Bollettino U. M. I.
(6) **1**-B (1982), 1029-1038

The Dirichlet Problem and the Perron-Frobenius Theorem.

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Sunto. – *Viene presentato un metodo di soluzione per il problema di Dirichlet per l'operatore di Laplace Δ in una regione convessa. Il metodo conduce anche a una soluzione «costruttiva» del problema di Neumann.*

1. – Notations.

Let Ω be an open, bounded, convex region with C^∞-smooth boundary $\partial\Omega$. We shall study the equation:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{in } \partial\Omega \end{cases} \tag{1.1}$$

with $f \in C^\infty(\partial\Omega)$ and $u \in C^\infty(\bar{\Omega})$. Δ is the Laplace operator.

We look for a solution of the form:

$$u(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\partial\Omega} z(\boldsymbol{\xi})\,\frac{\boldsymbol{n}(\boldsymbol{\xi})\cdot(\boldsymbol{x}-\boldsymbol{\xi})\,dS_{\boldsymbol{\xi}}}{|\boldsymbol{x}-\boldsymbol{\xi}|^3}, \qquad \boldsymbol{x}\in\Omega \tag{1.2}$$

with $z \in C^\infty(\partial\Omega)$, $\boldsymbol{n}(\boldsymbol{\xi})$ = exterior normal to $\partial\Omega$, and the space dimensionality has been chosen, for definiteness, equal to three.

It is well known that there is a unique $C^\infty(\partial\Omega)$-solution to (1.1), $\forall f \in C^\infty(\partial\Omega)$ and furthermore such solution admits the representation (1.2), [1].

It is also well known that z can be determined as the unique $C^\infty(\partial\Omega)$-solution of the equation:

$$2f(\boldsymbol{x}) = z(\boldsymbol{x}) - \frac{1}{2\pi}\int_{\partial\Omega} z(\boldsymbol{\xi})\,\frac{\boldsymbol{n}(\boldsymbol{\xi})\cdot(\boldsymbol{x}-\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|^3}\,dS_{\boldsymbol{\xi}} \tag{1.3}$$

$x \in \partial\Omega$, usually written as:

$$2f = z + Kz \tag{1.4}$$

where K is the operator on $C^\infty(\partial\Omega)$ with kernel:

$$K(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{2\pi}\frac{\boldsymbol{n}(\boldsymbol{\xi})\cdot(\boldsymbol{x}-\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|^3} \tag{1.5}$$

which, [1], is an operator on $C^\infty(\Omega)$ with range in $C^\infty(\partial\Omega)$.

Usually the proof of solubility of (1.4) passes through the proof that K is, in some space, a compact operator which does not contain -1 in its spectrum [1]: this is a non constructive argument as long as nothing is said on how close to -1 the spectrum can go.

In this paper we give a simple but self contained and constructive theory of the equation (1.4) based on the classical theorem of Perron-Frobenius in the modern version of Ruelle [2].

Arguments somewhat close in spirit to those in this paper can be found in [3].

2. – Double layer potentials.

We call K and K^* the operators on $C^\infty(\partial\Omega)$ with kernels $K(\boldsymbol{x}, \boldsymbol{\xi})$ given by (1.5) and $K^*(\boldsymbol{x}, \boldsymbol{\xi}) = K(\boldsymbol{\xi}, \boldsymbol{x})$. It is immediate to check that the operators K, K^* are continous on $C^\infty(\partial\Omega)$ in the maximum norm so that they can also be thought (by extension) as operators on $C(\partial\Omega)$ regarded as a Banach space with the maximum norm.

The main properties of the operators K, K^* are summarized by the following classical lemma, [1]:

LEMMA. – *The operators K, K^* map $C^\infty(\partial\Omega)$ into itself and:*

i) *For all $a \in [0, +\infty)$ the operators K, K^* map $C^{(a)}(\partial\Omega)$ into $C^{(a+1/2)}(\partial\Omega)$, «regularization or compactness property».*

ii) *The integrals*

$$(K^{(\sigma)})^n(\boldsymbol{x}, \boldsymbol{\xi}) \equiv \tag{2.1}$$

$$\equiv \int_{(\partial\Omega)^{n-1}} K^{(\sigma)}(\boldsymbol{x}, \boldsymbol{\xi}_1) K^{(\sigma)}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \dots K^{(\sigma)}(\boldsymbol{\xi}_{n-1}, \boldsymbol{\xi})\, dS_{\boldsymbol{\xi}_1} \dots dS_{\boldsymbol{\xi}_{n-1}},$$

where $K^{(0)} = K$ and $K^{(1)} = K^*$, are absolutely convergent and coincide *with the kernels of the operators* K^n *and* $(K^*)^n$, $\forall n \in Z_+$. *Furthermore they are continous in both* $\boldsymbol{x}$ *and* $\boldsymbol{\xi}$ *for* $\boldsymbol{x} \neq \boldsymbol{\xi}$.

iii) *There exists a computable geometric constant* $g(\Omega)$ *such that*

$$\int_{\partial\Omega} |(K^{(\sigma)})^4(\boldsymbol{x}, \boldsymbol{\xi}) - (K^{(\sigma)})^4(\boldsymbol{y}, \boldsymbol{\xi})|\, dS_{\boldsymbol{\xi}} \leqslant g(\Omega)|\boldsymbol{x} - \boldsymbol{y}| \tag{2.2}$$

where $\sigma = 0$ *or* 1 *and* $K^{(0)} = K$, $K^{(1)} = K^*$.

iv) *There exist two computable geometric constants* $\alpha(\Omega)$, $\beta(\Omega)$ *such that* $K^n(\boldsymbol{x}, \boldsymbol{\xi}) > 0$, $(K^*)^n(\boldsymbol{x}, \boldsymbol{\xi}) > 0$, $\forall \boldsymbol{x} \neq \boldsymbol{\xi} \in \partial\Omega$, $\forall n \geqslant 4$ *and:*

$$0 < \alpha(\Omega) \leqslant K^4(\boldsymbol{x}, \boldsymbol{\xi}),\ (K^*)^4(\boldsymbol{x}, \boldsymbol{\xi}) \leqslant \beta(\Omega) \tag{2.3}$$

for all $\boldsymbol{x} \neq \boldsymbol{\xi} \in \partial\Omega$.

The above properties, with the exception of the last (which is a consequence of the convexity of Ω) are necessary ingredients to the classical theory of the equation (1.4), [1].

COROLLARY. – *If* $f \in C^\infty(\partial\Omega)$ *and* z *is a continous solution to* (1.4), $2f = z + Kz$, *then* $z \in C^\infty(\partial\Omega)$.

PROOF. – In fact item i) of the lemma implies togheter with the equation that z is a $C^{(1/2)}$-function and, again item i) implies that, therefore, z is a $C^{(1)}$-function etc.

Therefore the existence and uniqueness problem in $C^\infty(\partial\Omega)$ for the equation (1.4) is equivalent to the same problem in $C(\partial\Omega)$ if $f \in C^\infty(\partial\Omega)$: this is the problem that we investigate by proving:

PROPOSITION. – *If* Ω *is convex the equation* (1.4) *admits one and only one continous solution, for all* f *in* $C(\partial\Omega)$.

The following section is devoted to the proof of the proposition above (well known) by a method whose discussion is the main purpose of this paper.

3. – The Perron-Frobenius-Ruelle theorem as applied to the present problem.

If $u \in C^\infty(\bar{\Omega})$ and $\Delta u = 0$ in Ω, $\forall \boldsymbol{x} \in \Omega$:

$$u(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\partial\Omega} \frac{\partial_n u(\boldsymbol{\xi})}{|\boldsymbol{x} - \boldsymbol{\xi}|}\, dS_{\boldsymbol{\xi}} - \frac{1}{4\pi}\int_{\partial\Omega} u(\boldsymbol{\xi})\, \frac{\boldsymbol{n}(\boldsymbol{\xi})\cdot(\boldsymbol{x} - \boldsymbol{\xi})}{|\boldsymbol{x} - \boldsymbol{\xi}|^3}\, dS_{\boldsymbol{\xi}} \tag{3.1}$$

which for $u \equiv 1$ yields:

$$1 = -\frac{1}{4\pi}\int_{\partial\Omega}\frac{\boldsymbol{n}(\boldsymbol{\xi})\cdot(\boldsymbol{x}-\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|^3}\,dS_{\boldsymbol{\xi}}\,,\qquad \forall \boldsymbol{x}\in\partial\Omega \tag{3.2}$$

and by taking the limit as $\boldsymbol{x}\to\partial\Omega$ it gives, [1], (see also (1.2), (1.3)):

$$1 = \tfrac{1}{2} + \tfrac{1}{2}K1 \Rightarrow K1 = 1 \tag{3.3}$$

where 1 denotes also the function identically equal to one.

Furthermore, $\forall f \geqslant 0$, $f\in C(\partial\Omega)$, $f\not\equiv 0$, $\forall n \geqslant 8$:

$$\frac{\big((K^*)^n f\big)(\boldsymbol{x})}{\big((K^*)^n f\big)(\boldsymbol{y})} = \frac{\int_{\partial\Omega}(K^*)^4(\boldsymbol{x},\boldsymbol{\xi})\big((K^*)^{n-4}f\big)(\boldsymbol{\xi})\,dS_{\boldsymbol{\xi}}}{\int_{\partial\Omega}(K^*)^4(\boldsymbol{y},\boldsymbol{\xi})\big((K^*)^{n-4}f\big)(\boldsymbol{\xi})}\,dS_{\boldsymbol{\xi}}\,. \tag{3.4}$$

The integrals involve positive functions, as $n \geqslant 8$, which are continous for $\boldsymbol{\xi}\neq\boldsymbol{x},\boldsymbol{y}$. And we can infer, $\forall n\geqslant 8$, from the inequality:

$$\min_j\frac{a_j}{b_j} \leqslant \frac{\sum_j a_j}{\sum_j b_j} \leqslant \max_j \frac{a_j}{b_j}\,,\qquad \forall a_j, b_j > 0 \tag{3.5}$$

conveniently applied to the Riemann sums approximating the integrals in (3.2) that, if $C=\beta(\Omega)/\alpha(\Omega)$ and if (2.3) is used:

$$C^{-1} \leqslant \frac{\big((K^*)^n f\big)(\boldsymbol{x})}{\big((K^*)^n f\big)(\boldsymbol{y})} \leqslant C\,,\qquad \forall f\geqslant 0,\ f\in C(\partial\Omega),\ \forall n\geqslant 8,\ \forall \boldsymbol{x},\boldsymbol{y}\in\partial\Omega\,. \tag{3.6}$$

There must exist, $\forall n\geqslant 8$, points $\boldsymbol{y}_n$ such that $\big((K^*)^n f\big)(\boldsymbol{y}_n) \geqslant \|f\|_1$ and points $\boldsymbol{y}'_n$ such that $\big((K^*)^n(\boldsymbol{y}'_n) \leqslant \|f\|_1$ where

$$(f, g) = \int_{\partial\Omega} f(\boldsymbol{\xi})g(\boldsymbol{\xi})\,\frac{dS_{\boldsymbol{\xi}}}{S(\Omega)}\,,\qquad f, g\in C(\partial\Omega)\quad\text{and}\quad \|f\|_1 = (|f|, 1) \tag{3.7}$$

if $S(\Omega) =$ surface area of $\partial\Omega$; this is because when $f\geqslant 0$:

$$\int_{\partial\Omega}\big((K^*)^n f\big)(\boldsymbol{\xi})\,\frac{dS_{\boldsymbol{\xi}}}{S(\Omega)} = \big((K^*)^n f, 1\big) = (f, K^n 1) = (f, 1) = \|f\|_1 \tag{3.8}$$

by (3.3).

This remark and the arbitrariness of y in (3.6) allow to deduce, by choosing $y = y_n$ or $y = y_n'$:

$$C^{-1}\|f\|_1 \leqslant ((K^*)^n f)(\xi) \leqslant C\|f\|_1 \tag{3.9}$$

for all $f \geqslant 0$, $f \in C(\partial\Omega)$, $\forall n \geqslant 8$.

Hence by taking $f = 1$ we see that the sequence $(K^*)^n 1$ is an equibounded sequence of functions in $C(\partial\Omega)$ which turns out to be also an equicontinous sequence, because by making use of (3.9) and (2.3), $\forall f \in C(\partial\Omega)$:

$$\begin{aligned} |((K^*)^n f)(x) - ((K^*)^n f)(y)| &= \Big|\int_{\partial\Omega} [(K^*)^4(x, \xi) - \\ &- (K^*)^4(y, \xi)]((K^*)^{n-4} f)(\xi)\, dS_\xi\Big| \leqslant C\|f\|_1 \int_{\partial\Omega} |(K^*)^4(x, \xi) - \\ &- (K^*)^4(y, \xi)|\, dS_\xi \leqslant Cg(\Omega)\|f\|_1 |x - y|\,. \end{aligned} \tag{3.10}$$

Hence also the sequence in $C(\partial\Omega)$:

$$0 \leqslant F_N(x) = N^{-1}\sum_{j=0}^{N-1}((K^*)^j 1)(x) \tag{3.11}$$

is an equicontinous and equibounded sequence of functions in $C(\partial\Omega)$. If h is an accumulation point in $C(\partial\Omega)$ of such a sequence (existing by the Ascoli-Arzelà criterion) one has: $h \geqslant 0$ and

$$(K^*)h = h \tag{3.12}$$

because K^* is continous on $C(\partial\Omega)$.

Since

$$(F_N, 1) = N^{-1}\sum_{j=0}^{N-1}((K^*)^j 1, 1) = N^{-1}\sum_{j=0}^{N-1}(1, K^n 1) \equiv 1 \tag{3.13}$$

it follows that $\|h\|_1 = 1$ so $h \not\equiv 0$.

Furthermore the same argument used in the corollary of § 2 says that $h \in C^\infty(\partial\Omega)$ and by (3.6):

$$C^{-1} \leqslant h(x) \leqslant C\,, \qquad \forall x \in \partial\Omega\,. \tag{3.14}$$

The properties analyzed so far for K^* have analogues for K and they are likewise proved interchanging the role of the function 1, (in the theory of K^*), with that of h.

In particular by the same argument used to obtain (3.9), (3.10) it follows that $\forall f \geqslant 0$, $f \in C(\partial\Omega)$:

$$\begin{cases} C^{-1}(h, f) \leqslant (K^n f)(\xi) \leqslant C(h, f)\,, & \forall n \geqslant 8\,, \\ |(K^n f)(x) - (K^n f)(y) \leqslant Cg(\Omega)\|f\|_1 |x - y|\,, & \forall n \geqslant 8\,. \end{cases} \tag{3.15}$$

We wish now to show that h verifying (3.12) is unique in $C(\partial\Omega)$.

Notice that if $g \in C(\partial\Omega)$ and $(h, |K^p g|) \xrightarrow[p\to\infty]{} 0$ it follows that $K^p g \to 0$ uniformly: in fact $\forall p \geqslant 8$, by (3.6):

$$|(K^p g)(x)| \leqslant \big(K^8 |K^{p-8} g|\big)(x) \leqslant C\big(K^8 |K^{p-8}|\big)(y)\,, \qquad \forall x, y \in \partial\Omega \tag{3.16}$$

and by multiplying both sides of (3.16) by $h(y)$ and by then integrating with respect to y we see that:

$$|(K^p g)(x)| \leqslant C\big(h, |K^{p-8} g|\big) \xrightarrow[p\to\infty]{} 0 \tag{3.17}$$

by the assumption.

Notice next that if $(h, g) = 0$ for $g \in C(\partial\Omega)$ and if we set $g_+ = (|g| + g)/2$ and $g_- = (|g| - g)/2$ then $(h, g_+) = (h, g_-)$, $g_\pm \in C(\partial\Omega)$ and:

$$\begin{aligned} (h, |K^n g|) &= (h, |K^n g_+ - K^n g_-|) = (h, |K^n g_+ - C^{-1}(h, g_+) + \\ &+ C^{-1}(h, g_+) - K^n g_-|) \equiv \big(h, |(K^n g_+ - C^{-1}(h, g_+)) - \\ &- (K^n g_- - C^{-1}(h, g_-))|\big) \leqslant (h, |K^n g_+ - C^{-1}(h, g_+)|) + \\ &+ (h, |K^n g_- - C^{-1}(h, g_-)|) \end{aligned} \tag{3.18}$$

but if $n \geqslant 8$, (3.15) shows that the two functions in the r.h.s. of (3.18) are actually non negative: hence the moduli in the r.h.s. of (3.21) can be suppressed, yielding:

$$\begin{aligned} (h, |K^n g|) &\leqslant \big(h, (K^n g_+ - C^{-1}(h, g_+))\big) + \\ &+ \big(h, (K^n g_- - C^{-1}(h, g_-))\big) = (1 - C^{-1})\big((h, g_+) + (h, g_-)\big) \equiv \\ &\equiv (1 - C^{-1})\big(h, |g|\big) \end{aligned} \tag{3.19}$$

by (3.12), for all $g \in C(\partial\Omega)$, with $(h, g) = 0$.

Therefore, since $(h, g) = 0$ implies $(h, K^{n-8} g) = 0$, $\forall n \geqslant 8$:

$$\begin{aligned} (h, |K^n g|) &\leqslant (1 - C^{-1})\big(h, |K^{n-8} g|\big) \leqslant \\ &\leqslant (1 - C^{-1})^{[n/8]}\big(h, |K^{n-8[n/8]} g|\big) \leqslant (1 - C^{-1})^{[n/8]}\big(h, K^{n-8[n/8]}|g|\big) = \\ &= (1 - C^{-1})^{[n/8]}\big(h, |g|\big)\,. \end{aligned} \tag{3.20}$$

This togheter with (3.17) shows that, for all $g \in C(\partial\Omega)$:

$$(3.21)\qquad (h, g) = 0 \Rightarrow |K^n g(\boldsymbol{x})| \leqslant C(1 - C^{-1})^{[n/8]-1} \cdot \big(h, |g|\big)\,.$$

In a symmetric fashion one also proves that for all $g \in C(\partial\Omega)$:

$$(3.22)\qquad (g, 1) = 0 \Rightarrow |(K^*)^n g(\boldsymbol{x})| \leqslant C(1 - C^{-1})^{[n/8]-1}\big(|g|, 1\big)\,.$$

Hence if $g \in C(\partial\Omega)$ and if we consider that $g = \big(g - (h, g)\big) + (h, g)$ or $g = \big(g - (g, 1)h + (g, 1)h\big)$ we deduce from (3.21), (3.22) respectively that:

$$(3.23)\qquad \begin{cases} |K^n g - (h, g)| \leqslant 2C(1 - C^{-1})^{[n/8]-1}\big(h, |g|\big) \\ |(K^*)^n g - (g, 1)h| \leqslant 2C(1 - C^{-1})^{[n/8]-1}\big(|g|, 1\big) \end{cases}$$

which also immediately imply that if $Kf = f$ then f must be a constant and that if $K^* f = f$ then f must be a constant multiple of h: i.e. they imply uniqueness of the solutions to the equations $Kf = f$ or $K^* f = f$ in $C(\partial\Omega)$.

We can now quickly complete the analysis of the equation (1.4) $2f = z + Kz$, $f \in C(\partial\Omega)$, as an equation in $C(\partial\Omega)$. A solution is in fact, $\forall f \in C(\partial\Omega)$:

$$(3.24)\qquad z = (h, f) + \sum_{n=0}^{\infty} (-1)^n K^n 2\big(f - (h, f)\big)$$

as it can be checked by substitution, noticing that the series is uniformly convergent by (3.23).

On the other hand if z_1, z_2 are two solutions to (1.4) $q = z_1 - z_2$ verifies $Kq = -q$, i.e. $K^n q = (-1)^n q$ which is impossible, as $K^n q \xrightarrow[n\to\infty]{} (h, q)$, unless $q \equiv 0$: so (3.24) is the only solution to (1.4) in $C(\partial\Omega)$.

4. – Physical meaning of h. the Neumann's problem and constructiveness.

The function h has a well known meaning: it is the surface charge density producing a constant potential inside $\partial\Omega$. This follows by observing that:

$$(4.1)\qquad V(\boldsymbol{x}) = \frac{1}{4\pi} \int \frac{h(\boldsymbol{\xi})}{|\boldsymbol{x} - \boldsymbol{\xi}|} \frac{dS_{\boldsymbol{\xi}}}{S(\Omega)}\,, \qquad \boldsymbol{x} \in \Omega$$

is a harmonic function in $C^\infty(\bar{\Omega})$, [1], and its normal derivative on $\partial\Omega$ is given by (see also (1.2), (1.3) which are proved in the same way, [1]):

$$\frac{1}{2}h(\boldsymbol{x})-\frac{1}{4\pi}\int_{\partial\Omega}\frac{h(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|^3}\,\boldsymbol{n}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{\xi})\,\frac{dS_{\boldsymbol{\xi}}}{S(\Omega)}= \\ =\tfrac{1}{2}(h-K^*h)\equiv 0\,, \quad \forall \boldsymbol{x}\in\partial\Omega\,. \tag{4.2}$$

Hence by the Neumann's problem uniqueness theorem it follows that $V(\boldsymbol{x})$ is a constant:

$$c^{-1}=\frac{1}{4\pi}\int_{\partial\Omega}\frac{h(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}\,\frac{dS_{\boldsymbol{\xi}}}{S(\Omega)}\,,\qquad \forall \boldsymbol{x}\in\Omega\,, \tag{4.3}$$

and the interpretation of c is that it is the electrostatic capacity of $\partial\Omega$ as a conductor and $h/S(\Omega)$ is the charge density distribution which is generated by a unit charge deposited on the conductor's surface. The equation $K^*h=h$ has the interpretation of Gauss' law as (by (4.2)) it can be read as saying that electric field outside Ω generated by the charge distribution density h is just h (because a simple calculation allows to deduce from (4.1) that the electric field generated outside Ω by h is given by the integral in (4.2) if it is computed on the surface of Ω and projected along the outer normal).

We can now easily solve the Neumann's problem:

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega \\ \partial_n u &= f \quad \text{in } \partial\Omega \end{aligned} \tag{4.4}$$

with $f\in C^\infty(\partial\Omega)$, $u\in C^\infty(\bar{\Omega})$.

We look for a solution to (4.4) of the form:

$$u(\boldsymbol{x})=\frac{1}{4\pi}\int_{\partial\Omega}\frac{z(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}\,dS_{\boldsymbol{\xi}}\,. \tag{4.5}$$

Then, [1], the computation of the normal derivative of (4.5) for a given $z\in C^\infty(\partial\Omega)$ leads to the equation, [1]:

$$2f=z-K^*z \tag{4.6}$$

which can be immediately solved (uniquely by the same argument used at the end of § 3 for the Dirichlet problem) by:

$$z = \sum_{n=0}^{\infty} (K^*)^n 2f + (\text{constant})h \tag{4.7}$$

provided the series converges: this is the case if $(f, 1) = 0$ by (3.23). However the condition $(f, 1) = 0$ is a well known condition, necessary for the solubility of (4.4): therefore $(f, 1) = 0$ is not a restrictive condition. Inserting (4.7) into (4.5) and using (4.3) we see that (4.4) has a solution determined up to a constant, if $(f, 1) = 0$.

Finally we can notice that the formulae (3.24) provide a constructive solution to the Dirichlet's problem as it follows from the inequalities:

$$\begin{cases} |h - (K^*)^n 1| \leqslant C(1 - C^{-1})^{[n/8]-1} \\ \left|z - \left(\sum_{n=0}^{2N+1} (-1)^h K^h 2f\right) - (h, f)\right| \leqslant 2C^2(1 - C^{-1})^{[(2N+2)/8]-1} \end{cases} \tag{4.8}$$

and a similar comment could be made for the (4.7) in connection with the Neumann's problem.

Even though (4.8), (4.9) provide explicit error estimates in terms of geometric constants it is clear that actual numerical programming cannot be directly based on (4.8), (4.9) by the instability caused by the fact that K, K^* do have one eigenvalue equal to 1: hence (4.8) do not provide directly a numerical algorithm in spite of the fact that they given a « constructive » solution (i.e. without compactness arguments).

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Pervenuta alla Segreteria dell' U. M. I.
il 17 giugno 1981