can be described as follows.

Imagine replacing (14), regarded as a map of a function $\lambda \equiv \lambda_h$ into $\lambda' = \lambda_{h-1}$, with the relation obtained in the limit $h \rightarrow -\infty$ by neglecting the remainder.

One finds, using the computations in [BG, §13], that the exchange graph part of B does not contribute and, in the limit $h \rightarrow -\infty$, the direct graph contribution can be written:

$$\lambda'(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) = \lambda(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) - \beta \int d\vec{\omega} d\vec{\omega}' \quad (18)$$

$$\delta^{(2)}(\vec{\omega}_1 + \vec{\omega}_2 - \vec{\omega} - \vec{\omega}') \lambda(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}, \vec{\omega}') \lambda(\vec{\omega}, \vec{\omega}', \vec{\omega}_3, \vec{\omega}_4)$$

where $\beta > 0$ and $\delta^{(2)}(\vec{\omega}_1 + \vec{\omega}_2 - \vec{\omega} - \vec{\omega}')$ is a distribution, parameterized by $\vec{\omega}_1 + \vec{\omega}_2$, defined by:

$$\delta^{(2)}(\vec{\omega}_1 + \vec{\omega}_2 - \vec{\omega} - \vec{\omega}') = \lim_{n \rightarrow -\infty} 2^{-2h} \gamma(2^{-h} (\vec{\omega}_1 + \vec{\omega}_2 - \vec{\omega} - \vec{\omega}')) \quad (19)$$

with γ being a fixed function on R^3 fastly decaying at ∞ , or:

$$\int f(\vec{\omega}, \vec{\omega}') \delta^{(2)}(\vec{\omega}_1 + \vec{\omega}_2 - \vec{\omega} - \vec{\omega}') d\vec{\omega} d\vec{\omega}' = \delta_{\vec{\omega}_1 + \vec{\omega}_2} \int f(\vec{\omega}, -\vec{\omega}) d\vec{\omega} \quad (20)$$

where $\delta_{\vec{\omega}_1 + \vec{\omega}_2} = 1$ if $\vec{\omega}_1 + \vec{\omega}_2 = \vec{0}$ and $\delta_{\vec{\omega}_1 + \vec{\omega}_2} = 0$ otherwise.

Hence we see that only if $\vec{\omega}_1 + \vec{\omega}_2 = \vec{0}$ the evolution of λ_h is non trivial (in the above " $h = -\infty$ " approximation); and (18) yields a closed equation for $\tilde{\lambda}_h(\vec{\omega}_1; \vec{\omega}_3) = \lambda_h(\vec{\omega}_1, -\vec{\omega}_1, \vec{\omega}_3, -\vec{\omega}_3)$. Note that the (18) also implies that the knowledge for all h of $\tilde{\lambda}_h(\vec{\omega}_1; \vec{\omega}_3)$ allows us to evaluate a closed equation for $\lambda_h(\vec{\omega}_1, -\vec{\omega}_1, \vec{\omega}_3, \vec{\omega}_4)$.

Hence we are led to conjecture that λ_h evolves by staying essentially constant provided $\lambda_h(\vec{\omega}_1, -\vec{\omega}_1, \vec{\omega}_3, -\vec{\omega}_3)$, *i.e.* the interaction between pairs of Cooper pairs does not blow up. The latter seems in fact the only source of instability in (14).

Furthermore the (18), regarded as a recursion relation determining the Cooper pairs interaction $\lambda_{h-1}(\vec{\omega}_1; \vec{\omega}_3)$ in terms of the $\lambda_h(\vec{\omega}_1; \vec{\omega}_3)$ is *diagonal* in the angular momentum representation of the Cooper pairs interaction:

$$\tilde{\lambda}_h(\vec{\omega}_1; \vec{\omega}_3) = \sum_{\substack{l=1 \\ l \text{ odd}}}^{\infty} (2l+1) \lambda_h(l) P_l(\vec{\omega}_1 \cdot \vec{\omega}_3) \quad (21)$$

where P_l are the Legendre polynomials and one can check, also, that $\lambda_0(l) \geq 0$, see [BG1], if the initial interaction potential is ≥ 0 , *i.e.* is repulsive.

In fact it is easy to check, see [BG1], that the recursion (18) for the $\lambda_h(l)$ is:

$$\lambda_{h-1}(l) = \lambda_h(l) - \beta \lambda_h(l)^2 \quad (22)$$

with the same $\beta > 0$, for all h, l : which seems a rather remarkable recursion.

Hence we see that $\lambda_h(l) \rightarrow 0$ as $h \rightarrow -\infty$ for every angular momentum l , if the interaction is repulsive (*i.e.* $\lambda_h(l) \geq 0$) and the basic picture that one should try to prove is as follows.

Assuming that $\lambda_h(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4)$ stays bounded uniformly and smooth on scale 2^h in the $\vec{\omega}$ -variables, it becomes constant as $h \rightarrow -\infty$ at fixed $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4$, if $\vec{\omega}_1 + \vec{\omega}_2 \neq \vec{0}$.

If $|\vec{\omega}_1 + \vec{\omega}_2| \simeq 2^{h_0}$ then $\lambda_h(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4)$ becomes constant for $h \ll h_0$, while for $h \gg h_0$ it evolves as $\lambda_h(\vec{\omega}_1, -\vec{\omega}_1, \vec{\omega}_3, \vec{\omega}_4)$. The latter evolves according to the recursion relation:

$$\lambda'(\vec{\omega}_1, -\vec{\omega}_1, \vec{\omega}_3, \vec{\omega}_4) = \lambda(\vec{\omega}_1, -\vec{\omega}_1, \vec{\omega}_3, \vec{\omega}_4) - \beta \int d\vec{\omega} \tilde{\lambda}(\vec{\omega}_1; \vec{\omega}) \lambda(\vec{\omega}, -\vec{\omega}, \vec{\omega}_3, \vec{\omega}_4) \quad (23)$$

Therefore the whole flow of the running couplings is controlled by the function $\tilde{\lambda}_h(\vec{\omega}; \vec{\omega}')$, which evolves according to (18) as:

$$\tilde{\lambda}'(\vec{\omega}_1; \vec{\omega}_3) = \tilde{\lambda}(\vec{\omega}_1; \vec{\omega}_3) - \beta \int \tilde{\lambda}(\vec{\omega}_1; \vec{\omega}) \tilde{\lambda}(\vec{\omega}; \vec{\omega}_3) d\vec{\omega} \quad (24)$$

which can be written in the angular momentum representation (21),(22).

So we have some chance for hoping that:

- 1) the (14) truncated to second order implies that $\lambda_h(\vec{\omega}_1, -\vec{\omega}_1, \vec{\omega}_3, -\vec{\omega}_3) \rightarrow 0$ as $h \rightarrow -\infty$, at least if the interaction is repulsive, and, as a consequence of the linear evolution (23), also $\lambda_h(\vec{\omega}_1, -\vec{\omega}_1, \vec{\omega}_3, \vec{\omega}_4) \rightarrow 0$.
- 2) the relevant part of the interaction is just the part due to the Cooper pairs;
- 3) the higher orders do not change the picture.

It is unfortunately hard, it seems, to show that (14) truncated to second order admits a bounded solution, when one tries to study it rigorously without the above discussed Cooper pairs approximation. The remarks on what becomes (14), once one makes the approximation $\lambda_h \sim P_C \lambda_h$ and one neglects the exchange graph part of B , is only a hint at some structural properties that have yet to be understood.

If $d = 1$ and the spin is zero, the fact that $\lambda_h \sim P_C \lambda_h$ is not an assumption but is a consequence of the exclusion principle. The situation looks therefore much better. But in this case the beta function is different and the cancellations that take place in $d > 1$ no longer occur. Although the problem can be studied in detail, [BG1], and one can reach a much more satisfactory set of conclusions, it turns out that the theory of the running constants flow is deeply different (and the Fermi surface is generically anomalous, but insensitive to the attractivity or repulsivity of the interaction); for this reason we do not discuss it here, see also [BG2].

Finally one can say that the approximation in which the relevant coupling between the quasi particles is $\lambda_h(\vec{\omega}; \vec{\omega}') \equiv \lambda_h(\vec{\omega}, -\vec{\omega}, \vec{\omega}', -\vec{\omega}')$ is in some sense an extension of the ideas behind the formulation of the Luttinger model to dimensions higher than 1, see also [A].

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