LECTURES ON THE BILLIARD

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Abstract

These lectures together with the appendices contain all the essential steps for a proof of the ergodicity of the dynamical system described below as (M,T,v) or "periodic billiard's section" when the obstacles are so situated that no collisionless paths are possible.

1. Introduction

A point mass moves, without friction, on a table; the particle is either elastically reflected by the boundary of the table (reflecting table) or, upon hitting the boundary, disappears to reappear on the opposite side with the same velocity (toric table).

The surface of the table contains some obstacles: upon collision with the obstacles the particle's velocity changes its direction according to the laws of elastic reflection (equal angle of impact and departure). The motion of the particle is clearly very complicated and we shall deal with the problem of discussing some of its qualitative features.

This problem hardly needs motivation: many authors have already noticed its interest and relevance for many applications.

Here we shall content ourselves with mentioning its relation to the Lorentz model.

The model consists of a particle that moves on a plane covered by randomly thrown obstacles. The quantity of interest is the value of the square of the distance $r(t)$, from a starting point, of a particle moving with unit speed on the plane and elastically colliding with the obstacles

We denote the quantity by $r(t)^2$ and define $D(t)$ as

$$t\, D(t) = E(r(t)^2)$$

where the average has to be taken with respect to the distribution of the random obstacles.

All the attempts to obtain rigorous estimates for D(t) have, so far, failed even for the simplest obstacles' distribution. There is, however, considerable work done on this problem by theoretical physicists and nonrigorous results as well as computer experiments are available.

For instance, assuming that the obstacles are distributed according to the Gibbs' distribution for a hard discs gas, it is believed that

$$\lim_{t\to\infty} D(t) = D_\rho < +\infty$$

where D_ρ is a value which depends on the density ρ of the distribution of the hard discs and $D_\rho > 0$ if ρ is small enough.

It is also conjectured that $\rho\, D_\rho$ is not analytic near $\rho = 0$. It can be seen, at least formally, that the value $D(t)$ is given by

$$D(t) = t^{-1} \int_0^1 d\tau \int_0^\tau E(\underline{v}(0)\cdot\underline{v}(t'))\, dt' = t^{-1} \int_0^t \tilde{D}(\tau)\, d\tau$$

where $\underline{v}(0)$ denotes the initial velocity of the test particle and $\underline{v}(t)$ its velocity at time t; the average is over the obstacles' distribution.

The above formula should make clear that the real problem one is facing is to control, in some sense, the average behavior of a trajectory in the long time region: for fixed time it is clear that the integrand in the above formula is analytic in the density ρ of the scatterers near $\rho = 0$ (at least if the scatterers are Poisson distributed with density ρ).

A similar but simpler problem can be obtained by assuming that the obstacles are periodically and not randomly distributed: so we have a nonrandom lattice structure for the centers of the obstacles and we look for $E(r(t)^2)$ or $E(\underline{v}(0) \cdot \underline{v}(t))$, where the average is, now, over the possible initial unit velocities and position in the unit cell of the obstacles' lattice. It is assumed that the initial positions are equally distributed in the "free" region of a basic cell of the lattice, while the velocity is homogeneously distributed over the angles.

The above problem is clearly a problem concerning a toric billiard and the statement that the toric billiard is ergodic implies, among other things that

$$t^{-1}\int_0^t E(\underline{v}(0)\cdot v(t'))\,dt' \xrightarrow[t\to\infty]{} 0$$

which means that $D(t)$ does not grow as fast as t in this simplified version of the Lorentz model.

On the other hand there are no more general results on $D(t)$ even in this simple case and, amazingly, the problem is still completely open.

We shall be mainly concerned in an exposition of the main steps of the long proof of the basic ergodic properties of toric billiards.

2. Geometry.

Z^2 is a square lattice in R^2 and its basic unit cell contains s inequivalent points: therefore Z^2 is divided into s sublattices of equivalent points. Each point of Z^2 is covered by an obstacle with a shape and orientation and relative position which is the same for all the obstacles covering equivalent points. The obstacles are assumed to be open strictly convex sets with a C^3-boundary furthermore their closures are supposed to be mutually disjoint.

We can regard, in a natural way, the space R^2 covered by the set O of the obstacles as a torus Q with obstacles. A particle, moving in R^2/O freely except when it collides elastically on the surface ∂O, can be thought of as moving on the torus Q.

We call "phase space" the manifold V (with boundary) consisting in the points (q,θ), $q \in Q/O$, $\theta \in [0,2\pi]$: q will represent the position of the "billiard ball" and θ the angle between its unit velocity and a fixed axis.

So V is a 3-dimensional closed manifold with boundary $\partial V = \{(q,\theta) \mid q \in \partial O\}$. V consists of "arrows".

If $x = (q,\theta) \in \partial V$ we call x "colliding" if the angle $\phi(x)$ which x forms with the outer normal to ∂O in q is between $\pi/2$ and $3\pi/2$ (counted counterclockwise). The points of the set $M = \{x \mid x \in \partial V_0 , x \text{ collides}\}$ can be parametrized by three numbers: if $x = (q,\theta) \in \partial V$ then $x = (i,r,\phi)$ where

(1) i is an index, $i = 1,2,\ldots,s$, denoting the obstacle O_i of O to which q belongs.

(2) r is a coordinate measuring the curvilinear abscissa on ∂O_i of $q \in \partial O_i$ counted, clockwise, from a fixed point of ∂O_i. If L_i = length of ∂O_i then two points on ∂O_i whose abscissae differ by L_i are identical.

(3) ϕ is an angle, $\pi/2 \leq \phi \leq 3\pi/2$, measuring the angle between the arrow x and the outer normal to ∂O_i in q, measured counterclockwise.

Therefore M splits naturally into a disjoint union of s cylinders $M_1, M_2, \ldots, M_s$. The billiards flow S_t: $V \to V$ is defined in the obvious way and conserves the measure

$$\mu(dq\, d\theta) = \frac{dq\, d\theta}{\int_V dq\, d\theta}$$

This completes the description of the billiards' dynamical system (V,S_t,μ). The system (V,S_t,μ) can also be realigned as a "special flow" in a natural way which, in some sense (see below), isolates the collisions as the main object of interest.

Let $M = \{x \mid x \in \partial V, x \text{ collides}\}$ be parametrized as above and let $T: M \to M$ be the transformation which maps $x \in M$ into the collision $Tx \in M$ which precedes it and let $\tau(x) < 0$ be the time interval between x and Tx: our assumptions imply that there is a lower bound $\tau_0 > 0$ to the flight time $-\tau(x)$.

If $y \in V$ we can clearly use as one of the coordinates of y the time $\tau(y) \leq 0$ which elapses between the instant the particle is in y and the last former collision $x(y) \in M$ on the past trajectory of y. Then y is uniquely determined, outside a set of μ-measure zero, by $(x(y),\tau(y))$ i.e. by a set (i,r,ϕ,t), $i = 1,2,\ldots,s$, $0 \leq r \leq L_i$, $\pi/2 \leq \phi \leq 3\pi/2$, $0 \geq t \geq \tau(T^{-1}x(y))$.

It is easy to check that for $t \geq 0$:

$$S_{-t}y \equiv S_{-t}(x(y),\tau(y)) = (x(y),\tau(y)+t)$$

if $\tau(y)+t \leq 0$ and

$$S_{-t}y = (Tx(y),\ t+\tau(y)+\tau(x(y)))$$

if $\tau(y) + t > 0$ but $\tau(y) + t + \tau(x(y)) < 0$.

Furthermore simple trigonometric considerations show that

$$\left\| \frac{\partial(q,\theta)}{\partial(r,\phi,t)} \right\| = -\cos\phi$$

so that

$$\mu(dr\ d\phi\ dt) = \frac{-\cos\phi\ dr\ d\phi\ dt}{\text{normalization}}$$

The S_t invariance of μ implies then the T-invariance of the measure ν on M defined by

$$\nu(dz\ d\phi) = \frac{-\cos\phi\ dz\ d\phi}{\text{normalization}} = \frac{-\cos\phi\ dz\ d\phi}{2\sum_i L_i}$$

We are therefore led to the dynamical system (M,T,ν) which, above the function $\tau(x)$ generates (V,S_t,μ) as a special flow [1], [2].

We shall discuss the proof that the section (M,T,ν) is ergodic [1], [2]: it can actually be proved further. Namely it can be proved that (M,T,ν) and (V,S_t,μ) are, respectively, a Bernoulli automorphism and a Bernoulli flow: the proof of such properties will not be considered here [12, [2], [3].

To proceed let us study the geometry of the collisions. The reader is advised to spend some time in checking the geometrical meaning of the statements below in the case of a billiard with only one circular obstacle and with the help of the geometric relations derived in the next lecture.

First we observe that T is not smooth and the space M is divided by the singularity lines into several disconnected parts on each of which T acts (very) smoothly (i.e. where T is as many times differentiable as the boundaries ∂O), see Figure 1.

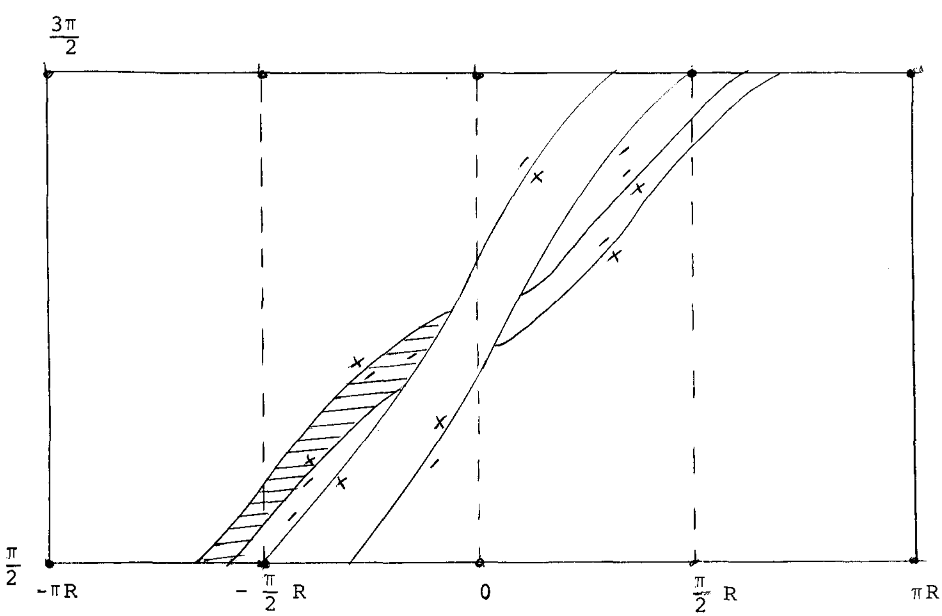


Figure 1

It is easy to see that the singularity set of T (which consist in the set $\partial M \cup T^{-1}\partial M$) which is outside ∂M consists in several (finitely many) connected disjoint sets. Each connected part can be thought of as a union of smooth "singularity lines" (at most in countable number); more precisely it consists in at least one line Σ_0, smooth and whose equation $\phi = \phi(r)$ monotonically increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, and in several other monotonic smooth lines merging into each other. A closer investigation (cf. next section) shows that the slope $d\phi/dr$ of the equation of a singularity line is bounded away from zero and ∞ and, furthermore, if x is a point in which two singularity lines merge ("confluence" point) the angle α between the tangents of the two lines at x is strictly positive. When a connected family of discontinuity lines is denumerable then the singularity lines accumulate at the intersection of Σ_0 with $\phi = \pi/2$ or $3\pi/2$ and nowhere else.

The singularity lines have "two sides": the side on which T is continuous is denoted +, the other -.

The accumulation points discussed above are points $x \in \partial M$ such that $\tau(y)$ is unbounded in the neighborhood of x.

As an example of the above structure we draw a few of the singularity lines of the periodic billiards in which there is only one obstacle of circular shape with radius R large enough (compared to the side of the torus). In this case there are 8 denumerable families of discontinuity lines. The curves of each family accumulate around one of the 8 points which are marked in the picture.

The 8 points marked in the picture correspond to the eight possible ways a particle can "escape" to ∞ without colliding. We shall restrict our attention to the case of billiards in which there is no free straight path to ∞: such billiards have only finitely many discontinuity lines and finitely confluence points in which they intersect.

The reason for this restriction is twofold: these billiards contain a large part of the general problem and mainly only leave out technical difficulties which tend to obscure the ideas of the proofs, on the other hand there seems to be no detailed treatment of the additional difficulties, present in the more general billiards, in the published literature.

To get an idea of the transformation T it is useful to see the image under T of the region shaded in Figure 1. It is easy to see that such an image has a shape which is a mirror image, around $\phi = \pi$, of the initial region. However the images of the short sides are the long sides and vice versa: i.e., the transformation T stretches in one direction the region under investigation and compresses it in another direction. This happens for all the regions in which the singularity lines partition M.

Let $\mathcal{P}$ be this partition: if we remember that the slopes of the "long sides" of the atoms of $\mathcal{P}$ have a slope away from zero and infinity, we can remark that the atoms of $T\mathcal{P}$ have a slope transversal to that of the atoms of $\mathcal{P}$. This means that the atoms of $T\mathcal{P}$ will be cut by singularity lines of T, and therefore, split upon action of T and subjected to a new stretching and contracting. Thus the transformation T acts on $\mathcal{P}$ in a way which vaguely resembles the baker's transformation, at least locally.

To investigate in more detail the action of T is is natural to study its action on smooth curves γ which are cut by singularity lines.

Let $\phi = \phi(r)$, $\bar{r} \leq r \leq \bar{\bar{r}}$, define a smooth ($C^1$) curve in M. And assume that γ is not cut by the singularity lines of T.

So the image $T\gamma$ will be a smooth curve, since T is a diffeomorphism in the complement of the singularity lines.

From the picture it is easy to obtain a differential relation between the functions $\phi = \phi(r)$ and $\phi' = \phi'(r')$ representing the curves γ and $T\gamma$ respectively.

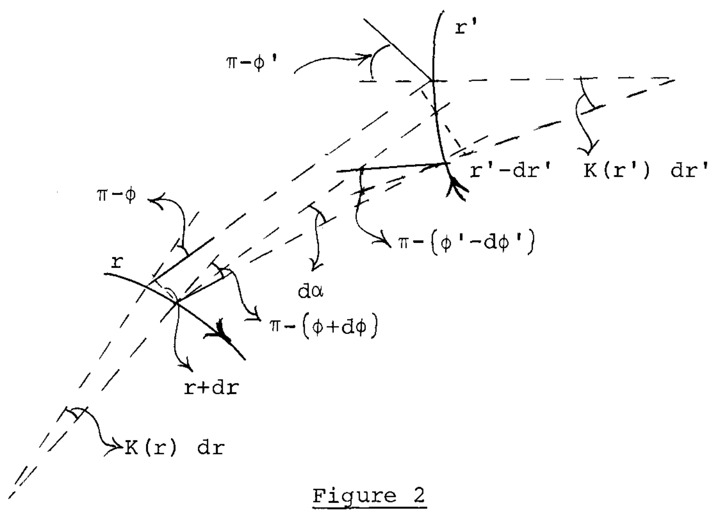


Figure 2

It is simple to deduce from the picture in Figure 2 the following relations:

$$-\cos\phi\ dr + d\alpha(-\tau(r,\phi)) = -\cos\phi'\ dr'$$

$$d\alpha = K(r)\ dr - d\phi$$

$$d\alpha = -K(r')\ dr' - d\phi'$$

By eliminating $d\alpha$ we can deduce:

$$\frac{-\cos\phi'\ dr'}{-\cos\phi\ \ dr} = \left(1 + (K(z) - \frac{d\phi}{dr})\ \frac{\tau(r,\phi)}{\cos\phi}\right) = \left(1+(K(r')+\frac{d\phi'}{dr'})\frac{\tau(r,\phi)}{\cos\phi'}\right)^{-1}$$

$$\frac{1}{-\cos\phi'}\ \frac{d\phi'}{dr'} = \frac{K(r')}{\cos\phi'} + \frac{1}{\tau(r,\phi) + \cfrac{1}{\cfrac{1}{-\cos\phi}\left(\cfrac{d\phi}{dr} - K(r)\right)}}$$

$$\frac{1}{\cos\phi}\ \ \frac{d\phi}{dr} = \frac{K(r)}{\cos\phi} + \frac{1}{\tau(r,\phi) + \cfrac{1}{\cfrac{1}{\cos\phi'}\left(\cfrac{d\phi'}{dr'} + K(r')\right)}}$$

Remarks

(1) If γ is a decreasing curve with equation $\phi = \phi(r)$, $\frac{d\phi}{dr} \leq 0$, then $\gamma' = T\gamma$ is also decreasing and

$$R_+^{-1} \leq -\frac{d\phi'}{dr'} \leq R_-^{-1} + \tau_0^{-1}$$

where $R_+^{-1} = \min K(r)$, $R_-^{-1} = \max K(r)$, $\tau_0 = \min |\tau(x)|$.

(2) Similarly if γ is an increasing curve then the curve $T^{-1}\gamma = \gamma'$ is also increasing and

$$R_+^{-1} \leq \frac{d\phi'}{dr'} \leq R_-^{-1} + \tau_0^{-1}$$

(3) If γ and $T\gamma$ are both decreasing

$$\frac{\cos\phi \quad dr}{-\cos\phi' \ dr'} \leq \left(1 + \frac{\tau_0}{R_+}\right)^{-1} \equiv \lambda_0^{-1}$$

where λ_0 is defined by the last equation

(4) If $\gamma' = T^{-1}\gamma$ and γ are both increasing

$$\frac{-\cos\phi' \ dr'}{\cos\phi \quad dr} \geq \lambda_0$$

It is therefore natural to introduce the notion of p-length of a curve γ,

$$p(\gamma) = -\int_\gamma \cos\phi(r)\, dr$$

as well as the notion of r-length

$$r(\gamma) = \int_\gamma dr$$

(3') If γ, $T^{-1}\gamma$, ..., $T^{-k}\gamma$ is a sequence of smooth curves such that $T^{-k}\gamma$ is decreasing (so that $T^{-i}\gamma$ is decreasing for all $0 \leq i \leq k$) then

$$p(T^{-i}\gamma) \leq \lambda_0^{-1}\, p(T^{-i+1}\gamma) \leq \cdots \leq \lambda_0^{-i}\, p(\gamma)$$

$$r(T^{-i}\gamma) \leq \frac{\lambda_0^{-(i-1)}}{\lambda_0^{-1}}\, r(\gamma) \ ,$$

$i = 0,1,\ldots,k$.

(4') If $\gamma, T\gamma, \ldots, T^k\gamma$ is a sequence of smooth curves such that $T^k\gamma$ is increasing (so that $T^i\gamma$ increases for $0 \le i \le k$) then

$$p(T^i\gamma) \le \lambda_0^{-1} p(T^{i-1}\gamma) \le \ldots \le \lambda_0^{-i} p(\gamma),$$

$$r(T^i\gamma) \le \frac{\lambda_0^{-(i-1)}}{\lambda_0 - 1} r(\gamma) \qquad ,$$

$i = 0,1,\ldots,k.$

Finally we remark that if $\phi = \phi(r)$ is the equation of $T^{-k}\gamma$ in (3') or of $T^k\gamma$ in (4') then $T^{-(k-1)}\gamma,\ldots,\gamma$ or $T^{k-1}\gamma,\ldots,\gamma$ verify the remarkable inequality

$$R_+^{-1} \le \left|\frac{d\phi}{dr}\right| \le R_-^{-1} + \tau_0^{-1} ,$$

so that, on such curves, the r-length is equivalent to the Euclidean length.

The above considerations show how natural it is to try to define a "contracting fiber" as a curve γ_c such that $T^{-1}\gamma_c, T^{-2}\gamma_c, \ldots,$ are a sequence of decreasing curves; similarly one should naturally look for an expanding curve γ_e as an increasing curve such that $T\gamma_e, T^2\gamma_e , \ldots,$ are all increasing.

It is also worth noting that if such contracting curves exist they must obey the following differential equations:

$$\frac{1}{-\cos\phi}\frac{d\phi}{dr} = f^{(\infty)}(a_1,b_1,a_2,b_2,\ldots)$$

where $f^{(k)}(\ldots)$ is the continued fraction

$$f^{(k)}(a_1,b_1,a_2,b_2,\ldots,b_{k-1},a_k) = \frac{a_1}{2} + \cfrac{1}{b_1 + \cfrac{1}{a_2 + \cfrac{1}{b_2 + \ldots \cfrac{1}{b_{k-1} + \cfrac{1}{a_k}}}}}$$

and

$$a_i = \frac{2K(T^{-(i-1)}x)}{\cos\phi(T^{-(i-1)}x)} \le -R_+^{-1}; \qquad b_i = \tau(T^{-i}x) \le -\tau_0.$$

Similarly an expanding curve should verify some analogous equation. It is, however, clear that the r.h.s. of the above differential equations is not continuous in $x = (r,\phi)$ but, instead, it is expected to have a dense set of singular (discontinuity) points.

3. The Fibers

Let x be a point in M and define

$d_T(x)$ = distance of x from the singularity lines of T

$d_{T^{-1}}(x)$ = distance of x from the singularity lines of T^{-1}

$d(x) = \inf (d_T(x), d_{T^{-1}}(x))$.

1 <u>Definition</u>. Let $q > 1$ and let k be a nonnegative integer. Put

$$M_q^{(K)+} = \{x \mid x \in M; -\cos \phi(T^{-h}x) \text{ and } d(T^{h}x) \geq q^{-1}(1+h^2)^{-1}, \quad \forall h = 0,\ldots,k\}$$

$$M_q^{(k)-} = \{x \mid x \in M, -\cos \phi(T^{-h}x) \text{ and } d(T^{-h}x) \geq q^{-1}(1+h^2)^{-1}, \quad \forall h = 0,\ldots,k\}$$

$$M_q^+ = \bigcap_{k=0}^{\infty} M_q^{(k)+}, \qquad M_q^- = \bigcap_{k=0}^{\infty} M_q^{(k)-},$$

$$M_q^{(k)} = M_q^{(k)+} \cap M_q^{(k)-}, \qquad M_q = M_q^+ \cap M_q^- .$$

The following lemma is a consequence of the Borel-Cantelli lemma.

2 <u>Lemma</u>. $\nu(\bigcup_q M_q) \equiv \lim_{q\to\infty} \nu(M_q) = 1.$

The following definition is fundamental.

3 <u>Definition</u>. If $x \in M$, $\cos \phi(x) > 0$, we define the 0-order contracting fiber $\hat{\gamma}_c(x)$ and the 0-order expanding fiber $\hat{\gamma}_e(x)$ as the curves with respective equations:

$$\frac{d\phi}{dr} = K(r), \qquad \phi(r(x)) = \phi(x), \qquad \text{for } \hat{\gamma}_e(x),$$

$$\frac{d\phi}{dr} = -K(r), \qquad \phi(r(x)) = \phi(x), \qquad \text{for } \hat{\gamma}_c(x),$$

and the above differential equations are meant to define $\phi(r)$ in an interval $[r',r'']$ around $r(x)$ which is maximal in the sense that the values of $\phi(r')$ and $\phi(r'')$ are either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

It should be noted that this definition has a very simple geometrical meaning: $\hat{\gamma}_c(x)$ is the equation of a set of colliding arrows which, after collision, are parallel, while $\hat{\gamma}_e(x)$ is a set of colliding arrows which are parallel (i.e. which, before collision, are parallel).

4 Proposition. Given $x \in M_q^{(k)^+}$ there exists a $\Delta > 0$ such that the connected part of $T^k\hat{\gamma}_c(T^{-k}x)$ which contains x is a smooth curve lying, at least, above the interval $[r(x)-\Delta, r(x)+\Delta]$.

The following theorem is the first remarkable result.

5 Theorem. Given $q > 1$, there is $\Delta_q > 0$ such that if $x \in M_q^{(k)^+}$ the connected part of the set $T^k\hat{\gamma}_c(T^{-k}x)$ which contains x is a smooth curve above the interval $[r(x)-\Delta_q, r(x)+\Delta_q] \equiv J_x$ at least. We call $\gamma_c^{(k)}(x)$ this curve above J_x. Its equation $\phi = \phi_c^{(k)}(r)$ has a negative derivative bounded away from zero and infinity $(R_+^{-1} \leq -d\phi_c^{(k)}/dr \leq R_-^{-1} + \tau_0^{-1})$. Furthermore Δ_q can be so chosen that if $y \in \gamma_c^{(k)}(x)$ then $y \in M_{2q}^{(k)^+}$.

The above theorem allows us to define the germ of a contracting fiber:

6 Definition. If $x \in M_q^{(k)^+}$ the curve $\gamma_c^{(k)}(x)$ will be called a "germ" of a contracting fiber of order k.

Notice the difference between $\gamma_c^{(0)}(x)$ and $\hat{\gamma}_c(x)$.

Similar theorems and definitions hold for the expanding fibers: we shall not write them but we shall refer to them as statements (5'), (6').

Notice also that the geometrical meaning of $\gamma_e^{(k)}(x)$ and $\gamma_c^{(k)}(x)$: they represent a set of arrows which, if followed k steps steps in the past or in the future, respectively, become parallel. They can also be defined by the differential equation

$$\frac{1}{-\cos\phi}\frac{d\phi}{dr} = f^{(k)}\left(\frac{2K(x)}{\cos\phi(x)}, \tau(T^{-1}x), \frac{2K(T^{-1}x)}{\cos\phi(T^{-1}x)}, \ldots\right.$$
$$\left.\ldots, \tau(T^{-k+1}x), \frac{2K(T^{-k}x)}{\cos\phi(T^{-k}x)}\right)$$

for $\gamma_c^{(k)}(x)$ and, similarly for $\gamma_e^{(k)}(x)$.

The above theorem tells us that, if $x \in M_q^{(k)+}$, a solution to the differential equation just written can be found locally and extended at least to the interval $[r(x)-\Delta_q, r(x)+\Delta_q]$.

Let us give an idea of the proof of the above theorem: The points $T^{-\ell}x$ approach "very slowly" the singularity lines of T as ℓ varies between 0 and k, because $x \in M_q^{(k)+}$.

Suppose we consider the smooth part γ_k' of $T^k\hat{\gamma}_c(T^{-k}x)$ defined on an interval of size 2Δ around $r(x)$. Then

$$r(T^{-i}\gamma_k') \leq \frac{\lambda_0^{-i}}{\lambda_0-1}\, 2\Delta, \qquad i = 0,1,\ldots,k,$$

but we also know that $d(T^{-i}x) \geq q^{-1}/(1+i^2)$ and $-\cos\phi(T^ix) > q^{-1}/(1+i^2)$ so that if Δ is chosen small enough (depending on q but not on k) we can obtain that, if $y \in T^{-i}\gamma_k'$, the values $-\cos\phi(y)$, $d(y)$ stay larger than $(2q)^{-1}/(1+i^2)$ for $i = 0,1,\ldots,k$.

This is so because the curves $T^{-i}\gamma_k'$ contract exponentially rapidly while their centers approach the singularity with polynomial speed. So the "difficulties" (i.e. having a piece of

$T^{-i}\gamma_k'$ cut by a singularity line of T) can arise only for small values of i and will not arise at all if Δ is chosen small enough at the beginning (roughly so that $q^{-1}(1+i^2)^{-1} - \lambda_0^{-i}(\lambda_0-1)^{-1}\Delta > (2q)^{-1}(1+i^2)^{-1}$, $\forall i = 0,\dots,\infty$, i.e. $\Delta \leq \Delta_q \sim q^{-1} \cdot$ const.).

It is very easy to make the above ideas into a rigorous proof by using an inductive procedure, see Appendix A.

4. Smoothness

The smoothness properties of the approximate fibers are of two types. The first type concerns the smoothness of functions defined on the fibers as, for instance, the slope of $\gamma_c^{(k)}(x)$ at y as a function of $y \in \gamma_c^{(k)}(x)$ or $\cos\phi(y)$ or $\tau(y)$ or $K(y)$ etc. The second type of smoothness concerns the x-dependence of $\gamma_c^{(k)}(x)$. Since the sets $M_q^{(k)\pm}$ are open and T is differentiable on them it is clear that $\gamma_c^{(k)}(x)$ has very good smoothness properties of the above types (for instance, if the boundary of the obstacles is C^∞, $\gamma_c^{(k)}(x)$ is C^∞ in any reasonable sense).

Therefore the above problems have to be interpreted as questions concerning the uniformity in k of the smoothness. While the problem of the first type of smoothness is rather well defined, the second is susceptible of several interesting interpretations.

The guide to a more precise formulation has to come from the use we wish to make of the smoothness properties. Consider a point $x \in M_q$ and draw through x the fibers $\gamma_c^{(k)}(x)$ and $\gamma_e^{(k)}(x)$, see Figure 3. Then consider a point $y \in M_q^{(k)} \cap U$ where U is a small neighborhood of x, so small that the fibers $\gamma_e^{(k)}(y)$ and $\gamma_c^{(k)}(y)$ intersect $\gamma_c^{(k)}(x)$ and $\gamma_e^{(k)}(x)$, respectively, in points b and a. Observe also that the restriction on the diameter of U that this implies depends solely on q. Call α, β the r-lengths of the arcs ax and bx respectively.

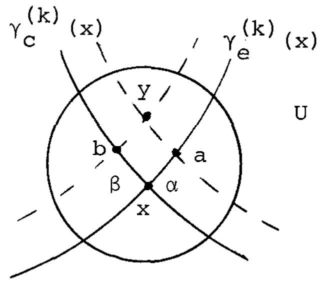


Figure 3

Our purpose is to use the coordinates (α,β) to describe the points of $U \cap M_q^{(k)}$ or of a large subset of it, in the sense that we wish to write, for all $E \subset M_q^{(k)} \cap U$ and with an obvious abuse of notation:

$$\nu(E) = \int_E \rho^{(k)}(\alpha,\beta)\, d\alpha\, d\beta \ .$$

Our smoothness property will be the statement that $\rho^{(k)}(\alpha,\beta)$ can be uniformly bounded, above and below, on a subset of $M_q^{(k)} \cap U$ which is a "sizable" fraction of it and has a simple geometric structure; i.e., it can be thought of, in the (α,β) coordinates, as the intersection of U with a rectangle. Furthermore the ratio between the upper and lower bounds of $\rho^{(k)}(\alpha,\beta)$ on the good subset of $U \cap M_q^{(k)}$ tends to 1, as the diameter of U tends to zero, uniformly in k and $x \in M_q$.

Let us make the above description more concrete: let $\tilde{\gamma}_c^{(k)}(x) = \gamma_c^{(k)}(x) \cap U \cap M_{q/2}^{(k)-}$ and $\tilde{\gamma}_e^{(k)}(x) = \gamma_e^{(k)}(x) \cap U \cap M_{q/2}^{(k)+}$. Let (α,β) be, respectively, the natural coordinates (r-arc lengths) on $\tilde{\gamma}_c^{(k)}(x)$ or $\tilde{\gamma}_e^{(k)}(x)$.

Pick a point $y_e \in \tilde{\gamma}_e^{(k)}(x)$ and $y_c \in \tilde{\gamma}_c^{(k)}(x)$; then define $y = \gamma_c^{(k)}(y_e) \cap \gamma_e^{(k)}(y_c)$, which certainly makes sense if y_e and y_c are in a suitably small neighborhood U of x (so that the fibers of the points $y \in M_{q/2}^{(k)} \cap U$ are longer than diameter (U)).

It is clear that the set $\hat{U}$ of y's so defined is, in a natural sense, the intersection of a "rectangle" with U, and, furthermore,

$$U \cap M_{q/4}^{(k)} \subseteq \hat{U} \subseteq M_q^{(k)} \cap U \subseteq U$$

We wish to show that $\rho^{(k)}(\alpha,\beta)/\rho^{(k)}(\alpha',\beta')$, (α,β), $(\alpha',\beta') \in \hat{U}$ differs from 1 by a quantity which approaches 1 as the diameter of $U \to 0$, uniformly in k and in (α,β), $(\alpha',\beta') \in U$.

Of course sometimes the set $U \cap M_{q/4}^{(k)}$ may be empty or very small, but this does not affect the proof: if q is large $M_{q/4}^{(k)}$ has large measure and so for many U's the content of the result will be nontrivial. There are also other natural systems of coordinates on suitable subsets of U: let $x \in U \cap M_q$, draw $\gamma_c^{(k)}(x)$; let $\tilde{\gamma}_c^{(k)}(x) = \gamma_c^{(k)}(x) \cap M_{q/2}^{(k)-}$; let $\hat{U}' = \bigsqcup_{y \in \tilde{\gamma}_c^{(k)}(x)} \{\gamma_e^{(k)}(y)\}$; then

$$U \cap M_{q/4}^{(k)-} \subset \hat{U}' \subset M_{q/2}^{(k)-} \cap U \subset U$$

and if $z \in \hat{U}'$ we can attach to z the coordinates (α,β) where

α = r-length along $\gamma_c^{(k)}(x)$ of the segment between x and $\gamma_c^{(k)}(x) \cap \gamma_e^{(k)}(z)$;

β = r-length along $\gamma_e^{(k)}(z)$ of the segment between z and $\gamma_c^{(k)}(x) \cap \gamma_e^{(k)}(z)$.

Similarly we can define a set $\hat{U}''$ and introduce on it coordinates in such a way that the roles of the expanding and contracting fibers are interchanged.

We shall be interested in the Radon-Mykodim derivatives of the measures $d\alpha\, d\beta$ on $\hat{U}'$, $\hat{U}''$ with respect to ν in the same sense discussed in the preceding case of $\hat{U}$. This derivative will be denoted with the same symbol $\rho^{(k)}(\alpha,\beta)$: when we shall talk of natural coordinates and of the associated $\rho^{(k)}(\alpha,\beta)$ it will be clear from the context which of the three types is meant.

To understand which are the properties that have to be proven let us calculate $\rho^{(k)}(\alpha,\beta)$: it is of great help to note that $\hat{U}$ is open, so that we can easily calculate $\rho^{(k)}(\alpha,\beta)$ by some local geometric constructions. Let $y \in \hat{U}$ and let δ_1,δ_2 be two "very small"

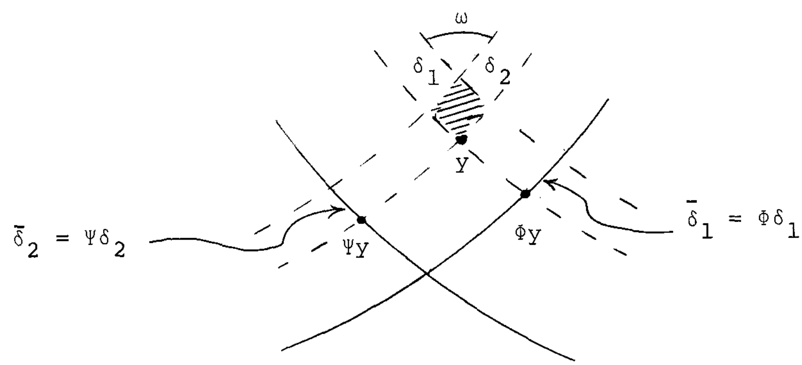


Figure 4

segments of $\gamma_e^{(k)}(y)$, $\gamma_c^{(k)}(y)$ respectively. Let $\bar{\delta}_1$, $\bar{\delta}_2$ be their images on $\tilde{\gamma}_e(x)$, $\tilde{\gamma}_c(x)$ under the mappings

$$\Phi:\ \hat{U} \cap \gamma_e^{(k)}(y) \to \tilde{\gamma}_e^{(k)}(x) \quad ; \qquad \Psi:\ \hat{U} \cap \gamma_c^{(k)}(y) \to \tilde{\gamma}_c^{(k)}(x)$$

which associate to a point $z \in \gamma_e^{(k)}(y) \cap \hat{U}$ the point $\Phi z = \gamma_e^{(k)}(x) \cap \gamma_c^{(k)}(z)$ and to $z \in \hat{U} \cap \gamma_c^{(k)}(y)$ the point $\Psi z = \gamma_c^{(k)}(x) \cap \gamma_e^{(k)}(z)$.

It is easy to see that Φ and Ψ are, in the natural sense, diffeomorphisms in the neighborhood of y.

Let E be the shaded region: some elementary geometrical considerations on Figure 4 allow us to check that:

$$\nu(E) = \frac{-\cos\ \phi(r)\ \text{area}\ E}{\text{normalization}} = - \frac{(\text{length}\ \delta_1)\cdot(\text{length}\ \delta_2)\,|\sin\ \omega|\cos\ \phi(y)}{\text{normalization}}$$

$$= \frac{\cos\ \phi(\Phi y)\ \cos\ \phi(\Psi y)}{-\cos\ \phi(y)} \left(\frac{d\phi_e}{dr} - \frac{d\phi_c}{dr}\right)_y \frac{f_c^{(k)}(y)\ f_e^{(k)}(y)}{\text{normalization}}\ r(\bar{\delta}_1)\cdot r(\bar{\delta}_2)$$

where $\phi_e(r)$, $\phi_c(r)$ are, respectively, the equations of $\gamma_e^{(k)}(y)$ and $\gamma_c^{(k)}(y)$ and $f_c^{(k)}(y)$, $f_e^{(k)}(y)$ are the Radon-Nykodim derivatives

$$\frac{p(\delta_1)}{p(\bar{\delta}_1)} \quad \text{and} \quad \frac{p(\delta_2)}{p(\bar{\delta}_2)}$$

Our problem is, thus, "reduced" to the search for a bound for $f_e^{(k)}(y)$ and $f_c^{(k)}(y)$ in a sufficiently small neighborhood of points $x \in M_q^{(k)}$ and to an estimate of $\frac{d\phi_e}{dr}$ and $\frac{d\phi_c}{dr}$.

The problem of studying $p(\delta_1)/p(\bar{\delta}_1)$ (say) can be naturally attacked by observing that, if we apply the transformation T^{-k} to $\gamma_c^{(k)}(z)$, $z \in \delta_1$, we obtain a family of curves which form a very smooth foliation, i.e. the foliation generated by the smooth differential equation $d\phi/dr = -K(r)$. This means that $p(T^{-k}\delta_1)/p(T^{-k}\ \bar{\delta}_1)$ is very easy to compute and is given, in the simple case $K(r) = R^{-1}$ = const., by

$$\frac{p(T^{-k}\ \ \delta_1)}{p(T^{-k}\ \Phi\delta_1)} = \frac{\cos\ \phi(T^{-k}\ \ \Phi y)}{\cos\ \phi(T^{-k}\ \ \ y)}\ \ \frac{R^{-1} + \frac{d\phi_e^{(k)}}{dr}\ (T^{-k}\Phi y)}{R^{-1} + \frac{d\phi_c^{(k)}}{dr}\ (T^{-k}\ y)} \equiv F_1^{(k)}(y)\ ,$$

where $\phi_e^{(k)}$ denoted the equations of $T^{-k}\ \gamma_e^{(k)}(y)$ or $T^{-k}\ \gamma_e^{(k)}(\Phi y)$.

Then the ratio $p(\delta_1)/p(\Phi\delta_1)$ can be written as

$$\frac{p(\delta_1)}{p(\Phi\delta_1)} = F_1^{(k)}(y) \prod_{i=0}^{k-1} \frac{\dfrac{p(T^{-i}\delta_1)}{p(T^{-i-1}\ \delta_1)}}{\dfrac{p(T^{-i}\ \ \Phi\delta_1)}{p(T^{-i-1}\Phi\delta_1)}} \equiv F_1^{(k)}(y)\ F_2^{(k)}(y)\ .$$

The expressions for the $\rho^{(k)}(\alpha,\beta)$ for the other natural systems of coordinates on $\hat{U}'$ or $\hat{U}''$ can be studied in a similar way: e.g. in the case of $\hat{U}'$ the $\rho^{(k)}(\alpha,\beta)$ has an expression which can be conveniently derived along the same lines as in the case of U by introducing an auxiliary smooth foliation which contains $\gamma_c^{(k)}(x)$ and which consists of decreasing curves with slope verifying the usual bounds (i.e. between $-(R_-^{-1}+\tau_0)^{-1}$ and $-R_+^{-1}$). We shall discuss the details of a slightly more complicated discussion in the proof of Lemma 13 and therefore we omit the details here.

As we have seen in the geometrical considerations the expansion coefficient at a point w of the p-length of a curve γ depends only on quantities like $\cos\phi(w)$, $\cos\phi(T^{-1}w)$, $k(w)$, $K(t^{-1}w)$, $\tau(w)$, $d\phi(w)/dr$: therefore it is through the study of the variation of these quantities as w varies along the curves $T^{-i}\gamma_c^{(k)}(y)$ and $T^{-i}\gamma_e^{(k)}(y)$ that the desired bounds can be obtained.

The key theorem is the following:

7 Lemma. Given $q > 1$, $k > 0$, $x \in M_q^{(k)}$ there exist constants $c_q > 0$, $\lambda > 1$ such that if $y,y' \in \gamma_c^{(k)}(x)$ and $\gamma_{yy'}$ denotes the arc between y,y' on $\gamma_c^{(k)}(x)$, then:

(i) $p(T^{-\ell}\gamma_{yy'}) \le \lambda^{-\ell} p(\gamma_{yy'})$; $r(T^{-\ell}\gamma_{yy'}) \le \dfrac{\lambda^{-\ell+1}}{\lambda-1} r(\gamma_{yy'})$

for $\ell = 0,1,\dots,k$.

(ii) if $\phi = \phi^{\ell}(r)$ is the equation of $T^{-\ell}\gamma_c^{(k)}(x)$ then

$$R_+^{-1} \le -\frac{d\phi^{\ell}}{dr} \le R_-^{-1} + \tau_0^{-1} \quad \text{for } \ell = 0,1,\dots,k.$$

(iii) the quantities

$$\frac{\cos\phi(T^{-\ell}y)}{\cos\phi(T^{-\ell}y')}, \quad \frac{\tau(T^{-\ell}y)}{\tau(T^{-\ell}y')}, \quad \frac{K(T^{-\ell}y)}{K(T^{-\ell}y')}$$

are between $\exp \pm c_q\lambda^{-\ell} r(\gamma_{yy'})$.

(iv) the quantities

$$\frac{d\phi^{\ell}}{dr}(y) \Big/ \frac{d\phi^{\ell}}{dr}(y')$$

are between $\exp \pm c_q\lambda^{-\ell} r(\gamma_{yy'})$.

(v) $$\sup_{|r-r(x)|\leq\Delta_q} |\phi^{(\ell)}(r) - \phi^{(\ell+1)}(r)| \leq c_q\lambda^{-\ell}, \quad \ell = 0,1,\ldots,k.$$

Proof: See Appendix A.

8 Corollary. Let $x \in M_q^+$ and let $\phi_c^{(k)}(r)$ denote the equation of $\gamma_c^{(k)}(x)$ defined on $[r(x)-\Delta_q, r(x)+\Delta_q]$. Then $\lim_{k\to\infty} \phi_c^{(k)}(r) = \phi_c(r)$ exists in the C^1-sense and the properties (i), (ii), (iii) and (iv) in the above lemma hold for the curve $\gamma_c(r)$ defined by $\phi_c(r)$ for all $\ell = 0,1,\ldots$. The limit is reached exponentially rapidly in the C^0-sense.

The above lemma and corollary tell us that each single k-contracting germ of a fiber is very smooth and does not behave wildly as $k \to \infty$.

The following theorem provides the estimates needed for the bounds on $\rho^{(k)}(\alpha,\beta)$:

9 Theorem. Let $k > 0$, $q > 1$. Let U be a neighborhood with diameter $D(U)$ so small that a k-fiber's germ (contracting or expanding) through any point of $M_q^{(k)} \cap U$ (if such a set is not empty), has a length larger than $D(U)$. Let $x \in M_q^{(k)} \cap U$ and let $\hat{U}$ $(\hat{U}',\hat{U}'')$ be the set defined above and introduce on it the natural coordinates (α,β). Then there is a θ_q: $(0,+\infty) \to (0,+\infty)$, infinitesimal near zero, and a $C_q > 0$ such that

$$\frac{\rho^{(k)}(\alpha,\beta)}{\rho^{(k)}(\alpha',\beta')}, \qquad (\alpha,\beta),\ (\alpha',\beta') \in \hat{U},\ (\hat{U}',\hat{U}'')$$

is between $\exp \pm \theta_q(D(G))$ and $\rho^{(k)}(\alpha,\beta) \leq C_q$, $\forall(\alpha,\beta) \in \hat{U},\ (\hat{U}',\hat{U}'')$.

Proof: See Appendix A.

The above theorem and corollary are enough to deduce that the system (M,T,ν) is a B-system (i.e. isomorphic to a Bernoulli shift) once it is known that it is a K-system [3].

The reader may notice that the statements of the last theorem would make sense also if $K = +\infty$: It would be interesting to have an explicit proof of such a result (see concluding remarks).

In the remaining lectures we sketch the proof of the ergodicity of (M,T,ν).

5. Ergodicity

As we saw in the preceding sections the billiards possess approximate foliations of contraction and expansion which are absolutely continuous in the sense of the last theorem.

It is therefore tempting to try the classical Hopf method to prove ergodicity.

A first step is described by the following theorem and corollary which express rather weak ergodic properties.

Given a continuous function $f \in C(M)$, put

$$W^f_{k,\rho} = \{x \mid x \in M, \ |h^{-1} \sum_{i=0}^{h-1} (f(T^{\,i}x) - \bar{f}^{\pm}(x))| < \rho, \ \forall h \geq k\},$$

where $\bar{f}^+$, $\bar{f}^-$ denote the future and the past averages of f. Birchoff-Von Neumann's theorem tells us that

$$\lim_{k\to\infty} \nu(W^f_{k,\rho}) = 1 , \qquad \forall \rho > 0 .$$

10 Lemma. Let $x \in M_q$ and $U \ni x$ be a neighborhood of x of sufficiently small diameter. Let $V = M_{q/4} \cap U$ and let $f \in C(M)$.

Given $\theta > 0$, $\rho > 0$, $\exists\, K_{\rho,\theta}$ such that if $K > K_{\rho,\theta}$ there is $\bar{W} \subset W^f_{k,\rho} \cap V$, and

(1) $\nu(\bar{W}) > (1-\theta)\nu(V)$;

(2) if $x,y \in \bar{W}$ there is a sequence $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ of alternatingly ℓ-expanding and ℓ-contracting fibers such that $x \in \gamma_1$, $y \in \gamma_4$, $\gamma_i \cap \gamma_{i+1} \in W^f_{k,\rho}$, $i = 1,2,3$, where ℓ is an arbitrary integer.

This lemma easily implies that $\bar{f}^{\pm}$ has an "essential oscillation" $\leq 4\rho$ on V i.e., that $\bar{f}^{\pm}$ is a constant on V. Hence

11 Corollary. The transformation T is ergodic on $\bigcup_{n=-\infty}^{+\infty} T^n V$.

By an application of Vitali's covering theorem the following can be deduced.

12 Proposition. The dynamical system (M,T,V) break in, at most, denumerably many ergodic components.

The proof of the above lemma shall not be explicitly given since Lemma 10, Corollary 11, Proposition 12 will not be used in the proof of global-ergodicity. It is not hard and involves only a few technical difficulties: it can be deduced from the information contained in statements 8 and 9 above without "further" use of the structure of the billiards: it is also the strongest result that we can obtain without further investigation of the special properties of our system. In fact, in the theory of the Anosov diffeomorphisms, the ergodicity is obtained from a local ergodic property through a covering argument. In our case the family of the sets of type V, of the above theorem and corollary, is a priori too small to allow such a covering argument (notice that the sets M_q are "badly" disconnected). It may just not be possible to construct, for all couples, x,y chosen outside a set of zero measure, a sequence $V_1, V_2, \ldots, V_s$ of finitely many sets of a form to which Proposition 11 applies and such that $x \in V_1$, $y \in V_s$ and $\nu(V_i \cap V_{i+1}) \neq 0$, $i = 1,2,\ldots,s-1$.

The following Theorem 13 will allow us to make a construction (see Lemma 17) of a chain of alternatingly ℓ-expanding and ℓ-contracting fibers connecting two points x,x' chosen outside a subset of measure zero of a sufficiently small neighborhood of a point x_0

such that $d(T^i x_0) \neq 0$ for all but one integer i.

The number $K(x,x')$ of elements of such a chain will now depend upon the two points x,x' but, if x,x' are outside a suitable set of measure zero, will not depend on ℓ.

Furthermore the fibers can be chosen to have the property that their successive intersections always take place in points in which, an a priori given continuous function, attains its past and future averages within $\rho K(x,x')^{-1}$ in $\bar{K}$-steps, provided $\bar{K}$ is large enough. This allows us to prove that the average of every continuous function is constant in U.

Now we can deduce global ergodicity by means of a simple covering argument. The complement of the set of the x 's whose trajectory is at most once tangent is denumerable: therefore the intersections of this set with the connected components of M are linearly connected. This means that each connected component of M is entirely contained in an ergodic component of T. This, of course, implies that T is ergodic. We shall indicate some details of the derivation of the just described results from the main Theorem 13 (below) after describing its proof in the next lecture. The complete proof is however in Appendix C.

The main lemma for the proof of global ergodicity is the following.

13 Theorem. Let x_0 be a point whose trajectory $x_0, Tx_0, T^2x_0, \ldots$ is never "tangent". Given $\varepsilon > 0$, $C > 0$ there is a neighborhood $U(\varepsilon,C)$ of x_0 such that if $\gamma \subset U$ is a decreasing curve with $r(\gamma) = \delta_0$ and with slope between $-(R_-^{-1} + \tau_0^{-1})$ and R_+^{-1} there is a quadrilateral G whose boundary consists of $\gamma_\ell \equiv \gamma, \gamma_r, \gamma_\mu, \gamma_d$ where $\gamma_r, \gamma_\ell \equiv \gamma$ are two disjoint decreasing curves while γ_d, γ_μ are two disjoint increasing curves with slope between R_+^{-1}

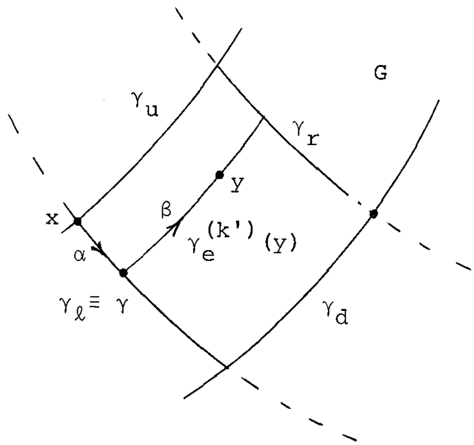


Figure 5

and $(R_-^{-1}+\tau_0^{-1})$. Furthermore $U(\varepsilon,C) \supset U(\varepsilon,C')$ if $C < C'$ and, if k is an arbitrary integer:

(i) there is an open subset $\hat{G}_k \subset G$ which admits a foliation with k'-expanding curves ($k' \geq k$) which join the opposite sides γ_ℓ, γ_r without intersecting γ_u, γ_d and have an r-length $\geq C\delta_0$.

(ii) $\nu(\hat{G}_k) > (1-\varepsilon)\nu(G)$

(iii) Let $y \in \hat{G}_k$ be parametrized by the r-coordinates (α,β) of $\bar{y} = \gamma_e^{(k')}(y) \cap \gamma_\ell$ along γ_ℓ and of y along $\gamma_e^{(k')}(y)$ from $\bar{y}$ (see Figure 5).

Then the Radon-Nykodim derivative $-\cos\phi\ dr\ d\phi/d\alpha\ d\beta$ is a function $\rho^{(k)}(\alpha,\beta)$ with the property that for all (α,β), $(\alpha',\beta') \in \hat{G}_k$ the ratio $\rho^{(k)}(\alpha,\beta)/\rho^{(k)}(\alpha',\beta')$ is between $\exp \pm \eta(\delta_0)$ where $\eta(\delta_0)$ is a suitable real function.

(iv) The side γ_r can be chosen to be a curve contained in a parallel translate, along γ_u of the curve γ_ℓ (extended, if too short (see Figure 5), beyond its lowest extreme in an arbitrary way which keeps it smooth and with slope between $-(R_-^{-1}+\tau_0^{-1})$ and $-R_+^{-1})$.

A similar result holds if $d(T^{-i}x_0) > 0$, $i.= 0,1,\ldots,$ and the role of the expanding and contracting fibers is inter changed. Similarly it is possible to prove an almost identical lemma in which $\gamma = \gamma_r$ instead of $\gamma = \gamma_\ell$. When we shall refer to Theorem 13 we shall also mean these last results.

Strictly speaking the properties of absolute continuity of the approximate fibers are not needed in the proof of this theorem sketched in the next lecture. However a crucial step of the proof relies upon a lemma whose proof is, essentially, a repetition of the arguments needed to prove the theorem 8 on the absolute continuity (see next lecture).

6. About the Main Lemma

The proof of this theorem (statement 13 above) proceeds in a very natural way. It needs a little extra information about the structure of the singularity lines of the transformations $T, T^2, \ldots T^L$. This information is contained in the next lemma.

14 Lemma. Consider the set, $S_L = \partial M \cup T^{-1}\,\partial M \cup \ldots \cup T^{-L}\,\partial M$ (which is the set of singularity lines of $\tilde{T} = T^L$). Suppose the billiard is "without horizon" (i.e. there is no straight path to ∞). Then the set S_L consists of finitely many smooth curves. The set X_L of points of S_L belonging to different curves (confluence points) is finite and the lines meet at $x \in X_L \setminus \partial M$ with a nonzero angle; if $x \in X_L \cap \partial M$ they have a contact of second order. So the ν-measure of the area trapped in a circle with radius $\sqrt{\delta}$ centered at $x \in X_L \cap \partial M$ by the two merging lines is of the order δ^2 (while its Lebesgue measure would be $\sim\delta^{3/2}$).

The proof is based on the differential expression of the singularity lines γ which is of the form

$$\frac{d\phi}{dr} = -K(r) - \frac{\cos\phi}{\tau_s(r,\phi)}$$

and $\tau_s(r,\phi) = \tau(r,\phi)$ if $\gamma \subset T^{-1}\partial M$ or, more generally, $\tau_s(\cdot)$ is given by a finite continued fraction (see formula in Lecture 2). The main remark is that the fact that the two curves merging in $x \in X_L$ are different implies that the limit of $\tau_s(r,\phi)$ as $(r,\phi)\to x$ is different along the two lines. This easily implies the Lemma 14.

We now come to the proof of Theorem 13. It will be convenient to put

$$Z = R_+(R_-^{-1}+\tau_0^{-1}) > 1$$

Consider a power $\tilde{T} = T^L$ so large that

$$\lambda^{Lm}(\lambda-1) > \lambda_1^m > 2\ , \qquad m = 1,2,\ldots,$$

(see Lemma 7 for the definition of λ).

One starts with a neighborhood U of a point x_0, such that $-\cos\phi(T^i x_0) > 0$, $i = 0,1,\ldots,$ so small that it is possible to draw through every point of U a k_0L-expanding fiber with r-length exceeding $4ZCD(U)$. The number k_0 will be at least such that $4CZ\ \lambda_1^{-k_0} < \lambda_2^{-k_0}$ where λ_2 is arbitrarily chosen so that $\lambda_1 > \lambda_2 > 2$; however the final choice of k_0 will be made at the end of the proof (to give its value now would be too complicated since it involves a large number of different geometrical constants which would have to be defined before their interest really appears).

Let $\gamma_0 \subset U$ be a decreasing curve and $r(\gamma_0) = \delta_0$. Draw through the points of γ_0 the k_0L-expanding fibers with r-length exceeding $4ZCD(U)$. Pick the point situated on the expanding fiber through the upper vertex of γ_0 and at an r-distance $2ZCD(U)$ from it (see Figure 6).

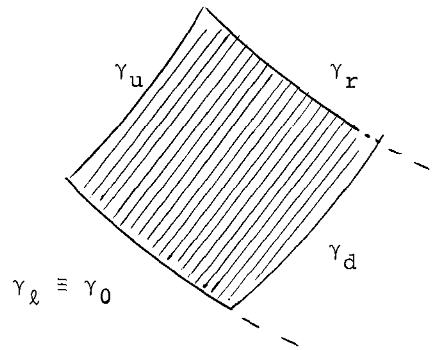


Figure 6

If the above construction makes sense the lines $\gamma_u, \gamma_d, \gamma_r, \gamma_\ell$ determine a quadrilateral G: in order that the above construction makes sense we may be forced to further reduce D(U), ahd hence U, and possibly extend γ_0 slightly beyond the lower vertex (in an arbitrary way which keeps the slope bounded between $-(R_-^{-1}+\tau_0^{-1})$ and and $-R_+^{-1})$ so that γ_r is long enough to cross γ_d.

Choosing U even smaller, if necessary, we can and shall assume that for $b > 0$ to be fixed later:

$$-\cos\phi\Big|_{T^k G} > \delta_0\left(\frac{K}{L}\right)^{-4}, \qquad k = 1,\dots,k_0 L$$

$$d_{\tilde{T}}(x)\Big|_{\tilde{T}^k G} > \delta_0\, K^{-16b}, \qquad k = 1,2,\dots k_0-1$$

the exact value of b is irrelevant provided it is large enough (from the estimates below $b = 9$ will be large enough); $d_{\tilde{T}}$ is the distance from the singularities of $\tilde{T}$.

If $k \le k_0$ the set $\hat{G}_k$ will coincide with G

If $k > k_0$ we shall obtain $\hat{G}_k$ by subtracting from G points in such a way that $\hat{G}_k$ is the union $\hat{G}_k = \bigsqcup_{s=1} \hat{G}_{k,s}$ of quadrilaterals which are unions of intersections of G with KL-expanding fibers (i.e. are fibered by kL-expanding fibers).

Furthermore,

$$\cos\phi\,\Big|_{T^i\hat{G}_{k,s}} > \delta_0\left(\frac{i}{L}\right)^{-4}, \qquad i = 1,\dots,kL\ ;$$

$$d_{\tilde{T}}(x)\,\Big|_{\tilde{T}^i G_{k,s}} > \delta_0\, i^{-16b}, \qquad i = 1,\dots,k-1.$$

Suppose $\hat{G}_{k-1,m}$ have been constructed. Put $\tilde{G}_{k-1,m} = \tilde{T}^{k-1}\,\hat{G}_{k-1,m}$. The set $\tilde{G}_{k-1,m}$ will, in general, be cut into "components" $O_{k-1,m,\ell}$ by the singularity lines of $\tilde{T}$. The sets $\tilde{T}^{-1}\tilde{G}_{k,s}$ will be subsets of some of these components which will be called the "good" components.

We immediately declare "bad" components all those which are contained in the triangular regions defined by the lines merging into a point $x \in X_L$ and a circle around x with radius $\delta_0(k-1)^{-b/2}$, if $x \in X_L \setminus \partial M$, and with radius $(\delta_0(k-1)^{-b/2})^{1/2}$, if $x \in X_L \cap \partial M$.

Lemma 14 above expresses that the ν-measure of the set of points so far discarded does not exceed

$$\text{const.}\ \frac{\delta_0^2}{(k-1)^b}$$

Divide the remaining components into regular and irregular. Regular will be those which have both bases on singularity lines.

It follows from the definition that the regular components 0 are such that

$$r(\gamma_r(0)) \geq \text{const.}\ \delta_0\ (k-1)^{-b/2}$$

(provided the D(U) had been chosen small compared to the minimal distance between disjoint singularity lines of $\tilde{T}$).

Let us call "big" a component 0 such that

$$r(\gamma_r(0)) \geq \delta_0(k-1)^{-4b}\ .$$

The regular components are all big if the k_0 had been chosen large enough.

If a component is big it is possible to draw the L-expanding fibers which cut $\gamma_r(0)$ in points at a distance $\delta_0(k-1)^{-8b}$ from the bases. This is easy to see if account is taken of the fact that the r-length of an increasing line joining $\gamma_r(0)$ to $\gamma_\ell(0)$ and verifying the usual bound on the slope is bounded by $\lambda_2^{-k}\delta_0$. We call $0'$ the set obtained by cutting, off 0, the two sets between the bases of 0 and the just drawn fibers. We discard 0 as bad if

$T\,0' \subset \mathcal{V}_{k-1+1/L,4} = \{x \mid -\cos\phi(x) \leq \delta_0(k-1+\frac{1}{L})^{-4}\}$ or if

$T0' \cap \mathcal{V}_{k-1+1/L,4} \neq \emptyset$ but $T0' \cap \mathcal{V}_{k,3/2} = \{x| -\cos\phi(x) \leq \delta_0 k^{-3/2}\}$.

If $T\,0' \cap \mathcal{V}_{k-1+1/L,4} \neq \emptyset$ and $T0' \not\subset \mathcal{V}_{k,3/2}$ we draw through $\gamma_r(T0') \cap \partial\mathcal{V}_{k-1+1/L,4}$ and through $\gamma_\ell(T0') \cap \partial\mathcal{V}_{k-1+1/L,4}$ an $(L-1)$-expanding fiber and cut out of $T0'$ the points between the bases of $T0'$ and the just drawn fibers. We repeat this process $(L-1)$ more times using for the ith stage the set $\mathcal{V}_{k-1+i/L,4}$ instead of $\mathcal{V}_{k-1+1/L,4}$. Let $\tilde{G}_{k,m}$ be the sets obtained in this way from the big 0's and let $\bar{0} = \tilde{T}^{-1}\tilde{G}_{k,m}$: $\bar{0} \subset 0$.

Lemma 15 and Corollary 16 below allow, after simple considerations, the deduction that

$$\frac{\nu(0 \setminus \bar{0})}{\nu(0)} \leq \text{const.}\ (k-1)^{4b} ,$$

in fact the Lemma 15 allows us to estimate the above ratio in terms of $r(\gamma_r(0 \setminus \bar{0}))/r(\gamma_r(0))$.

Consider now the irregular small components. They will sometimes be kept and sometimes be discarded.

To decide whether to keep them or not we define $\kappa(0)$ to be the number of times $\tilde{T}^{-1}$ has to be applied to an irregular small component 0 before seeing it contained as a part of a big component $0_{k-1-\kappa(0),m_1,\ell_1}$ cut out of some $\tilde{G}_{k-1-\kappa(0),m_1}$.

Define also $n(k)$ so that $(\frac{\lambda_2}{2})^{n(k)} = k^3$. Then if $\kappa(0) \geq n(k)$ we declare 0 bad. If $\kappa(0) < n(k)$ we declare it bad if also

$$\frac{r(\gamma_r(\tilde{T}^{-\kappa(0)}\ 0_{k-1,m,\ell}))}{r(\gamma_r(0_{k-\kappa(0)-1,m_1,\ell_1}))} < k^{-3}$$

Notice that

(1) If $0 = 0_{k-1,m,\ell}$ is irregular and if $\tilde{T}^{-\kappa(0)}\ 0 \subset 0_{k-\kappa(0)-1,m_1,\ell_1}$, then all the sets $\tilde{T}^{-i}\ 0$, $0 \leq i < \kappa(0)$, are contained in irregular components and, therefore, the component $0_{k-\kappa(0)-1,m_1,\ell_1}$ can contain $\tilde{T}^{-\kappa(0)}\ 0$ only for $2^{\kappa(0)}$ different 0's with given value of $\kappa(0)$.

(2) If $0_{k-1,m,} = 0$ is irregular and small it follows that

$$r(\tilde{T}^{-\kappa(0)}\ \gamma_r(0)) \leq \lambda_2^{-\kappa(0)} r(\gamma_r(0)) \leq \lambda_2^{-\kappa(0)}\ \delta_0 (k-1)^{-4b}$$
$$\leq \lambda_2^{-\kappa(0)}\ r(\gamma_r(0_{k-\kappa(0)-1,m_1,\ell_1}));$$

hence, if $\hat{0} = 0_{k-\kappa(0)-1,m_1,\ell_1}$, the Lemma 15 below shows

$$\frac{\nu(0)}{\nu(\hat{0})} \leq \lambda_2^{-\kappa(0)}\ \text{const.}$$

(3) If $0_{k-1,m,\ell}$ is irregular but not yet bad it follows that

$$r(\gamma_r(0_{k-1,m,\ell})) \geq \lambda_2^{\kappa(0)}\ r(\tilde{T}^{-\kappa(0)}\ \gamma_r(0_{k-1,m,\ell}))$$
$$\geq \frac{\lambda_2^{\kappa(0)}}{k^3}\ r(\gamma_r(0_{k-1-\kappa(0),m_1,\ell_1})) \geq \delta_0 (k-1-\kappa(0))^{-4b}\ k^{-3}$$

so that

$$r(\gamma_r(0_{k-1,m,\ell})) \geq \delta_0 (k-1)^{-5b}$$

(if k_0, b had been chosen large enough).

We continue our construction by looking at those irregular small components which are not yet bad according to the criteria so far exposed. Let 0 be such a component: We apply to it the same treatment applied to the big components and construct out of 0 $0'$ and then $\bar{0}$ (if $T0'$ is not such that 0 itself has to be declared bad, see case of the big components). If 0 is good the same arguments used for the case of big 0's allows us to say, via simple considerations based on Lemma 15 and Corollary 16 below, that

$$\frac{\nu(0 \setminus \bar{0})}{\nu(0)} \leq \text{const. } (k-1)^{-3b}.$$

The sets $\tilde{T}\bar{0} = \tilde{G}_{k,s}$ are added to the similar sets obtained from the big components and the union of this family of sets is $\tilde{G}_k = \tilde{T}^k \hat{G}_k$.

Remarking that $\nu(\mathcal{V}_{k,3/2}) \leq \text{const. } \delta_0^2/k^3$ and collecting all the above estimates we realize that the amount of points in $G \setminus \hat{G}_k$ has a measure not exceeding

$$\text{const. } \delta_0^2 \sum_{k \geq k_0} K^{-3} .$$

Thus Lemma 13 is proved, at least as far as the first two items are concerned. The proof of the third item is briefly commented upon after Lemma 15 below.

The measures of the little pieces of G discarded in the inductive construction have been estimated above by the following lemma which, essentially, tells us that, in the natural coordinates, a connected subset $\bar{G} \subset \tilde{G}_{k,m}$ consisting of expanding fibers of kL-order and embedded with the measure ν can be treated as a rectangle with ordinary Lebesgue measure in the natural coordinates (α,β), provided its side is not too long: so that the ratio of the ν-measures of two strips 0 and $\tilde{0}$ in $\bar{G}$ is close to the ratio of the r-lengths of their sides. More precisely,

15 Lemma. Given $C > 0$, $\delta_0 > 0$ let G_0 be a quadrilateral such that all the increasing lines which join its two decreasing sides $\gamma_\ell(G_0)$, $\gamma_r(G_0)$ have r-length $\geq C\,\delta_0$ but $\leq 4CZ\delta_0$. Assume

also that the slope of $\gamma_\ell(G_0)$ is between $-(R_-^{-1}+\tau_0^{-1})$ and $-R_+^{-1}$.

Let k_0 be such that $\lambda_1^{-k_0}$ 4 C Z $\leq \lambda_2^{-k_0}$, where λ_1,λ_2 are the same defined at the beginning of this lecture.

Suppose that for some $k \geq k_0$ the sets $\tilde{T}G_0,\ldots,\tilde{T}^kG_0$ are quadrilaterals and $\tilde{T}^kG_0$ consists of 0-expanding fibers (i.e. the 0-expanding fiber through every point of $\tilde{T}^kG_0$ joins the opposite decreasing sides of $\tilde{T}^kG_0$ without crossing the other two sides) and:

$$r(\gamma_r(\tilde{T}^hG_0)) < \delta_0(1+h)^{-b}, \qquad h = 0,1,\ldots,k$$

$$-\cos\phi|_{T^hG_0} \geq \delta_0(1+\tfrac{h}{L})^{-4}, \qquad h = 0,1,\ldots,kL$$

Then we introduce on $\tilde{T}^hG_0$ the natural coordinates (α,β) the Radon-Nykodim derivative of the Lebesgue measure $d\alpha\, d\beta$ on $\tilde{T}^hG_0$ with respect to the ν-measure is constant within $\exp \pm \theta(c)/(1+h)^{6-4}$, $\forall h \leq k$, where $\theta(\cdot)$ is a suitable function.

Proof: See Appendix B.

16 Corollary. In particular if $0_1 \supseteq 0_2$ are two connected subsets of $\tilde{T}^hG_0$, $h = 0,\ldots,k-1$, which are unions of (k-h)L-expanding fibers the ratio $\nu(0_1)/\nu(0_2)$ is bounded by const. $r(\gamma_r(0_1))/r(\gamma_r(0_2))$ if $r(\gamma_r(0_2)) \geq \delta_0(1+h)^{-5b}$.

The proof of the above lemma is an easy repetition of the absolute continuity proof sketched in Section 3): one has to follow the same chain of inequalities taking some care in not being too generous in the estimates and making full use of the strong convergence properties of the continued fraction which appears in the game. This proof yields also, implicitly, the third part of the main lemma.

The proof of Lemma 15 also yields, as a byproduct, item (iii) of the main lemma (actually this part can be proved directly and is much simpler).

In the Appendix B, Lemma 15 is derived in detail.

7. More About Ergodicity

Let $f \in C(M)$. Let $\bar{f}^+, \bar{f}^-$ be the pointwise future and past averages of f. Let

$$M_f = \{x \mid \bar{f}^+(x) = \bar{f}^-(x)\}$$

$$W^f_{k,\rho} = \{x \mid x \in M, \quad h^{-1} \sum_{i=0}^{h-1} f(T^{\pm i}x) - \bar{f}^{\pm}(x) \mid < \rho, \ \forall h \geq k\}.$$

Theorem 13 allows us to prove the following lemma.

17 Lemma. Let the trajectory $T^i x_0$, $i = 0, \pm 1, \ldots$ be never tangent or once tangent.

There is a neighborhood U of x_0 such that if $f \in C(M)$ $\exists$ a subset $V_f \subset U$ with full measure with the property that if $x,y \in V_f$ and $\rho > 0$ then one can find $k(x,y)$ and $s(x,y,f,\rho)$ such such that for all $s > s(x,y,f,\rho)$ there is a sequence of alternatingly ℓ-expanding and ℓ-contracting fibers $\gamma_1, \ldots, \gamma_k$ with $k \leq k(x,y)$ and $x \in \gamma_1$, $g \in \gamma_k$, $\gamma_i \cap \gamma_{i+1} \in W^f_{s,\rho}$ where ℓ can be arbitrarily chosen.

This lemma implies the

18 Corollary. The transformation T is ergodic on $\bigcup_{n=-\infty}^{+\infty} T^n U$, hence it is ergodic. In fact the lemma implies that the function $\bar{f}^+(x) = \bar{f}^-(x) =$ constant on V_f i.e. almost everywhere in U. This implies global ergodicity through the already mentioned covering argument.

The proof of Lemma 17 is quite simple in principle but very messy in its details. The main reason for the origin of complications is the fact that our fibers are only approximate. Of course they can be as approximated as one wishes and their absolute continuity is in some sense "uniform" and this allows us to overcome the problems that arise.

To give an idea of the proof we assume that the natural version for $K = +\infty$, i.e. for exact fibers, of Lemma 13 is true, and also that for almost all points $x \in M_q$ $(q = 1,2,\ldots)$, the fibers $\gamma_{c,e}(x)$ intersect in a set of full arc-length a fixed set of full measure . Then we show how the analogous version of Lemma 17 would arise.

This argument is interesting because it shows the path to follow in our more complicated case in which only approximate fibers are available.

Let $f \in C(M)$ and $U = U(\varepsilon, C)$ where the value of ε, C will appear later and let x,y be two points in $M_f \cap \left(\bigsqcup_{q=1}^{\infty} (M_q \setminus M_{q-1})\right)$ $\cap\, U$ such that $\gamma_c(x)$, $\gamma_e(y)$ have full intersection, with respect to their arc lengths, with M_f. Let $x \in M_{q_x} \setminus M_{q_x - 1}$ and $y \in M_{q_y} \setminus M_{q_y - 1}$.

Draw through x the lower segment γ_1 of $\gamma_c(x)$ (which extends

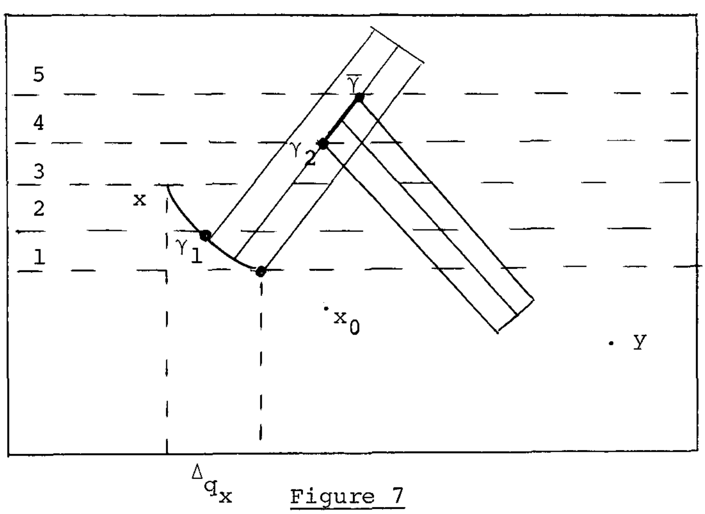


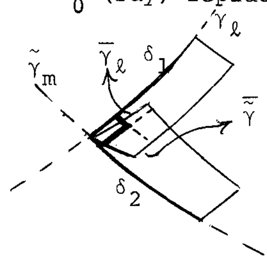
Figure 7

on the r-axis by an r-length Δ_{q_x}, if $x \in M_{q_x} \setminus M_{q_x - 1}$). Let γ_1' be its intersection with the lowest strip of the four strips with equal width drawn in Figure 7 (i.e. with region 1).

We apply to it the main lemma and find an expanding line $\bar{\gamma}$ which extends beyond region 4 and has full intersection with M_f and intersects γ_1 in a point of M_f (remember that the slopes of the fibers are uniformly bounded away from zero and ∞ and C can be suitably chosen (say, C = 10 Z would be large enough)). Let $\bar{\gamma}_2$ = intersection of $\bar{\gamma}$ with the region $(1 \cup 2 \cup 3 \cup 4)$ in Figure 7. We call $\gamma_2' = \gamma_2 \cap$ (region 4) and apply again the main lemma to γ_2', etc.

A similar construction is made starting from y. Clearly two sequences of alternatingly expanding and contracting curves with successive intersections in M_f and containing at most const. $(\Delta_{q_x}^{-1} + \Delta_{q_y}^{-1})$ lines can be constructed using the above procedure. Let γ_ℓ and $\tilde{\gamma}_m$ be the first two fibers of the strings starting in x and y, respectively, such that $\gamma_\ell \cap \tilde{\gamma}_m = z \neq \emptyset$. Let z be a point (if it is an arc the argument is much simpler).

We consider two little segments δ_1, δ_2 of equal r-arc length having z as common vertex and contained in γ_ℓ, $\tilde{\gamma}_m$ respectively. We construct two quadrilaterals G and G' by applying the main lemma to δ_1 and δ_2 (with $Z^{-1} = {}_0$ (say) replacing C and the same ε) .



Then, if ε had been conveniently chosen, close to zero, the sets $\hat{G}$, $\hat{G}'$ have an intersection $\hat{G} \cap \hat{G}'$ such that $\nu(\hat{G} \cap \hat{G}') > 0$: we can assume that $\hat{G}$ and $\hat{G}'$ are constituted by fibers with full intersection with M_f and intersecting δ_1 and δ_2 (respectively) in M_f: in this way we can construct two expanding contracting segments $\bar{\gamma}_\ell$ and $\bar{\tilde{\gamma}}_m$ such that $\bar{\gamma}_\ell \cap \gamma_\ell$, $\bar{\gamma}_\ell \cap \bar{\tilde{\gamma}}_m$, $\bar{\tilde{\gamma}}_m \cap \tilde{\gamma}_m$ are in M_f. This completes the construction if x_0 is never tangent.

The case when x_0 is once tangent can be treated in a similar way: the reason the proof does not fail is linked to the fact that the main lemma requires only that $T^i x_0$ is never tangent or $T^{-i} x_0$ is never tangent if $i \geq 0$: so one has to be careful in constructing the string of curves in such a way that when the string has to cross the singularity line through x_0 the segment that does the crossing is the one whose existence can be guaranteed by the lemma.

The proof of Lemma 17 which is of interest to us, just follows the above ideas: it is of course complicated by the fact that we have to deal with approximate fibers and therefore the role of M_f has to be played by $W^f_{k,\rho}$ which does not have full measure. See Appendix C for a complete proof.

8. Concluding Remarks

To go beyond ergodicity one can proceed along the general lines discussed in Weiss' lectures. The easiest way would be to prove the validity of the $k = \infty$ version of Theorem 9. Such a result would provide us with a set V of positive measure which is foliated by germs of contracting and expanding fibers which on V are absolutely continuous. Then a smooth partition of P, thought of as a parition of $\hat{M} = \bigsqcup_{i=1}^{\infty} T^i V$ would have a tail of σ-algebra whose sets are trivial and, therefore, T restricted to $\hat{M}$ is a K-system. Ergodicity implies that $\nu(\hat{M}) = 1$, so (M,T,ν) is a K-system. This, in turn, implies that (M,T,ν) is isomorphic to a Bernoulli scheme.

The version for $K = \infty$ of Theorem 9 (and of Lemma 13) does not involve too much work after the proof for $K < \infty$. The details of this proof are, however, missing in the literature. This is the reason why we have not relied on such a result in these lectures.

The mixing character of (M,T,ν) insures that (V,S_t,μ) is an ergodic flow (in the sense that there is no S_t-invariant $(\forall t \in (-\infty,+\infty))$ nontrivial set in V).

To prove that (V,S_t,μ) is a Bernoulli (or a K-) flow one has to construct foliations of expansion and contraction in (V,S_t,μ) and prove their absolute continuity and their "nonintegrability" in the sense discussed in [4] or in Weiss' lectures.

The existence and absolute continuity is easily obtained without much extra work while the nonintegrability does not seem to have been derived in detail in the published literature.

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Appendix A. Proof of Theorems 5, and 9, and Lemma 7.

(i) If $\bar{\gamma}$ is smooth and has equation $\phi = \bar{\phi}(r)$ and $d\bar{\phi}/dr \le 0$ and $T, T^2, \ldots, T^k$ are smooth on $\bar{\gamma}$, then a repeated application of the formulas expressing the equation of the curve in terms of its T-image yield

$$\frac{1}{-\cos\bar{\phi}(r)}\frac{d\bar{\phi}}{dr}(x) = f^{(k)}\left(\frac{2\,K(x)}{\cos\phi(x)},\ \tau(T^{-1}x),\ldots,\tau(T^{-k}x),\right.$$

$$\left.\left(-\left.\frac{d\bar{\phi}}{dr}\right|_{T^{-k}x} + K(T^{-k}x)\quad\frac{1}{\cos\phi(T^{-k}\,x)}\right)\right)$$

where $d\bar{\phi}/dr|_{T^{-k}x}$ abbreviates the r-derivative of the equation $T^{-k}\gamma$ computed at $r(T^{-k}x)$ and

$$f^{(k)}(a_1,b_1,\ldots,b_{h-1},a_h) = \frac{a_1}{2} + \cfrac{1}{b_1 + \cfrac{1}{a_2 + \ddots \cfrac{1}{b_{h-1} + \cfrac{1}{a_h}}}}$$

(i) The only property of $f^{(h)}$ that we shall really need is that if h is large, then $f^{(h)}$ depends very little on the variables with large index provided the entries of the continued fraction are not too small; more precisely, we shall need the following statement: if $0 < \sigma < \min\limits_{1\le i\le k-1} |b_i\; a_{i+1}|$ and $0 < \sigma < \min\limits_{1\le i\le k} |a_i\; b_i|$ and if the entries a_i, b_i all have the same sign there is a constant Q such that

$$\left|\frac{\partial\log f^{(k)}}{\partial a_i}\right| \le \frac{Q}{(1+\sigma)^{2i}}\frac{1}{a_i}, \qquad i = 1,2,\ldots,k$$

$$\left|\frac{\partial\log f^{(k)}}{\partial b_i}\right| \le \frac{Q}{(1+\sigma)^{2i}}\frac{1}{b_i}, \qquad i = 1,2,\ldots k-1$$

(Q could actually be chosen as $2(1+\sigma^2)$). In the case a_i, b_i have the values in (i) above the parameter σ could be $\tau_0 R_+^{-1}$.

(iii) From simple trigonometric considerations it is possible to see that $\exists\ c > 0$ such that

$$- \cos \phi(x) \leq C\ d(x)$$

$$- \cos \phi(T^{\pm 1}x) \geq C\ d(x)$$

$$|\tau(x)| \leq \frac{C}{d(x)}$$

(iv) The proof of Theorem 5 proceeds as follows. Let $x \in M_q^{(k)+}$ and put $d_i = c_i = q^{-1}(1+i^2)^{-1}$. Let $\tilde{\gamma}_c^{(k)}(x) = T^k\tilde{\gamma}_c(T^{-k}x)$. The set $\tilde{\gamma}_c^{(k)}(x)$ consists of a union of smooth curves. Let $\approx{\gamma}^{(k)}$ be the smooth curve which contains x. Let $\Delta > 0$ be so small that the curve $\approx{\gamma}^{(k)}(x)$ is at least above all the points of the interval $I = [r(x)-\Delta,\ r(x)+\Delta]$. Call $\tilde{c}_h = \min_{y\in I} |\cos \phi(T^{-h}y)|$ and $\tilde{d}_h = \min_{y\in I} d(T^{-h}y)$. The formula for the contraction coefficient $p(T\ d\gamma)\ /\ p(d\gamma)$ in Section 2 implies (see the remarks in section 2):

$$p(T^{-h}\ \approx{\gamma}^{(k)}) \leq \lambda^{-h} \quad (\approx{\gamma}^{(k)}) \ , \qquad h = 0,\ldots,k,$$

where $\lambda > (1+\tau_0/R_+) > 1$ (note that we are using that, by construction, the curves $T^{-h}\ \approx{\gamma}^{(k)}$ are decreasing $\forall h = 0,\ldots,k$).

The a priori bound on the slope of $T^{-h}\ \approx{\gamma}^{(k)}$ imply that the Euclidean length $|T^{-h}\ \approx{\gamma}^{(k)}|$ is related to $p(T^{-h}\ \approx{\gamma}^{(k)})$ by (if $\phi(r)$ is the equation of $T^{-h}\ \approx{\gamma}^{(k)}(x)$):

$$|T^{-h}\ \approx{\gamma}^{(k)}| = \int_{T^{-h}\approx{\gamma}^{(k)}} dr\sqrt{1+(\frac{d\phi}{dr})^2} \leq C' \int_{T^{-h}\,'\approx{\gamma}^{(k)}} dr \leq \frac{C'}{c_h}\, p(T^{-h}\ \approx{\gamma}^{(k)}) \ ,$$

for some suitable C'.

Hence

$$\tilde{d}_{h+1} \geq d_{h+1} - |T^{-h-1}\ \approx{\gamma}^{(k)}| \geq d_{h+1} - \frac{2\ C'}{\tilde{c}_{h+1}}\, \lambda^{-(h+1)}\Delta \ ,$$

and (iii) above implies for suitable " > 0:

$$\tilde{d}_{h+1} \geq d_{h+1} - \frac{c''}{c_h} \lambda^{-h-1} \Delta.$$

Similarly for some c" > 0,

$$\tilde{c}_{h+1} \geq c_{h+1} - \int_{T^{h-1}\tilde{\tilde{\gamma}}(k)} |\sin \phi| \left|\frac{d\phi}{dr}\right| dr \geq c_{h+1} - \frac{c''}{\tilde{c}_h} \lambda^{-h-1} \Delta;$$

from the last two formulas it is easy to infer by induction that there is $\Delta_q > 0$ such that if $\Delta < \Delta_q$ then $\tilde{c}_h > c_h/2$, $\tilde{d}_h > d_h/2$. Clearly this implies Theorem 5.

(v) Suppose that $k' > k \geq 1$ and consider the two curves $\gamma_c^{(k)}(x)$ and $\gamma_c^{(k')}(x)$; they are two decreasing curves stemming from x and defined "above" the interval $[r(x)-\Delta_q, r(x)+\Delta_q] = J_x$.

Let $\bar{r} \in J_x$ and define the vertical segment $\gamma_{\bar{r}} = \{r,\phi \mid r = \bar{r},\ \phi \text{ is between } \phi_c^{(k)}(\bar{r}) \text{ and } \phi_c^{(k')}(\bar{r})\}$: this segment can be regarded as an increasing curve; so that $T^{-k}\gamma_{\bar{r}}$ is increasing and with slope between R_+^{-1} and $R_-^{-1} + \tau_0^{-1}$. Furthermore $T^{-k}\gamma_{\bar{r}}$ is connected: otherwise, if $\bar{h} < k$ is the maximum h for which $T^{-h}\gamma_{\bar{r}}$ is connected, there would exist a singularity line for T^{-1} which cuts $T^{-\bar{h}}\gamma_{\bar{r}}$. This would imply that there is a singularity line which crosses either $T^{-\bar{h}}\gamma_c^{(k)}$ or $T^{-\bar{h}}\gamma_c^{(k')}$ which is impossible impossible because $\tilde{d}_h \geq (2q)^{-1}(1+h^2)$, $\forall h = 0,\dots,k$ on such curves.

The r-length of $T^k\gamma_{\bar{r}}$ has an upper bound $B = \max_{i \leq i \leq s} L_i$. (Where $L_i = \text{length } \partial O_i$); therefore

$$r(T\gamma_{\bar{r}}) \leq \frac{\lambda_0^{-(k-2)}}{\lambda_0 - 1} B$$

which easily implies that there is a geometric constant B' such that

$$|\gamma_{\bar{r}}| = |\phi^{(k)}(\bar{r}) - \phi^{(k')}(\bar{r})| \leq \lambda_0^{-k} B' \qquad \forall k' \geq k.$$

This proves (v) of Lemma 7.

(vi) The statements (i), (ii) and (iii) of Lemma 7 are either already implicitly proved or are a simple consequence of the exponential contraction of the length of $T^{-h}\gamma_c^{(k)}(x)$ as a function of h and the remark that on $T^{-h}\gamma_c^{(k)}(x)$ the quantities c_h, d_h are bounded below by $\frac{1}{2}c_h$ and $\frac{1}{2}d_h$ respectively. Statement (iv) of Lemma 7 is also a simple consequence of (i), (ii) and (iii) in Lemma 7 and of the continued fraction formula for the slope $d\phi^{(h)}/dr$ of $T^{-h}\gamma_c^{(k)}(x)$ and of the convergence property of the continued function discussed in (i), (ii) of this appendix.

(vii) Theorem 9 for the case of $\hat{U}$ now easily follows from Lemma 7, Corollary 8 along the lines discussed in Section (iv) from the explicit expressions for $\rho^{(k)}(\alpha,\beta)$. The cases of $\hat{U}'$, $\hat{U}''$ can be treated along the same lines as $\hat{U}$: i.e. it is easy to find for $\rho^{(k)}(\alpha,\beta)$ an explicit expression which has essentially the same structure as the expression for $\rho^{(k)}$ in the case of $\hat{U}$.

We do not give the details of this part of the proof, not only because of the similarity between the proof for the case of $\hat{U}$ and the case of $\hat{U}'$, $\hat{U}''$ but also because in Appendix B a detailed proof of a more difficult similar result is given.

The expression for $\rho^{(k)}$ in the cases $\hat{U}'$, $\hat{U}''$ can be easily found with the help of an auxiliary smooth foliation ξ which contains $\gamma_c^{(k)}(x)$ or $\gamma_e^{(k)}(x)$ respectively: the formal expression for $\rho^{(k)}$ in this case is the same found in Appendix B for the quantity that, there, is called $\rho^{(k)}$.

The proofs of this appendix follow ref. [3], [1].

Appendix B. Proof of Lemma 15

Let ξ be the smooth foliation of G_0 with lines parallel to $\gamma_\ell(G_0)$ along $\gamma_u(G_0)$ (if $\gamma_\ell(G_0)$ is too short we continue it a little keeping its slope smooth and still between R_+^{-1} and $(R_-^{-1} + \tau_0^{-1}))$.

If $\gamma \in \xi$ we denote $\phi_\xi^h(r)$ the equation of $T^h\gamma$. If $z \in G_0$ we denote $(d\phi_\xi^h/dr)(T^h z)$ as the slope at $T^h z$ of the curve of $T^h\xi$ which contains $T^h z$. Our assumption on the slope of $\gamma_\ell(G_0)$ allows us to deduce that

$$Z^{-1} \leq \frac{\frac{d\phi_\xi^h}{dr}(T^h z)}{\frac{d\phi_\xi^h}{dr}(T^h z')} \leq Z , \qquad \forall z, z' \in G_0 \ ; \ \forall h = 0,\ldots,k,$$

where, we remember, $Z = R_+(R_-^{-1} + \tau_0^{-1})$.

Assume, for simplicity, that $L = 1$. Then the sides $\gamma_r(G_0)$, $\gamma_\ell(G_0) \in \xi$ and $\gamma_u(G_0), \gamma_d(G_0)$ are contained in k-expanding fibers (by assumption).

If $z \in G_0$ we define the intrinsic coordinates of z to be the r-arc lengths (α,β) of the arcs xz and $\bar{z}z$ along, respectively, $\gamma_\ell(G_0)$ and the k-expanding fiber through z.

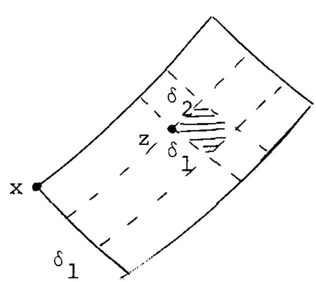

Figure 8

Let E be the set dashed in Figure 8 where the decreasing dotted lines represent close elements of ξ.

Let $|\cdot|$ denote the Euclidean length of an arc of a curve; then

$$r(\delta_1) = \frac{|\delta_1|}{\left[1 + \left(\frac{d\phi_\xi}{dr}(z)\right)^2\right]^{1/2}}, \quad z(\delta_2) = \frac{|\delta_2|}{\left[1 + \left(\frac{d\phi_e}{dr}(z)\right)^2\right]^{1/2}};$$

and the angle ω between δ_1 and δ_2 is

$$\sin\omega = \frac{(d\phi_e/dr)(z) - (d\phi_\xi/dr)(z)}{\left[1 + \left(\frac{d\phi_\xi}{dr}(z)\right)^2\right]^{1/2} \left[1 + \left(\frac{d\phi_e}{dr}(z)\right)^2\right]^{1/2}}$$

so that (remembering that, by definition, $-\cos\phi\, r(\delta) = p(\delta)$ for an infinitesimal arc $\delta \ni (r,\phi)$):

$$\begin{aligned}\rho^{(k)}(\alpha,\beta) &= \frac{\nu(E)}{r(\bar\delta_1)\, r(\delta_2)} = -\frac{|\delta_1|\,|\delta_2|}{r(\bar\delta_1)\, r(\delta_2)} \sin\omega \cos\phi(z) \\ &= -\cos\phi(\bar z)\, \frac{p(\delta_1)}{p(\bar\delta_1)} \left(\frac{d\phi_e}{dr}(z) - \frac{d\phi_\xi}{dr}(z)\right).\end{aligned}$$

An identical construction can be performed on $T^h G_0$ around $T^h z$ using the system of intrinsic coordinates similar to the one used for G_0: i.e. defining the α-coordinate as the r-length of the image under T^h of the arc $\widehat{x\bar z}$ along $T^h \gamma_\ell(G_0)$ and the β-coordinate as the r-length of the image (under T^h) of the arc $\widehat{z\bar z}$ on the k-expanding fiber through z.

We find that, in this case, the Radon-Mykodim derivative of ν with respect to $d\alpha\, d\beta$ can be written as

$$(*) \qquad -\cos\phi(T^h \bar z)\, \frac{p(\delta_1)}{p(\bar\delta_1)} \left\{\frac{d\phi_e^h}{dr}(T^h z) - \frac{d\phi_\xi^h}{dr}(T^h z)\right\},$$

where δ_1 and $\bar\delta_1$ are infinitesimal arcs on the fibers of the foliation through, respectively, $T^h x$ and $T^h z$; the relation between δ_1 and $\bar{\delta}_1$ being that they are cut, on the fibers of $T^h\xi$ on which they lie, by the same two (k-h)-expanding fibers (as in Figure 8). We observe that the above infinitesimal arguments make sense beca because of the smoothness of the foliation $T^h\xi$ and of the (k-h)-expanding foliation.

We observe next that

$$-\frac{1}{\cos\phi(T^h z)}\frac{d\phi_\xi^h}{dr}(z) = \frac{k(T^h z)}{\cos\phi(T^h z)}$$

$$+\cfrac{1}{\tau(T^{h-1}z)+\cfrac{1}{\cfrac{2k(T^{h-1}z)}{\cos\phi(T^{h-1}z)}+\cdots\cfrac{1}{\tau(z)+\cfrac{1}{\cfrac{1}{-\cos\phi(z)}\left(\cfrac{d\phi_\xi}{dr}(z)-K(r)\right)}}}}$$

$$\frac{1}{\cos\phi(T^h z)}\frac{d\phi_e^h}{dr}(z) = \frac{k(T^h z)}{\cos\phi(T^h z)}$$

$$+\cfrac{1}{\tau(T^h z)+\cfrac{1}{\cfrac{2k(T^{h+1}z)}{\cos\phi(T^{h+1}z)}+\cdots\cfrac{1}{\tau(T^{k-1}z)+\cfrac{1}{\cfrac{2k(T^k z)}{\cos\phi(T^k z)}}}}}$$

We now consider the various elements that enter into the expression (*) and study how much they vary inside $T^h G_0$.

Clearly

$$\left|\frac{\cos\phi(T^h z)}{\cos\phi(T^h z')}-1\right| \le \text{const.}\ (1+h^{b-4})^{-1}, \quad \forall z,z' \in G_0,\ h=0,\dots,k;$$

$$\left|\frac{K(T^h z)}{K(T^h z')}-1\right| \le \text{const.}\ (1+h^{b})^{-1}, \quad \forall z,z' \in G_0,\ h=0,\dots,k;$$

where the const. depends upon C but not on δ_0; if we were only interested in $k \geq k_0$ then the constant could be chosen C-independent (for the definition of k_0 see Section 5).

It is also not difficult to see that

$$\left|\frac{\tau(T^h z)}{\tau(T^h z')} - 1\right| \leq \text{const. } (1+h^{b-4})^{-1}, \quad \forall z,z' \in G_0,\ h=0,\dots,k-1.$$

This, in fact, immediately follows from the easy trigonometric estimates

$$\left|\frac{\partial \log \tau(x)}{\partial r}\right| \leq - \frac{\text{const.}}{\cos \phi(Tx)} \quad \forall x \in M ;$$

$$\left|\frac{\partial \log \tau(x)}{\partial \phi}\right| \leq - \frac{\text{const.}}{\cos \phi(Tx)} \quad \forall x \in M .$$

The above estimates on the maximum variations of the quantities τ, $\cos \phi$, k on $T^h G_0$ will be considered with the estimate of the logarithmic derivatives of the continued fraction expressing $d\phi^h/dr$ or $d\phi_e^h/dr$ (see above) considered in Appendix A.

We then easily obtain, using the continued fraction expressions for $d\phi_\xi^h/dr$ and $d\phi_e^h/dr$ that $h = 0,1,\dots,k$,

$$\left|\frac{\frac{1}{-\cos \phi(T^h z)} \frac{d\phi_\xi^h}{dr}(z)}{\frac{1}{-\cos \phi(T^h z')} \frac{d\phi_\xi^h}{dr}(z')} - 1\right| \leq \text{const. } \sum_{i=0}^{h} (1+\sigma)^{-2i}(1+(h-i)^{b-4})^{-1}$$

$$\leq \text{const.}/(1+h^{b-4}) ;$$

$$\left|\frac{\frac{1}{\cos \phi(T^h z)} \frac{d\phi_e^h}{dr}(z)}{\frac{1}{\cos \phi(T^h z')} \frac{d\phi_e^h}{dr}(z')} - 1\right| \leq \text{const. } \sum_{i=0}^{h} (1+\sigma)^{-2i}(1+(h+i)^{b-4})^{-1}$$

$$\leq \text{const.}/(1+h^{b-4}) ;$$

where σ can be taken τ_0/R_+ and the constant depends on C (but not on δ_0). It remains, therefore, to estimate the variation on $T^h G_0$ of $p(\delta_1)/p(\bar{\delta}_1)$.

As already remarked in the lectures

$$\frac{p(\delta_1)}{p(\bar{\delta}_1)} = \left\{ \prod_{i=0}^{k-h-1} \frac{\dfrac{p(T^{h+i}\,\delta_1)}{p(T^{h+i+1}\,\delta_1)}}{\dfrac{p(T^{h+i}\,\bar{\delta}_1)}{p(T^{h+i+1}\,\bar{\delta}_1)}} \right\} \frac{p(T^k\,\delta_1)}{p(T^k\,\bar{\delta}_1)}$$

and, assuming for simplicity $k(r) = R_e^{-1} = $ const. on the obstacle $\partial 0_\ell$,

$$= \left\{ \prod_{i=0}^{k-h-1} \frac{1 + \left(K(T^{h+1}\bar{z}) - \dfrac{d\phi_\xi^{h+i}}{dr}(T^{h+i}\,\bar{z})\right)\dfrac{\tau(T^{h+i}\,\bar{z})}{\cos\,\phi(T^{h+i}\,\bar{z})}}{1 + \left(K(T^{h+i}z) - \dfrac{d\phi_\xi^{h+i}}{dr}(T^{h+i}\,z)\right)\dfrac{\tau(T^{h+i}\,z)}{\cos\,\phi(T^{h+i}\,z)}} \right\}$$

$$\cdot \frac{R^{-1} + \dfrac{d\phi_\xi^{(k)}}{dr}(T^k\,z)}{R^{-1} + \dfrac{d\phi_\xi^{(k)}}{dr}(T^k\,\bar{z})}$$

where R^{-1} is the curvature of the obstacle on which $T^k z$, $T^k \bar{z}$ collide.

The just derived estimates on the variations of K, τ, cos ϕ, and $d\phi_\xi^h/dr$ on $T^h G_0$ allow us to immediately conc ude that the quantity appearing in the r.h.s. of the last equation differs from 1 by at most const.$\cdot(1+h^{b-4})^{-1}$, $h = 0,1,\ldots,k$ and the constant is δ_0-independent.

This completes the proof of the lemma when L = 1 and the obstacles are circular: the general case is a simple adaptation of the above arguments and we do not insist on the few obvious changes of symbols and of the numerical values of the constants (remember that L does not depend on C, δ_0 etc. but only on geometrical constants).

We notice that the above proof is essentially a repetation of the proof of the absolute continuity.

Appendix C. Proof of Lemma 17.

Let $C = 5R_+(R_-^{-1} + \tau_0^{-1}) = 5Z$. Let ε be a certain geometrical constant whose choice will be clear later. Consider the neighborhood $U = U(\varepsilon, C)$ of x_0.

Let $\frac{1}{2} > \eta > 0$ be a fixed positive number. Determine $q = q_\eta$ so that

$$\nu(U \setminus U \cap M_{q/4}) < \eta^2 \, \nu(U) .$$

Let (say) $\xi = 10^{-1} \arctan R_+^{1}$, and let $x, y \in M_{q/4} \cap U$ such that y is in the horizontal cone with vertex x and opening angle α such that $\tan \alpha = \xi$. Assume also that x, y are so close that it makes sense to speak of the quadrilateral G obtained by drawing through x, y the expanding and contracting germs. We also assume that the diameter of G is so small that Theorem 9 applies in a neighborhood of x containing G and, furthermore, the function $\theta_q(D(G))$ defined by Theorem 9 is such that

$$|\exp \pm \theta_q(D(G)) - 1| < \eta^2 .$$

Let Σ be the family of the (closed) quadrilaterals constructed as above. The family Σ covers the Lebesgue set of $M_{q/4} \cap U$ in the sense of Vitali.

Let $\{G_i\}_0^\infty$ be a sequence of pairwise disjoint sets $G_i \in \Sigma$, $i = 0,1,2,\ldots,$ so that $G_i \subset U$, $\bigcup_i G_i \supset M_{q/4} \cap U \pmod 0$. Then $\nu\left(\bigcup_{i=0}^{\infty} G_i \setminus M_{q/4} \cap U\right) < \eta^2 \, \nu(U)$.

Let I be the set of indices $i = 0,1,\ldots$ such that

(a) $\qquad \nu(G_i \cap (M_{q/4} \cap U)) > (1-\eta)\, \nu(G_i) , \quad i \in I$

then

(b) $\qquad \sum_{i \notin I} \nu(G_i) < \eta \, \nu(U) .$

We can choose a set J, finite and still satisfying (a), (b) above when J replaces I and, by relabeling, we can put $J = (1,2,\dots,N)$.

Let $2d_\eta > 0$ be the smallest between the r-lengths of arcs joining the opposite sides of $G_0,\dots,G_N$ and having slope bounded, in modulus, between R_+^{-1} and $(R_-^{-1}+\tau_0^{-1})$. Let $c_0(\eta)$ be the smallest between the measures $\nu(G_i)$, $i \in J$.

Let K_η be an integer such that for all $k \geq K_\eta$ the quadrilaterals $G_i^{(k)}$ obtained by drawing hrough x_i,y_i the k-germs of expanding and contracting fibers are so close to the respective G_i that $G_i^{(k)} \subset U$ and property (a) above holds when $G_i^{(k)}$ replaces G_i. This also means that

$$\nu\Big(\bigsqcup_{i=1}^{N} G_i^{(k)} \setminus M_{q/4} \cap U)\Big) < 2\eta\, \nu(U) \;.$$

It is by virtue of Corollary 8 that the above K_η exists. Corollary 8 also implies that K_η can and will be so chosen that the smallest of the r-lengths of the segments of arcs joining opposite sides of $G_i^{(k)}$ and with absolute slope between R_+^{-1} and $(R_-^{-1} + \tau_0^{-1})$ is $> d_\eta$.

Consider $G_i^{(k)}$ and let $z \in M_q^{(k)^{\pm}} \cap G_i^{(k)}$, draw the k-contracting fiber $\gamma_e^{(k)}(z)$ or the k-expanding fiber $\gamma_c^{(k)}(z)$ through z. Then

$$r\big(\gamma_c^{(k)}(z) \cap G_i^{(k)}\big) > d_\eta$$

$$r\big(\gamma_e^{(k)}(z) \cap G_i^{(k)}\big) > d_\eta \;.$$

Let us open a side discussion in order to define like constants $c_i(\eta)$, $i = 1,2,\dots,5$.

Consider a decreasing segment $\gamma \subset U$ with slope between $-(R_-^{-1} + \tau_0^{-1})$ and $-R_+^{-1}$ and such that $r(\gamma) > \frac{1}{GZ}\, d$ (say).

Draw the five equispaced horizontal lines as in Figure 10. Their spacing is $\Delta\phi \sim$ order of d_η.

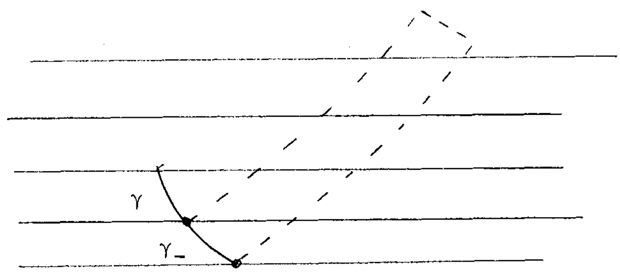

Figure 10

Apply Lemma 13 to the lower "half" γ_- of γ to construct a quadrilateral G for a $k \geq K_\eta$. The choice of C guarantees that the right side γ_r of G is above the uppermost line. Let D be the dashed region (i.e. the intersection of G with the strip between the fourth and the fifth line).

If we introduce on $\hat{G}^{(k)}$ the intrinsic coordinates mentioned in Lemma 13 and use that $\rho^{(k)}(\alpha,\beta)$ is bounded away from zero and infinity we see that $\exists\, c_1(\eta)$ such that

$$\nu(\hat{G}^{(k)} \cap D) > c_1(\eta) \qquad \forall\, k$$

where $c_1(\eta)$ is a suitable η-dependent (but k-independent) constant.

Also $\exists\, c_2(\eta)$, k-independent, such that

$$r(\hat{G}^{(k)} \cap \gamma_-) > c_2(\eta)\ r(\gamma_-) > 0\ .$$

Similar constructions can be performed when γ_- is taken to be the right side of the quadrilateral G or when γ is assumed increasing (with the usual slope restriction) and is taken as the upper or lower side of G: we may, and shall, assume also that the constants $c_1(\eta)$, $c_2(\eta)$ are good also for the corresponding statements in these similar cases.

We consider next two segments $\gamma_+,\gamma_- \subset U$ such that

$\emptyset \neq \gamma_+ \cap \gamma_- = z \in U$. Let $\bar{\gamma}_+$, $\bar{\gamma}_-$ be the two longest (in the r-length sense) parts into which z cuts γ_+ and γ_- respectively. We assume that γ_+, γ_- are respectively increasing and decreasing with absolute slope between R_+^{-1} and $R_-^{-1} + \tau_0^{-1}$. Suppose also that $r(\bar{\gamma}_+)$, $r(\bar{\gamma}_-) >$ const. d_η where the constant is suitably small (say const. = 1/6 Z). Finally suppose that $r(\bar{\gamma}_+) = r(\bar{\gamma}_-)$.

We can apply the constructions allowed by Lemma 13 to $\bar{\gamma}_+$ or $\bar{\gamma}_-$ with C = 2 (say) and the same ε fixed at the beginning (and not yet exhibited) and arbitrary $k \geq K_\eta$. We obtain, in this way, two quadrilaterals G_+, G_-.

It is easy to see that there are geometrical constants $\varepsilon(x_0)$, $A(x_0)$ (η and k independent) so that

$$\varepsilon(x_0) \leq \frac{\nu(G_+ \cap G_-)}{\nu(G_\pm)} < \varepsilon(x_0)^{-1}$$

$$A(x_0) \leq \frac{\nu(G_+)}{\nu(G_-)} \leq A(x_0)^{-1}$$

Such inequalities follow from simple geometric considerations on the slopes of the sides of G_+, G_-. Then

$$\nu\left((G_+ \setminus \hat{G}_+^{(k)}) \cap G_+ \cap G_-\right) < \nu\left(G_+ \setminus \hat{G}_+^{(k)}\right) < \varepsilon\ \nu(G_+) \ .$$

Therefore

$$\nu(G_+ \cap G_-) - \nu(\hat{G}_+^{(k)} \cap G_+ \cap G_-) < \varepsilon\ \nu(G_+)$$

or

$$\nu(\hat{G}_+^{(k)} \cap (G_+ \cap G_-)) > \left(\varepsilon(x_0) - \varepsilon\right)\nu(G_+) \ ;$$

and similarly

$$\nu(\hat{G}_-^{(k)} \cap (G_+ \cap G_-)) > \left(\varepsilon(x_0) - \varepsilon\right) \nu(G_-) \ .$$

Hence

$$\nu\left((G_+ \setminus \hat{G}_+^{(k)}) \cap \hat{G}_-^{(k)} \cap (G_+ \cap G_-)\right\} \leq \nu\left(G_+ \setminus G_+^{(k)}\right\} < \varepsilon\ \nu(G_+) \ ,$$

or

$$\nu(\hat{G}_-^{(k)} \cap G_+ \cap G_-) - \nu(\hat{G}_+^{(k)} \cap \hat{G}_-^{(k)} \cap G_+ \cap G_-) < \varepsilon\ \nu(G_+)$$

or

$$\nu(\hat{G}_+^{(k)} \cap \hat{G}_-^{(k)}) \equiv \nu(\hat{G}_+^{(k)} \cap \hat{G}_-^{(k)} \cap G_+ \cap G_-) > (\varepsilon(x_0)-\varepsilon)\nu(G_-)-\varepsilon\nu(G_+)$$

$$> \left(\frac{\varepsilon(x_0)-\varepsilon}{A(x_0)} - \varepsilon\right)\nu(G_+)$$

it is now clear that ε should have been chosen so that

$\frac{\varepsilon(x_0)-\varepsilon}{A(x_0)} - \varepsilon > 0.$

With this choice of ε we also have

$$\nu(\hat{G}_+^{(k)} \cap \hat{G}_-^{(k)}) > c_3(\eta) > 0$$

$$\nu(\hat{G}_+^{(k)} \cap G_-) > c_3(\eta)$$

$$(\hat{G}_-^{(k)} \cap G_+) > c_3(\eta)$$

where $c_3(\eta)$ is a suitable k-independent number, $\forall k \geq K_\eta$ (one could take $c_3(\eta) = \frac{\varepsilon(x_0)}{A(x_0)} c_1(\eta))$.

Using intrinsic coordinates on $\hat{G}_+^{(k)}$, as in Lemma 13, we also find

$$r(\hat{G}_+ \cap \bar{\gamma}_+) > c_4(\eta)\ r(\bar{\gamma}_+)$$

$$r(\hat{G}_- \cap \bar{\gamma}_-) > c_4(\eta)\ r(\bar{\gamma}_-)$$

where $c_4(\eta)$ is a suitable k-independent number, $\forall k \geq K_\eta$. These inequalities imply the inequality that we describe in the next proposition.

Let E_1 be a subset of $\bar{\gamma}_+$ and $E_2 \subset \bar{\gamma}_-$; then if E_1, E_2 fill a large enough fraction of $\bar{\gamma}_+, \bar{\gamma}_-$, respectively, the set

$$\hat{E}_1 \times \hat{E}_2 = \{z \mid z=\gamma_e^{(k)}(x_2) \cap \gamma_c^{(k)}(x_1),\ x_1 \in \bar{\gamma}_+ \cap E_1 \cap \hat{G}_+^{(k)},$$
$$x_2 \in \bar{\gamma}_- \cap E_2 \cap \hat{G}_-^{(k)}\},$$

is such that

$$\nu\left((\hat{E}_1 \times \hat{E}_2) \cap (\hat{G}_+^{(k)} \cap \hat{G}_-^{(k)})\right) > c_5(\eta)$$

the fraction of $\bar{\gamma}_+$ and $\bar{\gamma}_-$ that has to be covered in order that the above inequality holds depends solely on η (and not on $k \geq K_\eta$). We may and shall assume that $c_5(\eta)$ measures the fraction of $\bar{\gamma}_+, \bar{\gamma}_-$ that can be left out of $\bar{\gamma}_+ \cap E_1$ or $\bar{\gamma}_- \cap E_2$ still keeping the last inequality valid.

Let $c_\eta = \min_{i=0,\ldots,5} c_i(\eta)$.

We are now ready to complete the proof of Lemma 17.

Given $\eta > 0$ we choose $\bar{k}$ so large that all subsets $\Delta \subset U$ such that

$$\nu(\Delta) >> \frac{1}{2} c(\eta)$$

have the property

$$\nu(W^f_{\bar{k},p} \cap \Delta) > (1 - \sigma(\eta))\, \nu(\Delta)$$

where f is a continuous function on M, $W^f_{\bar{k},p}$ is the set of points where the Birchoff average $\bar{f}$ is attained within $\rho > 0$ after $\bar{k}$ steps at most; the function $\sigma(\eta)$ will be determined later.

Fix $K > \bar{K} + K_\eta$.

Consider the quadrilaterals $G_i^{(k)}$. Let

$$\hat{G}_i^{(k)} = \bigsqcup_{z \in M_{q/4} \cap G_i^{(k)}} \left(\gamma_c^{(k)}(z) \cap G_i^{(k)}\right). \quad \text{Then}$$

$$M_{q/4} \cap G_i^{(k)} \subset \hat{G}_i^{(k)} \subset G_i^{(k)} \cap M_{q/2}^{(k)+} \subset G_i^{(k)}$$

and

$$\nu(\hat{G}_i^{(k)} \cap W^f_{\bar{k},\rho}) > (1 - \sigma(\eta))\, \nu(\hat{G}_i^{(k)}) .$$

So, using intrinsic coordinates as in Lemma 13, we can write this inequality as

$$\int_0^{\bar{\alpha}} d\alpha \int_{\alpha(\hat{G}_i^{(k)} \cap W^f_{\bar{k},\rho})} \rho^{(k)}(\alpha,\beta)\, d\beta > (1-\sigma(\eta)) \int_0^{\bar{\alpha}} d\alpha \int_0^{\beta(\alpha)} \rho^{(k)}(\alpha,\beta)\, \Lambda\beta ,$$

where $\bar{\alpha}$, $\beta(\alpha)$ are the natural integraion bounds and $\alpha(\hat{G}_i^{(k)} \cap W^f_{\bar{k},\rho})$ denotes the r-measure of the intersection between $W^f_{\bar{k},\rho} \cap G_i^{(k)}$ and the arc $\gamma_c^{(k)}$ which has coordinate α.

This implies that the set N of the α's such that

$$\int_{\alpha(\hat{G}_i^{(k)} \cap W^f_{\bar{k},\rho})} \rho^{(k)}(\alpha,\beta)\, d\beta < \left(1 - \sqrt{\sigma(\eta)}\right) \int_{\alpha(\hat{G}_i^{(k)})} d\beta\, \rho^{(k)}(\alpha,\beta)$$

has the property

$$\int_N d\alpha \int_{\alpha(\hat{G}_i^{(k)})} \rho^{(k)}(\alpha,\beta)\, d\beta < \sqrt{\sigma(\eta)}\, \nu(\hat{G}_i^{(k)}) \ .$$

If

$$\int_{\alpha(\hat{G}_i^{(k)} \cap W^f_{\bar{k},\rho})} \rho^{(k)}(\alpha,\beta)\, d\beta \geq 1-\sqrt{\sigma(\eta)} \int_{\alpha(\hat{G}_i^{(k)})} d\beta\, \rho^{(k)}(\alpha,\beta)$$

it follows

$$\alpha(\hat{G}_i^{(k)}) - \alpha(\hat{G}_i^{(k)} \cap W^f_{\bar{k},\rho}) \geq 2\sqrt{\sigma}\, \alpha(G_i^{(k)})$$

since

$$\left|\frac{\rho^{(k)}(\alpha,\beta)}{\rho^{(k)}(\alpha',\beta')} - 1\right| < \eta^2 < \frac{1}{2} \quad \text{on} \quad \hat{G}_i^{(k)} \ ,$$

by our construction of the $G^{(k)}$.

This means that the set V'^f of the points $z \in U$ through which it is possible to draw a k-contracting germ of fiber with r-length at least d_η and filled (in the r-length sense) by $W^f_{\bar{k},\rho}$ within $2\sqrt{\sigma(\eta)}$ has measure $\geq (1-(2\eta^2+2\sqrt{\sigma(\eta)}+2\eta))\nu(U)$.

Let x,x' be in $V_\eta^f = V_\eta'^f \cap W^f_{\bar{k},\rho}$ (then $\nu(V_\eta^f) > (1 - 2\sigma(\eta) - 2\eta^2 - 2\sqrt{\sigma(\eta)} - 2\eta)\ \nu(U)$);
we show that there is a number N_η and a chain $\gamma_0,\gamma_1,\ldots,\gamma_{N_\eta}$ of alternatingly k-increasing and k-decreasing semgnets of k-fibers

such that $x \in \gamma_0$, $y \in \gamma_{N_\eta}$, $\gamma_i \cap \gamma_{i+1} \in W^f_{\bar{k},\rho}$, $i = 0,1,\dots,N_\eta-1$.

Let γ_0' = k-contracting fiber segment hrough x which is filled within $2\sqrt{\sigma}$ by $W^f_{\bar{k},\rho}$.

Draw the five equispaced parallel lines, as in the above discussion, relative to the arc γ_0'.

Notice that the lower half $\gamma_0'^{-}$ of γ_0' must be filled within $c_1\sqrt{\sigma(\eta)}$ by $W^f_{\bar{k},\rho}$ (where c_1 is a suitable geometric constant which could be taken as 4Z).

Next we apply Lemma 13 to the segment $\gamma_0'^{-}$ with ε and C chosen as in the preliminary discussion and $K > K_\eta + \bar{K}$. Notice, also, that $r(\gamma_{0-}) >$ const. d_η (with const. = 1/3Z (say)). Also, by our choice of $\bar{K}$:

$$\nu(\hat{G} \cap D \cap W^f_{\bar{k},\rho}) > (1 - \sigma(\eta))\ \nu(\hat{G} \cap D)$$

where D is the dashed region in Figure 10. On the other hand the relation

$$\int_{\gamma'\cap\hat{G}^{(k)}} d\alpha \int_{\alpha(\hat{G}^{(k)}\cap D\cap W^f_{\bar{k},\rho})} \rho^{(k)}(\alpha,\beta)\ d\beta > (1-\sigma(\eta)) \int_{\gamma_0'\cap\hat{G}} d\alpha \int_{\alpha(\hat{G}\cap D)} \rho^{(k)}(\alpha,\beta)\ d\beta$$

which is a rewriting of the preceding equation using the natural coordaintes (α,β) on $\hat{G}^{(k)}$ and symbols consistent with the preceding notations. This means that

$$\alpha(\hat{G} \cap D \cap W^f_{\bar{k},\rho}) > (1 - \sqrt{\sigma}\ B_\eta^2)\ \alpha(\hat{G} \cap D)$$

except for a set J of α's such that

$$\int_J d\alpha \int_{\alpha(\hat{G}\cap D)} \rho^{(k)}(\alpha,\beta)\ d\beta < \sigma^{1-\omega/2}\ \nu(\hat{G} \cap D)$$

where $0 < \omega < 1$ and B_η is the bound on $\rho^{(k)}(\alpha,\beta)/\rho^{(k)}(\alpha',\beta')$ on G.

So if $\bar{W} \subset \hat{G}^{(k)} \cap D \cap W^f_{\bar{k},\rho}$ is the set of points with intrinsic coordinates (α,β) the k-expanding segment through (α,β) is, inside D, filled within $\sqrt{\sigma(\eta)}^{\omega}\, B_{\eta}^{2}$ by points of $W^{f}_{\bar{k},\rho}$ has measure

$$\nu(\bar{W} \cap \hat{G} \cap D) \equiv \nu(\bar{W}) > \left(1 - \sqrt{\sigma(\eta)}^{2-\omega}\right) \nu(\hat{G} \cap D)$$

which implies that, if $E \subset \hat{\gamma}_0'$ is such that $z \in E \Rightarrow$ the k-expanding fiber $\gamma_e^{(k)}(z)$ has nonempty intersection with $\bar{W}$, we must have

$$r(\hat{\gamma}_0'^{-} \setminus E) < \text{const.}\ B_{\eta}^{2}\ \sqrt{\sigma(\eta)}^{2-\omega}\ r(\hat{\gamma}_0'^{-})$$

where the const. is a suitable geometric constant (which is an upper bound for the ratio of the r-lengths of the arcs obtained by intersecting D with two k-expanding fibers of $\hat{G}^{(k)}$).

Therefore

$$r(E) > \left(1 - B_{\eta}^{2}\ \sqrt{\sigma(\eta)}^{2-\omega}\ \text{const.}\right) r(\hat{\gamma}_0^{-})$$

$$> \left(1 - B_{\eta}^{2}\ \sqrt{\sigma(\eta)}^{2-\omega}\ \text{const.}\right) c_4(\eta)\ r(\gamma_0^{-1}) \ .$$

Remembering that $\gamma_0'^{-}$ was filled within $c_1\sqrt{\sigma(\eta)}$ by $W^{f}_{\bar{k},\rho}$ we realize that (if ω and $\sigma(\eta)$ so chosen that

$$\frac{1}{4}\, c_4(\eta) > c_1\ \sqrt{\sigma(\eta)}$$

$$B_{\eta}^{2}\ \sqrt{\sigma(\eta)}^{2-\omega}\ \text{const.} < \frac{1}{2}$$

$$B_{\eta}^{2}\ \sqrt{\sigma(\eta)}^{\omega} < c_1\ \sqrt{\sigma(\eta)}$$

there will be a point $x_1 \in \hat{\gamma}_0' \cap \hat{G}^{(k)}$ whose k-expanding fiber has intersection with D filled within $B_{\eta}^{2}\sqrt{\sigma(\eta)}^{\omega}$ by points of $W^{f}_{\bar{k},\rho} \cap D$.

Let γ_1' be the expanding fiber through x_1 cut at the intersection with the uppermost of the horizontal lines and let $\gamma_1'^{+}$ be $\gamma_1' \cap D$. The third property required on $\sigma(\eta)$ allows us to deduce that γ_1^{+} is filled within $c_1\sqrt{\sigma(\eta)}$ by points of $W^{f}_{\bar{k},\rho}$ and $r(\gamma_1^{-}) \geq \frac{1}{2Z}\, d_{\eta}$. Hence we can iterate the construction.

A similar construction can be performed starting from x'.

In this way we easily obtain a chain $\gamma_0,\ldots,\gamma_n$ and $x \in \gamma_0$, $x' \in \gamma_n$, $\gamma_i \cap \gamma_{i+1} \in W^f_{\bar{k},\rho}$ with only one possible exception occurring at some $i = i_0$; furthermore, the lines γ_i are alternatingly k-expanding and k-contracting segments (this sequence of arcs is obtained from the sequence of segments, constructed in the above interactive scheme, $\gamma_0',\gamma_1',\ldots,\gamma_N'$ by cutting off each γ_i' the pieces that "stick out of the chain").

The exception occurs, possibly, where the chain starting in x and one starting in y meet.

Let us consider the first step in the above construction before continuing the proof. It is clear that

$$\nu(\hat{G}^{(k)} \cap W^f_{\bar{k},\rho}) > (1-\sigma(\eta))\ \nu(\hat{G}^{(k)})$$

by our choice of $\bar{K}$. Which means that the set $\bar{\bar{W}} \subset \hat{G}^{(k)}$ consisting of the points $(\alpha,\beta) \in \hat{G}^{(k)}$ such that the k-expanding fiber through them is filled within $\sqrt{\sigma(\eta)}^{\omega}\ B^2_\eta$ by $W^f_{\bar{k},\rho}$ has measure exceeding $(1 - \sqrt{\sigma(\eta)}^{2-\omega})\ \nu(\hat{G}^{(k)}) > (1 - \sqrt{\sigma}^{2-\omega})\ c_1(\eta) > \frac{1}{2}\ c_1(\eta)$ if $(1 - \sqrt{\sigma(\eta)}^{2-\omega}) > 1/2$.

So the set $\bar{\bar{W}}$ has the property that

$$\nu(\bar{\bar{W}} \cap W^f_{\bar{k},\rho}) > (1-\sigma(\eta))\ \nu(\bar{\bar{W}}))$$

Also

$$r(\bar{\bar{W}} \cap \hat{\gamma}_0^-) > (1 - \sqrt{\sigma(\eta)}^{2-\omega}\ B^2_\eta\ \text{const.})\ r(\gamma_0^-) > \frac{c_4(\eta)}{2}\ r(\gamma_0^-)$$

if $1 - \sqrt{\sigma(\eta)}^{2-\omega}\ B^2_\eta$ const. $> 1/2$.

So we could replace in the above construction of $\gamma_1',\gamma_2',\ldots$ $\hat{G}^{(k)}$ by $\bar{\bar{W}}$: this is the reason why we have used the generous factor 1/4 (instead of 1/2) in the first equation defining $\sigma(\eta)$.

Using this refinement of the above arguments we can construct the chain $\gamma_0,\gamma_1,\ldots,\gamma_N$ so that also the property that γ_i is filled by $W^f_{\bar{k},\rho}$ within const. $\sqrt{\sigma(\eta)}^{\,\omega}\ B^2_\eta$ holds (the geometrical constant coming from the fact that γ'_i has to be chopped at one extreme to give γ_i).

We shall henceforth assume that the segments γ_i are chosen to also have this last property.

Consider γ_{i_0} and γ_{i_0+1}. We notice that we can always assume without loss of generality that $\gamma_{i_0} \cap \gamma_{i_0+1}$ is just a point (and not a segment): this comes from the fact that every time a new element of the chain is built it is chosen out of infinitely many possibilities.

The point Z divides on each of the segments γ_{i_0} and γ_{i_0+1} two segments. Let $\bar{\gamma}_{i_0}$, $\bar{\gamma}_{i_0+1}$ be the longest between them (in the r-length sense).

By construction $r(\bar{\gamma}_{i_0})$, $r(\bar{\gamma}_{i_0+1}) \geq 1/4Z\ d_\eta$. Furthermore $\bar{\gamma}_{i_0}$, $\bar{\gamma}_{i_0+1}$ are filled, in the r-arc length sense, within $\frac{c_1}{2}\sqrt{\sigma(\eta)}$, at least, by points $W_{\bar{k},\rho}$.

We can cut the longest between $\bar{\gamma}_{i_0}$ and $\bar{\gamma}_{i_0+1}$ to obtain two segments with equal arc length which we shall still call $\bar{\gamma}_{i_0}, \bar{\gamma}_{i_0+1}$, but which are now filled only within const. $\sqrt{\sigma(\eta)}$ by $W^f_{\bar{k},\rho}$ where the constant coujld be taken to be $c_1/4$ Z.

By choosing $\sigma(\eta)$ smaller, if necessary, we may assume that the sets $E_1 = \bar{\gamma}_{i_0} \cap W^f_{\bar{k},\rho}$ and $E_2 = \bar{\bar{\gamma}}_{i_0+1} \cap W^f_{\bar{k},\rho}$ fill "enough" of $\bar{\gamma}_{i_0}$ and $\bar{\gamma}_{i_0+1}$ respectively that the set $\hat{E}_1 \times \hat{E}_2$ has measure $> c_5(\eta)$ (we use here the same symbol $\hat{E}_1 \times \hat{E}_2$ introduced in the preliminaries to our proof).

Then $W^f_{\bar{k},\rho} \cap (\hat{E}_1 \times \hat{E}_2)$ must be nonempty and if $\bar{z} \in W^f_{\bar{k},\rho} \cap (\hat{E}_1 \times \hat{E}_2)$ the k-fibers $\gamma_c^{(k)}(\bar{z})$ and $\gamma_e^{(k)}(\bar{z})$ will intersect E_2 and E_1, respectively, in $W^f_{\bar{k},\rho}$. It is clear that we have in this way constructed a sequence $\tilde{\gamma}_0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{N+2}$ of alternatingly k-expanding and k-contracting fibers which have successive intersections in

$W^f_{\bar{k},\rho}$ and such that $x \in \tilde{\gamma}_0$, $x' \in \tilde{\gamma}_{N+2}$.

This completes the proof since we can choose

$$V^f = \bigcup_{n=1}^{\infty} V^f_{1/n} .$$