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Renormalization Theory and Group in Mathematical Physics

GIOVANNI GALLAVOTTI

1. Introduction. Starting in 1978 new techniques have been introduced in field theory in the attempt to transform the successful scale invariance ideas, developed in theoretical physics, into an algorithm useful in the mathematical problems of field theory ([**1**, **2**]; for a more complete list of references see [**26**]).

In this way several problems received new solutions and new perspectives were opened. I mention here first a new derivation of the ultraviolet stability for superrenormalizable theories which led to the attack and solution of some new problems, including the first case of a three-dimensional gauge theory [**3**–**6**].

To be fair it should be stressed that the new approach was not born independently of the classical work on constructive field theory, where the ideas of scaling played a basic, although not very explicit and systematic, role [**7**–**10**].

Also the theory of renormalization received new impetus from new derivations of the basic results [**11**], of the recent $n!$-bounds, and of the convergence of the planar φ^4-theory [**12**–**14**].

In particular the theory of the convergence of the planar models led us to the understanding that the beta function could be defined in a mathematically rigorous way and thus used to construct a field theory which is not superrenormalizable (but renormalizable and asymptotically free).

The notion of beta function in [**15**, **16**]* can in principle be extended to the planar nonrenormalizable theories to study their nontrivial realizations, or to the Gross-Neveu model in slightly more than 2 dimensions: two cases in which the above extensions of the beta function have a well-defined meaning, being expressed by convergent series, in the domain of interest [**18**–**20**].

The novelty of the approach even with respect to the classical problems of perturbation theory is that it made it possible to produce a rigorous proof of renormalizability for quantum electrodynamics (in 4 dimensions) together with natural bounds on the perturbation series coefficients: at least the possibility of

*See also §20 in the review paper [**26**]. Another model for which the beta function has the same properties of convergence as in the planar theory is the 2-dimensional Gross-Neveu model (discussed in [**17**]).

a renormalizability proof was well known, but a true proof was missing because the technical problems were hard to handle with classical tools [**21**].

Finally, the implications of the above techniques for the theory of the critical point and more generally for statistical mechanics seem to be under active investigation and far from being exhausted in their potentialities, even though statistical mechanics was the first field of mathematical physics where scaling ideas were applied [**22–25**].

2. The beta function. I cannot enter into too many details here but I wish to provide at least some of the ideas behind the above cited works recently dedicated to the theory of the beta function in the case of φ^4-theories in 4 dimensions (nonplanar or planar).

In Euclidean field theory the basic object is the free field: i.e., a gaussian random field on $\mathbf{R}^d$ with covariance operator

$$C = (1-\Delta)^{-1}, \qquad \Delta = \text{Laplacian}, \tag{2.1}$$

which has a rather singular kernel (so that the sample fields of the corresponding process are distributions on $\mathbf{R}^d$ in $H^{\text{loc}}_{-d/2+1-\varepsilon}$, i.e., "far from ordinary functions").

Such singular fields φ admit a "scaling decomposition" into regular fields:

$$\varphi_x = \lim_{N\to\infty} \sum_{k=0}^{N} \varphi_x^{(k)} \equiv \lim_{N\to\infty} \varphi_x^{(\le N)}, \qquad x \in \mathbf{R}^d, \tag{2.2}$$

where $\varphi_x^{(k)}$ has the same distribution as $\gamma^{(d-2)k/2}\varphi^{(0)}_{\gamma^k x}$, but is very smooth and essentially independently distributed on the scale γ^{-k}: here γ is an arbitrary prefixed parameter (usually one takes $\gamma = 2$), and furthermore the fields $\varphi^{(h)}, \varphi^{(k)}$ are independently distributed if $h \neq k$.

Intuitively one should think of $\varphi^{(k)}$ as a random field of large size $O(\gamma^{(d-2)k/2})$ but constant over cubes with side length γ^{-k}, i.e. "on scale" γ^{-k}, and furthermore with values independently distributed over different cubes.

The problem of field theory (for scalar fields) is to give a meaning to probability measures on the space of fields on $\mathbf{R}^d$ (i.e., on the space of distributions on $\mathbf{R}^d$) such as

$$\left(\exp \int_\Lambda V(\varphi_x)\,dx\right) P(d\varphi) \equiv \lim_{N\to\infty} \left(\exp \int_\Lambda V^{(N)}(\varphi_x^{(\le N)})\,dx\right) P(d\varphi^{(\le N)}), \tag{2.3}$$

with Λ being a fixed volume, say a cube, and $V^{(N)}$ some suitable sequence of functions; P is the free field distribution.

We restrict our attention here to "φ^4-theories," i.e., to special V's which are fourth-order polynomials:

$$\begin{aligned} V^{(N)}(\varphi_x^{(\le N)}) &= \overline{\lambda}_N \varphi_x^{(\le N)4} + \overline{\mu}_N \varphi_x^{(\le N)2} + \overline{\nu}_N + \overline{\alpha}_N (\partial \varphi_x^{(\le N)})^2 \\ &= \gamma^{4N}(\lambda_N H_4(X_x) + \mu_N H_2(X_x) + \nu_N + \alpha_N H_2(\partial X_x)) \end{aligned} \tag{2.4}$$

where $\overline{\lambda}_N, \ldots, \overline{\alpha}_N$ are arbitrary constants, $X_x = \varphi_x^{(\leq N)}/\sqrt{\gamma^{2N}}$ is a "normalized" field, and H_n are the Hermite polynomials ($H_0 = 1,\ H_1 = x,\ H_2 = x^2 - 1/2, \ldots$). The second way of writing the fourth-order polynomial, in terms of Hermite polynomials, is natural as is well known ("Wick ordering"). The factor γ^{4N} is inserted for convenience and is canceled when the integration over x is done if X_x is regarded as constant on the volume element, i.e., on the cubes of scale γ^{-N}.

The first basic idea is to find bounds on (2.3) by introducing the "effective potentials"

$$\exp A_k^{(N)}(\varphi^{(\leq k)}) \equiv \int \exp\left(\int_\Lambda V^{(N)}(\varphi_x^{(\leq N)})\,dx\right) P(d\varphi^{(k+1)}) \cdots P(d\varphi^{(N)}) \tag{2.5}$$

with the purpose of proving the existence of the limit $\lim_{N\to\infty} A_k^{(N)}(\varphi^{(\leq k)})$, $k = 0, 1, \ldots$.

The second idea is that although $A_k^{(N)}$ is a considerably complicated functional of $\varphi \equiv \varphi^{(\leq k)}$, it consists in fact of a "simple" "relevant" part of the form (L stands for "local"):

$$A_k^{(N)L} \equiv \int_\Lambda (\overline{\lambda}_k \varphi_x^4 + \overline{\mu}_k \varphi_x^2 + \overline{\nu}_k + \overline{\alpha}_k (\partial\varphi_x)^2)\,dx, \tag{2.6}$$

which we always think of as written in Wick ordered form, i.e. in terms of the Hermite polynomials of $X_x = \varphi_x^{(\leq k)}/\sqrt{\gamma^{2k}}$ as in (2.4), plus a remainder $R_k^{(N)} = A_k^{(N)} - A_k^{(N)L}$.

The remainder is "irrelevant" in various senses; here we simply mean that it is expressible in terms of the "form factors" $\mathbf{c}_k = (\lambda_k, \alpha_k, \mu_k, \nu_k)$ which, in turn, are "self-sufficient" because they satisfy a recursion relation:

$$\mathbf{c}_k = L\mathbf{c}_{k+1} + B(\mathbf{c}_{k+1}) \equiv T(\mathbf{c}_{k+1}) \tag{2.7}$$

where $B(\mathbf{c})$ is a formal power series in $\mathbf{c}$ and L is a diagonal matrix with diagonal $\operatorname{diag}(L) = (1, 1, \gamma^2, \gamma^4)$.

The main result of the renormalization group approach to renormalization theory is that the coefficients of B can be bounded; if B is

$$B(\mathbf{c}) = \sum_{p=2}^{\infty} \sum_{\substack{\mathbf{m}\in Z_+^4 \\ |\mathbf{m}|=p}} \beta(\mathbf{m}; N)\mathbf{c}^{\mathbf{m}}, \tag{2.8}$$

there is a constant $\beta > 0$ such that

$$\begin{aligned} &|\beta(\mathbf{m}; N)| \leq (p-1)!\,\beta^{p-1}, \qquad |\mathbf{m}| = p,\ \forall N, \\ &\lim_{N\to\infty} \beta(\mathbf{m}; N) = \beta(\mathbf{m}) \quad \text{exists } \forall \mathbf{m} \in Z_+^4. \end{aligned} \tag{2.9}$$

The above bounds embody all the results of perturbation theory; once proved they imply the finiteness, as well as explicit bounds (the natural $n!$-bounds of [**12**]), of the coefficients of the formal power series expressing the "irrelevant" remainders in terms of the form factors $\mathbf{c}_0$ [**15**, **16**, **26**].

It seems reasonable (no proof, however, exists) that if one could overcome the problems of convergence of the series (2.8) for B, then "by the same argument," the corresponding problems for the series for R_k should disappear.

This indicates that the key questions seem to be:

(1) Give a summation rule for (2.7) which is meaningful for $\mathbf{c} \in \mathcal{D} \subset \mathbf{R}^4$ where $\mathcal{D}$ is some (a priori unknown) suitable domain.

(2) Show that $\mathcal{D}$ is invariant for the flow of the "renormalization map": $T^{-1}\mathcal{D} \subset \mathcal{D}$.

Without an answer to questions (1) and (2) the theory remains a purely formal perturbation theory, finite to every order but with open convergence problems (nevertheless even this order by order statement is rather nontrivial; it is completely solved by the above approach, which also yields explicit and best (to date) bounds).

(3) Check the compatibility of the resulting stochastic process, with the axioms that it should fulfill in order to be interpreted as a quantum field theory.

The last question is essential for the interest of the theory: it is in fact quite clear that the results in [**27**, **28**] can be interpreted as solutions to (1) and (2) above which do not satisfy (3).

There are very few cases in which the above program can be carried through: basically they coincide with the cases where the series (2.8), or the corresponding one for models other than φ^4, admit bounds so much better than (2.9) that one is allowed to define unambiguously the beta function B because the series (2.8) converges.

In the planar φ^4-theory and in the 2-dimensional Gross-Neveu model the $(p-1)!$ can be replaced by 1; hence the series is convergent (see [**15**, **16**, **26**] for planar φ^4 and [**17**] for Gross-Neveu) for $|\mathbf{c}|$ small, thus answering unambiguously (1) with $\mathcal{D} = \{\mathbf{c} \,|\, |\mathbf{c}| < \delta\}$.

One can, in the latter cases, pass to the analysis of question (2) above (the third does not make sense for planar φ^4, while it is not hard in the case of the Gross-Neveu model because of the lack of ambiguity in answering (1)).

We then look for a set $\sigma \subset \mathcal{D}$ such that $T^{-k}\sigma \subset \mathcal{D}$, $\forall k \geq 0$. For instance a surface σ such that $T^{-k}\mathbf{c}_0 \underset{k\to\infty}{\longrightarrow} 0$ if $\mathbf{c}_0 \in \sigma$.

The great advantage of having a convergent beta function is that such a question can be easily answered by standard perturbation and bifurcation theory by truncating the series (2.8) to its second-order terms.

For instance, for planar φ^4-theory, $d = 4$, the series (2.8) truncated to second order becomes:

$$\begin{aligned}
\lambda_k &= \lambda_{k+1} + \beta\lambda_{k+1}^2 + \delta\lambda_{k+1}\mu_{k+1}, \\
\alpha_k &= \alpha_{k+1} + \beta'\lambda_{k+1}^2 + \delta'\lambda_{k+1}\mu_{k+1}, \\
\mu_k &= \gamma^2\mu_{k+1} + \beta''\lambda_{k+1}^2 + \varsigma''\mu_{k+1}^2 - \theta''\alpha_{k+1}\mu_{k+1}, \\
\nu_k &= \gamma^4\nu_{k+1} + \beta'''\lambda_{k+1}^2 + \varsigma'''\mu_{k+1}^2 + \theta'''\alpha_{k+1}\mu_{k+1} + \varepsilon'''\alpha_{k+1}^2,
\end{aligned} \tag{2.10}$$

where $\beta, \delta', \ldots, \varepsilon'''$ are positive, easily computable (see [**26**, (20.20), (20.21)]) constants.

Then an elementary analysis shows that for $\lambda_0 > 0$, $\alpha_0 = O(\lambda_0^2)$ suitably chosen, and λ_0 small, one can choose μ_0, ν_0 so that $\lambda_k = O(\lambda_0/(1+\beta k\lambda_0))$, $\mu_k = O(\lambda_k^2)$, $\nu_k = O(\lambda_k^2)$, $\alpha_k = O(\lambda_k^2)$, solving (2) with $\mathcal{D}$ becoming now, for instance, restricted to the sequence $\mathbf{c}_k$.

In this case the check of the convergence of the remainders R_k is indeed, as expected, a very simple technical matter and one obtains in this way a complete construction of the planar φ^4-theory [**15**, **16**]; unfortunately this is a little unsatisfactory because, a priori, the planar φ^4-theory is known to be unphysical and it is meaningless to ask question (3). The same ideas, however, can be applied to the physically meaningful Gross-Neveu model in 2 dimensions [**17**], and in this case question (3) is easily answered using the lack of ambiguity in problem (1).

The above technique can be extended to nonrenormalizable theories. However, it turns out that it is no longer possible to introduce even a formal recursion relation linking only $\mathbf{c}_k$ and $\mathbf{c}_{k+1}$: for instance, the simplest renormalizable theory is obtained by replacing the free field operator (2.1) by $C = (1-\Delta)^{-1+\varepsilon/2}$, which also admits a scaling decomposition like (2.2) with $\varphi^{(k)}$ of order $O(\gamma^{(d-2+\varepsilon)k/2})$. Considering the φ^4-theory in 4 dimensions with this free field one finds that (2.7) is replaced by

$$\mathbf{c}_k = L\mathbf{c}_{k+1} + B(\mathbf{c}_{k+1}, \mathbf{c}_{k+2}, \dots, \mathbf{c}_N), \tag{2.11}$$

where B is a formal power series and L is a diagonal matrix,

$$\operatorname{diag}(L) = (\gamma^{2\varepsilon}, \gamma^{\varepsilon}, \gamma^{2+\varepsilon}, \gamma^4).$$

The relation (2.11) does not, of course, uniquely fix B: nevertheless B can be defined in a natural way and the coefficients of its formal power series can be bounded uniformly in N.

The main use of such series is again in the case of theories for which B is convergent when $\sup_k |\mathbf{c}_k|$ is small: the above φ^4-theory in the planar version is an interesting example. Felder [**19**] finds a nontrivial planar theory by simply proving that the equation

$$\mathbf{c} = L\mathbf{c} + B(\mathbf{c}, \mathbf{c}, \mathbf{c}, \dots) \tag{2.12}$$

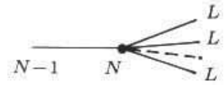
has a solution $\mathbf{c}_0$ within the domain of convergence of the series for B (uniformly in N, of course). A similar situation is met in the Gross-Neveu model in $2+\varepsilon$ dimensions [**20**].

3. The beta function and the tree expansion. The mathematical definition of the beta function is easily formulated in terms of the "tree expansion" [**15**, **16**].

The recursive evaluation of the integrals defining the effective potential (2.5) can be represented graphically by suitably interpreting the formal expansion (Taylor expansion)

$$\begin{gathered}\int e^{A_N^{(N)}} P(d\varphi^{(N)}) = \exp \sum_{n=1}^{\infty} \frac{\mathcal{E}_N^T(A_N^{(N)}; n)}{n!} \equiv \exp A_{N-1}^{(N)}, \\ \mathcal{E}_N^T(A_N^{(N)}; n) \equiv \frac{\partial^n}{\partial\theta^n} \log \int (\exp\theta A_N^{(N)}) P(d\varphi^{(N)})|_{\theta=0}.\end{gathered} \tag{3.1}$$

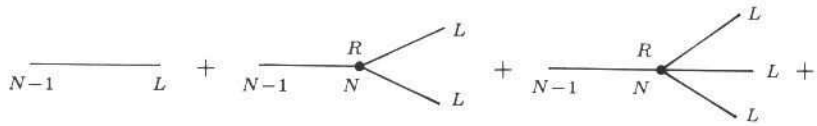
One represents $A_N^{(N)}$ graphically as $_N$——$_L$ and $n!\mathcal{E}_N^T(A_N^{(N)};n)$ as



i.e., by a vertex with subscript N, "scale index," representing $\mathcal{E}_N^T$, and n lines, recalling the order n, emerging from it; the extra line ending with the scale index $N-1$ reminds us that the object represented graphically is a functional of $\varphi^{(\le N-1)}$. The L reminds us that $A_N^{(N)}$ is "purely local", i.e., an integral of a function of $\varphi_x^{(\le N)}$: see (2.6).

By means of a projection operator $\mathcal{L}_{N-1}$ we "extract" from each of the terms in the sum in (3.1) the "local part," i.e., a part which has the same form as (2.6), and collect all the results in a term which, of course, will have again the same form (2.6) with coefficients $(\lambda_{N-1}, \alpha_{N-1}, \mu_{N-1}, \nu_{N-1})$ and which we shall denote graphically $_{N-1}$——$_L$.

In this way we can write the sum in (3.1) in a graphical form:

 (3.2)

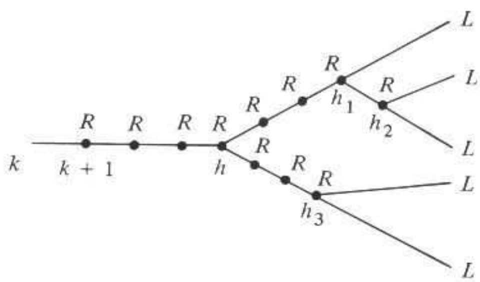
where the term $_{N-1}$—$\frac{R}{N}$—$_L$ is missing because $_{N-1}$—$\frac{R}{N}$—$_L$ is already "local" by definition, and $\frac{R}{N}$ symbolizes the operation $(1-\mathcal{L}_{N-1})\mathcal{E}_N^T$.

There is some ambiguity in the choice of the projection operator $\mathcal{L}_{N-1}$: one can select it in such a way as to simplify the formalism. Basically it turns out that there is only one natural choice, and precisely one should select a projection $\mathcal{L}_{N-1}$ which "commutes" with the integrations over different scales; i.e., such that:

$$\mathcal{L}_h\mathcal{E}_{h+1}\cdots\mathcal{E}_q \equiv \mathcal{E}_{h+1}\cdots\mathcal{E}_q\mathcal{L}_q, \qquad \mathcal{E}_q(\cdot) \equiv \int \cdot P(d\varphi^{(q)}), \tag{3.3}$$

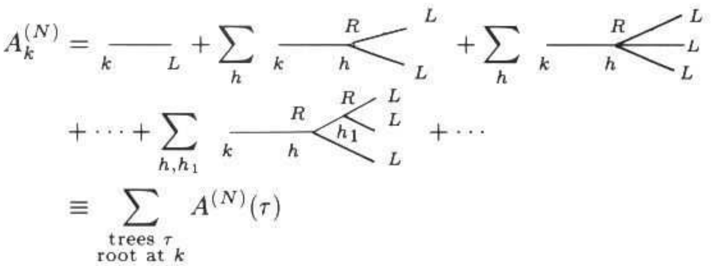
as can be done (see [**15**, **16**], and [**26**]).

Then, iterating the expansion (3.1), one reaches a representation of $A_k^{(N)}$ in terms of objects like the following:



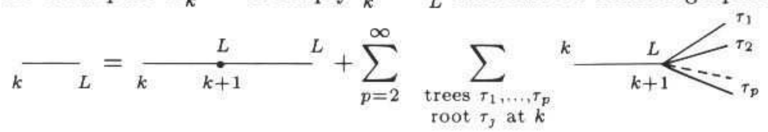
where each vertex bears a scale label representing an operation $\mathcal{E}_h^T$. The trivial vertices —$\frac{R}{h}$— are obviously redundant and can be eliminated.

Thus one obtains a representation of $A_k^{(N)}$ in terms of "trees" τ with root scale k:



(3.4)

and the local part $A_k^{(N)L}$ is simply ${}_k$—${}_L$ and can be defined graphically by:



(3.5)

where ${}_{k+1}^{L}$ represents the operation $\mathcal{L}_k \mathcal{E}_{k+1}^T$ and the terms with $p \geq 2$ are distinguished from those with $p = 1$ because the latter contribute the linear part in the recursion relation (2.6).

If one writes (3.5) explicitly, one finds a relation between the four coefficients $\mathbf{c}_k = (\lambda_k, \alpha_k, \mu_k, \nu_k)$ in $A_k^{(N)L}$ and the coefficients $\mathbf{c}_{k+1}, \mathbf{c}_{k+2}, \ldots$ in the form of a formal power series:

$$\mathbf{c}_k = L\mathbf{c}_{k+1} + B(\mathbf{c}_{k+1}, \mathbf{c}_{k+2}, \ldots, \mathbf{c}_N),$$

L is linear diagonal and $\operatorname{diag}(L) = (\gamma^{\varepsilon}, \gamma^{2\varepsilon}, \gamma^{2+\varepsilon}, \gamma^4)$,

$$B(\mathbf{c}_{k+1}, \ldots) = \sum_{s=1}^{\infty} \sum_{\substack{h_1, \ldots, h_s \\ h_i \geq k+1}} \sum_{\mathbf{m}_1, \ldots, \mathbf{m}_s \in Z_+^4} \beta(h_1, \ldots, h_s, \mathbf{m}_1, \ldots, \mathbf{m}_s) \mathbf{c}_{h_1}^{\mathbf{m}_1} \cdots \mathbf{c}_{h_1}^{\mathbf{m}_s}, \tag{3.6}$$

with the β's independent on N and such that:

$$\sum_{s} \sum_{h_1, \ldots, h_s} \sum_{\substack{\mathbf{m}_1, \ldots, \mathbf{m}_s \\ \sum |\mathbf{m}_j| = p}} |\beta(h_1, \ldots, \mathbf{m}_s)| \leq \beta^{p-1}(p-1)!. \tag{3.7}$$

Then the theory is renormalizable when the relation (3.6) can be solved recursively allowing us to express $\mathbf{c}_{k+1}, \mathbf{c}_{k+2}, \ldots$ as a formal power series in $\mathbf{c}_k$ with coefficients uniformly bounded in N; nonrenormalizable otherwise. This happens respectively when the free field covariance is $(1 - \Delta)^{-1}$, i.e. $\varepsilon = 0$, or $(1 - \Delta)^{-1-\varepsilon/2}$, with $\varepsilon > 0$.

4. Applications to statistical mechanics. The tree expansion is useful in statistical mechanics too. In fact a beginning of the tree expansion can be found already in the proof of the Debye screening in the 3-dimensional Coulomb gas [**29**]. It has been applied to the identification of the phase transitions heralding the Kosterlitz-Thouless regime in the 2-dimensional Coulomb gas, in studying the smoothness properties of the pressure $p(\beta, \lambda)$ as a function of the inverse temperature β and of the activity λ, and in various problems on the "massive Yukawa gas."

If

$$V(x) \underset{x\to\infty}{\cong} \frac{1}{2\pi}\log\left(\frac{|x|}{r_0}\right)^{-1}$$

is the Coulomb potential and the interaction between the charges (assumed to be ± 1) is regularized at short distances, then one can show, by a tree expansion technique [**30**–**32**], that $p(\beta,\lambda)$ is $2n$ times continuously differentiable, at $\lambda = 0$, in λ if:

$$\beta > \beta_n = 8\pi(1 - 1/2n), \qquad n = 1, 2, \dots. \tag{4.1}$$

Hence for $\beta > 8\pi$ it is C^∞ at the origin in λ; and in all cases the "Mayer expansion" (of $p(\beta,\lambda)$ in series of λ at $\lambda = 0$) is asymptotic up to the order for which it makes sense ($2n$ if $\beta > \beta_{2n}$).

The physical interpretation of the thresholds (4.1) is probably in terms of a sequence of phase transitions: for $\beta > \beta_n$ the gas contains a macroscopic fraction of charges bound into "stable" molecules with $2, 4, \dots, 2n$ atoms. A long way, however, still remains towards a rigorous proof of the above picture; basically what is still missing is a macroscopic description of the equilibrium states above the thresholds (4.1).

Another open problem is whether the series for $p(\beta,\lambda)$ in powers of λ is in fact convergent (and not just asymptotic) for $\beta > 8\pi$. This has been sometimes suggested as possible. It is in fact true in a related model which is a hierarchical version of the Coulomb gas [**33**].

Some progress in the techniques that may be helpful in such a question, particularly if it turns out to have a positive answer, has been achieved in the recent work of Benfatto [**35**], where it is shown how to use the tree expansion to prove the analyticity of the Mayer expansion for $p(\beta,\lambda)$ in the Yukawa gas in 2 dimensions.

The latter gas is like the Coulomb gas except that now

$$V(x) \cong \frac{1}{2\pi}\log\left(\frac{|x|}{r_0}\right)^{-1} \qquad \text{for } x \to 0,$$

while $V(x)$ decays exponentially at ∞.

For this gas one can show [**34**] that $p(\beta,\lambda)$ is analytic at $\lambda = 0$ if $\beta < 4\pi$, by using classical cluster expansion techniques. In [**35**] the same result is derived by using the tree expansion; however, as remarked by T. Kennedy (private communication), more is proved in the paper [**35**], namely

$$p_2(\beta,\lambda) \equiv p(\beta,\lambda) - \frac{1}{2!}\frac{\partial^2 p}{\partial\lambda^2}(\beta,\lambda)|_{\lambda=0}\lambda^2 \tag{4.2}$$

is in fact analytic in β through the threshold $\beta_1 = 4\pi$ up to (but excluding) the next threshold at $\beta_2 = 6\pi$. This extension is obviously implicit in [**35**], but further extensions seem to involve problems of the same nature that one meets in trying to understand whether the Coulomb gas pressure is analytic at $\lambda = 0$ for $\beta > 8\pi$. The related question for the Yukawa gas seems to be: consider

$$p_{2n}(\beta, \lambda) = p(\beta, \lambda) - \sum_{j=0}^{n} \frac{1}{(2j)!} \frac{\partial^{2j} p(\beta, \lambda)}{\partial \lambda^{2j}}|_{\lambda=0} \lambda^{2j}; \tag{4.3}$$

is it analytically continuable in β up to $\beta < \beta_n$ for λ small?

It seems plausible that one could prove the analyticity in λ of p_4 at $\beta = \beta_2 = 6\pi$ included by using techniques similar to those in [**35**], combined perhaps with ideas based on the beta function; but the question for $\beta > 6\pi$ seems considerably harder.

The whole situation is quite open and slightly frustrating: for instance it is not clear whether the tree expansion technique is really suited for the above problems. There are cases where it does not seem to be capable of reproducing results known by other methods. I refer here to the case of the 2-dimensional dipole gas where [**6**] it is shown that $p(\beta, \lambda)$ is analytic in λ at small λ. The latter result does not seem to follow from the tree expansion in a straightforward way, as might be expected, which means that in some sense the tree expansion may be an "over-expansion" hiding some cancellations, even in models where things seem to be quite simple, like the hierarchical models.

It would be desirable to understand such cancellation mechanisms to incorporate them in the tree expansion techniques.

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