Probabilistic aspects of critical phenomena*

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I. Generalities

To be definite we shall consider the problem of critical fluctuations with the purpose of illustrating their relevance in the analysis of other difficult problems like field theory.

The Ising model on the d–dimensional lattice $\mathbb{Z}^d$ is described by the Hamiltonian

1 $$H(\underline{\sigma}) = -J \sum_{\substack{|i-j|=1\\ i,j\in\Lambda}} \sigma_i\sigma_j - h\sum_i \sigma_i - B_\Lambda(\underline{\sigma}) \tag{1.1}$$

where $\underline{\sigma} = (\sigma_i)_{i\in\Lambda}$, $\sigma_i = \pm 1$, is a "finite spin configuration" in the cube $\Lambda \subset \mathbb{Z}^d$ centered at the origin and $B_\Lambda(\underline{\sigma})$ is a function which describes the "boundary conditions", *i.e.* defines a model for the walls of the box. For instance

2 $$B_\Lambda(\underline{\sigma}) = J\sum_{i\in\partial\Lambda} \sigma_i \tag{1.2}$$

has the physical interpretation that the box Λ is surrounded by an array of spins $+1$.

The Hamiltonian allows us to define the "equilibrium state in Λ" of the system as the family:

3 $$\begin{aligned}
\langle\sigma_x\rangle_\Lambda &= \frac{\sum_{\underline{\sigma}} \sigma_x\, e^{-\beta H_\Lambda(\underline{\sigma})}}{\sum_{\underline{\sigma}} e^{-\beta H_\Lambda(\underline{\sigma})}}\\
\langle\sigma_x\sigma_y\rangle_\Lambda &= \frac{\sum_{\underline{\sigma}} \sigma_x\,\sigma_y\, e^{-\beta H_\Lambda(\underline{\sigma})}}{\sum_{\underline{\sigma}} e^{-\beta H_\Lambda(\underline{\sigma})}}\\
\langle\sigma_x\sigma_y\sigma_z\rangle_\Lambda &= \frac{\sum_{\underline{\sigma}} \sigma_x\,\sigma_y\,\sigma_z\, e^{-\beta H_\Lambda(\underline{\sigma})}}{\sum_{\underline{\sigma}} e^{-\beta H_\Lambda(\underline{\sigma})}}\ \ldots
\end{aligned} \tag{1.3}$$

where the sums over the $2^{|\Lambda|}$ spin configurations in Λ ($|\Lambda|$ = number of points in Λ).

If we assign reasonable sequence $B = (B_\Lambda)_{\Lambda\in\, set\ of\ cubes}$ of boundary terms we can try to take the limit

4 $$\lim_{\Lambda\to\infty} \langle\sigma_{x_1}\ldots\sigma_{x_n}\rangle_\Lambda = \langle\sigma_{x_1}\ldots\sigma_{x_n}\rangle. \tag{1.4}$$

If, for given values of (β, h), this limit depends upon the sequence B we say that the model has a phase transition at (β, h).

Of course we should specify better which are the allowed choices of the sequence B: a possible definition is that $B = (B_\Lambda)_{\Lambda\in\, set\ of\ cubes}$ should be such that

5 $$\begin{aligned}
&(i)\quad \max_{\underline{\sigma}} \frac{|B_\Lambda(\underline{\sigma})|}{|\Lambda|} \xrightarrow[\Lambda\to\infty]{} 0\\
&(ii)\ \exists\,\delta_\Lambda \text{ such that } \frac{\delta_\Lambda}{|\Lambda|^{1/d}} \xrightarrow[\Lambda\to\infty]{} 0
\end{aligned} \tag{1.5}$$

and $B_\Lambda(\underline{\sigma}) \equiv B_\Lambda(\underline{\sigma}')$ if $\sigma_\xi = \sigma'_\xi$ for all ξ closer than δ_Λ to the boundary of Λ.

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The existence of a phase transition at given (β, h) means that the system is so sensitive to boundary perturbations that their influence does not disappear even in the limit $\Lambda \to \infty$.

Possibly passing to a subsequence we can and shall assume that the sequences B, which we consider, are such that all the limits of the correlation functions

6 $$\lim_{\Lambda\to\infty} \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_\Lambda = \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle \tag{1.6}$$

exist simultaneously: the set of all the functions $\langle \sigma_\xi \rangle \langle \sigma_\xi \sigma_\eta \rangle, \ldots$ obtained as limits of the homonymous finite volume functions for a given sequence B and given (β, h) is called an "equilibrium state" for the model at given (β, h).

A possible interpretation of an "equilibrium state" is that of a probability measure on the space of "infinite configurations". The measurable events are those which can be approximated by "local events" described by "cylinders":

7 $$E\binom{\overline{\sigma}_1 \ldots \overline{\sigma}_n}{\xi_1 \ldots \xi_n} = \big\{ \underline{\sigma} \mid \underline{\sigma} = (\sigma_\xi)_{\xi\in\mathbb{Z}^d},\, \sigma_{\xi_i} = \overline{\sigma}_{\xi_i},\ i = 1, \ldots, n \big\} \tag{1.7}$$

where $\underline{\sigma} = (\sigma_\xi)_{\xi\in\mathbb{Z}^d}$, $\sigma_\xi = \pm 1$, is what we call an infinite configuration and $\overline{\sigma}_1 \ldots \overline{\sigma}_n$ are assigned values ± 1: say that $\gamma_+ = (i_1, \ldots, i_p)$ are the sites with $\overline{\sigma}_{i_j} = +1$ while the others $\overline{\sigma}_j$ are -1. The probability of a cylinder is, obviously

8 $$\mu(E\binom{\overline{\sigma}_1 \ldots \overline{\sigma}_n}{\xi_1 \ldots \overline{\xi}_n}) = \langle \prod_{j\in\gamma_+} \frac{1+\sigma_{\xi_j}}{2} \prod_{j\notin\gamma_+} \frac{1-\sigma_{\xi_j}}{2} \rangle \tag{1.8}$$

As mentioned above the cylinders are not the only measurable sets: also those sets which can be obtained by finitely many union and complementation operations are measurable and receive the natural value for their probability. Actually all the sets in the smallest $\sigma-$algebra generated by the cylinders can be measured: *i.e.* the measure μ can be extended to a countably additive probability measure on the sets which are in the smallest family of sets which is closed under the operations of complementation and countable union contain the cylinders. This fact, although not immediately obvious, is easy to prove but we shall not need such a result.

Therefore the equilibrium states can be regarded as probability measures (on the space of infinite configurations) obtained through the above described process of thermodynamic limit.

It should be stressed that conceptually it is not necessary to take really the limit $\Lambda \to \infty$: all systems are actually finite and only those properties of infinite systems which can be interpreted in terms of properties of finite systems are physically meaningful.

Nevertheless it is sometimes simpler to pose the problems for infinite systems: for instance when, as we shall do shortly, we wish to talk of macroscopic regions inside a macroscopic system it is convenient (but by no means absolutely necessary) to have a really infinite system and consider large regions in it. In this way the relative importance of the orders of magnitude is more easily taken into account.

The situation resembles that which arises when we decide to make free use of the irrational numbers even though we are aware that in no application shall we ever need an irrational number: everything could be done (and is done) using only rationals.

Let μ be a probability measure which describes the equilibrium state of an infinite Ising model at temperature-field (β, h). We shall be interested in investigating the distribution of the magnetization in large, macroscopically large, regions.

There are several ways of formulating the problem, depending on which are the questions really interesting us.

We can proceed as follows: divide $\mathbb{Z}^d$ into squares of side L. We parameterize these squares by indices $(\underline{n}, L)$:

9 $$(\underline{n}, L) = \big\{ \underline{\xi} \mid \underline{\xi} \in \mathbb{Z}^d,\ n_i L \le \xi_i < (n_i + 1)L,\ i = 1, 2, \ldots, d \big\} \tag{1.9}$$

where $\underline{n} = (n_1, \ldots, n_d)$ is a point in $\mathbb{Z}^d$ itself.

We then define the deviation from the total average magnetization of the box $(\underline{n}, L)$ divided by a normalization factor to be chosen later:

10 $$V_{\underline{n}} = \frac{\sum_{\underline{\xi} \in (\underline{n}, L)} \big(\sigma_{\underline{\xi}} - \langle \sigma_{\underline{\xi}} \rangle\big)}{L^{d\rho/2}} \tag{1.10}$$

where $0 < \rho$ has to be chosen. The variable $V_{\underline{n}}$ will be called a "block spin".

The new variables $(V_{\underline{n}})_{\underline{n} \in \mathbb{Z}^d}$ will have a probability distribution μ' which is obtained from the probability distribution of the original variables $(\sigma_{\underline{n}})_{\underline{n} \in ZZZ^d}$: this defines a transformation $K_{\rho,L}$:

11 $$\mu' = K_{\rho,L}(\mu) \tag{1.11}$$

It is clear that there is no need for the original variables to take only the values $\pm$. The transformation $K_{\rho,L}$ can be defined likewise, as we shall assume, if the variables $\sigma_{\underline{n}}$ can take arbitrary real values.

Then it makes sense to iterate $K_{\rho,L}$ and

12 $$K_{\rho,L}\big(K_{\rho,L}\big) = K_{\rho,L \cdot L'} \tag{1.12}$$

Given (β, h) and a corresponding equilibrium state μ of the Ising model, we now ask the question: *can we fix ρ so that the limit*

13 $$\lim_{L \to \infty} K_{\rho,L}(\mu) \tag{1.13}$$

exists and is non-trivial?

The existence and non triviality of the above limit can be defined to mean that there is a probability measure μ_∞ such that

14 $$\langle V_{\xi_1} \ldots V_{\xi_n} \rangle_{\mu_\infty} = \int \mu_\infty(d\underline{V}) V_{\xi_1} \ldots V_{\xi_n} = \lim_{L \to \infty} \langle V_{\xi_1} \ldots V_{\xi_n} \rangle_{K_{\rho,L}(\mu)} \tag{1.14}$$

with the obvious meaning of the symbols and, furthermore, μ_∞ is not the δ–function on the 0-configuration (*i.e.* $\langle V_{\xi_1} \ldots V_{\xi_n} \rangle_{\mu_\infty} \neq 0$ for some n and $\xi_1, \ldots \xi_n \in \mathbb{Z}^d$).

It is easy to convince oneself that if a value ρ is "good" (*i.e.* $K_{\rho,L}(\mu)$ converges to a nontrivial distribution) then any other value is not good.

The following basic result on the theory of the Ising model holds and is useful to clarify the meaning of the next theorem on the transformation $K_{\rho,L}$.

Theorem A: [1], *If $h \neq 0$ there is no phase transition for the model (β, h) and, furthermore, there is an upper bound for the decay of the correlation functions of the form:*

15 $$|\langle \sigma_{\xi_1} \ldots \sigma_{\xi_n} \rangle^T| \leq \frac{C}{n_1! \ldots n_\ell!} e^{-\kappa \Delta(x_1, \ldots, x_p)} \tag{1.15}$$

where $C, \kappa > 0$ and $n_1, \ldots, n_\ell$ are the multiplicities of the different sites among $x_1, \ldots, x_p$ and $\Delta(x_1, \ldots, x_p)$ = length of the shortest graph connecting $x_1 \ldots, x_p$ as a set in $\mathbb{R}^d$. The superscript T denotes "truncation".

The truncated correlation functions are

16 $$\begin{aligned} \langle \sigma_{x_1} \rangle^T =& \langle \sigma_{x_1} \rangle \\ \langle \sigma_{x_1} \sigma_{x_2} \rangle^T =& \langle \sigma_{x_1} \sigma_{x_2} \rangle - \langle \sigma_{x_1} \rangle \langle \sigma_{x_2} \rangle \\ \langle \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \rangle^T =& \langle \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \rangle - \langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \rangle - \langle \sigma_{x_1} \sigma_{x_3} \rangle \langle \sigma_{x_2} \rangle \\ & - \langle \sigma_{x_2} \sigma_{x_3} \rangle \langle \sigma_{x_1} \rangle + 2 \langle \sigma_{x_1} \rangle \langle \sigma_{x_2} \rangle \langle \sigma_{x_3} \rangle \\ \ldots =& \ldots \end{aligned} \tag{1.16}$$

The general truncated function is defined as follows: let f be a function on $\mathbb{Z}^d$ and put

17
$$Z_\Lambda(\beta, h+f) = \sum_{\underline{\sigma}} e^{-\beta H_\Lambda(\underline{\sigma}) - \beta \sum_{\xi\in\Lambda} f(\xi)\sigma_\xi} \tag{1.17}$$

i.e. Z_Λ is the partition function when the external field is modified from h to $h+f$. Then, provided $x_1, \ldots, x_n \subset \Lambda$:

18
$$\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_\Lambda^T = \frac{1}{\beta^n} \frac{\partial^n \log Z_\Lambda(\beta, h+f)}{\partial f(x_1) \ldots \partial f(x_n)}\Big|_{f=0} \tag{1.18}$$

Clearly $\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_\Lambda^T$ is a well defined combination of products of correlation functions with coefficients which are Λ-independent: hence it makes sense to define by the same combinations the truncated functions for the measure μ.

The following theorem completes theorem A:

Theorem B: [2], *If $h = 0$ and β is small enough, say $< \overline{\beta}$, there is no phase transition for the model $(\beta, 0)$ and the same estimate as above holds for the truncated functions.*

Furthermore there exists β_c, $\beta_c > \overline{\beta}$ such that there is a phase transition at $(\beta, 0)$ when $\beta > \beta_c$. If the dimension $d = 2$ it can be shown that the truncated functions decay exponentially although not necessarily with a bound like that in theorem A as far as the combinatorial factors are concerned.

Coming back to the block-spin distribution w can quote the following result:

Theorem C: *If (β, h) is such that $h \neq 0$ or $h = 0$ but $\beta < \overline{\beta}$ (or just $\beta < \beta_c$ if $d = 2$), the limit $\lim_{L\to\infty} K_{\rho,L}(\mu) = \mu_\infty$ exists for $\rho = 1$ and if*

19
$$\chi^2 \stackrel{def}{=} \sum_{\xi\in\mathbb{Z}^d} \big(\langle\sigma_O\sigma_\xi\rangle - \langle\sigma_O\rangle\langle\sigma_\xi\rangle\big) \tag{1.19}$$

it turns out

20
$$\mu_\infty\Big(\prod_{\xi\in\mathbb{Z}^d} dV_\xi\Big) = \prod_{\xi\in\mathbb{Z}^d} \frac{e^{-V_\xi^2/2\chi^2}}{\sqrt{2\pi\chi^2}}\, dV_\xi\,. \tag{1.20}$$

This theorem tells us that outside the region of phase transition the magnetization fluctuations on a large scale are independently distributed and the distribution of each of them is essentially the same as that relative to a set of independent spins (see next exercise).

Exercise: [3], *Prove that if $\mu(\prod_{\xi\in\mathbb{Z}^d} d\sigma_\xi) = \prod_{\xi\in\mathbb{Z}^d} f(\sigma_\xi)\, d\sigma_x$ with $f(\sigma)d\sigma$ is a probability distribution with finite second moment*

21
$$\alpha^2 = \int \sigma^2 f(\sigma) d\sigma \tag{1.21}$$

zero mean and finite third moment (i.e. $\int f(\sigma)\,\sigma\, d\sigma = 0$ and $\int f(\sigma)\,|\sigma|^3\, d\sigma < \infty$) then

22
$$\lim_{L\to\infty} K_{1,L}(\mu) = \mu_\infty = \prod_{\xi\in\mathbb{Z}^d} \frac{e^{-V_\xi^2/2\alpha^2}}{\sqrt{2\pi\alpha^2}}\, dV_\xi. \tag{1.22}$$

This is a simple instance of the well known central limit theorem and the hint for the proof is to analyze the characteristic functions or the Fourier transforms of the joint block spin distributions.

The physical reason for the above theorem is that away from the phase transition region of the parameters (β,h) the correlations of the equilibrium states decay exponentially fast and this essentially means that far spins are independently distributed around their average value and therefore large collections of them behave as a collection of independent random variables with values ± 1 and their distribution verifies the central limit theorem (see exercise above).

I shall try, in the next lectures, to sketch a proof of the above theorem: however I shall present it in such a way that it will be adaptable to the case when $\beta=\beta_c$, $h=0$ where the correlations between far spins decay more slowly than exponentially. To prepare the discussion about the block spin distributions at the critical point let me remind you of the conjectured properties of the equilibrium measures near the critical point $(\beta_c,0)$, see for instance [2].

Let us agree that the correlation functions of the equilibrium state (β,h) will bear the subscript (β,h) and c will simply replace $(\beta_c,0)$.

It is assumed, on the basis of the phenomenology of the critical point, that

(i) $\langle\sigma_0\sigma_{\xi_1}\ldots\sigma_{\xi_{2n-1}}\rangle_c \propto \chi_{2n}(0,\xi_1,\ldots,\xi_{2n-1})$, $n=1,2,3,\ldots$, where χ_{2n} is a *homogeneous* function of degree ω_{2n} of the coordinates $(\xi_1,\ldots,2n-1)$ regarded as points in $\mathbb{R}^d$:

23 $$\chi_{2n}(0,\frac{\xi_1}{\lambda},\ldots,\frac{\xi_{2n-1}}{\lambda})=\lambda^{\omega_{2n}}\,\chi_{2n}(0,\xi_1,\ldots,\xi_{2n-1})\qquad 0<\lambda\in\mathbb{R}\tag{1.23}$$

The symbol $\propto$ means that the sums over the sites $(\xi_1,\ldots,\xi_{2n-1})$ in $\mathbb{Z}^d$ such that $|\xi_i|<R$ agree to leading order in R as $R\to\infty$. If $n=1$

24 $$\chi_2(0,\xi)=\frac{C}{|\xi|^{\omega_2}}\tag{1.24}$$

and ω_2 is usually written as

25 $$\omega_2=d+\eta-2\tag{1.25}$$

(ii)

26 $$\langle\sigma_\xi\rangle_{(\beta_c,h)}\stackrel{def}{=}m(h)=m_0\,h^{\delta^{-1}}(1+O(h))\tag{1.26}$$

and here the formula is meant to signify that

27 $$\frac{d^n}{dh^n}m(h)=m_0\,(\frac{1}{\delta})(\frac{1}{\delta}-1)\ldots(\frac{1}{\delta}-n+1)h^{\delta^{-1}-n}(1+O(h))\tag{1.27}$$

(iii) There are homogeneous functions $(x_1,\ldots,x_n)\to\Delta(x_1,\ldots,x_n)$ of degree 1 on $(\mathbb{R}^d)^n$ such that, if $\Delta_0(x_1,\ldots,x_n)$ is the sum of the lengths of the lines of the shortest graph connecting $(x_1,\ldots,x_n)$, then

28 $$0<a\le\frac{\Delta(x_1,\ldots,x_n)}{\Delta_0(x_1,\ldots,x_n)}\le b<\infty\tag{1.28}$$

for some a,b and

29 $$\langle\sigma_0\sigma_{\xi_1}\ldots\sigma_{\xi_{2n-1}}\rangle_{(\beta_c,h)}\propto e^{-\Delta(0,\xi_1,\ldots,\xi_{2n-1})/L(h)}\chi_{2n}(0,\xi_1,\ldots,\xi_{2n-1})\tag{1.29}$$

(iv) Similarly if $\beta<\beta_c$

30 $$\langle\sigma_0\sigma_{\xi_1}\ldots\sigma_{\xi_{2n-1}}\rangle_{(\beta,0)}\propto e^{-\Delta(0,\xi_1,\ldots,\xi_{2n-1})/\lambda(\beta)}\chi_{2n}(0,\xi_1,\ldots,\xi_{2n-1})\tag{1.30}$$

The symbol $\propto$ in (iii), (iv), means that the sum of both sides over $\xi_1,\ldots,\xi_{2n-1}$ agree to leading order in h^{-1} or $(\beta-\beta_c)^{-1}$.

(v) There exist $\overline{\nu},\nu$ such that

31 $$L(h)=\ell_o\,h^{-\overline{\nu}},\qquad \lambda(\beta)=\lambda_0\,(\beta-\beta_c)^{-\nu}. \tag{1.31}$$

Because of assumptions (iii),(ii),(i) and the definition of the truncated functions it follows that:

32 $$\frac{d^{2n-1}}{dh^{2n-1}}m(\beta_c,h)=\sum_{\xi_1,\ldots,\xi_{2n-1}}\langle\sigma_0\sigma_{\xi_1}\ldots\sigma_{\xi_{2n-1}}\rangle^T_{(\beta_c,h)} \tag{1.32}$$

Hence there must be a simple relation between $\omega_{2n},\delta,\overline{\nu}$:

33 $$\overline{\nu}\,((2n+1)d-\omega_{2n+2})=2n+1-\delta^{-1} \tag{1.33}$$

The derivation of this formula is a useful exercise.

We can eliminate $\overline{\nu}$ in terms of the more familiar exponent η by using $\omega_2=d+\eta-2$ (by definition of η): it turns out

34 $$\overline{\nu}=\frac{\delta-1}{\delta}\,\frac{1}{2-\eta} \tag{1.34}$$

and also we can rewrite the formula for ω_{2n}:

35 $$\omega_{2n+2}=(2n+1)\,(d-(2-\eta)\frac{\delta}{\delta-1})+\frac{2-\eta}{\delta-1} \tag{1.35}$$

Assumption (i) for $n=1$ can be verified only in one interesting instance: in the two dimensional Ising model and when 0 and ξ_1 are on the same row, [4]. The theory of Kadanoff on the two dimensional Ising model would allow us to prove (i) when all points lie on the same row, [4].

Assumptions (ii) have never been verified except perhaps in the cases $n=0,1$ in $d=2$, [5]. Assumption (i) is a strict interpretation of the absence of characteristic length at $b=\beta_c,h=0$, while assumptions (iii),(iv), which seem the most audacious, give a very strict interpretation of the following loose but frequent statements which are the basic ingredients for many theories of the scaling laws: *there is only one correlation length near* $\beta=\beta_c,h=0$ and *within the correlation length the spins are are at criticality.* Of course (ii),(iv) are not the only possible interpretations of the above statements which are usually made (and used) only in connection with the single spin and pair correlation functions.

The way in which the correlation length is introduced in (iv) is inspired by recent rigorous results on the cluster property in the Ising model in the $h\ne0$ or in the high temperature region, [1,2].

II: Macroscopic fluctuation theory

We shall first outline the simple calculations, [2], necessary to derive theorem C and the we shall apply them to discuss the critical fluctuations.

Let $\underline{n}_1,\ldots,\underline{n}_s\in\mathbb{Z}^d$ and let $\underline{n}_i\ne\underline{n}_j$ for $i\ne j$ and

2.1 $$E(\omega_1,\ldots,\omega_s)=\int e^{i\sum_{k=1}^s\omega_k y_{\underline{n}_k}}\,d\mu_{L,\rho}(y) \tag{2.1}$$

where $\mu_{L,\rho}=K_{L,\rho}(\mu)$ and μ is the Ising model equilibrium state at values (β,h) of its parameters. This quantity is the Fourier transform of the joint distribution of s block spins.

By the definition of $Z_\Lambda(\beta,h+f)$, (1.17), we see immediately that

2.2
$$E_\Lambda(\omega_1,\ldots,\omega_s)=\frac{e^{-i\sum_{k=1}^s \frac{\omega_k}{L^{\rho d/2}}\sum_{\xi\in(\underline{n}_k,L)}\langle\sigma_\xi\rangle}\,Z_\Lambda(\beta,h+i\sum_{j=1}^s\frac{\omega_j\chi_j}{L^{\rho d/2}})}{Z_\Lambda(\beta,h)} \tag{2.2}$$

where E_Λ is the characteristic function of the block spins computed in the approximate equilibrium state obtained by regarding the system confined in a finite cube Λ (so $E_(\omega_1,\ldots,\omega_s)=\lim_{\Lambda\to\infty}E_\Lambda(\omega_1,\ldots,\omega_s)$); furthermore the functions χ_j are $\chi_j(\xi)=1$ if $\xi\in(\underline{n}_j,L)$, $j=1,2,\ldots,s$ and $\chi_j(\xi)=0$ otherwise. Therefore

2.3
$$E_\Lambda(\omega_1,\ldots,\omega_s)=e^{-i\sum_{k=1}^s \frac{\omega_k}{L^{\rho d/2}}\sum_{\xi\in(\underline{n}_k,L)}\langle\sigma_\xi\rangle}\exp\Big[\int_0^1 dt\frac{d}{dt}\log Z_\Lambda(\beta,h+ti\sum_{j=1}^s\frac{\omega_j\chi_j}{L^{\rho d/2}})\Big] \tag{2.3}$$

and, using the definition of the truncated functions to write the Taylor expansion of the derivative inside the above integral, we find after a straightforward computation:

2.4
$$E_\Lambda(\omega_1,\ldots,\omega_s)=\exp\Big[\sum_{\substack{k_1,\ldots,k_s\\ \sum k_i\ge2}} J^{(\Lambda)\,k_1\ldots k_s}_{\underline{n}_1\cdots\underline{n}_s}\frac{(i\omega_1)^{k_1}\ldots(i\omega_s)^{k_s}}{k_1!\ldots k_s!}\frac{1}{L^{\rho d\sum_j k_j/2}}\Big] \tag{2.4}$$

where

2.5
$$J^{(\Lambda)\,k_1\ldots k_s}_{\underline{n}_1\cdots\underline{n}_s}=\sum_{\substack{X_i\subset(\underline{n}_i,L),|X_i|=k_i\\ i=1,\ldots,s}}\langle\sigma_{X_1}\ldots\sigma_{X_s}\rangle_\Lambda \tag{2.5}$$

with X_i a subset of $(\underline{n}_i,L)$ with possibly repeated sites and σ_{X_1} shortens the product $\sigma_{\xi_1}\ldots\sigma_{\xi_\ell}$ if $X_i=(\xi_1,\ldots,\xi_\ell)$; the subscript Λ recalls that the correlation functions are relative to the finite volume equilibrium state.

Passing to the limit $\Lambda\to\infty$, formally, we find

2.6
$$E(\omega_1,\ldots,\omega_s)=\exp\Big[\sum_{\substack{k_1,\ldots,k_s\\ \sum k_i\ge2}} J^{k_1\ldots k_s}_{\underline{n}_1\cdots\underline{n}_s}\frac{(i\omega_1)^{k_1}\ldots(i\omega_s)^{k_s}}{k_1!\ldots k_s!}\frac{1}{L^{\rho d\sum_j k_j/2}}\Big] \tag{2.6}$$

where $J^{\cdots}_{\cdots}$ is defined as $J^{(\Lambda)\,\cdots}_{\cdots}$ above by eliminating the subscript Λ on the right hand side.

It is now easy to see that if $\sum_{X_1,\ldots,X_\ell}|\langle\sigma_0\sigma_{X_1}\ldots\sigma_{X_\ell}\rangle^T|<\infty$ then, by choosing $\rho=1$ the above series converges, term by term, to zero except the terms $k_j=2,k_i=0$ for $j=1,\ldots,s$ and $i\ne j$. And since

2.7
$$J^{2,0,0,\ldots,k_s}_{\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_s}=\sum_{x_1,x_2\in(\underline{n}_1,L)}(\langle\sigma_{x_1}\sigma_{x_2}\rangle-\langle\sigma_{x_1}\rangle\langle\sigma_{x_2}\rangle) \tag{2.7}$$

we see that

2.8
$$L^{-d}J^{2,0,0,\ldots,k_s}_{\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_s}\xrightarrow[L\to\infty]{}\sum_{x\in\mathbb{Z}^d}(\langle\sigma_0\sigma_x\rangle-\langle\sigma_0\rangle\langle\sigma_x\rangle)=\chi^2 \tag{2.8}$$

and therefore $\lim_{L\to\infty}E(\omega_1,\ldots,\omega_s)=e^{-\sum_{j=1}^s\chi^2\,\omega_i^2}$.

The problem of showing that convergence term by term implies actual convergence (as well as the problem of taking first the limit $\Lambda\to\infty$) are not too hard in he case in hand and are based on correlation inequalities or on analyticity arguments (see [2]). That the convergence of the Fourier transforms of probability distributions implies the convergence of the probability distributions is standard in probability theory, [3]. The above argument also shows that at the critical point the above calculations fail: in fact χ^2, which is physically proportional to the susceptibility, is infinite at $(\beta_c, 0)$.

However, if we accept the assumptions about the critical behavior of the last section, we realize that the above term by term argument can easily be made even at $\beta = \beta_c$ and we leave as an exercise to the reader to check that the result of the calculation, if ρ is suitably chosen as $\rho = 1 + (2-\eta)d^{-1}$ is:

2.9
$$\begin{aligned} E(\omega_1, \ldots, \omega_s) = \exp\Big(\sum_{0 \neq \sum_i k_i} & \frac{(i\omega_1)^{k_1}}{k_1!} \ldots \frac{(i\omega_s)^{k_s}}{k_s!} L^{(\frac{1}{2}\sum_i k_i - 1)\zeta} \cdot \\ & \cdot \int_{[\,\underline{n}_1]^{k_1} \times [\,\underline{n}_2]^{k_2} \times \ldots \times [\,\underline{n}_s]^{k_s}} \chi_{|\,\underline{k}\,|}(X_1, \ldots, X_s)\, d^{k_1}X_1 \ldots d^{k_s}X_s \Big) \end{aligned} \tag{2.9}$$

where $[\,\underline{n}\,]$ denotes the unit square of $\mathbb{R}^d$ centered around the point $\underline{n} \in \mathbb{R}^d$, $|\,\underline{k}\,| = \sum_i k_i$, $dX = \prod_{x \in X} d^d x$, and

2.10
$$\zeta = \frac{\delta + 1}{\delta - 1}\,(2 - \eta) - d \tag{2.10}$$

This formula is remarkable in view of the BUCKINGHAM–GUNTON scaling law, [5] ($\omega_{2n} = n\omega_2 = (d - 2 + \eta)n$ which yields $\zeta = 0$), which holds under assumptions weaker than (i)–(v) above: $\zeta \le 0$. This shows that with the chosen ρ the limit term by term exists and is *Gaussian* if $\zeta < 0$ (*i.e.* only the terms with $\sum_i k_1 = 2$ do not tend to zero), while all the terms may be non zero if $\zeta = 0$ and the limit distribution of the macroscopic magnetization is not Gaussian (unless infinitely many integrals vanish).

Notice that even in the Gaussian case terms like $k_1 = 1, k_2 = 1, k_3 = \ldots = k_s = 0$ may be now present (while away from the critical point they were zero in the limit $L \to \infty$): this means that the block spins may be not independently distributed.

Of course the above argument works term by term only: before trying to make a rigorous proof of the convergence, however, it seems more important to try to justify rigorously the assumptions although we did not try to do anything in this direction here.

Let me finish the argument by remarking that $\zeta = 0$ if $d = 2$, as it stems from the exact solutions, [6]. If $d = 3$ it is not clear how to interpret the numerical experiments, [7]. If $d = 4$ the exponent ζ is 0 in the mean field theory and it is not known if, in the true theory, the value of ζ is zero: it is widely believed that at $d = 4$ the critical exponent cannot be defined naively. One should in this case reformulate the whole discussion, allowing for an L-dependence of the various exponents (to take into account the so-called "logarithmic corrections").

III. Applications to field theory [8]

Euclidean field theory tries to construct a probability measure on the space of functions on $\mathbb{R}^d$ which is of the following formal form:

3.1
$$\mu(d\varphi) = C\,\big\{ \exp - \int (z_1 |\nabla\varphi(x)|^2 + z_3\varphi(x)^2 + z_2\varphi(x)^4)\, d^dx \big\}\, \text{“}d\varphi\text{”} \tag{3.1}$$

where $z_1, z_3 > 0$, $z_3 \in \mathbb{R}$.

This measure is subject to the requirement that

3.2
$$\begin{aligned} &\int_{\mathbb{R}^d} \langle \varphi(0)\varphi(x) \rangle\, d^dx = m^{-2} > 0 \\ &\int_{\mathbb{R}^d} \langle \varphi(0)\varphi(x) \rangle\, |x|^2\, d^dx = Z^2 > 0 \\ -&\int_{\mathbb{R}^d} \langle \varphi(0)\varphi(x)\varphi(y)\varphi(z) \rangle^T\, dx\, dy\, dz \;=\; \lambda^2 > 0 \end{aligned} \tag{3.2}$$

where the values m, z, λ are, respectively, the vales of the *renormalized* mass, wave function and coupling constant.[1]

The above theory should be constructed in a non-perturbative way by introducing infrared and ultraviolet cut-offs.

The infrared cut-off is introduced by confining the fields in a box Λ and the ultraviolet one by discretizing the space (for instance). So if L is the number of lattice spacings into which the side of Λ is divided, let a be the size of the lattice step. The volume of Λ is then $V(a,L) = a^d L^d$ and we shall also introduce the quantity

3.3
$$(aL)^{-d} \sum_{|n_1|,\ldots,|n_d| \le \frac{1}{2}L} |\,\underline{n}\,|^2 \;\propto\; a^2 L^2 \tag{3.3}$$

which measures the size of Λ. We shall denote by C_L the box in $\mathbb{Z}^d$ with side L. Then

3.4
$$\begin{aligned}&\int (z_1 |\nabla\varphi(x)|^2 + z_3\varphi(x)^2 + z_2\varphi(x)^4)\, d^d x\} \simeq\\ &\simeq a^d \sum_{\underline{n}\in C_L} (z_1 \sum_{\underline{e}} \frac{\varphi_{\underline{n}+\underline{e}} - \varphi_{\underline{n}}}{a^2} + z_3\varphi^2_{\underline{n}} + z_2\varphi^4_{\underline{n}})\end{aligned} \tag{3.4}$$

where $\underline{e}$ is a unit step along the lattice directions and $\varphi_{\underline{n}}$ is the value of the field $\varphi(\underline{n}\,a)$.

The above measure μ, to be constructed, is by definition "approximated" by the "regularized model measure":

3.5
$$\mu^{(L,a)}(\prod_{\underline{n}\in C_L} d\varphi_{\underline{n}}) = const\ \exp\big(\sum_{\substack{\underline{n},\underline{m}\in C_L\\ |\underline{n}-\underline{m}|=1}} \alpha_1\varphi_{\underline{n}}\varphi_{\underline{m}} - \sum_{\underline{n}\in C_L}(\alpha_2\varphi^4_{\underline{n}} - \alpha_3\varphi^2_{\underline{n}})\big) \prod_{\underline{n}\in C_L} d\varphi_{\underline{n}} \tag{3.5}$$

where $\alpha_1, \alpha_2, \alpha_3$ are simply related to z_1, z_2, z_3 and $z_1, z_2 > 0$ implies $\alpha_1, \alpha_2 > 0$.

The values of $\alpha_1, \alpha_2, \alpha_3$ are to be determined by imposing that the quantities y_1, y_2, y_3 defined as

3.6
$$\begin{aligned} y_1 &= a^d \sum_{\underline{n}\in C_L} \langle\varphi_{\underline{0}}\varphi_{\underline{n}}\rangle\\ y_2 &= a^{d+2} \sum_{\underline{n}\in C_L} \langle\varphi_{\underline{0}}\varphi_{\underline{n}}\rangle\, |\,\underline{n}\,|^2\\ y_3 &= -a^{3d} \sum_{\underline{m},\underline{p},\underline{q}\in C_L} \langle\varphi_{\underline{0}}\varphi_{\underline{m}}\varphi_{\underline{p}}\varphi_{\underline{q}}\rangle^T \end{aligned} \tag{3.6}$$

have the appropriate values m^{-2}, Z^2, λ, respectively.

The theory of correlation inequalities implies that $m^2, Z^2, \lambda \ge 0$, [8,9,10].

The question now is to find which are the triples (m^2, Z^2, λ), "allowed values" of the "renormalized constants", which can be fixed so that for all L, a there exist values $\alpha_1, \alpha_2, \alpha_3$, "bare constants"), which correspond to them via the(3.5),(3.6) formulae.

If (m^2, Z^2, λ) are allowed, then the correlation inequalities can be used, [8,10], to guarantee that, passing possibly to a subsequence of $L \to \infty$, $a \to 0$ so that $La \to \infty$ too, the measures $\mu^{(L.a)}$ converge (in some sense which we do not discuss here) to a limit measure with all correlation functions bounded in terms of the 2-fields correlation. This

[1] These names are not quite appropriate: for instance in the physical interpretation m^{-2} is not the inverse of the square of the physical mass of the "particles". It is, however, the sum of the "mass diagrams".

is a statement about renormalizability of the above φ^4 theory: it proves that by conveniently fixing the "bare" constants $\alpha_1, \alpha_2, \alpha_3$ s functions of the cut-offs L, a we can make all correlation functions uniformly bounded with respect to the value of the cut-offs.

The correlation inequalities of interest here are the LEBOWITZ-NEWMAN inequalities which bound the n–fields correlations in terms of the pair correlation: for more details see [8] and its references. The discussion of the allowed values of Z^2, λ, m^2 can be made as follows.

The boundary of the set in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ of the allowed values of $\alpha_1, \alpha_2, \alpha_3$ will contain the following four pieces

3.7
$$\begin{aligned} \mathcal{M}_1 &= \{\, \underline{\alpha} \,|\, \alpha_1 = 0\} \\ \mathcal{M}_2 &= \{\, \underline{\alpha} \,|\, \alpha_2 = 0\} \\ \mathcal{M}_3 &= \{\, \underline{\alpha} \,|\, \alpha_1 = +\infty\} \\ \mathcal{M}_4 &= \{\, \underline{\alpha} \,|\, \alpha_2 = +\infty\} \end{aligned} \tag{3.7}$$

Let $F : (\alpha_1, \alpha_2, \alpha_3) \to (y_1, y_2, y_3)$ be the map defined by the (3.5),(3.6) above. It is an easy task to find the set $F(\mathcal{M}_1) \cup F(\mathcal{M}_2) \cup F(\mathcal{M}_3) \cup F(\mathcal{M}_4)$: it will turn out that it is the boundary of a certain region in the space (y_1, y_2, y_3).

The conjecture is that this region is the region of the allowed values for (y_1, y_2, y_3). This is not obvious because the map F is not necessarily one-to-one, at least this is not known, nor it has necessarily an differentiable inverse; also one may wonder what happens to the other pieces of the boundary of $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ (*e.g.* $\alpha_3 = +\infty$).

We shall assume that the above region is the region of the allowed values and refer the reader to [8] for a deeper discussion of the "hidden" assumptions just mentioned.

To avoid talking about 3–dimensional sets we shall fix the value $y_1 = m^{-2}$ and we look at the intersection of $F(\mathcal{M}_1) \cup F(\mathcal{M}_2) \cup F(\mathcal{M}_3) \cup F(\mathcal{M}_4)$ with the plane $y_1 = m^{-2}$.

(i) *Study of* $F(\mathcal{M}_1)$: If $\alpha_1 = 0$ the variables $\varphi_{\underline{n}}$ are independent in the distribution $\mu^{(L,a)}$ and, therefore,

3.8
$$y_2 = 0, \quad y_1 = a^d \langle \varphi^2_{\underline{0}} \rangle = m^{-2}, \quad y_3 = -a^{3d}(\langle \varphi^4_{\underline{0}} \rangle - 3\langle \varphi^2_{\underline{0}} \rangle^2) \tag{3.8}$$

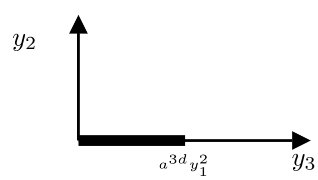

Fig.1: The boundary $F(\mathcal{M}_1)$.

and the value of y_3 varies between the values of the variable

3.9
$$y = -a^{3d}\Big(\frac{\int e^{-\alpha_2\varphi^4 + \alpha_3\varphi^2} \varphi^4 \, d\varphi}{\int e^{-\alpha_2\varphi^4 + \alpha_3\varphi^2} \, d\varphi} - 3y_1^2\Big) \tag{3.9}$$

where α_2, α_3 vary in such a way as to keep

3.10
$$y_1 = \frac{\int e^{-\alpha_2\varphi^4 + \alpha_3\varphi^2} \varphi^2 \, d\varphi}{\int e^{-\alpha_2\varphi^4 + \alpha_3\varphi^2} \, d\varphi} = m^{-2} \tag{3.10}$$

The minimum is $y_3 = 0$ (when $\alpha_2 = 0, \alpha_3 < 0$) and the maximum is $2y_1^2 a^{3d}$ which is realized when $\alpha_3, \alpha_2 \to \infty$ in such a way that the distribution *const* $e^{\alpha_3\varphi^2 - \alpha_2\varphi^4}$ approaches $\frac{1}{2}(\delta(\varphi - \sqrt{y_1}) + \delta(\varphi + \sqrt{y_1}))$.

(ii) On $\mathcal{M}_2$ the measure $\mu^{(a,L)}$ is a Gaussian distribution: hence the four point truncated correlations vanish and $y_3 \equiv 0$. If the matrix $G_{\underline{n}, \underline{m}} = -\alpha_1 \delta_{|\underline{n} - \underline{m}|, 1} + \alpha_3 \, \delta_{\underline{n}, \underline{m}}$, $\underline{n}, \underline{m} \in C_L$ has $g_{\underline{n}, \underline{m}}$ as inverse matrix then

3.11 $$y_1 = a^d \sum_{\underline{n}\in C_L} g_{\underline{0},\underline{n}}, \qquad y_2 = a^{d+2} \sum_{\underline{n}\in C_L} g_{\underline{0},\underline{n}} \,|\underline{n}|^2. \tag{3.11}$$

Notice that this makes sense only if $-\alpha_3 > d\,\alpha_1$: in the other cases what follows should be properly interpreted by fixing $\alpha_2 > 0$ and the letting it tend to 0

The extreme cases at fixed y_1 are

3.12 $$y_2 = 0, \qquad \text{(independent fields)}, \quad (\alpha_1 = 0) \tag{3.12}$$

or $g_{\underline{n},\underline{m}} = const$ corresponding to $\alpha_1 = +\infty$ (meaning that all fields have the same value: *i.e.* are *coherent*), which gives $y_1 = a^d L^d const$ and $y_2 = a^{d+2} const \sum_{\underline{n}} |\underline{n}|^2$, *i.e.* setting $D(L,a) = a^2 L^{-d} \sum_{\underline{n}} |\underline{n}|^2$

3.13 $$y_2 = y_1 D(L,a), \qquad \text{(coherent fields)}, \quad (\alpha_1 = +\infty) \tag{3.13}$$

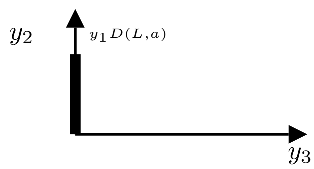

Fig.2: The boundary $F(\mathcal{M}_2)$.

(iii) On $\mathcal{M}_3$ the $\alpha = +\infty$ means that all fields are coherent so that again $y_2 \equiv y_1 D(L,a)$ while y_3 varies from 0 (Gaussian case with $\alpha_2 = 0$) to the limiting case when the coherent value of the field is distributed as $\frac{1}{2}\big(\delta(\varphi - \sqrt{\langle\varphi^2\rangle}) + \delta(\varphi + \sqrt{\langle\varphi^2\rangle})\big)$ when

3.14 $$y_3 = 2 y_1^2 a^d L^d. \tag{3.14}$$

(iv) This part of the boundary is obtained by letting $\alpha_2 \to +\infty$: however, to get a well defined limit, we need to tie α_2 and α_3 so that the ratio $\alpha_3/2\alpha_2 \stackrel{def}{=} \gamma^2$ is kept constant and > 0. In this case $e^{-\alpha_2 \varphi^4_{\underline{n}} - \alpha_3 \varphi^2_{\underline{n}}}$ is replaced by

3.15 $$\frac{1}{2}\big(\delta(\varphi_{\underline{n}} - \gamma) + \delta(\varphi_{\underline{n}} + \gamma)\big) \tag{3.15}$$

leading to an Ising model at inverse temperature $\alpha_1\gamma^2$, and

3.16 $$\begin{aligned} y_1 &= a^d \gamma^2 \sum_{\underline{n}\in C_L} \langle \sigma_{\underline{0}}\, \sigma_{\underline{n}} \rangle_{(\alpha_1\gamma^2,0)} \\ y_2 &= a^{d+2}\gamma^2 \sum_{\underline{n}\in C_L} \langle \sigma_{\underline{0}}\, \sigma_{\underline{n}} \rangle_{(\alpha_1\gamma^2,0)} \,|\underline{n}|^2 \\ y_3 &= -a^{3d}\gamma^4 \sum_{\underline{n},\underline{m},\underline{p}\in C_L} \langle \sigma_{\underline{0}}\, \sigma_{\underline{n}}\, \sigma_{\underline{m}}\, \sigma_{\underline{p}} \rangle^T_{(\alpha_1\gamma^2,0)} \end{aligned} \tag{3.16}$$

So the final picture of the (conjectured) allowed region that we get is

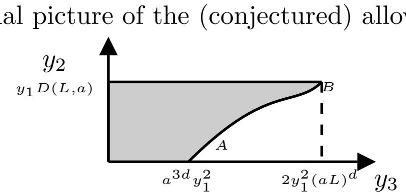

Fig.3: The boundaries of $F(\mathcal{M}_3)$ (horizontal) and of $F(\mathcal{M}_4)$ (from B to A).

where the equation of the AB line is given parametrically in terms of $\beta \stackrel{def}{=} \alpha_1\gamma^2 \in (0,+\infty)$:

3.17

$$\begin{aligned}\frac{y_2}{y_1} =& \frac{a^2 \sum_{\underline{n}\in C_L} \langle \sigma_{\underline{0}}\,\sigma_{\underline{n}}\rangle_{(\beta,0)}\,|\,\underline{n}\,|^2}{\sum_{\underline{n}\in C_L}\langle \sigma_{\underline{0}}\,\sigma_{\underline{n}}\rangle_{(\beta,0)}}\\ \frac{y_3}{y_1^2} =& -\frac{a^d \sum_{\underline{n},\underline{m},\underline{p}\in C_L}\langle \sigma_{\underline{0}}\,\sigma_{\underline{n}}\,\sigma_{\underline{m}}\,\sigma_{\underline{p}}\rangle^T_{(\beta,0)}}{(\sum_{\underline{n}\in C_L}\langle \sigma_{\underline{0}}\,\sigma_{\underline{n}}\rangle_{(\beta,0)})^2}\end{aligned} \tag{3.17}$$

We can also easily see that, when $a\to 0, L\to\infty$ (and $aL\to\infty$), the part of the curve which comes from the interval $|\beta-\beta_c|>\varepsilon>0$, for any $\varepsilon>0$ fixed (*before* taking the limit $L\to\infty, a\to 0$) collapse either into the lowest point A ($\beta<\beta_c$) or into the point B ($\beta>\beta_c$): an exercize (*hint: use the exponential clustering in the non critical Ising model*).

Therefore, removing the cut-offs, we realize that the whole curve is determined by the critical behavior of the Ising model!

It is even more remarkable that the assumptions about the critical behavior permit us to write down axplicitly the equation of "half" of the curve AB in the limit $a\to 0, L\to\infty$. We can proceed as follows. We consider only the case $\beta<\beta_c$ because we did not describe the critical behavior for $\beta>\beta_c$. The description of it along the same lines would lead to the other half of AB.

Applying the assumption made earlier we find, for $\beta>\beta_c$

3.18

$$\begin{aligned}\sum_{\underline{n}\in C_L}\langle \sigma_{\underline{0}}\,\sigma_{\underline{n}}\rangle_{(\beta,0)} \propto& \sum_{\underline{n}\in C_L} c_2\frac{e^{-\varepsilon(\beta)|\,\underline{n}\,|}}{|\,\underline{n}\,|^{d-2+\eta}}\\ \sum_{\underline{n}\in C_L}\langle \sigma_{\underline{0}}\,\sigma_{\underline{n}}\rangle_{(\beta,0)}\,|\,\underline{n}\,|^2 \propto& \sum_{\underline{n}\in C_L} c_2\frac{e^{-\varepsilon(\beta)|\,\underline{n}\,|}}{|\,\underline{n}\,|^{d-4+\eta}}\\ \sum_{\underline{n},\underline{m},\underline{p}\in C_L}\langle \sigma_{\underline{0}}\,\sigma_{\underline{n}}\,\sigma_{\underline{m}}\,\sigma_{\underline{p}}\rangle^T_{(\beta,0)} \propto& \sum_{\underline{n},\underline{m},\underline{p}\in C_L} e^{-\varepsilon(\beta)\Delta(\,\underline{0}\,,\,\underline{n}\,,\,\underline{m}\,,\,\underline{p}\,)}\,\chi_4(\,\underline{0}\,,\,\underline{n}\,,\,\underline{m}\,,\,\underline{p}\,).\end{aligned} \tag{3.18}$$

Furthermore $\varepsilon(\beta)=\kappa_0\,(\beta_c-\beta)^\nu$, hence

3.19

$$y_2 \underset{a\to 0}{\propto} y_1\, \frac{\int_{|x|\le La} e^{-\kappa_0\xi|x|}\frac{d^dx}{|x|^{d+\eta-4}}}{\int_{|x|\le La} e^{-\kappa_0\xi|x|}\frac{d^dx}{|x|^{d+\eta-2}}} \tag{3.19}$$

where ξ is a new name for $\xi \stackrel{def}{=} a^{-1}|\beta-\beta_c|$, and

3.20

$$y_3 \underset{a\to 0}{\propto} \frac{y_1^2\, a^{\omega_4-2(d+\eta-2)}\,\int_{|x|\le La} e^{-\kappa_0\xi\Delta(0,x_1,x_2,x_3)}\chi_4(0,x_1,x_2,x_3)dx_1\,dx_2\,dx_3}{\big(\int_{|x|\le La} e^{-\kappa_0\xi|x|}\frac{d^dx}{|x|^{d+\eta-2}}\big)^2} \tag{3.20}$$

which, as ξ varies between 0 and ∞ parameterizes the part AB of the curve, associated with $\beta-\beta_c<0$.

Notice that

3.21

$$\omega_4 - 2(d+\eta-2) = d-(2-\eta)\frac{\delta-1}{\delta+1} = -\zeta \tag{3.21}$$

hence if $\zeta<0$, *i.e.* if the Buckingham–Gunton scaling law holds with inequality sign (which we recall corresponded to a Gaussian distribution of the block spins at the cricial point), we find that the curve AB collapses into the line $y_3=0$ (*i.e. zero* *renormalized coupling constant* or *trivial theory*). If $\zeta = 0$ the line AB stays non trivial when $a \to 0, aL \to \infty$ and this indicates that non trivial theories exist in such case.

To reach really the above conclusions one should perform a similar analysis of the other part of the curve AB corresponding to $\beta > \beta_c$ (as already mentioned). This is not yet done and we shjall not enter into this problem.

It is quite clear that the above discussion can be used to approximate in terms of the critical properties of the Ising model the field theories wose renormalized parameters lie close to the AB line whose equation was just derived.

The discussion just made shows in a very concrete way the deep connection between two very interesting and unsolved problems: the critical point in the Ising model and the relativistic quantum field theory of type $z_3\varphi^4$.

IV. Remarks

(1) If $\zeta = 0$ the critical point recedes to ∞ in the cut-off removal process. So the allowed region of (y_1, y_2, y_3) depends only on the critical region where $\beta < \beta_c$ (high temperature side). This is probably due to the fact that by imposing $\int \langle\varphi(0)\varphi(x)\rangle\, dx < \infty$ we have excluded spontaneous symmetry breakdown, *i.e.* the possibility that $\langle\varphi(0)\varphi(x)\rangle \to m^2 > 0$. A similar theory to the one developed above should be possible allowing a symmetry breakdown: this is an interesting question.

In such a theory the "allowed region" will probably be determined by the critical region where $\beta > \beta_c$ (low temperature side).

(2) The above theory assumes perfect scaling: it would be interesting to see what would happen if scaling is not perfect (*i.e.* logarithmic corrections or more length scales are allowed).

(3) $\lambda\varphi^4$ field theory is known to exist in $d = 1, 2, 3$. It is an open problem to derive from this fact information about the critical behavior of the $d = 2, 3$ Ising model (*converse problem*).

(4) Let us sketch the renormalizability question. Let f be a smooth function with compact support on $\mathbb{R}^d$. Define

4.1
$$\varphi(f) \stackrel{def}{=} a^d \sum_{\underline{n}\in C_L} f(\underline{n}\, a)\varphi_{\underline{n}} \tag{4.1}$$

then, since $\alpha_1, \alpha_2 > 0$ the Newman inequalities, [10], *i.e.*

4.2
$$\langle\varphi(f)^{2n}\rangle \le (2n+)!!\langle\varphi(f)^2\rangle^n, \qquad \forall L, a \tag{4.2}$$

but

4.3
$$\begin{aligned}\langle\varphi(f)^2\rangle &= a^{2d} \sum_{\underline{n}\,,\,\underline{m}\in C_L} f(\underline{m}\, a) f(\underline{n}\, a)\langle\varphi_{\underline{n}}\varphi_{\underline{m}}\rangle \le\\ &\le (\max|f|)a^d \sum_{\underline{n}\in C_L} |f(\underline{n}\, a)| \big(a^d \sum_{\underline{m}}\langle\varphi_{\underline{n}}\varphi_{\underline{m}}\rangle\big) \propto_{\substack{a\to 0\\ aL\to\infty}}\\ &\propto_{\substack{a\to 0\\ aL\to\infty}} (\max|f|)(\int_{\mathbb{R}^d} |f(x)|\, dx) < \infty\end{aligned} \tag{4.3}$$

hence all Schwinger functions make sense and are uniformly bounded as $a \to 0, L \to \infty$ (at least if tested with smooth functions). So renormalization is achieved by fixing $0 < y_1 < \infty$.

(4) Exercise: derive the equation of the curve AB when $d = 1$ (*hint*: if $\xi_1 < \xi_2 < \ldots < \xi_{2n}$ the exact solution of the 1-dimensional Ising model is

4.5
$$\begin{aligned} &(a) \qquad \langle\sigma_{\xi_1}\ldots\sigma_{\xi_{2n}}\rangle = \langle\sigma_{x_1}\sigma_{\xi_2}\rangle\langle\sigma_{x_3}\sigma_{\xi_4}\rangle\ldots\langle\sigma_{x_{2n-1}}\sigma_{\xi_{2n}}\rangle \\ &(b) \qquad \langle\sigma_{\xi_1}\ldots\sigma_{\xi_{2n}}\rangle = (\tanh\beta)^{|\xi_1-\xi_2|} \end{aligned} \tag{4.4}$$

The result is $\frac{y_2}{y_3^2} = \frac{1}{18}y_1^3$.

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