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# The Integrability Problem and the Hamilton–Jacobi Equation: A Review

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There are very few Hamiltonian systems which are understood in a qualitative and quantitative way: the integrable ones play a prominent role among them.

Therefore the problem of solving the Hamilton–Jacobi equation, which can be regarded as the problem of finding whether a system is integrable or not, is considered important.

Given a Hamiltonian system $(H, \mathrm{W})$ with Hamiltonian $H$ analytic on the phase space region W

$$H(\boldsymbol{p}, \boldsymbol{q}) = \tfrac{1}{2}\boldsymbol{p}^2 + V(\boldsymbol{q}) \tag{1}$$

the Hamilton–Jacobi equation requires to find two functions $S$ and $F$ such that

$$\frac{1}{2}\left(\frac{\partial S}{\partial \boldsymbol{q}}(\boldsymbol{p}', \boldsymbol{q})\right)^2 + V(\boldsymbol{q}) = F(\boldsymbol{p}') \tag{2}$$

and such that the relations

$$\boldsymbol{p} = \frac{\partial S}{\partial \boldsymbol{q}}(\boldsymbol{p}', \boldsymbol{q}), \qquad \boldsymbol{q}' = \frac{\partial s}{\partial \boldsymbol{p}'}(\boldsymbol{p}', \boldsymbol{q})$$

define a regular (analytic or $C^\infty$) correspondence $\mathscr{C}$: $(\boldsymbol{p}, \boldsymbol{q}) \leftrightarrow (\boldsymbol{p}', \boldsymbol{q}')$ mapping W onto $\mathrm{V} \times \mathrm{T}^l$ where V is an open set in $\mathrm{R}^l$ and $\mathrm{T}^l$ an $l$-dimensional torus.

Usually one requires also that

$$\boldsymbol{p} \cdot \mathrm{d}\boldsymbol{q} = \boldsymbol{p}' \cdot \mathrm{d}\boldsymbol{q}' + \mathrm{d}\Phi \tag{3}$$

with $\Phi$ defined on the graph of $\mathscr{C}$: $\mathrm{G}(\mathscr{C}) = \{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p}', \boldsymbol{q}' | (\boldsymbol{p}, \boldsymbol{q}) = \mathscr{C}(\boldsymbol{p}', \boldsymbol{q}')\}$ and regular (i.e. analytic or $C^\infty$): "action preserving canonical map".

If such a solution of the Hamilton–Jacobi equation exists we find that in the new variables the motion is:

$$\boldsymbol{p}' = \boldsymbol{p}'_0 = \text{const.}, \qquad \boldsymbol{q}' = \boldsymbol{q}'_0 + \boldsymbol{\omega}(\boldsymbol{p}'_0)\, t \tag{4}$$

i.e. it is quasi-periodic (and $\boldsymbol{\omega}(\boldsymbol{p}') = \operatorname{grad} F(\boldsymbol{p}')$). More generally given two Hamiltonian systems $(H, \mathrm{W})$, $(H', \mathrm{W}')$ we wish to know if there exists a regular canonical map $\mathscr{C}$ of W onto W′ and a regular function $F$ such that:

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$$H'(\mathscr{C}^{-1}(p, q)) = F(H(p, q), A_2(p, q), \dots, A_s(p, q)) \tag{5}$$

where $A_2, \dots, A_s$ are the other smooth prime integrals of $H$ (if any) and W, W′ are invariant sets for the corresponding Hamiltonian flows.

If we require $\mathscr{C}$ to be canonical and action preserving we can turn easily the above equation onto a partial differential equation for its generating function $\Phi$ (see (3)). A particular context where the above problem arises is when $H'$ is

$$H_\varepsilon(p, q) = H(p, q) + \varepsilon f(p, q) \tag{6}$$

with $H, f$ analytic on W = ($H$-invariant open set in phase space) and in $\varepsilon$ for $\varepsilon$ close to zero. In this context we look for a family of maps $\mathscr{C}_\varepsilon$ and functions $F_\varepsilon$ such that

$$H_\varepsilon(\mathscr{C}_\varepsilon(p, q)) = F_\varepsilon(H(p, q), A_2(p, q), \dots, A_s(p, q)) \tag{7}$$

where $\mathscr{C}_\varepsilon(p, q)$ is an action preserving and $F_\varepsilon(E, A_2, \dots, A_s)$ is a function and such objects are analytic in their variables and in $\varepsilon$ for $\varepsilon$ near zero; furthermore they differ from the identity map and from the function $E$, respectively, up to terms divisible by $\varepsilon$ and $\mathscr{C}_\varepsilon$, $F_\varepsilon$ should be defined in a domain such that (7) makes sense in a region $W^{(\varepsilon)} \subset W$ large (e.g. such that $\partial W^{(\varepsilon)}$ is within $O(\varepsilon)$ of $\partial W$).

Writing $\Phi_\varepsilon$ the generating function of $\mathscr{C}_\varepsilon$ (i.e. $S_\varepsilon = p' \cdot q + \Phi_\varepsilon$) and

$$\begin{aligned} f(p, q, \varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k f^{(k)}(p, q), \qquad \Phi_\varepsilon(p', q) = \sum_{k=1}^{\infty} \varepsilon^k \Phi^{(k)}(p', q) \\ F_\varepsilon(E, A_2, \dots, A_s) &= E + \sum_{k=1}^{\infty} \varepsilon^k F^{(k)}(E, A_2, \dots, A_s) \end{aligned} \tag{8}$$

the equation (7) becomes:

$$H_\varepsilon'\left(p', q + \frac{\partial \Phi_\varepsilon}{\partial p'}(p, q)\right) = F_\varepsilon\left(H\left(p' + \frac{\partial \Phi_\varepsilon}{\partial q}(p', q), q\right), A_2\left(p' + \frac{\partial \Phi_\varepsilon}{\partial q}(p', q), q\right), \dots\right) \tag{9}$$

and, everything being analytic, we can transform (9) into a hierarchy of equations by developing in powers of $\varepsilon$ both sides and by equating the coefficients of equal order. One finds easily that:

$$\begin{aligned} \{H, \Phi^{(k)}\} &= P^{(k)} - \langle P^{(k)} \rangle, \qquad P^{(1)} = f^{(0)} \\ F^{(k)} &= \langle P^{(k)} \rangle \end{aligned} \tag{10}$$

where

$$\begin{aligned} P^{(k)} &= \{\text{polynomial in the derivatives of } \Phi^{(j)}, f^{(j)}, 0 \le j \le k-1\} \\ \langle \cdot \rangle &= \{\text{average over the surface } H(p, q) = H,\ A_2(p, q) = A_2, \dots\} \end{aligned} \tag{11}$$

assuming that the surfaces $\sigma$ of constant $H, A_2, \dots, A_s$ have a finite area. The $\{\cdot, \cdot\}$ denotes the Poisson bracket, as usual.

Since $\{H, \cdot\}$ is the derivative along the solutions of the Hamilton equations with Hamiltonian $H$ we see that if the surfaces $\sigma$ contain a dense orbit or a periodic orbit then the equations (10) may not be soluble as their right-hand side must verify some compatibility relations. Therefore we shall say that the perturbation $f$ is "formally integrable" if all the equations (10) can be solved recursively for all $k \geq 1$ by solving (recursively) the equation of order $k$ and choosing the solutions $\Phi^{(k)}$ such that $\langle \Phi^{(k)} \rangle = 0$ (note, in this respect, that $\Phi^{(k)}$ is defined up to a function of $H, A_2, \ldots, A_s$ so that this choice is always possible if a solution exists). We also say, alternatively, that $f$ "admits a perturbation theory". In this case we can ask: does the perturbation series converge? Basically the cases studied in the literature can be reduced (sometimes via a non-trivial work) to two:

(1) $$H_0(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{2} H_i(\boldsymbol{p}, \boldsymbol{q}) \equiv \sum_{i=1}^{2} \frac{p_i^2 + \omega_i^2 q_i^2}{2}, \qquad \mathrm{W} = \{0 < E_- < H_i < E_+ < +\infty\}$$

(2) $$H_0(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{2} \frac{p_i^2 - \omega_i^2 q_i^2}{2}, \qquad \mathrm{W} = \{0 < E_- < H_0 < E_+ < +\infty\}.$$

In case (1), "harmonic oscillators", one considers only the "non-resonant cases": i.e. when for all $r_1$, $r_2$ integers and for some $c > 0$

$$|\boldsymbol{\omega} \cdot \boldsymbol{r}| \equiv |\omega_1 r_1 + \omega_2 r_2| > c^{-1}(|r_1| - |r_2|)^{-2}, \qquad |r_1| + |r_2| > 0 \tag{12}$$

which is enough to insure that perturbation theory is well defined permitting a formal solution of the Hamilton–Jacobi equations. (This can be easily seen by changing variables into action angle variables for the harmonic oscillator which transform the equations (10) into

$$\begin{aligned} &\boldsymbol{\omega} \cdot \frac{\partial \Phi^{(k)}}{\partial \boldsymbol{\varphi}}(\boldsymbol{A}', \boldsymbol{\varphi}) = P^{(k)}(\boldsymbol{A}', \boldsymbol{\varphi}) - F^{(k)}(\boldsymbol{A}') \\ &F^{(k)}(\boldsymbol{A}') = (2\pi)^{-2} \int_{\mathrm{T}^2} P^{(k)}(\boldsymbol{A}', \boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{\varphi} \end{aligned} \tag{13}$$

which can be solved trivially by Fourier transform: "Birkhoff theorem" [1].)

However the example (see [1]):

$$\tfrac{1}{2}(p_1^2 + 2q_1^2) + \tfrac{1}{2}(p_2^2 + q_2^2) + \varepsilon\left(\tfrac{1}{2}(p_2^2 + q_2^2) + f\left(\frac{p_1}{\sqrt{p_1^2 + 2q_1^2}}, \frac{\sqrt{2}q_1}{\sqrt{p_1^2 + 2q_1^2}}, \frac{p_2}{\sqrt{p_2^2 + q_2^2}}, \frac{q_2}{\sqrt{p_2^2 + q_2^2}}\right)\right) \tag{14}$$

which becomes in action-angle coordinates for the two oscillators simply $\sqrt{2}A_1 + A_2 + \varepsilon(A_2 + \bar{f}(\varphi_1, \varphi_2))$ yields the simplest example of a non-convergent perturbation series if the Fourier transform of $\bar{f}$ never vanishes (it is easy to compute explicitly $\Phi^{(k)}$, $f^{(k)}$ and one finds that the series for $\Phi_\varepsilon$ does not converge).

The intricacy of the convergence of perturbation theory problems can be seen by the following positive result (which is a version of the well-known "KAM theorem" [2, 3, see also 8]) which we formulate by studying the Hamiltonian $H_\varepsilon = H_0 + \varepsilon f$ in the action-angle coordinates of the unperturbed harmonic oscillator, i.e. by writing $H_\varepsilon$ as:

$$H_\varepsilon(\boldsymbol{A}, \boldsymbol{\varphi}) = \boldsymbol{\omega} \cdot \boldsymbol{A} + \varepsilon f(\boldsymbol{A}, \boldsymbol{\varphi}) \tag{15}$$

and by introducing the following parameters which characterize the physical scales of the problem:

($\alpha$) $\rho, \theta, \xi > 0$ — "analyticity parameters" describing how far in the complex the function $f$ can be extended in the variables $\boldsymbol{A}$, $\varepsilon$, $\boldsymbol{\varphi}$ resp.

($\beta$) $\Omega = |\omega_1| + |\omega_2|$ — "inverse time scale"

($\gamma$) $E = \sup\left(\left|\frac{\partial f}{\partial \boldsymbol{A}}\right| + \frac{1}{\rho}\left|\frac{\partial f}{\partial \boldsymbol{\varphi}}\right|\right)$ — "perturbation scale"

($\delta$) $\eta = \sup\left|\frac{\partial^2 \bar{f}^{(0)}}{\partial \boldsymbol{A}\, \partial \boldsymbol{A}}(\boldsymbol{A})^{-1}\right|$ — "resonance mobility"

where we suppose, for simplicity, $\xi < 1$, $\theta < 1$, $E/\Omega < 1$, and the max is evaluated in the regions of analyticity of $f$ defined by the parameters $\rho$, $\theta$, $\xi$, and $\bar{f}^{(0)} \equiv (2\pi)^{-2} \int_{T^2} f^{(0)}\, d\boldsymbol{\varphi}$.

Then, assuming for simplicity $\theta$, $\xi$, $\Omega/c < 1$:

*Theorem*: There exist constants $a_1, \ldots, a_5$, $G > 0$ such that, given $\lambda \in (0, 1)$, if:

$$\varepsilon/\theta < G\, (E/\Omega)^{-a_1}\, (E\eta\rho^{-1})^{-a_2}\, \xi^{a_3} (C\Omega)^{-a_4}\, \lambda^{a_5} \tag{16}$$

(i.e. if $\varepsilon$ is small compared to "all the dimensionless quantities")

(1) There exist a $C^\infty$-map $\mathscr{C}_\varepsilon$ and a $C^\infty$-function $F_\varepsilon$ ($C^\infty$ in $\varepsilon$, $\boldsymbol{A}$, $\boldsymbol{\varphi}$) such that $\mathscr{C}_\varepsilon$: $V \times T^2 \to V \times T^2$ and is canonical and, for some $V^{(\varepsilon)} \subset V$:

$$H_\varepsilon(\mathscr{C}_\varepsilon(\boldsymbol{A}', \boldsymbol{\varphi}')) = F_\varepsilon(\boldsymbol{A}'), \qquad \forall (\boldsymbol{A}', \boldsymbol{\varphi}') \in V^{(\varepsilon)} \times T^2 \tag{17}$$

(2) $\mathrm{vol}(V^{(\varepsilon)} \times T^2) \geq \mathrm{vol}(V \times T^2)(1 - \lambda)$

(3) $V^{(\varepsilon)}$ = set obtained from V by taking out of it a layer of width of order $O(\varepsilon)$ near $\partial V$ and of order $|\boldsymbol{r}|^{-3}\, O(\varepsilon)$ around the surface

$$\sigma_{\boldsymbol{r}} = \left\{\boldsymbol{A}' | \boldsymbol{A}' \in V, \left|\boldsymbol{r} \cdot \frac{\partial F^{(\varepsilon)}}{\partial \boldsymbol{A}'}(\boldsymbol{A}')\right| = 0\right\}, \qquad \forall \boldsymbol{r} \neq \boldsymbol{0}\,,\ \boldsymbol{r} \in Z^2\,.$$

(4) If $\Phi^{(k)}$, $F^{(k)}$ are the perturbative solutions of Hamilton–Jacobi equations mentioned above, the generating function $\Phi_\varepsilon$ of $\mathscr{C}_\varepsilon$ and $F_\varepsilon$ are such that

$$\begin{aligned} \Phi_\varepsilon(\boldsymbol{A}', \boldsymbol{\varphi}) &= \sum_{k=1}^{N} \varepsilon^k\, \Phi^{(k)}(\boldsymbol{A}', \boldsymbol{\varphi}) + O(\varepsilon^{N+1}) \\ F_\varepsilon(\boldsymbol{A}') &= \boldsymbol{\omega} \cdot \boldsymbol{A}' + \sum_{k=1}^{N} \varepsilon^k\, F^{(k)}(\boldsymbol{A}') + O(\varepsilon^{N+1}) \end{aligned} \tag{18}$$

hold with a remainder independent on $A', \varphi$ provided $(A', \varphi') \in V^{(\varepsilon)} \times T^2$ (i.e. "Birkhoff's series are asymptotic").

(5) Finally the relations obtained by differentiating both sides of (17) arbitrarily many times hold as identities on $V^{(\varepsilon)} \times T^2$ (note that $V^{(\varepsilon)}$ has a dense open complement by (3) above so that this does not follow from (17)).

In other words the Hamilton–Jacobi equation is soluble on a large set and its solution is given by perturbation theory (the interpretation of the above version of KAM theorem in terms of the Hamilton–Jacobi equation is due to Pöshel (thesis) who proves the above theorem in a differentiable case [2]; the analytic case result is in [3]).

A simple consequence of the above theorem is the following. Consider a one-parameter family of initial data $\tau \to (\boldsymbol{p}(\tau), \boldsymbol{q}(\tau))$. In the $(A', \varphi)$-variable (defined by $\mathscr{C}_\varepsilon$) this line will be generically transversal to the smooth resonant surfaces appearing in the statement (3) of the above theorem.

Suppose we study the motions evolving from the various initial data as a function of $\tau \in [0, 1]$ by measuring the "two basic frequencies"* close to $(\omega_1, \omega_2)/2\pi$ of the motions, if they are quasi-periodic with frequencies close to $(\omega_1, \omega_2)/2\pi$: if the motions are chaotic or quasi-periodic with very different "basic frequencies" we say that the result of the measurements is zero (by convention). Then from the above result we have to expect a priori a result of the following type: if

$$\tau \to \mu_\varepsilon(\tau) \equiv \left(\frac{\partial F_\varepsilon}{\partial A_1'}(A'(\tau)) \Big/ \frac{\partial F_\varepsilon}{\partial A_2'}(A'(\tau))\right)$$

is the rotation number that the motions with initial data $(A(\tau), \varphi(\tau))$, $\tau \in [0, 1]$, would exhibit if (17) held on the whole $V \times T^2$ we shall see a measured rotation number which will vary smoothly outside a (dense) set of open intervals corresponding to data that fall too close to the resonant bands in $V \times T^2$: in fact there are arguments suggesting that the bands are not an artefact of the proof but should, rather, be generically present. In actual numerical experiments one shall, of course, see only finitely many bands because of resolution problems (related among other things to the fact that the time scale necessary to reveal that the motion is not close to the unperturbed (quasi-periodic) motion is proportional to $\varepsilon^{-1}$ and depends on the order $\boldsymbol{r}$ of the resonance involved).

We turn now to the case 2, of the perturbations of the hyperbolic oscillator which we prefer to describe, when $\omega_1 = \omega_2 = 1$, in a system of coordinates rotated of 45° as

$$H_0(p_1, p_2, q_1, q_2) = \tfrac{1}{2}(p_1 q_1 + p_2 q_2)\,. \tag{19}$$

The hyperbolic oscillators arise in a natural way when one wishes to study the perturbations of the motions with data close to data producing an unstable periodic orbit.

To model a linearly stable orbit of a general Hamiltonian system V, with period $T = 2\pi/\omega_1$ and Lyapunov exponent $\exp(i\omega_2 T)$ it is possible to use a harmonic oscillator Hamiltonian

$$H_0 = \sum_{i=1}^{2} \tfrac{1}{2}(p_i^2 + \omega_i^2 q_i^2) \tag{20}$$

and this means that it is possible to set up, near the orbit, a system of canonical coordinates in which the Hamiltonian takes the form (20) "up to higher order terms".

* Suitably selecting the two basic frequencies in some empirical way.

Similarly we can use the hyperbolic oscillator (19) to model a hyperbolic orbit of period $T$ and Lyapunov exponent $\exp\lambda$, $\lambda > 0$. It is enough to identify the points of the phase space of (19), $R^4$, modulo a canonical map so that a periodic orbit of period $T$ and Lyapunov exponent $\lambda$ appears in the new phase space.

For instance if

$$g = \begin{pmatrix} p_1 & -q_2 \\ p_2 & q_1 \end{pmatrix}$$

denotes the point $(p_1, p_2, q_1, q_2) \in R^4$ in the phase space and if we define the map (canonical and $H_0$-preserving)

$$g \to g' = \gamma g \equiv e^{\lambda \sigma_2} g \tag{21}$$

where $\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we introduce on the surface $H_0 = 1$ a periodic orbit of Lyapunov exponent $\lambda$ and period $\lambda$ (more generally if we replace $H_0$ by $H_0 c$ we find a periodic orbit of exponent $\lambda$ and period $\lambda/c$): then if a Hamiltonian system has a periodic orbit of similar characteristics (period and Lyapunov exponent) we can introduce a system of coordinates which are canonical and good near the given orbit and in which the system's Hamiltonian appears as a perturbation of the Hamiltonian (19) regarded with a phase space $R^4$ modulo the canonical map (21).

More generally one is led to study dynamical systems described by Hamiltonians

$$Q(H_0(\boldsymbol{p}, \boldsymbol{q})) + \varepsilon f(\boldsymbol{p}, \boldsymbol{q}, \varepsilon) \equiv H_\varepsilon(\boldsymbol{p}, \boldsymbol{q}) \tag{22}$$

where the points of phase space are identified modulo the actions of a discrete subgroup $\Gamma$ of the ($H_0$-preserving) canonical linear maps of $R^4$ into itself and $Q$ is an arbitrary function (e.g. $Q(x) = x$). If the group $\Gamma$ has more than one generator, one can find interesting examples of systems of type (21) with compact surfaces of constant energy. For instance if $\Gamma$ consists of maps of the form $g \to g' = \gamma g$ with $\gamma \in SL(2, R)$ and if such matrices form a hyperbolic Fuchsian group $\bar{\Gamma}$, then the Hamiltonian system on $\Gamma/R^4$ with

$$Q(H_0(g)) = \tfrac{1}{8}(p_1 q_1 + p_2 q_2)^2 \tag{23}$$

is isomorphic to a geodesic flow on the surface of constant negative curvature generated by the given Fuchsian group $\bar{\Gamma}$, [4]. (The reader familiar with hyperbolic geometry can readily check that the map of the flow generated by (23) on $\Gamma/R^4$ and the geodesic flow on the surface of constant negative curvature on the phase space of the coordinates $(p_x, p_y, x, y)$ describing the flow in the Lobatchewsky plane is [4]:

$$p_x + i p_y = \tfrac{1}{2} i (p_1 + i q_2)^2$$

$$x + i y = (p_2 + i q_1)/(p_1 - i q_2)$$

which is a canonical map.) We now analyze perturbation theory for (22) when $Q(x) = x$.

*Theorem*: Consider the system (22) on a phase space of the form $\Gamma/R^4$ where $\bar{\Gamma}$ is a hyperbolic discrete subgroup of SL(2, R) generating a smooth compact surface of constant negative curvature and suppose that $f$ is real analytic in $\varepsilon$ and in $g \in \Gamma/\mathrm{R}^4$. Suppose also that the perturbation theory is well defined on $\mathrm{W}(E_-, E_+) = \{g | 0 < E_- < Q(H_0(g)) < E_+ < +\infty\}$, then, given $\delta > 0$, it is convergent in $\mathrm{W}(E_- + \delta, E_+ - \delta)$ for $\varepsilon$ small enough.

This result has to be confronted with the similar but negative results mentioned above on the non-convergence of perturbation theory of harmonic oscillators.

Another result in [4] is:

*Theorem*: Under the same assumptions of the preceding theorem label by $(E, n, \varepsilon)$ the periodic orbits on the surface $H_\varepsilon = E$ of the Hamiltonian system (22). Assume that the labelling has been done so that as $\varepsilon$ varies the orbit $(E, n, \varepsilon)$ varies continuously* for small $\varepsilon$. Then a necessary and sufficient property for the existence (hence convergence) of perturbation theory is that for all $\varepsilon$ small enough

$$\frac{\int_{(E,n,\varepsilon)} \boldsymbol{p}\cdot \mathrm{d}\boldsymbol{q}}{\int_{(E,n',\varepsilon)} \boldsymbol{p}\cdot \mathrm{d}\boldsymbol{q}} \equiv \frac{\oint_{(E,n,0)} \boldsymbol{p}\cdot \mathrm{d}\boldsymbol{q}}{\oint_{(E,n',0)} \boldsymbol{p}\cdot \mathrm{d}\boldsymbol{q}} \qquad \forall n, n'. \tag{24}$$

At this point it is natural to formulate the following question: suppose that the closed periodic orbits on the surface of the system

$$H_f = Q(H_0) + f \tag{25}$$

(with $f$ so small that the Anosov's stability theorem applies) are labelled as $(n, E, f)$ in such a way that they depend continuously on $f$ as $f$ varies close to zero together with its first 2-derivatives (see [5]) and define

$$A(n, E, f) = \oint_{(E,n,f)} \boldsymbol{p}\cdot \mathrm{d}\boldsymbol{q}\,. \tag{26}$$

*Problem*: Is it true that if $A(n, E, f) = A(n, E, f')$ then the Hamiltonians $H_f$ and $H_{f'}$ are canonically conjugated on some invariant subset of their phase spaces? In other words do the adiabatic invariants (26) suffice to classify the perturbations of the hyperbolic systems of the form $Q(H_0)$ on the phase space $\Gamma/\mathrm{R}^4$ (with $\Gamma, \bar{\Gamma}$ introduced above)? The answer might be negative in analogy with KAM theory.

To conclude this lecture let me mention another problem which arises in connection with the existence of divergent perturbation series: Is it possible to "renormalize" such divergences, in analogy with renormalization in field theory, turning them into convergent series?

Let me briefly examine this question in the case of the harmonic oscillators. We ask the following: can we find a function $N_f(\boldsymbol{A}, \varepsilon)$ analytic in $\boldsymbol{A}$ and in $f$ near zero such that

* This is possible by Anosov's stability theorem see [5] and by the denumerability of such orbits.

$$\boldsymbol{\omega} \cdot \boldsymbol{A} + \varepsilon f(\boldsymbol{A}, \boldsymbol{\varphi}) - N_f(\boldsymbol{A}, \varepsilon) \tag{27}$$

leads to a perturbation theory which converges and allows to conjugate canonically (27) with the unperturbed Hamiltonian $\boldsymbol{\omega} \cdot \boldsymbol{A}$?

Of course, contrary to what happens in field theory, there is no need to modify the Hamiltonian in such a way that the new motions will be so similar to the unperturbed ones that they can be described by a convergent perturbation theory. Hence the problem (27) must have a different interpretation from the one which arises in field theory where renormalization is necessary even to give a meaning to the theory. We can think that the above question may arise in two ways:

(i) to modify a perturbation by "simple" terms which will suppress the possibility of development of motions too different from the unperturbed ones (e.g. chaotic); in this interpretation the above problem appears as a "control problem";

(ii) to provide a class of systems with fixed rotation number: if the series in $\varepsilon$ defining $N_f(\boldsymbol{A}, \varepsilon)$ has a radius of convergence larger than the one of the perturbation series associated with (27), then this leads to the possibility that the break-down of integrability is a sudden phenomenon due to singularities on perturbation theory and therefore with well defined threshold in $\varepsilon$.

The operation of modifying $\varepsilon f$ into

$$:\varepsilon f: = \varepsilon f - N_f \tag{28}$$

will be called, when possible, the "Wick ordering" of the perturbation $\varepsilon f$ with respect to $\boldsymbol{\omega} \cdot \boldsymbol{A}$.

By making use of the following interesting theorem of Rüssmann [6]:

*Theorem*: A sufficient condition for the convergence of perturbation theory of a harmonic (non-resonant) oscillator (15) is that the sequence of functions $F^{(k)}$ which are associated with the Hamilton–Jacobi equation (as explained above) be such that $F^{(k)}(\boldsymbol{A}')$ depends on $\boldsymbol{A}'$ only via $\boldsymbol{\omega} \cdot \boldsymbol{A}'$,

it is possible to show that [7]:

*Proposition*: The function $N_f$ if existing, is unique: its Taylor coefficients in $\varepsilon$ are computable independently on the convergence of the Taylor series itself.

In fact in [7] the preceding convergence criterion is also proved as the author was unaware of ref. [6] (and the two proofs differ only from a notational point of view). It is not known if the formal series for $N_f$ built in the last proposition converges under suitable assumptions on $f$.

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