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Smooth Prime Integrals for Quasi-Integrable Hamiltonian Systems.

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Summary. — A Hamiltonian with N degrees of freedom, analytic perturbation of a canonically integrable strictly nonisochronous analytic Hamiltonian, is considered. We show the existence of N functions on phase space and of class C^∞ which are prime integrals for the perturbed motions on a suitable region whose Lebesgue measure tends to fill locally the phase space as the perturbation's magnitude approaches zero. An application to the perturbations of isochronous nonresonant linear oscillators is given.

1. – Introduction.

Although it follows from the small-denominator theorem proof that the small perturbations of integrable Hamiltonian systems are not ergodic for the Liouville measure and that, therefore, there must be some prime integrals for their motions, the prime integrals' properties have not been investigated in detail.

Here we show that the small-denominator theory contains all information needed to deal with the above questions, at least as far as existence and basic regularity are concerned.

We shall only consider analytic Hamiltonian systems integrable by analytic canonical transformations.

Calling $\boldsymbol{A} \in R^N$ the N action variables and $\boldsymbol{\varphi} \in T^N$ the N conjugate angles (*), we suppose that the unperturbed Hamiltonian h_0 and the perturbation f_0 are

(*) T^N = standard torus in N dimensions = $\{[0, 2\pi]^N$ with opposite sides identified$\}$.

defined on a set of the form $V \times T^N$, $V \subset R^N$ open sphere with radius $r > 0$.

Thus the Hamiltonian will be written as

$$H_0(\boldsymbol{A}, \boldsymbol{\varphi}) = h_0(\boldsymbol{A}) + f_0(\boldsymbol{A}, \boldsymbol{\varphi}) . \tag{1.1}$$

We assume h_0 and f_0 analytic: more precisely, if T^N is *identified* with a subset of C^N via the map

$$\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_N) \leftrightarrow \boldsymbol{z} = (z_1, \dots, z_N) = \big(\exp[i\varphi_1], \dots, \exp[i\varphi_N]\big)$$

and if V is also regarded as a subset of C^N, we suppose that, if we put

$$C(\varrho_0, \xi_0; \boldsymbol{A}) = \big\{(\boldsymbol{A}', \boldsymbol{z}) | (\boldsymbol{A}', \boldsymbol{z}) \in C^{2N},\ |A_i' - A_i| < \varrho_0, \\ \exp[-\xi_0] < |z_i| < \exp[\xi_0],\ i = 1, 2, \dots, N\big\}, \tag{1.2}$$

$$W(\varrho_0, \xi_0; V) = \bigcup_{\boldsymbol{A} \in V} C(\varrho_0, \xi_0; \boldsymbol{A}) \supset V \times T^N, \tag{1.3}$$

then the functions h_0, f_0 regarded as functions on $W(\varrho_0, \xi_0, V) \cap V \times T^N$ extend to functions holomorphic in $W(\varrho_0, \xi_0, V)$ which will be denoted with the *same* symbols.

We can extend the differentiation with respect to $\boldsymbol{\varphi}$ in a natural way by setting

$$\frac{\partial}{\partial \varphi_k} \equiv - i z_k \frac{\partial}{\partial z_k}, \qquad k = 1, \dots, N.$$

We denote

$$\frac{\partial}{\partial \boldsymbol{A}} = \left(\frac{\partial}{\partial A_1}, \dots, \frac{\partial}{\partial A_N}\right), \qquad \frac{\partial}{\partial \boldsymbol{\varphi}} = \left(\frac{\partial}{\partial \varphi_1}, \dots, \frac{\partial}{\partial \varphi_N}\right).$$

We shall define, for $\boldsymbol{\zeta} \in C^N$, $|\boldsymbol{\zeta}| = \sum_{i=1}^N |\zeta_i|$ and, for $M = N \times N$ matrix, $|M| = \sum_{i,j=1}^N |M_{ij}|$. Let

$$\boldsymbol{\omega}_0(\boldsymbol{A}) = \frac{\partial h_0}{\partial \boldsymbol{A}}(\boldsymbol{A}), \tag{1.4}$$

$$E_0 \geqslant \sup |\boldsymbol{\omega}_0(\boldsymbol{A})|, \tag{1.5}$$

$$\varepsilon_0 \geqslant \sup \left|\frac{\partial f_0}{\partial \boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{z})\right| + \varrho_0^{-1} \left|\frac{\partial f_0}{\partial \boldsymbol{\varphi}}(\boldsymbol{A}, \boldsymbol{z})\right|, \tag{1.6}$$

$$\eta_0 \geqslant \sup \left|\left(\frac{\partial \boldsymbol{\omega}_0}{\partial \boldsymbol{A}}(\boldsymbol{A})\right)^{-1}\right|, \tag{1.7}$$

where $\partial\boldsymbol{\omega}_0/\partial\boldsymbol{A}$ is the Jacobian matrix of $\boldsymbol{\omega}_0$ (see (1.4)) and $(\boldsymbol{A}, \boldsymbol{z}) \in W(\varrho_0, \xi_0; V)$.

We say that the unperturbed system with Hamiltonian h_0 is strictly anisochronous if $\eta_0 < +\infty$.

For simplicity, we take $\varrho_0 < r$, $\xi_0 < 1$.

Our result is

Theorem. For fixed N, there exist $B > 1$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\varkappa > 0$ such that, for all $C_0 > E_0^{-1}$ verifying

$$B\varepsilon_0 C_0 (E_0 C_0)^{\alpha} (\eta_0 \varrho_0^{-1} E_0)^{\beta} \xi_0^{-\gamma} < 1\,, \tag{1.8}$$

it is possible to construct N functions $A_1', A_2', \ldots, A_N'$ and a subset $\Gamma \subset V \times T^N$ such that

(1.9) i) Volume $\bar{\Gamma} \geqslant \left(1 - \varkappa \left[\dfrac{(\eta_0 E_0 \varrho_0^{-1})^{2N}}{\sqrt[2N]{E_0 C_0}}\right]\right)$ volume $V \times T^N$.

ii) The $A_1', \ldots, A_N'$ are prime integrals for the perturbed motions starting in $\bar{\Gamma}$.

iii) The $\boldsymbol{A}$'s are «independent» on $\bar{\Gamma}$, *i.e.* their Jacobian determinant with respect to the $\boldsymbol{A}$ variables is not zero (actually equal to 1); furthermore, they are in involution on $\bar{\Gamma}$.

iv) Any other function $\tilde{\boldsymbol{A}} \in C^{\infty}(V \times T^N)$ which is a prime integral on $\bar{\Gamma}$ is, on $\bar{\Gamma}$, a function of $A_1', \ldots, A_N'$.

Our proof is based on the version of the Kolmogorov-Arnold-Moser theorem (1) given in (2), although it is in principle self-contained and the KAM theorem is a corollary of the following proof.

After completing this work, we received a preprint (3) in which essentially the same results are proved in the differentiable case: they do not imply immediately our results in the analytic case and it seemed to us worth publishing our proof which might help the readers to compare the analytic case with the differentiable case.

The proof of the theorem's last statement is only sketched.

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(1) V. ARNOLD: *Russ. Math. Surv.*, **18**, 85 (1963); J. MOSER: *Stable and random motions in dynamical systems*, in *Hermann Weyl Lectures* (Princeton, N. J., and Tokyo, 1973).
(2) G. GALLAVOTTI: *Meccanica elementare* (Torino, 1980).
(3) J. PÖSCHEL: *Über differenzierbare Faserungen invarianter Tori*, preprint ETH, Zürich (1981),.

2. – Proof.

Denote $z^{\nu} = \prod_{i=1}^{N} z_i^{\nu_i}$ for $z \in C^N$, $\nu = (\nu_1, \ldots, \nu_N) \in Z^N$ and let $|\nu| = \sum_{i=1}^{N} |\nu_i|$. Moreover, let $\exp[z] = (\exp[z_1], \ldots, \exp[z_N])$.

We consider the expansion

$$f_0(A, z) = \sum_{\nu \in Z^N} f_{0\nu}(A) z^{\nu} \tag{2.1}$$

and, by the assumed analiticity and Laurent's theorem, we infer from (1.6)

$$|\nu| |f_{0\nu}(A)| \leqslant \varepsilon_0 \varrho_0 \exp[-\xi_0|\nu|], \qquad \left|\frac{\partial f_{0\nu}}{\partial A}(A)\right| \leqslant \varepsilon_0 \exp[-\xi_0|\nu|]. \tag{2.2}$$

Proceeding as in the small-denominator theorem's proof (2), p. 444, we fix a sequence of « analyticity loss » parameters, quite arbitrarily, $\delta_j = \xi_0/16(1+j)^2$, $j = 0, 1, \ldots$: $4\sum_{j=0}^{\infty} \delta_j < \xi_0$:

We try now to remove the perturbation by a canonical transformation defined by a generating function Φ_0. Following perturbation theory (2), p. 426, we take

$$\Phi_0(A', z) = \sum_{0<|\nu|\leqslant N_0} f_{0\nu}(A') \frac{z^{\nu}}{-i\omega_0(A')\cdot\nu}, \tag{2.3}$$

which makes sense only if the denominator does not vanish: this is imposed by thinking

$$(A', z) \in W(\tilde{\varrho}_0, \xi_0; V_{C_0}^{(0)}) = \bigcup_{A \in V_{C_0}^{(0)}} C(\tilde{\varrho}_0, \xi_0; A),$$

where $\tilde{\varrho}_0$ is chosen small enough, *i.e.* for a suitably chosen $B_1 > 1$, as

$$\tilde{\varrho}_0 = \tfrac{1}{2}\varrho_0(B_1 C_0 E_0 N_0^{N+1})^{-1}, \qquad N_0 = 2\delta_0^{-1}\log(C_0 \varepsilon_0 \delta_0^N)^{-1} \tag{2.4}$$

and, if $S(A, \varrho)$ = open sphere in R^N with centre A and radius ϱ, $V_{C_0}^{(0)}$ is conveniently defined, for reasons which will appear clear, as union of spheres:

$$V_{C_0}^{(0)} = \bigcup_{A \in \bar{V}_{C_0}^{(0)}} S\left(A, \frac{\tilde{\varrho}_0}{2}\right), \tag{2.5}$$

and the set $\bar{V}_{C_0}^{(0)}$ is taken to be a « nonresonant » set

$$\bar{V}_{C_0}^{(0)} = \{A | A \in V(\varrho_0, -\tilde{\varrho}_0),\ |\omega_0(A)\cdot\nu|^{-1} \leqslant C_0|\nu|^N,\ 0 < |\nu| \leqslant N_0\} \tag{2.6}$$

and the set $V(\varrho_0, -\tilde{\varrho}_0)$ is constructed via the following strange-looking operation: consider all the points $\boldsymbol{A} \in V$ such that $S(\boldsymbol{A}, \varrho_0) \subset V$ and take the union of the spheres $S(\boldsymbol{A}, \varrho_0 - \tilde{\varrho}_0)$: clearly, if V is a sphere (as supposed) and $\tilde{\varrho}_0 < \varrho_0 <$ $<$ radius of V, one simply obtains all the points at a distance larger than $\tilde{\varrho}_0$ from ∂V. However, this construction makes sense for any open set V and we shall use it many times to build new sets out of other sets.

The above choice of $\tilde{\varrho}_0$ stems from the requirement that, $\forall \boldsymbol{A}' \in \pi_1 W(\tilde{\varrho}_0, \xi_0; V^{(0)}_{C_0})$ (π_1 being the «projection over the action variables» $\pi_1(\boldsymbol{A}, \boldsymbol{z}) = \boldsymbol{A}$, $\pi_2(\boldsymbol{A}, \boldsymbol{z}) = \boldsymbol{z}$) the denominator in (2.3) is «nonresonant» (see also ([2]), p. 446):

$$|\boldsymbol{\omega}_0(\boldsymbol{A}')\cdot\boldsymbol{\nu}|^{-1} \leqslant 2C_0|\boldsymbol{\nu}|^N, \qquad \forall 0 < |\boldsymbol{\nu}| \leqslant N_0, \tag{2.7}$$

imposed via the obvious estimates based on (2.2), while the definition of N_0 has been made so that the «ultraviolet part» of the perturbation, defined as

$$f_0^{[>N_0]}(\boldsymbol{A}, \boldsymbol{z}) = \sum_{|\boldsymbol{\nu}|>N_0} f_{0\boldsymbol{\nu}}(\boldsymbol{A})\,\boldsymbol{z}^{\boldsymbol{\nu}}, \tag{2.8}$$

be such that, $\forall(\boldsymbol{A}, \boldsymbol{z}) \in W(\varrho_0, \xi_0 - \delta_0; V)$,

$$\left|\frac{\partial f_0^{[>N_0]}}{\partial \boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{z})\right| + \frac{1}{\varrho_0}\left|\frac{\partial f_0^{[>N_0]}}{\partial \boldsymbol{\varphi}}(\boldsymbol{A}, \boldsymbol{z})\right| \leqslant B_2 \varepsilon_0^2 C_0 \tag{2.9}$$

for some $B_2 > 1$, obtained in the obvious way via (2.2), (2.1).

By (2.2), (2.3) it easily follows a bound for Φ_0 on the set $W(\tilde{\varrho}_0, \xi_0 - \delta_0; V^{(0)}_{C_0})$ on which Φ_0 turns out to be holomorphic:

$$\sup\left(\left|\frac{\partial \Phi_0}{\partial \boldsymbol{A}'}\right| + \frac{1}{\varrho_0}\left|\frac{\partial \Phi_0}{\partial \boldsymbol{\varphi}}\right|\right) \leqslant B_3 \varepsilon_0 C_0 (E_0 C_0)\, \delta_0^{-x_1}, \tag{2.10}$$

where $B_3 > 1$, $x_1 > 1$ are suitable constants: in the derivation of (2.10), (2.7) plays a crucial role too.

Thus we can try to put

$$\begin{cases} A_j = A'_j + \dfrac{\partial \Phi_0}{\partial \varphi_j}(\boldsymbol{A}'\boldsymbol{z}), \\[2mm] z'_j = z_j \exp\left[i\,\dfrac{\partial \Phi_0}{\partial A'_j}(\boldsymbol{A}', \boldsymbol{z})\right], \end{cases} \qquad j = 1, \ldots, N, \tag{2.11}$$

and use this map of $W(\tilde{\varrho}_0, \xi_0 - \delta_0; V^{(0)}_{C_0})$ into C^{2N} to generate a canonical transformation $\mathscr{C}^{(0)}$ and its inverse.

We have to invert the second of (2.11) with respect to $\boldsymbol{z}'$ or the first with respect to $\boldsymbol{A}$.

Essentially one has to use some implicit functions' theorem. The applicability condition is, of course, that $\partial\Phi_0/\partial A'_j$ has very small $\boldsymbol{\varphi}$-derivative (to invert the second) or that $\partial\Phi_0/\partial\varphi_j$ has very small $\boldsymbol{A}'$-derivative (to invert the first).

Actually we wish to invert the second of (2.11) in the nice form

$$z_j = z'_j \exp\left[i\Delta_j(\boldsymbol{A}', \boldsymbol{z}')\right] \tag{2.12}$$

with $\boldsymbol{\Delta}$ analytic enough: to do this we must naturally give up some analyticity in $\boldsymbol{z}'$ trying to define $\boldsymbol{\Delta}$ only in a smaller set, say $W(\tilde{\varrho}_0, \xi_0 - 2\delta_0;\, V^{(0)}_{C_0})$, compared to the analyticity region $W(\tilde{\varrho}_0, \xi_0 - \delta_0;\, V^{(0)}_{C_0})$ for $\partial\Phi_0/\partial\boldsymbol{A}'$.

Similarly we wish to invert the first of eqs. (2.11) in the nice form

$$\boldsymbol{A}' = \boldsymbol{A} + \Xi'(\boldsymbol{A}, \boldsymbol{z}) \tag{2.13}$$

with Ξ' analytic: again one must renounce to some analyticity in $\boldsymbol{A}$ trying to define Ξ' in a smaller region, *e.g.* $W(\tilde{\varrho}_0/2, \xi_0 - \delta_0;\, V^{(0)}_{C_0})$.

For instance, if $\boldsymbol{\Delta}$ and Ξ' are required to exist and to be holomorphic in the above-mentioned regions, the sufficient condition for this to happen can be derived by some standard implicit-function theorems (see, for instance, ([2]), p. 437, proposition XX); they have the form (see also (2.10))

$$B_4(B_3\,\varepsilon_0\,C_0\,E_0\,C_0\,\delta_0^{-x_1})\,\delta_0^{-x_2} < 1 \tag{2.14}$$

to define $\boldsymbol{\Delta}$ or

$$B_4(B_3\varepsilon_0\,C_0E_0C_0\,\delta_0^{-x_1}\varrho_0)(\tilde{\varrho}_0/2)^{-1}\,2^{x_2} < 1 \tag{2.15}$$

to define Ξ', where $B_4 > 4$, $x_2 \geqslant 1$ are constants depending on the particular implicit-function theorem used. It should be noted that the above conditions have a « dimensional interpretation » and they can immediately be guessed (*).

Under assumptions (2.14), (2.15) the functions $\boldsymbol{\Delta}$, Ξ' in (2.12), (2.13) also verify the bounds (see (2.10), (2.11)),

$$\begin{cases} |\boldsymbol{\Delta}| \leqslant B_3\,\varepsilon_0\,C_0\,E_0\,C_0\,\delta_0^{-x_1} < \delta_0\,, \\ |\Xi'| \leqslant B_3\,\varepsilon_0\,C_0\,E_0\,C_0\,\delta_0^{-x_1}\,\varrho_0 < \tilde{\varrho}_0/8 \end{cases} \tag{2.16}$$

in their analyticity domains $W(\tilde{\varrho}_0, \xi_0 - 2\delta_0;\, V^{(0)}_{C_0})$ and $W(\tilde{\varrho}_0/2, \xi_0 - \delta_0;\, V^{(0)}_{C_0})$, respectively.

(*) A dimensional estimate, as the physics nomenclature wishes, is basically a bound on the derivative of a holomorphic function by its maximum divided by the distance to the definition domain boundary.

This allows us to define, for $(\boldsymbol{A}', \boldsymbol{z}') \in W(\tilde{\varrho}_0, \xi_0 - 2\delta_0;\ V^{(0)}_{C_0})$,

$$\Xi(\boldsymbol{A}', \boldsymbol{z}') = \frac{\partial \Phi_0}{\partial \boldsymbol{\varphi}}(\boldsymbol{A}', \boldsymbol{z}' \exp[i\boldsymbol{\Delta}(\boldsymbol{A}', \boldsymbol{z}')]) \tag{2.17}$$

and, for $(\boldsymbol{A}, \boldsymbol{z}) \in W(\tilde{\varrho}_0/2, \xi_0 - \delta_0;\ V^{(0)}_{C_0})$,

$$\boldsymbol{\Delta}'(\boldsymbol{A}, \boldsymbol{z}) = \frac{\partial \Phi_0}{\partial \boldsymbol{A}'}(\boldsymbol{A} + \Xi'(\boldsymbol{A}, \boldsymbol{z}), \boldsymbol{z})\,. \tag{2.18}$$

In this way, we can consider the map $\mathscr{C}^{(0)}$

$$\begin{cases} \boldsymbol{A} = \boldsymbol{A}' + \Xi'(\boldsymbol{A}', \boldsymbol{z}')\,, \\ \boldsymbol{z} = \boldsymbol{z}' \exp[i\boldsymbol{\Delta}(\boldsymbol{A}', \boldsymbol{z}')]\,, \end{cases} \qquad (\boldsymbol{A}', \boldsymbol{z}') \in W(\tilde{\varrho}_0, \xi_0 - 2\delta_0;\ V^{(0)}_{C_0}), \tag{2.19}$$

and $\tilde{\mathscr{C}}^{(0)}$

$$\begin{cases} \boldsymbol{A}' = \boldsymbol{A} + \Xi'(\boldsymbol{A}, \boldsymbol{z})\,, \\ \boldsymbol{z}' = \boldsymbol{z} \exp[i\boldsymbol{\Delta}'(\boldsymbol{A}, \boldsymbol{z})]\,, \end{cases} \qquad (\boldsymbol{A}, \boldsymbol{z}) \in W\left(\frac{\tilde{\varrho}_0}{2}, \xi_0 - \delta_0;\ V^{(0)}_{C_0}\right), \tag{2.20}$$

and Ξ, Ξ', $\boldsymbol{\Delta}$, $\boldsymbol{\Delta}'$ verify in their domains of definition and holomorphy the bounds

$$\begin{cases} |\Xi|,\ |\Xi'| \leqslant B_3 \varepsilon_0 C_0 E_0 C_0 \delta_0^{-\alpha_1} \varrho_0 < \tilde{\varrho}_0/8\,, \\ |\boldsymbol{\Delta}|,\ |\boldsymbol{\Delta}'| \leqslant B_3 \varepsilon_0 C_0 E_0 C_0 \delta_0^{-\alpha_1} < \delta_0\,, \end{cases} \tag{2.21}$$

which imply

$$\mathscr{C}^{(0)} W\left(\frac{\tilde{\varrho}_0}{4}, \xi_0 - 3\delta_0;\ V^{(0)}_{C_0}\right) \subset W\left(\frac{\tilde{\varrho}_0}{2}, \xi_0 - 2\delta_0;\ V^{(0)}_{C_0}\right), \tag{2.22}$$

$$\tilde{\mathscr{C}}^{(0)} W\left(\frac{\tilde{\varrho}_0}{4}, \xi_0 - 3\delta_0;\ V^{(0)}_{C_0}\right) \subset W\left(\frac{\tilde{\varrho}_0}{2}, \xi_0 - 2\delta_0,\ V^{(0)}_{C_0}\right), \tag{2.23}$$

and, by construction,

$$\mathscr{C}^{(0)}\tilde{\mathscr{C}}^{(0)} = \tilde{\mathscr{C}}^{(0)}\mathscr{C}^{(0)} = \text{identity on } W\left(\frac{\tilde{\varrho}_0}{4}, \xi_0 - 3\delta_0;\ V^{(0)}_{C_0}\right). \tag{2.24}$$

It is also easy to see that $\boldsymbol{\Delta}$, $\boldsymbol{\Delta}'$, Ξ, Ξ' are real for $(\boldsymbol{A}, \boldsymbol{z})$ or $(\boldsymbol{A}', \boldsymbol{z}')$ in $R^N \times T^N$.

It follows from the general theory of the canonical transformations that $\mathscr{C}^{(0)}$ and $\tilde{\mathscr{C}}^{(0)}$ are completely canonical maps of $V^{(0)}_{C_0} \times T^N$ onto their images (and, therefore, their Jacobian determinant in the $(\boldsymbol{A}, \boldsymbol{\varphi})$ variables must be 1).

Using the first of (2.4), one sees that (2.14), (2.15) can be imposed by requiring the stronger but simpler condition

$$B_5\,\varepsilon_0\,C_0(E_0\,C_0)^2\,N_0^{N+1}\,\delta_0^{-x_3}<1 \tag{2.25}$$

with $B_5>1$, $x_3>1$ suitably chosen.

Hence, if (2.25) holds, we can use (2.19) to describe the Hamiltonian motions taking place in the image of $V^{(0)}_{C_0}\times T^N$ in the new variables $(\boldsymbol{A}', \boldsymbol{z}')$.

In the new variables the Hamiltonian is

$$H_1(\boldsymbol{A}', \boldsymbol{z}') = h_0\big(\boldsymbol{A}'+\Xi(\boldsymbol{A}', \boldsymbol{z}')\big) + f_0\big(\boldsymbol{A}'+\Xi(\boldsymbol{A}', \boldsymbol{z}'), \boldsymbol{z}'\exp[i\boldsymbol{\Delta}(\boldsymbol{A}', \boldsymbol{z}')]\big)\,. \tag{2.26}$$

Then, as in formal perturbation theory and as in the small-denominator theorem's proof ((2), p. 430, 448, [5.10.28], [5.12.37]), we write

$$H_1(\boldsymbol{A}', \boldsymbol{z}') = h_1(\boldsymbol{A}') + f_1(\boldsymbol{A}', \boldsymbol{z}')\,,$$

where $f_1 = H_1 - h_1$ and h_1 is defined as

$$h_1(\boldsymbol{A}') = h_0(\boldsymbol{A}') + f_{\boldsymbol{00}}(\boldsymbol{A}')\,.$$

A long but straightforward calculation based on the Cauchy formula for the holomorphic functions and on the basic estimates (2.2), (2.9), (2.10), (2.21) allows us after some labour to show that, if we define

$$\varrho_1 = \frac{\tilde{\varrho}_0}{8}\,, \qquad \xi_1 = \xi_0 - 4\delta_0\,, \qquad \boldsymbol{\omega}_1(\boldsymbol{A}) = \frac{\partial h_1}{\partial \boldsymbol{A}}(\boldsymbol{A})\,, \tag{2.27}$$

one has

$$\mathscr{C}^{(0)}W(\varrho_1, \xi_1; V^{(0)}_{C_0}) \subset W(\varrho_0, \xi_0; V)\,, \qquad \mathscr{C}^{(0)}(V^{(0)}_{C_0}\times T^N) \subset V\times T^N \tag{2.28}$$

and, for a suitable $B_6>1$, $x_4>1$, one can take

$$\begin{cases} \sup\left|\dfrac{\partial h_1}{\partial \boldsymbol{A}}\right| \leqslant E_0+\varepsilon_0 \equiv E_1 & \text{in } W(\varrho_0, \xi_0; V^{(0)}_{C_0})\,,\\[2ex] \sup\left|\left(\dfrac{\partial^2 h_1}{\partial \boldsymbol{A}\,\partial \boldsymbol{A}}\right)^{-1}\right| \leqslant \eta_0(1+B_6\eta_0\varepsilon_0\varrho_0^{-1}) \equiv \eta_1 & \text{in } W\left(\dfrac{\varrho_0}{2}, \xi_0; V^{(0)}_{C_0}\right),\\[2ex] \sup\left(\left|\dfrac{\partial f_1}{\partial \boldsymbol{A}}\right| + \dfrac{1}{\varrho_1}\left|\dfrac{\partial f_1}{\partial \boldsymbol{\varphi}}\right|\right) \leqslant B_6\,C_0\,\varepsilon_0^2\,N_0^{N+1}(E_0C_0)^3\,\delta_0^{-x_4} \equiv \varepsilon_1 & \\ & \text{in } W(\varrho_1, \xi_1; V^{(0)}_{C_0})\,, \end{cases} \tag{2.29}$$

provided (2.25) holds together with $B_6\,\eta_0\,\varepsilon_0\,\varrho_0^{-1}<1$: these two conditions can be implied by the simpler one

$$\sigma_0 = B_7 \varepsilon_0 C_0 (E_0 C_0)^2 N_0^{N+1} \delta_0^{-x_3} (\eta_0 E_0 \varrho_0^{-1}) < 1 , \tag{2.30}$$

if one notices that the holomorphy of h forces $\eta_0 E_0 \varrho_0^{-1} \geqslant 1$.

The « harder » estimate is the last of eqs. (2.29) and its derivation can be found also in (2) (p. 451-453), but it is simpler to derive it by oneself: it is again a « dimensional inequality ».

We define now

$$\left\{\begin{array}{ll} C_1 = 2^2 C_0 , & N_1 = 2\delta_1^{-1} \log (C_1 \varepsilon_1 \delta_1^N)^{-1} , \\ \tilde{\varrho}_1 = \dfrac{\varrho_1}{2} (B_1 C_1 E_1 N_1^{N+1})^{-1} , & \sigma_1 = B_7 \varepsilon_1 C_1 (E_1 C_1)^2 N_1^{N+1} \delta_1^{-x_3} (\eta_1 E_1 \varrho_1^{-1}) , \\ V_{C_1}^{(1)} = \bigcup\limits_{A \in \overline{V}_{C_1}^{(1)}} S\left(A, \dfrac{\tilde{\varrho}_1}{2}\right), & \\ \overline{V}_{C_1}^{(1)} = \{A | A \in V_{C_0}^{(0)}(\varrho_1, -\tilde{\varrho}_1), \; |\boldsymbol{\omega}_1(A) \cdot \boldsymbol{\nu}|^{-1} \leqslant C_1 |\boldsymbol{\nu}|^N, \; 0 < |\boldsymbol{\nu}| \leqslant N_1\} & \end{array}\right. \tag{2.31}$$

and $V_{C_0}^{(0)}(\varrho_1, -\tilde{\varrho}_1)$ is the set constructed as described after (2.6), which now is no longer as trivial as there. Notice that $V_{C_1}^{(1)} \subset V_{C_0}^{(0)}$.

The argument can now be iterated with $W(\varrho_1, \xi_1; V_{C_1}^{(1)})$ replacing $W(\varrho_0, \xi_0; V_{C_0}^{(0)})$.

Call

$$\tilde{\sigma}_k = (B_7 + B_6)^2 C_k \varepsilon_k (E_k C_k)^6 N_k^{2(N+1)} (\eta_k E_k \varrho_k^{-1}) \delta_k^{-2(x_3+x_4)} , \tag{2.32}$$

notice that $\tilde{\sigma}_k > \sigma_k$; then, assuming inductively

$$\left\{\begin{array}{l} \xi_k = \xi_0 - 4 \sum\limits_{j=0}^{k-1} \delta_j , \qquad C_k = C_0 (1 + k)^2 , \\ E_k < 2E_0 , \qquad \eta_k < 2\eta_0 , \qquad \tilde{\sigma}_k < 1 , \\ (C_0 \varepsilon_0)^{2^k} \leqslant C_k \varepsilon_k \leqslant (C_0 \varepsilon_0)^{(\frac{4}{3})^k} , \\ \varrho_k \geqslant \varrho_0 \left[(\log (C_0 \varepsilon_0)^{-1})^{-2k^2} \xi_0^k \right]^{(N+1)} (E_0 C_0)^{-k} , \end{array}\right. \tag{2.33}$$

one easily finds that, if $\tilde{\sigma}_0$ is small enough,

$$B_8 \tilde{\sigma}_0 < 1 , \tag{2.34}$$

eqs. (2.33) hold, $\forall k \geqslant 0$.

We can then define the canonical transformations $\mathscr{C}^{(n)}$, $\widetilde{\mathscr{C}}^{(n)}$ and (see (2.31), (2.16))

$$\mathscr{C}^{(n)}: \quad W(\varrho_{n+1}, \xi_{n+1}; V^{(n+1)}_{C_{n+1}}) \to W(\varrho_n, \xi_n; V^{(n)}_{C_n}) . \tag{2.35}$$

Call

$$\begin{cases} W_n = \mathscr{C}^{(0)} \dots \mathscr{C}^{(n-1)}\big(W(\varrho_n, \xi_n; V^{(n)}_{C_n})\big) \subset W_{n-1}, \\ \Gamma_n = \mathscr{C}^{(0)} \dots \mathscr{C}^{(n-1)}(V^{(n)}_{C_n} \times T^N) \subset \Gamma_{n-1} . \end{cases} \tag{2.36}$$

It can be easily checked that the first inclusion follows trivially from definitions (2.31) and from (2.16); the second inclusion follows from the intermediate inequality in (2.16) and from the fact that $C_n \varepsilon_n \varrho_n E_n C_n \delta_n^{-x_1} (\tilde{\varrho}_{n+1}/2)^{-1} \xrightarrow[n\to\infty]{} 0$: hence, possibly increasing the value of the constant B_8 in (2.34), we can and shall assume that $|\Xi_{(n)}| < \tilde{\varrho}_{n+1}/2$, if $\Xi_{(n)}$ is the analogue of Ξ for $\mathscr{C}^{(n)}$, $\forall n \geqslant 0$, strengthening the r.h.s. in (2.16).

It is also easy to see that

$$\mathscr{C}^{(k)} W(\varrho_{k+1}, \xi_{k+1}; V^{(k+1)}_{C_{k+1}}) \subset W\left(\frac{\tilde{\varrho}_k}{4}, \xi_k - 3\delta_k, V^{(k)}_{C_k}\right). \tag{2.37}$$

It can be easily checked that the limits

$$\big(A'_\infty(A, z), z'_\infty(A, z)\big) = \lim_{n\to\infty} \widetilde{\mathscr{C}}^{(n-1)} \dots \widetilde{\mathscr{C}}^{(0)}(A, z)$$

exist $\forall (A, z) \in W_\infty = \bigcap_{n=0}^{\infty} W_n$.

In fact, the map $\widetilde{\mathscr{C}}^{(k)}$ differs from the identity map, together with its derivatives of order M with respect to A and Q with respect to z, by a quantity that on $W(\tilde{\varrho}_k/4, \xi_k - 3\delta_k, V^{(k)}_{C_k})$ can be estimated by (see (2.20), (2.21))

$$\begin{cases} |\partial^{M+Q}(\pi_1 \widetilde{\mathscr{C}}^{(k)} - \pi_1)| \leqslant B_9 \varepsilon_k C_k E_k C_k \delta_k^{-x_1} \varrho_k \dfrac{M!}{\tilde{\varrho}_k^M} \dfrac{Q!}{\delta_k^Q} 2^{M+Q}, \\ |\partial^{M+Q}(\pi_2 \widetilde{\mathscr{C}}^{(k)} - \pi_2)| \leqslant B_9 \varepsilon_k C_k E_k C_k \delta_k^{-x_1} \dfrac{M!}{\tilde{\varrho}_k^M} \dfrac{Q!}{\delta_k^Q} 2^{M+Q} \end{cases} \tag{2.38}$$

with natural notation and for some $B_9 > 1$: notice that eqs. (2.38) are again dimensional estimates.

The convergence of (A'_∞, z'_∞) on W_∞ is clearly guaranteed by (2.38) and (2.33) (implying that the r.h.s. of (2.38) converges to zero as $k \to \infty$ faster than any exponential). Actually (2.38) gives much more: it shows that the functions

$$\begin{cases} A'_n(A, z) = \pi_1 \widetilde{\mathscr{C}}^{(n-1)} \dots \widetilde{\mathscr{C}}^{(0)}(A, z), \\ z'_n(A, z) = \pi_2 \widetilde{\mathscr{C}}^{(n-1)} \dots \widetilde{\mathscr{C}}^{(0)}(A, z) \end{cases} \tag{2.39}$$

have, on W_n, derivatives bounded as

$$(2.40)\qquad \begin{cases} \left|\dfrac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|}}{\partial \boldsymbol{A}^{\boldsymbol{a}}\partial \boldsymbol{z}^{\boldsymbol{b}}} \boldsymbol{A}'_n(\boldsymbol{A},\boldsymbol{z})\right| \leqslant B(\boldsymbol{a},\boldsymbol{b})\,, & \forall(\boldsymbol{A},\boldsymbol{z})\in W_n\,,\\ \left|\dfrac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|}}{\partial \boldsymbol{A}^{\boldsymbol{a}}\partial \boldsymbol{z}^{\boldsymbol{b}}} \boldsymbol{z}'_n(\boldsymbol{A},\boldsymbol{z})\right| \leqslant B'(\boldsymbol{a},\boldsymbol{b})\,, & \forall(\boldsymbol{A},\boldsymbol{z})\in W_n\,,\end{cases}$$

with B, B' depending « on everything » but not on n. Furthermore, the derivatives appearing in (2.40) converge, on W_∞, to some limits, which we decide to call

$$(2.41)\qquad \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|}}{\partial \boldsymbol{A}^{\boldsymbol{a}}\partial \boldsymbol{z}^{\boldsymbol{b}}}\boldsymbol{A}'_\infty \quad\text{or}\quad \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|}}{\partial \boldsymbol{A}^{\boldsymbol{a}}\partial \boldsymbol{z}^{\boldsymbol{b}}}\boldsymbol{z}'_\infty$$

with the natural meaning of the symbols, and the convergence, as $n\to\infty$, is faster than any exponential in n.

Our next task is to show that (2.41) are functions with the « correct properties » that one would expect from their symbolic notations.

To do this, we must be sure that the sets W_n are not too small if $\Gamma_\infty = \bigcap_{n=0}^{\infty}\Gamma_n \neq \emptyset$.

Let $(\boldsymbol{A},\boldsymbol{z})\in\Gamma_\infty$ and let

$$(2.42)\qquad \mu=\prod_{j=0}^{\infty}\big(1+(8NB_9)\,\varepsilon_j\,C_j\,E_j\,C_j\,\delta_j^{-\alpha_1}\,\delta_j^{-1}(\varrho_j/\tilde{\varrho}_j)\big)\equiv\prod_{j=0}^{\infty}(1+\theta_j)\,,$$

then, for each $n\geqslant 0$, there is $(\boldsymbol{A}_n,\boldsymbol{z}_n)\in V^{(n)}_{C_n}\times T^N$ such that

$$(2.43)\qquad (\boldsymbol{A},\boldsymbol{z})=\mathscr{C}^{(0)}\ldots\mathscr{C}^{(n-1)}(\boldsymbol{A}_n,\boldsymbol{z}_n)\,.$$

We wish to show that, if $|(\boldsymbol{A},\boldsymbol{z})-(\boldsymbol{A}',\boldsymbol{z}')|\equiv|\boldsymbol{A}-\boldsymbol{A}'|+\varrho_0|\boldsymbol{z}'-\boldsymbol{z}|$ and if $|(\boldsymbol{A},\boldsymbol{z})-(\boldsymbol{A}',\boldsymbol{z}')|<\xi_0\,\varrho_n/4\mu$, then the point $(\boldsymbol{A}',\boldsymbol{z}')$ is in W_n: in other words, W_n contains the complex sphere with radius $\xi_0\,\varrho_n/4\mu$ around $(\boldsymbol{A},\boldsymbol{z})\in\Gamma_\infty$.

In fact, eqs. (2.38) immediately imply for $(\boldsymbol{A}_1,\boldsymbol{z}_1)$, $(\boldsymbol{A}_2,\boldsymbol{z}_2)$ in $W(\tilde{\varrho}_p/4, \xi_p-3\delta_p; V^{(p)}_{C_p})$, $(\boldsymbol{A}_1,\boldsymbol{z}_1)$ in $V^{(p)}_{C_p}\times T^N$ and $|(\boldsymbol{A}_1,\boldsymbol{z}_1)-(\boldsymbol{A}_2,\boldsymbol{z}_2)|<\tilde{\varrho}_p/4$ that

$$(2.44)\qquad |\widetilde{\mathscr{C}}^{(p)}(\boldsymbol{A}_1,\boldsymbol{z}_1)-\widetilde{\mathscr{C}}^{(p)}(\boldsymbol{A}_2,\boldsymbol{z}_2)|\leqslant(1+\theta_p)|(\boldsymbol{A}_1,\boldsymbol{z}_1)-(\boldsymbol{A}_2,\boldsymbol{z}_2)|$$

with θ_p defined in (2.42).

Hence, by induction, it follows that

$$(2.45)\qquad \begin{aligned}|(\boldsymbol{A}_n,\boldsymbol{z}_n)-\widetilde{\mathscr{C}}^{(n-1)}\ldots\widetilde{\mathscr{C}}^{(0)}(\boldsymbol{A}',\boldsymbol{z}')|&\equiv\\ &\equiv|\widetilde{\mathscr{C}}^{(n-1)}\ldots\widetilde{\mathscr{C}}^{(0)}(\boldsymbol{A},\boldsymbol{z})-\widetilde{\mathscr{C}}^{(n-1)}\ldots\widetilde{\mathscr{C}}^{(0)}(\boldsymbol{A}',\boldsymbol{z}')|\leqslant\\ &\leqslant\prod_{j=0}^{n-1}(1+\theta_j)|(\boldsymbol{A},\boldsymbol{z})-(\boldsymbol{A}',\boldsymbol{z}')|\leqslant\mu\xi_0\,\varrho_n/4\mu=\varrho_n\,\xi_0/4\,;\end{aligned}$$

taking $(A'_n, z'_n) = \widetilde{\mathscr{C}}^{(n-1)} \dots \widetilde{\mathscr{C}}^{(0)}(A', z')$, we see that

$$(2.46) \qquad \begin{cases} |A'_n - A_n| < \varrho_n , \\ |(z'_n)_j| = |(z'_n)_j - (z_n)_j + (z_n)_j| = \left|1 + \dfrac{(z'_n)_j - (z_n)_j}{(z_n)_j}\right| ; \end{cases}$$

since $|(z_n)_j| = 1$, as $z_n \in T^N$, hence, if $\xi_\infty = \xi_0 - 4 \sum_{j=0}^{\infty} \delta_j$,

$$(2.47) \qquad \exp[-\xi_\infty] < 1 - \xi_0/4 \leqslant 1 - \xi_0 \varrho_n/4\varrho_0 \leqslant |(z'_n)_j| \leqslant \leqslant 1 + \xi_0 \varrho_n/4\varrho_0 \leqslant 1 + \xi_0/4 < \exp[\xi_\infty] ,$$

because the choice of δ_j has been such that $\xi_\infty > \xi_0/2$. Therefore, $(A'_n, z'_n) \in W(\varrho_n, \xi_n; V^{(n)}_{C_n})$ and (A', z') is consequently in W_n.

Let (A, z), $(\widetilde{A}, \widetilde{z}) \in \Gamma_\infty$, $\widetilde{z} = \exp[i\widetilde{\varphi}]$, suppose that

$$(2.48) \qquad \xi_0 \varrho_{n+1}/4\mu \leqslant |(A, z) - (\widetilde{A}, \widetilde{z})| < \xi_0 \varrho_n/4\mu .$$

Then the whole set of points parametrized by $t \in [0, 1]$: $A(t) = At + (1-t)\widetilde{A}$, $\varphi(t) = \varphi t + (1-t)\widetilde{\varphi}$ is in W_n if we suppose, as we obviously may without loss of generality, that the shortest path in T^N connecting φ with $\widetilde{\varphi}$ is the above segment.

We can, therefore, apply the Lagrange-Taylor formula to estimate $|A'_{(n)}(A, z) - A'_{(n)}(\widetilde{A}, \widetilde{z})|$ or, more generally, to estimate the difference between two arbitrary derivatives of order a_0 in the action variables and b_0 in the angles: given $M > 0$,

$$(2.49) \qquad \left| \frac{\partial^{|a_0|+|b_0|} A'_n}{\partial A^{a_0} \partial z^{b_0}} (A, z) - \frac{\partial^{|a_0|+|b_0|} A'_{(n)}}{\partial A^{a_0} \partial z^{b_0}} (\widetilde{A}, \widetilde{z}) - \sum_{\substack{a, b \in Z_+^N \\ 0 \leqslant |a|+|b| \leqslant M}} \frac{\partial^{|a_0+a|+|b_0+b|} A'_{(n)}}{\partial A^{a_0+a} \partial z^{b_0+b}} (\widetilde{A}, \widetilde{z}) \frac{(A - \widetilde{A})^a}{a!} \frac{(z - \widetilde{z})^b}{b!} \right| \leqslant \leqslant \Big(\max_{|a|+|b|=M+1} B(a_0 + a, b_0 + b) \Big) D(a_0, b_0, M) |(A, z) - (\widetilde{A}, \widetilde{z})|^{M+1} ,$$

where D is a suitable combinatorial factor: notice that the r.h.s. does not explicitly depend on n.

On the other hand, the limits (2.41) are reached at very high speed by (2.38) and in (2.49) we can replace the index n by ∞ with an error that can be explicitly controlled by (2.38).

If

$$\zeta_n(|\boldsymbol{a}_0|, |\boldsymbol{b}_0|) = B_9|\boldsymbol{a}_0|!|\boldsymbol{b}_0|!\sum_{k=n}^{\infty} C_k\varepsilon_k E_k C_k \delta_k^{-x_1}\varrho_k\left(\frac{8N\mu\varrho_0}{\xi_0\varrho_k}\right)^{|\boldsymbol{a}_0|+|\boldsymbol{b}_0|}, \tag{2.50}$$

we can remark that $\zeta_n \xrightarrow[n\to\infty]{} 0$ faster that any exponential in $(\frac{4}{3})^n$ or, see (2.33), faster than any power in ϱ_n. Hence from (2.49), (2.50) we get, suitably choosing $\tilde{D}$,

$$\left|\frac{\partial^{|\boldsymbol{a}_0|+|\boldsymbol{b}_0|}\boldsymbol{A}'_\infty}{\partial\boldsymbol{A}^{\boldsymbol{a}_0}\partial\boldsymbol{z}^{\boldsymbol{b}_0}}(\boldsymbol{A},\boldsymbol{z}) - \sum_{\substack{\boldsymbol{a},\boldsymbol{b}\in Z_+^N\\ 0\leqslant|\boldsymbol{a}|+|\boldsymbol{b}|\leqslant M}}\frac{\partial^{|\boldsymbol{a}_0+\boldsymbol{a}|+|\boldsymbol{b}_0+\boldsymbol{b}|}\boldsymbol{A}'_\infty}{\partial\boldsymbol{A}^{\boldsymbol{a}_0+\boldsymbol{a}}\partial\boldsymbol{z}^{\boldsymbol{b}_0+\boldsymbol{b}}}(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{z}})\frac{(\boldsymbol{A}-\tilde{\boldsymbol{A}})^{\boldsymbol{a}}}{\boldsymbol{a}!}\frac{(\boldsymbol{z}-\tilde{\boldsymbol{z}})^{\boldsymbol{b}}}{\boldsymbol{b}!}\right| \leqslant \tag{2.51}$$

$$\leqslant \tilde{D}(|\boldsymbol{a}_0|, |\boldsymbol{b}_0|, M)|(\boldsymbol{A},\boldsymbol{z})-(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{z}})|^{M+1} + \bar{\zeta}(|(\boldsymbol{A},\boldsymbol{z})-(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{z}})|, |a_0|, |b_0|, M)$$

and $\bar{\zeta}(x;p,q,s,)$ tends to zero, as $x\to 0$, faster than any power in x, $\forall p, q, s$ integers and $\bar{\zeta}$ can be chosen as

$$\bar{\zeta}(x;p,q,s) = \sum_{0\leqslant|\boldsymbol{a}|+|\boldsymbol{b}|\leqslant s}\frac{\zeta_n(p+|\boldsymbol{a}|, q+|\boldsymbol{b}|)}{\boldsymbol{a}!\boldsymbol{b}!}\varrho_0^{-p}\left(\frac{\xi_0\varrho_n}{4\mu\varrho_0}\right)^{|\boldsymbol{a}|+|\boldsymbol{b}|} \tag{2.52}$$

if $\varrho_{n+1}\xi_0/4\mu\leqslant x<\varrho_n\xi_0/4\mu$.

Identical arguments and conclusions hold for the angle variables $\boldsymbol{z}'_\infty(\boldsymbol{A},\boldsymbol{z})$, $(\boldsymbol{A},\boldsymbol{z})\in\Gamma_\infty$.

Hence $(\boldsymbol{A}'_\infty, \boldsymbol{z}'_\infty)$ are $2N$ functions on Γ_∞ extendible to its closure, by continuity, $\bar{\Gamma}_\infty$ and their extensions are in $C^\infty(\bar{\Gamma}_\infty)$ in the sense of Whitney [4], *i.e.* essentially in the sense of (2.51).

It appears from the above analysis that the Jacobian matrix

$$\left(\frac{\partial(\boldsymbol{A}'_\infty,\boldsymbol{z}'_\infty)}{\partial(\boldsymbol{A},\boldsymbol{z})}\right)$$

is a matrix close to the identity if (2.34) holds.

The variables $\boldsymbol{A}'_\infty$, $\boldsymbol{z}'_\infty$ verify the canonical commutation rules on $\bar{\Gamma}_\infty$ since $\boldsymbol{A}_n$, $\boldsymbol{\varphi}_n$ do, being canonical variables by construction: in particular, the $\boldsymbol{A}'_\infty$'s are in involution on Γ_∞. It also follows from the canonicity of the maps $\mathscr{C}^{(0)}, \ldots, \mathscr{C}^{(n)}, \ldots$ that $\det\left(\partial(\boldsymbol{A}'_\infty,\boldsymbol{z}'_\infty)/\partial(\boldsymbol{A},\boldsymbol{z})\right)\equiv 1$ on $\bar{\Gamma}_\infty$.

Another consequence of the above arguments and estimates is the existence of the limit

$$\lim_{n\to\infty}\boldsymbol{\omega}_n(\boldsymbol{A}) = \boldsymbol{\omega}_\infty(\boldsymbol{A}), \qquad \forall\boldsymbol{A}\in V^{(\infty)} = \bigcap_{n=0}^{\infty} V_{C_n}^{(n)}. \tag{2.53}$$

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(4) H. WHITNEY: *Trans. Am. Math. Soc.*, **36**, 63 (1934).

In fact, a repetition of the argument leading to (2.51) allows us to show, $\forall n \geqslant m$, $\forall (\boldsymbol{A}, \boldsymbol{z}) \in W(\varrho_n, \xi_n; V^{(n)}_{C_n})$,

$$\left|\frac{\partial^{|\alpha|}\boldsymbol{\omega}_n}{\partial \boldsymbol{A}^\alpha}(\boldsymbol{A}) - \frac{\partial^{|\alpha|}\boldsymbol{\omega}_m}{\partial \boldsymbol{A}^\alpha}(\boldsymbol{A})\right| \leqslant B(\boldsymbol{\alpha}) \sum_{k=n}^{\infty} \frac{\varepsilon_k}{\varrho_k^{|\alpha|}} \tag{2.54}$$

with $B(\boldsymbol{\alpha}) > 1$ suitably chosen. From this one immediately deduces, proceeding as before, that $\boldsymbol{\omega}_\infty \in C^\infty(\tilde{V}^{(\infty)})$ in the sense of Whitney and, more explicitly, if $\varrho_n/2 \leqslant |\boldsymbol{A} - \boldsymbol{A}| < \varrho_{n-1}/2$, $\tilde{\boldsymbol{A}} \in V^{(m)}_{C_m}$, $\tilde{\boldsymbol{A}} \in \boldsymbol{R}^N$,

$$\begin{aligned} |\boldsymbol{\omega}_m(\boldsymbol{A}) - \boldsymbol{\omega}_m(\tilde{\boldsymbol{A}})| &\leqslant B_{10}\Big(E_0\varrho_0^{-1} + \sum_{k=0}^{\infty} \varepsilon_k\varrho_k^{-1}\Big)|\boldsymbol{A} - \tilde{\boldsymbol{A}}| + 2\sum_{k=n}^{\infty}\varepsilon_k \leqslant \\ &\leqslant B_{10}\Big(E_0\varrho_0^{-1} + 4\sum_{k=0}^{\infty}\varepsilon_k\varrho_k^{-1}\Big)|\boldsymbol{A} - \tilde{\boldsymbol{A}}|\,, \end{aligned} \tag{2.55}$$

$$|\boldsymbol{\omega}_m(\boldsymbol{A}) - \boldsymbol{\omega}_0(\boldsymbol{A}) - \boldsymbol{\omega}_m(\tilde{\boldsymbol{A}}) + \boldsymbol{\omega}_0(\tilde{\boldsymbol{A}})| \leqslant B_{10} 4 \sum_{k=0}^{\infty} \varepsilon_k \varrho_k^{-1} |\boldsymbol{A} - \tilde{\boldsymbol{A}}|\,. \tag{2.56}$$

This means that, if ε_0 is small enough and if $\hat{r}$ is small enough, the function $\boldsymbol{\omega}_m$ is one to one on $V^{(m)}_{C_m} \cap \{\text{any sphere with radius } \hat{r}\}$; since $|M_0(\boldsymbol{A})\,\boldsymbol{v}| \geqslant \eta_0^{-1}|\boldsymbol{v}|$,

$$\begin{aligned} |\boldsymbol{\omega}_m(\boldsymbol{A}) - \boldsymbol{\omega}_m(\boldsymbol{A}')| &\equiv \\ &\equiv |\boldsymbol{\omega}_0(\boldsymbol{A}) - \boldsymbol{\omega}_0(\boldsymbol{A}') + \boldsymbol{\omega}_m(\boldsymbol{A}) - \boldsymbol{\omega}_0(\boldsymbol{A}) - \boldsymbol{\omega}_m(\boldsymbol{A}') + \boldsymbol{\omega}_0(\boldsymbol{A}')| \geqslant \\ &\geqslant |\boldsymbol{\omega}_0(\boldsymbol{A}) - \boldsymbol{\omega}_0(\boldsymbol{A}')| - 4\sum_{k=0}^{\infty}\varepsilon_k\varrho_k^{-1}|\boldsymbol{A} - \boldsymbol{A}'| = \\ &= |M_0(\boldsymbol{A}')(\boldsymbol{A} - \boldsymbol{A}') + \boldsymbol{\omega}_0(\boldsymbol{A}) - \boldsymbol{\omega}_0(\boldsymbol{A}') - M_0(\boldsymbol{A}')(\boldsymbol{A} - \boldsymbol{A}')| - \\ &- 4\sum_{k=0}^{\infty}\varepsilon_k\varrho_k^{-1}|\boldsymbol{A} - \boldsymbol{A}'| \geqslant \\ &\geqslant \eta_0^{-1}\Big(1 - \bar{B}_{10}E_0\eta_0\varrho_0^{-1}\frac{\hat{r}}{\varrho_0} - \eta_0 4\sum_{k=0}^{\infty}\varepsilon_k\varrho_k^{-1}\Big)|\boldsymbol{A} - \boldsymbol{A}'| \geqslant \frac{1}{2}\eta_0^{-1}|\boldsymbol{A} - \boldsymbol{A}'|\,; \end{aligned} \tag{2.57}$$

if ε_0 is small enough and if $\hat{r} = \frac{1}{4}\varrho_0(\bar{B}_{10}\eta_0E_0\varrho_0^{-1})^{-1}$, the condition on ε_0 can and shall be met by possibly increasing the constant B_8 in (2.34).

We can use the above remarks to estimate the measure of Γ_∞.

In fact, observe that, if a set $G \subset R^N$ is a union of open spheres of equal radius and each sphere contains some subset filling it up to a fraction $1 - \alpha$ of its volume, then the union of such subsets fills G up to a fraction $1 - B_{11}\sqrt[2N]{\alpha}$, say, of its volume, B_{11} being a G-independent constant.

Hence, if $\tilde{V}$ is a union of open spheres with radius ϱ and we consider the set $\tilde{V}(\varrho, -\tilde{\varrho})$ obtained by taking out of each of the covering spheres the outer shell of width $\tilde{\varrho}$, it follows that

$$\text{vol}\,\tilde{V}(\varrho, -\tilde{\varrho}) \geqslant (1 - B_{12}\sqrt[2N]{\tilde{\varrho}/\varrho})\,\text{vol}\,\tilde{V}\,. \tag{2.58}$$

We can now consider $V^{(n)}_{C_n} \supset \bar{V}^{(n)}_{C_n}$ (see (2.31)). We estimate its volume by that of $\bar{V}^{(n)}_{C_n}$: this is a set obtained from $V^{(n-1)}_{C_{n-1}}$, which is a union of spheres of radius $\varrho_n = \tilde{\varrho}_{n-1}/8$ by first taking out of each of the spheres an outer shell of width $\tilde{\varrho}_n$ and, secondly, depriving the remaining set of the « resonant points ».

In the first step we obtain the set $V^{(n-1)}_{C_{n-1}}(\varrho_n, -\tilde{\varrho}_n)$ whose volume may be bounded by using (2.58) by

$$\text{vol}\,V^{(n-1)}_{C_{n-1}}(\varrho_n, -\tilde{\varrho}_n) \geqslant \left(1 - B_{12}\sqrt[2N]{\frac{\tilde{\varrho}_n}{\varrho_n}}\right)\text{vol}\,V^{(n-1)}_{n-1}\,. \tag{2.59}$$

To estimate the measure of the set of the resonant points in $V^{(n-1)}_{C_{n-1}}(\varrho_n, -\tilde{\varrho}_n)$, *i.e.* the measure of the set V'_n of points in $V^{(n-1)}_{C_{n-1}}(\varrho_n, -\tilde{\varrho}_n) \subset V$, such that the inequality

$$|\boldsymbol{\omega}_n(\boldsymbol{A})\cdot\boldsymbol{\nu}|^{-1} \leqslant C_n|\boldsymbol{\nu}|^N$$

is not true for some $\boldsymbol{\nu}$, $0 < |\boldsymbol{\nu}| \leqslant N_n$, we notice that

$$\text{vol}\,V'_n = \int\limits_{V'_n} \mathrm{d}\boldsymbol{A}' \leqslant T \int\limits_{\boldsymbol{\omega}_n(V'_n)} \left|\det\frac{\partial\boldsymbol{A}}{\partial\boldsymbol{\omega}_n}\right| \mathrm{d}\boldsymbol{\omega}\,, \tag{2.60}$$

where T is an estimate on the maximum number of points $\boldsymbol{A}' \in V^{(n)}_{C_n}$, where the function $\boldsymbol{\omega}_n$ takes the same value: by (2.57) we can take

$$T = \left(\eta_0 E_0 \varrho_0^{-1}\frac{r}{\varrho_0}\right)^N B_{13} \tag{2.61}$$

for some $B_{13} > 1$.

Hence

$$\begin{aligned}\text{vol}\,V'_n &\leqslant B_{13}\left(E_0\eta_0\varrho_0^{-1}\frac{r}{\varrho_0}\right)^N \eta_n^N \int\limits_{\boldsymbol{\omega}_n(V'_n)} \mathrm{d}\boldsymbol{\omega} \leqslant \\ &\leqslant B_{13}\left(E_0\varrho_0^{-1}\eta_0\frac{r}{\varrho_0}\right)^N \eta_n^N \sum_{\boldsymbol{\nu}\neq\boldsymbol{0}} \int\limits_{\substack{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|<C_n^{-1}|\boldsymbol{\nu}|^{-N}\\ \boldsymbol{\omega}\in\boldsymbol{\omega}_n(V'_n)}} \mathrm{d}\boldsymbol{\omega} \leqslant B_{13}\left(E_0\eta_0\varrho_0^{-1}\frac{r}{\varrho_0}\right)^N \eta_n^N \sum_{\boldsymbol{\nu}\neq\boldsymbol{0}} \int\limits_{\substack{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|<C_n^{-1}|\boldsymbol{\nu}|^{-N}\\ |\boldsymbol{\omega}|<E_n}} \mathrm{d}\boldsymbol{\omega} \leqslant \\ &\leqslant B_{13}\left(E_0\eta_0\varrho_0^{-1}\frac{r}{\varrho_0}\right)^N \eta_n^N \frac{(2E_n)^{N-1}}{C_n} \sum_{\boldsymbol{\nu}\neq\boldsymbol{0}} \frac{2}{|\boldsymbol{\nu}|^{N+1}} \leqslant \\ &\leqslant B_{14}(\eta_0 E_0\varrho_0^{-1})^N \left(\frac{r}{\varrho_0}\right)^N \frac{(E_0\eta_0\varrho_0^{-1})^N}{E_0 C_n}\,\text{vol}\,V \leqslant B_{14}\frac{(\eta_0 E_0\varrho_0^{-1})^{2N}}{E_0 C_n}\,\text{vol}\,V\,.\end{aligned} \tag{2.62}$$

Therefore,

$$\operatorname{vol} V^{(n)}_{C_n} \geqslant \operatorname{vol} \bar{V}^{(n)}_{C_n} \geqslant (1 - B_{12}\sqrt[2N]{\tilde{\varrho}_n/\varrho_n}) \operatorname{vol} V^{(n-1)}_{C_{n-1}} - B_{14}\frac{(\eta_0 E_0 \varrho_0^{-1})^{2N}}{E_0 C_n} \operatorname{vol} V .$$

Hence inductively

$$(2.63)\qquad \operatorname{vol} V^{(n)}_{C_n} \geqslant$$

$$\geqslant \left\{\left(\prod_{k=1}^{\infty}(1 - B_{12}\sqrt[2N]{\tilde{\varrho}_k/\varrho_k})\right)\left[(1 - N\tilde{\varrho}_0/r) - B_{14}\frac{(\eta_0 E_0 \varrho_0^{-1})^{2N}}{E_0 C_0}\sum_{k=0}^{\infty}\frac{1}{(1+k)^2}\right]\right\} \operatorname{vol} V \geqslant$$

$$\geqslant \left(1 - B_{15}\frac{(\eta_0 E_0 \varrho_0^{-1})^{2N}}{\sqrt[2N]{E_0 C_0}}\right) \operatorname{vol} V ,$$

where the last factor in the first intermediate term arises from the fact that $V^{(0)}_{C_0}$ has to be treated differently from $V^{(n)}_{C_n}$; the second inequality is an easy consequence of the relation among $\tilde{\varrho}_0$, $E_0 C_0$, ϱ_0, r (see (2.33) and recall that $r > \varrho_0$).

Since $\Gamma_n = \mathscr{C}^{(0)} \dots \mathscr{C}^{(n-1)}(V^{(n)}_{C_n} \times T^N)$ is a canonical image of $V^{(n)}_{C_n} \times T^N$, it has the same Liouville measure:

$$(2.64)\qquad \operatorname{vol} \Gamma_\infty \geqslant \left(1 - B_{15}\frac{(\eta_0 E_0 \varrho_0^{-1})^{2N}}{\sqrt[2N]{E_0 C_0}}\right) \operatorname{vol}(V \times T^N) .$$

The continuability of A'_∞, z'_∞ to functions in $C^\infty(V \times T^N)$ defined on the whole phase space is an immediate consequence of Whitney's theorem [4] and of the uniformity of (2.51) with respect to (A, z) (allowing the extension by continuity of A'_∞, z'_∞ and of their derivatives to the closure $\bar{\Gamma}_\infty$ of Γ_∞).

It remains to prove that the A'_∞ are prime integrals in $\bar{\Gamma}_\infty$.

We notice that the above analysis and estimates immediately imply that the map $(A, z) \to (A'_\infty, z'_\infty)$ is one to one as a map between Γ_∞ and $\bigcap_{n=0}^{\infty} V^{(n)}_{C_n} \times T^N$ (recall that for each canonical map $\mathscr{C}^{(k)}$ we constructed also its inverse $\tilde{\mathscr{C}}^{(k)}$): hence it is possible to define a system of C^∞ co-ordinates in a neighbourhood of Γ_∞ for a neighbourhood of $V^{(\infty)} \times T^N$ and *vice versa* by using the above-mentioned C^∞ extension of the map $(A, z) \to (A'_\infty, z'_\infty)$. This is so because the Jacobian determinant of the map $(A, z) \to (A'_\infty, z'_\infty)$ is 1 in the $(A, \boldsymbol{\varphi})$, $(A_\infty, \boldsymbol{\varphi}_\infty)$ variables, as already noticed.

We also noticed that from the above analysis it immediately follows that

$$(2.65)\qquad \boldsymbol{\omega}_n(A'_n(A, z)) \xrightarrow[n\to\infty]{} \boldsymbol{\omega}_\infty(A_\infty(A, z)) , \qquad \forall (A, z) \in \Gamma_\infty ,$$

faster than any power of ϱ_n/ϱ_0.

It is now easy to complete the proof.

First show that A'_∞ are prime integrals: this follows from the fact that the evolution « commutes » with the canonical transformations.

Let $(\boldsymbol{A}, \boldsymbol{z}) \in \Gamma_\infty$ and fix $t > 0$. Then

$$\tilde{\mathscr{C}}^{(n-1)} \dots \tilde{\mathscr{C}}^{(0)}\big(S_t^{(0)}(\boldsymbol{A}, \boldsymbol{z})\big) = S_t^{(n)} \tilde{\mathscr{C}}^{(n-1)} \dots \tilde{\mathscr{C}}^{(0)}(\boldsymbol{A}, \boldsymbol{z}) \equiv S_t^{(n)}(\boldsymbol{A}_n', \boldsymbol{z}_n'), \tag{2.66}$$

where $S_t^{(n)}$ is the Hamiltonian flow with Hamiltonian H_n: hence from the form of the Hamilton's equations it follows that

$$\big|S_t^{(n)}(\boldsymbol{A}_n, \boldsymbol{z}_n) - (\boldsymbol{A}_n, \boldsymbol{z}_n \exp[i\boldsymbol{\omega}_n(\boldsymbol{A}_n)t])\big| \leqslant \varrho_0\big(\exp[\varrho_n^{-1}\varepsilon_n t] - 1\big) + \varepsilon_n t, \tag{2.67}$$

at least as long as the motion stays in $W(\varrho_n, \xi_n; V_{C_n}^{(n)})$, *i.e.* as long as the r.h.s. of (2.67) is $\leqslant \varrho_n$: this is certainly true for n large since $\varepsilon_n \to 0$ much faster than ϱ_n. Hence (2.65)-(2.67) imply, by taking the limit as $n \to \infty$ in (2.67), $\forall (\boldsymbol{A}, \boldsymbol{z}) \in \Gamma_\infty$,

$$\big(\boldsymbol{A}'_\infty\big(S_t^{(0)}(\boldsymbol{A}, \boldsymbol{z})\big), \boldsymbol{z}'_\infty\big(S_t^{(0)}(\boldsymbol{A}, \boldsymbol{z})\big)\big) = \big(\boldsymbol{A}'_\infty(\boldsymbol{A}, \boldsymbol{z}), \boldsymbol{z}'_\infty \exp[i\boldsymbol{\omega}_\infty(\boldsymbol{A}_\infty)t]\big), \tag{2.68}$$

which clearly means that $\boldsymbol{A}'_\infty$ are prime integrals on Γ_∞.

The last statement of the theorem follows from the remark that, by construction,

$$|\boldsymbol{\omega}_\infty(\boldsymbol{A}'_\infty) \cdot \boldsymbol{\nu}|^{-1} \leqslant 4C_n|\boldsymbol{\nu}|^N \qquad \text{if } 0 < |\boldsymbol{\nu}| \leqslant N_n. \tag{2.69}$$

Thus, if $\tilde{A} \in C^\infty(V \times T^N)$ is a prime integral on Γ_∞, it can be expressed in a small neighbourhood of Γ_∞ as

$$\tilde{A}(\boldsymbol{A}, \boldsymbol{z}) = \tilde{a}\big(\boldsymbol{A}'_\infty(\boldsymbol{A}, \boldsymbol{z}), \boldsymbol{z}'_\infty(\boldsymbol{A}, \boldsymbol{z})\big) \tag{2.70}$$

because $\boldsymbol{A}'_\infty$, $\boldsymbol{z}'_\infty$ are a co-ordinate system in a neighbourhood of $V^{(\infty)} \times T^N$ representing the points of a neighbourhood of Γ_∞.

Then, by (2.68) and since $\tilde{A}$ is a prime integral on Γ_∞,

$$\tilde{A}(\boldsymbol{A}, \boldsymbol{z}) \equiv \tilde{A}\big(S_t^{(0)}(\boldsymbol{A}, \boldsymbol{z})\big) \equiv \tilde{a}\Big(\boldsymbol{A}'_\infty(\boldsymbol{A}, \boldsymbol{z}), \boldsymbol{z}'_\infty(\boldsymbol{A}, \boldsymbol{z}) \exp\big[i\boldsymbol{\omega}_\infty\big(\boldsymbol{A}'_\infty(\boldsymbol{A}, \boldsymbol{z})\big)t\big]\Big), \tag{2.71}$$

but the N pulsations $\boldsymbol{\omega}_\infty(\boldsymbol{A}'_\infty)$ are rationally independent by (2.69): hence (2.71) and the arbitrariness of $t > 0$ imply that $\tilde{a}$ must be $\boldsymbol{z}'_\infty$-independent on Γ_∞, *i.e.* $\tilde{A}(\boldsymbol{A}, \boldsymbol{z}) = b\big(\boldsymbol{A}'_\infty(\boldsymbol{A}, \boldsymbol{z})\big)$ for some $b \in C^\infty(\Gamma_\infty)$.

3. – A simple application to the harmonic oscillators.

Consider N harmonic nonresonant oscillators which in action angle variables are described by the Hamiltonian

$$\bar{h}(\boldsymbol{A}) = \boldsymbol{\omega}_0 \cdot \boldsymbol{A}, \qquad (\boldsymbol{A}, \boldsymbol{\varphi}) \in R^N \times T^N \tag{3.1}$$

with

$$|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|^{-1} < C|\boldsymbol{\nu}|^{\alpha}, \qquad C>0,\ \alpha>0. \tag{3.2}$$

Let $\bar{f}(\boldsymbol{A}, \boldsymbol{\varphi})$ be analytic in $W(1, 1, S_1)$ and assume that its average over T^N, $\bar{f}_0(\boldsymbol{A})$ is such that

$$\left|\left(\frac{\partial^2 \bar{f}_0}{\partial \boldsymbol{A}\,\partial \boldsymbol{A}}(\boldsymbol{A})\right)^{-1}\right| \leqslant \bar{\eta} < +\infty \tag{3.3}$$

in the whole analyticity domain.

Consider the Hamiltonian

$$\bar{h}(\boldsymbol{A}) + \varepsilon \bar{f}(\boldsymbol{A}, \boldsymbol{\varphi}) . \tag{3.4}$$

We can apply to it a Birkhoff transformation to write it in new variables $(\boldsymbol{A}', \boldsymbol{\varphi}')$ as

$$h_0(\boldsymbol{A}') + \varepsilon^p f_0(\boldsymbol{A}, \boldsymbol{\varphi}, \varepsilon) , \tag{3.5}$$

where

$$h_0(\boldsymbol{A}') = \boldsymbol{\omega}_0\cdot\boldsymbol{A} + \varepsilon \bar{f}_0(\boldsymbol{A}) + \varepsilon^2 \tilde{f}(\boldsymbol{A}, \varepsilon) , \tag{3.6}$$

all functions being analytic as $(\boldsymbol{A}, \boldsymbol{z}) \in W(\frac{1}{2}, \frac{1}{2}, S_\beta)$ with $\beta < 1$ as close to 1 as wished (see [2], p. 442, proposition XXI) and as ε varies near 0. Furthermore,

$$\frac{|\boldsymbol{\omega}_0|}{2} < \left|\frac{\partial h_0}{\partial \boldsymbol{A}'}\right| < 2|\boldsymbol{\omega}_0| , \qquad \frac{\varepsilon^{-N}\bar{\eta}}{2} \leqslant \left|\left(\frac{\partial^2 h_0}{\partial \boldsymbol{A}'\,\partial \boldsymbol{A}'}(\boldsymbol{A}')\right)^{-1}\right| \leqslant 2\varepsilon^{-N}\bar{\eta} \tag{3.7}$$

and

$$\varepsilon_0 \geqslant \left(\sup\left|\frac{\partial \varepsilon^p f_0(\boldsymbol{A}', \boldsymbol{\varphi}', \varepsilon)}{\partial \boldsymbol{A}'}\right| + 2\left|\frac{\partial \varepsilon^p f_0(\boldsymbol{A}', \boldsymbol{\varphi}', \varepsilon)}{\partial \boldsymbol{\varphi}'}\right|\right) \leqslant G_p \varepsilon^p \tag{3.8}$$

if ε is small enough.

Thus, by taking $p > N$ and ε small enough, we can apply our theorem to prove that $S_1 \times T^N$ is covered, up to a set of measure as small as we wish for $\varepsilon \to 0$, by invariant tori and locally such tori can be thought as level surface of some C^∞-functions on $S_1 \times T^N$.

The idea for the above application is taken from [5]: it was suggested to us by GALGANI.

* * *

We owe to J. MOSER the information about the work of J. PÖSCHEL. We are indebted to L. GALGANI for many discussions and suggestions.

We are indebted to B. SOUILLARD for some ideas for the proof in the appendix.

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[5] T. NISHIDA: *Mem. Fac. Eng. Kyoto Univ.*, **33**, 27 (1971).

APPENDIX

Volume estimate.

Notice first that, if V is a union of unit spheres and if each of them is dilated by a factor $1+\eta$, keeping its centre fixed we obtain a new set V_η such that $\operatorname{vol} V_\eta \leqslant (1+\eta)^N \operatorname{vol} V$.

Let V be a union of spheres, of unit radius, all intersecting one of them $S(\boldsymbol{x}_0, 1)$. Suppose that each of them is covered up to a fraction $1-\alpha$ of its volume by some points which we call the $(1-\alpha)$ points. Call α points in V those which are out of the complement of the union of the $(1-\alpha)$ points.

Let $\delta > 0$ and consider a maximal set in the set of the centres of the spheres consisting of points at mutual distance not smaller than α^δ. Call G this set and let V_G be the union of the unit spheres centred at the points of G. Clearly V_{G,α^δ} covers V and $\operatorname{vol} V_{G,\alpha^\delta} \leqslant (1+\alpha^\delta)^N \operatorname{vol} V_G$.

We can estimate the volume of the $(1-\alpha)$ points in V as the fraction

$$\left(\operatorname{vol} V_G - 2B(\alpha^{-\delta})^N \alpha \operatorname{vol} V_G\right)/\operatorname{vol} V_{G,\alpha^\delta} \geqslant (1-2B\alpha^{1-N\delta})/(1+\alpha^\delta)^N,$$

where $n = B(\alpha^{-\delta})^N \operatorname{vol} V$ is an estimate of the number of elements of G and use has been made of the fact that the volume of each sphere is less than $\operatorname{vol} V$.

Let now V be an arbitrary union of spheres and let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p$ be a maximal set of the set of the centres consisting of points such that $|\boldsymbol{x}_i - \boldsymbol{x}_j| \geqslant 2$, $i \neq j$, and associate each of the other spheres to one (and only one) of the spheres of the maximal set which intersect it. The set V will consist of the p disjoint spheres with the centre in the selected maximal set, plus the set W of the $(1-\alpha)$ points outside this union plus the set of the α points outside this union:

$$\operatorname{vol} V \leqslant p \operatorname{vol} S(\mathbf{0}, 1) + \operatorname{vol} W + \left(1 - (1-B\alpha^{1-N\delta})/(1+\alpha^\delta)^N\right) 2^N p \operatorname{vol}\left(S(\mathbf{0}, 1)\right),$$

while the $(1-\alpha)$ points in V have a volume $p(1-\alpha)\operatorname{vol} S(\mathbf{0},1) + \operatorname{vol} W$, hence the fraction of $(1-\alpha)$ points in V is bounded below by $\left(1 - B_{11}\max(\alpha, \alpha^{1-N\delta}, \alpha^{N\delta})\right)$. Then choose $\delta = 1/2N$.

● RIASSUNTO

Si considera un sistema hamiltoniano a N gradi di libertà, perturbazione analitica di un sistema analiticamente e canonicamente integrabile e strettamente non isocrono. Si mostra l'esistenza di N funzioni definite sullo spazio delle fasi e ivi di classe C^∞ che sono integrali primi per il moto perturbato su opportune regioni la cui misura di Lebesgue tende a riempire localmente lo spazio delle fasi al tendere a zero della perturbazione. S'illustra un'applicazione alle perturbazioni di oscillatori isocroni non risonanti.

Резюме не получено.