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SOME PROBLEMS ON THE HAMILTON-JACOBI EQUATION

Giovanni Gallavotti*
Dipartimento di Matematica
II^a Università di Roma
Via Raimondo, 00173 Roma, Italia

I shall devote attention to some old, but still open, problems on the Hamilton-Jacobi equation reviewing the context in which they arise.

One can formulate the Hamilton-Jacobi equation as a rather general conjugacy problem between dynamical systems.

Suppose given two regular (i.e. real analytic) Hamiltonian systems (H,W) and (H_0,W_0), defined respectively by the Hamilton functions H and H_0, regular on some W and W_0 in phase space (i.e. W and W_0 are open subsets in the cotangent bundles to regular Riemannian manifolds). We consider the problem of the existence of a regular completely canonical map C, mapping W_0 onto W, and of a regular function F, defined on $H_0(W)$ = {range of H_0}, such that:

i) $C : W_0 \leftrightarrow W$

ii) $$H(C(p',q')) = F(H_0(p',q')) \quad (p',q') \in W_0 \tag{1}$$

iii) there is a regular function Φ defined on the graph $G(C) = \{p,q,p',q' | (p,q) \times (p',q') \in W \times W_0,$ $(p,q) = C(p',q')\}$ such that:

$$p \cdot dq = p' \cdot dq' + d\Phi \quad \text{on} \quad G(C) \tag{2}$$

i.e. C is "action preserving".

The properties i), ii), iii) are a global way of giving a meaning to the Hamilton-Jacobi equation, so that it is relevant for applications to mechanical problems. It is easy to check that the function on $G(C)$

$$S(p',q) = p' \cdot q + \Phi(p',q)$$

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defined wherever this makes sense (i.e. (p',q) determine a point $(p,q) \times (p',q') \in G(C)$), verifies the classical Hamilton-Jacobi equation

$$H\left(\frac{\partial S}{\partial q}(p',q),q\right) = F\left(H_0\left(p', \frac{\partial S}{\partial q}(p',q)\right)\right) \tag{3}$$

and one can regard i), ii), iii) as a precise way of stating the requirements on S to consider it an interesting solution of (3). Often one considers only the special case in which H_0 depends only on p'; then (3) has a slightly similar form [1].

The meaning of the equation (3), or better of i), ii), iii), is easy to understand. It simply implies that the Hamiltonian flows (H,W) and (H_0,W_0) are pointwise isomorphic, up to a trivial time rescaling (in fact C maps orbits into orbits together with the motion on them, up to a time rescaling equal to $\frac{dF}{dE}(E)$ if E is the orbit energy, $(H_0 \equiv E$ on the orbit)).

The above formulation of (3) is not very natural when (H_0,W_0) has nontrivial regular constants of motion $A_1(p',q') \equiv H_0(p',q)$, $A_2(p',q'),\ldots,A_r(p',q')$ independent and in involution. In this case the equation (3) is more naturally generalized to a form which, interpreted more precisely, amounts at replacing ii) above by:

ii') $$H(C(p',q')) = F(A_1(p',q'),\ldots,A_r(p',q')) \tag{4}$$

with F now regular on the domain $A(W_0)$ image of $(p',q') \to (A_1(p',q'),\ldots,A_r(p',q'))$. The replacement of ii) by ii') in i), ii), iii) means that (H,W) is a flow which is a linear combination with "constant coefficients" of the (commuting) flows generated by $A_1,\ldots,A_r$.

Here $r \leq n$, if n is the number of degrees of freedom: so if (H_0,W_0) is ergodic on the energy surfaces it is necessarily $r = 1$, while if (H_0,W_0) is integrable and $A_1,\ldots,A_n$ are the n action variables, it is $r = n$ and the solubility of i), ii'), iii) means that (H,W) is also integrable and its motions are quasiperiodic with frequencies and invariant tori trivially related to those of (H_0,W_0).

Let me recall here that (H_0,W_0) is said to be integrable if in suitable canonical coordinates W_0 can be given the form $V \times T^n$ with V open in R^n and T^n = n-dimensional torus, so that if $(A,\varphi) \in V \times T^n$ it is $H_0(A,\varphi) = h_0(A)$ for some h_0 [1].

A concrete example of the case when $r = 1$ is the Hamiltonian flow corresponding to the geodesic flow on a compact surface of constant negative curvature: in this case $H_0(p,q) = \frac{1}{2} g(q)p \cdot p$ is the kinetic energy (g being the metric) and for $0 < E_- < E_+ < \infty$

$$W_0 = \{(p,q)\,|\,H_0(p,q) \in (E_-,E_+)\} .$$

We consider now the case in which $H \equiv H_\varepsilon = H_0 + \varepsilon f_\varepsilon$ where f_ε is regular in ε too for $|\varepsilon|$ small: i.e. we consider the situation

arising in "perturbation theory" in which one is given a one parameter family of Hamiltonians.

One can look for solutions of i), ii'), iii) of the form of a power series in ε:

$$\begin{aligned} C &= \text{identity} + \varepsilon C_1 + \varepsilon^2 C_2 + \cdots \quad \text{on} \quad W_\varepsilon \subset W_0 \\ F &= A_1 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots \qquad \text{on} \quad J_\varepsilon \subset A(W_0) \end{aligned} \tag{5}$$

with domains W_ε, J_ε being also unknown.

One says that perturbation theory is "well defined" on W_0 if one can find a <u>formal</u> power series (5) solving i), ii'), iii) on W_0 and $J_0 = A(W_0)$.

In general perturbation theory is not well defined.

For instance the condition i), ii) and iii) of existence of a globally defined generating function Φ implies that, if $\gamma_i(\varepsilon,E),\gamma_j(\varepsilon,E)$ are two isoenergetic periodic solutions to the system (H,W) with energy $H = E$ and depending analytically on ε for ε small enough, then (if i), ii'), iii) are verified):

$$\frac{\oint_{\gamma_i(\varepsilon,E)} p \cdot dq}{\oint_{\gamma_j(\varepsilon,E)} p \cdot dq} = \text{very special } \varepsilon\text{-dependence} \tag{6}$$

as it is easy to check. Expanding (6) in powers of ε one finds a family of conditions on (H,W) which have to be necessarily verified, if formal perturbation theory exists.

An application arises, for instance, in the above mentioned case of the geodesic flow on a surface of constant negative curvature: in this case (H_0,W_0) is, for every $H_0 = E$, an Anosov flow and admits a dense set of periodic orbits which can be numbered $\gamma_1(E),\gamma_2(E),\ldots$: they have nonzero Lyapunov exponents and, by structural stability, persist under perturbation for $|\varepsilon|$ small enough. Hence they can be continued analytically in ε into families of orbits $\gamma_i(\varepsilon,E)$ periodic for (H,W).

Then, in this case, (6) becomes

$$\delta_{ij}(\varepsilon,E) = \frac{\oint_{\gamma_j(\varepsilon,E)} p \cdot dq}{\oint_{\gamma_i(\varepsilon,E)} p \cdot dq} = \varepsilon\text{-independent} = \frac{\ell_i}{\ell_j} \tag{7}$$

if ℓ_i is the length of the closed geodesic on which the i-th periodic orbit runs with energy E (in fact $\oint_{\gamma_j(0,E)} p \cdot dq = \sqrt{2E}\,\ell_j$, as is easy to compute).

In fact one can prove [2]:

<u>Proposition 1</u>: (Collet, Epstein, G.): If (H_0,W_0) is a geodesic flow on a surface of constant negative curvature then the condition

that for every pair i,j of periodic geodesics the nontrivial Taylor coefficients of the expansion in ε of (6) vanish is necessary and sufficient for the existence of perturbation theory for a given family (H,W) of perturbations.

A comparably simple result in the case in which (H_0,W_0) is integrable is not easy to formulate: if we use action-angle coordinates so that

$$W_0 = V \times T^n \quad V \text{ open in } R^n$$
$$H_0(A,\varphi) = h_0(A) \quad (A,\varphi) \in V \times T^n \tag{8}$$

the only simple criterion of existence of perturbation theory is, ("Birkhoff theorem"):

$$h_0(A) = \omega_0 \cdot A \text{ and } \omega_0 \text{ such that for some } C_0,\delta_0 > 0:$$
$$|\omega_0 \cdot v| \geq \frac{1}{C_0|v|^{\delta_0}}, \quad \forall v \in z^n,\ v \neq 0\ , \tag{9}$$

which physically corresponds to a "harmonic oscillator" of "nonresonant type" [3,1].

The nonintegrability theorem of Poincaré can be formulated, on the other hand, as a criterion for nonexistence of perturbation theory: if (H_0,W_0) has the form (8) and

$$\det \frac{\partial^2 h_0(A)}{\partial A \partial A} \neq 0 \tag{10}$$

then, "generically" on the perturbation f_ε (e.g. if $f_0(A,\varphi)$ has nonvanishing Fourier coefficients in φ), perturbation theory does not exist, ([4], and for an expository article [5]).

We come now to the question of convergence of perturbation theory. It is well known that, even if existent, perturbation series (5) need not be convergent. For instance in the above nonresonant harmonic oscillators one easily builds a counter example by choosing $n = 2$ and

$$f(A,\varphi) = (A_1 + g(\varphi_1)g(\varphi_2)) \tag{11}$$

which can be shown to be nonintegrable unless g is very special, and yields formal perturbation series which can be formally summed into functions which exhibit a simple but remarkable structure of dense singularities in ε, [6,5]. For a general class of results on nonconvergence, see [3].

There is, however, a (restrictive) convergence criterion working in the above nonresonant oscillators case [7]:

Proposition 2: (Rüssmann): if $F_0,F_1,F_2,\ldots$ in (5) have the form $F_j(A) = \psi_j(\omega_0,A)$, for some ψ_j, then the series in (5) converge in domains $W_\varepsilon,J_\varepsilon$ with boundaries close within $O(\varepsilon)$ to those of W_0,J_0.

As we shall see later it is remarkable that the above criterion is really relevant for some nontrivial applications. Unfortunately no satisfactory necessary and sufficient criterion is known for convergence, of perturbation theory of integrable systems even in the above nonresonant harmonic oscillators case.

The situation has to be contrasted with the following result [2]:

Proposition 3: (Collet, Epstein, G.): in the case of proposition 1 perturbation theory is convergent, whenever it exists, for ε small.

In other words the adiabatic invariants (6) associated with the pairs of isoenergetic periodic orbits form a complete set of invariants for the conjugacy problem posed by the Hamilton-Jacobi equation, i.e. by i), ii'), iii) above.

The above result extends a rigidity theorem of Guillemin-Kazhdan dealing with the geodesic flows on (variable) negative curvature compact surfaces perturbed by a (very) special perturbation, namely by a perturbation quadratic in the p-variables [8]. It has been extended to surfaces of variable negative curvature, with rather different methods, by De la Llave, Marco, Moryon [9].

A natural problem related with the above proposition is the following: suppose that $\varepsilon \equiv 1$, i.e. that (H,W) is not a family of Hamiltonians but, rather, it is a given fixed system. Assume that the periodic orbits of (H,W), (H_0,W_0) can be labeled by $i = 1,2,\ldots$ and by their energy as $\gamma_i(E)$. Suppose that

$$\frac{\oint_{\gamma_i(E)} p \cdot dq}{\oint_{\gamma_j(E)} p \cdot dq} = \frac{\ell_i}{\ell_j}, \quad i,j = 1,2,\ldots \tag{12}$$

where $\gamma_i(E)$ is the i-th periodic orbit on the surface $H = E$, for (H,W), and ℓ_i is the length of the closed geodesic on which the corresponding periodic orbit for (H_0,W_0) runs. Then: are (H,W), (H_0,W_0) conjugated canonically in the sense i), ii), iii)?

This problem is very hard as it cannot be attacked by perturbation theory even if H is very close to H_0 (but is not, a priori, a member of a one parameter family of perturbations for which perturbation theory exists), and really new ideas seem necessary.

De la Llave has made progress in questions analogous to the latter in the case of conjugacy problems for maps [10].

It is amusing to remark that one can think of a lot of other necessary conditions for the solubility of the conjugacy problem i), ii), iii), for instance it is clear that special ε-dependence has to be fulfilled also by the ratios of the periods or of the Lyapunov exponents of corresponding isoenergetic periodic orbits [2]: however such invariants may not be sufficient to ensure the solubility of i), ii), iii): a counterexample is discussed in [2].

The problem i), ii)', iii) is only one example of a class of problems related to the Hamilton-Jacobi equation.

I wish to mention here a further extension of the equation (4), which is inspired by renormalization theory in the theory of fields [11].

Suppose (H,W) to be a family of perturbations of (H_0,W_0), as above, for which perturbation theory does not exist or, even if existent, is not convergent.

Then given a subset $C \subset$ {analytic functions on W_0} one can ask: can one find a family $N_\varepsilon \in C$, regular in ε too, such that $(H + N_\varepsilon, W)$ admits a convergent perturbation theory with respect to (H_0,W_0) in the sense of i), ii'), iii) and (5). And one can either prescribe F or leave it free; thus defining two related problems.

We say that the above problems are well posed in C if, up to an additive constant, there is one and only one formal power series solution for N_ε.

To show the interest of the above question let me present an interesting case in which it arises naturally (which, unfortunately, is also the only case known to me).

Suppose that (H_0,W_0) is a nonresonant harmonic oscillator and C = {set of linearly A-dependent functions on N_0}. Then the above problem becomes whether or not one can find a function $N_\varepsilon(A)$ analytic on W_0 and ε for ε small such that

$$\omega_0 \cdot A + \varepsilon F_\varepsilon(A,\varphi) + N_\varepsilon(A) \quad \text{on} \quad V \times T^n w_0 \tag{13}$$

is conjugate to $\omega_0 \cdot A$ in the sense i), ii), iii) (here F_ε is prescribed).

It can be shown, easily, that the problem is well-posed (in the above sense): however, the question of convergence is rather unclear.

It can be shown, in fact:

Proposition 4: (G., Chierchia): The above problem is well posed and yields convergent power series solutions for $N \in C$, as well as for C and F. Therefore N can be written

$$N_\varepsilon(A) = A \cdot a(\varepsilon) \tag{14}$$

and $a(\varepsilon)$ is analytic in ε near $\varepsilon = 0$.

The proof of the above proposition [6,5,12] is, in essence, a repetition of the proofs of the Rüssmann proposition 2 quoted above, and of the main proofs in the theory of Dinaburg, Sinai, [13], Rüssmann [14] for the (apparently unrelated) quasiperiodic one-dimensional Schrödinger equation.

Basically one determines $N_\varepsilon(A)$ by imposing that the criterion of proposition 2 for convergence of Birkhoff series is verified order by order in ε and, simultaneously, one proves the convergence of the algorithm [6,5,12].

The interest of the result is its mentioned relation with the Schrödinger equation [6].

Consider the problem

$$-q'' + (\varepsilon V(\omega_0 t + \varphi) - E)q = 0, \qquad t \in R \tag{15}$$

where V is periodic, analytic, on T^n, $\varphi \in T^n$ (e.g. $\varphi = 0$) and $\omega_0 \in R^n$ verifies the nonresonance condition (9): $q'' = \frac{d^2}{dt^2}$.

It is easy to check that (15) are the Hamilton equations for a Hamiltonian system described by canonically conjugated pairs of variables $(p,q),(B_1,\varphi_1),\dots,(B_n,\varphi_n)$ in R^2 for (P,φ) and in $R^n \times T^n$ for (B,φ). The Hamiltonian is, in fact:

$$\frac{p^2}{2} + \frac{1}{2}(E + \varepsilon V(\varphi))q^2 + \omega_0 \cdot B \ . \tag{16}$$

Replacing (P,q) with the action angle variables (A,ψ) for the oscillator $(P^2 + Eq^2)/2$ one describes (16) via the Hamiltonian

$$\sqrt{E}\, A + \omega_0 \cdot B + \frac{A}{\sqrt{E}}\, \varepsilon V(\varphi)\cos^2\psi \ . \tag{17}$$

It is easy to see that an invariant torus for the system (17) corresponds to a quasiperiodic solution of (15), i.e. the value E is in the continuum spectrum of the Schrödinger operator.

However, (17) needs not be integrable: nevertheless the above proposition 4, applied to the present case [6,5], yields the existence of a function $a(\varepsilon,E,\lambda)$, analytic in ε such that

$$\lambda A + \omega_0 \cdot B + \frac{A\varepsilon}{\sqrt{E}} V(\varphi)\cos^2\psi + Aa(\varepsilon,E,\lambda) \tag{18}$$

is integrable and conjugated to $\lambda A + \omega_0 \cdot B$ if $(\lambda,\omega_0) = \omega \in R^{n+1}$ also verifies a nonresonance condition like (9).

It follows that for such λ's the value E_ε such that

$$\sqrt{E_\varepsilon} = \lambda + a(\varepsilon,E_\varepsilon,\lambda) \tag{19}$$

is in the continuous spectrum of the Schrödinger operator. On this remark one can build the theory of the continuous spectrum of the quasiperiodic Schrödinger equation developed by Dinaburg, Sinai, Rüssmann (see Chierchia [12]).

It seems plausible that the type of questions like "is it possible to add to H a term δH of specified form so that $H + \delta H$ becomes conjugate to a given H_0" may arise in applications other than the above and therefore it would be nice to know more results in this direction.

REFERENCES

[1] G. Gallavotti, The Elements of Mechanics, Springer, N. Y., 1983.

[2] P. Collet, H. Epstein and G. Gallavotti, 'Perturbations of geodesic flows on surfaces of constant negative curvature and their mixing properties', Comm. Math. Phys. **95** (1984), 61-112.

[3] J. Moser, 'Stable and Random Motions', Princeton University Press, Annals of Math. Studies, vol. **77** (1973).

[4] H. Poincaré, Methodes Nonvelles de la Mécanique Celeste, Geuthier-Villar, 1897.

[5] G. Gallavotti, Les Houches Notes, 1984; 'Quasi integrable mechanical systems', In Phénomenes Critiques, Systèmes aleatoires, théorie de Jange, K. Ostervalder, R. Stora, and Les Houches, eds., session XLIII, part II, pagine 539-623, Reidel, 1986.

[6] G. Gallavotti, 'Classical Mechanics and Renormalization Group', In Regular and Chaotic Motions in Dynamic Systems, G. Velo and A. Wightman, eds., Erice, 1983, pp. 185-232, Plenum, 1985.

[7] H. Russmann, 'Uber normalform analytischer hamiltonscher differentialgleitchungen in der nähe einer gleichgewichtlösung', Math. Annalen **169** (1980), 55.

[8] V. Guillemin, A. 'Kazhdan', Topology **19** (1980), 291-299 and 301-312.

[9] R. De la Llave, J. Marco and R. Moryon, Ann. Math. (1985).

[10] R. De la Llave, private comm.

[11] G. Gallavotti, 'A criterion of integrability', Comm. Math. Phys. **87** (1982), 365-383.

[12] L. Chierchia, Thesis, published in Quaderni del C.N.R.-G.N.F.M. (1986).

[13] E. Dinaburg and Y. Sinai, 'The one dimensional Schrödinger equation with a quasi periodic potential', Funct. Analysis Appl. **9** (1975), 279.

[14] H. Rüssmann, 'On the one dimensional Schrödinger equation with a quasi periodic potential', Ann. N.Y. Acad. Sci. **90** (1979), 197.