

Quantum Fields – Algebras, Processes

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In these notes I collect, together with the talk that I gave at the meeting (represented by §6,7) some other introductory topics like the theory of the Bernoulli fields [1],[2] in a form which takes into account the results in [3], [4], [5].

It seems to me that the Bernoulli field theory already contains the problems typical of the euclidean field theory (i.e., the renormalization of the ultraviolet divergences) in a form essentially identical to the one in which they appear in the more interesting euclidean fields. However, such models are simpler as they do not present some additional difficulties which do not seem to me to have much to do with the ultraviolet stability. See concluding remarks for references and historical comments.

1. The Hierarchical Model

The role that the hierarchical model will play in these lectures is that of providing a guide to the understanding of the mechanism which allows the cancellation of the various divergent contributions to the ground state energy to actually occur, in a controllable way. The hierarchical model is a model of field theory in which the free field is replaced by some new free field of simpler structure but which leads to essentially the same ultraviolet problems.

Let $(Q_n)_{n=k}^{\infty}$ be a sequence of pavements of R^d with cubic tesserae: the tesserae of Q_k have side size 2^{-k} and the pavements will be supposed compatible.

To each tessera Δ we associate a gaussian random variable z_{Δ} normalized so that $E(z_{\Delta}^2) \equiv 1, \forall \Delta \in U Q$; we shall also suppose that z and z_{Δ} are independent unless $\Delta, \Delta' \in Q_k$ for some $k = 0, 1, \dots$

The variables associated with the same pavement Q_k of R^d will

be supposed distributed with the following formal density

$$P(\prod_{\Delta \in Q_k} dz_{\Delta}) \propto \exp\left(-\frac{\alpha^2}{2} \sum_{\Delta, \Delta' \text{ n.n.}} (z_{\Delta} - z_{\Delta'})^2 - \frac{\beta^2}{2} \sum_{\Delta} z_{\Delta}^2\right) \prod_{\Delta} dz_{\Delta} \quad (1.1)$$

i.e. $z_{\Delta}, \Delta \in Q_k$, is the gaussian process on the lattice of the centers of the tesserae with invariance operator

$$= \alpha^2 D + \beta^2 \quad (1.2)$$

when D is the second difference.

We define the "free field":

$$\phi_{\xi} = \sum_{\substack{\Delta \ni \xi \\ \Delta \in Q_k}} \sqrt{2^{(d-2)k}} z_{\Delta} \quad \xi \in R^d \quad (1.3)$$

and the cut-off field

$$\phi_{\xi}^{[\leq N]} = \sum_{k=0}^N \sqrt{2^{(d-2)k}} z_{\Delta_k} \quad \xi \in R^d \quad (1.4)$$

where $\Delta_0, \Delta_1, \dots, \Delta_N$ are in Q_0, \dots, Q_N and contain ξ . To resolve the ambiguities at the boundaries let us suppose, for instance, that the Δ 's are open. The field $\phi^{[\leq N]}$ is normalizable, in contrast to the field ϕ . Its normalized copy

$$X_{\xi}^{(N)} = \sum_{k=0}^N \frac{\sqrt{2^{(d-2)k}} z_{\Delta_k}}{\sqrt{\sum_{k=0}^N 2^{(d-2)k}}} \quad \xi \in R^d \quad (1.5)$$

verifies a recursion relation which stems from its definition:

$$X_{\Delta}^{(N)} = \frac{z_{\Delta}^{(N)} + \sqrt{\Gamma_N} X_{\Delta'}^{(N-1)}}{\sqrt{1 + \Gamma_N}} \quad (1.6)$$

where $\Delta \subset Q_N, \Delta' \subset Q_{N-1}, \Delta' \supset \Delta$ and $X_{\Delta}^{(N)}$ denotes the constant value of $X_{\xi}^{(N)}$ on Δ , finally

$$\Gamma_N = \sum_{k=0}^{N-1} \sqrt{2^{(d-2)(k-N)}} = \begin{cases} N & \text{if } d = 2 \\ \frac{1-2^{-N(d-2)}}{2^{d-2}-1} & \text{if } d > 2 \end{cases} \quad (1.7)$$

so if $d = 3$, $\Gamma_N \cong 1$, if $d = 4$, $\Gamma_N \cong \frac{1}{3}$ etc..

The "interaction" that we wish to consider is defined by

$$V_{0,I}^{(N)}(\phi) = -\lambda \int_I \phi_{\xi}^{[\leq N]} d\xi \quad (1.8)$$

where I is a cube paved by Q_0 .

The above quantity is the "nonrenormalized" interaction.

We define for an arbitrary gaussian random variable x the random variables $:x^n$: by

$$:x^n: = \sqrt{2c}^{-n} H_n\left(\frac{x}{\sqrt{2c}}\right), \quad c = E(x^2) \quad (1.9)$$

where H_n = n -th hermite polynomial (e.g. $H_2(y) = y^2 - 1/2$, $H_4(y) = y^4 - 3y^2 + 3/2$ etc.). The "renormalized interaction" to first order in λ is

$$V_{1,I}^{(N)}(\phi) = -\lambda \int_I :(\phi_{\xi}^{[\leq N]})^4: d\xi. \quad (1.10)$$

We shall later define the higher order renormalized interactions $V_{t,I}^{(N)}(\phi)$, $t \geq 2$. The ultraviolet problem to order t is to show the existence (or nonexistence) of a function $E_t(\lambda)$ such that

$$\exp - E_t(\lambda)(I) \leq \int \exp V_{t,I}^{(N)}(\phi) P(\phi) \leq \exp E_t(\lambda)(I)$$

$$\lim_{\lambda \rightarrow 0} \lambda^{-t} E_t(\lambda) = 0.$$

We shall consider only $d = 2$ or $d = 3$. The formulation of the ultraviolet stability in $d \geq 4$ would be different and, anyway, it is still an open problem.

The $d = 2$, $d = 3$ cases are special because the models are "superrenormalizable" in such dimensions. The mathematical aspect of superrenormalizability can be seen by writing $V_{1,I}^{(N)}$, say, as a sum using that $x_{\xi}^{(N)}$ is piecewise constant: since

$$:(\phi_{\xi}^{[\leq N]})^4: \cong 2^{2(d-2)N} (1+\Gamma_N)^2 H_4(X_{\Delta}^{(N)})$$

we find

$$V_{1,I}^{(N)} = -\lambda \sum_{\Delta \in Q_N} 2^{(d-4)N} (1+\Gamma_N) H_4(X_{\Delta}^{(N)})$$

i.e. if we look at the interaction as a function of the normalized fields we see that if $d < 4$ the "effective coupling constant" is $\lambda 2^{(d-4)N}$: this will allow the use of perturbation theory for N large: the fact that $\lambda 2^{(d-4)N} \xrightarrow{N \rightarrow \infty} 0$ is also called "asymptotic freedom".

2. Analysis of the Ultraviolet Problem in the Bernoulli Model to Order 1 in $d = 2$. The Lower Bound

The Bernoulli field is defined by setting $a = 0$ (i.e. all the z 's become independently distributed), see (1.1).

The theory of the ultraviolet problem for the Bernoulli field is quite easy but very instructive, already in the $d = 2$, $t = 1$ case. We first find a lower bound for

$$Z(I) = \int P_N(dz) \exp V_{1,I}^{(N)}(\phi) \quad (2.1)$$

by saying

$$Z(I) \geq \int P_N(dz) \left(\prod_{i=0}^N \prod_{\substack{\Delta \in Q_i \\ \Delta \subset I}} \chi_{\Delta}^B(X^{(i)}) \right) e^{V_{1,I}^{(N)}} \quad (2.2)$$

where χ_{Δ}^B are suitable characteristic functions: χ_{Δ}^B is the c.f. of the event, if $\Delta \in Q_k$:

$$\{X \mid |\chi_{\Delta}^{(k)}| \leq B(1+k)^4 \log(e+\lambda^{-1})\} \quad (2.3)$$

where $B > 0$ will be chosen later and is supposed at least such that: $H_4(b) \geq -\min(H_4) = 3/2$, $\forall b \geq B$. The $\log(e+\lambda^{-1})$ has been inserted to improve the estimates, automatically, at λ small, and is not the optimal choice.

To estimate the above integral we notice

$$Z(I) \geq \int P_{N-1}(dz) \left(\prod_{i=0}^{N-1} \prod_{\substack{\Delta \in Q_{N-1} \\ \Delta \subset I}} \chi_{\Delta}^B \right) \times$$

$$\times \prod_{\Delta \in Q_{N-1}} \int_{X_{\Delta}^{B_N}} (X_{\Delta}^{(N)}) e^{-\lambda(1+\Gamma_N)^2 2^{-2N} H_4(X_{\Delta})} e^{-\frac{z_{\Delta}^2}{\sqrt{\pi}}} dz_{\Delta} \quad (2.4)$$

Hence the estimate can be obtained applying the following easy lemma

Lemma1: Let $|X^1| \leq B^1$ and let $X = \frac{z + \sqrt{\Gamma} X^1}{\sqrt{1+\Gamma}}$, $B \geq B^1$; $B^1 \geq 1$, let $\bar{B} = B\sqrt{1+\Gamma} - \sqrt{\Gamma} B^1 \geq 1$, then:

$$\int \exp - \lambda(1+\Gamma)^2 H_4(X) [\chi^B(X) + \sigma(1-\chi^B(X))] e^{-\frac{z^2}{\sqrt{\pi}}} dz = \exp - \lambda \Gamma^2 H_4(X^1) + \epsilon_{\Gamma, \sigma}(\lambda, B, \bar{B}) \quad \sigma = 0, 1 \quad (2.5)$$

and ϵ can be estimated by

$$|\epsilon_{\Gamma, \sigma}(\lambda, B, \bar{B})| \leq (\lambda \Gamma^2 B^4 e^{C_1 \lambda (1+\Gamma^2) B^4})^2 + C_2 e^{-C_3 \bar{B}^2} + C_4 (\lambda \Gamma^2 B^4) \equiv \eta_{\Gamma}(\lambda, B, \bar{B}) \quad (2.6)$$

uniformly in $|X^1| \leq B^1$.

Proof: it is an easy exercise.

The application is immediate: the integrals in (2.4) are bounded by

$$\int_{X_{\Delta}^{B_N}} (X_{\Delta}^{(N)}) e^{-\lambda(1+\Gamma_N)^2 2^{-2N} H_4(X_{\Delta})} e^{-\frac{z_{\Delta}^2}{\sqrt{\pi}}} dz_{\Delta} \geq \exp - \lambda 2^{-2N} \Gamma_N H_4(X_{\Delta}^{(N-1)}) + \eta_{\Gamma_N}(\lambda 2^{-2N}, B_N, \bar{B}_N) \quad (2.7)$$

where

$$\bar{B}_N = \sqrt{1+\Gamma_N} B_N - \sqrt{\Gamma_N} B_{N-1} = O(N^{3+1/2} B). \quad (2.8)$$

Since in (2.4) we are only interested in the case $|X_{\Delta}^{(N-1)}| < B_{N-1}$.

Next notice that $\Gamma_N = (1+\Gamma_{N-1})$ (recall $d = 2$) and $4 \cdot 2^{-2N} = 2^{-2(N-1)}$ so that (2.7), (2.4) imply, denoting

$$\chi_N^B = \prod_{i=0}^N \prod_{\Delta \in Q_N} \prod_{\Delta \subset I} \chi_{\Delta}^{B_i}$$

$$Z(I) \geq \int \chi_N^B e^{V_{1,I}^{(N)}} P_N(d\phi) \geq$$

$$\geq \int \chi_{N-1}^B \prod_{\Delta \in Q_{N-1}} \exp - \lambda H_4(X_{\Delta}^{(N-1)}) 2^{-2N(1+\Gamma_N)} P_{N-1}(d\phi) \cdot \exp - \eta_{\Gamma_N}(\lambda 2^{-2N}, B_N, \bar{B}_N) 2^{2N|I|} = \int \chi_{N-1}^B e^{V_{1,I}^{(N-1)}} P_{N-1}(d\phi) \exp - \eta_{\Gamma_N}(\lambda 2^{-2N}, B_N, \bar{B}_N). \quad (2.9)$$

Hence

$$Z(I) \geq \exp - |I| \sum_{j=0}^{\infty} \eta_{\Gamma_j}(\lambda 2^{-2j}, B_j, \bar{B}_j) \quad (2.10)$$

provided B is so chosen that $\forall j, \bar{B}_j \geq 1$ (e.g. $B = 1$).

It is clear that the above series converges and, from (2.6), we see that the sum can be bounded by

$$G \lambda^2 (\log(e+\lambda^{-1}))^8 \quad (2.11)$$

for λ small enough.

Before continuing let us interpret the above calculation:

i) It is clear that

$$\int V_{1,0}^{(N)} (\phi^{(N)}) P_N(dz) \equiv V_{1,0}^{(N-1)} (\phi^{(N-1)}) \quad (2.12)$$

and it is natural to attempt to use this information to prove the bounds.

ii) The naive bound $Z(I) \leq \exp \max V_{0,1}^{(N)}$ gives $Z(I) \leq \exp + \lambda(1+\Gamma_N) 2^{-2N} \frac{3}{2} 2^{2N} |I| + \infty$. However, we can take better advantage of the asymptotic freedom and of the fact that the $V_{1,I}^{(N)}$ is a sum

of random variables little correlated.

We try to write

$$\int \exp v_1^{(N)} \hat{P}_N(dz) = \exp\left\{\hat{E}_N(v_z^{(N)}) + \frac{1}{2!} \hat{E}_N^T(v_I^{(N)}; 1) + \dots + \frac{1}{t!} \hat{E}_N^T(v_1^{(N)}; t)\right\} + (\text{error}) \quad (2.13)$$

where \hat{E}_N is the average with respect to $(\hat{P}_N dz) \equiv$ distribution of $(z_\Delta)_{\Delta \in Q_N}$, and $\hat{E}_N^T(\cdot; s)$ truncated expectation of order s .

The fact that the z_Δ variables are not too correlated (in our case they are actually independent) will tell us that the error is of the order of the volume of I in units of 2^{-N} , i.e. $\text{const } |I| 2^{dN}$ and not $\text{const } \{|I| 2^{dN}\}^{t+1}$ as it would be for strongly correlated fields $(z_\Delta)_{\Delta \in Q_N}$. The asymptotic freedom says that the constant is of the order of $(\lambda 2^{(d-4)N} (1+\Gamma_N))$.

Hence we see that the errors in (2.13) will be of the order

$$\lambda^{t+1} 2^{(t+1)(d-4)} 2^{dN} (1+\Gamma_N)^{2(t+1)} \quad (2.14)$$

and therefore summable over N as soon as $(t+1)(d-4) > d$, i.e. if $d = 2$ already for $t = 1$ (if $d = 3$ we should take $t = 3$).

iii) The calculation made before was just an attempt to estimate the error of (2.13) from below: the attempt was successful even though the result was not as good as the naive estimate (2.14) (e.g. we have not found a bound like λ^2 but only like $\lambda^2 (\log(e+\lambda^{-1}))^4$, which is, however, of order larger than 1 as $\lambda \rightarrow 0$).

3. The Upper Bound in the Bernoulli Case

We proceed, again, recursively. The basic remarks are the following.

i) Let $\phi = (z_\Delta)_{\Delta \in U_1 Q_1}$ be a random field.

Let

$$\bar{B}_N = \sqrt{1+\Gamma_N} B_N - \sqrt{\Gamma_N} B_{N-1} = O(B \log(e+\lambda^{-1}) (1+N)^{3+1/2}) \quad (3.1)$$

$$\tilde{B}_N = B_N / (1+N)^2 = O(B \log(e+\lambda)^{-1} (1+N)^2)$$

and define

$$D_N(\phi) = \{\Delta \in Q_N, |x_\Delta| > B_N\} \quad (3.2)$$

$$R_N(\phi) = \{\Delta \in Q_N, |z_\Delta| > \tilde{B}_N\} \quad (3.3)$$

and notice that $\forall \phi$:

$$R_N \supset D_N \setminus D_{N-1} \quad (3.4)$$

$$\Delta \in D_{N-1} \setminus (R_N \cup D_N) \Rightarrow |x_\Delta^{(N)}| \geq \frac{B_{N-1} \sqrt{\Gamma_N} - \hat{B}_N}{\sqrt{1+\Gamma_N}} \geq \frac{B_{N-1}}{2}. \quad (3.5)$$

ii) The positivity of $H_4(b)$, $b \geq B_N$, implies, c.f.i. (1.10):

$$V_{1,I}^{(N)}(\phi) \geq V_{1,I \setminus D_N}^{(N)}(\phi) \quad (3.6)$$

hence

$$Z(I) \geq \int e^{V_{1,I \setminus D_N}^{(N)}(\phi)} P_N(d\phi) \equiv Z_O^{(N)}(I). \quad (3.7)$$

Also, from (1.10)

$$V_{1,I \setminus D_N}^{(N)}(\phi) \leq V_{1,I \setminus (D_N \cup \hat{R}_N)}^{(N)} + 4 \#(\hat{R}_N) \frac{3}{2} \lambda 2^{-2N} (1+\Gamma_N) \quad (3.8)$$

where $\#(R_N)$ = number of tesserai in R_N , and \hat{R}_N is the smallest set paved by Q_{N-1} containing R_N . Remark then that (3.8), (1.10), (3.5) imply

$$V_{1,I \setminus D_N}^{(N)}(\phi) \leq V_{1,I \setminus (R_N \cup D_{N-1})}^{(N)} + 4 \#(\hat{R}_N) \frac{3}{2} \lambda 2^{-2N} (1+\Gamma_N) + \dots \quad (3.9)$$

$$- \lambda 2^{-2N} (1+\Gamma_N) H_4(\bar{B}_N \sqrt{1+\Gamma_N}) \#(D_{N-1}).$$

Hence

$$\int e^{V_{1, I \setminus D_N}^{(N)}(\phi)} dP_N \leq \int e^{V_{1, I \setminus D_{N-1} \cup \hat{R}_N}^{(N)}} e^{4\#(R_N) \frac{3}{2} \lambda 2^{-2N} (1+\Gamma_N)} dP_N \quad (3.10)$$

$$\leq \int \prod_{\Delta \in R_N} \bar{\chi}_\Delta^{-B}(z) \prod_{\Delta \in R_N} (1 - \bar{\chi}_\Delta^{-B}(z)) e^{V_{1, I \setminus D_{N-1} \cup \hat{R}_N}^{(N)}} e^{4\#(R_N) \frac{3}{2} \lambda 2^{-2N} (1+\Gamma_N)} dP_N$$

where $\bar{\chi}_\Delta^{-B}$ is the c.f. of the event

$$|z_\Delta^{(N)}| \geq \tilde{B}_N. \quad (3.11)$$

Hence if R is a set paved by Q_N :

$$\leq \sum_{R \subset I} \int \prod_{\Delta \in R} \bar{\chi}_\Delta^{-B}(z) \prod_{\Delta \in R} (1 - \bar{\chi}_\Delta^{-B}(z)) e^{V_{1, I \setminus D_{N-1} \cup \hat{R}_N}^{(N)}} e^{4\#(R) \frac{3}{2} \lambda (1+\Gamma_N) 2^{-2N}} dP_N$$

$$\leq \sum_{R \subset I} \int \prod_{\Delta \in R} \bar{\chi}_\Delta^{-B}(z) e^{V_{1, I \setminus D_{N-1} \cup \hat{R}_N}^{(N)}} e^{4\#(R) \cdot \frac{3}{2} \cdot \lambda (1+\Gamma_N) 2^{-2N}} dP_N \quad (3.12)$$

$$= \sum_{R \subset I} \int \prod_{\Delta \in R} \bar{\chi}_\Delta^{-B}(z) e^{4\#(R) \frac{3}{2} \lambda (1+\Gamma_N) 2^{-2N} \frac{-z_\Delta^2}{\sqrt{\pi}}} \cdot$$

$$\cdot \left[\prod_{\Delta' \in I \setminus R \cup D_N} \left(\int \exp - \lambda (1+\Gamma_N)^2 H_4(x_\Delta^{(N)}) 2^{-2N} \frac{e^{-z^2}}{\sqrt{\pi}} dz \right)^4 \right] dP_{N-1}$$

applying the lemma of the preceding section the last integral can be bounded by

$$\sum_{R \subset I} \int \prod_{\Delta \in R} \bar{\chi}_\Delta^{-B}(z) e^{\frac{-z_\Delta^2}{\sqrt{\pi}}} \cdot \exp - V_{1, I \setminus D_{N-1} \cup \hat{R}_N}^{(N-1)}. \quad (3.13)$$

$$\cdot \exp + \eta_{\Gamma_N} (\lambda 2^{-2N}, B_N, B_{N-1}) |I| \exp + 4\#(R) \frac{3}{2} \lambda (1+\Gamma_N) 2^{-2N}$$

Applying

$$V_{1, I \setminus D_{N-1} \cup \hat{R}_N}^{(N-1)} \leq V_{1, I \setminus D_{N-1}}^{(N-1)} + 4\#(R_N) (1+\Gamma_N)^2 \lambda 2^{-2N} H_4(B_{N-1}) \quad (3.14)$$

we have found:

$$z_0^{(N)}(I) \leq z_0^{(N-1)}(I) \cdot \sum_{R \subset I} (e^{8\#(R_N) (1+\Gamma_N)^2 2^{-2N} \lambda H_4(B_N)} \exp - \tilde{c} B_N^2 \#(R_N)) \exp \eta_{\Gamma_N} (\lambda 2^{-2N}, B_N, B_{N-1}) |I| \quad (3.15)$$

$$= z_0^{(N-1)}(I) \exp(\eta_{\Gamma_N} + (\lambda 2^{-2N}, B_N, B_{N-1}) + \eta^1(\lambda, N)) |I|$$

where

$$\eta^1(\lambda, N) = \log(1 + \exp - \tilde{c} B_N^2 + 8(1+\Gamma_N)^2 \lambda 2^{-2N} H_4(B_N)). \quad (3.16)$$

Hence

$$z_0^{(N)}(I) \leq \exp I \sum_{N=0}^{\infty} (\eta_{\Gamma_N} (\lambda 2^{-2N}, B_N, B_{N-1}) + \eta^1(\lambda, N)) \quad (3.17)$$

and it is easily realized that the series in parenthesis behaves as $\lambda \rightarrow 0$ as (2.11) with a different G, say G'. Hence:

$$E(\lambda) \leq (G+G') \lambda^2 \log(e+\lambda^{-1})^8. \quad (3.18)$$

4. Hierarchical Model to Order d = 2

We define the renormalized interaction to order t, for d = 2, as:

$$V_{t; I}^{(N)} = V_{1; I}^{(N)} - \sum_{k=2}^t \frac{\epsilon^k (V_{1; I}^{(N)}; k)}{k!}. \quad (4.1)$$

Consider for example t = 2.

It is natural to proceed as before: however, it will no longer be so easy because

$$V_{2;I}^{(N-1)} \neq \hat{E}_N^T(V_{2;I}^{(N)}) + \frac{1}{2} \hat{E}_N^T(V_{2;I}^{(N)}; 2). \quad (4.2)$$

Therefore we introduce, recursively

$$\tilde{V}_{2;I}^{(N-k)} = \left[\hat{E}_{N-k+1}^T(\tilde{V}_{2;I}^{(N-k+1)}) + \frac{1}{2} \hat{E}_{N-k+1}^T(\tilde{V}_{2;I}^{(N-k+1)}; 2) \right]_{(2)} \quad (4.3)$$

$$\tilde{V}_{2;I}^{(N)} = V_{2;I}^{(N)} \quad (4.4)$$

where $[]_{(2)}$ = truncation of a polynomial in λ to order λ^2 .

To proceed as before one needs to first get an idea on how $\tilde{V}_{2;I}^{(N-k)}$ looks like: it is easy to see that (by induction)

$$\tilde{V}_{2;I}^{(N-k)} = \left[\hat{E}_{>N-k}^T(V_{2;N}^{(N)}) + \frac{1}{2} \hat{E}_{>N-k}^T(V_{2;N}^{(N)}; 2) \right]_{(2)} \quad (4.5)$$

hence if $\tilde{\phi} = \phi^{[\leq N]}$, $\phi = \phi^{[\leq N-k]}$:

$$\tilde{V}_{2;I}^{(N-k)} = \int_I -\lambda : \phi_{\xi}^4 : d\xi - \frac{\lambda^2}{2} 4! \int_I C_{\xi\eta}^{(N)}{}^4 d\xi d\eta \quad (4.6)$$

$$+ \frac{1}{2} \int_I \lambda^2 \int (: \phi_{\xi}^4 : : \phi_{\eta}^4 :) dP_{>N-k} d\xi d\eta - \frac{1}{2} \int \lambda^2 : \phi^4 : : \phi^4 : d\xi d\eta.$$

The integral

$$\int : \phi_{\xi}^4 : : \phi_{\eta}^4 : dP_{>N-k} \quad (4.7)$$

can be graphically represented as the integral of the graph

$$\begin{aligned} & \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)_{\xi} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)_{\eta} = \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)_{\xi} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)_{\eta} + 4^2 : \begin{array}{c} | \\ | \\ | \end{array} :_{\xi} : \begin{array}{c} | \\ | \\ | \end{array} :_{\eta} + \\ & + 2 \binom{4}{2} : \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)_{\xi} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)_{\eta} : + 3! 4^2 : \left(\begin{array}{c} \circ \\ \circ \end{array} \right)_{\xi} :_{\xi} : \left(\begin{array}{c} \circ \\ \circ \end{array} \right)_{\eta} :_{\eta} + \\ & + 4! \left(\begin{array}{c} \circ \\ \circ \\ \circ \end{array} \right)_{\xi} :_{\xi} \end{aligned} \quad (4.8)$$

where each line represents $\tilde{\phi}$ and each internal line represents $C^{(N)}$. Integration over $\tilde{\phi}$ simply gives the same result described by (4.8) but with the lines interpreted as ϕ , now. The last term in (4.6) can be treated likewise and yields the same result as (4.8) with the external lines interpreted as $\tilde{\phi}$ and internal ones also interpreted as $C^{(N-k)}$.

Therefore we can write (4.5) as

$$\begin{aligned} V_{2;I}^{(N-k)} &= V_{2;I}^{(N-k)} + 4^2 : \begin{array}{c} | \\ | \\ | \end{array} :_{\xi} : 2 \binom{4}{2}{}^2 : \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)_{\xi} :_{\xi} : \\ &+ 3! 4^2 : \left(\begin{array}{c} \circ \\ \circ \end{array} \right)_{\xi} :_{\xi} : \end{aligned} \quad (4.9)$$

when a bar on a graph means that it has to be computed as the difference of two graphs with the internal lines once considered as $C^{(N)}$ and the other time as $C^{(N-k)}$.

Since the model is hierarchical it is clear that the structure of $\tilde{V}_{2;I}^{(N-k)}$ will be

$$\begin{aligned} \tilde{V}_{2;I}^{(N-k)} &= \sum_{\Delta \in Q_{N-k}} \{ -\lambda 2^{-2(N-k)} (1+\Gamma_{N-k}) H_4(X_{\Delta}^{(N-k)}) \\ &+ \lambda^2 C_{N,N-k} f_{N,N-k}(X_{\Delta}^{(N-k)}) \} \end{aligned} \quad (4.10)$$

and we are now concerned by finding some expressions for $f_{N,N-k}$.

We want to show that $f_{N,N-k}(X_{\Delta}^{(N-k)})$ can be written as

$$\begin{aligned} & \lambda^2 (\bar{C}_{N,N-k}^{(6)} H_6(X_{\Delta}^{(N-k)}) + \bar{C}_{N,N-k}^{(4)} H_4(X_{\Delta}^{(N-k)}) + \\ & + \bar{C}_{N,N-k}^{(2)} H_2(X_{\Delta}^{(N-k)}) + \bar{C}_{N,N-k}^{(0)} 2^{-4(N-k)} (1+\Gamma_{N-k})^6 \end{aligned} \quad (4.11)$$

and $\exists \bar{C}$

$$|\bar{C}_{N,N-k}^{(j)}| \leq \bar{C} \quad \forall N, k. \quad (4.12)$$

If this were true we would have proven the lower bound: we could

in fact proceed recursively using the following lemma:

Lemma 2: Let $|X^1| \leq B^1$ and let $X = \frac{z + \sqrt{1+X^1}}{\sqrt{1+1}} B^1 \geq 1, \bar{B} = B\sqrt{1+\Gamma} - B^1\sqrt{1+\Gamma} \geq 1$ then if V has the form

$$V(X) = \sum_{n=1}^D a^n X^n \quad (4.13)$$

and if $\|a\| = \sup_{n,\Delta} |a^n|$ then

$$\int e^{-z^2} \frac{dz}{\sqrt{\pi}} e^V X^B(X) = \exp\left\{ \sum_{j=0}^t \frac{E^T(V; j)}{j!} + \epsilon_{\Gamma}(B, B^1, \|a\|) \right\}$$

where, for suitable constants G_1, C_1, C_2, C_3, C_0

$$\begin{aligned} |\epsilon_{\Gamma}(B, B^1, \|a\|)| &\leq G_1 \{ (C_1 B^D \|a\| e^{2B^D} \|a\| C_2)^{t+1} + \\ &+ C_2 e^{-C_3 \bar{B}^2} + C_4 B^D \|a\| \} \end{aligned} \quad (4.14)$$

which allows to deduce

$$\begin{aligned} \int e^{\tilde{V}_{2,I}^{(N-k)}} X_{N-k}^B P_{N-k}(dz) &= \int \exp\left\{ \sum_{j=1}^2 \frac{E_{N-k}^T(\tilde{V}_{2,I}^{(N-k)}; j)}{j!} \right. \\ &\left. + \bar{n}(\lambda, N) |I| \right\} X_{N-k-1}^B P_{N-k-1}(dz) \end{aligned} \quad (4.15)$$

where $\bar{n}(\lambda, N)$ is

$$\begin{aligned} G_1 (C_1 B_N^G \lambda^{2-2N} \bar{C} e^{2B_N^6 \lambda^{2-2N} C_2 \bar{C}})^3 + \\ + C_2 e^{-C_3 \bar{B}_N^2} + C_4 \bar{B}_N^6 \lambda^{2-2N} \bar{C} \end{aligned} \quad (4.16)$$

which can be summed over N giving a result $\propto O(\lambda^3 (\lg e + \lambda^{-1})^{18})$.

(4.15), (4.16) are not yet the result that we want because we have still to replace

$$\sum_{j=1}^2 \frac{\tilde{E}_{N-k}^{(N-k)}(V_{2,I}; j)}{j!}$$

by its second order truncation with respect to λ . This is done easily, since, clearly, the difference between these two quantities is

$$\leq (\lambda \bar{C} B_{N-k}^6 2^{-(N-k)})^3 \quad (4.17)$$

and, therefore, it can be put into the error.

It remains to prove the bounds (4.11), (4.12). Consider for

instance $\frac{3!4}{2} : \bigcirc$: which is the most difficult and dangerous:

$$\begin{aligned} \frac{3!4}{2} \lambda^2 \int_I d\xi d\eta (C_{\xi, \eta}^{(N)} - C_{T, \eta}^{(N-k)})^3 : \phi_{\xi} \phi_{\eta} : = \\ = \sum_{\substack{\Delta, \Delta_1 \subset I \\ \Delta, \Delta_1 \in Q_{N-k}}} \frac{4! \lambda}{2} \int_{\Delta \times \Delta_1} (C_{\Delta, \Delta_1}^{(N-k)}) (C_{\xi, \eta}^{(N)} - C_{T, \eta}^{(N-k)})^3 : (X_{\Delta} X_{\Delta_1}) : d\xi d\eta \end{aligned} \quad (4.18)$$

notice that $C_{\xi, \eta}^{(N)}$ and $C_{\xi, \eta}^{(N-k)}$ must be equal, unless ξ, η are in the same tessera of Q_{N-k} . Then the above term is

$$\sum_{\substack{\Delta \in Q_{N-k} \\ \Delta \subset I}} \frac{4! \lambda}{2} 2^{(d-2)(N-k)} (1 + \Gamma_{N-k}) H_2(X_{\Delta}) \left(\int_{\Delta^2} (C_{\xi, \eta}^{(N)})^3 - C_{\xi, \eta}^{(N-k)3} \right) d\xi d\eta. \quad (4.19)$$

So let us study

$$\begin{aligned} 0 \leq \int_{\Delta^2} (C_{\xi, \eta}^{(N)})^3 - C_{\xi, \eta}^{(N-k)3} d\xi d\eta = \sum_{i=N-k+1}^N \int_{\Delta^2} (C_{\xi, \eta}^{(i)})^3 - C_{\xi, \eta}^{(i-1)3} d\xi d\eta \\ = \sum_{i=N-k+1}^N \sum_{\substack{\Delta^1 \in Q_{i-1} \\ \Delta^1 \subset \Delta}} \int_{(\Delta^1)^2} (C_{\xi, \eta}^{(i)})^3 - C_{\xi, \eta}^{(i-1)3} d\xi d\eta \leq \end{aligned}$$

$$\leq \sum_{i=N-k+1}^N (1+\Gamma_i)^3 \underbrace{2^{3(d-2)i}}_{\text{co-variance}^3} \underbrace{2^{-2d(i-1)}}_{\text{Volume element of } (\Delta^1)^2} \underbrace{2^{[(i-1)-(N-k)]d}}_{\text{\# elements in } \Delta} C$$

constant

which can be bounded by

$$4 \sum_{i=N-k+1}^N (1+\Gamma_i)^3 2^{(2d-6)i} 2^{-d(N-k)} \quad (4.20)$$

Hence the coefficient of $H_2(X_\Delta)$, called before $C_{N,N-k}$, can be bounded by

$$\frac{4! \lambda^2 4C}{2} (1+\Gamma_{N-k}) \sum_{i=N-k+1}^N 2^{(2d-6)i} (1+\Gamma_i) 2^{-2(N-k)} \quad (4.21)$$

and this is bounded by

$$\bar{C} \lambda^2 (1+\Gamma_{N-k})^4 2^{+2(d-4)(N-k)} \quad d = 2 \quad (4.22)$$

$$\bar{C} \lambda^2 (1+\Gamma_{N-k})^4 2^{+2(d-4)(N-k)} \left(\sum_{i=N-k+1}^N 1 \right) \quad d = 3$$

hence, if $d = 2$, (4.11), (4.12) are proven.

In three dimensions this point would be the only different point in the whole proof. In order to avoid this problem the definition of renormalized hamiltonian will be changed so that the coefficient of $H_2(X_\Delta)$, just computed, turns out to be just zero.

This can be achieved by introducing in $V_{O,I}^{(N)}$ a "mass counterterm", i.e.

$$V_{O,I}^{(N)} = \lambda \int_I : \phi_\xi^{(\leq N)} : d\xi - \lambda^2 \frac{4 \cdot 3!}{2} \int_{I \times I} C_{\xi\eta}^{[\leq N]} : \phi_\xi^{[\leq N]} : \phi_\eta^{[\leq N]} : d\xi \quad (4.23)$$

Then the interaction renormalized to order t becomes

$$V_{t,I}^{(N)} = \left[V_{O,I}^{(N)} - \sum_{j=1}^t \frac{1}{j!} E^T(V_{O,I}^{(N)}; j) \right]_{(t)} \quad (4.24)$$

and the above scheme works for all $t \geq 3$. The only thing that has really to be checked is the bound on the coefficients $C_{N,N-k}$. This is long but straightforward.

5. The Upper Bound in the Case $d = 2, t = 2$

This is essentially a repetition of the argument in §3. However, some comments seem to be important.

In fact, the presence of very high order polynomials in $\tilde{V}^{(N)}$ puts some doubts about the possibility of removing the field cut-offs.

The reason why such terms do not cause problems is that at the beginning one can make use of the locality of the interaction to "eliminate" the interaction from the regions where the fields are large.

Let us recall the structure of $V_I^{(N)}$:

$$\begin{aligned} V_I^{(N)} &= -\lambda (1+\Gamma_N)^2 2^{-2N} \sum_{\Delta \subset I} H_4(X_\Delta) - \frac{\lambda^2}{2} E^T(V_{O,I}^{(N)}; 2) = \\ &= -\lambda (1+\Gamma_N)^2 2^{-2N} \sum_{\Delta \subset I} H_4(X_\Delta) - \lambda^2 2^{-2N} \sum_{\Delta \subset I} Y_\Delta \end{aligned} \quad (5.1)$$

and Y_Δ has a bound like

$$Y_\Delta \leq \frac{1}{2} 4! \int_{\Delta \times I} (C_{\xi\eta}^{[\leq N]})^4 \frac{d\xi d\eta}{2^{-2N}} \leq C < \infty. \quad (5.2)$$

Therefore, if we introduce the constants

$$\begin{aligned} \bar{B}_N &= \sqrt{1+\Gamma_N} B_N - \sqrt{\Gamma_N} B_{N-1} \\ \tilde{B}_N &= B_N / (1+N)^2 \end{aligned} \quad (5.3)$$

and the regions

$$D_N(\phi) = \{\Delta \mid \Delta \in Q_N, |x_\Delta| > B_N\} \quad (5.4)$$

$$R_N(\phi) = \{\Delta \mid \Delta \in Q_N, |z_\Delta| > B_N\}$$

It is then, as already remarked, true that

$$R_N(\phi) \supset D_N(\phi) \setminus D_{N-1}(\phi) \quad (5.5)$$

$\forall \phi$, because in $D_N \setminus D_{N-1}$ one has $|z| > \bar{B}_N \gg B_N$; also:

$$\Delta \in D_{N-1}(\phi) \setminus R_N(z) \Rightarrow |x_\Delta^{(N)}| \geq \frac{B_{N-1} \sqrt{1+\Gamma_N} - B_N}{\sqrt{1+\Gamma_N}} \geq \frac{1}{2} B_{N-1} \gg 1.$$

Therefore

$$V_I^{(N)} \leq \hat{V}_I^{(N)} = V_{I \setminus D_N}^{(N)} \quad (5.6)$$

and to obtain an upper bound it is enough to study

$$\int \exp \hat{V}_I^{(N)} P_N(dz). \quad (5.7)$$

Let us call χ_Δ^B the characteristic function of

$$\{z \mid |z_\Delta| < B \log(e+\lambda^{-1})\} \quad (5.8)$$

and put $\overset{\circ}{\chi}_\Delta^B = 1 - \chi_\Delta^B$. Then

$$\int \exp V_I^{(N)} P_N(dz) = \sum_{R \subset Q_N} \int \prod_{\Delta \in R} \overset{\circ}{\chi}_\Delta^{B_N} \prod_{\Delta \notin R} \chi_\Delta^{B_N} e^{\hat{V}_I^{(N)}}. \quad (5.9)$$

We are still bothered by the fact that $\hat{V}_I^{(N)}$ contains "interaction" inside R where something still "goes wrong".

Therefore, we make a "generous" estimate: if

$$\hat{V}_I^{(N)} \leq \bar{C} \lambda e^{\lambda(1+N)} H_4(B_N) 2^{-2N} \#(R) + V_{I \setminus D_N \cup R_N}^{(N)} \quad (5.10)$$

and if \hat{R} denotes the smallest set paved by Q_{N-1} containing R

$$\hat{V}_I^{(N)} \leq \bar{C}^{-1} \lambda e^{\lambda(1+N)} H_4(B_N) 2^{-2N} \#(R_N) + V_{I \setminus D_N \cup \hat{R}_N}^{(N)} \quad (5.11)$$

when $\prod_{\Delta \in R} \overset{\circ}{\chi}_\Delta^{B_N} \prod_{\Delta \in \hat{R}} \overset{\circ}{\chi}_\Delta^{B_N} = 1$ and, under this condition, we see that

$$\hat{V}_I^{(N)} \leq \bar{C}^{-1} \lambda e^{\lambda(1+N)} H_4(B_N) 2^{-2N} \#(R_N) + V_{I \setminus D_{N-1} \cup \hat{R}_N}^{(N)}. \quad (5.12)$$

It should be stressed, perhaps, that v_Δ is a "non local" term (involving an integral over I) therefore modifications of the region where there is interaction produce errors due to the change of v_Δ : such errors are very large, being proportional to the areas of the deleted regions. They have been absorbed in the errors as indicated above because of positivity or because $H_4(B_N)$ is large (this explains why we have not set $\bar{C}^1 = 4\bar{C}$, as it would be the case if this fact had been overlooked). Notice, however, $V_{I \setminus S}^{(N)} = \sum_{\Delta \subset I \setminus S} V_\Delta^{(N)}$, $\forall S$ paved by R_{N-1} .

We can now perform the integration over $z^{(N)}$; we proceed as usual

$$\begin{aligned} & \int P_N(dz) e^{V_{I \setminus D_{N-1}}^{(N)} \overset{\circ}{\chi}_R^{B_N} \overset{\circ}{\chi}_R^{B_N} \overset{\circ}{\chi}_R^{B_N}} \leq \\ & \leq \int P_{N-1}(dz) \overset{\circ}{\chi}_R^{B_N}(z^{(N)}) \left(\prod_{\Delta \in R} \frac{e^{-(z_\Delta^{(N)})^2}}{\sqrt{\pi}} dz_\Delta^{(N)} \right) \cdot \prod_{\Delta' \notin R \cup D_{B-1}} \overset{\circ}{\chi}_{\Delta'} \\ & \quad \cdot e^{\frac{V_\Delta^{(N)}(x_\Delta^{(N-1)} \sqrt{1+\Gamma_N} + z)}{\sqrt{1+\Gamma_N}}} \frac{e^{-(z)^2}}{\sqrt{\pi}} dz)^4 \leq \\ & \leq \int P_{N-1}(dz) e^{(g_1 - g_2 \bar{B}_N^2) \#(R)} \prod_{\Delta' \notin R \cup D_{N-1}} \overset{\circ}{\chi}_{\Delta'}^{(N-1)} e^{V_{I \setminus R \cup D_{N-1}}^{(N-1)}} = \\ & = e^{(g_1 - g_2 \bar{B}_N^2) \#(R)} \int P_{N-1}(dz) \exp \hat{V}_{I \setminus D_{N-1} \cup \hat{R}}^{(N-1)} \exp \varepsilon_-(N) |I| \end{aligned} \quad (5.13)$$

where $\varepsilon_-(N)$ is the estimate of the error in the lower bound (changed in sign). However, inside $I \setminus D_{N-1}$ the field $x^{(N-1)}$ is small ($< B_{N-1}$) hence the high order terms in $\hat{V}^{(N-1)}$ which are not constant are really "negligible" compared to the leading

term: so

$$\tilde{V}_{I \setminus D_{N-1}}^{(N-1)} \hat{U}_R \leq \tilde{V}_{I \setminus D_{N-1}}^{(N-1)} + \#(R) 2^{-2N} \hat{C}^{(1+N)^2} H_4(B_{N-1}) \lambda e^\lambda \quad (5.14)$$

and since

$$\begin{aligned} \sum_R e^{(g_1 - g_2 B_N^2) \#(R_N)} \#(R) 2^{-2N} \hat{C}^{(1+N)^2} H_4(B_{N-1}) \lambda e^\lambda &= \\ = (1 + e^{g_1 - g_2 B_N^2} \lambda e^\lambda 2^{-2N} \hat{C}^{(1+N)^2} H_4(B_{N-1})) |I| 2^{2N} & \quad (5.15) \end{aligned}$$

we have shown

$$\int e^{\hat{V}_I^{(N)}} P_N(dz) \leq \exp(\epsilon_{-}(N) + \eta(N)) |I| \cdot \quad (5.16)$$

$$\cdot \int \exp \tilde{V}_{I \setminus D_{N-1}}^{(N-1)} P_N(dz).$$

It is now natural to define

$$\hat{V}_I^{(h)} = \tilde{V}_{I \setminus D_h}^{(h)} \quad h = 0, 1, \dots, N \quad (5.17)$$

and, of course

$$\tilde{V}_{I \setminus D_h}^{(h)} \leq \tilde{V}_{I \setminus D_h \cup R_h}^{(h)} + \#(R_h) \hat{C}^1 \lambda e^\lambda (1+h)^2 2^{-2h} H_4(B_h) \quad (5.18)$$

because the high orders do not matter since we are removing the fields only from regions where $|X^{(N)}| < B_N!$

Similarly

$$\begin{aligned} \tilde{V}_{I \setminus D_h}^{(h)} &\leq \tilde{V}_{I \setminus D_h \cup R_h}^{(h)} + \#(R_h) \hat{C}^1 \lambda e^\lambda (1+h)^2 2^{-2h} H_4(B_h) \\ &\leq \tilde{V}_{I \setminus D_{h-1} \cup R_h}^{(h)} + \#(R_h) \hat{C}^1 \lambda e^\lambda (1+h)^2 2^{-2h} H_4(B_h) \end{aligned} \quad (5.19)$$

where the last step is perhaps the most interesting. It is possible because in $D_{h-1} \setminus D_h \cup R_h$ the field $X^{(h)}$ though not too large ($X^{(h)} < B_h$) is still very large ($|X^{(h)}| > B_h / (1+h)^2$): hence the higher

order terms are really small compared to the leading term which is positive because $B_h / (1+h)^2$ is very large. This is true $\forall \lambda$ if h is large enough: $h \geq h_\lambda$. Since the higher orders are of higher order in λ this will be true for all $h \geq 0$ if λ is small enough.

Finally integrating over $z^{(h)}$ the $\exp \tilde{V}_{I \setminus Q_{h-1} \cup R}^{(h)}$ when $\frac{\tilde{B}_h}{R} \frac{\tilde{B}_h}{R} = 1$ gives us a result

$$\leq \exp(g_1 - g_2 B_h^2) \#(R) \int P_{h-1}(dz) \exp \tilde{V}_{I \setminus D_{h-1}}^{(h-1)} \exp \epsilon_{-}(h) |I| \quad (5.20)$$

by the very definition of $\tilde{V}^{(h-1)}$. Since outside D_{h-1} the higher order terms are really "higher order" we immediately realize that

$$\begin{aligned} \tilde{V}_{I \setminus D_{h-1} \cup R}^{(h-1)} &\leq \tilde{V}_{I \setminus D_{h-1}}^{(h-1)} + (\#(R) \hat{C}^{(1+h)^2} 2^{-2h} H_4(B_{h-1})) \\ &= \hat{V}_I^{(h-1)} + (\text{error}). \end{aligned} \quad (5.21)$$

Therefore we are in an inductive position and we have achieved the estimate:

$$E_+(\lambda) = \sum_{h \geq h_\lambda} (\epsilon_{-}(h) + \eta(h)) + \tilde{E}(\lambda) \quad (5.22)$$

where

$$\tilde{E}(\lambda) = \max_{\phi} \hat{V}_I^{(h_\lambda)}(\phi) |I|^{-1} \quad (5.23)$$

and $\tilde{E}(\lambda) = 0$ if λ is small enough.

Since $E_+(\lambda)$ is easily seen to have the wanted form (i.e. $\lambda^{-2} E_+(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$) the proof is finished.

We have now all the basic ingredients and mechanisms to attack more complex problems like the hierarchical Markov field or the Euclidean field.

6. The Euclidean Field Theory

I shall now directly consider the case of the three dimensional ϕ^4 theory without illustrating the case of the Markov field (see [3]).

We define a gaussian field Z distributed with covariance operator

$$A = C \frac{(1-D)(\gamma^2-D)}{(\gamma^2-1)} \quad (6.1)$$

where D is the Laplace operator and the constant C is chosen so that

$$E(Z_\xi^2) = \frac{1}{2} \quad (6.2)$$

i.e.

$$\frac{1}{(2\pi)^3} \frac{1}{C} \int_R \frac{(\gamma^2-1) d^3k}{(1+k^2)(\gamma^2+k^2)} = \frac{1}{2} \quad (6.3)$$

Then the euclidean field ϕ is defined as

$$\phi_\xi = \sum_{N=0}^{\infty} \sqrt{\gamma^N} Z_\xi^{(N)} \quad (6.4)$$

where $Z^{(N)}$ are gaussian fields independently distributed so that $Z_\xi^{(N)}$ has the same distribution of $Z_{2^N \xi}$

$$Z_\xi^{(N)} \stackrel{\sim}{\text{Prob.}} Z_{2^N \xi} \quad (6.5)$$

The field (6.4) has covariance operator $A_0 = C^{-1}(1-D)$ and (6.4), (6.5) is the probabilistic interpretation of the obvious relation

$$\frac{1}{1-D} = \sum_{N=0}^{\infty} \left(\frac{1}{\gamma^{2N-D}} - \frac{1}{\gamma^{2(N+1)-D}} \right) \quad (6.6)$$

It is natural to introduce the cut-off and the normalized fields

$$\phi_\xi^{[\leq N]} = \sum_{k=0}^N \sqrt{\gamma^k} Z_\xi^{(k)} \quad N = 0, 1, \dots \quad (6.7)$$

$$X_\xi^{(N)} = \frac{\phi_\xi^{[\leq N]}}{\sqrt{2E((\phi_\xi^{[\leq N]})^2)}} = \frac{Z_\xi^{(N)} + \sqrt{\Gamma_N} X_\xi^{(N-1)}}{\sqrt{1+\Gamma_N}} \quad (6.8)$$

We then define the "third order renormalized interaction" $V_I^{(N)}$ as

$$V_{O,I}^{(N)} = -\lambda \int_I : (\phi_\xi^{[\leq N]})^4 : d\xi \quad (6.9)$$

$$V_I^{(N)} = V_{O,I}^{(N)} - \frac{1}{2!} \langle (V_{O,I}^{(N)})^2 \rangle_{(2)} - \frac{1}{2!} \langle (V_{O,I}^{(N)})^2 \rangle_{(0)} - \frac{1}{3!} \langle (V_{O,I}^{(N)})^3 \rangle_{(0)} \quad (6.10)$$

where

$$\begin{aligned} \frac{1}{2} \langle (V_{O,I}^{(N)})^2 \rangle_{(2)} &= \frac{\lambda^2 4 \cdot 3!}{2} \int_{I^2} d\xi d\eta (C_{\xi\eta}^{[\leq N]})^2 : (\phi_\xi^{[\leq N]})^2 : \\ \frac{1}{2} \langle (V_{O,I}^{(N)})^2 \rangle_{(0)} &= \frac{\lambda^2 4!}{2} \int_{I^2} d\xi d\eta (C_{\xi\eta}^{[\leq N]})^4 \quad (6.11) \\ \frac{1}{3!} \langle (V_{O,I}^{(N)})^3 \rangle_{(0)} &= \frac{\lambda^3}{3!} \left(\frac{4}{2}\right)^3 (2!)^3 \int_{I^3} d\xi d\eta d\zeta (C_{\xi\eta}^{[\leq N]})^2 (C_{\eta\xi}^{[\leq N]})^2 (C_{\zeta\xi}^{[\leq N]})^2 \end{aligned}$$

and

$$C_{\xi\eta}^{[\leq N]} = \int \phi_\xi^{[\leq N]} \phi_\eta^{[\leq N]} P_N(dZ) \quad (6.12)$$

and $P_N = \prod_{i=0}^N \hat{P}_i$ denotes the probability distribution of $Z^{(0)}, Z^{(1)}, \dots$

Our problem is to show the existence of $E(\lambda)$ such that

$$\int \exp V_I^{(N)} P_N(dZ) = \exp \theta E(\lambda) |I| \quad (6.13)$$

where $\theta \in [-1, 1]$ and

$$\lim_{\lambda \rightarrow 0} \lambda^{-3} E(\lambda) = 0 \quad (6.14)$$

This problem contains a new difficulty with respect to the already treated models: the field $Z^{(N)}$ is not built with simpler fields which are "independent on a scale γ^{-N} ": however, the covariance of $Z^{(N)}$ decays exponentially over a scale γ^{-N} . This means, in colourful language, that the ultraviolet problem is now coupled with an infrared problem: the volume of I "felt by the high frequencies" will always be very large ($\sim \gamma^{3N} |I|$).

The idea of the proof of (6.13), (6.14) is the same as the one used in the former case. One integrates over the high frequencies showing that the result has the right form in order to apply a recursion argument.

Therefore it is natural to introduce, as in the hierarchical case, the quantity $\tilde{V}_I^{(h)}$ defined as

$$\tilde{V}_I^{(N)} = \tilde{V}_I^{(N)} \tag{6.15}$$

$$\tilde{V}_I^{(h)} = [E_{h+1}^T(\tilde{V}_I^{(h+1)}; 1) + \frac{1}{2!} E_{h+1}^T(\tilde{V}_I^{(h+1)}; 2) + \frac{1}{3!} E_{h+1}^T(\tilde{V}_I^{(h+1)}; 3)] \tag{3}$$

with the notation naturally following the one previously used.

Before continuing it is good to get a feeling, and also a workable form, for $\tilde{V}_I^{(N)}$. This is the so called "small diagram theory".

The result is

$$\begin{aligned} \tilde{V}_I^{(h)} = & -\lambda \int_I \gamma^{2h} (1+\gamma_h)^2 : (X_\xi^{(h)})^4 : d\xi - \\ & - \lambda^2 \int_{I \times I} \gamma^{4h} \bar{\Lambda}_{\xi\eta} \frac{:(X_\xi^{(h)} - X_\eta^{(h)})^2:}{(\gamma^h |\xi - \eta|)^{1/2}} e^{-\gamma^h d(\xi, \eta)} \end{aligned} \tag{6.16}$$

$$\begin{aligned} & + \sum_{p=0}^3 \sum_{q=0}^1 \sum_{r=0}^1 (\lambda \gamma^{2h})^p (\lambda^2 \gamma^h)^q (\lambda^3)^r \sum_{n_1 \dots n_p} \sum_{m_1 \dots m_q, m_r} \\ & \quad 0 \leq \sum_i n_i + \sum_g m_g + m_r \leq 8 \end{aligned}$$

$$\begin{aligned} & \int \bar{\Lambda}_{\xi_1 \dots \xi_p}^{n_1 \dots n_p} \bar{\Lambda}_{\eta_1 \dots \eta_q}^{m_1 \dots m_q} \bar{\Lambda}_{\eta_1}^{m_r} e^{-\gamma^h d(\xi_1, \dots, \xi_p, \eta_1, \dots, \eta_q)} (d\xi_1 \dots d\xi_p d\eta_1 \dots d\eta_q) \times \\ & \quad \times X_{\xi_1}^{n_1} \dots X_{\xi_p}^{n_p} X_{\eta_1}^{m_1} \dots X_{\eta_q}^{m_q} X_{\eta_1}^{m_r} \end{aligned}$$

where in the above formula the choices of the indices which give rise to meaningless expressions (e.g. if $r = 0$ or $n = 0$ η_1 or ξ_1 can be defined) should be interpreted by simply ignoring the indices which cannot make sense.

The coefficients $\bar{\Lambda} \dots$ turn out to have the following properties: $\exists \bar{\Lambda} > 1$ such that

$$\bar{\Lambda}^{-1} \leq \bar{\Lambda}_{\xi\eta} (\gamma^h |\xi - \eta|)^{3-1/2} \cdot \min(1, \gamma^h |\xi - \eta|) \leq \bar{\Lambda} \tag{6.17}$$

$$\int_{\Delta_1 \times \Delta_2 \times \dots} |\bar{\Lambda}_{\xi_1 \dots \xi_p}^{n_1 \dots n_p}| d\xi_1 d\xi_2 \dots \leq \bar{\Lambda} (\gamma^{-3h})^{p+q+r}$$

The calculation leading to (6.16) is a long but simple algebraic analysis. It can be most easily done using diagrams. Represent $\tilde{V}_I^{(N)}$ as

$$\begin{aligned} & \begin{array}{c} \diagup \quad \diagdown \\ \xi \end{array} : \quad \begin{array}{c} \text{---} \circ \text{---} \\ \xi \end{array} : \quad \begin{array}{c} \circ \\ \xi \end{array} : \quad \begin{array}{c} \odot \\ \xi \end{array} \end{aligned} \tag{6.18}$$

Then $\tilde{V}_I^{(h)}$ will have a representation like

$$\begin{aligned} & \begin{array}{c} \diagup \quad \diagdown \\ \xi \end{array} : \quad \begin{array}{c} \text{---} \circ \text{---} \\ \xi \end{array} : \quad \begin{array}{c} \circ \\ \xi \end{array} : \quad \begin{array}{c} \odot \\ \xi \end{array} : \\ & + \begin{array}{c} \text{---} | \text{---} \\ \xi \end{array} : + \begin{array}{c} \text{---} \times \text{---} \\ \xi \end{array} : + \begin{array}{c} \text{---} \circ \text{---} \\ \xi \end{array} : + \begin{array}{c} \text{---} | \text{---} \\ \xi \end{array} : + \\ & + \begin{array}{c} \text{---} | \text{---} \\ \xi \end{array} : + \begin{array}{c} \text{---} \circ \text{---} \\ \xi \end{array} : + \begin{array}{c} \text{---} \times \text{---} \\ \xi \end{array} : + \begin{array}{c} \text{---} \circ \text{---} \\ \xi \end{array} : + \\ & + \begin{array}{c} \text{---} \circ \text{---} \\ \xi \end{array} : + \begin{array}{c} \text{---} \times \text{---} \\ \xi \end{array} : + \begin{array}{c} \text{---} \circ \text{---} \\ \xi \end{array} : + \left\{ \begin{array}{c} \text{---} \circ \text{---} \\ \xi \end{array} : + \begin{array}{c} \text{---} | \text{---} \\ \xi \end{array} : \right\} \end{aligned} \tag{6.19}$$

where the T means that "there is a hard line connecting all the points of the graph": the combinatorial coefficients and the actual combination of covariances entering into the above expressions have to be found by more careful analysis (this is reminded by the index R). See, for details, appendix B of [3], [5].



Look for instance at : $\text{---} \bigcirc \text{---}$: it gives

$$+ \frac{\lambda^2}{2} 3!4^2 \int_{I \times I} (C_{\xi\eta}^{[\leq N^3]} - C_{\xi\eta}^{[\leq h]^3}) : (\phi_\xi \phi_\eta - \phi_\xi^2) : d\xi d\eta = \quad (6.20)$$

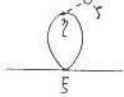
$$= - \frac{\lambda^2 3!4^2}{4} \gamma^h (1+\Gamma_h) \int_{I \times I} (C_{\xi\eta}^{[\leq N^3]} - C_{\xi\eta}^{[\leq h]^3}) : (X_\xi^{(h)} - X_\eta^{(h)})^2 : d\xi d\eta$$

so

$$\begin{aligned} \bar{A}_{\xi\eta} &= \frac{3!4^2}{4} (1+\Gamma_h) \gamma^{3h} \frac{(\gamma^h |\xi-\eta|)^{1/2}}{|\xi-\eta|^3} (e^{-|\xi-\eta|} e^{-\gamma^N |\xi-\eta|} - e^{-|\xi-\eta|} + e^{-\gamma^h |\xi-\eta|}) = \\ &= \frac{3!4^2}{4} (1+\Gamma_h) (\gamma^h |\xi-\eta|)^{-3+1/2} e^{-\gamma^h |\xi-\eta|} (1 - e^{-(\gamma^N - \gamma^h) |\xi-\eta|}) \geq \\ &\geq \frac{3!4^2}{4} \gamma (\gamma^h |\xi-\eta|)^{-3+1/2} e^{-\gamma^h |\xi-\eta|} (1 - e^{-\gamma}) \min(1, \gamma^N |\xi-\eta|) \end{aligned} \quad (6.21)$$

hence

$$\int_{\Delta^2} \bar{A}_{\xi\eta} \leq \left[\frac{3!4^2}{4} \gamma (1 - e^{-\gamma}) \right] \gamma^{-6h} \int_{\mathbb{R}^3} \frac{e^{-|x|}}{|x|^{3-1/2}} d^3x. \quad (6.22)$$

As another example we consider the graph:  which gives rise to (if $C_i \equiv C^{[\leq i]}$):

$$\frac{\lambda^3}{2} 4^2 3! \binom{2}{0} \binom{4}{0} 2! \sum_{h+1}^N \int_{I^3} d\xi d\eta d\zeta (C_i^2(\xi, \eta) - C_{i-1}^2(\xi, \eta)) C_i^3(\eta, \zeta) : \phi_\xi^2 : \quad (6.23)$$

and this corresponds to $r = 1$, $m'_r = 2$ and the A coefficient is

$$\frac{\lambda^3}{2} 4^2 3! \binom{2}{0} \binom{4}{0} \sum_{i=h+1}^N (1+\Gamma_h) \gamma^h \int_{I^2} d\eta d\zeta (C_i^2(\xi, \eta) - C_{i-1}^2(\xi, \eta)) C_i^3(\eta, \zeta)$$

$$\leq \frac{\lambda^3}{2} 4^2 3! (1+\Gamma_h) \gamma^h \sum_{i=h+1}^N \int_{I^2} d\eta d\zeta \frac{5i}{2} \frac{(1 - e^{-\gamma^i |\xi-\eta|})}{(\gamma^i |\xi-\eta|)^2} e^{-\gamma^{i-1} |\xi-\eta|} \frac{(1 - e^{-\gamma^i |\zeta-\eta|})}{(\gamma^i |\zeta-\eta|)^3}$$

$$\leq \frac{\lambda^3 4^2 3!}{2} \frac{\gamma}{\gamma-1} \sum_{i=h+1}^N \gamma^{-i} \int_{\mathbb{R}^3} \frac{e^{-|x|}}{|x|^2} dx \int_{\mathbb{R}^3} \frac{1 - e^{-|x|}}{|x|^3} e^{-\gamma^{-i} |x|} dx$$

$$\leq C \lambda^3 \gamma^h \sum_{i=h+1}^N i \gamma^{-i} \leq \bar{C} \lambda^3 h. \quad (6.24)$$

So we shall be able to bound the interaction $\tilde{V}^{(N)}$ when

$$\begin{aligned} |X_\xi^{(N)}| &\leq B_N & \forall \xi \in I \\ |X_\xi^{(N)} - X_\eta^{(N)}| &\leq B_N (\gamma^N |\xi-\eta|)^{1/4} & \forall \xi, \eta \in I, \gamma^N |\xi-\eta| < 1. \end{aligned} \quad (6.25)$$

It is therefore natural, following the scheme of the Bernoulli case to introduce the following regions

$$\begin{aligned} D_N^g(\phi) &= \{ \xi \mid |X_\xi^{(N)}| > B_N (1 + \gamma^N d(\xi, I)) \} \\ D_N^r(\phi) &= \{ \xi \mid \exists \eta, |\gamma^N (\xi-\eta)| < 1 \text{ s.t. } |X_\xi^{(N)} - X_\eta^{(N)}| > \\ &> (1 + \gamma^N d(\xi, I)) |\xi-\eta|^{1/4} \} \end{aligned} \quad (6.26)$$

$$\begin{aligned} R_N(\phi) &= \{ \Delta \mid \Delta \in Q_N \text{ such that } \exists \xi \in \Delta \text{ where } \\ &|Z_\xi^{(N)}| > B_N^1 (1 + \gamma^N d(\xi, I)) \text{ or } \exists \xi, \eta \in \Delta \text{ where } |Z_\xi^{(N)} - Z_\eta^{(N)}| \geq \\ &\geq B_N^1 (\gamma^N |\xi-\eta|)^{1/4} (1 + \gamma^N (d(\xi, I))) \text{ and } |\xi-\eta| \gamma^N \leq 1 \}. \end{aligned}$$

Notice that D_N^g , D_N^r are "arbitrary" open sets while $R_N(\phi)$ is pavable by Q_N , by definition: i.e. R_N is defined in a substantially different way from D_N^g and D_N^r . Alternatively we could also define D_N^g and D_N^r to be the smallest Q_N -pavable sets containing the above defined sets: this would not be, however, a natural definition as it was, "by accident", in the hierarchical case.

We also define

$$\hat{V}_I^{(h)}(\phi) = \{\text{same as (6.16) with } I \setminus D_h^g(\phi) \text{ replacing every-} \tag{6.27}$$

where I , and $(I \setminus D_h^g(\phi))^2 \setminus D_h^r(\phi)$ replacing $I \times I$.

Calling $\tilde{I} = I \setminus D_h^g(\phi)$ this means that we replace I by \tilde{I} everywhere and, then, $I \times I$ by $(\tilde{I} \times \tilde{I}) / D_h^r(\phi)$ or $I \times I \times I$ by the set obtained by taking out of $I \times I \times I$ all triples containing a pair in D_h^r . The A-coefficients that are put into the integrals defining (6.27) are those relative to the region I .

The above definition of \hat{V} is slightly simpler than the one of ref. [5]: it seems, however, as good. On the other hand it makes the proof of the basic inequalities (6.28) ÷ (6.30) below slightly harder, (see [5]). The functions \hat{V} verify some inequalities, as in the Bernoulli case: suppose that B is large enough, (i.e. B larger than the largest zero of $H_4(b)$), then $\exists h_\lambda, \bar{C}$, and

$$i) \quad V_I^{(N)} \leq \hat{V}_I^{(N)} \tag{6.28}$$

$$ii) \quad \hat{V}_I^{(h)}(\phi) \leq (\bar{C})^\lambda e^{\bar{C}\lambda} \gamma^{2h} B_{h-1}^{\bar{C}} \gamma^{-3h} \neq (R_h(\phi) \cap I) + H_S^{(h)} \tag{6.29}$$

where $\hat{R}_h(\phi) = \{\Delta \mid \Delta \in Q_h, d(\Delta, R_h(\phi)^{(h)}) \leq (B_h^1)^3 \gamma^h\}$ and $H_S^{(h)}$ is defined as (6.27), replacing I everywhere in the integration regions, by $S \setminus D_{h-1}^g(\phi)$ and, then, $(I \setminus D_h^g(\phi))^2$ by $(S \setminus D_{h-1}^g(\phi))^2 \setminus D_{h-1}^r$ (in the same sense), and using A-coefficients associated with the region S .

$$iii) \quad \left\{ \sum_{k=1}^3 \frac{E_h^{\pi}(H_I \setminus \hat{R}_h(\phi); K)}{k!} \right\} - \hat{V}_I^{(h-1)} \leq (\bar{C})^\lambda e^{\bar{C}\lambda} B_{h-1}^{\bar{C}} \gamma^{-h} 4 \gamma^{3h} |I| \tag{6.30}$$

iv) $h_\lambda \equiv 0$ if λ is small enough.

All the above properties of $\hat{V}_I^{(h)}, \tilde{V}_I^{(h)}$ have obvious analogues in the Bernoulli model and are what we need to copy the argument already explained in the Bernoulli case to the Field theory of Euclidean fields: the only missing tool is an algorithm for

estimating the error involved in the integration of the highest frequency field.

Such algorithm is provided by the following lemma of probability theory which allows to conclude that even the results of the integrations are as if the model was hierarchical. Let Z be the standard field with (6.1) as covariance operator and let:

$$U_J(Z) = \sum_{p=1}^S \sum_{\substack{\xi_1 \\ n_1 \leq D}} \int_{J^p} A_{\xi_1 \dots \xi_p}^{n_1 \dots n_p} e^{-\kappa d(\xi_1, \dots, \xi_p)} \tag{6.31}$$

$$Z_{\xi_1}^{n_1} \dots Z_{\xi_p}^{n_p} d\xi_1 \dots d\xi_p +$$

$$+ \sum_{q=1}^S \sum_{\substack{\xi_1' \\ m_1 \leq D}} \int_{J^{2q}} A_{\xi_1' \dots \xi_q'}^{m_1 \dots m_q} e^{-\kappa d(\xi_1' \dots \xi_q')} \prod_{j=1}^q \left(\frac{Z_{\xi_j} - Z_{\xi_j'}}{|\xi_1 - \xi_j'|} \right)^{m_j}$$

where the A's are such that

$$\|A\| = \sup_{\Delta_1 \times \dots \times \Delta_p} \int |A_{\xi_1 \dots \xi_p}^{n_1 \dots n_p}| d\xi_1 \dots d\xi_p + \tag{6.32}$$

$$+ \sup_{\Delta_1 \times \Delta_1' \times \dots \times \Delta_q \times \Delta_q'} \int |A_{\xi_1' \dots \xi_q'}^{m_1 \dots m_q}| d\xi_1' \dots d\xi_q'$$

Let $\chi_\Delta^B(Z)$ be the c.h.f. of the event

$$\{|Z| |Z_\xi| < B \quad \forall \xi \in \Delta, \quad |Z_\xi - Z_n| < B |\xi - n|^{1/4}, \tag{6.33}$$

$$\forall \xi, n \in \Delta, \quad |\xi - n| \leq |\Delta|\}$$

$\Delta \in Q_0$. Let $\chi_\Delta^B = 1 - \chi_\Delta^B$. Then

Lemma: Let γ be close enough to 1. Given $t \geq 0$ integer $\exists B^*, G, G', \rho, \rho', \rho^N, \gamma_1, \gamma_2$ depending only on t, D, D', κ such that if $B > B^*$ and if $R \subset Q_0, J \subset I$ and $H_{J \setminus R}$ is defined from (6.31) as after (6.29)

$$\int \chi_R^B \chi_{Q \setminus R}^B e^{H_{J \setminus R}} P(dZ) \leq \exp(|I| \delta(B, \|A\|) + \delta'(B, \|A\|)|R|)$$

Finally

$$Z_+(h_\lambda) \leq \exp \max_{\phi} \hat{V}^{(h_\lambda^{-1})}(\phi). \quad (7.13)$$

Hence

$$Z_+(N) \leq \exp |I| \left\{ \sum_{h=h_\lambda}^{\infty} (\varepsilon(h) + \eta(h)) + \tilde{e}(\lambda) \right\} \quad (7.14)$$

where $\tilde{e}(\lambda) |I|$ is an estimate for $\sup \hat{V}^{(h_\lambda^{-1})}(\phi)$. Since $h_\lambda \equiv 0$, if λ is small enough, we see that one can take:

$$E_+(\lambda) = \sum_{h=h_\lambda}^{\infty} (\varepsilon(h) + \eta(h)) \quad (7.15)$$

and this concludes the proof since it is easy to see that

$$\lim_{\lambda \rightarrow 0} \lambda^{-3} E_+(\lambda) = 0. \quad (7.16)$$

So it only remains to prove the basic lemma, last of the § 6.

8. Concluding Remarks

The proof of the lemma of § 6 is discussed in ref. [5] where it is reduced to a general support property of free fields roughly saying that the probability that a gaussian field, associated with an elliptic operator "with positive mass", is everywhere in a region I bounded by B times the square root of its covariance is bounded below by

$$\exp - C_1 e^{-C_2 B^2} |I|$$

$\forall I \subset \mathbb{R}^d$, if C_1, C_2 are suitably chosen and B is larger than B^* (I-independent). This support property, very easy to prove in the hierarchical models using techniques of statistical mechanics or gaussian processes, is much harder in the continuum cases. It is proven in ref. [4].

As it is well known the results presented here were all well known see the basic references quoted in [3], [5].

The method, however, seems to be somewhat different (although

the basic idea is, of course, the phase space cell decomposition of Glimm-Jaffe, [6]).

It seems to me that this idea is closely related to that in the analysis of the structure of the functional integrals developed by Wilson [7]: the relation should appear clear in our proof.

The proof discussed here is in the works [3], [5].

The above presentation is mainly taken out of my notes from lectures given at the Ecole Polytechnique in October-November 1978, in Palaiseau.

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