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SOME PROBLEMS IN HAMILTONIAN PERTURBATION THEORY

Abstract.

We present a general review of the basic facts in classical perturbation theory with a few conjectures.

1.

Consider an integrable hamiltonian system described by the hamiltonian

$$H_0(\underline{A}, \underline{\varphi}) = h_0(\underline{A}) \qquad (\underline{A}, \underline{\varphi}) \in R^l \times T^l \tag{1}$$

the $(\underline{A}, \underline{\varphi})$ being "action-angle" canonical variables, T^l being the standard l-dimensional torus, i.e. $[0, 2\pi]^l$ with "periodic boundary conditions".

For symplicity we suppose that h_0 is real analytic and strictly non isochronous

$$\det \frac{\partial^2 h}{\partial \underline{A}\, \partial \underline{A}} (\underline{A}) \neq 0 \tag{2}$$

All the motions of this system are quasi periodic

$$\left\{\begin{array}{ll} \dot{\underline{A}} = 0 & \underline{A} = \underline{A}_0 \\ \dot{\underline{\varphi}} = \dfrac{\partial h_0}{\partial \underline{A}} (\underline{A}) \equiv \underline{\omega}(\underline{A}) & \Rightarrow \quad \underline{\varphi} = \underline{\varphi}_0 + \underline{\omega}(\underline{A}_0)\, t \end{array}\right. \qquad t \geqslant 0 \tag{3}$$

every torus $\{\underline{A}\} \times T^l$ is invariant and therefore the energy surface $h(\underline{A}) = E$ is not ergodically run by the hamiltonian flow for $l \geqslant 2$.

Let now f be a "perturbation", $(\underline{A}, \underline{\varphi}) \to f(\underline{A}, \underline{\varphi})$, which, for simplicity, we suppose to be analytic i.e. if $\underline{\varphi} = (\varphi_1, \ldots, \varphi_l)$ is identified with $\underline{z} = (e^{i\varphi_1}, \ldots, e^{i\varphi_l})$ the function $\bar{f}(\underline{A}, \underline{z}) = f(\underline{A}, \underline{\varphi})$, defined on $R^l \times T^l$ (where

T^l is now regarded as a product of l unit circles in C^l), can be extended to a holomorphic function in a complex civinity of $R^l \times T^l$ in C^{2l}.

One now wishes to know whether the hamiltonian

$$h_0(\underline{A}) + \epsilon f(\underline{A}, \underline{\varphi}) \tag{4}$$

still describes quasi periodic motions or whether the hamiltonian flow associated with

$$\left\{ \begin{aligned} \dot{\underline{A}} &= -\epsilon \frac{\partial f}{\partial \underline{\varphi}} (\underline{A}, \underline{\varphi}) \\ \dot{\underline{\varphi}} &= \underline{\omega}(\underline{A}) + \epsilon \frac{\partial f}{\partial \underline{A}} (\underline{A}, \underline{\varphi}) \end{aligned} \right. \tag{5}$$

is ergodic on the energy surface.

Historically opinions seem to have oscillated between mantaining that for "generic" f one should have ergodicity and, alternatively, mantaining the contrary. Boltzmann [1], Poincaré [2], Fermi [4] had or hinted to the first opinion while Weierstrass held the opposite view (see Moser [6]). It is fair to say that Boltzmann, to explain his ideas, gave an example of a perturbation (strongly non analytic) which turned out later to exhibit the appropriate ergodicity properties,Ehrenfest [3], Boltzmann [1], Sinai [6]).

Also Poincaré always stressed that any statement on this field needed, in any case, a complete mathematical proof to be really reliable and, finally, Fermi realized in a fundamental work with Pasta and Ulam (Fermi [4]) that the "ergodic hypothesis" could not be taken in a too strict sense.

The main mathematical result on the subject is the following Kolmogorov, Arnold, Moser theorem (KAM-theorem).

Theorem. Let $W = U \times T^l \subset R^l \times T^l$ be a bounded phase space region. There is a subset $W_\epsilon \subset W$ and a subset $U_\epsilon \subset U$ such that

$$W_\epsilon = \bigcup_{\underline{A}' \in U_\epsilon} \mathscr{T}(\underline{A}') \tag{6}$$

where $\mathscr{T}(\underline{A}')$ are l-dimensional tori with parametric equations

$$\left\{ \begin{aligned} \underline{A} &= \underline{A}' + \underline{\alpha}(\underline{A}', \underline{\varphi}') \\ \underline{\varphi} &= \underline{\varphi}' + \underline{\beta}(\underline{A}', \underline{\varphi}') \end{aligned} \right. \qquad \underline{\varphi}' \in T^l \tag{7}$$

such that there is $a > 0$, $\epsilon_0 > 0$ for which $\forall |\epsilon| < \epsilon_0$:

i) the relation

$$\|\underline{\alpha}\|_\infty + \|\underline{\beta}\|_\infty \leqslant \epsilon^a \tag{8}$$

ii) the tori $\mathscr{T}(\underline{A}')$ are invariant with respect to the perturbed hamiltonian flow (5) and the motion on $\mathscr{T}(\underline{A}')$ is quasi periodic and described simply as

$$\underline{\varphi}' \to \underline{\varphi}' + \underline{\omega}_\infty(\underline{A}')\, t \tag{9}$$

where $\underline{A}' \to \underline{\omega}_\infty(\underline{A}')$ are suitable R^l-valued functions on U_ϵ

iii) it is

$$\begin{cases} |\underline{\omega}_\infty(\underline{A}') - \underline{\omega}(\underline{A}')| < \epsilon^a \\ |\underline{\omega}_\infty(\underline{A}')\cdot\underline{v}|^{-1} < C(\epsilon)\,|\underline{v}|^l \qquad \forall\, \underline{v} \in Z^l,\ \underline{v} \neq 0 \end{cases} \tag{10}$$

for a suitably chosen function $C(\epsilon)$, $C(\epsilon) \underset{\epsilon\to 0}{\to} \infty$

iv) the tori $\mathscr{T}(\underline{A}')$ fill most of phase space:

$$\text{vol}\, W_\epsilon \geqslant (1-\epsilon^a)\,\text{vol}\, W \tag{11}$$

v) furthermore if $\underline{A} \in U$, $d(\underline{A}, \partial U) > \epsilon$, $|\underline{\omega}(\underline{A})\cdot\underline{v}|^{-1} < \dfrac{C(\epsilon)}{2}\,|\underline{v}|^l$ for all $\underline{v} \neq \underline{0}$, there is $\underline{A}' \in U_\epsilon$ such that $\underline{\omega}_\infty(\underline{A}') = \underline{\omega}(\underline{A})$.

The above theorem means that for ϵ small most of phase space is covered by quasi periodically run invariant tori of dimension l. Furthermore most of the "irrationally run" tori survive perturbation.

The above theorem has, as complement, the following ("Sinai-Kubo theorem"):

Theorem. If f is only piecewise analytic there are examples of choices of f which makes the hamiltonian flow ergodic on the energy surfaces.

Observation: in the actual example the gradient of f has some discontinuity so we are talking here of generalized Hamilton equations which should be supplemented by some prescriptions to solve the non uniqueness questions.

Another important complement is that (Poincaré):

Theorem. One can find examples of integrable systems like (1) which have an invariant torus $\{\underline{A}\} \times T^l$ run with pulsations $\underline{\omega}(\underline{A})$ with rational components ("resonating torus") such that the perturbed system (4) no longer has an invariant torus close within ϵ^a to $\{\underline{A}\} \times T^l$ run quasi periodically with pulsations $\underline{\omega}(\underline{A})$.

This can be proven by an example. Just consider:

$$\frac{A_1^2}{2} + \frac{A_2^2}{2} + \epsilon \cos(\varphi_1 - \varphi_2) \tag{12}$$

For $\epsilon = 0$ the torus $A_1 = 1,\ A_2 = 1$ is run with pulsations $\underline{\omega} = (1,1)$. However it is easy to see that there cannot exist two functions $\underline{\alpha}, \underline{\beta}$ on T^2 such that the torus

$$\begin{cases} A_1 = 1 + \alpha_1(\underline{\varphi}') \\ A_2 = 1 + \alpha_2(\underline{\varphi}') \qquad \underline{\varphi}' \in T^l \\ \underline{\varphi} \ = \underline{\varphi}' + \underline{\beta}(\varphi') \end{cases} \tag{13}$$

is invariant under the flow generated by (12) and $\|\underline{\alpha}\|_\infty + \|\underline{\beta}\|_\infty \leqslant \epsilon^a$ for some $a > 0,\ \forall \epsilon \geqslant 0$.

If one is unhappy because (12) is integrable (as it has $A_1 + A_2$ as a second prime integral) another identically discussed example would be

$$\frac{A_1^2}{2} + \frac{A_2^2}{2} + \epsilon \cos(\varphi_1 - \varphi_2) + \epsilon(\cos\varphi_1)^2 \tag{14}$$

2. Quasi periodic motions and non quasi periodic or "random" motions.

The above discussion shows that the "nonresonant tori" of an integrable system survive after a small perturbation while the resonant ones don't.

It will be discussed later and it is generally believed that the resonant tori not only disappear after a perturbation is introduced but also that they are responsible for the appearance of a totally different behaviour of the solutions of (4) (compared to the unperturbed quasi periodic behaviour). Basically one expexts that if $\{\underline{A}_0\} \times T^l$ is a resonant torus, e.g. if the components of $\underline{\omega}(\underline{A})$ are rational (*), then the perturbation greatly affects the motions of initial data too close to it, i.e. those with

$$|\underline{A} - \underline{A}_0| < B \frac{\sqrt{\epsilon}}{|\underline{v}|^b} \tag{15}$$

if $B > 0$, $b > l/l - 1$, are suitable and:

(*) Usually a resonant torus is a torus $\{\underline{A}\} \times T^l$ such that either $\underline{\omega}(\underline{A}) \cdot \underline{v} = 0$ for some $\underline{v} \neq 0$, $\underline{v} \in Z^l$, or such that $|\underline{\omega}(\underline{A}) \cdot \underline{v}|^{-1} \to \infty$ too fast as $\underline{v} \to \infty$.

$$\underline{\omega}(\underline{A}_0) = \frac{1}{N}(v_1, \ldots, v_l) \qquad \underline{v} \in Z^l \tag{16}$$

(where N is real), provided $\underline{v}$ is not too large.

One in fact expects that for such data the time evolution $t \to \underline{A}(t)$, $\underline{\varphi}(t)$ is such that the Fourier-transform of $\underline{A}(t)$ has a continuoum component (and not just finitely many Dirac's deltas and their harmonics as in the ordinary quasi periodic motions).

Notice however that the volume occupied by the points (15) with $\underline{\omega}(\underline{A}_1)$ verifying (16) and also $0 < \Omega_1 < |\underline{\omega}(\underline{A}_0)| < \Omega_2 < +\infty$ has volume not exceeding const $\sqrt{\epsilon^{l-1}}$ since the series

$$\sum_{\underline{v} \neq 0} \frac{1}{|\underline{v}|^{b(l-1)}} < +\infty \tag{17}$$

Calling "random" any motion $t \to (\underline{A}(t), \underline{\varphi}(t))$ for which the Fourier's transform of $\underline{A}(t)$ has a continuoum component we see that random behaviour should quite generally appear in a dense open set "containing all the resonances" but having a small measure. Clearly this is not incompatible with the KAM theorem which only says that in the complement of a set of small measure the motion is quasi periodic. The annoying feature emerging from the above discussione in that the random motion takes place in a dense set inside the non random behaviour. One may think that this situation is very ankward and hard to see in experiments particularly as far as the quasi periodic motions are concerned. It is, however, a remarkable and well known fact that the quasi periodic motions are nevertheless very easy to observe numerically for small ϵ and, for such ϵ's, there seems to be no trace of random behaviour. The reason is that the sets on which the quasi periodic motions take place vary smoothly in phase space even through they cannot toach a dense open set. This is in fact the content of the following theorem (proven first by Poshel and later by Chierchia, Gallavotti) essentially already contained in Arnold's work.

Theorem. There exist l C^∞ functions $A'_1, \ldots, A'_l$ on W which are independent on W_ϵ and such that the invariant tori in W_ϵ can be represented as level surfaces of the $\underline{A}'$'s:

$$\underline{A}'(\underline{A}, \underline{\varphi}) = \underline{a} \tag{18}$$

Observation:

1) Since the tori of W_ϵ are invariant the (18) says that the A''s are prime integrals on W_ϵ.

2) It makes no sense to say that in a numerical experiment one "sees" a quasi periodic motion. Rather it makes sense to say that one sees a quasi periodic motion within a certain approximation η (*) for a time of lenght T: "(η, T)-tori". In the numerical experiments η and T cannot be too large. On the other hand the proof of the KAM theorem is perfectly suited to answer questions such as: which is the phase space volume covered by (η, T)-tori? (once the meaning of the "approximation within η for a time T" is declared). It can be easily seen that, whatever reasonable meaning one attaches to the approximation within η for a time T, one shall forsee, using the methods of the KAM theorem's proof, that the (η, T)-tori fill an *open* region in phase space.

This region can be defined in terms of approximations to the prime integrals $A'_1, \dots, A'_l$ and this method of discussing, the quasi periodic motions has been actually and successfully employed in concrete numerical experiments (Giorgilli et alii [8]).

I think that not only qualitative predictions but also rather sharp quantitative predictions could be obtained by a wise use of the above idea. The general belief being, apparently, that on the contrary one could only, by using, rigorous methods, get an understanding of the motions in ridiculously small regions in phase space.

3. Randomness mechanisms. Two possible ways.

It may be of some interest to analyse the instability mechanisms which produce the above mentioned complicated structure of a phase space covered by a large set of invariant tori with an open dense, though small, complement.

There are two basic different mechanisms for randomness. The first is a rather general mechanism working for systems with more than two degrees of freedom.

1) To understand this mechanism in the general non isochronous case it is good to first consider how it operates in isochronous systems where its structure is very clear. We do this by considering a simple example. Let:

$$h_0(\underline{A}) = A_1 \qquad \text{so} \quad \underline{\omega}(\underline{A}) = (1, 0, \dots, 0) \tag{19}$$

(*) According to some a priori explained approximation arterion.

This is a very strongly resonating system, *everywhere* in phase space.

Consider a perturbation $f(A_2, \dots, \varphi_2)$ which, if we denote $(A_1, A_2, \dots, A_l)$ as $(A, \underline{B})$ and $\underline{\varphi} = (\varphi_1, \dots, \varphi_l) = (\varphi, \underline{\psi})$, yields the following equations of motion

$$(20) \qquad \begin{cases} \dot{A} = 0 & \dot{\underline{B}} = -\,\epsilon \dfrac{\partial f}{\partial \underline{\psi}}(\underline{B}, \underline{\psi}) \\[2ex] \dot{\varphi} = 1 & \dot{\underline{\psi}} = \epsilon \dfrac{\partial f}{\partial \underline{\psi}}(\underline{B}, \underline{\psi}) \end{cases}$$

Therefore the motion of this perturbed system is very simple: for $\epsilon = 0$ it is 2π-periodic while for $\epsilon > 0$ it has the form

$$(21) \qquad A = A_0 \ , \quad \varphi = \varphi_0 + t \ , \quad \underline{B}(t) = \underline{b}(\epsilon t) \ , \quad \underline{\psi}(t) = \gamma(\epsilon t)$$

where $\underline{b}, \gamma$ verify the hamiltonian system

$$(22) \qquad \begin{cases} \dot{\underline{b}} = -\dfrac{\partial f}{\partial \underline{\gamma}}(\underline{b}, \underline{\gamma}) \\[2ex] \dot{\underline{\gamma}} = \dfrac{\partial f}{\partial \underline{b}}(\underline{b}, \underline{\gamma}) \end{cases}$$

as it immediately follows changing scale $t \to \epsilon t$ in the (20) for $\underline{B}, \underline{\varphi}$.

If (22) describes a non integrable hamiltonian we see that the motion is composed of a "periodic" fast motion on time scale 1 and of a non integrable but slow motion on scale $1/\epsilon$. So we see that in a fully resonating system a perturbation, no matter how small, can change completely the asymptotic hehaviour of the system producing randomness (i.e. non integrability).

In some sense if there is a resonance some of the system's coordinates (the $\underline{B}, \underline{\psi}$) do not move at all in the unperturbed situation and therefore their evolution is entirely governed by the perturbation.

The defect of the above example is that it is based on a strongly resonant isochronous system and one may think that the situation is drastically different in non isochronous system.

However this is not so as we show by considering, heuristically, the hamiltonian:

$$(23) \qquad \frac{A^2}{2} + \frac{B^2}{2} + \epsilon f(A \underline{B} \varphi \underline{\psi})$$

and its dynamics near a resonant torus $A = 1$, $\underline{B} = \underline{0}$.

Let us set

$$(24)\qquad \begin{cases} A(t) = 1 + \sqrt{\epsilon}\, a_\epsilon(\sqrt{\epsilon}\, t) \\ \underline{B}(t) = \sqrt{\epsilon}\, \underline{b}_\epsilon(\sqrt{\epsilon}\, t) \\ \underline{\psi}(t) = \underline{\gamma}_\epsilon(\sqrt{\epsilon}\, t) \\ \varphi(t) = \delta_\epsilon(\sqrt{\epsilon}\, t) \end{cases}$$

Then one finds the following equations for the hamiltonian flow generated by (23):

$$(25)\qquad \begin{cases} \dot{a}_\epsilon = -\dfrac{\partial f}{\partial \varphi}(1 + \sqrt{\epsilon}\, a_\epsilon\,, \sqrt{\epsilon}\, \underline{b}_\epsilon\,, \delta_\epsilon\,, \underline{\gamma}_\epsilon) \\ \dot{\underline{b}}_\epsilon = -\dfrac{\partial f}{\partial \underline{\gamma}}(1 + \sqrt{\epsilon}\, a_\epsilon\,, \sqrt{\epsilon}\, \underline{b}_\epsilon\,, \delta_\epsilon\,, \underline{\gamma}_\epsilon) \\ \dot{\underline{\gamma}}_\epsilon = \underline{b}_\epsilon + \sqrt{\epsilon}\,\dfrac{\partial f}{\partial \underline{B}}(1 + \sqrt{\epsilon}\, a_\epsilon\,, \sqrt{\epsilon}\, \underline{b}_\epsilon\,, \delta_\epsilon\,, \underline{\gamma}_\epsilon) \\ \dot{\underline{\delta}}_\epsilon = a_\epsilon + \dfrac{1}{\sqrt{\epsilon}} + \sqrt{\epsilon}\,\dfrac{\partial t}{\partial A}(1 + \sqrt{\epsilon}\, a_\epsilon\,, \sqrt{\epsilon}\, \underline{b}_\epsilon\,, \delta_\epsilon\,, \underline{\gamma}_\epsilon) \end{cases}$$

These equations show that δ is a very fastly verying variable for small ϵ and one can guess that the "slow" variables a_ϵ, $\underline{b}_\epsilon$, $\underline{\gamma}_\epsilon$ have a limit as $\epsilon \to 0$: $a, \underline{b}, \underline{\gamma}$, which should verify:

$$(26)\qquad \begin{cases} \dot{a} = 0 \\ \dot{\underline{b}} = -\dfrac{\partial \overline{f}}{\partial \underline{\gamma}}(1,0,\underline{\gamma}) \qquad \overline{f}(A\underline{B}\underline{\gamma}) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(A\underline{B}\varphi\underline{\gamma})\, d\varphi \\ \dot{\underline{\gamma}} = \underline{b} \end{cases}$$

The above equation should describe a good approximation to the motion even when $\epsilon \neq 0$ for a time $0(1/\epsilon)$ and for data of the form (24) at $t = 0$. However this equation, hamiltonian with hamiltonian:

$$(27)\qquad \frac{\underline{b}^2}{2} + \overline{f}(1,0,\underline{\gamma})$$

will in general be of a non integrable type. So comparing with the isocronous resonant case, we see that the "only" effect of the non isocrony is to reduce the time scale on which the non integrability becomes manifest (in the isocronous case in fact the time scale in $1/\epsilon$ while now it is $1/\sqrt{\epsilon}$).

The general case of a torus $\{A, \underline{B}\} \times T^l$ which resonates can be treated along the above lines. For instance if $\underline{\omega}(A, \underline{B}) \equiv (A, \underline{B})$ has l rationally related components $\underline{\omega} = \omega_0 \underline{v}$, $\underline{v} \in Z^l$, $MCD(v_1, \dots, v_l) = 1$, one shall first make a change of coordinates to transform the system into one for which the torus $(A, B) \times T^l$ becomes $(A', \underline{B}') \times T^l$ and $\underline{\omega}(A', \underline{B}') = (\omega_0, 0, 0, \dots, 0)$.

The change of coordinate will be taken of the form:

$$\begin{cases} (A', \underline{B}') = \mathcal{N}(A, \underline{B}) \\ (\varphi', \underline{\psi}') = (\mathcal{N}^T)^{-1}(\varphi, \underline{\psi}) \end{cases} \tag{28}$$

where $\mathcal{N}$ is an integer-entries matrix with determinant 1 (to map T^l into T^l in a $1-1$ way), such that

$$(\mathcal{N}^{-1})^T \underline{v} = (1, 0, \dots, 0) \tag{29}$$

This matrix always exist if $MCD(v_1, \dots, v_l) = 1$ and (28) is a completely canonical map. Of course in doing this change of variables we shall prove that data which are within $0(\sqrt{\epsilon})$ from the torus $(A', \underline{B}') \times T^l$ with $(A', \underline{B}') = = \mathcal{N} \underline{\omega}_0$ will behave non integrably for a time of the $0\left(\frac{1}{\sqrt{\epsilon}}\right)$: in the original variables this will mean that data which are close to the resonant torus within $0\left(\frac{1}{\|\mathcal{N}\|}\sqrt{\epsilon}\right)$ will behave non integrably for a time of the order of $0(\sqrt{\epsilon}^{\,-1})$.

This also gives a rough explanation of why one expects that the random motions will take place within the region (15) as $\|\mathcal{N}\| \sim 0(|\underline{v}|)$.

The b-value in (15) is an improvement which should come from a closer analysis making use of possible extra assumptions on f needed any way to discuss the asymptotic behaviour for large times $\gg 0(1/\sqrt{\epsilon})$.

2) Clearly the above randomness mechanism does not work if $l = 2$ as the "effective" secular motion Hamiltonian turns out to be itself integrable (as any 1-dimensional system), in general. There is however a second randomness mechanism which will in general operate in this case. Consider the same system as above and suppose that $\overline{f}(1, 0, \gamma)$ is not identically constant. Then the secular motion is a one degree of freedom sustem whose hamiltonian is:

$$\frac{b^2}{2} + \overline{f}(1, 0, \gamma) \tag{30}$$

As mentioned above this system can be integrated for most initial data: in fact all *except* those with energy:

$$\frac{b_0^2}{2} + \bar{f}(1,0,\gamma_0) = \max_{\gamma} \bar{f}(1,0,\gamma_0) \tag{31}$$

Consider for instance the simple case

$$\bar{f}(1,0,\gamma) = -\cos\gamma$$

Then the system is a pendulum and it is well known that one can find action-angle variables for a pendulum everywhere in phase space except near the equilibriul points.

Since all the intuition about preservation of quasi periodicity is just based on the possibility of using the action-angle variables we see that near the torus $(1,0) \times T^2$ there are points, hence motions, for which our intuition breaks down.

So we may expect that something unexpected may happen in phase space near $U_i \{1,0\} \times (T^1 \times \{\gamma_i\})$ where $\gamma_1, \gamma_2 \dots$ are the maxima points for the "secular potential" $\bar{f}(1,0,\gamma)$. Although numerical experiments precisely confirm this a detailed analysis of what happens is very hard even if one is content with a heuristic argument as the one given before to discuss the $l > 2$ case, (Benettin, Cercignani, Galgani, Eckmann).

To have an idea of how complicate the situation might be when perturbation theory cannot be applied (i.e. no action angle variably exist) see Moser [6].

4. Other comments and conjectures.

From the KAM theorem we see that there is a "treshold" ϵ_0 below which the theorem can guarantee the permanence of the ordered quasi periodic motions in large phase space regions. We call ϵ_0 the "ergodicity theshold", loosely speaking. The question arise on how does ϵ_0 to depend on the number of degrees of freedom and on f.

It we restrict only mildly f, say to be holomorphic in a region like

$$\underline{A}, \underline{z} \mid (\underline{A}, \underline{z}) \in C^{2l}, \; |A_i - A_{0i}| < \rho, \; e^{-\xi} < |z_i| < e^{\xi}, \quad i = 1, \dots, l \tag{32}$$

and to have there a bound like

$$\left| \frac{\partial f_0}{\partial \underline{A}} \right| + \frac{1}{C} \left| \frac{\partial f_0}{\partial \underline{\varphi}} \right| \leqslant 1 \tag{33}$$

and if we restrict h_0 also to be holomorphic in the same region (recall, however, that h_0 depends solely on $\underline{A}$) and to be such that

$$(34) \qquad \left|\frac{\partial h_0}{\partial \underline{A}}\right| \leqq E_0 \qquad \left\| \left(\frac{\partial^2 h_0}{\partial \underline{A}\, \partial \underline{A}}\right)^{-1} \right\| \leqslant \eta_0$$

then one can prove a bound for ϵ_0 which depends on l as $(l!)^{-\overline{a}}$ for some $\overline{a} > 0$. This is likely to be an optimal result in the general context.

However the consideration of physically interesting hamiltonian systems with many degrees of freedom puts severe further restrictions upon the hamiltonian.

A typical example is:

$$(35) \qquad \sum_{1=0}^{N} \frac{p_i^2}{2} + \epsilon \sum_{i=0}^{N-1} f(\varphi_{i+1} - \varphi_i)$$

where f is analytic on T^1, and this represents a system of N rotators coupled in chain.

I think that for such a system, for instance, one should be able to prive a bound for ϵ_0 like

$$(36) \qquad \epsilon_0 \propto l^{-\tilde{a}} \qquad \text{for some } \widetilde{a} > 0$$

greatly improving the former bound relative to the general case. Of course, in formulating the theorem, the phase space volume shall no longer be measured by the Liouville measure but rather by the canonical ensemble measure as prescribed by Statistical Mechanics, for large N. This proof should be feasible with the presently known perturbation techniques.

It would be interesting to see if the (36) could be improved to

$$(37) \qquad \epsilon_0 \propto (\log l)^{-\overline{\overline{a}}} \qquad \text{or} \qquad \epsilon_0 \propto 0(1)$$

Such results may not be absurd; however we do not have, so far, any idea on how to prove them. It is clear that one needs, beyond the KAM theorem's ideas, some new ones.

From a numerical point of view the above question seems very hard and there is controversial evidence for all the conceivable behaviours (see for instance Galgani [9]).

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