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Elliptic Equations and Gaussian Processes

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We consider a Gaussian process P on $\mathscr{S}'(R^d)$ generated by a polynomial in the Laplace operator. We prove some support properties for P. As a by-product we strenghten earlier results on the stochastic Dirichlet problem on bounded regions $\Lambda \subset R^d$. We describe in this way the conditional P-distribution of the restriction to Λ of $\varphi \in \mathscr{S}'(R^d)$, supposing φ is known outside Λ: a somewhat detailed description of the singularity of φ on Λ is given.

1. INTRODUCTION

The progress of constructive field theory has been so deep that it is too early to appreciate the implications of its results in other areas of mathematics.

In this paper we present a discussion of some problems which naturally arise in field theory and which, in our opinion, are interesting on their own.

Let A be a elliptic operator on R^d with constant coefficients. We shall actually assume that A has the form

$$A = \prod_{i=0}^{m-1} (\alpha_i^2 - D) \tag{1.1}$$

with $0 < \alpha_0^2 < \alpha_1^2 < \cdots < \alpha_{m-1}^2$ and $D = \sum_{i=1}^d \partial^2/\partial x_i^2$. Most of our results should be extendable to a general elliptic operator with constant coefficients and with the principal Green's function exponentially decaying at infinity. We shall consider only $d \geqslant 2$.

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We shall also consider bounded open regular[1] domains Λ and the formal Dirichlet problem

$$\begin{aligned} Au &= 0 && \text{in } \Lambda \\ \partial^l u &= z^{(l)} && \text{in } \partial\Lambda,\ l = 0, 1,\dots, m-1, \end{aligned} \tag{1.2}$$

where ∂^l denotes the lth inner normal derivative of u on $\partial\Lambda$ and $\underset{\sim}{z} = (z^{(0)}, z^{(1)},\dots, z^{(m-1)})$ is a given $m$ple of distributions on $\partial\Lambda$.

In field theory the above equation appears associated with the Gaussian probability measure, on the Borel sets $\mathscr{B}^0$ of $\mathscr{S}'(R^d)$ relative to the $\mathscr{S}(R^d)$-topology, with a covariance kernel given by the principal Green's function of A

$$C_{\xi\eta} = \int_{\mathscr{S}'(R^d)} z_\xi z_\eta P_A(dz) = \int_{R^d} \frac{d^dk}{(2\pi)^d} \frac{e^{ik(\xi-\eta)}}{\prod_{j=0}^{m-1} (\alpha_j{}^2 + k^2)}, \qquad \xi, \eta \in R^d, \tag{1.3}$$

to be thought as a distribution in $\mathscr{S}'(R^d \times R^d)$.

The boundary value which appears in (1.2) is then the "trace" on $\partial\Lambda$ of the first $m - 1$ normal derivatives of a distribution z chosen P_A-randomly in $\mathscr{S}'(R^d)$.

The reasons for the interest of such a stochastic differential equation are found in the following heuristic considerations.

Let $u = L(\underset{\sim}{z}) = L(z^{(0)}, z^{(1)},\dots, z^{(m-1)})$ be the "solution" of (1.2). Let ζ be a Gaussian random variable in $\mathscr{S}'(R^d)$ whose probability distribution is a Gaussian measure on $\mathscr{B}^0$ whose covariance kernel is the Green's function $\xi, \eta \to C^0_{\xi\eta}$ of the operator A with null boundary conditions on the complement Λ^c of Λ.[2] We shall denote such measure $P_A{}^0$, whenever Λ is understood: hence, if $\xi, \eta \in R^d$,

$$C^0_{\xi\eta} = \int_{\mathscr{S}'(R^d)} \zeta_\xi \zeta_\eta P_A{}^\circ(d\zeta). \tag{1.4}$$

[1] This means that there are a finite number of points $x \in \partial\Lambda$ such that, if they are chosen as the origin of a coordinate system in R^d in which the plane $x_d = 0$ is the tangent plane to $\partial\Lambda$ in x, $\exists \nu \in \mathscr{D}(R^{d-1})$ such that the surface $x_d = \nu(\underset{\sim}{x})$, $\underset{\sim}{x} \in R^{d-1}$, coincides with $\partial\Lambda$ in an open neighborhood of 0 in R^d. Furthermore the "surface elements" of $\partial\Lambda$ thus described cover $\partial\Lambda$. Later we shall impose some further regularity constraints.

[2] That is, we consider the Green's function of the minimal extension (Friedrichs' extension) of the operator A acting on $\mathscr{D}(\Lambda)$. Such function is a distribution in $\mathscr{D}'(\Lambda) \times \mathscr{D}'(\Lambda)$ which can be thought as a distribution in $\mathscr{S}'(R^d) \times \mathscr{S}'(R^d)$ if "extended by 0": we denote such extension $C^\circ : C^\circ_{\xi\eta} = 0$ unless $\xi, \eta \in \Lambda \times \Lambda$. The extendibility by zero is a well-known property of the Green's function of the operators A. It is also a consequence of our estimates in Appendix A which implicitly imply that, $\forall f \in \mathscr{S}(R^d)$, $\int_\Lambda C^\circ_{\xi\eta} f(\eta)\, d\eta$ is in $C^{(m-1)}(R^d)$, which permits us to define C° as a functional in $\mathscr{S}'(R^d) \times \mathscr{S}'(R^d)$. The appropriate continuity of such functional is also a consequence of the estimates in Appendix A.

Then consider the random field $z \in \mathcal{S}'(R^d)$ defined in terms of two independent random fields $\zeta, \bar{z} \in \mathcal{S}'(R^d)$ as

$$z_\xi = \zeta_\xi + u_\xi(\bar{z}), \qquad \xi \in R^d, \tag{1.5}$$

where ζ is $P_\Lambda{}^0$-randomly chosen in $\mathcal{S}'(R^d)$ and $\bar{z}$ is P_Λ-randomly chosen in $\mathcal{S}'(R^d)$ and

$$\begin{aligned} u_\xi(\bar{z}) &= L_\xi(\bar{z}, \partial\bar{z},..., \partial^{m-1}\bar{z}), \qquad && \xi \in \Lambda, \\ u_\xi(\bar{z}) &= \bar{z}_\xi\,, && \xi \notin \Lambda, \end{aligned} \tag{1.6}$$

where L denotes the "solution" to (1.2) with boundary values given by the first $m - 1$ inner normal derivatives of $\bar{z}$ on $\partial\Lambda$.

Heuristic considerations suggest that (1.5), (1.6) should really make sense and that the random field z "defined" by (1.5), (1.6) has the same distribution of a P_Λ-distributed random field; i.e., $\forall\xi, \eta \in R^d$,

$$\int_{\mathcal{S}'(R^d)\times\mathcal{S}'(R^d)} (\zeta_\xi + u_\xi(\bar{z}))(\zeta_\eta + u_\eta(\bar{z}))\, P_\Lambda{}^\circ(d\zeta)\, P_\Lambda(d\bar{z}) = C_{\xi\eta}\,. \tag{1.7}$$

In field theory (1.5), (1.7) would usually be written in a form more suggestive and more appropriate for the applications as follows. Define for an arbitrary open set $U \subset R^d$

$\mathscr{B}_U{}^\circ = \{$sub σ-algebra of $\mathscr{B}^\circ$ generated by the functions on

$\mathcal{S}'(R^d)$ of the form $z \to z(f) = \int_{R^d} f(\xi)\, z_\xi\, d\xi, f \in \mathscr{D}(u), z \in \mathcal{S}'(R^d)\}$

and, for an arbitrary closed set C,

$$\mathscr{B}_C{}^\circ = \bigcap_{U\supset C} \mathscr{B}_U{}^\circ.$$

Call $\mathscr{B}$, $\mathscr{B}_U$, $\mathscr{B}_C$ the completion P_Λ-modulo zero of $\mathscr{B}^\circ$, $\mathscr{B}_U{}^\circ$, $\mathscr{B}_C{}^\circ$. Furthermore, assuming that (1.5) makes sense and that the inverse images of sets in $\mathscr{B}^\circ$ via the map $(\zeta, \bar{z}) \to z$ defined by (1.5) are $P_\Lambda{}^\circ \times P_\Lambda$-measurable we call P_Λ' the measure on $\mathscr{B}^\circ$ which is the inverse image of $P_\Lambda{}^\circ \times P_\Lambda$. We call $\mathscr{B}'$, $\mathscr{B}_U'$, $\mathscr{B}_C'$ the completions P_Λ'-modulo zero of $\mathscr{B}^\circ$, $\mathscr{B}_U{}^\circ$, $\mathscr{B}_C{}^\circ$.

Then heuristic considerations suggest that $\mathscr{B} = \mathscr{B}'$, $\mathscr{B}_U' = \mathscr{B}_U$, $\mathscr{B}_C = \mathscr{B}_C'$.

and $P_\Lambda = P'_\Lambda$ on $\mathscr{B}$. Relation (1.7) can then be also written if G is $\mathscr{B}_{\Lambda^C}$-measurable:

$$\int_{\mathscr{S}'(R^d)} G(z)\, F(z)\, P_\Lambda(dz) = \int_{\mathscr{S}'(R^d)\times\mathscr{S}'(R^d)} G(\bar{z})\, F(\zeta + u(\bar{z}))\, P_\Lambda(d\bar{z})\, P_\Lambda{}^\circ(d\zeta) \quad (1.8)$$

which expresses the "Markov property" of P_Λ and is called the DLR equation for P_Λ.

For Gaussian processes like P_Λ the above formulas (1.5), (1.7), (1.8) have been proven in a suitable sense in [1, 2]: the first problem that we study is to give them a meaning strong enough for new applications to field theory.

Secondly we shall be interested in showing that, for a set E of distributions, whose P_Λ-measure can be well estimated to be "large" the distributions $z \in \mathscr{S}'(R^d)$ will have good smoothness properties together with the traces of their first $m - 1$ normal derivatives on regular surfaces.

The results on such "support properties" are obtained by using a technique, new as far as we know, which relies on the Markov property (1.8) and, rather heavily, on some fine details of the theory of elliptic equations with constant coefficients.

The results on the theory of the elliptic equations that we use should at least seem familiar to the specialists: they are, essentially, a constructive version of the main propositions of Chapter II of Ref. [3].[3] Unfortunately we have been unable to find a reference for such results in the amount of detail needed here. Since we feel that the proofs are not always straightforward we have given a brief outline of our method in Appendix A.

In the next section we set up some definitions needed in the proofs and take this chance to give a precise formulation of our results.

2. Definitions and Description of the Results

Our notation on the spaces of distributions will be the same as those of the book of Lions and Magenes [3, Chap. I].

It is convenient to start by describing the theorem on Eq. (1.2) and set up, at the same time, a convention for the nonconventional norms that we need.

We shall consider a bounded regular region Λ and its homothetic images $\lambda\Lambda$, $\lambda \geqslant 1$.

To define the norms in the spaces of distributions on $\partial\lambda\Lambda$ we shall associate

[3] By constructive we mean here a theory, like those in [4, 5], in which the explicit estimates for the isomorphism constants and for the dimensions of the defect spaces are "explicitly" given in terms of the geometric structure of the boundary of Λ (which are possible because of our constant coefficients and decay assumptions on the operator A, very special in the class considered in [3]).

to each set $\lambda\Lambda$, $\lambda \geqslant 1$, a covering $\sigma_1, \ldots, \sigma_{n_\lambda}$ of $\partial\lambda\Lambda$ by regular, regularly spaced (as $\lambda \to \infty$), surface elements.[4]

To the covering $\sigma_1, \ldots, \sigma_{n_\lambda}$ we associate a partition of unity $\alpha_1, \ldots, \alpha_{n_\lambda}$ with smooth, regular as $\lambda \to \infty$, functions on $\partial\lambda\Lambda$.[5]

If f is a distribution on $\partial\lambda\Lambda$ then $\alpha_i f$ has support in σ_i and can be represented in the local system of coordinates associated with σ_i by a distribution $\overline{\alpha_i f}$ on R^{d-1} with support in $\{|\underline{x}| < \frac{1}{2}\}$.

To measure the magnitude of f we introduce the following norms on the space of the distributions on R^{d-1}: for $g \in \mathscr{D}'(R^{d-1})$, $\epsilon \in (0, 1)$:

$$\| g \|_{\bar{C}^{(\epsilon)}(G)} = \sup_{\substack{\underline{x}, \underline{y} \in G \\ |\underline{x} - \underline{y}| \leqslant 1}} e^{(|\underline{x}|)^{1/2}} \left(|g(\underline{x})| + \frac{|g(\underline{x}) - g(\underline{y})|}{|\underline{x} - \underline{y}|^\epsilon} \right), \tag{2.1}$$

where $G \subset R^{d-1}$ is any set. So the space $\bar{C}^{(\epsilon)}(R^{d-1})$, associated naturally to (2.1), is the space of the Hölder continuous function with exponent ϵ whose Hölder continuity modulus decays at infinity as $\exp(-(|\underline{x}|)^{1/2})$.[6]

The basic space $\bar{C}^{(\epsilon)}(G)$ is used to define other spaces which we need.

The space $C_s^{(\epsilon)}(R^{d-1})$, $\epsilon \in (0, 1)$, $s \in R$, will be defined to be the set of the distributions g in $\mathscr{S}'(R^{d-1})$ such that

$$\| g \|_{C_s^{(\epsilon)}(R^{d-1})} = \| (1 - \underline{D})^{(s-\epsilon)/2} g \|_{\bar{C}^{(\epsilon)}(R^{d-1})} < \infty, \tag{2.2}$$

[4] This means that $\exists \lambda_0(\Lambda) \geqslant 1$ such that $\forall \lambda \geqslant \lambda_0(\Lambda)$: (i) $\exists \xi_1, \ldots, \xi_{n_\lambda} \in \partial\Lambda$ and n_λ functions $\nu_1, \ldots, \nu_{n_\lambda} \in \mathscr{D}(R^{d-1})$ such that, in the system of Cartesian coordinates in which ξ_i is the origin and $x_d = 0$ is the tangent plane π_i to $\partial\lambda\Lambda$ in ξ_i, the surface element σ_i is described by $x_d = \nu_i(\underline{x})$, $|\underline{x}| < \frac{1}{2}$, $\underline{x} = (x_1, \ldots, x_{d-1}) \in R^{d-1}$ and the set σ_i' described by $x_d = \nu_i(\underline{x})$, $\underline{x} \in S^1 = \{\underline{x} \mid |\underline{x}| < 1\}$ is also contained in $\partial\lambda\Lambda$. Also (ii) $\exists \delta_\Lambda^1$ such that $n_\lambda \leqslant \delta_\Lambda^1 \lambda^{d-1}$ (iii) $\exists \delta_\Lambda^2$ such that if $d(\sigma_i, \sigma_j) > 0$ then $d(\sigma_i, \sigma_j) \geqslant \delta_\Lambda^2$ (iv) $\exists r_\Lambda$ such that for each i the number of values of the index j such that $d(\sigma_i, \sigma_j) = 0$ is r_Λ. We shall call $\sigma_1', \ldots, \sigma_{n_\lambda}'$ the "enlarged covering" of $\partial\lambda\Lambda$ associated with the regular, regularly spaced as $\lambda \to \infty$, covering $\sigma_1, \ldots, \sigma_{n_\lambda}$. The system of coordinates used in (i) will be referred to as the "local system of coordinates" associated with σ_i.

[5] This means that supp $\alpha_i \subset \sigma_i$, $\sum_{i=1}^{n_\lambda} \alpha_i = 1$ and (i) if $\underline{x} \to \tilde{\alpha}_i(\underline{x})$ is the function which represents α_i in the local system of coordinates associated with σ_i then $\tilde{\alpha}_i \in \mathscr{D}(R^{d-1})$, $\tilde{\alpha}_i \geqslant 0$; (ii) $\exists$ a sequence $a_0, a_1, \ldots$ such that $\| \tilde{\alpha}_i \|_{C^{(p)}(R^{d-1})} \leqslant a_p$, $p = 0, 1, \ldots$; $\forall i$, $\forall \lambda \geqslant \lambda_0(\Lambda)$ (possibly readjusting the value $\lambda_0(\Lambda)$ previously defined); (iii) on each $\sigma_i' \supset \sigma_i$ we shall imagine defined a function α_i' with support on σ_i' and equal to 1 on σ_i and such that $\| \tilde{\alpha}_i' \|_{C^{(d)}(R^{d-1})} \leqslant a_p$ $\forall \lambda \geqslant \lambda_0(\Lambda)$, $\forall i$, $\forall p$; (iv) we shall extend α_i and α_i' to C^∞ functions on R^d to functions $\tilde{\alpha}_i$, $\tilde{\alpha}_i'$ with support within distance 1 from σ_i', and with all the normal derivatives of σ_i' vanishing and such that $\| \tilde{\alpha}_i \|_{C^{(p)}(R^d)}$, $\| \tilde{\alpha}_i' \|_{C^{(p)}(R^d)} \leqslant a_p$, $p = 0, 1, \ldots$. We can also suppose that such extensions are "canonically made."

[6] We choose $\exp(-(|\underline{x}|)^{1/2})$ as weight in (2.1) for simplicity: it is by no means a natural choice nor an optimal one. It will appear from our use of (2.1) that a natural and optimal choice would be an exponential decay $\exp(-\text{const}\,|\underline{x}|)$ with a constant proportional to λ_0.

i.e., $g \in C_s^{(\epsilon)}(R^{d-1})$ if after being "differentiated" $s - \epsilon$ times it is still Hölder continuous with exponent ϵ and well localized in space; in (2.2) $\underline{D} = \sum_{j=1}^{d-1} \partial^2/\partial x_j^2$.

We shall measure the magnitude of $f \in \mathscr{D}'(\partial\lambda\Lambda)$ by the norms defined for $\epsilon \in (0, 1)$, $s \in R$,

$$\| f \|_{C_s^{(\epsilon)}(\sigma_i, \alpha_i)} = \| \dot{\overline{\alpha_i f}} \|_{C_s^{(\epsilon)}(R^{d-1})} \tag{2.3}$$

$$\| f \|_{C_s^{(\epsilon)}(\partial\lambda\Lambda)} = \sup_{i=1,\dots,n_\lambda} \| f \|_{C_s^{(\epsilon)}(\sigma_i, \alpha_i)} . \tag{2.4}$$

For studying Eq. (1.2) it is also convenient to introduce the "space of boundary data":

$$\mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda) = \prod_{j=0}^{m-1} C_{s-j}^{(\epsilon)}(\partial\lambda\Lambda) \tag{2.5}$$

and put, for $\underline{z} = (z^{(0)}, z^{(1)}, \dots, z^{(m-1)})$,

$$\| \underline{z} \|_{\mathscr{C}_s^{(\epsilon)}(\sigma_i)} = \sum_{j=0}^{m-1} \| z^{(j)} \|_{C_{s-j}^{(\epsilon)}(\sigma_i, \alpha_i)} \tag{2.6}$$

$$\| \underline{z} \|_{\mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda)} = \sup_{i=1,\dots,n_\lambda} \| \underline{z} \|_{\mathscr{C}_s^{(\epsilon)}(\sigma_i)} . \tag{2.7}$$

Finally we shall associate to the above space of boundary data a space of distributions which is very convenient to describe distributions u near $\partial\lambda\Lambda$. To construct it, let σ_i be a surface element on $\partial\lambda\Lambda$ and let $(\underline{x}, x_d)$ be the coordinates of a point $\xi \in R^d$ in the system of coordinates associated with σ_i.

Let $\underline{j} = (j_1, \dots, j_d)$ be d nonnegative integers and

$$| \underline{j} | = \sum_{i=1}^{d} j_i , \qquad \partial^{(\underline{j})} = \partial^{|\underline{j}|}/\partial x_1^{j_1} \cdots \partial x_d^{j_d}.$$

Given $u \in \mathscr{S}'(R^d)$ and $\lambda \geqslant \lambda_0(\Lambda)$ define[7]

$$\zeta_\xi^{(\underline{j})}(\underline{x}, x_d) = \tilde{\alpha}_i(\underline{x}, x_d + \nu_i(\underline{x}))(\partial^{(\underline{j})} u)(\underline{x}, x_d + \nu_i(\underline{x})), \tag{2.8}$$

$$\theta_\xi^{(\underline{j})} = [(1 - \underline{D})^{(s-|\underline{j}|-\epsilon)/2} \zeta^{(\underline{j})}](\underline{x}, x_d), \tag{2.9}$$

where $\underline{D} = \sum_{j=1}^{d-1} \partial^2/\partial x_i^2$, $\epsilon \in (0, 1)$, $s \in R$. The function $\zeta^{(\underline{j})}$ at fixed x_d is the trace of $\partial^{(\underline{j})} u$ on the surface parallel to σ_i and translated by x_d from it, multi-

[7] Cf. footnote 5 for the definition of $\tilde{\alpha}_i$.

plicated by $\tilde{\alpha}_i$. Then define for $G = R^d$ or $G = \Lambda$ and $G' = R^d$ or $R^{d-1} \times [0, \infty)$, respectively,

$$\| u \|_{\mathscr{C}_s^{(\epsilon)}(G;\sigma_i,\tilde{\alpha}_i)} = \sum_{|\underline{j}| \leqslant m-1} \| \theta^{(j)} \|_{\bar{C}^{(\epsilon)}(G')}, \tag{2.10}$$

$$\| u \|_{\mathscr{C}_s^{(\epsilon)}(G;\partial\lambda\Lambda)} = \sup_{i=1,\ldots,n_\lambda} \| u \|_{\mathscr{C}_s^{(\epsilon)}(G;\sigma_i,\tilde{\alpha}_i)}. \tag{2.11}$$

The above notions (2.8)–(2.11) make sense, when $G = R^d$, more generally for an arbitrary surface σ described by equations like $x_d = \nu(\underline{x})$, $\nu \in \mathscr{S}(R^{d-1})$ to which is associated a function $\tilde{\alpha} \in \mathscr{D}(R^d)$: we shall denote $\| u \|_{\mathscr{C}_s^{(\epsilon)}(R^d;\sigma,\tilde{\alpha})}$ the number (2.11) thus constructed.

The space $\mathscr{C}_s^{(\epsilon)}(R^d; \partial\lambda\Lambda)$ consists of distributions which behave rather nicely on $\partial\lambda\Lambda$: for instance, the finiteness of the (semi)-norm (2.11) permits to define in a natural sense the trace of u on $\partial\lambda\Lambda$ and, at the same time, the trace $\underline{\partial} u$ of the first $m - 1$ normal derivatives of u on $\partial\lambda\Lambda$. Of course, the above definitions are interesting since

$$\| \underline{\partial} u \|_{\mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda)} \leqslant c(s) \| u \|_{\mathscr{C}_s^{(\epsilon)}(R^d;\partial\lambda\Lambda)} \tag{2.12}$$

for a suitably chosen $c(s)$, $\forall \lambda \geqslant \lambda_0(\Lambda)$; see Appendix B.

Therefore if makes sense to ask the following question: given $z \in \mathscr{C}_s^{(\epsilon)}(R^d; \partial\lambda\Lambda)$, find $u \in \bigcap_{s'<s} \mathscr{C}_{s'}^{(\epsilon)}(R^d; \partial\lambda\Lambda)$ such that

$$\begin{aligned} Au &= 0 \qquad \text{in } \lambda\Lambda \\ u &= z \qquad \text{in } (\lambda\Lambda)^C \end{aligned} \tag{2.13}$$

which means that $u \in C^\infty(\lambda\Lambda)$ and $Au = 0$ in $\lambda\Lambda$; furthermore $u = z$ as a distribution in $(\lambda\Lambda)^C \backslash \partial\lambda\Lambda$.

We shall refer to the problem of existence and uniqueness of such solutions u of (2.13) as the "Dirichlet problem in $\mathscr{C}_s^{(\epsilon)}(R^d; \partial\lambda\Lambda)$."

One expects that in $\lambda\Lambda \cup \partial\lambda\Lambda$ the solutions of (2.13), when existing, will depend only on $\underline{\partial} z$: we shall therefore denote them either $u(z)$ or $u(\underline{\partial} z)$: the second notation will be used only to refer to the solution of (2.13) restricted to $\lambda\Lambda \cup \partial\lambda\Lambda$. We shall also consider the distributions described in the local coordinates of σ_i by $\alpha_i(\underline{x}, x_d)\, z(\underline{x}, x_d)$, which have support near σ_i and denote $u(\tilde{\alpha}_i z)$ or $u(\underline{\partial}\tilde{\alpha}_i z)$, the solution in the whole R^d or in $\lambda\Lambda$, respectively.

We are now in a position to describe the main results on the theory of (2.13) or, loosely speaking, of (1.2).

PROPOSITION 1. *$\exists$ a continuous function $(\alpha, \alpha') \to \theta(\alpha, \alpha')$ null on the diagonal (i.e., $\theta(\alpha, \alpha) = 0$) with values in $(0, \pi/2)$ such that, given A as in (1.1), if Λ is*

$\theta(\alpha_0, \alpha_{m-1})$-conically regular[8] *and if $\Gamma \subset \Lambda$ is a regular region well situated in Λ*,[9] *then*:

(i) *$\exists$ a continuous function $(\epsilon, s) \to \lambda_1(\epsilon, s)$ on $(0, 1) \times R$ such that $\forall \lambda \geqslant \lambda_1(\epsilon, s)$ and $\forall z \in \mathscr{C}_s^{(\epsilon)}(R^d, \partial\lambda\Lambda)$ the Dirichlet problem in $\mathscr{C}_s^{(\epsilon)}(R^d, \partial\lambda\Lambda)$, cf.* (2.13), *has a unique solution $u(z)$ (which in $\lambda\Lambda$ depends only on $\underset{\sim}{\partial} z$, cf.* (2.12)). *It verifies $\forall \epsilon, s, s' < s$,*

$$\| u(z)\|_{\mathscr{C}_{s'}^{(\epsilon)}(R^d;\partial\lambda\Lambda)} = \overset{\circ}{C}_{s',s,\epsilon} \| z \|_{\mathscr{C}_s^{(\epsilon)}(R^d;\partial\lambda\Lambda)},$$
$$\| u(\underset{\sim}{\partial} z)\|_{\mathscr{C}_{s'}^{(\epsilon)}(\lambda\Lambda;\partial\lambda\Lambda)} = \overset{\circ}{C}_{s',s,\epsilon} \| \underset{\sim}{\partial} z \|_{\mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda)} \tag{2.14}$$

for a suitable continuous $\overset{\circ}{C}_{s',s,\epsilon}$ (λ-independent).

(ii) *$\exists \kappa > 0$ such that, $\forall \epsilon \in (0, 1)$, $s \in R$, $s > \epsilon$, $s - \epsilon$ not integer and if $s_\epsilon = [s - \epsilon] + \epsilon$, $\forall \Delta \subset \lambda\Lambda$, Δ open, $\forall \lambda \geqslant \lambda_1(\epsilon, s)$, $\forall i = 1,\ldots, n_\lambda$*

$$\| u(\underset{\sim}{\partial} \tilde{\alpha}_i z)\|_{C^{(s_\epsilon)}(\Delta)} \leqslant C_{s,\epsilon} e^{-\kappa d(\Delta,\sigma_i)} \| \underset{\sim}{\partial} z \|_{\mathscr{C}_s^{(\epsilon)}(\sigma_i,\alpha_i)}, \tag{2.15}$$

where $C_{s,\epsilon}$ is a suitable continuous function (λ-independent).[10]

(iii) *$\forall \epsilon \in (0, 1)$, $\forall s \in R$, $\forall s' < s$, $\forall \lambda \geqslant \lambda_1(\epsilon, s)$,*

$$\| \underset{\sim}{\partial} u(\underset{\sim}{\partial} \tilde{\alpha}_i z)\|_{\mathscr{C}_{s'}^{(\epsilon)}(\tau_j,\alpha_j{}^\tau)} \leqslant c(s', s, \epsilon)\, e^{-\kappa d(\tau_j,\sigma_i)} \| \underset{\sim}{\partial} z \|_{\mathscr{C}_s^{(\epsilon)}(\sigma_i,\alpha_i)} \tag{2.16}$$

$\forall$ choices of the surface elements $\sigma_i \subset \partial\lambda\Lambda$, $\tau_j \subset \partial\lambda\Gamma$, if $c(s', s, \epsilon)$ is a suitable continuous function and the $\alpha_j{}^\tau$ are defined as the α_j.

[8] A regular region Λ is "θ-conically regular" if for all $\xi \in \partial\Lambda$ the cone around the outer normal in ξ and opening θ has no intersections with $\partial\Lambda$ other than the apex ξ itself. Note that this is not a local condition.

[9] Given a regular region $\Gamma \subset \Lambda$ we say that Γ is well situated inside Λ if when $\partial\Lambda$ and $\partial\Gamma$ touch each other they do that on a not too large region and have a contact of infinite order. Precisely either $\partial\Gamma \cap \partial\Lambda = \varnothing$ or $\exists \xi \in \partial\Lambda \cap \partial\Gamma$ and ν_Γ, $\nu_\Lambda \in \mathscr{D}(R^{d-1})$ such that in a Cartesian system of coordinates in which $x_d = 0$ is the tangent plane to $\partial\Lambda$ in ξ and ξ is the origin: (i) $\exists$ a neighborhood U of the origin such that the equations: $x_d = \nu_\Lambda(\underset{\sim}{x})$, $\underset{\sim}{x} \in U$ and $x_d = \nu_\Gamma(\underset{\sim}{x})$, $\underset{\sim}{x} \in U$ describe two surfaces contained in $\partial\Lambda$ and $\partial\Gamma$ and which contain $\partial\Lambda \cap \partial\Gamma$. (ii) The contact is of infinite order: i.e., if $\partial^{(\underset{\sim}{h})}\nu$ is an $|\underset{\sim}{h}|$-th order derivative of $\nu = \nu_\Lambda - \nu_\Gamma$:

$$\sup_{\underset{\sim}{x} \in U} \frac{|\partial^{(\underset{\sim}{h})}\nu(\underset{\sim}{x})|}{|\nu(\underset{\sim}{x})|^{1-a}} < \infty \qquad \forall a \in (0, 1).$$

[10] $\forall \beta > 0$, β not integer, $\forall \Delta \subset R^d$, Δ open

$$\| u \|_{C^{(\beta)}(\Delta)} = \| u \|_{C^{(\beta)}(\bar{\Delta})} = \| u \|_{C^{([\beta])}(\bar{\Delta})} + \sum_{|\underset{\sim}{\alpha}|=[\beta]} \sup_{\substack{\underset{\sim}{x},\underset{\sim}{y} \in \bar{\Delta} \\ |\underset{\sim}{x}-\underset{\sim}{y}| \leqslant 1}} \frac{|\partial^{(\underset{\sim}{\alpha})}y(\underset{\sim}{x}) - \partial^{(\underset{\sim}{\alpha})}u(\underset{\sim}{y})|}{|\underset{\sim}{x} - \underset{\sim}{y}|^{\beta-[\beta]}},$$

where $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_d)$ are d nonnegative integers.

(iv) $\forall(\epsilon, s', s) \in (0, 1) \times R^2$, $\forall\lambda \geqslant \lambda_1(s)$,

$$\| \underline{\partial} u(\underline{\partial}\tilde{\alpha}_i z) \|_{\mathscr{C}_{s'}^{(\epsilon)}(\tau_j, \alpha_j \Gamma)} \leqslant \tilde{C}(s', s, \epsilon)\, e^{-\kappa d(\tau_j, \sigma_i)} \| \underline{\partial} z \|_{\mathscr{C}_s^{(\epsilon)}(\tau_j, \alpha_j)} \tag{2.17}$$

$\forall$ choices of the surface elements $\sigma_i \subset \partial\lambda\Lambda$, $\tau_j \in \partial\lambda\Gamma$ such that $d(\tau_j, \sigma_i) \geqslant 1$ if $\tilde{C}$ is a suitably chosen continuous function.

Remarks. (1) The important feature of this theorem is the uniformity in λ and the exponential decay property in the local, up-to-the-boundary, estimates (2.15), (2.16).

(2) To use apparently more general data $\underline{z} \in \mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda)$ would not actually be more general because such data can always be thought as the normal derivatives of a distribution $z \in \mathscr{C}_s^{(\epsilon)}(R^d; \partial\lambda\Lambda)$.

(3) The above proposition holds for operators A of the form (1.1) even if some of the α's coincide, provided $\alpha_0 > 0$ ("same proof"). The case $(1 - D)^m$ is particularly interesting: the main simplification, in this case, is that one finds that Proposition 1 holds without *any* conical regularity condition on $\partial\Lambda$.

Although our naive double-layer techniques do not, for technically not by-passable reasons, allow the removal of the conical regularity requirement in the general case, it seems plausible that this condition is not really necessary.

Also the loss in regularity in (2.14), (2.16) ($s' < s$) is probably not real: we have proofs in special cases with $s' = s$ but the technique is considerably more complex.

Propositions 2, 3, and 4 illustrate the applications, proposed in this paper, to the theory of the Gaussian measure P_A of Proposition 1.

Given a function $\nu \in \mathscr{D}(R^{d-1})$ and a function $\tilde{\alpha} \in \mathscr{D}(R^d)$ we regard ν as the equation of a surface σ in R^d ($x_d = \nu(\underline{x})$, $\underline{x} \in R^{d-1}$ describes σ) and assume for simplicity that all the normal derivatives of $\tilde{\alpha}$ vanish on σ.

Then we can introduce the distributions $\zeta^{(j)}$, $\theta^{(j)}$ like in (2.8), (2.9) replacing σ_i, $\tilde{\alpha}_i$ by σ, $\tilde{\alpha}$ and define the event, cf. (2.10),

$$E_{\tilde{\alpha},\nu}^{B,s,\epsilon} = \{ z \mid z \in \mathscr{S}'(R^d), \| z \|_{\mathscr{C}_s^{(\epsilon)}(R^d; \sigma, \tilde{\alpha})} < B \}. \tag{2.18}$$

The distributions in $E_{\tilde{\alpha},\nu}^{B,s,\epsilon}$ are quite nice on σ: it makes sense in a natural way to define the traces on σ of their first $m - 1$ normal derivatives multiplied by $\tilde{\alpha}$.

PROPOSITION 2. *Let $\epsilon, \epsilon' \in (0, \frac{1}{2})$, $\epsilon < \epsilon'$, $s = m - d/2 - (\epsilon' - \epsilon)$. Then $E_{\tilde{\alpha},\nu}^{B,s,\epsilon}$ is P_A-measurable and $\exists C_1, C_2 > 0$ such that*

$$P_A(E_{\tilde{\alpha},\nu}^{B,s,\epsilon}) \leqslant 1 - C_1 e^{-C_2 B^2} \tag{2.19}$$

if $\| \nu \|_{C^{(1)}(R^{d-1})} < \frac{1}{4}$.[11]

[11] $\frac{1}{4}$ is not optimal. Actually, optimally there should be no such restriction.

Furthermore the random variables (*cf.* (2.9))

$$\theta_{\xi}^{(\underline{j})} = [(1 - \underline{D})^{(s-|\underline{j}|-\epsilon)/2}\zeta^{(\underline{j})}](\underline{x}, x_d), \qquad \xi = (\underline{x}, x_d) \in R^d, \tag{2.20}$$

form a family of Gaussian random variables with respect to the measure P_A, $\forall \underline{j}$, $|\underline{j}| \leqslant m - 1$: *their covariance* $X^{(\underline{j})}(\xi, \eta)$ *verifies*, $\forall \epsilon'' \in (0, \epsilon')$

$$\sup_{\substack{|\xi-\xi'|\leqslant 1 \\ |\eta-\eta'|\leqslant 1, |\underline{j}|\leqslant m-1}} \left\{ \frac{|X^{(\underline{j})}(\xi, \eta) - X^{(\underline{j})}(\xi', \eta')|}{(|\xi - \xi'| + |\eta - \eta'|)^{2\epsilon''}} e^{\mu(|\xi|+|\eta|)} \right\} < \infty \tag{2.21}$$

if $\mu > 0$ *is small enough, provided* $\| \nu \|_{C^{(1)}(R^{d-1})} < \frac{1}{4}$.[11]

Remarks. (1) The considerations preceding Proposition 2 show that this is a "trace theorem."

The arbitrariness of ϵ, ϵ' in Proposition 2 and (2.12) allow us to conclude, by simple arguments, that Proposition 2 means that P_A-almost surely (see (2.19), (2.18)) we have

$$\| \underline{\partial} z \|_{\mathscr{C}_s^{(\epsilon)}(\sigma, \tilde{\alpha})} < +\infty, \qquad \| z \|_{\mathscr{C}_s^{(\epsilon)}(R^d; \sigma, \tilde{\alpha})} < +\infty \tag{2.22}$$

if $\epsilon \in (0, \frac{1}{2})$, $s < m - d/2$, and $\underline{\partial} z$ denotes the trace $(z, \partial z, ..., \partial^{m-1} z)$ of the first $m - 1$ normal derivatives of z on σ which exist in the natural sense, provided by (2.18), (2.19), when $\| \nu \|_{C^{(1)}(R^{d-1})} < \frac{1}{4}$.

(2) Clearly (2.22) and Proposition 1 allow us to give a rigorous "naive" (hence strong) meaning to the r.h.s. of (1.5) as a random field on R^d for regions of the form $\lambda \Lambda$ with Λ a region $\theta(\alpha_0, \alpha_{m-1})$-conically regular and with $\lambda \geqslant \lambda_1(\epsilon, s)$, given $\epsilon \in (0, \frac{1}{2})$, $s < m - d/2$.

In fact, assuming[12] that $\lambda_1(\epsilon, s) > \lambda_0(\Lambda)$ and that $\lambda_0(\Lambda)$ is so large that $\forall \lambda \geqslant \lambda_0(\Lambda)$, $\| \nu_i \|_{C^1(R^{d-1})} < \frac{1}{4}$, one chooses ζ and $\bar{z}$ in $\mathscr{S}'(R^d)$ with a probability distribution P_A° and P_A, respectively. Then one solves the Dirichlet problem (2.13) in $\mathscr{C}_s^{(\epsilon)}(R^d; \partial\lambda\Lambda)$: this is possible by Proposition 1. Call $u(\bar{z})$ the solution.

By (2.14) and a continuity argument it is easy to see that $u(\bar{z})$ is a P_A-measurable random field. Hence we can define the $P_A^{\circ} \times P_A$-measurable random field on R^d

$$z = \zeta + u(\bar{z}). \tag{2.23}$$

(3) Once the map $\zeta, \bar{z} \to \zeta + u(\bar{z}) = z$ has been $P_A^{\circ} \times P_A$-almost everywhere defined on $\mathscr{S}'(R^d) + \mathscr{S}'(R^d)$ as a $P_A^{\circ} \times P_A$-measurable map into $\mathscr{S}'(R^d)$ we can use it to define by inverse images a measure P'_A on the σ-algebra $\mathscr{B}^{\circ}$ (of the Borel sets of the $\mathscr{S}(R^d)$-topology on $\mathscr{S}'(R^d)$).

[12] As we may and shall do, possibly readjusting $\lambda_0(\Lambda)$ and $\lambda_1(\epsilon, s)$.

By standard arguments [1, 2, 8] it follows that any $\mathscr{B}_{\Lambda^C}$-measurable function depends only on the field in Λ^C.

To prove the Markov property (1.8) we have to show only (1.7) and to use then the uniqueness of the Gaussian measures of given covariance.

The relation (1.7) is proven in [1] for $m - d/2 > 0$ and in [2] in general. It can also be checked directly provided one has enough familiarity with the distributions C° and C°: one first shows that the real problem is to prove (1.7) for $\xi, \eta \in \lambda\Lambda$ (cf. Appendix C); then if $\xi, \eta \in \lambda\Lambda$ and (1.7) is explicitly written in terms of the covariances C° and C of ζ and z one realizes that it expresses a relation between C° and C well known in the theory of partial differential equations.[13] Proposition 2 can be called a "local support property."

Typically we call "global support properties" the "good" estimates of events having the form $\{z \mid z \in \mathscr{S}'(R^d), \| z \|_{H(I)} < B\}$ where I is a cube centred at the origin and $\| z \|_{H(I)}$ is some "local norm" on z: for instance,

$$E_I^{B,\beta} = \{z \mid z \in \mathscr{S}'(R^d), \| z \|_{C^{(\beta)}(I)} < B\}. \tag{2.24}$$

In the following we shall only consider sets of distributions measurable with respect to P_Λ, $P_\Lambda{}^\circ$, $P_\Lambda{}^\circ \times P_\Lambda$ (as it will appear from the context) leaving systematically the measurability problems to the reader: a very good and short guide to the analysis of such problems is the theory in Section II of Ref. [6].

Another typical global event which arises naturally in connection with the concrete exploitation of the Markov property (1.8) and of Propositions 1 and 2 is the following.

Let Q_1 be a pavement of R^d with cubic tesserae Δ of unit side size. Shrink each tessera about its center by $(1 - \delta)$, $\delta \in (0, 1)$, and then turn it into a regular convex box by smoothing, in the same way for all Δ's in Q_1, the edges and the corners. Call $\tilde{Q}_1$ the set of the modified tesserae thus obtained and call $\tilde{Q}_{1,l}$ the set of tesserae obtained by dilating by a factor l the whole quasi-pavement $\tilde{Q}_1$.

We denote $\square$ the tesserae of $\tilde{Q}_{1,l}$: they have the form of $l\Delta$, $\Delta \in \tilde{Q}_1$ and on each of them we consider a regular, regularly spaced (as $l \to \infty$), covering with surface elements (cf. footnotes 4–6). We call $\sum_{1,l}$ the family of such surface elements.

With this definition in mind it makes sense to consider the event

$$\bar{E}_\sigma^{B,s,\epsilon,l} = \{z \mid z \in \mathscr{S}'(R^d), \| \underline{\partial} z \|_{\mathscr{C}_s^{(\epsilon)}(\sigma)} < B\}, \tag{2.25}$$

[13] Let $f \in \mathscr{S}(R^d)$ and consider the equation $Au = f$; it has a unique solution in $\mathscr{S}(R^d)$. Restrict it to Λ and call $(z^{(0)},..., z^{(m-1)}) = \underline{z} = \underline{\partial} u$ the trace on $\partial\Lambda$ of the first $m - 1$ normal derivatives of u on $\partial\Lambda$. Then consider the problem $Av = f$ in Λ, $\underline{\partial} v = \underline{z}$ in $\partial\Lambda$, $v \in C^\infty(\bar{\Lambda})$: its solution v coincides with the restriction of u to $\bar{\Lambda}$. On the other hand u can be expressed in terms of C and f and v in terms of C°, f, $\underline{z}$ and the normal derivatives of C° on $\partial\Lambda$ of order $m, m + 1,..., 2m - 1$ (by the Green's formula). Choosing $f \in \mathscr{D}(\Lambda)$ and using its arbitrariness this implies a relation between C, C° which, if iterated once, yields (1.7).

where $B > 0$, $\epsilon \in (0, \frac{1}{2})$, $s < m - d/2$, $l \geqslant 1$ and $\sigma \in \sum_{1,l}$ and

$$\bar{E}_I^{B,s,\epsilon,l} = \bigcap_{\substack{\sigma \subset I \\ \sigma \in \Sigma_{1,l}}} \bar{E}_\sigma^{B,s,\epsilon,l} \tag{2.26}$$

which is another example of "global event."

As an example of good estimate of a global event we give the following proposition.

PROPOSITION 3. (i) *Let* $m - d/2 > 0$, $0 < \epsilon < \beta < s < m - d/2$ *and suppose also* $\beta < [m - d/2] - (-1)^d/2$ *and* $\epsilon \in (0, \frac{1}{2})$, $\beta - \epsilon$ *not integer. Let* $\beta_\epsilon = [\beta - \epsilon] + \epsilon$. *There exist five continuous functions of* ϵ, β, s, *denoted* $\bar{l}$, $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$, $\bar{c}_4$ *such that*

$$P_A(E_I^{B,\beta_\epsilon} \cap \bar{E}_I^{B,s,\epsilon,l}) \geqslant \exp(-\mid I \mid \bar{c}_1 e^{-\bar{c}_2 B^2}) \tag{2.27}$$

$\forall B^2 \geqslant \bar{c}_3 + \bar{c}_4 \log l$, $\forall l \geqslant \bar{l}$ *if* $|I|$ = *volume of* I.

(ii) *Let* $\epsilon \in (0, \frac{1}{2})$, $s < m - d/2$, $s \in R$, *There exist five continuous functions of* ϵ, s, *denoted* $\hat{l}$, $\hat{c}_1$, $\hat{c}_2$, $\hat{c}_3$, $\hat{c}_4$ *such that*

$$\begin{gathered} P_A(\bar{E}_I^{B,s,\epsilon,l}) \geqslant \exp(-\hat{c}_1 e^{-\hat{c}_2 B^2} \mid I \mid) \\ \forall B^2 \geqslant \hat{c}_3 + \hat{c}_4 \log l, \forall l \geqslant \hat{l}. \end{gathered} \tag{2.28}$$

Remark. This proposition is not the strongest we can prove and is not very useful in the applications. We shall obtain it as a special case of a theorem (the "integration grid existence theorem," Theorem I of Section 5) which we do not describe here because for its accurate formulation we need some involved geometric considerations. It will be written and proved in Sections 4, 5, and 6.

In the applications the above proposition must be coupled with its "opposite" which is much easier and standard and which is our last result.

Let $G \subset R^d$ be an open regular set and let G_Q be the smallest set of tesserae of Q_1 whose union covers G: let $G_Q = (\Delta_1, \Delta_2, ...)$. Finally call $\mathring{\chi}_G^{B,\beta}$ the characteristic function of the event

$$\{z \mid z \in \mathscr{S}'(R^d), \| z \|_{C^{(\beta)}(G)} > B\} \tag{2.29}$$

and $\mathring{\chi}_\sigma^{B,s,\epsilon,l}$ the characteristic function of

$$\{z \mid z \in \mathscr{S}'(R^d), \| \partial z \|_{\mathscr{C}_s^{(\epsilon)}(\sigma)} > B\}, \tag{2.30}$$

where σ is a surface element in $\sum_{1,l}$ (cf. footnotes 4–6). Then, with the above notation

PROPOSITION 4. *If $m - d/2 > 0$ and $\beta \in (0, [m - d/2] - (-1)^d/2)$ there exist two continuous functions of β, denoted g_1 and g_2, such that $\forall G \subset R^d$*

$$\int P_A(dz) \prod_{i=1}^{N} \chi_{G \cap \Delta_i}^{\circ B_i, \beta}(z) \leqslant \prod_{i=1}^{N} \exp(g_1 - g_2 B_i^2) \tag{2.31}$$

$\forall B_i \geqslant 0.$

Similarly if $m - d/2$ is arbitrary and $s < m - d/2$, $\epsilon \in (0, \frac{1}{2})$ there exist two continuous functions of ϵ and s, denoted g_3, g_4, such that for all choices of N different surface elements $\sigma_1, \ldots, \sigma_N \in \sum_{1,l}$

$$\int P_A(dz) \prod_{i=1}^{N} \tilde{\chi}_{\sigma_i}^{\circ B, s, \epsilon, l}(z) \leqslant \prod_{i=1}^{N} \exp(g_3 - g_4 B_i^2) \tag{2.32}$$

$\forall B_i \geqslant 0$, $\forall l \geqslant \lambda_0(\Lambda)$.

Finally the family $\sum_{1,l}$ of surface elements out of which $\sigma_1, \ldots, \sigma_N$ in (2.32) are taken can be replaced by the larger family $\sum_l$ introduced later in Section 4.

It is fair to associate some names to the above propositions: the Markov property (1.8) and the general theory of Gaussian Markov processes associated with elliptic operators is due to Pitt [1], see also [2]. The idea that the spaces $\mathscr{C}_s^{(\epsilon)}(R^d; \partial\Lambda)$ are the natural spaces for the support properties of P_Λ is due to Colella and Lanford [6] who proved Proposition 2 in the special case $d = 2$, $m = 1$, and $\partial\Lambda$ flat; Proposition 4 is essentially a refinement of Proposition 2 and is based on the well-known idea of Wiener for the proof of the continuity of the sample paths of the Brownian motion [7, 6]. Proposition 1 should, at least, sound familiar to the specialists in the theory of elliptic equations: our method of proof was inspired by [3, 4].

Proposition 3 is our main result (or, better, our main result is the extension of Proposition 3: the integration grid existence theorem (Theorem I of Section 5)): in this paper we prove Theorem I in full detail and, also in detail, the parts of the proofs of Proposition 2 and 4 which involve new problems with respect to the earlier literature.

The support properties, in the simple form considered in Proposition 2, have been studied by several authors in more general or different settings.

Actually one could derive Proposition 2 from our estimates of Section 3 (i.e., from the inequality (2.21)) using the elegant and general theory of measurable norms ([9], see also the list of references in [9]) instead of proceeding via the method of [6]. The analysis in Section 3 is, however, necessary to check the validity of the assumptions needed to apply the general theory to our case.

Many extensions or parallel developments to the important work in [1] exist in the literature. In particular attention has been devoted to support properties in other natural spaces of distributions or with respect to slightly non-Gaussian processes. For instance, local support properties have been considered with

Sobolev spaces H_s replacing our $\mathscr{C}_s^{(\epsilon)}$ spaces and with a $\lambda p(\phi)_2$-Markov process replacing the Gaussian process [10].

Also the stochastic Dirichlet problem has been considered in several papers, after [1], in Sobolev spaces and in a form weaker than ours [11].

We are indebted to the referee for pointing out to us some references thus giving us the opportunity to think again over our reference list and to revise it.

3. Proof of Proposition 2

The work [6] reduces the proof of Proposition 2 to the problem, trivial in the case considered in [6], of showing that the formal expression for the covariance $X^{(j)}(\xi, \eta)$ actually defines a distribution on $R^d \times R^d$ verifying (2.21), (see also Section 6, for details).

Hence we shall concentrate in proving (2.21) in our case (general d and m and more significantly general $\partial\Lambda$).

To express the covariance of the random field $\theta^{(j)}$ defined in (2.20) let us first recall two properties of the Green's functions of polynomials of the Laplacian. They follow by direct computation from the definitions:

(1) If $\sigma < 0$ is not an integer multiple of $\frac{1}{2}$ the operator $(1 - \underline{D})^\sigma$, $\underline{D} = \sum_{j=1}^{d-1} \partial^2/\partial x_j^2$ is a kernel operator on R^{d-1} and its kernel is a distribution verifying

$$N_\sigma(\underline{x} - \underline{x}') = (1 - \underline{D})^\sigma_{\underline{x}\underline{x}'} = \int \frac{d\underline{k}}{(2\pi)^{d-1}} \frac{e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}}{(1 + \underline{k}^2)^{-\sigma}}, \tag{3.1}$$

$$N_\sigma(\underline{x} - \underline{x}') = |\underline{x} - \underline{x}'|^{2(-\sigma-(d-1)/2)} I_\sigma((\underline{x} - \underline{x}')^2) + J_\sigma((\underline{x} - \underline{x}')^2), \tag{3.2}$$

$$|(\partial^{(p)} N_\sigma)(\underline{x} - \underline{x}')| \leqslant c_{p,\sigma} \exp(-\tfrac{2}{3} |\underline{x} - \underline{x}'|), \qquad |\underline{x} - \underline{x}'| \geqslant 1, \tag{3.3}$$

where I_σ, J_σ are real analytic in their arguments and $\partial^{(p)}$ denotes a pth-order derivative of N_σ, $p = 0, 1,\ldots$, and $C_{p,\sigma}$ are suitable constants.[14]

(2) Consider a pth-order derivative of the distribution C, on R^d:

$$C(\xi) = \int \frac{dk}{(2\pi)^d} \frac{e^{ik\xi}}{\prod_{j=0}^{m-1} (\alpha_j^2 + k^2)} \tag{3.4}$$

then $\partial^{(p)}C$ has the form of a finite sum of "singular parts" plus a "regular part" which is a real analytic function of the components of ξ, $\forall p = 0, 1,\ldots, 2m - 1$. The singular parts have the form

$$\frac{(\log |\xi|)^{a_d}}{|\xi|^{d-2m+p}} \prod_{i=1}^{d} \left(\frac{\xi_i}{|\xi|}\right)^{b_i} I_{b_1\cdots b_d}(\xi), \tag{3.5}$$

[14] The number $\frac{2}{3}$ in (3.3) could be replaced by another number <1.

where $b_1, b_2, \dots, b_d$ are nonnegative integers, $I_{b_1, b_2, \dots, b_d}$ is a real analytic function of the components of ξ and, finally, $a_d = 0$ if d is odd while $a_d = 1$ if d is even. Furthermore, for suitably chosen c_p, $p = 0, 1, \dots, 2m - 1$

$$| \partial^{(p)} C(\xi) | \leqslant c_p \exp(-\tfrac{2}{3}\alpha_0 | \xi |). \tag{3.6}$$

We shall now consider the random field $\theta^{(\underset{\sim}{j})}$ defined in (2.20) and we shall distinguish between the cases when $(m - | \underset{\sim}{j} | - 1)/2$ is integer or not.

Let $(m - | \underset{\sim}{j} | - 1)/2$ be integer. The form of (2.20) then immediately implies that it is enough to consider only the case $| \underset{\sim}{j} | = m - 1$.

The formal covariance of (2.20) is easily computed: let $\sigma = (s - (m - 1) - \epsilon)/2 < 0$ (by assumption on s). Then if $(\underset{\sim}{x}, x_d), (\underset{\sim}{y}, y_d) \in R^d$, $\underset{\sim}{x}, \underset{\sim}{y} \in R^{d-1}$

$$\int \theta^{(j)}_{\underset{\sim}{x}, x_d} \theta^{(j)}_{\underset{\sim}{y}, y_d} P_A(dz) = \int N_\sigma(\underset{\sim}{x} - \underset{\sim}{x}') N_\sigma(\underset{\sim}{y} - \underset{\sim}{y}') \tilde{\alpha}_i(\underset{\sim}{x}', x_d) \times \tilde{\alpha}_i(\underset{\sim}{y}', y_d) \partial^{(\tilde{j})}(C_{\underset{\sim}{x}' - \underset{\sim}{y}', \nu(\underset{\sim}{x}') - \nu(\underset{\sim}{y}') + x_d - y_d}) \, d\underset{\sim}{x}' \, d\underset{\sim}{y}', \tag{3.7}$$

where $\partial^{(\tilde{j})}$, $\tilde{j} = (\tilde{j}_1, \dots, \tilde{j}_{2d})$ is some derivative of order $| \tilde{j} | = 2(m - 1)$.[15]

The expression (3.7) after "performing" the derivatives becomes a finite linear combination of expression like

$$\int N_\sigma(\underset{\sim}{x} - \underset{\sim}{x}') N_\sigma(\underset{\sim}{y} - \underset{\sim}{y}') a(\underset{\sim}{x}', \underset{\sim}{y}', x_d, y_d)(\partial^{(j')} C)_{\xi' - \eta'}, \tag{3.8}$$

where a is a product of derivatives of $\tilde{\alpha}$ or ν and has support in a bounded region; we have put $\xi' = (\underset{\sim}{x}', x_d + \nu(\underset{\sim}{x}'))$, $\eta' = (\underset{\sim}{y}', y_d + \nu(\underset{\sim}{y}'))$, $| j' | \leqslant 2(m - 1)$.

We shall show that (3.7) verifies (2.21). To avoid repetitions of the same arguments we shall present here only the discussion of the Hölder continuity for $\underset{\sim}{x}, \underset{\sim}{y}, x_d, y_d$ in a bounded region, i.e., we shall multiply (3.7) by a function $\chi(\underset{\sim}{x}, \underset{\sim}{y}, x_d, y_d)$ with compact support and show that $\chi X^{(\underset{\sim}{j})}$ verifies (2.21): the other three cases to consider can be treated likewise (and actually turn out to be simpler). Also, for simplicity, we only consider the d odd case.

By the properties discussed in (2) above (cf. lines before (3.5)) it will be enough to study the Hölder continuity properties of expressions like

$$\int \bar{\chi}(\xi, \eta, \underset{\sim}{x}', \underset{\sim}{y}') N_\sigma(\underset{\sim}{x} - \underset{\sim}{x}') N_\sigma(\underset{\sim}{y} - \underset{\sim}{y}') \prod_{i=1}^{d} \left(\frac{\xi_i' - \eta_i}{| \xi' - \eta' |} \right)^{b_i} \frac{\tilde{I}(\xi' - \eta')}{| \xi' - \eta' |^{d-2-q}} \, d\underset{\sim}{x}' \, d\underset{\sim}{y}' \tag{3.9}$$

[15] We call (3.7) the "formal covariance" of $\theta^{(\underset{\sim}{j})}$ because strictly speaking we have not yet even shown that $\theta^{(\underset{\sim}{j})}$ makes sense as a P_A-measurable random field. As usual the proof that (3.7) has the property (2.21) also implicitly provides the proof (which we wish to leave to the reader) that $\theta^{(\underset{\sim}{j})}$ is a P_A-measurable random field with covariance (3.7).

where $\xi = (\underline{x}, x_d)$, $\eta = (\underline{y}, y_d)$, $q = 0, 1$, $\tilde{I}$ is a real analytic function of its arguments and $\tilde{\chi} \in \mathscr{D}(R^{4d-2})$ (because we are studying the Hölder continuity of (3.7) in a bounded region).

With some repeated applications of the Taylor series it is possible provided $\| \nu \|_{C^{(1)}(R^{d-1})} < \frac{1}{4}$[16] to express (3.9) as a series over $n_1, ..., n_d$, integers ≥ 0, and over $q = 0, 1$ of expression like

$$H_{n_1 \cdots n_d}(\xi, \eta) = \int d\underline{x}'\, d\underline{y}'\, | \underline{x} - \underline{x}' |^{-d/2+\epsilon'} \, | \underline{y} - \underline{y}' |^{-d/2+\epsilon'} \times \frac{a_{n_1 \ldots n_d}(\underline{x}, \underline{y}, \underline{x}', \underline{y}', x_d, y_d)}{| \xi' - \eta' |^{d-2-q}} \prod_{i=1}^{d} \left(\frac{\xi_i' - \eta_i'}{| \xi' - \eta' |} \right)^{n_i} \tag{3.10}$$

where we changed the notation setting $\xi' = (\underline{x}', x_d)$, $\eta' = (\underline{y}', y_d)$ and $\xi = (\underline{x}, x_d)$, $\eta = (\underline{y}, y_d)$; here $a_{n_1 \cdots n_d}$ are C^∞ functions with uniformly bounded support which have the property

$$\| a_{n_1 \ldots n_d} \|_{C^{(r)}(R^{4d-2})} \leq K_r \gamma^{n_1 + n_2 + \ldots + n_d} \tag{3.11}$$

when $\gamma = 4 \| \nu \|_{C^{(1)}(R^{d-1})} < 1$, and $K_r > 0$ is suitably chosen.

Therefore the problem of finding the modulus of Hölder continuity of (3.9) is now reduced to that of finding the modulus of Hölder continuity of (3.10) and of showing that when (3.11) holds, i.e., when $\gamma < 1$, this modulus can be summed over $n_1, ..., n_d$.

By using the support properties of the a's it is easy to see that we can replace, in our analysis, (3.10) by an expression which is more convenient for our calculation

$$\bar{H}_{n_1 \ldots n_d} = \int d\underline{x}'\, d\underline{y}' \frac{e^{-|\underline{x}-\underline{x}'|}}{| \underline{x} - \underline{x}' |^{d/2-\epsilon'}} \frac{e^{-|\underline{y}-\underline{y}'|}}{| \underline{y} - \underline{y}' |^{d/2-\epsilon'}} \frac{e^{-|\xi'-\eta'|}}{| \xi' - \eta' |^{d-2-q}} \times \prod_{i=1}^{d} \left(\frac{\xi_i' - \eta_i'}{| \xi' - \eta' |} \right)^{n_i} b_{n_1 \ldots n_d}(\underline{x}, \underline{y}, \underline{x}', \underline{y}', x_d, y_d), \tag{3.12}$$

where the b's verify the same bound (3.11) with a new constant $\bar{K}_r$ but the same γ; they also have the same support property of the a's: i.e., they have all support in a common bounded region.

The common support properties of the b's and (3.11) allow to represent them as Fourier integrals as

$$b_{\underline{n}}(\underline{x}, \underline{y}, \underline{x}', \underline{y}', x_d, y_d) = \int \exp[i(\underline{h}\underline{x} + \underline{h}'\underline{x}' + \underline{k}\underline{y} + \underline{k}'\underline{y}' + \omega x_d + \omega' y_d)] \times \hat{b}_{\underline{n}}(\underline{h}, \underline{k}, \underline{h}', \underline{k}', \omega, \omega')\, d\underline{h}\, d\underline{k}\, d\underline{h}'\, d\underline{k}'\, d\omega\, d\omega' \tag{3.13}$$

[16] The constant 4 is not optimal.

and $\forall\lambda > 0\ \exists C_\lambda$ such that

$$|\hat{b}_{\underline{n}}(\underline{h}, \underline{k}, \underline{h}', \underline{k}', \omega, \omega')| \leqslant C_\lambda(1 + |\underline{h}| + |\underline{h}'| + |\underline{k}| + |\underline{k}'| + |\omega| + |\omega'|)^{-\lambda}. \tag{3.14}$$

Furthermore, by first integrating over the radial variable $|\underline{x}|$ or $|\xi|$ it follows

$$\left| \int d\underline{x} e^{i\underline{k}\underline{x}} \frac{e^{-|\underline{x}|}}{|\underline{x}|^{d/2-\epsilon'}} \right| \leqslant \frac{D}{(1 + |\underline{k}|)^{d/2+\epsilon'-1}}, \qquad k \in R^{d-1}, \tag{3.15}$$

$$\left| \int d\xi e^{ik\xi} \frac{e^{-|\xi|}}{|\xi|^{d-2}} \prod_{i=1}^{d} \left(\frac{\xi_i}{|\xi|}\right)^{n_i} \right| \leqslant \frac{E}{1 + k^2}, \qquad k \in R^d, \tag{3.16}$$

$\forall n_1, \ldots, n_d$ and for suitably chosen D, E. Hence the integral of (3.12) can be written

$$\int d\underline{h}\, d\underline{k}\, d\underline{h}'\, d\underline{k}'\, d\underline{w}\, d\tilde{\omega}\, d\omega\, d\omega' \exp[i(\underline{h}\underline{x} + \underline{k}\underline{x} + (\omega + \tilde{\omega})x_d + (\omega - \tilde{\omega})y_d$$
$$+ (\underline{w} - \underline{h}')\underline{x} + (\underline{k}' - \underline{w})\underline{y})] \frac{D(\underline{w} + \underline{h}')}{(1 + |\underline{w} + \underline{h}'|)^{d/2-1+\epsilon'}} \frac{D(\underline{w} - \underline{h}')}{(1 + |\underline{w} - \underline{h}'|)^{d/2-1+\epsilon'}}$$
$$\times \frac{E(\underline{w}, \tilde{\omega})}{(1 + \underline{w}^2 + \tilde{\omega}^2)} \frac{C_\lambda(\underline{h}, \underline{k}, \underline{h}', \underline{k}', \omega, \omega')}{(1 + |\underline{h}| + |\underline{k}| + |\underline{h}'| + |\underline{k}| + |\omega| + |\omega'|)^\lambda}, \tag{3.17}$$

where $D(\cdot)$, $E(\cdot)$, $C(\cdot)$ are functions whose modulus is bounded by the constants with the corresponding name in (3.14), (3.15), (3.16).

If we choose λ very large (say, $d + 100$) the variables $|\underline{h}|$, $|\underline{k}|$, $|\underline{h}'|$, $|\underline{k}'|$, $|\underline{w}|$, $|\underline{w}'|$ are forced to "stay close to zero" and this allows us to find a bound on the Hölder continuity modulus of (3.17). In fact to compute the variation δ of (3.17) when $\underline{x}$ is varied between $\underline{x}_1$ and $\underline{x}_2$, $\underline{y}$ between $\underline{y}_1$ and $\underline{y}_2$ and x_d between t_1 and t_2 and y_d between τ_1 and τ_2, we can use the inequalities: $\forall\eta \in (0, 1)$, $\exists B_\eta$:

$$\frac{|e^{i\underline{h}\underline{x}} - e^{i\underline{h}\underline{y}}|}{|\underline{x} - \underline{y}|^\eta} \leqslant |\underline{h}|^\eta B_\eta \qquad \forall \underline{x}, \underline{y} \in R^{d-1}$$
$$\frac{|e^{i\omega t} - e^{i\omega\tau}|}{|t - \tau|^\eta} \leqslant |\omega|^\eta B_\eta \qquad \forall t, \tau \in R \tag{3.18}$$

and we see that (3.18), (3.17) imply, if $\eta = 2\epsilon'' < 1$, that $|\delta|$ is bounded by $(|\underline{x}_1 - \underline{x}_2|^{2\epsilon''} + |\underline{y}_1 - \underline{y}_2|^{2\epsilon''} + |t_1 - t_2|^{2\epsilon''} + |\tau_1 - \tau_2|^{2\epsilon''})$ times the integral

$$\gamma^{n_1+\cdots+n_d} B_{2\epsilon''} C_\lambda D^2 E \int \frac{d\underline{k}\, d\underline{h}\, d\underline{k}'\, d\underline{h}'\, d\omega\, d\omega'\, d\underline{w}\, d\tilde{\omega}}{(1 + |\underline{k}| + |\underline{h}| + |\underline{k}'| + |\underline{h}'| + |\omega| + |\omega'|)^\lambda}$$
$$\times \frac{|\underline{w} + \underline{h}'|^{2\epsilon''} + |\underline{w} - \underline{k}'|^{2\epsilon''} + |\omega + \tilde{\omega}|^{2\epsilon''} + |\omega' - \tilde{\omega}|^{2\epsilon''}}{(1 + |\underline{w} + \underline{h}'|)^{d/2-1+\epsilon'}(1 + |\underline{w} - \underline{k}'|)^{d/2-1+\epsilon'}(1 + |\underline{w}|^2 + \tilde{\omega}^2)} \tag{3.19}$$

and the integral converges if $\epsilon'' < \epsilon'$. Since the r.h.s. of (3.19) can be summed over n_1, the proof of the $2\epsilon''$-Hölder continuity is completed for the case when $(m - |\underline{j}| - 1)/2$ is integer and ξ, η are in a given bounded set: if either ξ or η or both are far from the curved part of the surface in R^d described by ν the proof proceeds along the same lines and, actually, is easier and, using (3.3) one also finds the exponential decay property mentioned above. We omit the details. It remains to study the cases $(m - |\underline{j}| - 1)/2$ not integer. The argument just given for the integer case can be easily adapted to cover this new case: i.e., one can prove that

$$\begin{aligned} &\zeta^{(\underline{j})} \in (1 - \underline{D})^{-(s-|\underline{j}|-\epsilon)/2+1/2}\ \bar{C}^{(\epsilon)}(R^d), \\ &\frac{\partial}{\partial x_k} \zeta^{(\underline{j})} \in (1 - \underline{D})^{-(s-|\underline{j}|-\epsilon)/2+1/2}\ \bar{C}^{(\epsilon)}(R^d) \end{aligned} \tag{3.20}$$

for $k = 1,\ldots, d$ and then use a general technique in the theory of distributions, that if

$$\begin{aligned} &\zeta \in (1 - \underline{D})^{a}\ \bar{C}^{(\epsilon)}(R^d), \\ &\frac{\partial}{\partial x_k} \zeta \in (1 - \underline{D})^{a}\ \bar{C}^{(\epsilon)}(R^d), \qquad k = 1,\ldots, d, \end{aligned} \tag{3.21}$$

then $\zeta \in (1 - \underline{D})^{a-1/2} \bar{C}^{(\bar{\epsilon})}(R^d)$, $\forall \bar{\epsilon} < \epsilon$.

4. Proof of Proposition 3. Part I: Geometry. The Integration Grid

Before proceeding to the proof we must do a complicated geometric construction, referred as "construction of an integration grid," which is needed if one wishes to use our Proposition 1 and does not want to get in involved in the rather difficult theory of the Dirichlet problem (1.2) in regions with corners and edges.

Let $Q_1, Q_2, \ldots, Q_{d+1}$ be $d + 1$ pavements of R^d with cubic tesserae with side size 1. We suppose that the centers of the tesserae of such pavements are on parallel lattices with step 1 and origin at the points

$$\bar{\xi}_k = \frac{2^{k-1} - 1}{2^{k-1}} (1, 1,\ldots, 1) \in R^d, \tag{4.1}$$

respectively, for $k = 1, 2,\ldots, d + 1$.

Now we turn each tessera into a smooth region by some deformations. Let $0 < \delta < (10 2^d)^{-1}$, say:

(i) We shrink each tessera about its center by a homothety factor $(1 - \delta)$: after this first operation the $d + 1$ pavements are no longer such because they leave unpaved corridors of width $<(5 \cdot 2^d)^{-1}$ between them.

(ii) Turn every corner into a smooth corner and, also, every edge, of any order, into a smooth edge

FIGURE 4.1

(iii) We now wish to modify more the boundaries of Q_2, Q_3,..., Q_{d+1} in such a way that they intersect in a smooth way between each other and with the boundaries of Q_1. We also want that the modified regions are conically regular for cones with opening 0, at least. We add the further condition that if two points belong to the same boundary Σ_1 which has been deformed in order to cross smoothly another boundary Σ_2 and if they lie on opposite sides of Σ_2 then their distance is larger than, say, $\delta/100$: in other words, upon crossing, the surfaces must stay adherent for a while.

The two-dimensional situation is easily described by pictures

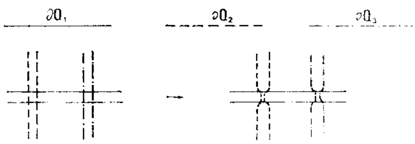


FIGURE 4.2

FIGURE 4.3

and the last condition means that if a crossing is enlarged in scale it looks like

FIGURE 4.4

In the deformations that we consider we only allow displacements of $\delta/2$ (at most), of each point.

(iv) Finally we suppose that the contacts between different surfaces are of infinite order in the same sense used in defining when a region is well situated into another in Section 2 (footnote 8).

After some meditation the reader will recognize that the above construction is possible also if $d > 2$. Actually such constructions may be done in many ways: e.g., if $d = 2$ we could do the following

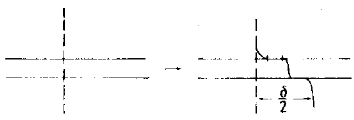


FIGURE 4.5

Note that the "sharper" are the deformed corners the less conically regular the region results: however, the above regions can certainly be constructed conically with opening 0 (i.e., such that their closures are never intersected again by their outer normals).

We call $\tilde{Q}_{1,1}, \tilde{Q}_{2,1}, ..., \tilde{Q}_{d+1,1}$ the sets of deformed tesserae. If we scale by a homothety factor l such assembly of strange boxes we obtain new families $\tilde{Q}_{1,l}, ..., \tilde{Q}_{d+1,l}$. We take l large enough (i.e., $l > l_0$, where l_0 is precisely defined below).

If $\square \in \tilde{Q}_{i,l}$, $l \geqslant 1$, we can construct over its boundary a covering with regular regularly spaced surface elements, for $l \to \infty$ (see Section 2, footnotes 4 and 5) and associate to every such covering a regular (for $l \to \infty$) partition of unity: the function associated with a surface element σ will be called α_σ (cf. footnotes 4, 5).

We shall naturally define coverings and partitions of unity on all the other tesserae $\square'$ of the same family $\tilde{Q}_{i,l}$ by "translation."

We shall call $\Sigma_{1,l}\Sigma_{2,l}, ..., \Sigma_{d+1,l}$ the family of surface elements obtained in this way and Σ_l will denote the $\bigcup_{i=1}^{d+1} \Sigma_{i,l}$.

We call each $\Sigma_{i,l}$ an "integration grid" and the family $\Sigma_{1,l}, ..., \Sigma_{d+1,l}$ a "complete integration grid of conical regularity 0."

Such "complete integration grids" will be needed to prove Proposition 3 for operators A whose conical regularity parameter $\theta(\alpha_0, \alpha_{m-1})$ is small enough. For operators with large conical regularity parameter the above grid is not very useful, but it is possible to reduce the problem to the preceding case, as explained in Appendix D.

From now on we shall, therefore, consider only operators for which $\theta(\alpha_0, \alpha_{m-1})$ is so small that the tesserae in $\tilde{Q}_{1,l}, ..., \tilde{Q}_{d+1,l}$ are all $\theta(\alpha_0, \alpha_{m-1})$-conically regular.[17]

We shall suppose always l so large that no surface element $\sigma \in \Sigma_l$ has points on opposite sides of a surface $\partial\square$ of some $\square \in \bigcup_{i=1}^{d+1} \tilde{Q}_{i,l}$: $\exists l_0$ such that this is true for all $l \geqslant l_0$ because of the last condition in (iii) (i.e., because of the flat

[17] The set of values θ for which a given region is θ-conically regular is open. Also the θ-conical regularity is a homothety invariant notion.

piece of Fig. 4.4) and because the surface elements have a maximum diameter D, l-independent, by definition: take $l_0(\delta/100) > D$, for instance. We shall also suppose l_0 so large that the construction of the surface elements and of the associated partitions of unity is actually possible in the sense of footnotes 4–6. Furthermore we shall suppose that l_0 is so large that the norms $\| \nu_i \|_{C^{(1)}(R^{d-1})}$ of all the surface elements $\sigma_i \subset \partial\square$ are, for $l \geq l_0$, smaller than $\frac{1}{4}$ so that Proposition 2 applies to them.

5. Proof of Proposition 3. Reduction to the Case $|I| = 1$

As said at the end of Section 4 we shall consider, for simplicity, only the case of operators A for which $\theta(\alpha_0, \alpha_{m-1})$ is so small that the integration grid Σ_l constructed in Section 4 is $\theta(\alpha_0, \alpha_{m-1})$-conically regular: concretely this means $(\alpha_{m-1} - \alpha_0)/\alpha_0 \ll 1$ (see Appendix A).

We consider the event associated with the surface element $\sigma \in \Sigma_l$:

$$\bar{E}_\sigma^{B,s} = \{z \mid z \in \mathscr{S}'(R^d), \| \partial z \|_{\mathscr{C}_s^{(\epsilon)}(\sigma)} < B\} \tag{5.1}$$

and in the l.h.s. of (5.1) ϵ does not appear because it will be fixed in $(0, \frac{1}{2})$ throughout the proof. Call $\bar{\chi}_\sigma^{B,s}$ the characteristic function of the event (5.1).

Proposition 3 will be a consequence of the following assertion which, as we shall see, is much stronger (using the same notation as Proposition 3):

THEOREM I. *Given $s < m - d/2$, $\epsilon \in (0, \frac{1}{2})$, there exist constants c_1, c_2, c_3, c_4, l_1 such that*

$$\int P_A(dz) \prod_{\sigma \in \Sigma_l} \bar{\chi}_\sigma^{B_\sigma, s}(z) \geq \exp(- |I| c_1 e^{-c_2 B^2}) \tag{5.2}$$

$\forall l \geq l_1$, $\forall B^2 > c_3 + c_4 \log l$ *if*

$$B_\sigma = B \log(e + d(\sigma, I)). \tag{5.3}$$

Furthermore c_1, c_2, c_3, c_4, l_1 can be chosen continuous in s, ϵ.

In this section we show that the above theorem is true "if it is true for small boxes I" and we shall also show that the theorem implies Proposition 3. More precisely, we shall prove the following lemma:

LEMMA 1. *Suppose that there exist positive continuous functions $\tilde{c}_1, \tilde{c}_2$, of* *the parameters* $(\epsilon, s) \in (0, \frac{1}{2}) \times (-\infty, m - d/2)$ *such that* (*see above*) $\forall B > 0$, $\forall \square \in \bigcup_{i=1}^{d+1} \tilde{Q}_{i,l}$,

$$\int P_A^\circ(d\zeta) \prod_{\substack{\sigma \in \Sigma_e \\ \sigma \subset \square}} \bar{\chi}_\sigma^{B,s}(\zeta) \geqslant 1 - \tilde{c}_1 l^d e^{-\tilde{c}_2 B^2} \tag{5.4}$$

then the above theorem is true.[18]

Remark. Here P_A° denotes the Gaussian process on $\mathscr{D}'(\square)$ with Dirichlet covariance introduced in Section 1. In the proof we shall use heavily the Markov property (1.8).

Proof. Let $B_0(s, l)$ be a continuous function of s, l such that

$$\tilde{c}_1 l^d e^{-\tilde{c}_2 B_0^2} \leqslant \tfrac{1}{2}, \tag{5.5}$$

i.e., $B_0^2 = \tilde{c}_3 + \tilde{c}_4 \log l$ for suitably chosen continuous $\tilde{c}_3$, $\tilde{c}_4$. Call (1), (2),... the corridors between the tesserae of $\tilde{Q}_{1,l}$, $\tilde{Q}_{2,l}$ $\cdots$ which we think as closed sets. We put $(12) = (1) \cap (2)$, $(123) = (1) \cap (2) \cap (3)$ $\cdots$. Note that $(1) \cap (2) \cap \cdots \cap (d+1) \equiv (12 \cdots d+1) = \varnothing$: in fact $d((12 \cdots d), (d+1)) \geqslant \delta l$, at least, as it follows from the construction of Σ_l.

Choose $2d + 1$ functions of s: $s_1' < s_2' < \cdots < s_d' = s < s_1 < s_2 < \cdots < s_{d+1} < m - d/2$ continuously dependent on s.

Let $l_1 = \lambda_1(\epsilon, s)$, we suppose always, below, $l \geqslant l_1 \geqslant l_0$.

Call $P(B, S)$ the integral in the l.h.s. of (5.2). Then using (1.8) and, then, (5.4),

$$\begin{aligned}
P(B, s) &= \int P_A(d\bar{z}) \left\{ \prod_{\sigma \subset (1)} \bar{\chi}_\sigma^{B_\sigma, s}(\bar{z}) \right\} \prod_{\square \in \tilde{Q}_{1,l}} \left[\int P_A^\circ(d\zeta) \prod_{\sigma \cap \square \neq \varnothing} \bar{\chi}_\sigma^{B_\sigma, s}(\zeta + u(\bar{z})) \right] \\
&\geqslant \int P_A(d\bar{z}) \left\{ \prod_{\sigma \subset (1)} \bar{\chi}_\sigma^{B_\sigma, s}(\bar{z}) \right\} \left[\prod_{\square \in \tilde{Q}_{1,l}} \left(\prod_{\sigma \cap \square \neq \varnothing} \bar{\chi}_\sigma^{B_\sigma/2, s}(u(\bar{z})) \right) \right] \\
&\quad \cdot \left[\prod_{\square \in \tilde{Q}_{1,l}} \int P_A^\circ(d\zeta) \prod_{\sigma \cap \square \neq \varnothing} \bar{\chi}_\sigma^{B_\sigma/2, s}(\zeta) \right] \\
&\geqslant \phi_1(B, s) \int P_A(d\bar{z}) \left\{ \prod_{\sigma \subset (1)} \bar{\chi}_\sigma^{B_\sigma, s}(\bar{z}) \right\} \prod_{\square \in \tilde{Q}_{1,l}} \left(\prod_{\sigma \cap \square \neq \varnothing} \bar{\chi}_\sigma^{B_\sigma/2, s}(u(\bar{z})) \right),
\end{aligned} \tag{5.6}$$

[18] Here we take the pragmatic attitude that there might be "false theorems" despite the contradiction.

where, if $B > 2B_0(s, l)$,

$$\phi_1(B, s) = \prod_{\square \in \tilde{Q}_{1,l}} \exp\{-2\tilde{c}_1 l^d e^{-\tilde{c}_2 (B^2/4)(\log(e+d(\square, I)))^2}\}$$
$$\geqslant \exp\{|-\tilde{c}_5 e^{-\tilde{c}_6 B^2} \mid I \mid\} \tag{5.7}$$

for suitably chosen continuous functions $\tilde{c}_5$, $\tilde{c}_6$.

We shall now use that, as it follows easily from the definition (2.2), (2.3), there is an immersion constant $N(s', s)$[19] such that

$$\| \partial z \|_{\mathscr{C}_s^{(\epsilon)}(\sigma)} \leqslant N(s, s') \| \partial z \|_{\mathscr{C}_{s'}^{(\epsilon)}(\sigma)} \tag{5.8}$$

if $s' > s$. The uniformity of N in σ and l follows from the assumed regularity of the partition of unity used to define the norms and from the regularity of the surface elements (cf. footnotes 5, 5; see in particular footnote 5, part (ii)). It also follows that $N(s, s')$ can be chosen continuous in s, s', ϵ, for $s' > s$, $\epsilon \in (0, \frac{1}{2})$.

From (5.6), (5.8) it follows that

$$P(B, s) \geqslant \phi_1(B, s) \int P_A(d\bar{z}) \prod_{\sigma \subset (1)} \bar{\chi}_\sigma^{B'_\sigma/N(s,s_1), s_1}(\bar{z})$$
$$\times \prod_{\square \in \tilde{Q}_{1,l}} \prod_{\sigma \cap \square \neq 0} \bar{\chi}_\sigma^{B_\sigma/2, s}(u(\bar{z})) \qquad \forall B' < B \tag{5.9}$$

On the other hand Proposition 1, (2.16), implies for each $\square \in \tilde{Q}_{1,l}$

$$\frac{\| \partial u(\partial z) \|_{\mathscr{C}_s^{(\epsilon)}(\sigma)}}{\log(e + d(\sigma, I))} \leqslant c(s, s', \epsilon) \sum_{\tau \subset \partial \square} e^{-\kappa d(\sigma, \tau)}$$
$$\times \frac{B'}{N(s, s_1)} \frac{\log(e + d(\tau, I))}{\log(e + d(\sigma, I))} \leqslant h_1(s, s_1) B', \tag{5.10}$$

where $h_1(s, s_1) > 1$ is a suitable continuous function of s, s_1, ϵ whose existence comes from the fact that the covering of $\partial\square$ has been made with regularly spaced surface elements (which guarantees that the sum in (5.10) can be bounded l-independently and σ-independently).

Therefore if we choose $B' = B/2h_1(s, s_1)$, (5.9) becomes, setting $K_1(s, s') = 2h_1(s, s') N(s, s') \equiv k_1$

$$P(B, s) \geqslant \phi_1(B, s) \int P_A(d\bar{z}) \prod_{\sigma \subset (1)} \bar{\chi}_\sigma^{B_\sigma/k_1, s_1}(\bar{z}). \tag{5.11}$$

[19] We do not add an index ϵ, on which $N(s, s')$ also depends since we are not interested in such dependence.

We now proceed as before by using the second integration grid $\tilde{Q}_{2,l}$: calling $P'(B; s)$ the integral on the r.h.s. of (5.11) we find if $B/k_1 \geqslant 2B_0(s_1, l)$:

$$\begin{aligned} P'(B, s) &= \int P_A(d\bar{z}) \left\{ \prod_{\sigma \subset (12)} \bar{\chi}_\sigma^{B_\sigma/k_1, s_1}(\bar{z}) \right\} \\ &\quad \times \prod_{\square \in \tilde{Q}_{2,l}} \left[\int P_A{}^\circ(d\zeta) \prod_{\sigma \cap (1) \cap \square \neq \varnothing} \bar{\chi}_\sigma^{B_\sigma/k_1, s_1}(\zeta + u(\bar{z})) \right] \\ &\geqslant \phi_1\left(\frac{B}{k_1}, s_1\right) \int P_A(d\bar{z}) \left\{ \prod_{\sigma \subset (12)} \bar{\chi}_\sigma^{B_\sigma/k_1, s_1}(\bar{z}) \right\} \\ &\quad \times \prod_{\square \in \tilde{Q}_{2,l}} \prod_{\sigma \cap (1) \cap \square \neq \varnothing} \bar{\chi}_\sigma^{B_\sigma/2k_1, s_1}(u(\bar{z})), \end{aligned} \tag{5.12}$$

where ϕ_2 can be supposed given by (5.7), possibly readjusting the values of $\tilde{c}_5$, $\tilde{c}_6$.

To estimate the last integral in (5.12) define

$$\begin{aligned} s_2(\sigma) &= s_2 \qquad \text{if} \quad d(\sigma, (1)) \leqslant 1, \\ s_2(\sigma) &= s_1' \qquad \text{if} \quad d(\sigma, (1)) > 1, \end{aligned} \tag{5.13}$$

and

$$\frac{B_{2\sigma}}{\log(e + d(\sigma, I))} = \frac{B}{k_1} \frac{\exp[(\kappa/2)\, d(\sigma, (12))]}{3h_{1,2}(s_1, s_2)\, N(s_1, s_2)} \equiv \frac{B}{k_2} e^{(\kappa/2) d(\sigma, (12))}, \tag{5.14}$$

where, cf. (2.16), (2.17) in Proposition 1[20]

$$h_{1,2}(s_1, s_2) > \sum_{\tau \subset \partial \square} c(s_1, s_2(\tau), \epsilon, d(\tau, \sigma))\, e^{-(\kappa/2) d(\sigma, \tau)} \frac{\log(e + d(\tau, I))}{\log(e + d(\sigma, I))} \tag{5.15}$$

and, by the regular spacing assumption, h_{12} can be supposed to be finite and continuous and l, σ-independent.

Then if $\prod_{\sigma \subset (2)} \bar{\chi}_\sigma^{B_{2\sigma}, s_2(\sigma)}(\bar{z}) = 1$, Proposition 1, (2.16), (2.17), and (5.15) imply

$$\| \partial u(\bar{z}) \|_{\mathscr{C}_{s_1}^{(\epsilon)}(\sigma)} \leqslant \frac{B_\sigma}{2k_1}$$

and therefore, (5.12) says

$$P(B, s) \geqslant \phi_1(B, s)\, \phi_2\left(\frac{B}{k_1}, s_1\right) \int P_A(d\bar{z}) \prod_{\sigma \subset (2)} \bar{\chi}_\sigma^{B_{2\sigma}, s_2(\sigma)}(\bar{z}) \tag{5.16}$$

[20] Where $c(s_1, s_2(\tau), \epsilon, d(\tau, \sigma))$ is $c(s_1, s_2(\tau), \epsilon)$ if $d(\tau, \sigma) \leqslant 1$ and $\tilde{c}(s_1, s_2(\tau), \epsilon)$ if $d(\tau, \sigma) > 1$.

We now iterate the procedure and after $(d+1)$ steps we reach the conclusion

$$P(B, s) \geqslant \phi_1(B, s) \prod_{i=1}^{d} \phi_{i+1}\left(\frac{B}{k_i}, s_i\right) \int P_A(d\bar{z}) \prod_{\sigma\subset(d+1)} \bar{\chi}_\sigma^{B_{d+1,\sigma}, s_{d+1}(\sigma)}(\bar{z}), \quad (5.17)$$

where, for a suitable continuous function k_{d+1} of s, ϵ,

$$s_{d+1}(\sigma) = s_d' = s,$$

$$\frac{B_{d+1,\sigma}}{\log(e + d(\sigma, I))} = \frac{B}{k_{d+1}} e^{(\kappa/2)d((d+1),(12\cdots d))} \geqslant B\frac{e^{\kappa(/2)\delta l}}{k_{d+1}} \quad (5.18)$$

by the remark, at the beginning of the proof, on $d((d+1), (12 \cdots d))$ (5.17) holds provided $B/k_d \geqslant 2B_0(s_d, l)$.

We now choose a continuous function of ϵ, s, denoted l_1 such that

$$e^{(\kappa/2)\delta l_1} \geqslant 2k_{d+1} \quad (5.19)$$

and we see that (5.17), (5.18), (5.19) imply $\forall l \geqslant l_1$

$$P(B, s) \geqslant \exp(-\tilde{c}_7 e^{-\tilde{c}_8 B^2} \mid I \mid) P(2B, s) \quad (5.20)$$

$\forall B^2 \geqslant 4k_d^2(\tilde{c}_3(s_d) + \tilde{c}_4(s_d) \log l)$ where $\tilde{c}_7$ and $\tilde{c}_8$ are suitably chosen continuous functions of s, ϵ.

Since, by Proposition4[21] it easily follows that

$$\lim_{n\to\infty} P(2^n B, s) = 1 \quad (5.21)$$

we have $\forall B^2 > 4K_d^2(\tilde{c}_3(s_d) + \tilde{c}_4(s_d) \log l)$

$$P(B, s) \geqslant \exp\left(- \mid I \mid \tilde{c}_7 \sum_{n=0}^{\infty} e^{-2^{2n}\tilde{c}_8 B^2}\right) \quad (5.22)$$

which, clearly, implies that the theorem is true and the proof of Lemma 1 is completed.

The next lemma shows which are the basic inequalities that still have to be checked to prove Proposition 3.

Denote $\chi_G^{B,\beta}$ the characteristic function of the event

$$E_G^{B,\beta} = \{z \mid z \in \mathscr{S}'(R^d), \| z \|_{C^{(\beta)}(G)} < B\} \quad (5.23)$$

for an arbitrary choice of the open set $G \subset R^d$ and for $\beta \geqslant 0$.

[21] Proposition 4 is proved in Section 7 independently of the content of Sections 5 and 6.

LEMMA 2. *Suppose valid the assumptions of Lemma* 1 (*hence Theorem* I). *Suppose also* $m - d/2 > 0$ *and that there exist two positive continuous functions* $\tilde{c}_9, \tilde{c}_{10}$ *of* $\beta \in (0, [m - d/2] - (-1)^d/2)$ *such that* $\forall B \geqslant 0$ *and* $\forall \square \in \bigcup_{i=1}^{d+1} \tilde{Q}_{i,l}$, $\forall l \geqslant l_0$,

$$\int \chi_{\square}^{B,\beta}(\zeta)\, P_A{}^{\circ}(d\zeta) \geqslant 1 - \tilde{c}_9 l^d e^{-\tilde{c}_{10} B^2} \tag{5.24}$$

then Proposition 3 *holds.*

Remark. (1) This means that in order to prove Proposition 3 we have to check (5.4), (5.24)

$$\int P_A{}^{\circ}(d\zeta)\, \bar{\chi}_{\sigma}^{B,s}(\zeta) \geqslant 1 - \tilde{c}_{11} e^{-\tilde{c}_{12} B^2}, \tag{5.25}$$

$$\int P_A{}^{\circ}(d\zeta)\, \chi_{\Delta\cap\square}^{B,\beta}(\zeta) \geqslant 1 - \tilde{c}_{13} e^{-\tilde{c}_{14} B^2} \tag{5.26}$$

$\forall B \geqslant 0$, $\forall \square \in \bigcup_{i=1}^{d+1} \tilde{Q}_{i,l}$, $\forall \sigma \subset \partial\square$, $\forall$ tesserae Δ of a unit pavement Q, and for suitably chosen continuous functions $\tilde{c}_{11}, \tilde{c}_{12}$ of $(\epsilon, s) \in (0, \frac{1}{2}) \times (-\infty, m - d/2)$ and $\tilde{c}_{13}, \tilde{c}_{14}$ of $\beta \in (0, [m - d/2] - (-1)^d/2)$ when $m - d/2 > 0$.

(2) The proof that follows is the first application of a rather general technique of integration based on Theorem I and justifies the name of "complete integration grid" given to Σ_l.

Proof. Note first that, if $\tilde{\beta} = [\beta - \epsilon] + \epsilon$, $\forall s < m - d/2$

$$\begin{aligned} P_A(E_I^{B,\tilde{\beta}}) &\geqslant P_A\Big(E_I^{B,\tilde{\beta}} \cap \bigcap_{\sigma\in\Sigma_{1,l}} \bar{E}_{\sigma}^{B',s}\Big) \\ &\geqslant P_A\Big(E_I^{B,\tilde{\beta}} \cap \bigcap_{\sigma\in\Sigma_l} \bar{E}_I^{B',s}\Big) \equiv P(B, \tilde{\beta}, B', s) \end{aligned} \tag{5.27}$$

so we only study the last term, choosing $s > \beta$ and $l \geqslant l_1(s, \epsilon)$.

Let $s < s_1 < \cdots < s_{d+1} < m - d/2$ and suppose s_i continuous function of β. If l is large enough $(2\delta l > 1)$:

$$\chi_I^{B,\tilde{\beta}}(z) \geqslant \chi_{(1)\cap I}^{B/2,\tilde{\beta}}(z) \prod_{\square\in\tilde{Q}_{1,l}\cap I} \chi_{\square\cap I}^{B/2,\tilde{\beta}}(z) \tag{5.28}$$

with P_A-probability 1 (see Lemma 2). Hence the l.h.s. of (5.27) is larger than

$$\begin{aligned} P(B, \tilde{\beta}, B', s) \geqslant \int & P_A(d\bar{z})\, \chi_{(1)}^{B/2,\tilde{\beta}}(\bar{z}) \prod_{\sigma\subset(1)\cap I} \bar{\chi}_{\sigma}^{B',s}(\bar{z}) \\ & \times \Bigg[\prod_{\square\in\tilde{Q}_{1,l}} \int P_A{}^{\circ}(d\zeta)\, \chi_{\square\cap I}^{B/4,\tilde{\beta}}(\zeta)\, \chi_{\square\cap I}^{B/4,\tilde{\beta}}(u(\bar{z})) \prod_{\sigma\cap\square\neq\varnothing} \bar{\chi}_{\sigma}^{B'/2,s}(\zeta) \\ & \times \prod_{\sigma\cap\square\neq\varnothing} \bar{\chi}_{\sigma}^{B'/2,s}(u(\bar{z}))\Bigg]. \end{aligned} \tag{5.29}$$

If we suppose that B is larger than $4B_0'$ defined by $l^d\tilde{c}_9 e^{-\tilde{c}_{10}B_0'^2} = \frac{1}{4}$ and $(B'/2)^2 > c_3 + c_4 \log l > B_0^2(s, l)$ (cf. Theorem I and (5.4) and (5.5)) we see that (5.29) takes the form

$$P(B, \tilde{\beta}, B', s) \geqslant \psi_1(B, \tilde{\beta}, B', s) \int P_A(d\bar{z})$$

$$\times \chi^{B/2,\tilde{\beta}}_{(1)\cap I}(\bar{z}) \prod_{\sigma\subset(1)\cap I} \bar{\chi}^{B',s}_\sigma(\bar{z}) \prod_{\square\in\tilde{Q}_{1,l}} \left\{\chi^{B/4,\tilde{\beta}}_\square(u(\bar{z})) \prod_{\sigma\cap\square\neq\phi} \chi^{B'/2,s}_\sigma(u(\bar{z}))\right\}, \quad (5.30)$$

where for suitable $\tilde{c}_{15} \div \tilde{c}_{18}$ continuous in ϵ, $\tilde{\beta}$, s

$$\psi_1(B, \tilde{\beta}, B', s) \geqslant \exp[-(\tilde{c}_{15}e^{-\tilde{c}_{16}B^2} + \tilde{c}_{17}e^{-\tilde{c}_{18}B'^2}) \, |I|] \quad (5.31)$$

We can find, by Proposition 1, a continuous function of s, $h_1(s)$, such that if $B' < B/h_1(s)$ all the characteristic functions in the curly bracket in (5.30) have value 1. Thus

$$P(B, \tilde{\beta}, B', s) \geqslant \psi_1(B, \tilde{\beta}, B', s) \int P_A(dz)\, \chi^{B/2,\tilde{\beta}}_{(1)\cap I}(z) \prod_{\sigma\subset(1)} \bar{\chi}^{B'/k_1,s_1}_\sigma(z)$$

$$\geqslant \psi_1(B, \tilde{\beta}, B', s) \int P_A(dz)\, \chi^{B/2,\tilde{\beta}}_{(1)\cap I}(z) \prod_{\sigma\in\Sigma_l} \bar{\chi}^{B'/k_1,s_1}_\sigma(z) \quad (5.32)$$

provided $(B'/2)^2 > c_3 + c_4 \log l$, $B \geqslant 4B_0'$, and $B' < B/h_1$; here K_1 is another continuous function of s, ϵ which is necessary to eliminate the characteristic functions in the last product of (5.30), via (2.16).

Clearly after $d + 1$ iterations one reaches the conclusion

$$P(B, \tilde{\beta}, B', s) \geqslant \psi_{d+1}(B, \tilde{\beta}, B', s) \int P_A(dz) \prod_{\sigma\in\Sigma_l} \chi^{B'/k_{d+1},s_{d+1}}_\sigma(z) \quad (5.33)$$

and ψ_{d+1} has an expression like (5.31) with constants $\tilde{c}_{15}' \div \tilde{c}_{18}'$ continuously dependent on β, ϵ, s provided

$$B^2 \geqslant h_{d+1}^2 B'^2 \geqslant \tilde{c}_{19} + \tilde{c}_{20} \log l, \quad (5.34)$$

where k_{d+1}, h_{d+1}, $\tilde{c}_{19}$, $\tilde{c}_{20}$ are continuous functions of ϵ, β, s as these parameters vary in the regions under consideration.

Hence Lemma 1 combined with (5.33) implies Proposition 3.

6. Local Estimates and Completion of the Proof of Theorem I and Proposition 3

As remarked in Section 5, we have only to show the validity of (5.25), (5.26) in order to complete the proof of Proposition 3 and of Theorem I.

The technique goes, essentially back to Wiener. It relies on the following general result: let θ_ξ, $\xi \in R^d$ be a Gaussian random field on R^d with distribution P. Suppose

$$\begin{aligned} &\int \theta_\xi^2 P(d\theta) \leqslant f_1(\xi), && \xi \in R^d, \\ &\int (\theta_\xi - \theta_\eta)^2 P(d\theta) \leqslant f_2(\xi, \eta) \mid \xi - \eta \mid^{2\delta}, && \xi, \eta \in R^d, \mid \xi - \eta \mid \leqslant 1, \end{aligned} \tag{6.1}$$

for some bounded continuous functions f_1, f_2 and for some $\delta \in (0, \frac{1}{2})$. Then the event

$$\begin{aligned} E^{\delta', B} = \{\theta \mid \theta \in \mathscr{S}'(R^d); \mid \theta_\xi - \theta_\eta \mid \geqslant B f_2(\xi, \eta)^{1/2} \log(e + \mid \xi \mid) \\ \times \mid \xi - \eta \mid^{\delta'} \text{ for some pair } \xi, \eta \in R^d, \mid \xi - \eta \mid \leqslant 1, \text{ or} \\ \mid \theta_\xi \mid \geqslant B f_1(\xi)^{1/2} \log(e + \mid \xi \mid) \text{ for some } \xi \in R^d\} \end{aligned} \tag{6.2}$$

has a probability

$$P(E^{\delta', B}) \leqslant \exp(\bar{k} - \bar{\bar{k}} B^2) \tag{6.3}$$

provided $0 < \delta' < \delta$ and $\bar{K}, \bar{\bar{K}}$ are two suitably chosen continuous functions of δ'. A simple proof can be found in Ref. [6]. Consider for instance (5.25). Let $\sigma \subset \square$, $\sigma \in \Sigma_l$ and consider the system of local coordinates $(\underline{x}, x_d)$ associated with σ. Let

$$\zeta_\sigma^{(\underline{j})}(\underline{x}) = \bar{\alpha}_\sigma(\underline{x}) \overline{(\partial^{(\underline{j})} z)(\underline{x})}, \qquad \underline{x} \in R^{d-1}, \tag{6.4}$$

be the distribution which represents $\alpha_0 \partial^{(\underline{j})} z$ in this system of coordinates; $h = \mid \underline{j} \mid = 0, 1, \ldots, m-1$. To compute $\| \partial^j z \|_{\mathscr{C}^{(\epsilon)}_{s-j}(\sigma)}$ we have to study the distribution

$$\theta = (1 - \underline{D})^{(s - |\underline{j}| - \epsilon)/2} \zeta_\sigma^{(\underline{j})}. \tag{6.5}$$

In order to apply the above theorem by Wiener we compute the covariance of $(\theta_{\underline{x}} - \theta_{\underline{y}})$ with respect to the probability measure $P_A{}^\circ$, for $\underline{x}, \underline{y} \in \square$.

We note that

$$\begin{aligned} \int P_A{}^\circ(dz)(\theta_{\underline{x}})^2 &= \int P_A(dz)(\theta_{\underline{x}})^2, \\ \int P_A{}^\circ(dz)(\theta_{\underline{x}} - \theta_{\underline{y}})^2 &\leqslant \int P_A(dz)(\theta_{\underline{x}} - \theta_{\underline{y}})^2 \end{aligned} \tag{6.6}$$

because inside $\square$ the Gaussian field P_A-distributed is the sum of two independent fields one of which is $P_A{}^\circ$-distributed (cf. (1.5), (1.8)).

Proposition 2 implies (cf. (2.21)) that the r.h.s. of (6.6) can be bounded by

$$e^{-\bar{k}|\underline{x}|} \quad \text{or} \quad e^{-\bar{k}(|\underline{x}|+|\underline{y}|)}\,|\,\underline{x}-\underline{y}\,|^{2\epsilon''} \tag{6.7}$$

for some $\bar{k} > 0$ and $\epsilon'' < \frac{1}{2}$ and all $\underline{x}, \underline{y}$'s. Hence (5.25) follows immediately from the result quoted at the beginning of this section.

Inequality (5.26) is proved along the same ideas as a consequence of the fact that the covariance of a derivative $\partial^{(p)}z$ of the random field z, P_A-distributed, is bounded by

$$\begin{aligned}\int P_A{}^{\circ}(dz)(\partial^{(p)}z_\xi - \partial^{(p)}z_\eta)^2 &\leqslant \int P_A(dz)(\partial^{(p)}z_\xi - \partial^{(p)}z_\eta)^2 \\ &= \int \frac{|\,e^{ik\xi} - e^{ik\eta}\,|^2}{\prod_{i=0}^{m-1}(\alpha_i{}^2 + k^2)}(\underset{\sim}{k})^{2p}\,dk,\end{aligned} \tag{6.8}$$

where $(\underset{\sim}{k})^{2p} = \prod_{i=1}^{d} k_i^{2\nu_i}$, $\sum_{i=1}^{d} \nu_i = |\,p\,|$ which has a singularity structure for $|\,\xi - \eta\,| \to 0$ easily deducible. One finds that the derivatives of order equal to the maximum integer $\nu_m < m - d/2$ have a covariance which ensures their Hölder continuity with exponent $<(m - d/2 - \nu_m)$[22] for d odd or $<(m - d/2 - \nu_m - \frac{1}{2})$ for d even.

As before this means for instance that, if $\underline{\nu}_m = (\nu_1, \ldots, \nu_d)$, $\sum_{i=1}^{d} \nu_i = \nu_m$ and if $\nu_m < \beta < [m - d/2] - (-1)^d/2$:

$$P_A{}^{\circ}(\{z\,|\,\|\,\partial^{(\underline{\nu}_m)}z\,\|_{C^{(\beta-\nu_m)}(\Delta)} > B\}) < \exp(k_1 - k_2B^2), \tag{6.9}$$

where k_1, k_2 are continuous functions of β. To show that an inequality like (6.9) holds also for the quantities

$$P_A{}^{\circ}(\{z\,|\,\|\,\partial^{(\underline{\nu})}z\,\|_{C^{(0)}(\Delta)} > B\}) \tag{6.10}$$

with $|\,\underline{\nu}\,| < \nu_m$ one can proceed in a similar way.

7. Proof of Proposition 4

The proof follows a classical scheme and we sketch it only, for completeness, for (2.32).

Let Q be a pavement of R^3 with unit tesserae Δ.

Consider the random field $\theta_\sigma^{(\lambda)}$ or R^{d-1} defined by (6.5).

[22] Our argument here is close to that used in Section 3 (cf. (3.18), (3.19)).

Given N different tesserae of Q: $\Delta_1, \Delta_2, \ldots, \Delta_N$ it is clear that, by using the Schwartz inequality the problem of estimating (2.32) can be reduced to that of estimating

$$\int \prod_{i=1}^{N} \mathring{\chi}_{\sigma}^{B,s,\epsilon,l}(z)\, P_A(dz) \tag{7.1}$$

where $\sigma_1, \ldots, \sigma_N$ are chosen intersecting $\Delta_1, \ldots, \Delta_N$, $\sigma_i \in \Sigma_l$ and $\mathring{\chi}_{\sigma}^{B,s,\epsilon,l}$ is the characteristic function of the event

$$E_{\sigma}^{B,s} = \{z \mid \| \theta_{\sigma}^{(j)} \|_{\bar{C}^{(\epsilon)}(R^{d-1})} > B\} \tag{7.2}$$

((cf.) (2.1), (2.18), (2.30)).

We may also suppose, by further use of the Schwartz inequality, that the distance between Δ_i and Δ_j is so large that $d(\sigma, \sigma') > 1$, say, if $\sigma \cap \bar{\Delta}_i \neq \varnothing$, $\sigma' \cap \bar{\Delta}_j \neq \varnothing$.

Again one follows the idea of Wiener's theorem.

Consider the tangent planes π_i to σ_i used to set up the Cartesian system of coordinates associated with σ_i. Image them dotted by the dense lattice of the dyadic points of R^{d-1}.

Choose N integers $p_1, p_2, \ldots, p_N \geqslant 0$ and on each π_i, $i = 1, \ldots, N$, two points $\underline{x}_i, \underline{y}_i$ lying on the dyadic lattice with step 2^{-p_i} and, there nearest neighbors

$$| \underline{x}_i - \underline{y}_i | = 2^{-p_i}. \tag{7.3}$$

The probability of the event in which, writing $\theta_{\underline{x}}$ for $\theta_{\underline{x}}^{(j)}$

$$\theta_{\underline{x}_i} - \theta_{\underline{y}_i} > B_i e^{-(1/2)((|\underline{x}|)^{1/2}+(|\underline{y}|)^{1/2})} \, | \underline{x}_i - \underline{y}_i |^{\epsilon} \equiv \lambda_i \tag{7.4}$$

$\forall i = 1, \ldots, N$, is bounded above by, $\forall \alpha_1, \ldots, \alpha_N \in [0, \infty)$:

$$\int \exp \left\{ \sum_{i=1}^{N} [(\theta_{\underline{x}_i} - \theta_{\underline{y}_i}) - \lambda_i]\alpha_i \right\} P_A(dz)$$

$$= \exp \left\{ \sum_{i,j=1}^{N} h(\underline{x}_i, \underline{y}_i; \underline{x}_j, \underline{y}_j)\alpha_i\alpha_j - \sum_{i=1}^{N} \lambda_i\alpha_i \right\}, \tag{7.5}$$

where h is a function which can be easily constructed from the covariance of the P_A process, and by using the methods developed in the proof of Proposition 2 it can be proved that the there are a continuous function of s, ϵ, f_1, and a positive constant $\bar{\kappa}$ (which can be taken the same as that in (6.7)) such that in general:

$$| h(\underline{x}, \underline{y}; \underline{x}', \underline{y}')| \leqslant f_1((| \underline{x} - \underline{y} | \cdot |\underline{x}' - \underline{y}' |)^{1/2})^{2\epsilon''} \cdot e^{-\bar{\kappa} d(\Delta, \Delta')} e^{-\bar{\kappa}(|\underline{x}|+|\underline{y}|+|\underline{x}'|+|\underline{y}'|)} \tag{7.6}$$

where $\underline{x}, \underline{y}$ are points on the tangent plane π associated with a surface element σ which intersect $\bar{\Delta}$ and $\underline{x}', \underline{y}' \in \pi'$ associated with σ' intersecting $\bar{\Delta}'$ and $d(\Delta, \Delta')$ is large so that $d(\sigma, \sigma') \geqslant 1$; we have put, as in Section 3, $s = m - d/2 - (\epsilon' - \epsilon)$ and supposed $2\epsilon' < 1$ and $\epsilon'' = (\epsilon + \epsilon')/2$.

We note that the quadratic form in the r.h.s. of (7.5) is just

$$\int \left[\sum_{i=1}^{N} \alpha_i (\theta_{\underline{x}_i} - \theta_{\underline{y}_i}) \right]^2 P_A(dz) \geqslant \int \left[\sum_{i=1}^{N} \alpha_i (\theta_{\underline{x}_i} - \theta_{\underline{y}_i}) \right]^2 P_A^{\circ}(dz), \tag{7.7}$$

where P_A° is a Dirichlet process associated with A and some region $A \supset \sigma, \sigma'$. Therefore the r.h.s. of (7.6) is, also, an estimate of the event (7.4) with respect to P_A°: this will be used to set up the final remark.

Then (7.6) implies that

$$\sum_{i,j=1}^{N} h(\underline{x}_i, \underline{y}_i; \underline{x}_j, \underline{y}_j)\, \alpha_i \alpha_j$$

$$\leqslant f_2 \sum_{i=1}^{N} \left\{ \alpha_i^2 \mid \underline{x}_i - \underline{y}_i \mid^{2\epsilon''} \exp\left[-\frac{k}{2} (\mid \underline{x}_i \mid + \mid \underline{y}_i \mid) \right] \right\} \tag{7.8}$$

and f_2 is a suitable continuous function of ϵ, s.

Hence, recalling (7.4) for the definition of λ_i, (7.8) imply that the r.h.s. of (7.5) is estimated by

$$\exp \left\{ \sum_{i=1}^{N} \left[f_2 2^{-p_i 2\epsilon''} e^{-(\bar{\kappa}/2)(|\underline{x}_i| + |\underline{y}_i|)} \alpha_i^2 - B_i \alpha_i 2^{-\epsilon p_i} e^{-(1/2)((|\underline{x}_i|)^{1/2} + (|\underline{y}_i|)^{1/2})} \right] \right\}. \tag{7.9}$$

Choosing α_i as

$$\alpha_i = \frac{1}{2} \frac{B_i}{f_2} 2^{-(\epsilon - 2\epsilon'') p_i} \frac{e^{-(1/2)((|\underline{x}_i|)^{1/2} + (|\underline{y}_i|)^{1/2})}}{e^{-(\bar{\kappa}/2)(|\underline{x}_i| + |\underline{y}_i|)}} \tag{7.10}$$

(7.9) can be bounded by

$$\exp \left[- \sum_{i=1}^{N} \frac{B_i^2}{4 f_2} 2^{2(\epsilon'' - \epsilon) p_i} \bar{c}(\underline{x}_i, \underline{y}_i) \right] \tag{7.11}$$

with

$$\bar{c}(\underline{x}, \underline{y}) = \exp \left[\frac{\bar{\kappa}}{2} (\mid \underline{x} \mid + \mid \underline{y} \mid) - ((\mid \underline{x} \mid)^{1/2} + (\mid \underline{y} \mid)^{1/2}) \right] \tag{7.12}$$

It is clear that (7.11) is an estimate of each of the 2^N events obtained by multiplying by -1 both sides of (7.4) for a subset of i's. Hence an estimate for the event

$$|\theta_{\underline{x}_i}-\theta_{\underline{y}_i}|>B_i e^{-(1/2)((|\underline{x}_i|)^{1/2}+(|\underline{y}_i|)^{1/2})}|\underline{x}_i-\underline{y}_i|^{\epsilon}\qquad \forall i=1,\dots,N \quad (7.13)$$

is 2^N times (7.11).

Since the event $|\theta_{\underline{x}}-\theta_{\underline{y}}|>Be^{-(1/2)((|\underline{x}|)^{1/2}+(|\underline{y}|)^{1/2})}|\underline{x}-\underline{y}|^{\epsilon}$ for some pair $\underline{x},\underline{y}$ in π, $|\underline{x}-\underline{y}|\leqslant 1$, implies that there exists a pair $\tilde{\underline{x}},\tilde{\underline{y}}$ of dyadic nearest neighbors, on some dyadic lattice of spacing 2^{-P}, such that

$$|\theta_{\tilde{\underline{x}}}-\theta_{\tilde{\underline{y}}}|>\tilde{K}Be^{-(1/2)((|\tilde{\underline{x}}|)^{1/2}+(|\tilde{\underline{y}}|)^{1/2})}|\tilde{\underline{x}}-\tilde{\underline{y}}| \quad (7.14)$$

(this simple fact is the reason for the validity of the Wiener's theorem [6]), where $\tilde{K}$ is a suitable number ($\tilde{K}=2(d-1)^{3/2}\sum_{j=0}^{\infty}2^{-\epsilon j}$) it is clear that the probability of the event

$$\sup_{\substack{\underline{x},\underline{y}\in\pi_i\\|\underline{x}-\underline{y}|\leqslant 1}}\frac{|\theta_{\underline{x}}-\theta_{\underline{y}}|}{|\underline{x}-\underline{y}|^{\epsilon}}e^{((|\underline{x}|)^{1/2}+(|\underline{y}|)^{1/2})}>B_i \qquad \forall i=1,\dots,N \quad (7.15)$$

can be bounded by

$$\prod_{i=1}^{N}\left[\sum_{p=0}^{\infty}\bar{c}_1 2^{p(d-1)}\exp\left(-\frac{\bar{c}_2 B_i^2}{4f_2\tilde{K}^2}2^{2p(\epsilon''-\epsilon)}\right)\right]\leqslant\exp\left[\sum_{i=1}^{N}(f_3-f_4B_i^2)\right] \quad (7.16)$$

with f_3, f_4 continuous functions of ϵ, s and $\bar{c}_1$, $\bar{c}_2$ are suitable constants.

To complete the proof of Proposition 4 we still have to estimate the probability of events like

$$\sup_{\underline{x}\in\mathbb{Z}^{d-1}}|\theta_{\underline{x}}|e^{(|\underline{x}|)^{1/2}}>B_i, \qquad i=1,\dots,N, \quad (7.17)$$

and combine the estimates with the preceding argument to complete the proof of (2.28): all this should by now be obvious: we omit therefore the details.

The inequality (2.31) is proved in exactly the same way.

Remark. The observation after (7.7) explains why Proposition 4 holds also when P_Λ is replaced by P_Λ°, where P_Λ° is the Gaussian measure associated with the Green's function of an operator A on any domain Λ and $\sigma_1,\dots,\sigma_N\subset\Lambda$ and $G\subset\Lambda$.

Appendix A: Proof of Proposition 1 (Sketch)

For definiteness we shall only consider the case $d = 3$.

A.1. *The Dirichlet Problem in* R_+^3

Let $\xi = (\underline{x}, x_3) \in R^3$, $x_3 > 0$. Let $\underline{z} = (z^{(0)},..., z^{(m-1)}) \in \mathscr{D}(R^2)^m$, where R^2 is identified with the $x_3 = 0$ plane.

The solution of $Au = 0$ in R_+^3, $\partial^j u = z^{(j)}$ can be written

$$u_\eta = \sum_{l=0}^{m-1} \int_{R^2} b_l(\eta, \xi)\, z_\xi^{(l)}\, d\sigma_\xi \tag{A.1.1}$$

if $(\eta - \xi) = (\underline{x}, \epsilon) \in R_+^3$, $\underline{x} \in R^2$, $\epsilon > 0$, we write $b_l(\eta, \xi) \equiv b_l(\epsilon, \underline{x})$. The Fourier transform has the form

$$\hat{b}_l(\epsilon, \underline{k}) = \frac{\det V((\alpha_0^{\,2} + \underline{k}^2)^{1/2},..., (\alpha_{m-1}^2 + \underline{k}^2)^{1/2};\, e^{-\epsilon(\alpha^2+\underline{k}^2)^{1/2}}, l)}{\det V((\alpha_0^{\,2} + \underline{k}^2)^{1/2},..., (\alpha_{m-1}^2 + \underline{k}^2)^{1/2})} \tag{A.1.2}$$

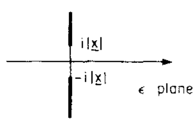
where $V(z_0, ..., z_{m-1})$ is the Vandermonde matrix of $z_0, ..., z_{m-1}$ and $V(z_0, ..., z_{m-1}; f(\underline{z}), l)$ is the matrix obtained by replacing the lth row of $V(z_0, ..., z_{m-1})$ by $(-1)^l (f(z_0), ..., f(z_{m-1}))$.

$$b_l(\epsilon, \underline{x}) = \int \frac{d\underline{k}}{(2\pi)^2}\, \hat{b}_l(\epsilon, \underline{k})\, e^{i\underline{k}\underline{x}}. \tag{A.1.3}$$

Then

Lemma A.1. (i) *The functions* (A.1.3) *extend, at fixed* $|\underline{x}| > 0$, *to holomorphic functions in the cut complex* ϵ*-plane,* $\Gamma(\pm i\,|\underline{x}|)$.

(ii) *The function* $(\epsilon, \underline{x}) \to b_l(\epsilon, \underline{x})$ *is of class* C^∞ *in the whole* $R^3 \backslash \{|\underline{x}| = 0, \epsilon < 0\}$ *and*



$$Ab_l = 0 \qquad in \quad R^3 \backslash \{|\underline{x}| = 0, \epsilon < 0\} \tag{A.1.4}$$

(iii) $\exists$ *constants* $\bar{\kappa}$, $\bar{\gamma}_p$, $\bar{\sigma}$ *such that if* $\partial^{(p)}$ *denotes any* $|\bar{p}|$*-th order derivative of* b_l *with respect to* ϵ *and* $\underline{x}$

$$|\partial^{(p)} b_l(\underline{x}, \epsilon)| \leqslant \bar{\gamma}_p\, e^{-\bar{\kappa}(\epsilon^2 + \underline{x}^2)^{1/2}} \tag{A.1.5}$$

if

$$\epsilon^2 + \underline{x}^2 > 1, \qquad -\epsilon < \frac{\bar{\sigma}\alpha_0}{(\alpha_{m-1}^2 - \alpha_0^{\,2})^{1/2}} \, | \underline{x} | \tag{A.1.5}$$

Define

$$\theta(\alpha_0, \alpha_{m-1}) = \arctan \frac{(\alpha_{m-1}^2 - \alpha_0^{\,2})^{1/2}}{\bar{\sigma}\alpha_0}.$$

(iv) *The following representation of $b_l(\epsilon, \underline{x})$ holds:*

$$b_l(\epsilon, \underline{x}) = \epsilon^m h_l(\epsilon, r^2) + \frac{\epsilon^{l+1}}{r^3} \sum_{\substack{\beta=0 \\ \alpha+\beta \geqslant [(m-l)/2]}}^{m(m-1)-l} \sum_{\alpha=0}^{[(m-l)/2]} \left(\frac{\epsilon}{r}\right)^{2\beta} \epsilon^{2\alpha}$$

$$\times\, h_l^{\alpha,\beta}(\epsilon^2, r^2) + \epsilon^l \epsilon^{2[(m-l)/2+1/2]} k_l(\epsilon^2, r^2) \log(\epsilon + r) \tag{A.1.6}$$

$l = 0, 1,..., m-1$, $r^2 + \underline{x}^2$ and h_l, $h_l^{\alpha,\beta}$, k_l are analytic near zero in their arguments.

Proof. It is a long exercise in the theory of Fourier transforms and on analytic functions; the basic ideas of our proof are the following.

(1) Let ν, δ be two small positive numbers. Define

$$\begin{aligned} \alpha_j'^2 &= \alpha_0^{\,2} + w(\alpha_j^{\,2} - \alpha_0^{\,2}), \qquad & \mu^2 &= \alpha_0^{\,2} - \delta^2 \\ \beta_j^{\,2} &= \alpha_j'^2 - \mu^2, & \bar{\beta}^2 &= \alpha_{m-1}^2 - \alpha_0^{\,2} \end{aligned} \tag{A.1.7}$$

choose $\nu = (\delta^2/2\bar{\beta}^2)^2$ and $\delta = \min(\bar{\beta}, \alpha_0/100)$ and let $w \in U = \{w \mid |\operatorname{Im} w| < \nu, -\nu < \operatorname{Re} w < 1 + \nu\}$. Let $z = ((\mu^2 + \underline{k}^2)^{1/2})^{-1}$, $N_l = m(m-1) - l$ and

$$\varphi_{\epsilon,w}(z) = (-1)^l \sum_{j=0}^{m-1} \frac{1}{1+\beta_j^{\,2} z^2} \prod_{r<s} \left(\left(\frac{1+\beta_r^{\,2}z^2}{1+\beta_j^{\,2}z^2}\right)^{1/2} + \left(\frac{1+\beta_s^{\,2}z^2}{1+\beta_j^{\,2}z^2}\right)^{1/2} \right) \Big/ (\beta_r^{\,2} - \beta_s^{\,2})$$

$$\times\, V_j^{\,l} \left(\left(\frac{1+\beta_0^{\,2}z^2}{1+\beta_j^{\,2}z^2}\right)^{1/2}, ..., \left(\frac{1+\beta_{m-1}^2 z^2}{1+\beta_j^{\,2}z^2}\right)^{1/2} \right) \exp\left[-\epsilon z \beta_j^{\,2} \left(\frac{(1+\beta_j^{\,2}z^2)^{1/2} - 1}{\beta_j^{\,2} z^2} \right) \right]. \tag{A.1.8}$$

The cuts of the square rooths start from $\pm i/\beta_j$ and go to ∞ parallel to the imaginary axis, on opposite directions. Therefore:

$$b_l(\epsilon, \underline{x}) = (-1)^{N_l} \frac{d^{N_l+2}}{d\epsilon^{N_l+2}} f_w(\epsilon, \underline{x}) \Big|_{w=1}, \tag{A.1.9}$$

where

$$f_w(\epsilon, \underline{x}) = \int d\underline{k}\, e^{i\underline{k}\underline{x}} \frac{e^{-\epsilon(\mu^2+\underline{k}^2)^{1/2}}}{\mu^2 + \underline{k}^2} \varphi_{\epsilon,w}\left(\frac{1}{(\mu^2+\underline{k}^2)^{1/2}}\right). \tag{A.1.10}$$

(2) A formal power series for $\varphi_{\epsilon,w}$ and the formula

$$\frac{e^{-\mu(\underline{x}^2+y^2)^{1/2}}}{(\underline{x}^2+y^2)^{1/2}} = \frac{1}{2\pi}\int d\underline{k}\, e^{i\underline{k}\underline{x}}\, \frac{e^{-y(\mu^2+\underline{k}^2)^{1/2}}}{(\mu^2+\underline{k}^2)^{1/2}} \tag{A.1.11}$$

show that

$$f_w(\epsilon, \underline{x}) = \int_\epsilon^{+\infty} dy\, \frac{e^{-\mu(y^2+\underline{x}^2)^{1/2}}}{(y^2+\underline{x}^2)^{1/2}}\hat{\varphi}_{\epsilon,w}(y-\epsilon) \tag{A.1.12}$$

and $\hat{\varphi}_{\epsilon,w}$ is the Borel transform of (A.1.6).

(3) It is immediate to see that the properties of $\varphi_{\epsilon,w}$ imply that both sides of (A.1.12) make sense and are analitic in $U\backslash\{0\}$ as functions of w. Furthermore they coincide for w small because the various interchanges of integrals and sums needed to deduce (A.1.12) from (A.1.10) are permitted if w is small. Hence, $\forall\epsilon > 0$

$$f_1(\epsilon, \underline{x}) = \int_\epsilon^{+\infty} dy\, \frac{e^{-\mu(y^2+\underline{x}^2)^{1/2}}}{(y^2+\underline{x}^2)^{1/2}}\hat{\varphi}_{\epsilon,1}(y-\epsilon) \tag{A.1.13}$$

(4) The function $\hat{\varphi}_{\epsilon,1}(z)$ is entire in both its arguments and, therefore, we can use (A.1.13) for analytic continuation to $\Gamma(\pm i\,|\,\underline{x}\,|)$ in the obvious way. Also property (ii) easily follows from (A.1.13) and from its validity for $\epsilon > 0$.

(5) The exponential decay follows for $\epsilon < 0$ from the bound on $\hat{\varphi}_{\epsilon,1}(z)$:

$$|\,\hat{\varphi}_{\epsilon,1}(z)| \leqslant c\exp(\bar{\beta}\,|\,\epsilon\,| - \delta\,|\,z\,|) \tag{A.1.14}$$

valid for ϵ, z real.

For $\epsilon > 0$ it follows from the usual argument on the shift of the integration over k_1, k_2 to an hyperbola $\operatorname{Im} k_i^2 = \alpha_0^2/2 + \operatorname{Re} k_i^2$, $i = 1, 2$, $\operatorname{Im} J_i > 0$.

(6) All the detailed analysis of the structure of the coefficients b_l can be read out of a patient analysis of the integral (A.1.8) or, better, of its singular part

$$\int_\epsilon^R \hat{\varphi}_{\epsilon,1}(y-\epsilon)\,\frac{e^{-\mu(y^2+\underline{x}^2)^{1/2}}}{(y^2+\underline{x}^2)^{1/2}}\,dy \tag{A.1.15}$$

with $0 < R < \infty$.[23]

[23] One should first prove, by suitable Taylor expansions the basic relation

$$\int_\epsilon^b (y-\epsilon)^n\,\frac{e^{-\mu(y^2+\underline{x}^2)^{1/2}}}{(y^2+\underline{x}^2)^{1/2}}\,dy = \varphi_n(\epsilon,\underline{x}^2) + \Big(\sum_{2j\leqslant n}\epsilon^{n-2j}\underline{x}^{2j}\varphi_{j,n}(\underline{x}^2)\Big)\log(\epsilon+(\epsilon^2+\underline{x}^2)^{1/2})$$
$$\Big(\sum_{2j\leqslant n}\epsilon^{n-2j}\underline{x}^{2j}\tilde{\varphi}_{j,n}(\underline{x}^2)\Big)\frac{\epsilon}{(\epsilon^2+\underline{x}^2)^{1/2}} + \Big(\sum_{2j+1\leqslant n}\epsilon^{n-2j-1}\underline{x}^{2j}\bar{\varphi}_{j,n}(\epsilon^2,\underline{x}^2)\Big)(\epsilon^2+\underline{x}^2)^{1/2},$$

where the φ's are real analytic.

From this one finds series expansions for the b_l's in powers of ϵ^2, $\underline{x}^2$ with coefficients which are either constants or constants times $\log(\epsilon + r)$ or $(1/r^n)\epsilon^s$.

Vast numbers of such coefficients are zero because the original expressions of $\hat{b}_l(\epsilon, \underline{k})$ (see (A.1.8)) were expressed as determinants (note, for instance, that $b_l(\epsilon, \underline{x})$ must have a zero of order m at $\epsilon = 0$ if $\underline{x} \neq 0$, cf. (A.1.2); but this is not the only "cancellation" that takes place: developing the determinants in (A.1.8) in powers of z other relations arise).

Another important property of the functions b_l is that of acting naturally as operators on $C_s^{(\epsilon)}(R^2)$ to analogous spaces of functions on $R_+{}^3$.

Such properties are based on the following lemma on the theory of distributions.

Let $\Delta_\psi = \{z \mid z \in \mathbb{C}, \mid \operatorname{Im} z \mid < \tan\psi \mid \operatorname{Re} z \mid\}$ *and* $\Delta_\psi{}^+ = \Delta_\psi \cap \{z \mid \operatorname{Re} z > 0\}$.

LEMMA A.2. *Let* $\xi = (\underline{x}, t) = (x_1, x_2, t) \in R_+{}^3$, $t > 0$, *and let* $\underline{n} = (n_{ij})$, $i \leqslant j$, $i, j = 1, 2, 3$, *be six nonnegative integers. Consider the functions*

$$f_{\underline{n}}(\underline{x}, t) = \frac{t}{(t^2 + \underline{x}^2)^{3/2}} \prod_{i \leqslant j}^{1,3} \left(\frac{\xi_i \xi_j}{t^2 + \underline{x}^2}\right)^{n_{ij}}. \tag{A.1.16}$$

(i) *There exists a family of combinatorial coefficients* $(\mu_\alpha)_{\alpha \in \mathcal{A}}$ *such that* $\sum_{\alpha \in \mathcal{A}} \mid \mu_\alpha \mid \leqslant c^{|\underline{n}|}$, $c > 0$, *and a family* $(\nu_1(\alpha), \nu_2(\alpha))_{\alpha \in \mathcal{A}}$ *of integer-valued functions such that the Fourier transform of* (A.1.16), *with respect to* $\underline{x}$, *has a representation like*

$$\sum_\alpha \mu(\alpha)(\cos\theta)^{\nu_1(\alpha)}(\sin\theta)^{\nu_2(\alpha)}\, \varphi_\alpha(kt), \tag{A.1.17}$$

where θ = *polar coordinate of* $\underline{k} = (k_1, k_2)$ *and* $k = (k_1{}^2 + k_2{}^2)^{1/2}$ *and* $\nu_1(\alpha) + \nu_2(\alpha) \leqslant 2 \mid n \mid = 2 \sum_{i \leqslant j} n_{ij}$.

(ii) *The function* $z \to \varphi_\alpha(z)$ *is entire in* z *and, given* $\psi \in (0, \pi/4)$, *verifies the bound*

$$\left| \frac{d^j}{dz^j} \varphi_\alpha(z) \right| < \gamma_j\, e^{-\delta |z|} c^{|\underline{n}|}, \qquad z \in \Delta_\psi{}^+, \tag{A.1.18}$$

where γ_j, δ *depend only on* ψ, $c > 0$ *is a constant.*

Proof. Note that

$$\hat{f}_{\underline{n}}(\underline{k}, t) = \int dx_1\, dx_2\, e^{i\underline{k}\underline{x}} \frac{t}{(t^2 + \underline{x}^2)^{3/2}} \left(\frac{t^2}{t^2 + \underline{x}^2}\right)^{n_{33}} \left(\frac{x_1{}^2}{t^2 + \underline{x}^2}\right)^{n_{11}}$$

$$\times \left(\frac{x_2{}^2}{t^2 + \underline{x}^2}\right)^{n_{22}} \left(\frac{x_1 x_2}{t^2 + \underline{x}^2}\right)^{n_{12}} \left(\frac{x_1 t}{t^2 + x^2}\right)^{n_{13}} \left(\frac{x_2 t}{t^2 + \underline{x}^2}\right)^{n_{23}}. \tag{A.1.19}$$

Rotating the x_1, x_2 axes by an angle θ equal to the polar angle of $\underline{k} = (k_1, k_2)$ we can rewrite (A.1.19) as a finite sum of $2^{2(n_{11}+n_{22}+n_{33})+n_{13}+n_{23}}$ terms each of which has the form

$$(\cos\theta)^a (\sin\theta)^b\, \varphi(kt) \tag{A.1.20}$$

where $k = (k_1{}^2 + k_2{}^2)^{1/2}$, a, b are nonnegative integers and

$$\varphi(z) = \int dx\, dy\, e^{iz(x/t)} \frac{t}{(t^2+x^2+y^2)^{3/2}} \left(\frac{t^2}{t^2+x^2+y^2}\right)^{\nu_{33}} \left(\frac{x^2}{t^2+x^2+y^2}\right)^{\nu_{11}}$$
$$\times \left(\frac{y^2}{t^2+x^2+y^2}\right)^{\nu_{22}} \left(\frac{xy}{t^2+x^2+y^2}\right)^{\nu_{12}} \left(\frac{xt}{t^2+x^2+y^2}\right)^{\nu_{13}} \left(\frac{yt}{t^2+x^2+y^2}\right)^{\nu_{23}} \tag{A.1.21}$$

with

$$2(\nu_{11}+\nu_{22}+\nu_{12}) + \nu_{13} + \nu_{23} = 2(n_{11}+n_{22}+n_{12}) + n_{13} + n_{23},$$
$$2\nu_{22} + \nu_{23} + \nu_{13} = 2n_{33} + n_{13} + n_{23}, \quad a + b = 2(n_{11}+n_{22}+n_{12}) + n_{13} + n_{23}.$$

This proves (A.1.17).

The function φ has several useful integral representation. Changing $\underline{x}/t \to \underline{x}$, (A.1.21) takes the form:

$$\varphi(z) = \int dx\, dy\, e^{izx} \frac{1}{(1+x^2+y^2)^{3/2}} \frac{1}{(1+x^2+y^2)^{\nu_{33}}} \left(\frac{x^2}{1+x^2+y^2}\right)^{\nu_{11}}$$
$$\times \left(\frac{y^2}{1+x^2+y^2}\right)^{\nu_{22}} \left(\frac{xy}{1+x^2+y^2}\right)^{\nu_{12}} \left(\frac{x}{1+x^2+y^2}\right)^{\nu_{13}} \left(\frac{y}{1+x^2+y^2}\right)^{\nu_{23}}$$
$$= \int dx\, e^{izx}\, \phi(x). \tag{A.1.22}$$

ϕ is implicitly defined in (A.1.22) and is a holomorphic function of x in the complex plane cut along the imaginary axis from i to $+i\infty$ and from $-i$ to $-i\infty$.

In polar coordinates the integral (A.1.21) takes the form

$$\varphi(z) = \int \rho\, d\rho\, d\varphi\, e^{i\rho\cos\varphi} \frac{z}{(z^2+\rho^2)^{3/2}} \left(\frac{z^2}{z^2+\rho^2}\right)^{\nu'}$$
$$\times \left(\frac{\rho^2}{z^2+\rho^2}\right)^{\nu''} \left(\frac{z\rho}{z^2+\rho^2}\right)^{\sigma} (\cos\varphi)^{\bar\nu} (\sin\varphi)^{\bar{\bar\nu}}, \tag{A.1.23}$$

where $2\nu' + \sigma = 2\nu_{33} + \nu_{23} + \nu_{13}$, $2\nu'' + \sigma = 2(\nu_{11} + \nu_{22} + \nu_{12}) + \nu_{23} + \nu_{13}$, $\bar\nu = 2\nu_{11} + \nu_{13}$, $\bar{\bar\nu} = 2\nu_{22} + \nu_{23}$; $\sigma = 0, 1$, so that σ has the same parity as $\bar\nu + \bar{\bar\nu}$.

Performing the integral over φ and calling

$$J(\zeta^2) = \zeta^\sigma \int d\varphi\, e^{i\zeta\cos\varphi}(\cos\varphi)^{\bar\nu}(\sin\varphi)^{\bar{\bar\nu}} \tag{A.1.24}$$

and changing variables $l^2 + z^2 \to r^2$

$$\varphi(z) = z \int_z^\infty \frac{dr}{r^2}\left(\frac{z^2}{r^2}\right)^{\nu'}\left(\frac{r^2 - z^2}{r^2}\right)^{\nu''}\left(\frac{z}{r^2}\right)^{\sigma} J_\sigma(r^2 - z^2). \tag{A.1.25}$$

The functions J_σ are entire in ζ and

$$\begin{aligned} \left|\frac{d^N}{d\zeta^N} J_\sigma(\zeta^2)\right| &= c_N \mid \zeta \mid^\sigma \exp(\mid \operatorname{Im} \zeta \mid), \qquad \forall N \geqslant 0, \\ \left|\frac{d^N}{d(\zeta^2)^N} J_\sigma(\zeta^2)\right| &\leqslant c_N(1 + \mid \zeta \mid)^{-N+\sigma} \exp(\mid \operatorname{Im} \zeta \mid), \qquad \forall N \geqslant 0. \end{aligned} \tag{A.1.26}$$

The second statement of the lemma follows now easily from (A.1.21), (A.1.25), and from the bounds (A.1.26) and

$$\mid \phi(x) \mid \leqslant \frac{c_1}{1 + \mid x \mid^2} \qquad \forall x = \frac{i}{2} + y,\ y \in \Delta_{\psi'}, \tag{A.1.27}$$

where ψ' is arbitrarily chosen in $(\psi, \pi/4)$.

The formulas (A.1.25), (A.1.26) can be used to continue analytically $\varphi(z)$ to complex z: we interpret the integral as a contour integral from z to 1 along a contour avoiding the origin plus a contour integral along the real axis from to $+\infty$.

The origin might be a branch point for φ: however, there is no singularity at $z = 0$ since the integrand in (A.1.25) has a nonnegative, integer, formal power in z and has no odd powers of r.

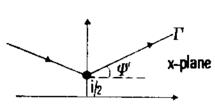
So φ is an entire function of z.

We now distinguish two cases

(i) $\mid z \mid > 1$, $z \in \Delta_\psi^+$

(ii) $\mid z \mid \leqslant 1$, $\operatorname{Re} z > 0$.

Case (i). We use the representation (A.1.22) and change the integration over x to a complex contour Γ drawn in the picture.



It is then easily seen that the integrand in the r.h.s. of (A.1.22) can be bounded for $z \in \Delta_\psi^+$ by

$$e^{-(1/2)|z||x|\sin(\psi'-\psi)}\, e^{-(1/2)\mathrm{Re}\, z} \frac{c_1}{1+|x|^2}$$

which obviously means that the jth derivative of φ can be bounded in Δ_ψ^+ by (A.1.18) if $|z| > 1$.

Case (ii). For Re $z > 0$ we interpret (A.1.25) as a contour integral along a contour from z to 1 plus the integral from 1 to $+\infty$ along the real axis.

It is easy to see that the integral from 1 to $+\infty$ together with its z-derivatives can be bounded as wanted by using (A.1.26).

The contribution to (A.1.25) from the integral from z ($|z| \leqslant 1$) to 1 comes from a contour integral on a contour close to the origin: we may choose the contour as the segment from z to 1 (since we have taken Re $z > 0$).

Then this integral can be calculated "explicitly" by developing $J(\zeta^2)$ in powers of ζ^2 and, then, bounding term by term the jth order z-derivatives of the resulting series. The result is, again, the wanted bound and (A.1.18) follows.

It is important to draw some corollaries about the functions (A.1.16).

COROLLARY A.3. *Define the maps*

$$(\bar{F}g)(\underline{y}, t) = e^t(1 - \underline{D}_y)^{-j/2} \frac{\partial^j}{\partial t^j} \int d\underline{x}\{t^l f_{\underline{n}}(\underline{y} - \underline{x}, t)e^{-t^2}\}\, g(\underline{x}), \tag{A.1.28}$$

$$(Fg)(\underline{y}, t) = (1 - \underline{D}_y)^{-j/2} \frac{\partial^j}{\partial t^j} \int d\underline{x}\{t^l f_{\underline{n}}(\underline{y} - \underline{x}, t)e^{-(t^2+x^2+y^2)}\}\, g(\underline{x}) \tag{A.1.29}$$

for $j, l = 0,..., m - 1$.

(i) *The operator* $\bar{F}$ *maps* $C_{s-l}^{(\epsilon)}(R^2)$ *into* $(1 - \underline{D})^{-(s'-\epsilon)/2}C^{(\epsilon)}(R_+^3) = \hat{C}_{s'}^{(\epsilon)}(R_+^3)$; *the operator* F *maps* $C_{s-l}^{(\epsilon)}(R^2)$ *into* $C_{s'}^{(\epsilon)}(R_+^3)$, $\forall s' < s$, $\forall \epsilon \in (0, 1)$.

(ii) *If a norm on* $\hat{C}_{s'}^{(\epsilon)}(R^2 \times [0, 1])$ *is introduced in the natural way* (*i.e., the norm of* $g = (1 - \underline{D})^{-(s'-\epsilon)/2}\bar{g}$ *with* $\bar{g} \in C^{(\epsilon)}(R_+^3)$ *is put equal to the norm of* $\bar{g}$) *then the operators* $\bar{F}, F$ *act continuously with a bound on the norms*

$$\|\bar{F}\|, \|F\| \leqslant \gamma(\epsilon, s, s')\, C^{|\underline{n}|}, \tag{A.1.80}$$

where c *is a constant and* γ *is a suitable continuous function of* $\epsilon \in (0, 1)$, $s \in R$, $s' \in R$, $s' < s$.

Proof. It is easy to see that the statements on $\bar{F}$ imply those on F. In fact if we define the operators

$$(Hg)(\underline{y}) = e^{-\underline{y}^2} g(\underline{y}), \tag{A.1.31}$$

$$(Tg)(\underline{y}, t) = e^{-t} g(\underline{y}, t) \tag{A.1.32}$$

we see that

$$F = (1 - \underline{D})^{-j/2} H(1 - \underline{D})^{j/2} T\bar{F}H \qquad \text{(A.1.33)}$$

and it is easy to see (cf. also Appendix B, estimate for (B.3)) that $\forall a, b \in R$ the operators

$$H_a = (1 - \underline{D})^a H(1 - \underline{D})^{-a}$$

map continously $C_b^{(\epsilon)}(R^2)$ into itself and TH_a maps the space $\hat{C}_b^{(\epsilon)}(R_+^3) = (1 - \underline{D})^{-(b-\epsilon)/2} C^{(\epsilon)}(R_+^3)$ into $C_b^{(\epsilon)}(R_+^3)$ with continuity if $\hat{C}_b^{(\epsilon)}(R_+^3)$ is endowed with the natural norm.

The continuity statement about F is then the statement that

$$H_{(s'-j-\epsilon)/2} T\{\bar{F}(1 - \underline{D})^{l/2}(1 - \underline{D})^{-(s-s')/2}\} H_{(s-l-\epsilon)/2} \qquad \text{(A.1.34)}$$

maps $\bar{C}^{(\epsilon)}(R^2)$ into $\bar{C}^{(\epsilon)}(R_+^3)$ continously.

The continuity statement about $\bar{F}$ means that the operator in curly brackets maps continously $\bar{C}^{(\epsilon)}(R^2)$ into $C^{(\epsilon)}(R_+^3)$: then the above continuity properties for H_a and H_aT together with (A.1.30) for $\bar{F}$ imply (A.1.30) for F (with new values for the constants).

We shall now prove the bound (A.1.30) for $\bar{F}$.

Since $\bar{F}$, by (A.1.17), can be written as a sum of operators of multiplication of the Fourier transform of g, it will suffice to consider instead of $\bar{F}$ the operator of multiplication of the Fourier transform of g by

$$e^t(1 + \underline{k}^2)^{-j/2} \frac{\partial^j}{\partial t^j} (t^l e^{-t^2} \varphi(kt))(\cos\theta)^{\nu}(\sin\theta)^{\bar{\nu}}. \qquad \text{(A.1.35)}$$

If $g = (1 - \underline{D})^{-(s-l-\epsilon)/2}\bar{g}$, $\bar{g} \in \bar{C}^{(\epsilon)}(R^2)$ we have then to consider the multiplication operator of the Fourier transform of $\bar{g}$ by

$$e^t(1 + \underline{k}^2)^{(l-j)/2} \frac{\partial^j}{\partial t^j} (t^l e^{-t^2} \varphi(kt)) \frac{(\cos\theta)^{\nu}(\sin\theta)^{\bar{\nu}}}{(1 + \underline{k}^2)^{(s-s')/2}}. \qquad \text{(A.1.36)}$$

Performing the derivative allows us to restrict the attention to multiplication operators like

$$\left(e^t \frac{\partial^p}{\partial t^p} e^{-t^2}\right) (1 + \underline{k}^2)^{(l-j)/2}\, t^{l-h} \underline{k}^{j-p-h} \varphi^{(j-p-h)}(kt) \frac{(\cos\theta)^{\nu}(\sin\theta)^{\bar{\nu}}}{(1 + \underline{k}^2)^{(s-s')/2}} \qquad \text{(A.1.37)}$$

with $l - h \geqslant 0$, $j - p - h \geqslant 0$. To eliminate some fractional powers we regard the above multiplication operator on Fourier transforms as a diagonal 2×2 matrix acting on the Fourier transforms of the elements of $\bar{C}^{(\epsilon)}(R^2) \times \bar{C}^{(\epsilon)}(R^2)$: we shall, eventually, be interested only in the action of such a duplicated version of (A.1.37) on the space of the elements of $\bar{C}^{(\epsilon)}(R^2) \times C^{(\epsilon)}(R^2)$ having the form $\underline{x} \to \binom{g(x)}{g(\underline{x})}$.

We also introduce on $\bar{C}^{(\epsilon)}(R^2) \times \bar{C}^{(\epsilon)}(R^2)$ the "linear Laplacian":

$$1 - \underset{\sim}{\sigma} \cdot \underset{\sim}{\partial} = 1 - \sum_{i=1}^{2} \sigma_i \frac{\partial}{\partial x_i} \tag{A.1.38}$$

with $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which has the property

$$(1 - \underset{\sim}{\sigma} \cdot \underset{\sim}{\partial})(1 + \underset{\sim}{\sigma} \cdot \underset{\sim}{\partial}) = 1 - \underset{\sim}{D} \tag{A.1.39}$$

as operators on $\bar{C}^{(\epsilon)}(R^2) \times \bar{C}^{(\epsilon)}(R^2)$.

Then we can rewrite (A.1.37) as (if $k = (k_1{}^2 + k_2{}^2)^{1/2}$)

$$(1 + i\underset{\sim}{\sigma}\underset{\sim}{k})^{l-h}\, t^{l-h} \varphi^{(j-p-h)}(kt) \left[e^t \frac{\partial^p}{\partial t^p} e^{-t^2}\right]$$

$$\times \left\{ \frac{k^{j-h-p}}{(1 + \underset{\sim}{k}^2)^{(j-h)/2}} \frac{(1 - i\underset{\sim}{\sigma} \cdot \underset{\sim}{k})^{l-h}}{(1 + \underset{\sim}{k}^2)^{(l-h)/2}} \frac{(\cos\theta)^{\bar{\nu}}(\sin\theta)^{\bar{\bar{\nu}}}}{(1 + \underset{\sim}{k}^2)^{(s-s')/2}} \right\}. \tag{A.1.40}$$

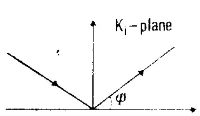
The operator in curly brackets maps $\bar{C}^{(\epsilon)}(R^2)$ into $C^{(\epsilon)}(R^2)$ if $s > s'$ with a norm bounded by $2^{\bar{\nu}+\bar{\bar{\nu}}}$ times a constant, since the Fourier transform

$$N(\underset{\sim}{x}) = \int dk_1\, dk_2 \frac{e^{ik_1x_1 + ik_2x_2}}{(1 + k_1{}^2 + k_2{}^2)^{(s-s')/2}} \frac{(k_1{}^2 + k_2{}^2)^{(j-h-p)/2}}{(1 + k_1{}^2 + k_2{}^2)^{(j-h)/2}}$$

$$\times \left(\frac{1 - i\sigma_1 k_1 - i\sigma_2 k_2}{(1 + k_1{}^2 + k_2{}^2)^{1/2}}\right)^{l-h} \left(\frac{k_1}{(k_1{}^2 + k_2{}^2)^{1/2}}\right)^{\bar{\nu}} \left(\frac{k_2}{(k_1{}^2 + k_2{}^2)^{1/2}}\right)^{\bar{\bar{\nu}}} \tag{A.1.41}$$

can be written, after a rotation which brings $\underset{\sim}{x}$ parallel to $(1, 1)/2^{1/2}$, as a sum of $2^{\bar{\nu}+\bar{\bar{\nu}}}$ integrals of the type (A.1.41) with σ_1, σ_2 replaced by $(\cos\bar{\theta}\sigma_{1,2} \pm \sin\bar{\theta}\sigma_{2,1})$ if $\bar{\theta}$ is the angle of the rotation and with $\exp(ik_1x_1 + ik_2x_2)$ replaced by $\exp[i(x/2^{1/2})(k_1 + k_2)]$, and, finally, with $\bar{\nu}, \bar{\bar{\nu}}$ replaced by two nonnegative integers with the same sum.

The Fourier transform (A.1.41) being in the sense of distributions, can be computed by deforming the k_1, k_2 integrations to the contour drawn in the picture (ψ is arbitrarily chosen in $(0, \pi/4)$).



This shows that $N(\underset{\sim}{x})$ can be bounded by a constant times $|\underset{\sim}{x}|^{-2+s-s'}$. Since the kernels N are locally summable they map $\bar{C}^{(\epsilon)}(R^2)$ continously into $C^{(\epsilon)}(R^2)$.

It remains to show (cf. (A.1.40)) that the operator of multiplication of the Fourier transforms by

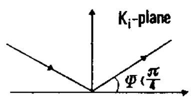
$$\left(e^t \frac{\partial^p}{\partial t^p} e^{-t^2}\right) t^{l'}\{(t\underline{\sigma} \cdot \underline{k})^{l''} \varphi^{(l''')}(kt)\}, \qquad l', l'', l''' \geqslant 0, l' + l'' + l''' \leqslant l + j, \tag{A.1.42}$$

map $C^{(\epsilon)}(R^2)$ into $C^{(\epsilon)}(R_+^3)$ continously. Obviously we can disregard the first two factors and prove the statement that, for $t \in [0, 1]$, the operator in curly brackets maps continously $C^{(\epsilon)}(R^2)$ into $C^{(\epsilon)}(R^2 \times [0, 1])$.

The Fourier transform ($k = (k_1^2 + k_2^2)^{1/2}$)

$$\mathscr{K}(\underline{x}) = \int d\underline{k}\, e^{i\underline{k}\underline{x}}(\underline{\sigma} \cdot \underline{k})^{l''} \varphi^{(l''')}(k) \tag{A.1.43}$$

is a C^∞ function since by Lemma A.2. $\varphi^{(j)}(k)$ decays exponentially at ∞. By shifting the integration contour to the one drawn in the picture (this is allowed by Lemma A.2), after rotating the axes so that $\underline{x} = |\underline{x}|((1, 1)/2^{1/2})$ we see that $\mathscr{K}(\underline{x})$ decays at ∞ as $|\underline{x}|^{-l''-2}$.



So the only case for which $\mathscr{K}(\underline{x})$ may not be summable in $\underline{x}$ is $l'' = 0$.

However $\varphi^{(l''')}(k)$ is entire in k and we can write

$$\varphi^{(l''')}(k) = e^{-\delta k}\varphi^{(l''')}(0) + (\varphi^{(l''')}(k) - e^{-\delta k}\varphi^{(l''')}(0)) \tag{A.1.44}$$

and we see that if $k \in \Delta_\psi^+$ both functions decay exponentially as $k \to \infty$ and have the "same" analyticity properties in Δ_ψ^+.

The second behaves as k at the origin: so its Fourier transform, by the previous argument, decays as $|\underline{x}|^{-3}$ as $|\underline{x}| \to \infty$. The first term has a Fourier transform which can be explicitly computed being proportional to the Poisson kernel:

$$\frac{1}{(\delta^2 + |\underline{x}|^2)^{1/2})^3} \tag{A.1.45}$$

hence it also decays as $|\underline{x}|^{-3}$.

It is now easy to estimate the norm of (A.1.42) (without the first two t-dependent factors) as operator from $C^{(\epsilon)}(R^2)$ to $C^{(\epsilon)}(R^2 \times [0, 1])$: it amounts to studying the Hölder continuity of

$$\tilde{f}(\underline{y}, t) = \int d\underline{x} \frac{1}{t^2} \mathscr{K}\left(\frac{\underline{x} - \underline{y}}{t}\right) f(\underline{x}) = \int \mathscr{K}(x) f(\underline{y} + t\underline{x})\, d\underline{x} \tag{A.1.46}$$

when $f \in C^{(\epsilon)}(R^2)$. We see that

$$\frac{|\tilde{f}(\underline{y}, t) - \tilde{f}(\underline{y}', t')|}{|\underline{y} - \underline{y}'|^\epsilon + |t - t'|^\epsilon} \leqslant \frac{|\tilde{f}(\underline{y}, t) - \tilde{f}(\underline{y}', t)|}{|\underline{y} - \underline{y}'|^\epsilon} + \frac{|\tilde{f}(\underline{y}', t) - \tilde{f}(\underline{y}', t')|}{|t - t'|^\epsilon}$$

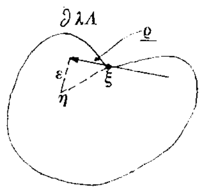
$$\leqslant \|f\|_{C^{(\epsilon)}(R^2)} \left[\int |\mathscr{K}(\underline{x})|\, dx + \int |\mathscr{K}(\underline{x})|\, |\underline{x}|^\epsilon\, d\underline{x}\right]. \tag{A.1.47}$$

This completes the proof of Corollary A.3.

A.2. *The Double-Layer Formula and Its Properties*

Let Λ be a conically regular region with conical regularity parameter larger than $\theta(\alpha_0, \alpha_{m-1}) = \arctan(\alpha_{m-1}^2 - \alpha_0^2)^{1/2}/\bar{\sigma}\alpha_0$, cf. (A.1.5). Given $\xi \in \partial\lambda\Lambda$ let $\underline{n}_\xi$ be the inner normal to $\partial\lambda\Lambda$ which we use to define the vector $\underline{l}$ and the quota ϵ. Define

$$B_l(\eta, \xi) = b_l(\epsilon, \underline{l}). \tag{A.2.1}$$



If $\underline{\zeta} = (\zeta^{(0)}, \ldots, \zeta^{(m-1)}) \in C^\infty(\partial\lambda\Lambda)^m$ the formula

$$u_\eta = \sum_{l=0}^{m-1} \int_{\partial\lambda\Lambda} B_l(\eta, \xi)\, \zeta_\xi^{(l)}\, d\sigma_\xi \tag{A.2.2}$$

defines a $C^\infty(\lambda\Lambda)$-function such that

$$Au = 0 \qquad \text{in } \lambda\Lambda \tag{A.2.3}$$

which for obvious reasons, is called the "double-layer" potential generated by the "double layer density" $\underline{\zeta} = (\zeta^{(0)}, \ldots, \zeta^{(m-1)})$.

The first property of (A.2.2) is summarized in the following lemma.

LEMMA A.4. *The double-layer potential* (A.2.2) *is a continuous map from* $\mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda)$ *to* $\mathscr{C}_{s'}^{(\epsilon)}(\lambda\Lambda; \partial\lambda\Lambda)$, $s' < s$, (*cf.* (2.11)) *if* λ *is large enough:*[24] $\lambda \geqslant \lambda(s', s, \epsilon)$. *Its norm is uniformly bounded in* λ.

Proof. This lemma is a consequence of Lemmas A.1, A.2, and of Corollary A.3.

[24] This restriction is not really necessary here. It has been made to make use of the geometric definitions stated in Section 2.

Let σ_i be a surface element on $\partial\lambda\Lambda$, σ_i' the associated enlarged surface element and α_i, α_i' the functions corresponding to them (cf. footnotes 4–6). We call E_i, E_i' the projections of σ_i, σ_i' on the tangent plane π_i to $\bar{\xi}_i$ (cf. Section 2) and define the quantities $\underline{x}'$, $\underline{y}'$, ϵ, $\underline{\rho}$, η, η_t, ξ as in the picture

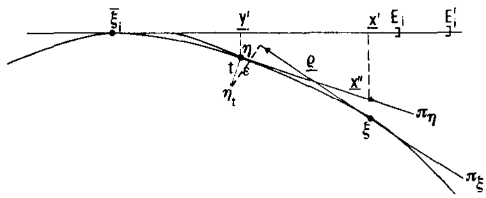


FIGURE A1

We then write $u = \sum_{i=1}^{n_\lambda} u^{(i)}$ where $u^{(i)}$ is the double-layer potential corresponding to the double-layer density $\alpha_i\zeta$, using the partition of unity on $\partial\lambda\Lambda$ associated with the covering $\sigma_1,\dots,\sigma_{n_\lambda}$ (cf. footnotes 4–6).

It is clear that the problem of the proof of the lemma reduces to the analysis of the behavior of $u^{(i)}$ near σ_i or near the neighboring surface elements. This means that we are really interested on the function obtained from (A.2.2) by replacing $\underline{\zeta}$ by $\alpha_i\underline{\zeta}$ and $B_l(\eta, \xi)$ by $\tilde{\alpha}_i'(\eta_t)\, B_l(\eta_t, \xi)\, \alpha_i'(\xi)$ or by $\tilde{\alpha}_i'(\eta_t)\, B_l(\eta_t, \xi)\, \alpha_j'(\xi)$ if $d(\sigma_i, \sigma_j) = 0$. We only discuss the first of such functions for λ large: the other can be treated in an identical manner.

Then $u^{(j)}$ can be written as a finite sum of addends each coming from one of the terms in the representation (A.1.6) for the b_l's. We consider one such expression and call it u dropping systematically the index j of the surface element. Let

$$u(\eta_t) = u(\underline{y}', t) = \int_{R^2} d\underline{x}'\, \tilde{\alpha}'(\underline{y}', t)\, \frac{\epsilon^{l+1}}{r^3} \left(\frac{\epsilon}{r}\right)^{2\beta} h(\epsilon, r^2)\, \tilde{\alpha}'(\underline{x}', 0)\, \bar{\alpha}(\underline{x}')\, \zeta(\underline{x}') \quad \text{(A.2.4)}$$

or

$$u(\eta_t) = \int_{R^2} d\underline{x}'\, \tilde{\alpha}'(\underline{y}', t)\, \epsilon^m \log(\epsilon + r)\, k(\epsilon, r^2)\, \tilde{\alpha}'(\underline{x}', 0)\, \bar{\alpha}(\underline{x}')\, \zeta(\underline{x}') \quad \text{(A.2.5)}$$

and suppose $\| \alpha\zeta \|_{C^{(\epsilon)}_{s-l}(R^{d-1})} < +\infty$ for some given $\epsilon \in (0, 1)$, $s \in R$.

Let us study (A.2.4). First we express ϵ, r in terms of the natural coordinates $\underline{y}'$, $\underline{y}' - \underline{x}'$, and t. If $\underline{y}', \underline{x}' \in E'$ and $t \in [0, 1]$ and λ is large enough:

$$\epsilon = t + \sum_{i,j=1}^{2} a_{ij}(\underline{x}', \underline{y}' - \underline{x}', t)(y_i' - x_i')(y_j' - x_j'), \quad \text{(A.2.6)}$$

$$r^2 = (\underline{y}' - \underline{x}')^2 + t^2 + \sum_{i,j=1}^{2} b_{ij}(\underline{x}', \underline{y}' - \underline{x}', t)(y_i' - x_i')(y_j' - x_j'), \quad \text{(A.2.7)}$$

$$\underline{\rho}^2 = r^2 - \epsilon^2 = (\underline{y}' - \underline{x}')^2 + \sum_{i,j=1}^{2} c_{ij}(\underline{x}', \underline{y}' - \underline{x}', t)(y_i' - x_i')(y_j' - x_j'), \quad \text{(A.2.8)}$$

and the functions a, b, c may and shall be assumed to be of class C^∞ with common bounded support (recall that $\tilde{\alpha}'$ has support in a neighborhood of σ), and also

$$\| a_{ij} \|_{C^{(k)}(R^5)}, \| b_{ij} \|_{C^{(k)}(R^5)}, \| c_{ij} \|_{C^{(k)}(R^5)} \leqslant \delta_k(\lambda) \xrightarrow[\lambda\to\infty]{} 0. \tag{A.2.9}$$

By a simple application of the Taylor series we find that the integration kernel in (A.2.4) can be rewritten as

$$\sum_{g=0}^{l+1} \sum_{\substack{n_{ij}=0 \\ i,j=1,2,i\leqslant j}}^{\infty} \frac{t^{l+1-g+2\beta}}{(t^2 + (\underline{x}' - \underline{y}')^2)^{3/2-g+\beta}} e^{-t^2-\underline{x}'^2-\underline{y}'^2}$$

$$\times \prod_{i\leqslant j}^{1,2} \left(\frac{\xi_i \xi_j}{t^2 + (\underline{x}' - \underline{y}')^2} \right)^{n_{ij}} \mathscr{K}_{\underline{n}}^g(\underline{y}', \underline{y}' - \underline{x}', t), \tag{A.2.10}$$

where $\xi_i = y_i' - x_i'$ and $\mathscr{K}_{\underline{n}}^g$ are suitable C^∞-functions with common bounded support such that

$$\| \mathscr{K}_{\underline{n}}^g \|_{C^{(k)}(R^5)} \leqslant (c_k \delta_k(\lambda))^{|\underline{n}|} c_k' \tag{A.2.11}$$

and the constants c_k, c_k' can be taken independently of the particular surface element under consideration (also the supports can be taken inside a given bounded region for λ large enough).

By using the idea of Section 3 we express the function $\mathscr{K}_{\underline{n}}^g$ by a Fourier transform (cf. (3.13)) and we can reduce the problem to that of the theory of the operators with Kernels (A.1.29).

It is an immediate consequence of Corollary A.3 that given $s' < s$ and $\epsilon \in (0, 1)$, there is a constant $\bar{C}$ such that the inequality

$$\sum_{|\underline{j}|\leqslant m-1} \| (1 - \underline{D})^{(s'-|\underline{j}|-\epsilon)/2} \partial^{(\underline{j})} u(\underline{y}', t) \|_{\bar{C}^{(\epsilon)}(R_+^3)}$$

$$\leqslant \bar{C} \sum_{\underline{n}} \gamma(s, s', \epsilon) c^{|\underline{n}|} \int d\underline{h}\, d\underline{h}'\, d\omega\, | \hat{\mathscr{K}}_{\underline{n}}^g(\underline{h}, \underline{h}', \omega)| (1 + |\underline{h}|^{|\underline{j}|} + |\underline{h}'|^{|\underline{j}|} + |\omega|^{|\underline{j}|})$$

$$\times \mathscr{N}_{s,s',\epsilon}(\underline{h}, \underline{h}') \| \alpha\zeta \|_{C_{s-l}^{(\epsilon)}(R^2)} \tag{A.2.12}$$

where $\hat{\mathscr{K}}_{\underline{n}}^g$ denotes the Fourier transform of $\mathscr{K}_{\underline{n}}^g$ and where $\mathscr{N}_{s,s',\epsilon}(\underline{h}, \underline{h}')$ is the product of the norms of the operators, depending on $\underline{h}$, $\underline{h}'$ on $\bar{C}^{(\epsilon)}(R^2)$ defined by

$$g \to g', g'(\underline{x}) = (1 - \underline{D})^{(s'-|\underline{j}|-\epsilon)/2} e^{i(\underline{h}+\underline{h}')\underline{x}} (1 - \underline{D})^{-(s'-|\underline{j}|-\epsilon)/2} g(\underline{x}), \tag{A.2.13}$$

$$g \to g', g'(\underline{x}) = (1 - \underline{D})^{(s-l-\epsilon)/2} e^{i\underline{h}\underline{x}} (1 - \underline{D})^{-(s-l-\epsilon)/2} g(\underline{x}). \tag{A.2.14}$$

Such norms are easy to estimate by noticing, considering (A.2.14), say, that

$$g'(\underline{x}) = e^{i\underline{h}\underline{x}}g(\underline{x}) + \int d\underline{x}'\, e^{i\underline{h}\underline{x}'} N(\underline{x} - \underline{x}')\, g(\underline{x}'), \tag{A.2.15}$$

$$N(\underline{x}) = \int d\underline{k}\, e^{i\underline{k}\underline{x}} \left(\frac{(1 + (\underline{k} - \underline{h})^2)^{-(s-l-\epsilon)/2}}{(1 + \underline{k}^2)^{-(s-l-\epsilon)/2}} - 1\right), \tag{A.2.16}$$

and by the method of the proof of Corollary A.3 it can be shown that there are constants ν_1, ν_2, ν_3 such that (see also Lemma A.4)

$$|N(\underline{x})| \leqslant \nu_1 \frac{|\underline{h}|^\nu}{|\underline{x}|} e^{-\nu_3|\underline{x}|} \tag{A.2.17}$$

with ν_1, ν_2, ν_3 depending on s.

Hence it follows from (A.2.17) that $\exists \nu, \nu'$, depending on s, s' such that

$$\mathscr{N}_{s,s',\epsilon}(\underline{h}, \underline{h}') \leqslant \nu(1 + |\underline{h}| + |\underline{h}'|)^{\nu'}. \tag{A.2.18}$$

The bounds in (A.2.9), (A.2.11) imply that one can find constants, depending only on s, s', ϵ, denoted $\tilde{c}$, $\tilde{c}'$ such that

$$|\mathscr{K}_{\underline{n}}^{g}(\underline{h}, \underline{h}', \omega)| \leqslant \tilde{c}'(\tilde{c}\delta_{\nu'+6}(\lambda))^{|\underline{n}|}(1 + |\underline{h}| + |\underline{h}'| + |\omega|)^{-6-\nu'}. \tag{A.2.19}$$

Therefore if $c\tilde{c}\delta_{\nu'+6}(\lambda) < \frac{1}{2}$ the series (A.2.12) converges proving the wanted result for the contribution (A.2.4) to the double-layer potential.

To treat (A.2.5) observe that (A.2.6), (A.2.8) imply that if we define

$$\tilde{\epsilon}_t(\tau) = \tau + \sum_{i,j=1}^{2} a_{ij}(x_i' - y_i')(x_j' - y_j'), \tag{A.2.20}$$

where the subscript t recalls that a_{ij} depend also on t we note that

$$\begin{aligned} \log(\epsilon + (\epsilon^2 + \underline{\rho}^2)^{1/2}) &= \log(\tilde{\epsilon}_t(1) + (\tilde{\epsilon}_t(1)^2 + \underline{\rho}^2)^{1/2}) - \int_t^1 \frac{d\tau}{(\tilde{\epsilon}_t(\tau)^2 + \underline{\rho}^2)^{1/2}} \\ &= \log(\tilde{\epsilon}_t(1) + (\tilde{\epsilon}_t(1)^2 + \underline{\rho}^2)^{1/2}) \\ &\quad - \int_t^1 \frac{d\tau}{[\tau^2 + (\underline{x}' - \underline{y}')^2 + \sum_{i,j=1}^2 f_{ij}(x_i' - y_i')(x_j - y_j')]^{1/2}} \end{aligned} \tag{A.2.21}$$

where f_{ij} are C^∞ functions of t, e, $\underline{x}'$, $\underline{y}$ which have $C^{(p)}$ norms tending to zero with $\lambda \to \infty$.

Clearly inserting (A.2.21) into (A.2.5) it is easily seen that, if λ is large enough, the first logarithm gives a contribution to (A.2.5) which fulfills the inequalities that we want to prove. In fact, if λ is large enough, it is a C^∞ function of all its arguments on the support of $\tilde{\alpha}(\underline{y}', t)\,\tilde{\alpha}'(\underline{x}', 0)$.

The contribution to (A.2.5) from the second term in (A.2.21) is treated by rewriting it as

$$\int_t^1 d\tau \int_\tau^1 d\theta \frac{\theta}{[\theta^2 + (\underline{x}' - \underline{y}')^2 + \sum_{i,j=1}^2 f_{ij}(x_i' - y_i')(x_j' - y_j')]^{3/2}}$$
$$- \int_t^1 d\tau \frac{1}{[1 + (\underline{x}' - \underline{y}')^2 + \sum_{i,j=1}^2 f_{ij}(x_i' - y_i')(x_j' - y_j')]^{1/2}} \quad \text{(A.2.22)}$$

and clearly the second integral here contributes to (A.2.5) in a trivial way (as already seen for the first logarithm in (A.2.21)).

Therefore we only have to study the contribution of the first term in (A.2.22) to (A.2.5).

However at fixed t, τ the operator corresponding to the first integral in (A.2.22) is of the type just discussed on the first part of the proof and the result that we are looking follows from the previous considerations.

To continue the analysis of the properties of the double-layer potentials consider now the enlarged surface elements $\sigma_i' \supset \sigma_i$ associated with a surface element σ_i on $\partial\lambda\Lambda$ (cf. Fig. A1 and footnotes 4–6). Let α_i', α_i be the two functions on $\partial\lambda\Lambda$ associated with σ_i', σ_i (cf. definitions in Section 2).

Let $\underline{j} = (j_1, j_2)$ be two nonnegative integers. Define on $(\partial\lambda\Lambda)^2$ a function $a_i^{(\underline{j})}$ which has support in $\sigma_i' \times \sigma_i'$ and has, in the local system of coordinates associated with σ_i, a representative $\bar{a}_i^{(\underline{j})}$ defined by

$$\bar{a}_i^{(\underline{j})}(\underline{x}', \underline{y}') = (x_1' - y_1')^{j_1}(x_2' - y_2')^{j_2}\,\bar{\alpha}_i'(\underline{x}')\,\bar{\alpha}_i'(\underline{y}'). \quad \text{(A.2.23)}$$

Let $(\underline{\nu}'(\underline{x}'))^2$ be the square of the gradient of the function ν_i describing σ_i in its local coordinates.

Then using Lemma A.4 it makes sense to define the "curvature coefficients" of $\partial\lambda\Lambda$ of order $(\underline{j}, l, p)$, l, p integers, as

$$C_{l,p}^{(\underline{j}),i}(\eta) = \lim_{t\to 0} \int d\sigma_\xi\, \alpha_i(\eta)\, \bar{a}_i^{(\underline{j})}(\eta, \xi)\{\partial_t{}^p B_l(\eta_t, \xi) - B_l{}^p(\eta_t, \xi)\}, \quad \text{(A.2.24)}$$

where $B_l{}^p(\eta_t, \xi)$ is defined as $\partial_t{}^p b_l(t, \underline{x}'')$, if $\underline{x}''$ is the projection of ξ on π_n (see Fig. A1).

Corollary A.5. *The curvature coefficients are in $C^\infty(\partial\lambda\Lambda)$ and $\forall k \geq 0$, $\forall p, l = 0,\ldots, m-1$, $\forall \underline{j} = (j_1, j_2)$,*

$$\lim_{\lambda\to\infty} \sup_{i=1,\ldots,n_\lambda} \| C_{l,p}^{(\underline{j}),i} \|_{C^{(k)}(\partial\lambda\Lambda)} = 0. \quad \text{(A.2.25)}$$

This is a corollary to the preceding lemma since the subtraction of $B_l{}^p$ in (A.2.24) amounts to removing from the various contributions to $\partial_t{}^p B_l$ from the various terms in (A.1.10) the formally most singular ones and in modifying slightly the others: direct computation shows that such differences do not matter and the corollary is proved just in the same way as the Lemma A.4.

Finally define, for $\zeta \in C^\infty(\partial\lambda\Lambda)$

$$\zeta^i_{\geqslant\nu}(\xi, \eta) = \left(\zeta_\xi - \sum_{|\underline{j}|<\nu} \frac{a^{(\underline{j}),i}(\xi, \eta)}{\underline{j}!} \zeta^{(\underline{j}),i}_\eta\right) \alpha_\eta{}^i, \qquad \text{(A.2.26)}$$

where $\underline{j}! = j_1! \, j_2!$ and $\zeta^{(\underline{j}),i}$ denotes the $\underline{j}$th derivative of ζ_η with respect to the coordinates $\underline{y}'$ of η in the local system of coordinates associated with σ_i.

Lemma A.6. *Let $\zeta \in C^\infty(\partial\lambda\Lambda)$, then*

$$(K_{pl}\zeta)_\eta = \lim_{t\to 0} \partial_t{}^p \int B_l(\eta_t, \xi)\zeta_\xi \, d\sigma_\xi, \qquad p, l = 0,..., m-1, \quad \text{(A.2.27)}$$

can be expressed as

$$\delta_{pl}\zeta_\eta + \sum_{i=1}^{n_\lambda} \sum_{|\underline{j}|\leqslant p-l-1} \frac{1}{\underline{j}!} C^{(\underline{j}),i}_{l,p}(\eta) \, \zeta^{(\underline{j}),i}_\eta \alpha^i_\eta$$

$$+ \sum_{i=1}^{n_\lambda} \int_{\partial\lambda\Lambda} [\partial_t{}^p B_l(\eta_t, \xi)]_{t=0} \, \zeta^i_{\geqslant\nu_{pl}}(\xi, \eta) \, d\sigma_\xi, \qquad \text{(A.2.28)}$$

where $\nu_{pl} = p - l$,[25] *provided λ is large enough.*[26]

For the proof one notices that $\zeta^i_{\geqslant\nu_{pl}}$ has a zero of order ν_{pl} at η, as a function of ξ, and if one looks at the structure of $\partial_t{}^p B_l(\eta_t, \xi) - B_l{}^p(\eta_t, \xi)$ one finds that *uniformly* in $t \to 0$

$$\begin{aligned} |\underline{x}' - \underline{y}'|^{p-l} \, |\partial_t{}^p B_l(\eta_t, \xi) - B_l{}^p(\eta_t, \xi)| &\leqslant \frac{c}{|\underline{x}' - \underline{y}'|}, \qquad p \geqslant l, \\ |\partial_t{}^p B_l(\eta_t, \xi) - B_l{}^p(\eta_t, \xi)| &\leqslant c, \qquad p < l, \end{aligned} \qquad \text{(A.2.29)}$$

and, from this remark, and from the fact that the b_l are the solution kernels for the Dirichlet problem in a half-space Lemma A.6 follows.

We shall write

$$K_{pl} = \delta_{pl} + K^\circ_{pl}, \qquad p, l = 0,..., m-1, \qquad \text{(A.2.30)}$$

[25] With the convention that meaningless summations give 0.

[26] See footnote 24.

and regard K° as an operator on $C^\infty(\partial\lambda\Lambda)^m$

$$(K^\circ\underset{\sim}{\zeta})_p = \sum_{l=0}^{m-1} K^\circ_{pl}\zeta^{(l)}, \qquad \underset{\sim}{\zeta} \in C^\infty(\partial\lambda\Lambda)^m, \tag{A.2.31}$$

and call it a "trace operator."

We call the formal equation for $\underset{\sim}{\zeta}$

$$\underset{\sim}{z} = \underset{\sim}{\zeta} + K^\circ\underset{\sim}{\zeta} \tag{A.2.32}$$

the "double-layer density equation."

A.3. *Trace Operators*

An immediate consequence of Lemma A.1, (A.2.10), and of the fact that if $\underset{\sim}{y}' \to \underset{\sim}{x}'$ then $\epsilon \to 0$ as $|\underset{\sim}{y}' - \underset{\sim}{x}'|^2$ is the following lemma.

LEMMA A.7. *Let $\xi, \eta \in \sigma_i' \times \sigma_i'$ and call $\underset{\sim}{x}', \underset{\sim}{y}'$ their local coordinates. Then for* $i = 1,\ldots, n_\lambda$,

$$\begin{aligned}[\partial_t{}^p B_l(\eta_t, \xi)]_{t=0} &= \frac{|\underset{\sim}{x} - \underset{\sim}{y}'|^{m-p+\epsilon_{m,l}}}{|\underset{\sim}{x}' - \underset{\sim}{y}'|^{2+\nu_{pl}}} \phi^{(i)}_{p,l}\left(\frac{\underset{\sim}{x}' - \underset{\sim}{y}'}{|\underset{\sim}{x}' - \underset{\sim}{y}'|}, \underset{\sim}{x}', \underset{\sim}{y}'\right) \\ &+ |\underset{\sim}{x}' - \underset{\sim}{y}'|^{2(m-p+1-\epsilon_{m,l})}\log|\underset{\sim}{x}' - \underset{\sim}{y}'|\,\psi^{(i)}_{p,l}\left(\frac{\underset{\sim}{x}' - \underset{\sim}{y}'}{|\underset{\sim}{x}' - \underset{\sim}{y}'|}, \underset{\sim}{x}', \underset{\sim}{y}'\right) \\ &+ \Omega^{(i)}_{p,l}\left(\frac{\underset{\sim}{x}' - \underset{\sim}{y}'}{|\underset{\sim}{x}' - \underset{\sim}{y}'|}, \underset{\sim}{x}', \underset{\sim}{y}'\right) \qquad \forall\xi, \eta \in \sigma_i \times \sigma_i,\end{aligned} \tag{A.3.1}$$

where $\epsilon_{m,l} = 1$ if $m - l$ is even, $\epsilon_{m,l} = 0$ if $m - l$ is odd; ö, ψ, Ω *are C^∞ functions of $\underset{\sim}{x}', \underset{\sim}{y}'$ and of the polar angle of $(\underset{\sim}{x}' - \underset{\sim}{y}')/|\underset{\sim}{x}' - \underset{\sim}{y}'|$ in which they are polarly even.*[27]

The functions ϕ, ψ, Ω have support in $\sigma_i' \times \sigma_i'$ (but they verify (A.3.1) only in $\sigma_i \times \sigma_i$). Furthermore the $C^{(k)}$-norms of ϕ, ψ, Ω tend to zero as $\lambda \to \infty$, $\forall k > 0$ uniformly in the surface element index i, $i = 1, 2,\ldots, n_\lambda$.

It is convenient to supplement the above lemma by a description of the analyticity properties of the Fourier transforms of functions of the type of those encountered in Lemma A.2:

[27] That is, their Fourier series in the polar angle of $(\underset{\sim}{x}' - \underset{\sim}{y}')/|\underset{\sim}{x}' - \underset{\sim}{y}'|$ contains only even powers (exp $2in\varphi$).

LEMMA A.8. *Consider the functions*

$$f_{\underline{n}}^{g,1}(\underline{x}) = \frac{1}{|\underline{x}|^{2+p-l-g}} \prod_{i\leqslant j}^{1,2} \left(\frac{x_i x_j}{|\underline{x}|^2}\right)^{n_{ij}} e^{-|\underline{x}|^2}, \tag{A.3.2}$$

$$f_{\underline{n}}^{g,2}(\underline{x}) = \frac{\log|\underline{x}|}{|\underline{x}|^{p-l-g}} \prod_{i\leqslant j}^{1,2} \left(\frac{x_i x_j}{|\underline{x}|^2}\right)^{n_{ij}} e^{-|\underline{x}|^2}, \tag{A.3.3}$$

where $\underline{n} = (n_{ij})_{i,j=1,2}$ *are some nonnegative integers and* $g \geqslant 1$, $g \leqslant 2m$, *say*; $l, p = 0,..., m-1$.

Call $\hat{f}_{\underline{n}}^{g,\sigma}(\underline{k})$ the Fourier transform of the distributions

$$\varphi \to \int f_{\underline{n}}^{g,\sigma}(\underline{x})\, \varphi_{\geqslant p-l}(\underline{x})\, d\underline{x}, \qquad \varphi \in \mathscr{S}(R^2). \tag{A.3.4}$$

Then $\hat{f}_{\underline{n}}^{g,\sigma}$ are holomorphic in k_1, k_2 in the $\Delta_{\psi}= \{z \in \mathbb{C} \mid |\operatorname{Im} z| \leqslant \tan\psi\, |\operatorname{Re} z|\}$ $\psi \in (0, \pi/4)$ and there they verify the bound

$$|\hat{f}_{\underline{n}}^{g,\sigma}(\underline{k})| \leqslant \gamma(\lambda)\, c(\lambda)^{|\underline{n}|}(1+|\underline{k}|^2)^{(p-l-1)/2} \log(1+|\underline{k}|^2) \tag{A.3.5}$$

$\forall(k_1, k_2) \in \Delta_{\psi} \times \Delta_{\psi}$ (if $|\underline{k}|^2 = |k_1|^2 + |k_2|^2$), and $\gamma(\lambda)$, $c(\lambda)$ are suitable positive constants.[28] Finally $\lim_{\lambda\to\infty} c(\lambda) = \lim_{\lambda\to\infty} \gamma(\lambda) = 0$.

The proof of this lemma is analogous to the proof of Lemma A.2. Actually the above lemma is slightly easier since there is no t around.

As Corollary A.3 and Lemma A.4 follow from Lemma A.2, the following lemma follows from Lemmas A.7, A.8 (we shall not prove it explicitly):

COROLLARY A.9. $\exists\kappa > 0$ *and* $\lambda_1(\Lambda)$ *such that* $\forall\lambda \geqslant \lambda_1(\Lambda)$

$$\| \alpha_{\sigma_i} K^\circ \alpha_{\sigma_j} \|_{\mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda)} \leqslant \gamma_{\epsilon,s}(\lambda)\, e^{-\kappa d(\sigma_i,\sigma_j)}, \tag{A.3.6}$$

where $\gamma_{\epsilon,s}(\lambda)$ *is a suitable function such that*

$$\gamma_{\epsilon,s}(\lambda) \xrightarrow[\lambda\to\infty]{} 0. \tag{A.3.7}$$

Hence $\exists\lambda_1(\epsilon, s)$ *such that* $\forall\lambda \geqslant \lambda_1(\epsilon, s)$

$$\| K^\circ \|_{\mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda)} \leqslant \tfrac{1}{2}. \tag{A.3.8}$$

[28] Note the i-independence of the r.h.s. of (A.3.5), $i = 1,..., n_\lambda$. All the constants are, however, ψ dependent.

Furthermore K° *maps* $\mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda)$ *continuously into* $\mathscr{C}_{s+1-\delta}^{(\epsilon)}(\partial\lambda\Lambda)$, $\forall\delta > 0$, *with a continuity modulus bounded in a way similar to* (A.3.6) *for each fixed* $\delta > 0$. *In other words* K° *"regularizes" and is compact in* $\mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda)$.

We remark that the exponential decay factor in (A.3.6) appears naturally because of the decay properties of the b_l's, cf. Lemma A.1. The case of the above corollary is the case $d(\sigma_i, \sigma_j) = 0$ when all the previous work becomes necessary.

This time there is no loss in regularity (as in Lemma A.4 where $s' < s$) because in (A.3.1) there is no term as singular as $|\underline{x}' - \underline{y}'|^{-2-\nu_{pl}}$. Property (A.3.7), of course, follows from Corollary A.5.

Corollary A.9, (A.3.8), show that if $\lambda \geqslant \lambda_1(\epsilon, s)$ the double-layer density equation has a unique solution in $\mathscr{C}_s^{(\epsilon)}(\partial\lambda\Lambda)$.

This fact together with Lemma A.6 and (A.3.6), (A.3.7) and Lemma A.4 permits us to prove statement (i) of Proposition 1. Statement (ii) has also been implicitly proved in the above series of lemmas and we do not discuss it explicitly.

Statement (iv) is clearly a consequence of the regularity, and of the exponential decay from the boundary, of the b_l's and of the exponential decay in (A.3.6). Statement (iii), however, is still nontrivial and some more work has to be done to prove it. It is a consequence of Corollary A.9 and of the following lemma:

LEMMA A.10. *Let* $\underline{\zeta} \in C^\infty(\partial\lambda\Lambda)^m$ *and let* $u(\alpha_0\underline{\zeta})$ *be the double-layer potential of* $\alpha_0\underline{\zeta}$. *Let* $\Gamma \subset \Lambda$ *be a well situated region in* Λ (*cf. footnotes* 8, 9).

Let σ, τ be two surface elements on $\partial\lambda\Lambda$, $\partial\lambda\Gamma$: $\exists\delta > 0$ such that $\forall\lambda \geqslant \lambda_1(\epsilon, s)$

$$\|\partial u(\underline{\partial}\alpha_\sigma \underline{z})\|_{\mathscr{C}_{s'}^{(\epsilon)}(\tau)}$$

$$\leqslant [c_{s',s,\epsilon} + c'_{s',s,\epsilon}(d(\tau,\sigma)^{-(s'-s)} \log(e + d(\tau,\sigma)))^\delta] \|\underline{\partial} \underline{z}\|_{\mathscr{C}_s^{(\epsilon)}(\sigma)}, \quad \text{(A.3.9)}$$

where $c_{s',s,\epsilon}$, $c'_{s',s,\epsilon}$ are suitable constants.

The proof is given in the next section.

A.4. *Proof of the Last Lemma*

The proof is based on the following steps which are described for simplicity in the case σ is flat.

We only have to consider the case when σ and τ are very close: consider for instance the case in the figure

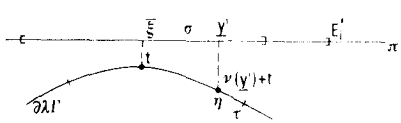


FIGURE A2

We imagine that τ is contained in the surface described in the local coordinates associated with π by the function $\nu \in \mathscr{D}(R^{d-1})$. Again for simplicity consider only the case in which τ is contained in the part of $\partial\lambda\Gamma$ "above" σ_i'.

A closer analysis of the Vandermonde determinants involved in (A.1.2) shows that $\partial_\epsilon{}^p b_l(\epsilon, \underline{x})$ has a Fourier transform of the form

$$\hat{b}_l^{(p)}(\epsilon, \underline{k})$$

$$= ((\alpha_0{}^2 + \underline{k}^2)^{1/2})^{p-l} \sum_{j=0}^{m-1} e^{-\epsilon(\alpha_j{}^2+\underline{k}^2)^{1/2}} H_j^{p,l} \left(\left(\frac{\alpha_1{}^2 + \underline{k}^2}{\alpha_0{}^2 + \underline{k}^2}\right)^{1/2}, \ldots, \left(\frac{\alpha_{m-1}^2 + \underline{k}^2}{\alpha_0{}^2 + \underline{k}^2}\right)^{1/2}\right) \tag{A.4.1}$$

where $H_j^{p,l}$ is a polynomial. Let $\hat{\beta}$ be one of the m terms in the above sum and let

$$\beta(\epsilon, \underline{x}) = \int d\underline{k}\, e^{i\underline{k}\underline{x}}((\alpha_0{}^2 + \underline{k}^2)^{1/2})^{p-l}\, \hat{\beta}(\epsilon, \underline{k})$$

$$= \int d\underline{k}\, e^{i\underline{k}\underline{x}}((\alpha_0{}^2 + \underline{k}^2)^{1/2})^{p-l}\, e^{-\epsilon(\alpha^2+\underline{k}^2)^{1/2}}\, H \left(\left(\frac{\alpha_1{}^2 + \underline{k}^2}{\alpha_0{}^2 + \underline{k}^2}\right)^{1/2}, \ldots, \left(\frac{\alpha_{m-1}^2 + \underline{k}^2}{\alpha_0{}^2 + \underline{k}^2}\right)^{1/2}\right), \tag{A.4.2}$$

where H is a polynomial.

Clearly our problem is to study the transformation

$$(\mathscr{L}f)(\underline{x}) = \int a(\underline{x})\, \beta(\nu(\underline{x}) + t, \underline{x} - \underline{y})\, a(\underline{y}) f(\underline{y})\, d\underline{y} \tag{A.4.3}$$

for supp $a \subset E_i'$, and we have to show that $\mathscr{L}\colon \mathscr{C}_{s-l}^{(\epsilon)} \to \mathscr{C}_{s'-p}^{(\epsilon)}$ and that the norm of $\mathscr{L}$ in such spaces can be bounded by a continuous function of $s, s', s' < s$. From the calculation the singular behaviour when $s' \geqslant s$ will also appear manifest. Of course we are interested only in estimates which are uniform in t, for small t. To compute the norm of $\mathscr{L}$ we consider the operator $\mathscr{B}$ on $\bar{C}^{(\epsilon)}$ defined by the kernel

$$B(\underline{x}, \underline{y}) = (1 - \underline{D}_x)^{(s'-p-\epsilon)/2}(1 - \underline{D}_y)^{-(s-l-\epsilon)/2}\, \beta(\nu(\underline{x}) + t, \underline{y} - \underline{x})$$

$$= \int d\underline{h}\, d\underline{k}\, e^{i\underline{k}(\underline{x}-\underline{y})+i\underline{h}\underline{x}}(\alpha_0{}^2 + \underline{k}^2)^{(p-l)/2}\, \frac{[1 + (\underline{h} + \underline{k})^2]^{(s'-p-\epsilon)/2}}{(1 + \underline{k}^2)^{(s-l-\epsilon)/2}}$$

$$\times \left\{\int d\underline{x}'\, e^{-i\underline{h}\underline{x}'} \hat{\beta}(\nu(\underline{x}') + t, \underline{k})\right\}. \tag{A.4.4}$$

We call $j(\underline{h}, \underline{k})$ the function in parenthesis and remark that at fixed $\underline{h}$ it is holomorphic in k_1, k_2 for k_1 and k_2 in the connected region obtained by considering the hyperbola $\mathrm{Re}(z^2 + (\alpha_0/3)^2) = 0$ and by translating it from $-|\underline{h}|$ to $|\underline{h}|$

parallel to the real axis (actually it is holomorphic in a much larger region): furthermore $j(\underline{h}, \underline{k})$ is bounded uniformly in $t > 0$, by

$$|j(\underline{h}, \underline{k})| \leqslant c_{B,\gamma} \frac{(1 + |\underline{k}|^2)^\gamma}{(1 + |\underline{h}|^2)^B} \tag{A.4.5}$$

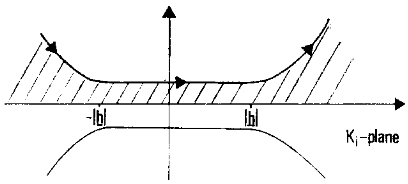


FIGURE A3

$\forall B, \gamma > 0$ and all k's in the region dashed in the Fig. A2 (which has to be thought a closed region).

The inequality (A.4.5) can be proved by integrating by parts suitably many times in (A.4.2) and use the ∞-order contact assumption for the contact of τ and σ when $t = 0$ (recall that Γ was supposed well situated in Λ).

We may, and shall also suppose that $\underline{x} - \underline{y} = (|\underline{x} - \underline{y}|/2^{1/2})(1, 1)$ (otherwise we change the axes, at the beginning). We can, then, deform the k_1, k_2 integrations to the upper path Γ in Fig. A2: the form of the contour shows that the term $[1 + (\underline{h} + \underline{k})^2]^{(s'-p)/2}$ has no singularity in the dashed region.

Then the expression for $B(\underline{x}, \underline{y})$

$$B(\underline{x}, \underline{y}) = \int_{R^2} d\underline{h} \int_{\Gamma\times\Gamma} dk_1\, dk_2\, e^{i\underline{h}\cdot x} \frac{[1 + (\underline{h} + \underline{k})^2]^{(s'-p-\epsilon)/2}}{(\alpha_0{}^2 + \underline{k}^2)^{(l-p)/2}(1 + \underline{k}^2)^{(s-l-\epsilon)/2}}$$

$$\times\, e^{i(|\underline{x}-\underline{y}|/2^{1/2}(k_1+k_2))} j(\underline{h}, \underline{k} \mid \underline{x} - \underline{y}), \tag{A.4.6}$$

where j has "become" a function also of $\underline{x} - \underline{y}$ by our new assumption that $\underline{x} - \underline{y} = (|\underline{x} - \underline{y}|/2^{1/2})(1, 1)$:

$$j(\underline{h}, \underline{k} \mid \underline{x} - \underline{y}) = \int d\underline{x}'\, e^{-i\underline{h}\underline{x}'} \beta[\nu[R_{\underline{x}-\underline{y}}\underline{x}') + t, \underline{k}], \tag{A.4.7}$$

where $R_{\underline{x}-\underline{y}}$ is the rotation which brings $\underline{x} - \underline{y}$ to $(|\underline{x} - \underline{y}|/2^{1/2})(1, 1)$.

It is easy to see, as in (A.4.5), that, $\forall\gamma, B > 0$

$$\sup_{\underline{x}-\underline{y}} |j(\underline{h}, \underline{k} \mid \underline{x} - \underline{y})| \leqslant c'_{B,\gamma} \frac{(1 + |\underline{k}|^2)^\gamma}{(1 + |\underline{h}|^2)^B} \tag{A.4.8}$$

for a suitable constant $C'_{B,\gamma} > 0$.

The result now follows by "power counting," using the inequalities (3.18), the formula (A.4.6), and the estimate (A.4.8).

Consider for instance the case $s' < s$. Let $\omega = s - s'$. Then if we denote $B(\underline{x}, \underline{y}) \equiv \tilde{B}(\underline{x}, \underline{x} - \underline{y})$ we see that if $\eta \leqslant 1$, (3.18), (A.4.6), (A.4.8) imply, for a suitable c:

$$\frac{|\tilde{B}(\underline{x}, \underline{v}) - \tilde{B}(\underline{x}', \underline{v})|}{|\underline{x} - \underline{x}'|^{\eta}} \leqslant \frac{c}{|\underline{v}|^{2-(\omega-\nu)}}, \qquad \forall \underline{v} \in R^2 \tag{A.4.9}$$

and

$$|\tilde{B}(\underline{x}, \underline{v})| \leqslant \frac{c}{|\underline{v}|^{2-(\omega-\nu)}}, \qquad \forall \underline{v} \in R^2 \tag{A.4.10}$$

which implies that if $f \in \bar{C}^{(\epsilon)}$, $\underline{x}, \underline{y} \in \Delta$,

$$\left| \int d\underline{y} \frac{\tilde{B}(\underline{x}, \underline{y} - \underline{x}) - \tilde{B}(\underline{x}', \underline{y} - \underline{x}')}{|\underline{x} - \underline{x}'|^{\epsilon}} f(\underline{y}) \right|$$

$$\leqslant \int d\underline{y} \frac{|\tilde{B}(\underline{x}, \underline{y} - \underline{x}) - \tilde{B}(\underline{x}', \underline{y} - \underline{x})|}{|\underline{x} - \underline{x}'|^{\epsilon}} |f(\underline{y})| + \int d\underline{y} \frac{|f(\underline{y}) - f(\underline{y} + \underline{x} - \underline{x}')|}{|\underline{x} - \underline{x}'|^{\epsilon}}$$

$$\times |\tilde{B}(\underline{x}', \underline{y} - \underline{x}')| \leqslant \bar{C}_{\Delta} \|f\|_{C^{(\epsilon)}(\Delta)} \tag{A.4.11}$$

for a suitable $\bar{C}_{\Delta}$.

It remains to study the "easy part" of the problem, i.e., the behavior as $|\underline{x}| \to \infty$: we do not give the details.

In the above calculation we have not been careful in keeping track of the dependence of the various constants on the parameters ϵ, s: it is however clear that this also can be done easily (to prove the assertions on the continuity). If $s' \geqslant s$ one proceeds in the same way.

Appendix B: Proof of (2.12)

By definition

$$\|\underline{\partial} u\|_{\mathscr{C}_s^{(\epsilon)}(\sigma i, \alpha i)} = \sum_{j=0}^{m-1} \|(1 - \underline{D})^{(s-j-\epsilon)/2} \overline{\alpha_i(\underline{x})(\partial^j u)(\underline{x})}\|_{\bar{C}^{(\epsilon)}(R^{d-1})}$$

$$= \sum_{j=0}^{m-1} \left\| \sum_{|\beta|=j} (1 - \underline{D})^{(s-j-\epsilon)/2} \bar{\alpha}_i(\underline{x}) \gamma_{j,\beta}(\underline{x})(\partial^{(\beta)} u)(\underline{x}, \nu(\underline{x})) \right\|_{\bar{C}^{(\epsilon)}(R^{d-1})}, \tag{B.1}$$

where $\gamma_{j,\underline{\beta}}$ are C^∞ functions depending only on the derivatives of $\nu_i(\underline{x})$ of order one; and from the hypotheses on the surface elements $\exists g_p$, $p = 0, 1,\ldots$ such that

$$\| \gamma_{j,\underline{\beta}} \|_{C^{(p)}(R^{d-1})} \leqslant g_p \, . \tag{B.2}$$

Therefore by (2.8), (2.10) we can take in (2.12)

$$C_s = \sup_{\substack{\underline{\beta}, j, \epsilon \\ 0 \leqslant j \leqslant m-1, |\underline{\beta}| \leqslant j}} \| (1 - \underline{D})^{(s-j-\epsilon)/2} \gamma_{j,\underline{\beta}} (1 - \underline{D})^{-(s-j-\epsilon)/2} \| \tag{B.3}$$

where the expression under the norm sign is regarded as an operator on $\bar{C}^{(\epsilon)}(R^{d-1})$.

Therefore we have only to show that, $\forall a \in R$, $(1 - \underline{D})^a \ \gamma (1 - \underline{D})^{-a}$ is bounded, as operator on $\bar{C}^{(\epsilon)}$, in terms of a continuous function of a and of the norm in (B.2) for $p = 0, 1,\ldots, p([a])$.

This follows from (A.2.14), by representing γ in terms of its Fourier transform $\hat{\gamma}(\underline{h})$ and then using that $\hat{\gamma}(\underline{h})$ decays at infinity faster than any power.

Appendix C: Some Details About The Proof of the Markov Property

Let Λ be a $\theta(\alpha_0, \alpha_{m-1})$-conically regular region.

Using the notation of Section 2 we shall first note that, as is well known, the Dirichlet covariance C° relative to the region $\lambda\Lambda$ is bounded in the operator sense by the free covariance C. Therefore since our support properties, as expressed in Proposition 2, are a consequence of moments inequalities it follows that Proposition 2 holds for the $P_\Lambda{}^\circ$-randomly chosen distributions.

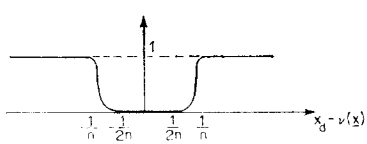
Let σ be a surface element on $\partial\lambda\Lambda$ and let $\tilde{\alpha}$ be the function associated with it. We have to consider the random variable $z(f) = \int f(\xi) \, z_\xi \, d\xi$ on $\mathscr{S}'(R^d)$ with z chosen P_Λ-randomly (given $f \in \mathscr{S}(R^d)$), and the random variable $u(\bar{z})(f) + \zeta(f)$ with $\bar{z}, \zeta$ chosen $P_\Lambda \times P_\Lambda{}^\circ$ randomly. We have to show that in the sense of probability distributions

$$z(f) \simeq u(\bar{z})(f) + \zeta(f) \tag{C.1}$$

assuming that (C.1) holds if $\operatorname{supp} f \cap \partial\lambda\Lambda = \varnothing$ (which can be shown to be true as explained in Section 2). Also it is nonrestrictive to suppose $\operatorname{supp} f \subset \{\xi \mid \tilde{\alpha}(\xi) = 1\}$.

The idea is that $u(\bar{z})(f) + \zeta(f)$ is a limit of functions of the form $u(\bar{z})(f^{(n)}) + \zeta(f^{(n)})$ with $\operatorname{supp} f^{(n)} \cap \partial\lambda\Lambda = \varnothing$.

Let $\chi^{(n)} \in C^\infty(R^d)$ be a function whose representative $\bar{\chi}^{(n)}$ in the local system of coordinates depends only on $x_d - \nu(\underline{x})$ and has the form described in the picture:



Put

$$f^{(n)} = \chi^{(n)} f \tag{C.2}$$

and, as we show below

$$\lim_{n\to\infty} u(\bar{z})(f^{(n)}) + \zeta(f^{(n)}) = u(\bar{z})(f) + \zeta(f) \tag{C.3}$$

$P_A{}^\circ \times P_A$-almost everywhere.

The limit exists by the support property of $u(\bar{z})$, ζ and by Proposition 1.

In fact, for instance, using the bars to denote representatives in the local system of coordinates of σ

$$u(\bar{z})(f^{(n)}) = \int dx\, dx_d \bar{f}(\underline{x}, x_d)\, \bar{\chi}^{(n)}(x_d)\, \overline{u(\bar{z})}(\underline{x}, x_d)$$

$$= \int d\underline{x}\, dx_d \bar{f}(\underline{x}, x_d)\, \bar{\chi}^{(n)}(x_d)\, \tilde{\alpha}(\underline{x}, x_d)\, \overline{u(\bar{z})}(\underline{x}, x_d) \tag{C.4}$$

$$= \int d\underline{x}\, dx_d [(1 - \underline{D}_x)^{-(s-\epsilon)/2} \bar{f}(\underline{x}, x_d)]\, \bar{\chi}^{(n)}(x_d) [(1 - \underline{D}_x)^{(s-\epsilon)/2} \overline{\tilde{\alpha}(\underline{x}, x_d)}\, \overline{u(\bar{z})}(\underline{x}, x_d)],$$

where $(1 - \underline{D}_x)^a g(\underline{x})$ denotes symbolically the result of the application of the operator $(1 - \underline{D})^a = (1 - \sum_{j=1}^{d-1} \partial^2/\partial x_j{}^2)^a$ to the distribution g; $s < m - d/2$, $\epsilon \in (0, \frac{1}{2})$. We have used that $\tilde{\alpha} = 1$ on supp f.

Then by Propositions 2 and 1 it follows that $[(1 - \underline{D}_x)^{(s-\epsilon)/2} \bar{\tilde{\alpha}} \bar{u}(\bar{z})](\underline{x}, x_d) \in \bar{C}^{(\epsilon)}(R^d)$, i.e., it is a Hölder continuous function, rapidly decreasing at ∞, of $\underline{x}, x_d$.

So by dominated convergence we have

$$\lim_{n\to\infty} u(\bar{z})(f^{(n)}) = \int d\underline{x}\, dx_d \bar{f}(\underline{x}, x_d) [\overline{\tilde{\alpha} u(\bar{z})}](\underline{x}, x_d) \equiv u(\bar{z})(f) \tag{C.5}$$

P_A-almost surely.

An identical argument can be set up to treat the convergence of $\zeta(f^{(n)})$ to $\zeta(f)$. We have only to show, now, that the random variables $u(\bar{z})(f^{(n)}) + \zeta(f^{(n)})$, which are distributed as $z(f^{(n)})$, by assumption, converge, say, in $L^2(P_A \times P_A{}^\circ)$ or, respectively, in $L_2(P_A)$ as $n \to \infty$. But, of course, this is once more a consequence of Proposition 2 and a dominated convergence theorem. We have

$$\delta_{nm} \equiv \int |z(f^{(n)}) - z(f^{(m)})|^2 P_A(dz)$$

$$= \int \bar{f}(\underline{x}, x_d)[\bar{\chi}^{(n)}(x_d) - \bar{\chi}^{(m)}(x_d)] \bar{f}(\underline{y}, y_d)[\bar{\chi}^{(n)}(y_d) - \bar{\chi}^{(m)}(y_d)]$$

$$\times\, \bar{\tilde{\alpha}}(\underline{x}, x_d)\, \bar{\tilde{\alpha}}(\underline{y}, y_d)\, C_{\underline{x}-\underline{y}, \nu(\underline{x})-\nu(\underline{y})+x_d-y_d}\, d\underline{x}\, d\underline{y}\, dx_d\, dy_d\,. \tag{C.6}$$

It can be shown that $K(\underline{x}, x_d; \underline{y}, y_d)$ defined by

$$(1 - \underline{D}_x)^{(s-\epsilon)/2}(1 - \underline{D}_y)^{(s-\epsilon)/2}\, \tilde{\bar{\alpha}}(\underline{x}, x_d)\, \tilde{\bar{\alpha}}(\underline{y}, y_d)\, C_{\underline{x}-\underline{y}, \nu(\underline{x})-\nu(\underline{y})-x_d-y_d} \quad \text{(C.7)}$$

is in $\bar{C}^{(\tilde{\epsilon})}(R^d)$, $\tilde{\epsilon} < \epsilon$ (cf. proof of Proposition 2 or (2.20), (2.21)).

Therefore, by rewriting (C.6) as

$$\delta_{nm} = \int [(1 - \underline{D}_x)^{(s-\epsilon)/2} \tilde{f}(\underline{x}, x_d)] \cdot [(1 - \underline{D}_y)^{(s-\epsilon)/2} \tilde{f}(\underline{y}, y_d)]$$

$$\times [\bar{\chi}^{(n)}(x_d) - \bar{\chi}^{(m)}(x_d)][\bar{\chi}^{(n)}(y_d) - \bar{\chi}^{(m)}(y_d)]\, R(\underline{x}, \underline{y}, x_d, y_d)\, d\underline{x}\, d\underline{y}\, dx_d\, dy_d$$

we can conclude by dominated convergence

$$\lim_{n,m\to\infty} \delta_{nm} = 0.$$

In a similar way one deals with $\zeta(f^{(n)})$. This completes the proof of the reduction of the check of (C.1) to the case $\operatorname{supp} f \cap \partial\lambda\Lambda = \varnothing$.

Appendix D: $\theta(\alpha_0, \alpha_{m-1})$ Large Case

Note that, $\forall a, b, c, b \neq a$

$$\frac{b-a}{(a+t)(b+t)} = \frac{b-c}{(c+t)(b+t)} + \frac{c-a}{(a+t)(c+t)}, \quad \text{(D.1)}$$

therefore if $a_1 < a_2 < a_3$

$$\frac{a_3 - a_1}{(a_1+t)(a_2+t)(a_3+t)} = \frac{a_3 - c}{(c+t)(a_2+t)(a_3+t)} + \frac{c - a_1}{(a_1+t)(a_2+t)(c+t)}. \quad \text{(D.2)}$$

Hence it is easy to show that, in general,

$$\frac{1}{(\alpha_0^2 + t)\cdots(\alpha_{m-1}^2 + t)} = \sum_{\substack{\beta_0,\ldots,\beta_{m-1} \\ \beta_0>\alpha_0, |\beta_i-\beta_j|<\delta}} \frac{c(\beta_0, \ldots, \beta_{m-1})}{(\beta_0^2 + t)\cdots(\beta_{m-1}^2 + t)} \quad \text{(D.3)}$$

with $c(\beta_0, \ldots, \beta_{m-1}) > 0$.

This means that, replacing t by $-D$, we can always represent the Gaussian process with covariance A^{-1} as a sum of processes, finitely many in number and *independently* distributed, associated with covariance operators of the type (1.1) with conical regularity parameter as small as we please.

It is then clear that Proposition 3 for operators of the form (1.1) holds in general if it holds in the case of operators with small conical regularity parameter. Hence the restriction in Sections 4, 5, and 6 to such operators does not affect the generality of the result.

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