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On the Analyticity of the Pressure in the Hierarchical Dipole Gas

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The convergence of the Mayer expansion is proved by estimating directly the convergence radius.

KEY WORDS: Coulomb systems; renormalization group; multiscale expansion; dipole gas.

1. INTRODUCTION

In ref. 1 the analyticity of the pressure in the dipole gas was proved by a somewhat indirect method, i.e., by methods not based on a direct C^n bound on the nth order coefficient of the Mayer expansion.

On the other hand, recently success has been met in finding direct C^n bounds on the nth-order coefficient of the Mayer series for the two-dimensional Yukawa gas in the region $\beta \in (0, 6\pi)$ [here $\beta \equiv \alpha^2$ will be symbols denoting the inverse temperature; see refs. 2 and 3 for the unit conventions; in ref. 2 the result is stated only for $\beta < 4\pi$, but it is obvious that the bounds obtained in fact imply that it holds for $\beta \in (0, 6\pi)$; in ref. 3 the result is obtained for $\beta \in (0, 16\pi/3)$ by a similar method].

With the above results in mind, we have tried to prove by direct methods also the convergence of the Mayer expansion for the dipole gas, with the aim of developing techniques eventually suitable to prove the often conjectured convergence of the Mayer expansion for the two-dimensional Coulomb gas at low temperature [i.e., for $\beta \in (8\pi, \infty)$].

So far we have been unable to understand the delicate cancellation mechanism necessary to obtain the desired bounds: we have, however, been

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able to find the mechanism for the much simpler case of the hierarchical dipole gas model introduced in ref. 4 and treated there by indirect methods. The ideas stem from our technique for sharp estimates on the truncated expectations for hierarchical models developed in ref. 3 and are exposed in Section 3 below after a brief review of the notations.

2. THE MODEL

Let $\mathscr{Z}_k$, $k=0, +1,...$, be a sequence of compatible pavements of $\mathbf{R}^d$ with tesserae of size γ^k, where γ is an integer greater than 1. If $\Delta \in \mathscr{Z}_k$, we imagine it divided in two halves and set $\mu_x^{(k)} = \pm 1$ if x is a point of Δ in the left or the right half. To each Δ we associate a Gaussian variable z_Δ so that $\mathscr{E}(z_\Delta^2)=1$, $\mathscr{E}(z_\Delta z_{\Delta'})=0$ for $\Delta \neq \Delta'$. Let

$$\varphi_x = \sum_{k=0}^{\infty} \gamma^{-kd/2} \mu_x^{(k)} z_{\Delta_x^{(k)}} \tag{2.1}$$

where $\Delta_x^{(k)}$ is the tessera of size γ^k containing x. This is the hierarchical dipole field introduced in ref. 4.

The hierarchical dipole gas partition function at volume Λ, activity λ, and inverse temperature $\beta = \alpha^2$ is[4]

$$Z_\Lambda = \int \prod_\Delta \left(\frac{\exp(-z_\Delta^2/2)}{(2\pi)^{1/2}}\, dz_\Delta \right) \exp\left(\lambda \int_\Lambda dx \cos \alpha \varphi_x \right) \tag{2.2}$$

If we define

$$\varphi_x^{(k)} = \gamma^{-kd/2} \mu_x^{(k)} z_{\Delta_x^{(k)}}$$

we can write

$$Z_\Lambda = \lim_{R \to \infty} \mathscr{E}_R \mathscr{E}_{R-1} \cdots \mathscr{E}_0 \left\{ \exp\left[\lambda \int_\Lambda dx \cos \alpha \varphi_x^{(\leq R)} \right] \right\} \tag{2.3}$$

where

$$\varphi_x^{(\leq R)} = \sum_{k=0}^{R} \varphi_x^{(k)} \tag{2.4}$$

and $\mathscr{E}_k$ is the integration over $\varphi^{(k)}$.

3. THE BOUND

By induction one easily proves (see refs. 2, 5, and 6, §12, for similar calculations) that

$$\log Z_\Lambda = \sum_{\vartheta} \sum_{\{h_v\}_{v \in \vartheta}} \left[\prod_{i=1}^{n} \frac{\lambda}{2} \exp\left(-\frac{\alpha^2}{2} U_i \right) \right] \times \sum_{\sigma_1,\dots,\sigma_n = \pm 1} \int dx_1 \cdots dx_n \left\{ \prod_{v \in \vartheta} \left[\exp\left(-\frac{\alpha^2}{2} U_{v'v} \right) \right] \frac{F_v}{s_v!} \right\} \tag{3.1}$$

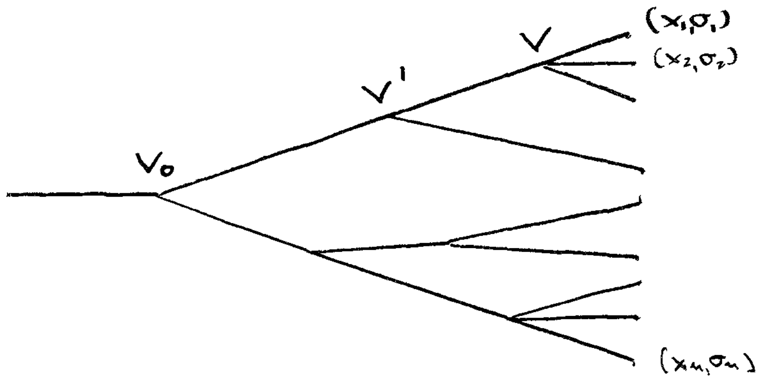

Fig. 1.

where ϑ is a tree with root r (see Fig. 1) and $n \geqslant 1$ endpoints labeled by (x_i, σ_i), $i = 1,..., n$; v are the tree vertices (excluding the root and the endpoints), out of which emerge s_v branches; v' is the vertex preceding v and, for each vertex v, there is an integer label h_v which is a summation index subject to the condition

$$0 \leqslant h_v < h_{v'} \tag{3.2}$$

We explain now the meaning of the other symbols used in Eq. (3.1). We say that the endpoint i belongs to v, $i \in v$, if i can be reached by climbing the tree from v upward; then we define

$$Q_v = \sum_{i \in v} \sigma_i, \qquad \varphi_v^{(h)} = \sum_{i \in v} \varphi_{x_i}^{(h)} \sigma_i \tag{3.3}$$

$$F_v = \mathscr{E}_{h_v}^T(\exp(i\alpha\varphi_1^{h_v}); ...; \exp(i\alpha\varphi_{v_s}^{h_v})) \tag{3.4}$$

where $v_1,..., v_s$ are the $s = s_v$ vertices which follow v in the tree and $\mathscr{E}_h^T$ is the truncated expectation with respect to $\varphi^{(h)}$, and

$$U_{v'v} = \sum_{h = h_v + 1}^{hv' - 1} \mathscr{E}(\varphi_v^{h^2}), \qquad U_i = \sum_{h = 0}^{h_i - 1} \mathscr{E}(\varphi_{x_i}^{h^2}) \tag{3.5}$$

where h_i is the label of the highest vertex containing the endpoint i and the rhs has to be interpreted as 0 if the sum is empty.

The integrals over the x_i in Eq. (3.1) can be easily performed using the independence of the z_Δ variables (which implies that $\prod_v F_v$ is identically zero unless all the points belonging to v are located inside one and the same box Δ of size γ^{h_v}) and one gets for the pressure $p(\lambda)$ the explicit expression

$$p(\lambda)=\sum_{\vartheta}\sum_{\{h_v\}_{v\in\vartheta}}\left[\prod_{i=1}^{n}\frac{\lambda}{2}\exp\left(-\frac{\alpha^2}{2}U_i\right)\right]$$
$$\times\sum_{\sigma_1,\ldots,\sigma_n=\pm 1}\left\{\prod_{v\in\vartheta}\gamma^{dh_v(s_v-1)}\left[\exp\left(-\frac{\alpha^2}{2}U_{v'v}\right)\right]\frac{F_v^*}{s_v!}\right\} \tag{3.6}$$

where, if $s=s_v$ and $h=h_v$,

$$F_v^*=\frac{1}{2^s}\sum_{\mu_1\cdots\mu_s}\mathscr{E}^T(\exp(i\alpha\gamma^{-dh/2}\mu_1 Q_{v_1}z);\ldots;\exp(i\alpha\gamma^{-dh/2}\mu_s Q_{v_s}z))$$
$$=\mathscr{E}^T(\cos\alpha\gamma^{-dh/2}Q_{v_1}z;\ldots;\cos\alpha\gamma^{-dh/2}Q_{v_s}z) \tag{3.7}$$

and $\mathscr{E}^T$ is the truncated expectation with respect to a Gaussian variable z with $\mathscr{E}(z^2)=1$. Therefore the problem is to find a good bound of F_v^*. Let $w_i=\alpha\gamma^{-dh/2}Q_{v_i}$ and remark that [see also ref. 6, Appendix A, (A.6)]

$$\mathscr{E}^T(\cos w_1 z;\ldots;\cos w_s z)$$
$$=\left(\prod_{i=1}^{s}w_i^2\right)\mathscr{E}^T\left(\frac{\cos w_1 z-1}{w_1^2};\ldots;\frac{\cos w_s z-1}{w_s^2}\right)$$
$$=\left(\prod_{i=1}^{s}w_i^2\right)\frac{\partial}{\partial\lambda_1\cdots\partial\lambda_s}\log\mathscr{E}\left(\exp\sum_{i=1}^{s}\lambda_i\frac{\cos w_i z-1}{w_i^2}\right)\Bigg|_{\lambda=0}$$
$$=\left(\prod_{i=1}^{s}w_i^2\right)\frac{\partial^s}{\partial\lambda_1\cdots\partial\lambda_s}\log\mathscr{E}\left(\prod_{i=1}^{s}\left[1+\lambda_i\frac{\cos w_i z-1}{w_i^2}\right]\right)\Bigg|_{\underline{\lambda}=0}$$

If $|\lambda_i|\leqslant\delta=1/3s$, $i=1,\ldots,s$, and if $\lambda^X\equiv\prod_{i\in X}\lambda_i$ when $X\subseteq\{1,\ldots,s\}$,

$$\left|\log\mathscr{E}\left(\prod_{i=1}^{s}\left[1+\lambda_i\frac{\cos w_i z-1}{w_i^2}\right]\right)\right|$$
$$=\left|\log\left[1+\sum_{\varnothing\neq X\subseteq(1,\ldots,s)}\lambda^X\mathscr{E}\left(\prod_{i\in X}\frac{\cos w_i z-1}{w_i^2}\right)\right]\right|$$
$$\leqslant\left|\log\left[1-\sum_{k=1}^{s}\binom{s}{k}\left(\frac{\delta}{2}\right)^k\mathscr{E}(z^{2k})\right]\right|$$
$$\leqslant\left|\log\left[1-\sum_{k=1}^{s}(s\,\delta)^k\right]\right|\leqslant\log 2$$

By this bound and Cauchy theorem we get

$$|F_v^*| \leqslant \left[\prod_{i=1}^{s} (\alpha\gamma^{-dh/2} Q_{v_i})^2 \right] (3s)^s \log 2 \tag{3.8}$$

Equations (3.6) and (3.8) and the positivity of $U_{v'v}$ and U_i imply

$$|p(\lambda)| \leqslant \sum_{\vartheta} \lambda^n \sum_{\{h_v\}_{v\in\vartheta}} \prod_{v\in\vartheta} \left[(C_0\alpha^2)^{s_v} \gamma^{-dh_v} \prod_{i=1}^{s_v} n_{v_i}^2 \right] \tag{3.9}$$

where n_v denotes the number of endpoints belonging to v and C_0 is a suitable positive constant.

Since $\sum_v s_v \leqslant 2(n-1)$ and the number of trees is less than 2^{4n}, we have, for some β-dependent $C_1 > 0$,

$$|p(\lambda)| \leqslant \sum_{n=1}^{\infty} (C_1\lambda)^n \sup_{\vartheta\in\Gamma_n} \prod_{\substack{v\in\vartheta \\ v\neq v_0}} \exp(n_v \gamma^{-dl_v/2}) \tag{3.10}$$

where Γ_n is the family of trees with n endpoints, l_v is the minimum value of the label h_v, resulting from the condition (3.2), and v_0 is the first vertex following the root.

We want now to show that, for some $C > 0$,

$$G(\vartheta) \equiv \prod_{\substack{v\in\vartheta \\ v\neq v_0}} \exp(n_v \gamma^{-dl_v/2}) \leqslant C^n \tag{3.11}$$

Given ϑ, let $l = l_{v_0}$; of course, $l > l_v$ for any $v \neq v_0$. The bound (3.11) immediately follows from the inequality

$$G(\vartheta) \leqslant e^{c_l n}, \qquad c_l = \sum_{k=0}^{l-1} \gamma^{-dk/2} \tag{3.12}$$

that we shall prove by induction on l.

If $l = 0$, then $G(\vartheta) = 1$, since there is no vertex following v_0; then $c_0 = 0$. Suppose that (3.11) is true for $l \leqslant m-1$; then, if $l = m$ and $s = s_{v_0}$,

$$G(\vartheta) \leqslant \prod_{i=1}^{s} \exp(c_{l_{v_i}} n_{v_i}) \exp(n_{v_i} \gamma^{-dl_{v_i}/2}) \leqslant \prod_{i=1}^{s} \exp(c_{l_{v_i}+1} n_{v_i}) \leqslant \exp(c_m n) \tag{3.13}$$

We want to stress that this result is strictly linked to the bound (3.8). The usual tree graph expansion[7] applied to the truncated expectation in (3.7) does not allow us to take into account efficiently the cancellations implied by the sum over the μ_i. The result would, in fact, be a bound of the type

$$|F_v^*| \leqslant (cs)^s \left(\prod_{i=1}^{s} |Q_i| \right) \left(\frac{\sum_{i=1}^{s} |Q_i|}{s} \right)^s \gamma^{-dhs} \tag{3.14}$$

which is much worse than (3.8) in the case of some trees.

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